Remarks on the Well-Posedness of the Nonlinear Cauchy Problem

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This paper is dedicated to François Treves.

Abstract. We show that hyperbolicity is a necessary condition for the well posedness of the noncharacteristic Cauchy problem for nonlinear partial differential equations. We give conditions on the initial data which are necessary for the existence of solutions and we analyze Hadamard's instabilities in Sobolev spaces. We also show that genuinely nonlinear equations raise new interesting problems.

1. Introduction

The question of the well-posedness of the Cauchy problem was first raised by Hardamard ([**8**], [**9**]) who proved that it is ill-posed in the case of linear second order elliptic equations. But the introduction in [**9**] clearly indicates that Hadamard was interested in nonlinear equations as well. In modern words, Hadamard's proof is based on the analytic regularity of linear elliptic boundary problems. This regularity has been extended to nonlinear elliptic equations by Morrey ([**22**]) so that Hadamard's argument also applies to general nonlinear elliptic equations.

For general *linear* equations, it is well known that hyperbolicity is a necessary condition for the well-posedness of the noncharacteristic Cauchy problem in C^∞, (see Lax [**16**], Mizohata [**21**] and Ivrii-Petkov [**13**] for a simplified proof and further developments; see also [**10**]). For *nonlinear* equations, Wakabayashi [**27**] has proved that the existence of a smooth *stable* solution implies hyperbolicity, stability meaning that one can perturb the initial data and the source terms in the equations. In a previous paper, Yagdjian obtained this result, with a much weaker definition of stability, in the sense of continuous dependence on the initial data, for the special case of "gauge invariant" equations ([**28**]). We also mention [**29**] for a particular case and [**11**] for first order scalar complex equations.

In this paper, we continue the analysis of Hadamard's instabilities for nonhyperbolic nonlinear equations in two directions. First, we give necessary conditions on the initial data for the existence of smooth solutions of a given equation, without perturbing the equation or the source terms. Next, we also want to point out

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that the nonlinear theory yields interesting and difficult new problems. There are many interesting examples, for instance in multi-phase fluid dynamics, where the equations are not *everywhere* hyperbolic. To mention one occurrence of this phenomenon, consider Euler's equations of gas dynamics in Lagrangian coordinates:

$$\begin{cases} \partial_t u + \partial_x v \;=\; 0\,, \\ \partial_t v + \,\partial_x p(u) \;=\; 0\,. \end{cases} \tag{1.1}$$

The system is hyperbolic [resp. elliptic] when $p'(u) > 0$ [resp. $p'(u) < 0$]. For van der Waals state laws, it happens that p is decreasing on an interval $[u_*, u^*]$. A mathematical example is

$$p(u) = u\,(u^2 - 1) \tag{1.2}$$

Hadamard's argument (see e.g. [**8**], [**9**]) shows that the Cauchy problem with data taking values in the elliptic region is ill-posed: if $u_{|t=0}$ is real analytic near $\underline{x}$ and $u(\underline{x})$ belongs to the elliptic interval, then any local C^1 solution is analytic (see e.g. [**22**]); thus the initial data $v_{|t=0}$ *must* be analytic for the initial value problem to have a solution.

However, there are other interesting questions about the system (1.1). There are classical solutions with values in one of the hyperbolic region $u < u_*$ or $u > u^*$, but there are also discontinuous solutions, modeling for instance phase transitions, which take values in both regions. They have been extensively studied, see e.g. [**25**], [**3**], [**7**]. Another remarkable fact is that (1.1) has a conserved energy. Let $P(u)$ satisfy $P' = p$. Then the energy:

$$E(t) \;:=\; \int \big(\frac{1}{2}\,v^2(t,x) + P(u(t,x))\big)\,dx \tag{1.3}$$

is conserved for solutions of (1.1). For the example (1.2), $P(u) = \frac{1}{4}\,u^2\,(u^2 - 2)$. If one consider the periodic problem, the L^2 norm is dominated by the L^4 norm on $[0, 2\pi]$, thus the boundedness of E controls the L^4 and L^2 norm of the solutions. Of course, this is formal, and the validity of *a priori* bounds does not prove the *existence* of solutions. However, this indicates that the nonexistence of solutions is much more subtle than in the linear case. In particular, there is no blow up phenomenon in L^p norms.

The equations (1.1) are thought as approximations or limits of more complicated models which may include for instance viscosity or capillarity (see e.g. [**3**]); numerical schemes have also been considered (see e.g. [**6**], [**12**]). In the case of periodic solutions, spectral methods lead to filter high frequencies and to consider the "approximate" system

$$\begin{cases} \partial_t u^\lambda + \partial_x v^\lambda = 0, \\ \partial_t v^\lambda + \partial_x S_\lambda p(u^\lambda) = 0, \end{cases} \qquad \begin{cases} u^\lambda{}_{|t=0} = S_\lambda h, \\ v^\lambda{}_{|t=0} = S_\lambda k, \end{cases} \tag{1.4}$$

where S_λ is the projector on Fourier modes of index $|n| \le \lambda$. For instance, when p is given by (1.2), the conservation of energy and the Cauchy-Lipschitz theorem imply that in the periodic case:

> *for all $h \in L^4$ and $k \in L^2$, the equations* (1.4) *have global solutions* (u^λ, v^λ) *which are uniformly bounded in* $C^0([0, \infty[; L^4 \times L^2)$.

Note that there are no conditions on h, which can take values in the elliptic region $u^2 < 1/3$. The question is to analyze the behavior of (u^λ, v^λ) as $\lambda \to +\infty$. Because of the bounds on (u^λ, v^λ) and $(\partial_t u^\lambda, \partial_t v^\lambda)$, there are subsequences which converge weakly and strongly in $C^0([0,1], H^{-\varepsilon})$. In particular, the weak limits $(\underline{u}, \underline{v})$ are bounded with values in L^2 and continuous in time for the weak tolopogy of L^2. Thus $\underline{u}(0) = h$ and $\underline{v}(0) = k$. Taking the weak limit $\underline{p}$ of $p(u^\lambda)$, there holds

$$\partial_t \underline{u} + \partial_x \underline{v} = 0\,, \quad \partial_t \underline{v} + \partial_x \underline{p} = 0\,.$$

The question is to express $\underline{p}$ in terms of $\underline{u}$ and $\underline{v}$. As mentioned above, the answer, $\underline{p} = p(\underline{u})$, $(\underline{u}, \underline{v})$ smooth, *cannot* be true in general when h takes values in the elliptic zone. The common idea is that the limits $(\underline{u}, \underline{v})$ "escape" from the elliptic region, as suggested by numerical calculations ([**6**], [**12**]), but no rigorous proof of this fact seems available in the literature. A detailed answer to the questions above remains a very interesting open problem. Motivated by this problem, we consider in Section 5 a modified nonlocal system:

$$(1.5)\qquad \begin{cases} \partial_t u = a(t)\partial_x v, \\ \partial_t v = |a(t)|\partial_x u, \end{cases} \qquad \text{with} \quad a(t) = \|u(t)\|^2_{L^2} - 1.$$

This a version of Kirchhoff equations ([**15**], [**18**] [**1**]), which is non hyperbolic when $a < 0$. As (1.1), this system has a natural (formal) energy:

$$(1.6)\qquad E(t) = \|v(t,\cdot)\|^2_{L^2} + \Big| \|u(t,\cdot)\|^2_{L^2} - 1 \Big|.$$

This implies that the equations with filtered initial data $(S_\lambda h, S_\lambda k)$ has global solutions (u^λ, v^λ) uniformly bounded in $L^\infty([0, +\infty[, L^2(\mathbb{T}))$. For large classes of "nonanalytic" initial data $(h,k) \in L^2$, with $\|h\|_{L^2} < 1$, we show that the limits are $\underline{u} = h$, $\underline{v} = k$, constant in time, remaining in the elliptic region. The limit equations, $\partial_t \underline{u} = \partial_t \underline{v} = 0$, have little to see with the original ones. This indicates that the answers to the questions above might be very delicate.

Now we review the results of Sections 2 to 4. To fix a framework, we consider first order square systems

$$(1.7)\qquad \partial_t u = F(t, x, u, \partial_x u)\,, \quad u_{|t=0} = h\,.$$

where F is a smooth function of $(t, x, u, v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^N \times (\mathbb{R}^N)^d$. The principal symbol of the equation reads

$$(1.8)\qquad \tau \mathrm{Id} - \xi \cdot \partial_v F(t, x, u, v)\,.$$

Hyperbolicity means that all the eigenvalues of $\partial_v F$ are real. We consider the local Cauchy problem (1.7) near $(0, \underline{x})$ and a given base point $(\underline{u}, \underline{v})$, assuming that the initial data satisfy

$$(1.9)\qquad h(\underline{x}) = \underline{u}\,, \quad \partial_x h(\underline{x}) = \underline{v}.$$

The results in Sections 2 to 4 illustrate the idea that if $\partial_v F(0, \underline{x}, \underline{u}, \underline{v})$ has a non real eigenvalue, then the Cauchy problem (1.7) (1.9) for classical solutions is ill-posed.

Well posedness means first *solvability*. Hadamard's counterexamples (see the example (1.1) above) prove that analyticity type conditions on the data are necessary for the existence of solutions of the elliptic Cauchy problem. In the same vein, consider the equation[1]:

$$(1.10)\qquad (\partial_x + i\partial_y)u = u^2\,, \quad x > 0, \qquad u_{|x=0} = h\,,$$

[1] This explicit elementary example was suggested by Nicolas Lerner.

with $h(0) \neq 0$. Any C^0 solution on $\{x \geq 0\}$ near the origin is C^1, does not vanish and $1/u$ satisfies the Cauchy-Riemann equation. $(\partial_x + i\partial_y)(1/u) = 0$. Therefore, $1/h$ is the trace of an holomorphic function in $\{x > 0\}$, implying that necessarily $1/h$ is microlocally analytic in the direction $+1$ at the origin. In particular, if h is real valued, h must be real analytic near the origin.

In Section 2, we extend this analysis to first order scalar complex equations. For such equations, is is proved in [**11**] that the existence of solutions for all complex data close to a given h, implies that the system must be semilinear and hyperbolic. We give a more precise result, showing that, in the non hyperbolic case, *microlocal analyticity conditions on the initial data are necessary for the existence of classical solutions*, hence that the Cauchy problem has no classical solution for most initial data. The proof is based on the analysis of [**19**] (see also [**2**]), which provides approximate integral representation of the C^1 or C^2 solutions. Taking real and imaginary parts, this provides examples of 2×2 systems where the Cauchy problem has no classical solutions.

This analysis does not extend to general systems: the representation and approximation theorems valid in the scalar case have no analogue; the local uniqueness of the Cauchy problem may be false ([**20**]); there are no microlocal analytic regularity theorem at elliptic directions for C^1 or C^2 solutions. However, there are results about $H^{s'}$ microlocal regularity for H^s solutions when $s' \leq 2s - s_0$ (see Bony [**4**] and Sablé-Tougeron [**23**]). In Section 4, we show that if $\partial_v F(0, \underline{x}, \underline{u}, \underline{v})$ has a nonreal eigenvalue, then for all H^s local solution, with $s > \frac{d}{2} + 1$, the polarized $H^{s'}$ wave front set of the initial data is not arbitrary, when $s' < 2s - \frac{d}{2} - 1$. In particular, for most data in $H^{s'}$, the Cauchy problem has no local H^s solution. Note that s' can be taken larger than s, and for any "loss" k, it applies to $s' = s + k$, if s is large enough. The restriction $s > \frac{d}{2} + 1$ is natural in order to have C^1 classical solution. The restriction $s' < 2s - d - 1/2$ is forced by the Theorem of microlocal ellipticity for nonlinear equations (see [**4**] [**23**]). Under additional assumptions on the equations, one can show that for arbitrary large s' there are $H^{s'}$ initial data such that the Cauchy problem has no local H^s solution.

For linear equations, standard functional analysis results convert well-posedness into estimates, and necessary conditions are found by contradicting the estimates. Solvability implies continuous dependence on the data (see also F.John [**14**] for general remarks on this notion). For nonlinear equations, there are no such abstract argument and it is reasonable to include the continuous dependence in the definition of the well posedness. In addition, because local uniqueness is not guaranteed, we also include it in the following definition of Hölder continuous solvability. In the next statement, B_r denotes the ball $\{|x - \underline{x}| < r\}$ and $\Omega_{r,\delta}$ the lens shaped domain

$$\Omega_{r,\delta} = \big\{(t, x) \ : \ 0 < t \,, |x - \underline{x}|^2 + \delta t < r^2\big\}$$

Definition 1.1. We say that the Cauchy problem (1.7) is Hölder well posed on H^σ, if there are constants $r_0 > r_1 > 0$, $\delta > 0$, $c > 0$, C and $\alpha \in]0, 1]$, such that for all $h \in H^\sigma(B_{r_0})$ satisfying $\|h - \underline{u} - x \cdot \underline{v}\|_{H^\sigma(B_{r_0})} \leq c$ and all $r \in]0, r_1]$, the Cauchy problem (1.7) has a unique solution in $C^1(\overline{\Omega}_{r,\delta})$, with norm bounded by C. Morever, given h_1 and h_2, the corresponding solutions satisfy for all $r < r_1$:

$$\|u_1 - u_2\|_{L^\infty(\Omega_{r,\delta})} \leq C\|h_1 - h_2\|^\alpha_{H^\sigma(B_{r_0})} \tag{1.11}$$

In Section 3, we show that *if F is real analytic and if $\partial_v F(0, x_0, u_0, v_0)$ has a nonreal eigenvalue, then the Cauchy problem* (1.7) *is not Hölder well posed on H^σ, for all σ.*

Note that the definition above differs strongly from the notion of *stable* solution introduced in [**27**] in the sense that we do not allow perturbations of the equations, while the stability used in [**27**] is related to the solvability of

$$\partial_t u + F(t, x, u, \partial_x u) = f\,, \quad u_{|t=0} = h + h'. \tag{1.12}$$

for all f and h' small. The analysis is based on the construction of asymptotic solutions using WKB or geometric optics expansions. But they are not exact solutions, yielding error terms f which are precisely the source terms chosen in [**27**]. In this analysis, the choice of f is dictated by the choice of h. It is interesting and much stronger to consider exact solutions of (1.7) (as in [**28**]), or to be able to choose h and f independently. In Section 3, we construct exact solutions close to the approximate solutions, by Cauchy-Kowalewski type arguments. This is where we use the analyticity of the equation. In this respect, the results of this section give a detailed account of the H^s instability of analytic solutions when hyperbolicity fails.

2. Necessary conditions for scalar complex equations

To simplify the discussion, consider a quasilinear scalar equation

$$\partial_t u + \sum_{j=1}^{d} a_j(t, x, u)\partial_{x_j} u + b(t, x, u) \;=\; 0\,, \quad u_{|t=0} = h\,. \tag{2.1}$$

where the a_j are holomorphic functions of (t, x, u) on a neighborhood of $(0, \underline{x}, \underline{u})$. The Cauchy data h is always assumed to satisfy $h(\underline{x}) = \underline{u}$. The nonhyperbolicity condition reads

$$\operatorname{Im} a(0, \underline{x}, \underline{u}) \neq 0\,. \tag{2.2}$$

Theorem 2.1. *If the Cauchy problem* (2.1) *has a C^1 solution for $t \geq 0$ on a neighborhood of $(0, \underline{x})$, then for all $\xi \in \mathbb{R}^d$ such that $\xi \cdot \operatorname{Im} a(0, \underline{x}, \underline{u}) > 0$, $(\underline{x}, \xi)$ does not belong to the analytic wave front set of h.*

For the definition of the analytic wave front set, we refer to [**24**] or [**10**]. In particular, it contains the C^∞ wave front set and the theorems implies that if the local Cauchy problem has a C^1 solution, then h must be C^∞ at $(\underline{x}, \underline{\xi})$ if $\underline{\xi} \cdot \operatorname{Im} a(0, \underline{x}, \underline{u}) > 0$. This means that for all C^∞ cut-off function χ supported in a sufficiently small neighborhood of $\underline{x}$, the Fourier transform of χh is rapidly decreasing in any small conical neighborhood of $\underline{\xi}$. For "most" functions h in H^s, $(\underline{x}, \underline{\xi})$ belongs to the the C^∞ wave front set. Theorem 2.1 implies that for most h, the Cauchy problem (2.1) has no C^1 solution.

Example 2.2. Taking real and imaginary parts of the unknowns yields nonexistence theorem for 2×2 real systems. With $\alpha_j(u, v) = \operatorname{Re} a_j(u + iv)$, $\beta_j(u, v) = \operatorname{Im} a_j(u + iv)$, the equation (2.1) with $b = 0$ is equivalent to:

$$\begin{cases} \partial_t u + \sum \partial_{x_j} \alpha_j(u, v) = 0\,, & u_{|t=0} = h, \\ \partial_t v + \sum \partial_{x_j} \beta_j(u, v) = 0\,, & v_{|t=0} = k. \end{cases} \tag{2.3}$$

Suppose that $\underline{\beta} = \beta(h(\underline{x}), k(\underline{x})) \neq 0$ and choose $\underline{\xi}$ such that $\underline{\xi} \cdot \underline{\beta} > 0$. If $h + ik$ is not microlocally analytic at $(\underline{x}, \underline{\xi})$, then the Cauchy problem (2.3) has no local C^1 solution near $(0, \underline{x})$.

For instance, this applies to the the system:

$$
\left\{
\begin{array}{ll}
\partial_t u + u\partial_x u - v\partial_x v + \partial_y u = 0\,, & u_{|t=0} = h, \\
\partial_t v + v\partial_x u + u\partial_x v + \partial_y v = 0\,, & v_{|t=0} = k,
\end{array}
\right.
\tag{2.4}
$$

when $k(\underline{x}) \neq 0$. For functions independent of y, or equivalently dropping the ∂_y, the system is elliptic for $v \neq 0$ and Hadamard's argument applies. The example (2.4) shows that Theorem 2.1 also applies to nonelliptic systems.

Proof of Theorem 2.1. **a)** The complex characteristic curves are integral curves of the holomorphic vector field:

$$
\mathcal{L} = \partial_t + \sum a_j(t,x,u)\partial_{x_j} \; - \; b(t,x,u)\partial_u\,.
$$

They are given by $Z_j(t,x,u) = c_j, U(t,x,u) = c_0$, where Z_j and U are local holomorphic solutions of

$$
\left\{
\begin{array}{ll}
\mathcal{L}Z_j = 0\,, & Z_{j\,|t=0} = x_j\,, \\
\mathcal{L}U = 0\,, & U_{|t=0} = u\,.
\end{array}
\right.
$$

We also introduce the additional variables $v = (v_1, \ldots, v_d)$, which are placeholders for $\partial_{x_j} u$, and the function

$$
J(t,x,u,v) := \det\Big(\frac{\partial Z_j(t,x,u)}{\partial x_k} + v_k \frac{\partial Z_j(t,x,u)}{\partial u}\Big).
$$

Let $G(t,x,u)$ be a holomorphic solution of $\mathcal{L}G = 0$ on a complex neighborhood $\mathcal{O}$ of $(0, \underline{x}, \underline{u})$. Suppose that u is C^1 solution of (2.1) on $[0,T] \times \Omega$ such that $(t,x,u(t,x)) \in \mathcal{O}$ for all $(t,x) \in [0,T] \times \Omega$. Then, by Lemma 2.2.2 of [**19**], there holds for all $s \in [0,T]$ and $\chi \in C^1_0(\Omega)$:

$$
\begin{aligned}
\int_\Omega G(0,x,h(x))\chi(x)\,dx = &\int_\Omega G(s,x,u(s,x))\,\chi(x)\,\widetilde{J}(t,x)\,dx \\
&- \int_{[0,s]\times\Omega} \sum_j \partial_{x_j}\chi(x)\,a_j(t,x,u(t,x))\,G(t,x,u(t,x))\,\widetilde{J}(t,x)\,dtdx
\end{aligned}
\tag{2.5}
$$

with $\widetilde{J}(t,x) := J(t,x,u(t,x), \partial_x u(t,x))$.

b) We use (2.5) with

$$
G_{\lambda,y}(t,x,u) := \Big(\frac{\lambda}{\pi}\Big)^{d/2} U(t,x,u)\, e^{-\lambda q(Z(t,x,u)-y)}\,.
\tag{2.6}
$$

where $q(y) = \langle Qy, y\rangle$ is a quadratic form, with real coefficients, positive definite on $\mathbb{R}^d$.

The $G_{\lambda,y}$ are defined and holomorphic for $|t| \leq T$, $|x - \underline{x}| \leq r$, $|u - \underline{u}| \leq \rho$, for some $T > 0$, $r > 0$ and $\rho > 0$. We can also assume that the given solution u of (2.1) is defined and C^1 for real $(t,x) \in [0,T] \times \overline{\Omega}$ where Ω is the ball $\{|x - \underline{x}| < r\}$ and that $|u(t,x) - \underline{u}| < \rho$ on this domain. We fix $\chi \in C_0^\infty(\Omega)$ equal to 1 on a smaller neighborhood of $\underline{x}$. Because $Z(t,x,u) = x + O\big(|t|\big)$, $\mathrm{Re} Z(t,x,u(t,x)) \neq 0$ for t small and x in the support of $d\chi$. Because $Z(0,x,u) - \underline{x} = x - \underline{x} \neq 0$ on the support of $d\chi$, there are $\Omega_0 \subset \Omega$, real neighborhood of $\underline{x}$, $\varepsilon > 0$, $\delta > 0$ and $T_0 > 0$, such that

$$\begin{aligned} &\forall y \in \Omega_0 + i[-\delta, \delta]^d,\ \forall t \in [0, T_0],\ \forall x \in \mathrm{supp} d\chi\ : \\ &\qquad \mathrm{Re} q(Z(t,x,u(t,x)) - y) \geq 2\varepsilon > 0. \end{aligned} \tag{2.7}$$

Consider

$$Th(y, \lambda) := \Big(\frac{\lambda}{\pi}\Big)^{d/2} \int_\Omega e^{-\lambda q(x-y)}\, h(x)\, \chi(x)\, dx. \tag{2.8}$$

We apply (2.5) to $G = G_{\lambda,y}$ given by (2.6). The estimate (2.7) shows that the second integral in the right hand side is $O(e^{-\varepsilon\lambda})$. Therefore, there is C such that for all $y \in \Omega_0 + i[-\delta, \delta]$ and $t \in [0, T_0]$,

$$\begin{aligned} \Big| Th(y, \lambda) - \Big(\frac{\lambda}{\pi}\Big)^{d/2} \int_\Omega \widetilde{U}(t,x) e^{-\lambda q(\widetilde{Z}(t,x) - y)} \chi(x) \widetilde{J}(t,x) dx \Big| \\ \leq C e^{-\varepsilon\lambda} \end{aligned} \tag{2.9}$$

where $\widetilde{Z}(t,x) := Z(t,x,u(t,x))$ and we use similar notations for $\widetilde{U}$ and $\widetilde{J}$ (see the estimate (4.3.1) in [**19**]).

c) We now make use of Assumption (2.2). Shrinking Ω if necessary, in addition to the previous requirements, we can further assume that

$$\forall x \in \Omega\ :\quad |\mathrm{Im}(a(0,x,h(x)) - \underline{a}| \leq \rho, \tag{2.10}$$

where $\underline{a} = \mathrm{Im}(a(0, \underline{x}, h(\underline{x}))$ and $\rho > 0$ to be chosen later on. We use the estimate (2.9) with

$$y = \underline{x} - it\mathrm{Im}\underline{a} + y',\quad y' \in \mathbb{C}^d,\quad |y'| \leq \rho t. \tag{2.11}$$

Because $Z(t,x,u) = x - ta(0,x,u) + O(t^2)$ and $u \in C^1([0,T] \times \overline{\Omega})$, there holds:

$$\mathrm{Im}\big(\widetilde{Z}(t,x) - y\big) = -\mathrm{Im} y' - t\big(\mathrm{Im} a(0,x,h(x)) - \underline{a}\big) + O(t^2)\,.$$

Thus, there is $T_1 > 0$ such that for $t \in [0, T_1]$:

$$|\mathrm{Im}\big(\widetilde{Z}(t,x) - y\big)| \leq 3\rho t.$$

Hence

$$q\big(\mathrm{Im}\big(\widetilde{Z}(t,x) - y\big)\big)| \leq 9\|Q\|\rho^2 t^2.$$

On the other hand: $\mathrm{Im} y = t\underline{a} - \mathrm{Im} y'$ and therefore if ρ is small enough and $|y'| \leq \rho t$,

$$q(\mathrm{Im} y) \geq t^2 q(\underline{a})/2.$$

Hence, if ρ small enough, for $t \in]0, T_1]$, y satisfying (2.11) and $x \in \Omega$ there holds:

$$-\mathrm{Re} q(\widetilde{Z}(t,x) - y) \leq q\big(\mathrm{Im}(\widetilde{Z}(t,x) - y)\big) \leq q\big(\mathrm{Im} y\big) - \frac{t^2}{4} q(\underline{a}) \tag{2.12}$$

We now fix $t > 0$, $t \leq \min(T_0, T_1)$, such that $y \in \Omega_0 + i[-\delta, \delta]^d$ for all y satisfying (2.11). Thus, the estimates (2.9) and (2.12) imply that there are $\varepsilon_1 > 0$ and $C > 0$ such that for all y in the complex ball of radius ρt centered at $\underline{x} - it\underline{a}$ and for all $\lambda \geq 1$, there holds

$$|Th(y, \lambda)| \leq C e^{\lambda (q(\mathrm{Im} y) - \varepsilon_1)}\,. \tag{2.13}$$

Since the quadratic form q is definite positive on $\mathbb{R}^d$, for $y \in \mathbb{C}^d$, the unique real critical point of $x \mapsto \mathrm{Re} q(y - x)$ is $x = \mathrm{Re} y$ and at this point $-\partial_x q(y - x)$ is equal to $-Q\mathrm{Im} y$. By Proposition 7.2 of Sjöstrand [**24**] (see also [**5**], section I.2), the estimate

(2.13) on a neighborhood of $\underline{x} - it\underline{a}$ implies that $(\underline{x}, tQ\underline{a})$ does not belong to the analytic wave front set of h.

d) For all ξ such that $\xi \cdot \underline{a} > 0$, there is a definite positive real symmetric Q such that $Q\underline{a} = \xi$. We apply the previous step to $q(x) = \langle Qx, x\rangle$ which implies that there is $t > 0$ such that $(\underline{x}, t\xi)$ does not belong to the analytic wave front set of h. Since the wave front is conic in ξ, the theorem is proved. $\square$

3. Hadamard's instabilities in Sobolev spaces

We consider systems, and for simplicity we state the results for quasi-linear systems:

$$\partial_t u = \sum_{j=1}^{d} A_j(t,x,u)\partial_{x_j} u + F(t,x,u), \quad u_{|t=0} = h. \tag{3.1}$$

We assume that the A_j and F are real valued and real analytic near $(0, \underline{x}, \underline{u}) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^N$. We want to compare two solutions of (3.1) with initial data h_1 and h_2. We can choose h_1 to be analytic, for instance $h_1(x) = \underline{u}$, and find an analytic local solution u_1 by Cauchy-Kowalewski theorem. Changing u to $u - u_1$, we get an equation similar to (3.1), with the additional information that 0 is a solution, that is:

$$F(t,x,0) = 0 \quad \text{or} \quad F(t,x,u) = F_1(t,x,u)u. \tag{3.2}$$

We look for solutions of (3.1) in lens shaped domains

$$\Omega_{r,\delta} = \big\{(t,x) \ : \ t \geq 0, \ |x - \underline{x}|^2 + \delta t < r^2\big\}, \tag{3.3}$$

assuming that the equation is not hyperbolic at $(0, \underline{x}, 0)$:

Assumption 3.1. There is $\underline{\xi} \in \mathbb{R}^d$ such that the matrix $\underline{A} := \sum \underline{\xi}_j A_j(0, \underline{x}, 0)$ has a nonreal eigenvalue.

The next theorem shows that the Cauchy problem is not Hölder well posed. We denote by B_r the ball of radius r centered at $\underline{x}$.

Theorem 3.2. *For all m, $\alpha \in]0,1]$, $r_0 > 0$ and $\delta > 0$, there are $r_\varepsilon \to 0$, families of initial data $h_\varepsilon \in H^m(B_{r_0})$ and solutions u_ε of* (3.1) *on $\Omega_{r_\varepsilon,\delta}$, such that*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^2(\Omega_{r_\varepsilon,\delta})} / \|h_\varepsilon\|^\alpha_{H^m(B_{r_0})} = +\infty. \tag{3.4}$$

Let λ_0 denote an eigenvalue of $\underline{\xi} \cdot A(0, \underline{x}, 0)$ such that $\gamma_0 = |\mathrm{Im}\lambda_0| > 0$ is maximum. Let $\underline{r}$ denote an eigenvector associated to λ_0. We consider initial data

$$h_\varepsilon(x) := \varepsilon^M \ \mathrm{Re}\big(e^{ix\cdot\underline{\xi}/\varepsilon}\underline{r}\big)\,. \tag{3.5}$$

We look for solutions

$$u_\varepsilon(t,x) = \mathrm{u}(t, x, t/\varepsilon, x \cdot \underline{\xi}/\varepsilon) \tag{3.6}$$

where $\mathrm{u}(t,x,s,\theta)$ is 2π periodic in θ. For u_ε to be solution of the equation, it is sufficient that u solves an equation of the form

$$\partial_s \mathrm{u} = \mathrm{A}(y,\mathrm{u})\partial_\theta \mathrm{u} + \varepsilon\big(\mathrm{B}(y,\mathrm{u})\partial_y u + F(y,\mathrm{u})\big), \tag{3.7}$$

with $y = (t, x - \underline{x})$ and $\mathrm{A}(y,u) = \sum \underline{\xi}_j A_j(t,x,u)$. In particular $\mathrm{A}(0,0) = \underline{A}$ and the equation reads

$$(\partial_s - \underline{A}\partial_\theta)\mathrm{u} = \mathrm{F}(\mathrm{u}) := (\mathrm{A} - \underline{A})\partial_\theta \mathrm{u} + \varepsilon\big(\mathrm{B}\partial_y u + F(y,\mathrm{u})\big). \tag{3.8}$$

The solution of the Cauchy problem is given by

$$\mathrm{u} = e^{s\underline{A}\partial_\theta}\mathrm{h} + \mathcal{T}(\mathrm{u}), \quad \mathcal{T}(\mathrm{u})(s) := \int_0^s e^{(s-s')\underline{A}\partial_\theta}\mathrm{F}(\mathrm{u}(s'))ds'. \tag{3.9}$$

We solve this equation following the method explained in Wagschall [**26**] (see also the references therein).

Function spaces and existence of solutions. Given power series $\mathrm{u} = \sum u_\alpha y^\alpha$ and $\Phi = \sum \Phi_\alpha y^\alpha$, we say that $\mathrm{u} \ll \Phi$ when $|u_\alpha| \le \Phi_\alpha$ for all α. Consider the series

$$\phi(z) = c_0 \sum_{n=0}^{+\infty} \frac{z^n}{n^2+1}$$

where c_0 is taken such that $\phi^2 \ll \phi$ (cf [**17**], [**26**]). For $y \in \mathbb{R}^{1+d}$, we denote by $Y = \sum y_j$, and we will consider power series $\mathrm{u}(y)$ such that there is a constant C such that

$$\mathrm{u}(y) \ll C\phi(RY + R_0).$$

$R > 0$ and $R_0 \in]0,1]$ are given parameters. These power series are convergent for $R\sum |y_j| + R_0 \le 1$.

Next we introduce the weight function on $\mathbb{Z}$:

$$\langle n\rangle = |n| \quad \text{when } n \neq 0, \qquad \langle 0\rangle = 2.$$

Note that for all p and q in $\mathbb{Z}$:

$$\langle p+q\rangle \le \langle p\rangle + \langle q\rangle. \tag{3.10}$$

Given positive parameters γ, κ, ε, R and ρ, we consider formal Fourier series

$$\mathrm{u}(s,\theta,y) = \sum_{-\infty}^{+\infty} \mathrm{u}_n(s,y)e^{in\theta} \tag{3.11}$$

where the $\mathrm{u}_n(s,y)$ are power series in y, with coefficient C^∞ in $s \in [0,\underline{s}]$, where

$$\underline{s} := \min\{\kappa/\gamma, 1/\varepsilon\rho\}. \tag{3.12}$$

We denote by $\mathbb{E}$ the space of u such that there is a constant C such that for all $s \in [0,\underline{s}]$:

$$\mathrm{u}_n(s,y) \ll C\frac{c_1}{n^2+1}e^{(\gamma s-\kappa)\langle n\rangle}\phi(RY + \varepsilon\rho s)\,. \tag{3.13}$$

The number c_1 is chosen such that

$$\sum_{p+q=n} \frac{c_1}{p^2+1}\frac{c_1}{q^2+1} \le \frac{c_1}{n^2+1}.$$

Elements $\mathrm{u} \in \mathbb{E}$ define smooth functions on the domain

$$\begin{aligned}\Delta = \big\{(s,\theta,y) \ : 0 \le s <\underline{s}, \\ \theta \in \mathbb{T},\ R\sum |y_j| + \varepsilon\rho s < 1\big\}.\end{aligned} \tag{3.14}$$

The best constant C in (3.13) defines a norm $\|u\|$ on $\mathbb{E}$. Equipped with this norm, $\mathbb{E}$ is a Banach space. The choice of c_0 and c_1 and (3.10) imply that $\mathbb{E}$ is a Banach algebra:

$$\|\mathrm{uv}\| \le \|\mathrm{u}\|\ \|\mathrm{v}\|. \tag{3.15}$$

When u is valued in $\mathbb{C}^N$, we denote by $\|u\|$ the sup of the norms of the components of u.

LEMMA 3.3. *If $F(y,u)$ is holomorphic on a neighborhood of the origin in $\mathbb{C}^{1+d}\times\mathbb{C}^N$, there are constants R_0, C_0 and a_0 such that for all parameters γ, κ, ε, $R \geq R_0$ and ρ, the mapping $\mathrm{u} \mapsto F(\cdot, \mathrm{u})$ maps the ball of radius a_0 of $\mathbb{E}$ into the ball of radius C_0 in $\mathbb{E}$.*

PROOF. There are constants R_0, C and a such that

$$F(y,u) \ll C\phi(R_0 Y)\prod_{j=1}^{N}\frac{1}{a-u_j},$$

in the sense of power series in (y,u). Substituting $u = u(y)$ in the expansion, using (3.15) as well as the identities $\phi^2 \ll \phi$ and $\phi(R_0Y) \ll \phi(RY + \varepsilon\rho s)$, yields for $|||\mathrm{u}||| < a$:

$$|||F(\cdot,\mathrm{u})||| \leq C\frac{1}{(a - |||\mathrm{u}|||)^N}.$$

□

We further denote by $|||\cdot|||'$ the norm obtained when ϕ is replaced by its derivative ϕ' in (3.13), and by $|||\cdot|||_1$ the norm obtained when $c_1/(n^2+1)$ is replaced by $c_1/\sqrt{n^2+1}$. In particular, there holds:

$$\begin{array}{rcl} |||\partial_y \mathrm{u}|||' & \leq & R|||\mathrm{u}|||, \\ |||\partial_\theta \mathrm{u}|||_1 & \leq & |||\mathrm{u}|||. \end{array}$$

Moreover, differentiating the estimate $\phi^2 \ll \phi$ implies that $2\phi\phi' \ll \phi'$, thus

$$2|||\mathrm{uv}|||' \leq |||\mathrm{u}|||\ |||\mathrm{v}|||'.$$

Similarly, there is c_2 independent of all the parameters such that:

$$|||\mathrm{uv}|||_1 \leq c_2|||\mathrm{u}|||\ |||\mathrm{v}|||_1.$$

Factoring out y and u in $\mathrm{A}(y,u) - \underline{A}$ and u in $F(y,u)$, using that $y \ll (2/c_0R)\phi(RY) \ll (2c_0/R)\phi(RY + \varepsilon\rho s)$ and that $\phi' \ll \phi$, we deduce from Lemma 3.3 and the estimates above that for $R \geq R_0$ and $|||\mathrm{u}||| \leq a_0$:

$$|||(\mathrm{A}(y,\mathrm{u}) - \underline{A})\partial_\theta \mathrm{u}|||_1 \leq C\big(R^{-1} + |||\mathrm{u}|||\big)|||\mathrm{u}|||, \tag{3.16}$$

$$|||\mathrm{B}(y,\mathrm{u})\partial_y \mathrm{u} + F(y,\mathrm{u})|||' \leq CR|||\mathrm{u}|||. \tag{3.17}$$

Next, we investigate the action of the operator

$$\mathrm{v}(s) = \mathcal{I}(\mathrm{f})(s) := \int_0^s e^{(s-s')\underline{A}\partial_\theta}\mathrm{f}(s')ds'.$$

On each Fourier component, it reads

$$\mathrm{v}_n(s,y) = \int_0^s e^{in(s-s')\underline{A}}\mathrm{f}_n(s',y)ds'.$$

By the definition of γ_0, for all $\gamma > \gamma_0$ there is a constant K_γ such that:

$$\forall n \in \mathbb{Z},\ \forall s \in [0,+\infty[: \quad e^{ins\underline{A}} \leq K_\gamma e^{|n|\gamma s}. \tag{3.18}$$

By definition of the norm $|||\mathrm{f}|||_1$, there holds:

$$\mathrm{f}_n(s',y) \ll \frac{c_1}{\sqrt{n^2+1}}|||\mathrm{f}|||_1 e^{(s'\gamma-\kappa)\langle n\rangle}\phi(RY + \varepsilon\rho s').$$

Using that for $s' \le s$, $\phi(RY+\varepsilon\rho s') \ll \phi(RY+\varepsilon\rho s)$, and integrating term by term the power series in y, implies that

$$\mathrm{v}_n(s,y) \ll \frac{c_1 K_{\gamma_1}}{\sqrt{n^2+1}} \|\mathrm{f}\|_1 \phi(RY+\varepsilon\rho s) \int_0^s e^{|n|\gamma_1(s-s')} e^{(s'\gamma-\kappa)\langle n\rangle} ds'$$

For $\gamma > \gamma_1 > \gamma_0$, the last integral is estimated by

$$e^{(s\gamma-\kappa)\langle n\rangle} \int_0^s e^{s'(|n|\gamma_1 - \langle n\rangle\gamma)} ds' \le \frac{C}{(\gamma-\gamma_1)\langle n\rangle} e^{(s\gamma-\kappa)\langle n\rangle}\,.$$

Choosing $\gamma_1 = (\gamma+\gamma_0)/2$, this shows that for all $\gamma > \gamma_0$, there is a constant K_γ such that

$$\|\mathcal{I}(\mathrm{f})\| \le K_\gamma \|\mathrm{f}\|_1.$$

Similarly, there holds

$$\mathrm{v}_n(s,y) \ll \frac{K_\gamma}{n^2+1} \|\mathrm{f}\|' e^{(s\gamma-\kappa)\langle n\rangle} \int_0^s e^{\gamma(s-s')(|n|-\langle n\rangle)} \phi'(RY+\varepsilon\rho s') ds'.$$

Since $|n| \le \langle n\rangle$, we can ignore the exponential in the integral. Moreover,

$$\varepsilon\rho \int_0^s \phi'(RY+\varepsilon\rho s') ds' \ll \phi(RY+\varepsilon\rho s) - \phi(RY) \ll \phi(RY+\varepsilon\rho s).$$

Therefore,

$$\|\mathcal{I}(\mathrm{f})\| \le \frac{K_\gamma}{\varepsilon\rho} \|\mathrm{f}\|'.$$

Using (3.16) (3.17), these inequalities yield estimates for the operator $\mathcal{T}(\mathrm{u})$ defined in (3.10). Similarly, one obtains estimates for increments $\mathcal{T}(\mathrm{u}) - \mathcal{T}(\mathrm{v})$:

PROPOSITION 3.4. *There are R_0 and a_0 and for all $\gamma > \gamma_0$ there is a constant K_γ, such that for all $R \ge R_0$, all $\kappa > 0$, all $\rho > 0$ and all $\varepsilon \in]0,1]$, there holds for all u and v in $\mathbb{E}$ such that $\|\mathrm{u}\| \le a_0$ and $\|\mathrm{v}\| \le a_0$:*

$$\begin{aligned} &\|\mathcal{T}(\mathrm{u})\| \le K_\gamma\big(R^{-1} + 2\|\mathrm{u}\| + R\rho^{-1}\big)\|\mathrm{u}\|, \\ &\|\mathcal{T}(\mathrm{u}) - \mathcal{T}(\mathrm{v})\| \le K_\gamma\big(R^{-1} + \|\mathrm{u}\| + \|\mathrm{v}\| + R\rho^{-1}\big)\|\mathrm{u}-\mathrm{v}\| \end{aligned}$$

COROLLARY 3.5. *With notations as above, if*

$$K_\gamma\big(R^{-1} + 4a + R\rho^{-1}\big) < \frac{1}{2},$$

then for all $\mathrm{f} \in \mathbb{E}$ with $\|\mathrm{f}\| \le a$, the equation

$$\mathrm{u} = \mathrm{f} + \mathcal{T}(\mathrm{u})$$

has a unique solution $\mathrm{u} \in \mathbb{E}$ such that $\|\mathrm{u}\| \le 2a$. Moreover,

$$\|\mathrm{u} - \mathrm{f}\| \le K_\gamma\big(R^{-1} + 4\|\mathrm{f}\| + R\rho^{-1}\big)\|\mathrm{f}\|.$$

Application. In accordance with (3.5), we solve the Cauchy problem (3.7) with initial data

$$\mathrm{u}_{|s=0} = \mathrm{h} := \varepsilon^M \mathrm{Re}(e^{i\theta}\underline{r}). \tag{3.19}$$

Let

$$\mathrm{f} = e^s \underline{A}\partial_\theta \mathrm{h} = \varepsilon^M \mathrm{Re}\big(e^{is\lambda_0 + i\theta}\underline{r}\big). \tag{3.20}$$

We consider only small values of the parameter ε, and we use the notation $\varepsilon^M = e^{-\kappa_1}$, that is $\kappa_1 = M|\ln\varepsilon|$.

Consider a small parameter $\beta > 0$, to be chosen later on, such that $\beta M < 1/2$. We fix

$$\left\{ \begin{array}{ll} \gamma = (1+\beta)\gamma_0, & \kappa = (1-\beta)\kappa_1, \\ R = e^{\beta\kappa_1} = \varepsilon^{-\beta M}, & \rho = R^2 = \varepsilon^{-2\beta M}. \end{array} \right. \tag{3.21}$$

Introduce $\sigma = (1-\beta)/(1+\beta) < 1$. For ε small enough, $\kappa/\gamma = \sigma\kappa_1/\gamma_0 \le (\varepsilon\rho)^{-1} = \varepsilon^{-1+2\beta M}$, thus, the end point (3.12) is $\underline{s} = \sigma\kappa_1/\gamma_0$.

Proposition 3.6. *There is a constant $c > 0$ such that for all $M \ge 1$ and $\beta \in]0, 1/2M[$, there is ε_0 such that for all $\varepsilon \in]0, \varepsilon_0]$, and parameters as in* (3.21)*, the Cauchy problem* (3.7), (3.19) *has a solution* $\mathrm{u} \in \mathbb{E}$*, and*

$$\forall (s, \theta, y) \in \Delta : \quad |\mathrm{u}(s, \theta, y)| \ge c e^{s\gamma_0 - \kappa_1}. \tag{3.22}$$

Proof. For ε small enough, there holds

$$|\!|\!|\mathrm{f}|\!|\!| = \frac{2}{c_1 c_0} |\underline{r}| \max_{s \in [0, \underline{s}[} e^{\kappa - \kappa_1 + s(\gamma_0 - \gamma)} \le C e^{-\beta\kappa_1}$$

for some constant C independent of ε and β. By Corollary 3.5, there is K, depending only on β, such that for $K\varepsilon^{\beta M} < 1$, the problem has a unique solution $\mathrm{u} \in \mathbb{E}$ and

$$|\!|\!|\mathrm{u} - \mathrm{f}|\!|\!| \le K e^{-2\beta\kappa_1}.$$

For $(s, \theta, y) \in \Delta$, there holds $R \sum |y_j| + \varepsilon\rho s \le 1$. Since the series $\phi(z)$ converges at $z = 1$,

$$|(\mathrm{u} - \mathrm{f})(s, \theta, y)| \le K e^{-2\beta\kappa_1} \sum_{n \in \mathbb{Z}} \frac{c_1}{n^2 + 1} \phi(1) e^{(s\gamma - \kappa)\langle n \rangle}.$$

Since $s\gamma - \kappa \le 0$ and $\langle n \rangle \ge 1$, this implies that there is K' such that

$$\begin{aligned} |(\mathrm{u} - \mathrm{f})(s, \theta, y)| &\le K' e^{-2\beta\kappa_1} e^{(s\gamma - \kappa)} \\ &\le K' e^{s\gamma_0 - \kappa_1} e^{-\beta\kappa_1} e^{\underline{s}\beta\gamma_0} = K' e^{s\gamma_0 - \kappa_1} e^{-\beta(1-\sigma)\kappa_1}. \end{aligned}$$

Because $\underline{r}$ and $\overline{\underline{r}}$ are eigenvectors associated to distinct eigenvalues λ_0 and $\overline{\lambda_0}$, they are linearly independent and there is $c > 0$ such that

$$|f(s, \theta)| \ge 2c e^{s\gamma_0 - \kappa_1}.$$

Since $\sigma < 1$, the two estimates above imply that for ε small enough (3.22) is satisfied. □

Proof of Theorem 3.2. The integer $m \ge 1$ and the Hölder exponent $\alpha \in]0, 1]$ are given, as well as the parameter $\delta > 0$. We fix M large enough, such that

$$\alpha' := \frac{M - m}{M} \alpha - \frac{1 + d}{2M} > 0. \tag{3.23}$$

Note that $\alpha' < \alpha \le 1$. Next we choose $\beta > 0$ such that

$$1 - \alpha' < \sigma := \frac{1 - \beta}{1 + \beta} \quad \text{and} \quad 2M\beta < 1 \tag{3.24}$$

and we fix the parameters γ, κ, R and ρ as in (3.21). By Proposition 3.6, for ε small enough, we have a solution u of (3.7) (3.19) on the domain Δ defined in (3.14). Thus

$$u_\varepsilon(t, x) = \mathrm{u}(\frac{t}{\varepsilon}, \frac{x \cdot \xi}{\varepsilon}, t, y)$$

is a solution of (3.1) (3.5) on the domain

$$\tilde{\Delta}_\varepsilon = \big\{(t,x) \ : 0 \le t < \varepsilon \underline{s}, \ \ \sum |x_j| + t + t\varepsilon^{-\beta M} < \varepsilon^{\beta M}\big\}.$$

Since $\underline{t}_\varepsilon := \varepsilon \underline{s} = \sigma \gamma_0^{-1} M \varepsilon |\ln \varepsilon|$ and $2\beta M < 1$, for ε small enough, this domain contains

$$\Delta_\varepsilon = \big\{(t,x) \ : 0 \le t < \underline{t}_\varepsilon, \ \ |x| < c\varepsilon^{\beta M}\big\}$$

with $c = 1/(2\sqrt{d})$. For ε small, it also contains the lens shaped domain $\Omega_{r_\varepsilon,\delta}$, with

$$r_\varepsilon = (t_\varepsilon/\delta)^{1/2}. \tag{3.25}$$

Moreover, for ε small, $\Omega_{r_\varepsilon,\delta}$ contains the cube

$$\big\{\underline{t}_\varepsilon - \varepsilon \le t \le \underline{t}_\varepsilon, \ |x - \underline{x}| \le \varepsilon\big\}\,.$$

Thus, Proposition 3.6 implies that there is $c > 0$ such that for all ε small enough:

$$\|u_\varepsilon\|_{L^2(\Omega_{r_\varepsilon,\delta})} \ge c e^{\underline{s}\gamma_0 - \kappa_1} \varepsilon^{1+d} = c\varepsilon^{M(1-\sigma)+(1+d)/2}.$$

On the other hand, the Sobolev norm of the initial data on a fixed ball B_{r_0} centered at $\underline{x}$ is of order:

$$\|h_{\kappa,\varepsilon}\|_{H^m(B_{r_0})} \le C\varepsilon^{M-m}\,.$$

Thus, using the notation (3.23),

$$\|u_\varepsilon\|_{L^2(\Omega_{r_\varepsilon,\delta})} / \|h_\varepsilon\|^\alpha_{H^m(B_{r_0})} \ge \frac{c}{C^\alpha} \varepsilon^{M(1-\sigma-\alpha')} \tag{3.26}$$

which, by (3.24), tends to $+\infty$ as ε tends to zero. □

4. Solvability in Sobolev spaces

In this section, we consider the fully nonlinear Cauchy problem in $\mathbb{R}^{1+d}$:

$$\begin{cases} \partial_t u = F(t,x,u,\partial_{x_1} u, \ldots, \partial_{x_d} u), \quad t \ge 0 \\ u_{|t=0} = h, \end{cases} \tag{4.1}$$

near $(0,\underline{x})$. We assume that F is C^∞ in a neighborhood of $\underline{p} := (0,\underline{x},\underline{u},\underline{v})$ in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^N \times (\mathbb{R}^N)^d$. The initial data h is smooth and satisfies

$$h(\underline{x}) = \underline{u}, \quad \partial_x h(\underline{x}) = \underline{v}. \tag{4.2}$$

Assumption 4.1. There is $\underline{\xi} \in \mathbb{R}^d$ such that the matrix $\underline{\xi} \cdot \partial_v F(\underline{p}) = \sum \underline{\xi}_j \partial_{v_j} F(\underline{p})$ has nonreal eigenvalues.

Because $\xi \cdot \partial_v F$ is real, this implies that there is at least one eigenvalue with positive imaginary part. Denote by $\underline{\Pi}$ the spectral projector of $\underline{\xi} \cdot \partial_v F(\underline{p})$ associated to eigenvalues in $\{\mathrm{Im}\mu > 0\}$.

Theorem 4.2. *Let $s > d/2 + 1$ and $s \le s' < 2s - 1 - d/2$. Suppose that the Cauchy data h satisfies* (4.2) *and*

$$(\mathrm{Id} - \underline{\Pi})h \in H^{s'} \quad \textit{near } \underline{x}. \tag{4.3}$$

If the Cauchy problem (4.1) *has a solution in $C^0([0,T]; H^s(\omega))$, then*

$$(\underline{x}, \underline{\xi}) \notin WF_{H^{s'}}(\underline{\Pi} h). \tag{4.4}$$

In this statement, $WF_{H^{s'}}$ denotes the $H^{s'}$ wave front set. In the spirit of Hadamard's argument and of Theorem 2.1, Theorem 4.2 shows that smoothness of part of the Cauchy data, here $\underline{\Pi}h$, implies smoothness of the other components. For all $s'' \in [s, s'[$, there are many Cauchy data h such that

$$h \in H^{s''}, \quad (\mathrm{Id} - \underline{\Pi})h \in H^{s'}, \quad (\underline{x}, \underline{\xi}) \in WF_{H^{s'}}(\underline{\Pi}h). \tag{4.5}$$

For such data, all $T > 0$ and all neighborhood ω of $\underline{x}$, Theorem 4.2 implies that (4.1) has no local solution in $C^0([0,T]; H^s(\omega))$. This implies that the Cauchy problem is not locally well posed from $H^{s''}$ to $C^0(H^s)$ for all $s'' = 2s - 1 - d/2 - \varepsilon > s$ when $s > 1 + d/2$.

The proof is an application of the results of Monique Sablé-Tougeron [**23**] about the propagation of microlocal singularities for nonlinear boundary value problems. For the convenience of the reader, we sketch a proof within the class of spaces $C^0(H^s)$ instead of the class $H^{s,s'}(\mathbb{R}^{1+d})$ used in [**23**].

Proof. Decreasing slightly s, we can assume that $\rho := s - 1 - d/2 \notin \mathbb{N}$. Suppose that $h \in H^s(\mathbb{R}^d)$ satisfies (4.2) and that $u \in C^0([0,T]; H^s(\omega))$ is a solution of (4.1).

a) The product is continuous from $H^{\sigma-\alpha} \times H^{\sigma-\beta}$ into $H^{\sigma-\alpha-\beta}$, when $\sigma > d/2$, $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta \leq 2\sigma$. By induction on k, (4.1) implies that

$$\partial_t^k u \in C^0([0;T]H^{s-k}(\omega)), \quad k \in \{0, \ldots, [2s-2]\}$$

Therefore, for all smooth function G,

$$\partial_t^k G(\,\cdot\,, \,\cdot\,, u, \partial_x u) \in C^0(H^{s-1-k}), \quad k \in \{0, \ldots, [2s-2]\}\,.$$

Since, $\rho = 2s - 2 - (s - 1 + d/2) < 2s - 2 - d$ this property is true up to $k = [\rho] + 1$. Denoting by $C^\alpha(\mathbb{R}^d)$ the usual Hölder space for $\alpha \in \mathbb{R}\backslash\mathbb{Z}$, this implies that $g = G(\cdot, u(\cdot), \partial_x u(\cdot))$ satisfies

$$\partial_t^k g \in C^0([0,T]; C^{\rho-k}(\omega)), \quad k \in \{0, \ldots, [\rho] + 1\}\,. \tag{4.6}$$

For $\rho > 0$, $\rho \notin \mathbb{N}$, we denote by $\widetilde{C}_\rho$ the set of functions g which satisfy this property.

b) Localizing near $(0, \underline{x})$, and using Bony's paralinearization theorem in x ([**4**], [**23**]), (4.1) implies that $\tilde{u} = \chi_1 u$ satisfies

$$\partial_t \tilde{u} - T_A(t, x, \partial_x)\, \tilde{u} = f, \tag{4.7}$$

where $\chi_1 \in C_0^\infty(\omega)$ is equal to one near $\underline{x}$ and $f \in C^0([0,T]; H^{s-1+\rho}(\omega'))$ for some smaller neighborhood ω' of $\underline{x}$. In this equation, T_A denotes a paradifferential operator in x of symbol

$$A(t, x, \xi) := \sum_{j=1}^{d} \xi \cdot \partial_v F(p(t,x)) + \partial_u F p(t,x) \tag{4.8}$$

with $p(t,x) = (t, x, u(t,x), \partial_x u(t,x))$. The coefficients belong to $\widetilde{C}^\rho$.

c) We perform a microlocal block diagonalization of A. Near $(0, \underline{x}, \underline{\xi})$, there are symbols

$$P = \sum_{j=0}^{[\rho]} P_j\,, \quad D = \sum_{j=0}^{[\rho]} D_j$$

such that

$$P \sharp A - \partial_t (P - P_{[\rho]}) = D \sharp P\,. \tag{4.9}$$

The terms $P_j(t,x,\xi)$ [resp. D_j] are C^∞ and homogeneous of degree $-j$ [resp. $1-j$] in ξ and $\widetilde{C}^{\rho-j}$ in (t,x). In (4.9), $\sharp$ denotes the composition of symbols (cf [**4**]):

$$\Big(\sum_{j\leq[\rho]} P_j\Big)\sharp\Big(\sum_{j\leq[\rho]} Q_j\Big) := \sum_{l+m+|\alpha|\leq[\rho]} \frac{1}{\alpha!}(\partial_\xi^\alpha P_l)\,((-i\partial_x)^\alpha Q_m)$$

Moreover, D is block diagonal and there is one block D^I associated to the spectrum of $A(0,\underline{x},\underline{\xi})$ in $\{\mathrm{Im}\mu>0\}$. In particular

$$\mathrm{Im}\ \mathrm{spec}(D_0^I(0,\underline{x},\underline{\xi}))>0 \tag{4.10}$$

The construction is classical. The principal terms P_0 and D_0 are chosen such that

$$P_0A_0(P_0)^{-1}=D_0,$$

where $A_0=\xi\cdot\partial_vF$ is the principal symbol of A. Next, one proceeds by induction on j, choosing P_j and D_j such that the terms of degree $1-j$ in the two sides of (4.9) are equal. In particular, (4.9) is an identity between symbols of degree 1, like D, and the degree of the last term, like $D_{[\rho]}$ is $1-[\rho]$. This is why, in ∂_tP, we can ignore the last term which would be $\partial_tP_{[\rho]}$, of degree $-[\rho]$.

In this computation, we only use symbols with positive degrees of smoothness $\rho-j$, with $j\leq[\rho]$. However, the term $\partial_tP_{[\rho]}$ which will appear in the remainders, requires one more derivative. This is why we took $k\leq[\rho]+1$ in (4.6).

d) Suppose that $\chi(t,x,\xi)$ is a microlocal cut-off function supported in a conical neighborhood of $(0,\underline{x},\underline{\xi})$ where (4.9) is satisfied. The equation (4.7) implies that

$$(\partial_t-T_{\chi D})T_{\chi P}\tilde{u}\;=\;T_{\chi^2P}(\partial_t-T_A\tilde{u})\;+\;T_{\chi\partial_tP_{[\rho]}}\tilde{u}\;+\;T_Q\tilde{u}\;+\;Ru$$

where R is a remainder in the $[\rho]$-calculus x. If $\rho<1$ then $Q=0$ and if $\rho>1$, Q is a symbol of degree zero, $(\rho-1)$ smooth in x and equal to zero near $(0,\underline{x},\underline{\xi})$. In particular, near $(0,\underline{x})$, $Ru\in C^0(H^{s-1+\rho})$ and $T_Qu(t,\cdot)\in H^{s-1+\rho})$ near $(\underline{x},\underline{\xi})$), uniformly in t. Moreover, since $\partial_tP_{[\rho]}$ is of degree $-[\rho]$ with smoothness $C^{\rho-[\rho]-1}$ in x, the operator $T_{\chi\partial_tP_{[\rho]}}$ is of order $-[\rho]+(1-\rho+[\rho])=1-\rho$. Therefore, we see that

$$w:=T_{\chi P}\tilde{u}$$

satisfies

$$\partial_tw-T_{\chi D}w\in C^0(H^{s-1+\rho})\,,\quad \text{near } (0,\underline{x},\underline{\xi})\,. \tag{4.11}$$

e) D is block diagonal. Denote by w_I the components of w which correspond to the bloc D^I. The equation (4.11) implies that

$$\partial_tw_I-T_{\chi D^I}w\in C^0(H^{s-1+\rho})\,,\quad \text{near } (0,\underline{x},\underline{\xi})\,. \tag{4.12}$$

By (4.10), this problem is elliptic and the backward Cauchy problem is well posed ([**23**]). This implies that $w\in C^0(H^{s+\rho})$ near $(0,\underline{x},\underline{\xi})$. By construction, $w_I=T_\Pi u$ where $\Pi=\sum_{j\leq[\rho]}\Pi_j$, with Π_j of degree $-j$ in ξ and $\widetilde{C}^{\rho-j}$ in (t,x). In particular,

$$w_{I\,|t=0}=T_{\Pi_{|t=0}}h\in H^{s+\rho}\,,\quad \text{near } (\underline{x},\underline{\xi})\,. \tag{4.13}$$

For (t,x,ξ) close to $(0,\underline{x},\underline{\xi})$, the principal symbol $\Pi_0(t,x,\xi)$ is the spectral projector of $\xi\cdot\partial_vF(t,x,u(t,x),\partial_xu(t,x))$ corresponding to eigenvalues in $\{\mathrm{Im}\mu>0\}$. In particular

$$\underline{\Pi}:=\Pi_0(0,\underline{x},\underline{\xi}). \tag{4.14}$$

Since the system $(\Pi_0, \mathrm{Id}-\underline{\Pi})$ is elliptic near $(\underline{x}, \underline{\xi})$, there are symbols $U = \sum_{j\leq[\rho]} U_j$ and $V_j = \sum_{j\leq[\rho]}$ of degree zero such that

$$\underline{\Pi} \;=\; U \sharp \Pi_{|t=0} \;+\; V \sharp (\mathrm{Id} - \underline{\Pi}), \quad \text{near } (\underline{x}, \underline{\xi})\,.$$

This implies that for χ_1 supported in a sufficiently small conical neighborhood of $(\underline{x}, \underline{\xi})$

$$\chi_1(x, D_x)\underline{\Pi} = T_U T_{\Pi_{|t=0}} + T_V(\mathrm{Id} - \underline{\Pi}) + R \tag{4.15}$$

with R of order $-\rho$.

Suppose that the initial data satisfies

$$h \in H^s(\mathbb{R}^d)\,, \quad (\mathrm{Id} - \underline{\Pi})h \in H^{s+\rho}(\mathbb{R}^d). \tag{4.16}$$

Then (4.15) and (4.13) imply that $\chi_1(x, D_x)\underline{\Pi}h \in H^{s+\rho}$, that is $\underline{\Pi}h \in H^{s+\rho}$ near $(\underline{x}, \underline{\xi})$ or $(\underline{x}, \underline{\xi}) \notin WF_{H^{s+\rho}}(\underline{\Pi}h)$. ☐

Remark 4.3. a) Note that only the condition $u \in C^0(H^s)$ is used to prove (4.13).

b) The proof only relies on the ellipticity $\partial_t - T_{\chi D_I}$. Thus, the Sobolev spaces H^s do not play any particular role and there are analogous results in the Hölder spaces C^μ.

c) For semilinear equation, the critical index $1 + d/2$ can be decreased to $d/2$ as usual. This is also the case for system of conservation laws, since we only need to paralinearize functions of u.

One can push a little further the analysis when Assumption 4.1 is strengthened.

Assumption 4.4. The real eigenvalues of $\xi \cdot \partial_v F(p(t,x))$ are semi-simple and have constant multiplicity, and there are nonreal eigenvalues.

In this case, the condition $s' < 2s - 1 - d/2$ in Theorem 4.2 can be relaxed.

Theorem 4.5. *Under Assumption* 4.4*, for all $\sigma > d/2 + 1$, there are Cauchy data $h \in H^\sigma(\mathbb{R}^d)$, satisfying* (4.2) *such that for all $s > 2 + d/2$, all $T > 0$ and all neighborhood ω of $\underline{x}$, the Cauchy problem* (4.1) *has no solution $u \in C^0([0, T] : H^s(\omega))$.*

The meaning is that one can take σ very large and s very close to $2 + d/2 + 2$, so that u will be of class C^2, but not much smoother, while the initial data is as smooth as we want.

Proof. **a)** Suppose that $u \in C^0([0, T]; H^s(\omega))$ solves (4.1). We show that $u \in C^0([0, T']; H^{s'}(\omega'))$ for $T' < T$, $\omega' \subset\subset \omega$ and $s' \leq \sigma$ such that $s' < 2s - 2 - d/2$. It is sufficient to prove that $u \in C^0([0, T']; H^{s'}(\omega'))$ with $s' = \min(\sigma, 2s - 2 - d/2)$ when $\rho := s - 1 - d/2 \notin \mathbb{N}$.

The analogue is proved in [**23**], for m-th order scalar equations, when the real roots of the principal symbol are simple. As in the proof of Theorem 4.2, near any $\xi \neq 0$, there is an elliptic symbol $P = \sum_{j\leq[\rho]}$ such that $w := T_P u$ satisfies

$$\partial_t w - T_D w \in C^0(H^{s'})\,, \quad w_{|t=t_0} \in H^\sigma \quad \text{near } (0, \underline{x}, \xi)\,.$$

The matrix D is block diagonal. By Assumption 4.4, the blocks of the principal symbol D_0 are either hyperbolic, that is of the form $i\lambda\mathrm{Id}$ with $\lambda(t, x, \xi)$ real, or elliptic, meaning that the imaginary part of the eigenvalues is either positive or negative.

Since the Cauchy data is H^σ, the equation implies that hyperbolic blocks are microlocally $H^{s'}$. The same result holds for negative elliptic blocks. For positive elliptic block, we use the backward elliptic regularity, and decreasing the interval of time, we see that the elliptic modes are $C^0(H^{s+\rho})$. This shows that $w \in C^0(H^{s'})$. Since P is elliptic, u has the same regularity.

b) Repeating the argument in a), we deduce that any solution $u \in C^0(H^s)$ with initial data in H^σ is necessarily in $C^0(H^\sigma)$ on a smaller domain. Therefore, by Theorem 4.2, if

$$h \in H^\sigma(\mathbb{R}^d)\,, \quad (\mathrm{Id} - \underline{\Pi})h \in H^{\sigma'+\rho'}(\mathbb{R}^d)\,, \quad \underline{\Pi}_0 h \notin H^{\sigma'+\rho'}(\underline{x}, \underline{\xi})\,,$$

the Cauchy problem has no solution in $C^0(H^{\sigma'})$, thus no solution $u \in C^0(H^s)$. $\square$

5. An example of problems with elliptic zones

In this section we consider a modified version of (1.1). This is a nonhyperbolic form of Kirchhoff equation. The advantage is that we can make explicit computations, the drawback is that the equation is nonlocal[2]. The modified system reads:

$$\begin{cases} \partial_t u = a(t)\partial_x v, \\ \partial_t v = |a(t)|\partial_x u, \end{cases} \qquad \text{with} \quad a(t) = \|u(t)\|^2_{L^2} - 1. \tag{5.1}$$

As (1.1), this system has a natural (formal) energy:

$$E(t) = \|v(t,\cdot)\|^2_{L^2} + \Big| \|u(t,\cdot)\|^2_{L^2} - 1 \Big|. \tag{5.2}$$

If u is $C^1(L^2)$, the mapping $t \mapsto U(t) := \|u(t,\cdot)\|^2_{L^2}$ is C^1, thus $|U(t) - 1|$ is Lipschitzean and

$$\frac{d}{dt}|U(t) - 1| = \mathrm{sign}\big(U(t) - 1\big)\frac{dU(t)}{dt} \qquad a.e.$$

If (u, v) is in addition $C^0(H^1)$, then

$$\frac{dE(t)}{dt} = |a| \int v\partial_x u dx + a\mathrm{sign}(a) \int u\partial_x v dx = |a| \int \partial_x(uv)dx = 0$$

In the spirit of (1.4), we considered the filtered system, with truncated frequencies. Since the system has constant coefficients in x, it is sufficient to filter the initial data:

$$\begin{cases} \partial_t u^\lambda = a^\lambda(t)\partial_x v^\lambda, \\ \partial_t v^\lambda = |a^\lambda(t)|\partial_x u^\lambda, \end{cases} \qquad \begin{cases} u^\lambda{}_{|t=0} = 0, \\ v^\lambda{}_{|t=0} = S_\lambda h, \end{cases} \tag{5.3}$$

with $a^\lambda(t) = \|u^\lambda(t)\|^2_{L^2} - 1$ and S_λ is defined on the Fourier side by:

$$\widehat{S_\lambda h}(\xi) = 1_{\{|\xi| \le \lambda\}}\hat{h}(\xi).$$

[2]Thierry Colin proposed the simpler example : $\partial_t u = -(1 - \|\partial_x u\|^2_{L^2})\,\partial_x^2 u$. The system (5.1) is first order and fits the general presentation of this paper.

For Fourier transforms, the system reads for $|\xi| \le \lambda$:

$$\left\{\begin{aligned} &\partial_t \hat u^\lambda = i\xi a^\lambda(t) \hat v^\lambda, \\ &\partial_t \hat v^\lambda = i\xi |a^\lambda(t)| \hat u^\lambda, \end{aligned}\right. \qquad \left\{\begin{aligned} &\hat u^\lambda{}_{|t=0} = 0, \\ &\hat v^\lambda{}_{|t=0} = \hat h, \end{aligned}\right. \tag{5.4}$$

and $\hat u^\lambda = \hat v^\lambda = 0$ for $|\xi| \ge \lambda$. This is a system of ordinary differential equations, and it has local solutions, C^1 in time with values in L^2. One can use the energy E to prove that the solutions are global in time, but we provide a direct proof.

Suppose that (u^λ, v^λ) is defined and that $U^\lambda(t) := \|u^\lambda(t)\|^2_{L^2} \le 1$ on $[0, T]$. This is certainly true for T small. Then, $|a^\lambda| = -a^\lambda$ on this interval and for $|\xi| \le \lambda$:

$$\left\{\begin{aligned} &u^\lambda(t,\xi) = i \sinh\big(\xi A^\lambda(t)\big) h(\xi) \\ &v^\lambda(t,\xi) = \cosh\big(\xi A^\lambda(t)\big) h(\xi) \end{aligned}\right. \qquad A^\lambda(t) = t - \int_0^t U^\lambda(s) ds. \tag{5.5}$$

Therefore

$$U^\lambda(t) = \frac{1}{2\pi} \int_{|\xi|\le\lambda} \sinh^2(\xi A^\lambda(t))\, |\hat h(\xi)|^2 \, d\xi,$$

and

$$\frac{dU^\lambda}{dt} = (1 - U^\lambda(t)) I^\lambda(t) \tag{5.6}$$

with

$$I^\lambda(t) = \frac{1}{2\pi} \int_{|\xi|\le\lambda} \xi \sinh(2\xi A^\lambda(t))\, |\hat h(\xi)|^2 \, d\xi.$$

Since $U^\lambda(0) = 0$, the equation (5.6) implies that $1 - U^\lambda$ does not vanish and remains positive on $[0, T]$. In particular $U^\lambda(T) < 1$. By (5.5), there holds

$$\|v^\lambda(t)\|^2_{L^2} = U^\lambda(t) + \|S_\lambda h\|^2_{L^2} \le 1 + \|h\|^2_{L^2}.$$

Therefore, by continuation, this implies that (5.3) has a unique global solution in $C^0([0, +\infty[; L^2(\mathbb{R})$ and that $U^\lambda(t) < 1$ for all time. Moreover, $A^\lambda \ge 0$ and the integral I^λ is positive, implying that U^λ is strictly increasing.

Since U^λ is increasing, there holds $A(t) \le t(1 - U^\lambda(t))$. Therefore

$$\begin{aligned} 1 \ge U^\lambda(t) &\ge \frac{1}{8\pi} \int_{C_\lambda} \big(e^{2|\xi| t(1-a^\lambda(t))} - 2\big) \, |h(\xi)|^2 \, d\xi \\ &\ge e^{t(1-U^\lambda(t))\lambda} \int_{C_\lambda} \frac{1}{8\pi} |h(\xi)|^2 \, d\xi - \frac{1}{2}\|h\|^2_{L^2}. \end{aligned}$$

where $C_\lambda := \{\lambda 2 \le |\xi| \le \lambda\}$. Hence

$$1 - U^\lambda(t) \le \frac{1}{t\lambda}(\mu(\lambda) + K). \tag{5.7}$$

where $K = \ln(8\pi(1 + \|h\|^2))$ and

$$\mu(\lambda) = -\ln\Big(\int_{C_\lambda} |h(\xi)|^2 \, d\xi\Big) \tag{5.8}$$

The condition $\mu(\lambda) \le C\lambda$ implies that h is real analytic. On the other hand, for general non analytic functions, there holds

$$\lim_{\lambda\to+\infty} \frac{\mu(\lambda)}{\lambda} = 0. \tag{5.9}$$

Typically, for general H^s functions which are not smoother than H^s, $\mu \le C \ln \lambda$ with C related to s.

PROPOSITION 5.1. *Suppose that $h \in L^2(\mathbb{R})$ is not analytic in the sense that it satisfies* (5.9). *Then solutions (u^λ, v^λ) of* (5.3) *converge weakly to* $(0, h)$.

This means that the weak limits satisfy

$$\begin{cases} \partial_t \underline{u} = 0, & \underline{u}_{|t=0} = 0, \\ \partial_t \underline{v} = 0, & \underline{v}_{|t=0} = h. \end{cases} \tag{5.10}$$

These "limit" equations have nothing to see with the original ones (5.1), implying that (5.3) are *not* approximations of (5.1).

PROOF. The estimate (5.7) and (5.9) imply that for all $t > 0$, $U^\lambda(t) \to 1$ when $\lambda \to \infty$. Since $U^\lambda < 1$ and is increasing, thins implies that $A^\lambda(t) \to 0$, uniformly on compacts subsets of $]0, +\infty[$. By (5.5)

$$\widehat{u}^\lambda(t, \xi) \;\to\; 0 \qquad \hat{v}^\lambda(t, \xi) \to \hat{(}\xi),$$

uniformly on compacts of $]0, +\infty[\times\mathbb{R}$. $\square$

REMARK 5.2. The same analysis applies to more general initial data. One can for instance take $u(0, \cdot) = h \neq 0$, with $\|h\|_{L^2} < 1$ and $v(0, \cdot) = 0$. This only amounts to interchange cosh and sinh in (5.5).

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