Invariance and Stability of the Profile Equations of Geometric Optics

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We dedicate this article to Peter Lax whose mathematical ideas have guided and inspired us throughout our careers and who has served as as model for a life well lived.

Abstract. The profile equations of geometric optics are described in a form invariant under the natural transformations of first order systems of partial differential equations. This allows us to prove that various strategies for computing profile equations are equivalent. We prove that if L generates an evolution on L^2 the same is true of the profile equations. We prove that the characteristic polynomial of the profile equations is the localization of the characteristic polynomial of the background operator at $(y, d\phi(y))$ where ϕ is the background phase. We prove that the propagation cones of the profile equations are subsets of the propagation cones of the background operator.

§1. Introduction.

In this paper we revisit the profile equations of geometric optics. In our earlier work we avoided the language of vector bundles and concentrated on the family of symmetric hyperbolic systems. There were three reasons. First, the symmetric systems dominate in applications. Second there are users in the scientific community for whom vector bundles are not familiar. Finally, the symmetric category has certain simplifying features.

For a system of first order operators $L(y, \partial)$ with principal symbol, L_1 and phase satisfying the eikonal equation, $L_1(y, d\phi(y)) = 0$, the leading profile is naturally a section of the bundle $\ker L_1(y, d\phi(y))$. The bundle $\operatorname{rg} L_1(y, d\phi(y))$ and quotient bundles by the kernel and range play central roles. It is no extra work to consider operators which are maps of vector bundles. The resulting economy of notation has advantages similar to considering linear transformations of vector spaces in lieu of matrices. In local coordinates and with a smooth choice of bases in the fibers, operators on bundles take the familiar form of differential expressions with smooth matrix valued coefficients.

For equation from physics and from geometry, the unknown v and the quantity Lv are often quantities of different natures. It is natural to have different spaces for the unknowns v and the the values of Lv.

In our treatments (with J.-L. Joly) of geometric optics for symmetric hyperbolic systems we lean heavily on spectral projectors associated with the symbol of the operators. This approach has at least three weaknesses;

1. The symmetry property is not preserved under multiplication of the system by an invertible matrix valued function.

2. In practice one often computes the profile equations by hand and without reference to projectors or the symmetric structure. It is important to show that this proceedure leads always to the same result.

3. There are hyperbolic systems which generate good L^2 evolutions and are neither symmetric nor symmetrised by multiplying by a matrix valued function.

The second item is the key hint. If there is an answer which is independent of the choices that one makes, then there should be a choice independent description. In §2, we present such a construction which has the advantage of working in general thereby overcoming defect **3**.

In §3 we introduce the operators L which generate evolutions on L^2. For operators which generate evolutions on L^2 and phases satisfying the smooth variety hypothesis, we prove in §4, in that the profile equation is essentially a transport equation at the group velocity. The smooth variety hypothesis is rarely violated in practice.

In §5, we show that when the original equations are symmetrisable in the sense of Friedrichs and the phase satisfies the constant rank hypothesis which is weaker than the smooth variety hypothesis, then the profile operator acting on sections of $\ker L_1(y, d\phi(y))$ is also symmetrisable.

For systems generating an evolution in L^2, and phases satisfying the constant rank hypothesis we prove in §6 that the profile equations generate an evolution on L^2 sections of $\ker L_1(y, d\phi(y))$. In other words, the profile equations inherit this strong type of hyperbolicity from the original operator L.

When the constant rank hypothesis is satisfied, we prove in §7, that the characteristic polynomial of the profile equations is the localisation of the characteristic polynomial of L. It follows that its propagation cones at each point are subsets of the propagation cones of the background hyperbolic system. In particular the domains of influence of the profile equations are subsets of the domains for the background equations. This was known in the symmetric case by a special argument.

The central ideas and methods of this article are readily adapted to nonlinear, resonant, dispersive, and diffractive geometric optics ([JMR1,2], [DR]. [DJMR]). We present the one phase, nondispersive, linear theory to highlight the new elements. We treat situations including multiple roots and most importantly variable multiplicity. It is in the latter situation that the profile equations can be nontrivial hyperbolic systems as in conical refraction.

§2. The profile equations, invariant description.

After recalling the WKB computation of asymptotic solutions this section has two results. The first extracts a pair of equations for the n^{the} coefficient a_n using the natural quotient spaces and their canonical projections. The second result shows that the standard way to derive equations for a_n involving *ad hoc* choices yield answers independent of the *ad hoc* choices.

Suppose that $L(y,\partial)$ is a first order linear system of partial differential operators with smooth coefficients mapping sections of a vector bundle V to sections of another vector bundle, W. Both bundles have fiber dimension N and lie above the same open set $\Omega \subset \mathbb{R}^{1+d}_{t,x}$. Even if one considers the case of vector spaces V and W, one immediately finds bundles with nonconstant fibers playing a central role. The issues we address are local in y so it suffices to consider bundles over a fixed open subset of space time.

Denote $y = (t,x)$ and by V_y and W_y the fibers over the point y. The principal symbol $L_1(y,\eta)$ is a linear map from V_y to W_y which is a linear function of η. The variable η belongs to the cotangent space $(\mathbb{R}^{1+d}_y)^*$.

Assumption. *Suppose that ϕ is a smooth real valued function on Ω satisfying the* **eikonal equation**

$$\forall y \in \Omega, \quad d\phi(y) \neq 0, \quad \text{and}, \quad \det L_1(y, d\phi(y)) \;=\; 0\,. \tag{2.1}$$

We impose the **constant rank hypothesis**,

$$k \;:=\; \dim\Big(\ker L_1(y, d\phi(y))\Big) \;\text{ is independent of }\; y \in \Omega\,. \tag{2.2}$$

Notation. *For smooth functions $f(\varepsilon,y)$ and $g(\varepsilon,y)$ the notation $f \sim g$ means that they have the same Taylor expansion at $\varepsilon = 0$. The notation*

$$f(\varepsilon,y) \;\sim\; a_0(y) + \varepsilon a_1(y) + \cdots,$$

means that the right hand side is the Taylor expansion of f at $\varepsilon = 0$. The notation $f = O(\varepsilon^\infty)$ is synonymous with $f \sim 0$.

Seek asymptotic solutions,

$$v^\varepsilon \;=\; e^{i\phi(y)/\varepsilon}\, V(\varepsilon,t,x), \qquad V(\varepsilon,t,x) \;\sim\; a_0(t,x) + \varepsilon\, a_1(t,x) + \cdots\,, \qquad L\, v^\varepsilon \sim 0\,. \tag{2.3}$$

Compute

$$L(y,\partial)\Big(e^{i\phi(y)/\varepsilon}\, V(\varepsilon,t,x)\Big) \;=\; e^{i\phi(y)/\varepsilon}\, L\Big(y, \partial + \frac{id\phi(y)}{\varepsilon}\Big) V\,.$$

$$\begin{aligned} L\Big(y, \partial + \frac{id\phi(y)}{\varepsilon}\Big) V \;&\sim\; L\Big(y, \partial + \frac{id\phi(y)}{\varepsilon}\Big)\Big(a_0(t,x) + \varepsilon\, a_1(t,x) + \cdots\Big) \\ &\sim\; \frac{L_1(y, d\phi(y)) a_0}{\varepsilon} \;+\; \sum_{n=0}^{\infty} \varepsilon^n \Big(L_1(y, id\phi(y))\, a_{n+1} \;+\; L(y,\partial)\, a_n\Big)\,. \end{aligned}$$

The residual is $O(\varepsilon^\infty)$ if and only if the coefficient of ε^n vanishes for all $n \geq -1$.

Proposition 2.1. *With $v^\varepsilon(y)$ defined by (2.3) one has $L\, v^\varepsilon = O(\varepsilon^\infty)$ if and only if,*

$$L_1(y, id\phi(y))\, a_n \;+\; L(y,\partial)\, a_{n-1} \;=\; 0\,, \qquad \text{for} \quad n = 0, 1, \ldots\,, \tag{2.4}$$

where by convention $a_{-1} := 0$.

When the coefficients a_n satisfy the equations (2.4), $L\,v^\varepsilon$ is infinitely small in ε. The leading coeffient a_0 gives the envelope of the wave packet $e^{i\phi(y)/\varepsilon}\,a_0(y)$. The coefficients a_n are called the **profiles** of v^ε. The profile a_n appears in two equations, (2.4) and

$$L_1(y, id\phi(y))\,a_{n+1} \;+\; L(y,\partial)\,a_n \;=\; 0. \tag{2.5}$$

The determination of the a_n is recursive. Given a_{n-1} with

$$L(y,\partial)\,a_{n-1}(y) \;\in\; \mathrm{rg}\,L_1(y, d\phi(y))\,, \qquad \text{(so (2.4) is not inconsistent)},$$

the determination of a_n goes as follows.

• Equation (2.4) determines $a_n(y)$ modulo the kernel of $L_1(y, d\phi(y))$.

• Projecting (2.5) along the range of L_1 gives an evolutionary differential equation expected to complete the determination of a_n and guaranteeing $L(y,\partial)\,a_n(y) \in \mathrm{rg}\,L_1(y, d\phi(y))$ for the next step.

• The exception to this rule is the leading coefficient a_0 for which one has $a_{-1} = 0$ which trivially satisfies $L(y,\partial)\,a_{-1}(y) \in \mathrm{rg}\,L_1(y, d\phi(y))$.

This section is devoted to analysing the "projecting" in the middle bullet. Projecting along the range onto any compliment will do, but involves an arbitrary choice. More generally, one can multiply by any matrix of rank equal to $\dim\ker L_1(y, d\phi(y))$ and annihilating the range. In the symmetric case, [JMR1,2,3, etc.] have systematically taken the orthogonal projection along the range onto the kernel. This has three advantages,

• It is well defined. That is, it requires no arbitrary choices.

• It has norm equal to one.

• It is self adjoint.

The latter two properties lead to natural *a priori* estimates.

There is an analogous map which depends neither on symmetry nor on *ad hoc* choices. It is the natural projection from W to $W/\mathrm{rg}\,L_1(y, d\phi(y))$. Using this projector yields a construction robust under the three natural transformations preserving first order linear systems, namely,

1. Smooth change of basis in V.

2. Smooth change of basis in W.

3. Mulitplication of L by an invertible matrix valued function of y.

It allows us to prove, in Proposition 2.2, that the profile equations are independent of the projection method that one employs.

Introduce four natural vector bundles over Ω,

$$\ker L_1(y, d\phi(y))\,, \qquad \mathrm{rg}\,L_1(y, d\phi(y))\,, \qquad \frac{V}{\ker L_1(y, d\phi(y))}\,, \quad \text{and,} \quad \frac{W}{\mathrm{rg}\,L_1(y, d\phi(y))}\,.$$

Denote by $i_{\rm rg}$ and i_{ker} the inclusions

$$\mathrm{rg}\, L_1(y, d\phi(y)) \quad \stackrel{i_{\rm rg}}{\longrightarrow} \quad W\,, \qquad \ker L_1(y, d\phi(y)) \quad \stackrel{i_{\rm ker}}{\longrightarrow} \quad V\,.$$

and by π_{rg} and π_{ker} the natural projections,

$$V \quad \stackrel{\pi_{ker}}{\longrightarrow} \quad \frac{V}{\ker L_1(y, d\phi(y))}\,, \qquad W \quad \stackrel{\pi_{rg}}{\longrightarrow} \quad \frac{W}{\mathrm{rg}\, L_1(y, d\phi(y))}.$$

To derive the profile equations use the two exact sequences,

$$0 \;\rightarrow\; \ker L_1(y, d\phi(y) \;\stackrel{i_{\rm ker}}{\longrightarrow}\; V \;\stackrel{L_1(y,d\phi(y)}{\longrightarrow}\; W \;\stackrel{\pi_{rg}}{\longrightarrow}\; \frac{W}{\mathrm{rg}\, L_1(y, d\phi(y))} \;\rightarrow\; 0\,,$$

$$0 \;\rightarrow\; \ker L_1(y, d\phi(y)) \;\stackrel{i_{\rm ker}}{\longrightarrow}\; V \;\stackrel{\pi_{ker}}{\longrightarrow}\; \frac{V}{\ker L_1(y, d\phi(y))} \;\rightarrow\; 0\,.$$

Even when the original differential operator $L(y, \partial)$ acts on vector spaces, the quotient bundles arise naturally. And, the quotient bundles generically have nonconstant fibers since $\ker L_1(y, d\phi(y))$ and $\mathrm{rg}\, L_1(y, d\phi(y))$ typically vary from point to point.

The bundle maps

$$\pi_{ker} \;:\; V \;\rightarrow\; \frac{V}{\ker L_1(y, d\phi(y))}\,, \qquad \text{and,} \qquad L_1(y, d\phi(y)) \;:\; V \;\rightarrow\; \mathrm{rg}\, L_1(y, d\phi(y)),$$

are surjective and have the same kernels. Therefore, there is a uniquely determined invertible bundle map,

$$\tilde{L}_1(y) \;:\; \frac{V}{\ker L_1(y, d\phi(y))} \quad \rightarrow \quad \mathrm{rg}\, L_1(y, d\phi(y))$$

yielding a commutative diagram,

$$\begin{array}{ccc} V & \stackrel{L_1(y,d\phi(y))}{\longrightarrow} & \mathrm{rg}\, L_1(y, d\phi(y)) \\ \pi_{ker} \searrow & & (\tilde{L}_1)^{-1} \swarrow\!\nearrow \tilde{L}_1 \\ & \frac{V}{\ker L_1(y,d\phi(y))} & \end{array}$$

Theorem 2.2. *The profile equations (2.4) are satisfied if and only if for* $n = 0, 1, \dots,$

$$\pi_{rg}\, L(y, \partial)\, a_n \;=\; 0, \qquad \text{and,} \qquad \pi_{ker} a_n \;=\; -(\tilde{L}_1(y))^{-1} L(y, \partial) a_{n-1}\,, \tag{2.6}$$

with the convention $a_{-1} = 0$.

Remark. In order for the second equation in (2.6) to make sense, one must know that $L(y,\partial)a_{n-1}$ takes values in $\operatorname{rg} L_1(y,d\phi(y))$. This is equivalent to the first equation in (2.6) for the the index $n-1$.

Proof. If the equations of Proposition 2.1 are satisfied, then the first equation in (2.6) is proved by applying π_{rg} to (2.5).

Equation (2.4) implies that $L(y,\partial)a_{n-1}$ lies in $\operatorname{rg} L_1(y,d\phi(y))$. The second equation in (2.6) is proved by applying $\tilde L_1(y)^{-1}$ to (2.4).

This proves that the equations (2.6) are necessary.

Conversely, If the equations (2.6) are satisfied, then the second equation of (2.6) with $n=0$ implies the case $n=0$ of (2.4)

To prove the case $n\geq 1$ of (2.4), multiply the second equation of (2.6) by $\tilde L_1(y)$ to prove (2.4). ∎

We next verify that the equations (2.6) are well behaved under the three natural transformations of the operator L. The equations (2.6) are expressed in term of bundle maps. The invariance under change of bases in V_y and W_y is automatic.

The behavior under multiplication of L by a a matrix is only slightly harder. If $M(y)$ is a smooth invertible bundle map from W to Y then replacing the operator L from V to W by the operator ML from V to Y yields equivalent differential equations. The map $M(y)$ induces an invertible bundle map

$$\frac{W_y}{\operatorname{rg} L_1(y,d\phi(y))} \quad\mapsto\quad \frac{Y_y}{\operatorname{rg}(ML)_1(y,d\phi(y))}.$$

defined by,

$$w+\operatorname{rg} L_1(y,d\phi(y)) \;\mapsto\; Mw + M\operatorname{rg} L_1(y,d\phi(y)) \;=\; Mw+\operatorname{rg}(ML)_1(y,d\phi(y))\,.$$

This bundle map is denoted $M(y)$ with little risk of confusion. With this notation, the preceding identity asserts that

$$M\big(w \bmod rg\, L_1(y,d\phi(y))\big) \;=\; \big(Mw\big) \bmod rg\,(ML)_1(y,d\phi(y))\,.$$

This identity is equivalent to,

$$M\pi_{rg\,L_1(y,d\phi(y))} \;=\; \pi_{rg\,(ML)_1(y,d\phi(y))}\,M\,.$$

This proves the following. *The operator, $\pi_{rg\,(ML)_1(y,d\phi)}\,ML$, associated to ML is equal to M times the operator, $\pi_{rg\,L_1}\,L$, associated to L. In particular the first equations from the pair (2.6) are either satisfied for both L and ML or for neither.*

The map π_{rg} is natural from the mathematical point of view. On the other hand, it is not the sort of object a scientist would normally employ. In scientific practice, what is usually done is to write down the N scalar equations represented by (2.5). One then seeks linear combinations of these equations which do not contain any a_{n+1} terms thereby yielding equations for a_n alone. This corresponds to to choosing a smoothly varying basis

$$\ell_1(y)\,,\ \ell_2(y)\,,\ \ldots\,,\ \ell_k(y)$$

of the k dimensional annihilator of $\mathrm{rg}\, L_1(y, d\phi(y))$. Applying $\ell_j(y)$ to (2.5) shows that

$$\ell_j(y)\Big(L(y,\partial)\, a_n\Big) \;=\; 0, \qquad j = 1, 2, \ldots, k\,.$$

These k scalar equations replace the first equation from (2.6). Defining

$$K(y)\, w \;:=\; \Big(\ell_1(y)(w), \ell_2(y)(w), \ldots, \ell_k(y)(w)\Big) \;\in\; \mathbb{C}^k,$$

shows that this practical construction is a special case of the next proposition which proves that replacing (2.6) in this way yields an equivalent system of partial differential equation. In the same way, the projection algorithms of [Me] which are more general than the spectral projections of [JMR] yield also equivalent descriptions.

Proposition 2.3. *Suppose that*

$$K \;:\; W \;\to\; Z$$

is a smooth bundle map defined for $y \in \Omega$. Suppose that K satisfies for all $y \in \Omega$,

$$K(y)\Big(\mathrm{rg}\, L_1(y, d\phi(y))\Big) \;=\; 0, \qquad \mathrm{rank}\, K(y) \;=\; k\,.$$

Then each of the equations

$$\pi_{rg}\, L(y,\partial)\, a \;=\; 0, \qquad \text{and,} \qquad K\, L(y,\partial)\, a \;=\; 0\,,$$

implies the other

Proof. Replacing Z by the subbundle $\mathrm{rg}\, K$ reduces to the case of maps which are surjective.

In that surjective case, K and π_{rg} have the same kernels and are surjective. Therefore, there is a unique invertible bundle map

$$M \;:\; \frac{W}{\mathrm{rg}\, L_1(y, d\phi(y))} \;\to\; Z,$$

so that the following diagram commutes,

$$W \qquad \xrightarrow{\pi_{\mathrm{rg}}} \qquad \frac{W}{\mathrm{rg}\, L_1(y,d\phi(y))} \quad \to 0$$

$$K \searrow \qquad\qquad M^{-1} \nearrow\!\!\!\swarrow M$$

$$Z$$

$$\searrow$$

$$0$$

Therefore,

$$K\,L(y,\partial) \;=\; M\,\pi_{rg}\,L(y,\partial)\,, \qquad \pi_{rg}\,L(y,\partial) \;=\; M^{-1}K\,L(y,\partial).$$

The Proposition follows. ∎

The profile equations lead to initial value problems. The coefficient a_n is determined by solving

$$\pi_{rg}\,L(y,\partial)\,a_n \;=\; 0, \qquad \pi_{ker}\,a_n \;=\; -(\tilde{L}_1(y))^{-1}L(y,\partial)a_{n-1}\,. \tag{2.7}$$

Once a_{n-1} is known one can choose a section H_n of V so that

$$\pi_{ker}\,H_n \;=\; -(\tilde{L}_1(y))^{-1}L(y,\partial)a_{n-1}\,.$$

Then $a_n - H_n$ takes values in $\ker L_1(y,d\phi(y))$ so there is a unique smooth section g_n of $\ker L_1(y,d\phi(y))$ so that

$$a_n \;=\; i_{\ker}(g_n) \;+\; H_n\,.$$

The number of equations is equal to $\dim\ker L_1(y,d\phi(y))$ which is the number of unknowns represented by g. The initial data for a_n is equivalent to prescribing the values of $g_n(t_0,x)$. In §5, we prove in great generality that the initial value problem for g_n is a good hyperbolic system.

Note that the value of g_n depends on the choice H_n. A convenient way is to choose a subbundle $J\subset V$ complementary to $\ker L_1(y,d\phi(y))$,

$$J_y \;\oplus\; \ker L_1(y,d\phi(y)) \;=\; V_y\,. \tag{2.8}$$

Denote by $P(y)$ the projection along J_y onto $\ker L_1(y,d\phi(y))$. Then in the above construction one can take

$$H_n \;=\; -(I-P)\,L_1(y)^{-1}L(y,\partial)\,a_{n-1}\,, \qquad g_n = P\,a_n\,. \tag{2.9}$$

With this choice the initial data that are required are the values of $P\,a_n|_{t=t_0}$. When there is a natural hermitian structure on V as in the symmetrisable case described in §4, one can choose J to be the orthogonal complement and P the orthogonal projection.

§3. Hyperbolic operators generating evolutions on L^2.

We introduce a family of hyperbolic operators for which the profile equations yield well posed initial value problems.

Denote by $L_1(t,x,\tau,\xi) = L_1(y,\eta)$ the prinicipal symbol. We assume that the Cauchy problem for L is well posed. This implies that the planes $t=$ constant are noncharacteristic. For, if there were a noncharacteristic point, the differential equation $Lu=0$ would represent a nontrivial constraint on the values of admissible initial data. In addition, when the Cauchy problem is at least weakly well posed, the theorem of Lax and Mizohata ([L], [M]) implies that one has hyperbolicity in the sense that for all real ξ and $y\in\Omega$ the roots τ of the equation

$$\det L_1(y,\tau,\xi) \;=\; 0, \tag{3.1}$$

are all real.

The maximal sound speed measured using the Euclidean norm in x (and dual norm in ξ) is defined by [1],

$$\sigma \;:=\; \sup\Big\{|\tau| \;:\; \exists (y,\xi)\in\Omega\times\mathbb{R}^d,\;\; |\xi|=1,\; \text{and},\;\; \det L_1(y,\tau,\xi) \;=\; 0\Big\}\,. \tag{3.2}$$

Definitions. *A* **forward influence cone** *is a subset of* Ω *of the form*

$$\Big\{|x-x_0| \;\le\; \sigma(t-t_0), \qquad t_0\le t\le t_1\Big\}\,. \tag{3.3}$$

The cone opens toward the future. A **backward influence cone** *is defined similarly but opening toward the past. For any such cone* Γ, $\Gamma(t)$ *denotes its* **section** *at time* t.

It is important that part of the definition is that the cone lies inside Ω where the phase ϕ is defined.

Defintion. *The operator* $L(y,\partial)$ *generates an evolution on* L^2 *if for every forward influence cone* Γ *there is a constant* C *so that for any* $g\in L^2(\Gamma(t_0))$ *supported in the interior of* $\Gamma(t_0)$ *there is a unique* $u\in L^\infty([t_0.t_1]\,;\,L^2(\Gamma(t)))$ *vanishing on a neighborhood of the lateral boundaries of* Γ *and solving the initial value problem,*

$$L\,v \;=\; 0 \;\text{ in }\; \Gamma \qquad v \;=\; g \;\text{ on }\; \Gamma(t_0)\,. \tag{3.4}$$

In addition there is a constant $C=C(t_0,t_1)$ *independent of* g *so that,*

$$\sup_{t_0\le t\le t_1} \|v(t)\|_{L^2(\Gamma(t)} \;\le\; C\,\|g\|_{L^2(\Gamma(t_0))}\,. \tag{3.5}$$

Remarks. 1. By a Duhamel construction, one solves inhomogeneous equations and the solutions satisfy the estimate

$$\sup_{t_0\le t\le t_1} \|v(t)\|_{L^2(\Gamma(t)} \;\le\; C\Big(\|g\|_{L^2(\Gamma(t_0))} \;+\; \int_{t_0}^{t_1} \|Lv(t)\|_{L^2(\Gamma(t))}\,dt\Big)\,. \tag{3.6}$$

A perturbation argument shows that changing the lower order terms does not affect whether one generates an evolution on L^2.

2. Using the weak=strong result of Friedrichs (see [Fr1], [LP]) together with (3.6) one shows that the solution is continous with values in $L^2(\Gamma(t))$ so the trace at time t is a well defined element of $L^2(\Gamma(t))$.

[1] The Euclidean norm can be avoided by using the more precise ideas associated with **influence curves** defined as lipschitzean curves whose tangents belong to the forward propagation cones of the constant coefficient hyperbolic operators $L(\underline{y},\partial)$ (see [Lr], [JMR4], [R2]).

3. Finite speed of propagation is built into this definition since the property is supposed for every forward influence cone.

4. *The frozen constant coefficient operators $L_1(\underline{y}, \partial)$ are generators of strongly continuous semigroups on $L^2(\mathbb{R}^d)$ with bounds uniform for $\underline{y} \in \Gamma$. That is,*

$$\exists C, \quad \forall \underline{y} \in \Gamma \quad \forall g \in L^2(\mathbb{R}^d\,;\, V_{\underline{y}})\,,$$

the constant coefficient Cauchy problems

$$L_1(\underline{y},\ \partial)v \ = \ 0, \qquad v|_{t=\underline{t}} \ = \ g, \tag{3.7}$$

are uniuqely solvable and the solutions satisfy

$$\sup_{t\in\mathbb{R}} \ \|v(t)\|_{L^2(W_{\underline{y}})} \ \leq \ C\,\|v(\underline{t})\|_{L^2(\mathbb{R}^d\,;\,W_{\underline{y}})}\,. \tag{3.8}$$

Sketch of Proof of 4. Blow up at $\underline{y}$. The solution v of (3.7) is constructed as a weak star limit in $L^\infty([\underline{t}, T]\ \ L^2(\mathbb{R}^d))$ of

$$v^\varepsilon \ := \ u^\varepsilon(\underline{t} + \varepsilon(t - \underline{t})\,,\, \underline{x} + \varepsilon(x - \underline{x})),$$

where u^ε is the solution of

$$L_1(y, \partial)u^\varepsilon \ = \ 0, \qquad u^\varepsilon\big|_{t=\underline{t}} \ = \ g\Big(\frac{x - \underline{x}}{\varepsilon}\Big)\,.$$

For ε small, the family v^ε is bounded in $L^\infty([\underline{t}, T]\,;\, L^2(\mathbb{R}^d))$, and satisfies

$$L_1(\underline{y} + \varepsilon(y - \underline{y}), \partial)\, v^\varepsilon \ = \ 0, \qquad v^\varepsilon\big|_{t=\underline{t}} = g\,.$$

The limit of any weak star convergent subsequence solves (3.7) with estimate (3.8). Uniqueness follows from Hölmgren's Theorem. ∎

5. When L generates an evolution on L^2, the hyperplanes $t = \textit{constant}$ are noncharacteristic so $A_0(y)$ is for each y an invertible linear map $V_y \to W_y$. Therefore, $A_0^{-1}L$ is a differential operator mapping sections of V to sections of V. It is automatic that $A_0^{-1}L$ generates an evolution on L^2 since the solutions of $Lv = 0$ and $A_0^{-1}\,Lv = 0$ are the same. Then **4.** implies that for all real τ, ξ, the matrix $A_0^{-1}\,L_1(y, \tau, \xi)$ is similar to a real diagonal matrix. In particular for characteristic (y, η),

$$\ker\,(A_0^{-1}L)_1(y, \eta) \ \oplus \ \mathrm{rg}\,(A_0^{-1}L)_1(y, \eta) \ = \ V_y\,. \tag{3.9}$$

6. *If L generates an evolution on L^2 then the transposed operator on the dual bundles*

$$L(y, \partial)^\dagger \,:\, W^\dagger \ \to \ V^\dagger$$

generates an evolution on L^2 with the same choice of σ

Sketch of proof of 6. The characteristic polynomial of the dual (see §7) differs by a constant factor from that of L proving the equality of the σ.

Existence for the dual problem follows from the esimtate (3.6) for the L by a standard duality.

Uniqueness and finite speed for the dual problem follow by a Hölmgren type argument from the existence and finite speed for L. ∎

§4. Transport at the group velocity.

Definition. *The phase ϕ satisfies the* **smooth variety hypothesis** *if in a conic neighorhood of the points $(y, d\phi(y))$ the characteristic variety of L is a smooth graph, $\{(y;\tau,\xi) : \tau = \tau(y,\xi)\}$. For such a ϕ, define the* **group velocity associated to** ϕ *by,*

$$\mathbf{v}(y) \ := \ -\nabla_\xi \tau(y, d\phi(y))\,. \tag{4.1}$$

When the smooth variety hypothesis is satisfied, the profile equations of geometric optics have principal part which is the directional derivative at the group velocity.

Proposition 4.1. *Suppose that the phase ϕ satisfies the constant rank hypothesis.*

i. *If $L_1(y,\partial)$ generates an evolution on L^2, then for any $(y;\tau,\xi)$ in the characteristic variety,*

$$\pi_{rg}(y,\tau,\xi)\, A_0(y)\, i_{ker}(y,\tau,\xi) \ : \ \ker L_1(y,\tau,\xi) \quad \to \quad \frac{W}{\operatorname{rg} L_1(y,\tau,\xi)} \tag{4.2}$$

is invertible.

ii. *If in addition, ϕ satisfies the smooth variety hypotheses, then the profile operator $\pi_{rg}\, L(y,\partial)\, i_{ker}$ differs from*

$$\pi_{rg}\, A_0\, i_{ker}\Big(\partial_t + \mathbf{v}.\partial_x\Big) \tag{4.3}$$

by an operator of order zero.

Remarks. 1. Part **i.** implies that when the constant rank hypothesis is satisfied, the hyperplanes $\{t = const.\}$ are noncharacteristic for the profile operator, $\pi_{rg}\, L(y,\partial)\, i_{ker}$.

2. Part **ii.** is equivalent to the assertion that the prinicipal symbol of $\pi_{rg}\, L(y,\partial)\, i_{ker}$ is equal to

$$\pi_{rg}\, A_0\, i_{ker}\Big(i\tau + \sum_{j=0}^{d} i\xi_j\Big)\,.$$

Proof. i. It is sufficient to show that it is injective. Suppressing the y dependence, it suffices to show that if

$$r \in \ker L_1(\tau,\xi) \qquad \text{and} \qquad \pi_{rg}(\tau,\xi) A_0 r \ = \ 0, \tag{4.4}$$

then $r = 0$.

Compute

$$L_1(\tau,\xi) \;=\; \big(L_1(\tau,\xi)\,A_0^{-1}\big)A_0\,,\qquad L_1(\tau,\xi)\,A_0^{-1} \;=\; A_0\big(A_0^{-1}L_1(\tau,\xi)\big)A_0^{-1}\,.$$

The second similarity together with (3.9) implies that

$$\ker L_1(\tau,\xi)A_0^{-1} \;\oplus\; \mathrm{rg}\, L_1(\tau,\xi)A_0^{-1} \;=\; W_y\,.$$

The conditions of (4.4) are equivalent to,

$$A_0\, r \;\in\; \mathrm{rg}\, L_1(\tau,\xi)\, A_0^{-1} \;\;\cap\;\; \ker L_1(\tau,\xi)\, A_0^{-1} \;\;=\;\; \{0\}\,.$$

Therefore, $A_0 r = 0$, so $r = 0$.

ii. The coefficients of ∂_t and ∂_j in the profile operator are respectively

$$\pi_{rg}(y,\tau(y,\xi),\xi)\, A_0(y)\, i_{ker}(y,\tau(y,\xi),\xi), \quad \text{and,} \quad \pi_{rg}(y,\tau(y,\xi),\xi)\, A_j(y)\, i_{ker}(y,\tau(y,\xi),\xi).$$

The assertion is equivalent to the identities,

$$\pi_{rg}\, A_j\; i_{ker} \;=\; -\frac{\partial \tau}{\partial \xi_j}\, \pi_{rg}\, A_0\; i_{ker}\,, \qquad j = 1,\dots,d\,. \tag{4.6}$$

Reason for y fixed and suppress the y dependence for ease of reading. First show that $\ker L_1(\tau(\xi),\xi)$ has dimension independent of ξ. The kernel is the eigenspace of the diagonalisable transformation $A_0^{-1}L_1(0,\xi)$ corresponding to the eigenvalue $-\tau(\xi)$. The smooth variety hypothesis implies that for δ small, the dimension is equal to

$$\mathrm{trace}\; \frac{1}{2\pi i}\; \oint_{|z+\tau(\xi)|=\delta} \Big(z - A_0^{-1}L_1(0,\xi)\Big)^{-1}\, dz\,.$$

It is a continuous integer valued function, hence locally constant.

Choose a smooth local basis

$$w_1(\xi)\,,\dots,\, w_k(\xi) \qquad \text{of} \qquad \ker L(\tau(\xi),\xi)\,.$$

Differentiate the identity $L(\tau(\xi),\xi)w_m(\xi) = 0$ with respect to ξ_j to find

$$\Big(A_0\frac{\partial \tau}{\partial \xi_j} \;+\; A_j\Big)w_m \;+\; L(\tau(\xi),\xi)\,\frac{\partial w_m}{\partial \xi_j} \;=\; 0\,.$$

Multiply by π_{rg} to find

$$\pi_{rg}\Big(A_0\frac{\partial \tau}{\partial \xi_j} \;+\; A_j\Big)w_m \;=\; 0, \qquad j = 1,\dots,d, \quad m = 1,\dots,k\,. \tag{4.7}$$

Equation (4.7) is equivalent to (4.6). ∎

§5. The profile equation in the symmetrisable case.

In this section we show that if the operator L is symmetrisable in the sense of Friedrichs [F2], then the profile operator $\pi_{rg}\, L(y,\partial)\, i_{ker}$ is also symmetrisable. This argument is a variant of those used by [JMR1,2,3] to show that the profile equations yield well posed initial value problems. It is pertinent when the smooth variety hypothesis is violated, since when the variety is smooth the profile equations are transport equations that are essentially trivial. A classic nontrivial example is conical refraction (see [Lu],[JMR2]).

Multiplying L by $A_0^{-1}(y)$ achieves two related goals. First it reduces to an operator with $V = W$. And second, the coefficient of ∂_t is the identity transformation on V_y.

Definition. *The operator L is symmetrisable in the sense of Friedrichs if and only if there is a smooth map $y \mapsto \gamma(y)$ with $\gamma(y)$ a scalar product on V_y (equivalently, the vector bundle V has a hermitian structure) so that $A_0^{-1}(y)A_j(y)$ is $\gamma(y)$-hermitian for all y.*

The standard energy estimate by integration by parts shows that this is equivalent to the existence of a $C > 0$ for each forward influence cone Γ, so that for all smooth solutions of $Lv = 0$ supported in Γ, one has,

$$\frac{d}{dt}\int (v(t,x)\,,\, v(t,x))_{\gamma(t,x)}\; dx \;\le\; C \int (v(t,x)\,,\, v(t,x))_{\gamma(t,x)}\; dx.$$

Equivalently, writing

$$A_0^{-1}\, L = \partial_t + G(y,\partial_x)$$

the operator G satisfies

$$G \;+\; G^* \;=\; \text{order zero}$$

where the adjoint is taken with respect the time dependent scalar product,

$$\int (\;\cdot\;,\;\cdot)_{\gamma(t,x)}\; dx\,,$$

on square integrable sections of V.

If $\phi(y)$ satisfies the eikonal equation and the constant rank hypothesis, (3.9) allows us to introduce **the spectral projection** $\Pi(y))$ along the range of $A_0^{-1}(y)L_1(y,d\phi(y))$ onto its kernel.

The scalar product $\gamma(y)$ induces a scalar product on

$$\frac{V}{\mathrm{rg}\,(A_0^{-1}(y)L_1)(y,d\phi(y))}$$

and a smooth family of unitary maps $U(y)$,

$$\frac{V}{\mathrm{rg}\,(A_0(y)^{-1}L_1)(y,d\phi(y))} \quad \xrightarrow{U(y)} \quad \Big(\mathrm{rg}\, A_0(y)^{-1}L_1(y,d\phi(y)\Big)^{\perp_{\gamma(y)}}\,.$$

In addition,

$$\Big(\mathrm{rg}\, A_0(y)^{-1} L_1(y, d\phi(y)\Big)^{\perp_{\gamma(y)}} \;=\; \ker\,(A_0(y)^{-1}L_1)(y, d\phi(y)) \;=\; \ker L_1(y, d\phi(y)).$$

When L is symmetrisable, $A_0^{-1}(y) L_1(y, d\phi(y))$ is $\gamma(y)$ hermitian so $\Pi(y)$ is the $\gamma(y)$-orthogonal projection of V_y onto $\ker L_1(y, d\phi(y))$. One has,

$$U(y)\, \pi_{\mathrm{rg}\, A_0^{-1} L_1(y, d\phi(y))} \;=\; \Pi(y)\,.$$

Proposition 5.1. *Suppose that L is symmetrisable in the sense of Friedrichs and ϕ is a phase satisfying the eikonal equation and the constant rank hypothesis.*
i. *The following two differential operators on the vector bundle* $\ker L_1(y, \partial))$ *are equal,*

$$\Pi\, (A_0^{-1} L)(y, \partial)\, \Pi\, i_{\ker} \;=\; U\, \pi_{rg}\, A_0^{-1} L(y, \partial)\, i_{\ker}\,.$$

ii. *The operator* $\Pi(y)\, (A_0^{-1}(y) L)(y, \partial)\, \Pi(y)$, *mapping sections of* $\ker L_1(y, d\phi(y))$ *to sections of the same bundle, is symmetrisable in the sense of Friedrichs.*

Proof. i. Follows from the identities, $\Pi\, i_{\ker} = \; i_{\ker}$ and $U\, \pi_{rg} = \Pi$.
ii. Since the projectors $\Pi(y)$ and $A_0^{-1} A_j$ are $\gamma(y)$-hermitian, it follows that the coefficient matrices

$$\Pi(y) A_0^{-1}(y) A_j(y) \Pi(y)$$

are $\gamma(y)$-hermitian. ∎

We do not know a a simple algebraic proof like this that the profile equation is well posed for more general problems. For example for systems symmetrised by a a family of pseudodifferential operator in x. The next example shows that it is not simply a question of having a good hyperbolic operator sandwiched between two nice projectors. In the next section we give a proof valid under very general hypotheses.

Example. Consider

$$A \;:=\; \begin{pmatrix} 0 & -\Lambda \\ 1 & 0 \end{pmatrix}, \qquad \Lambda > 0, \qquad \mathrm{spec}\, A \;=\; \big\{ \pm i \Lambda^{1/2} \big\}, \qquad \sup_{\xi \in \mathbb{R}} \| e^{\xi A} \| \;<\; \infty.$$

The projector $\Pi \;=\; \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, is orthogonal with respect to the standard scalar product on $\mathbb{C}^2$ with range equal to $\mathbb{C}(1,1)$. The sandwiched matrix satisfies

$$\Pi\, A\, \Pi \;=\; \frac{1-\Lambda}{2}\, \Pi, \quad \mathrm{spec}\, \Pi\, A\, \Pi \;=\; \big\{ (1-\Lambda)/2\,,\, 0 \big\}, \quad \Lambda \neq 1 \;\Rightarrow\; \sup_{\xi \in \mathbb{R}} \| e^{\xi\, \Pi A \Pi} \| \;=\; \infty.$$

This shows that $L := \partial_t + i A \partial_x$ generates an evolution on $L^2(\mathbb{R})$ while for $\Lambda \neq 1$, $\Pi L \Pi$ does not generate an evolution of the square integrable $\mathrm{rg}\, \Pi$ valued functions. It is not even weakly hyperbolic.

§6. Stability of the profile equations; general case.

In this section we show that the profile equations of geometric optics generate an L^2 evolution whenever L generates an L^2 evolution and the phase satisfies the constant rank hypothesis. For hyperbolic systems that are not symmetrisable in the sense of Friedrichs and phases not satisfying the smooth variety hypothesis, this results is new.

Theorem 6.1. *Suppose that L generates an evolution on L^2 and the phase ϕ satisfies the eikonal equation and the constant rank hypothesis. Then,*

$$\pi_{\rm rg} L(y, \partial)\, i_{\rm ker} \ :\ \ker L_1(y, d\phi(y)) \ \to \ \frac{W}{{\rm rg}\, L_1(y, d\phi(y))}\,,$$

is a first order operator between vector bundles that generates an evolution on L^2.

Proof. Suppose that Γ is a forward influence cone in Ω beginning at t_0 and ending at t_1. For a smooth section $g(x)$ of $\ker L_1(0, x), d\phi(0, x))$ supported in the interior of $\Gamma(t_0)$, we need to show that the Cauchy problem

$$\pi_{\rm rg} L(y, \partial)\, i_{ker}\, v = 0, \qquad v^\varepsilon(t_0, x) = g\,, \tag{6.1}$$

has a unique solution v that is a section of $\ker L_1(t, x), d\phi(t, x))$, $v \in L^\infty([t_0, t_1]\,;\, L^2(\Gamma(t))$, and satisfies estimate (3.6).

We construct such a solution as the weak limit

$$v \ = \ {\rm weak} - \lim_{\varepsilon \to 0}\ e^{-i\phi/\varepsilon}\ v^\varepsilon \tag{6.2}$$

where v^ε is the solution of the initial value problem,

$$L\, v^\varepsilon \ = \ 0, \qquad v^\varepsilon(t_0, x) \ = \ e^{i\phi(t_0, x)/\varepsilon}\, g(x)\,. \tag{6.3}$$

Define u^ε by

$$u^\varepsilon \ := \ e^{-i\phi(y)/\varepsilon}\, v^\varepsilon, \qquad \text{so,} \qquad v^\varepsilon \ := \ e^{i\phi(y)/\varepsilon}\, u^\varepsilon. \tag{6.4}$$

The equation $Lv^\varepsilon = 0$ is then,

$$0 \ = \ L\Big(y, \partial + \frac{id\phi(y)}{\varepsilon}\Big) u^\varepsilon \ = \ \frac{i\, L_1(y, d\phi(y))}{\varepsilon}\, u^\varepsilon \ + \ L(y, \partial)\, u^\varepsilon\,. \tag{6.5}$$

Since $L(y, \partial)$ generates a continuous evolution on L^2, v^ε is bounded in $L^\infty([t_0, t_1]\,;\, L^2)$. Therefore u^ε is bounded in $L^\infty([t_0, t_1]\,;\, L^2) \subset L^2([t_0, t_1] \times \mathbb{R}^d)$. Since $L(y, \partial)$ is first order,

$$L(y, \partial)\, u^\varepsilon \ \text{ is bounded in } \ H^{-1}(\Gamma)\,. \tag{6.6}$$

Equation (6.5) implies that

$$\frac{L_1(y, d\phi(y))}{\varepsilon}\, u^\varepsilon \ \text{ is bounded in } \ H^{-1}(\Gamma)\,. \tag{6.7}$$

The initial values,

$$u^\varepsilon(0,x) \;=\; g(x)\,, \tag{6.8}$$

are independent of ε.

Suppose that $u \in L^\infty([t_0,t_1]]\,;\,L^2(\Gamma(t))$ is a weak star limit point of the sequence u^ε. Choose a sequence $\varepsilon(k)\to 0$ so that $u^{\varepsilon(k)}$ tends weakly to u. *The next argument concerns only that subsequence.* Estimate (6.7) implies that $L_1(y,d\phi(y))u^\varepsilon$ tends to zero in $H^{-1}(\Gamma)$. Passing to the weak limit in (7) yields

$$L_1(y,d\phi(y))u \;=\; 0\,. \tag{6.9}$$

Introduce special coordinates in the fibers. Multiplying L by A_0^{-1} we may suppose that the coefficient of ∂_t is equal to the identity matrix.

The constant rank hypothesis shows that (3.9) is a smooth decomposition. It yields

$$u^\varepsilon(y) \;=\; u^\varepsilon_{\rm ker}(y) \;+\; u^\varepsilon_{\rm rg}(y).$$

With this representation, $L_1(y,d\phi(y))$ has the block form

$$\begin{pmatrix} 0 & 0 \\ 0 & M(y) \end{pmatrix}, \qquad M(y)\in {\rm Hom}\big({\rm rg} L_1(y,d\phi(y)\big) \ \ \text{invertible.}$$

Estimate (6.7) shows that

$$u^\varepsilon_{rg} \quad \overset{\text{weakly}}{\longrightarrow} \quad 0 \quad \text{in} \quad H^{-1}(\Gamma).$$

Thus the weak limit $u = u_{\rm ker}$.

Equation (6.5) in the current coordinates shows that

$$\partial_t u^\varepsilon_{\rm ker} \quad \text{is bounded in} \quad L^\infty([t_0,t_1]\,;\,H^{-1}(\Gamma(t))). \tag{6.10}$$

Therefore u^ε_{ker} is bounded in in ${\rm Lip}([t_0,t_1]\,;\,H^{-1}(\Gamma(t))$ with initial value equal to g by (6.8). It follow that $u^\varepsilon_{\rm ker}$ converges uniformly to a Lipschitz continuous function with values in H^{-1} whose initial value is g. In particular,

$$u|_{t=0} = g\,. \qquad (6.11).$$

Thus $v\in L^\infty([0,T]\,;\,L^2(\Gamma(t)))$ solves the initial value problem (6.1). And, v inherits the *a priori* bound

$$\big\|v\big\|_{L^\infty([t_0,t_1]\,;\,L^2(\Gamma(t))} \;\le\; C\,\|g\|_{L^2(\mathbb{R}^d)}.$$

Since the u^ε are supported uniformly in a strictly smaller influence cone the same is true of u showing that u vanishes on a neighborhood of the lateral boundary of Γ. This settles the existence part for the evolution in L^2.

To prove uniqueness, use a duality argument as in Hölmgren's Theorem. Remark **6** above shows that the operator between duals,

$$\big(\pi_{\rm rg} L(y,\partial) i_{ker}\big)^\dagger \ :\ \ker L_1(y,d\phi(y))^\dagger \quad\to\quad \big(W/{\rm rg}\, L_1(y,d\phi(y))\big)^\dagger\,,$$

generates an L^2 evolution. This shows that for each $T\in[t_0,t_1]$ and

$$\psi\ \in\ L^2\Big(\Gamma(T)\,;\,\big({\rm ker}L_1(T,x), d\phi(T,x))\big)^\dagger\Big), \qquad {\rm supp}\,\psi\ \subset\subset\ \Gamma(T)\,,$$

There is a $w\in L^\infty([t_0,T]\,;\,L^2(\Gamma(t))$ with values in ${\rm ker}L_1(y,d\phi(y))^\dagger$ satisfying the initial value problem

$$\big(\pi_{\rm rg} L(y,\partial) i_{ker}\big)^\dagger w\ =\ 0, \qquad w(T,\cdot)=\psi\,.$$

From the finite speed for $L^\dagger$, w vanishes on a neighborhood of the lateral boundaries of Γ. Therefore in the integrations by parts over Γ which follow the only boundary terms which arise are from the top and bottom.

Use Green's identity which is proved using Friedrich's weak=strong result,

$$\int_{t_0}^T \big\langle \pi_{\rm rg} L(y,\partial) i_{ker} v\,,\,w\big\rangle\ -\ \big\langle v\,,\,\big(\pi_{\rm rg} L(y,\partial) i_{ker} v\big)^\dagger w\big\rangle\ dx\,dt\ =\ \int \big\langle v(t)\,,\,w(t)\big\rangle\ dx\Big|_{t=t_0}^{t=T}\,.$$

To prove uniqueness of the Cauchy problem for the problem defining $v=v_{\rm ker}$ with initial value $g=0$, use this identity with,

$$\pi_{\rm rg} L(y,\partial) i_{ker} v\ =0,\quad \big(\pi_{\rm rg} L(y,\partial) i_{ker} v\big)^\dagger w=0,\quad v_{\rm ker}\big|_{t=0}\ =\ 0\quad w_{\rm ker}\big|_{t=T}=\psi\,,$$

to find,

$$\int \big\langle v(T)\,,\,\psi\big\rangle\ dx\ =\ 0\,.$$

Since T and ψ are arbitrary it follows that $v=0$. ∎

Combining this result with the discussion at the end of §2, yields the following existence and uniqueness result for asymptotic solutions.

Corollary 6.2. *Suppose that L and ϕ are as in Theorem 6.1 and Γ is a forward influence cone with base at $t=t_0$. Suppose that $J\subset V$ is a subbundle complementary to* $\ker L_1(y,d\phi(y))$ *as in (2.8) and that $P(y)$ is the projection along J_y onto* $\ker L_1(y,d\phi(y))$. *Then for any family of smooth initial sections $x\mapsto g_n(x)$ of* $\ker L_1\big((t_0,x),d\phi(t_0,x))\big)$ *supported in the interior of $\Gamma(t_0)$ there are uniquely determined smooth profiles a_n defined in Γ, vanishing on the lateral boundaries, satisfying the equations (2.6) together with the initial conditions*

$$P\,a_n\big|_{t=t_0}\ =\ g_n\,.$$

§7. Characteristic polynomial of the profile operator.

In this section we compute the characteristic polynomial of the operator, $\pi_{\rm rg}\, L(y,\partial)\, i_{\rm ker}$ when L generates an L^2 evolution when the constant rank hypothesis is satisfied The smooth variety hypothesis need not be satisfied.. A classic example is conical refraction in crystal optics (see [Lu], [JMR2]).

The symbol $L_1(y,\eta)$ is a linear map from V_y to W_y so does not have a well defined determinant. For fixed y choosing bases in V_y and W_y yields a matrix valued function $L_1(y,\eta)$ and therefore a determinant. Different choices of bases yield functions of η which differ at most by a multiplicative constant. Therefore, *the characteristic polynomial at y is well defined up to a nonvanishing constant factor.*

Definition. *Suppose that $P(\eta)$ is a polynomial in N variables and that $\underline{\eta} \in \mathbb{C}^N$. Define μ to be the unique integer so that for $|\alpha| < \mu$, $\partial^\alpha P(\underline{\eta}) = 0$ and there is an α with $|\alpha| = \mu$ and $\partial^\alpha P(\underline{\eta}) \neq 0$. The* **localization of** P **at** $\underline{\eta}$ *is the (nonzero) term homogenous of degree μ in the Taylor expansion of P at $\underline{\eta}$.*

Proposition 7.1. *Suppose that $L(y,\partial)$ generates an evolution on L^2 and that ϕ satisfies the constant rank hypothesis. For each $\underline{y}$, the characteristic polynomial of the profile operator $\det \pi_{rg}\, L_1(y,\eta)\, i_{ker}$ is a nonzero constant times the localization of the characteristic polynomial $\eta \mapsto \det L_1(y,\eta)$ at $\underline{\eta} := d\phi(\underline{y})$. That is, as $\eta \to 0$,*

$$\det L_1(\underline{y}, d\phi(\underline{y}) + \eta) \;=\; c(\underline{y})\, \det \Big(\pi_{rg} L_1(\underline{y},\eta)\, i_{ker}\Big) + O(|\eta|^{k+1})\,, \qquad c(\underline{y}) \;\neq\; 0\,.$$

Remark. Proposition 4.1.**i** shows that $\eta \mapsto \det\big(\pi_{rg}\, L_1(\underline{y},\eta)\, i_{ker}\big)$ is nonzero at $\eta = (\tau,\xi) = (1,0,\dots,0)$. That is, $(\underline{y},1,0,\dots,0)$ is noncharacteristic for $\pi_{rg}\, L(y,\partial)\, i_{ker}$.

Proof. From (3.9) one has,

$$\ker A_0^{-1} L_1(\underline{y},\underline{\eta}) \;\oplus\; {\rm rg}\, A_0^{-1} L_1(\underline{y},\underline{\eta}) \;=\; V_{\underline{y}}\,.$$

Corresponding to this direct sum decomposition, linear transformations from $V_{\underline{y}}$ to itself have block forms,

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

Choose a basis

$$b_1,\dots,b_k, \qquad \text{for} \qquad \ker L_1(\underline{y}, d\phi(\underline{y}))\,.$$

Then the equivalence classes

$$[b_j] \;:=\; b_j \;+\; {\rm rg}\, A_0^{-1}\, L_1(\underline{y}, d\phi(\underline{y}))\,,$$

form a basis for $W_{\underline{y}}/\,{\rm rg}\, L_1(\underline{y}, d\phi(\underline{y}))$. Choose a complimentary set so that

$$b_1,\dots,b_k, c_1, c_2\dots, c_{N-k} \qquad \text{is a basis for} \quad V_{\underline{y}}\,.$$

In these bases the matrices for $i_{ker A_0^{-1} L_1}$ and $\pi_{rg\, A_0^{-1} L_1}$ are

$$\begin{pmatrix} I \\ 0 \end{pmatrix} \qquad \text{and} \qquad (\, I \quad 0\,)\,,$$

respectively.
For η small and supressing the $\underline{y}$ dependence, the symbol has bloch form,

$$A_0^{-1} L_1(\underline{\eta} + \eta) \;=\; \begin{pmatrix} R(\underline{\eta} + \eta) & O(|\eta|) \\ O(|\eta|) & E(\underline{\eta}) + O(|\eta|) \end{pmatrix}\,,$$

where

$$R(\zeta) \;:=\; (\, I \quad 0\,)\, A_0^{-1} L_1(\zeta) \begin{pmatrix} I \\ 0 \end{pmatrix}$$

is the principal symbol of $\pi_{rg} A_0^{-1} L\, i_{ker}$, and $E(\underline{\eta})$ is invertible. Since the kernel of $A_0^{-1} L_1(y, \underline{\eta})$ has dimension k, it follows that $R(\underline{\eta}) = 0$.
Use the linearity of the symbol of $R(\zeta)$

$$R(\underline{\eta} + \eta) = R(\underline{\eta}) + R(\eta) = R(\eta), \quad \text{so,} \quad A_0^{-1} L_1(\underline{\eta} + \eta) = \begin{pmatrix} R(\eta) & O(|\eta|) \\ O(|\eta|) & E(\underline{\eta}) + O(|\eta|) \end{pmatrix}.$$

Hence,

$$\det A_0^{-1} L_1(\underline{\eta} + \eta) \;=\; \det \begin{pmatrix} R(\eta) & O(|\eta|) \\ O(|\eta|) & E(\underline{\eta}) + O(|\eta|) \end{pmatrix}.$$

Since $R(\eta)$ is a linear function of η with values in $\mathrm{Hom}\big(\ker A_0^{-1} L_1(\underline{\eta})\big)$ which has dimension k one has $\det R(\eta) = O(|\eta|^k)$. Therefore the determinant on the right is equal to

$$c \det R(\eta) + O(|\eta|^{k+1}), \qquad c \;=\; \det E(\underline{\eta}) \;\neq\; 0\,,$$

proving the proposition. ∎

Application. It is known that if L is hyperbolic with timelike direction $\mathbf{t}$ then the localisation is also hyperbolic with this timelike direction (see [G], [H]). Thus the timelike cone of the localisation contains the timelike cone of the original operator. By duality (see [JMR4]), the propagation cone of the localisation is a subset of the propagation cone of the background operaotor. Thus the speeds of propagation of the profile equations are no faster than those of the background operator.

References.

[DR] P. Donnat and J. Rauch, Dispersive nonlinear geometric optics, Dispersive nonlinear geometric optics, Jour. Math. Physics, **38**(1997), 1484-1523.

[DJMR] P. Donnat, J.-L. Joly, and G. Métiver, and J. Rauch, Diffractive nonlinear geometric optics, Seminaire Équations aux Derivées Partielles 1995-96, École Polytéchnique, XVII-1 to XVII-23.

[F1] K.-O. Friedrichs, On the identity of weak and strong extensions of differential operators, Trans. AMS **55**(1944), 132-151.

[F2] K.-O. Friedrichs, Symmetric hyperbolic linear differential equations, Comm. Pure Appl. Math. **7**(1954), 345-392.

[FL] K.-O. Friedrichs and P.D. Lax, Systems of conservation laws with a convex extension, Proc. Nat. Acad. Sci. **68** (1971), 1686-1688.

[G] L. Gärding, Linear hyperbolic partial differential equations with constant coefficients, Acta. Math. **85** (1951), 1-62.

[H] L. Hörmander, *The Analysis of Linear Partial Differential Operators vol.II*, Springer-Verlag, Berlin, 1983.

[JMR1] J.-L. Joly, and G. Métiver, and J. Rauch, Coherent and focusing multidimensional nonlinear geometric optics, Annales de L'École Normale Supérieur, **28**(1995), 59-113.

[JMR2] J.-L. Joly, and G. Métiver, and J. Rauch, Coherent nonlinear waves and the Wiener algebra, Annales de L'Institut Fourier, 44(1994), 167-196.

[JMR3] J.-L. Joly, and G. Métiver, and J. Rauch, Diffractive nonlinear geometric optics with rectification, Indiana U. Math. J. **47**(1998), 1167-1242.

[JMR4] J.-L. Joly, and G. Métiver, and J. Rauch, Hyperbolic domains of determination and Hamilton-Jacobi equations, Journal of Hyperbolic Partial Differential Equations **2**(2005), 713-744.

[Ln] D. Lannes, Dispersive effects for nonlinear geometrical optics with rectification, Aymptotic Anal. **18**(199811-146.

[Lx] P.D. Lax, Asymptotic solutions of oscillatory initial value problems, Duke Math. J. **24**(1957)627-646.

[LP] P.D. Lax and R. Phillips, Local boundary conditions for dissipative symmetric linear differential operators, Comm. Pure Appl. Math. **13**(1960)427-455.

[Lr] J. Leray, *Hyperbolic Differential Equations*, Institute for Advanced Study, 1953.

[Ld] D. Ludwig, Conical refraction in crystal optics and hydromagnetics, Comm. Pure Appl. Math. **14**(1961), 113-124.

[M] S. Mizohata, S, *Lectures on the Cauchy Problem*, Tata Institute Lectures on Mathematics and Physics No. 35, Tata Institute of Fundamental Research, Bombay 1965.

[Me] G. Métivier, *The Mathematics of Nonlinear Optics*, Handbook of differential equations: evolutionary equations. vol. **V** pg 169-313, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2009.

[R1] J. Rauch, Lectures on Geometric Optics, in *Hyperbolic Equations and Frequency Interactions* eds. Luis Caffarelli and Weinan E., IAS/Park City Mathematics Series, Volume V, AMS, (1999), 383-466.

[R2] J. Rauch, *Hyperbolic Partial Differential Equations and Geometric Optics*, available at `www.math.lsa.umich.edu/~rauch`.