# Chapter 3

# Fourier Synthesis

In this chapter we consider the Cauchy problem for constant coefficients equation, but with a slightly different approach. Our main goal is to give *stability estimates*. In particular, we show how the *maximal estimates* lead to the notions of *symmetrizability* and *strong hyperbolicity*.

We consider systems (2.1.1) $P(\partial)$, supposed to be hyperbolic in the direction $\nu$. The coefficients of $P$ are matrices in $\mathcal{L}(\mathbb{E})$ and $N$ denotes the dimension of $\dim \mathbb{E}$.

The *time* variable $\nu \cdot x$ plays a particular role, we call it $t$ and choose coordinates such that $x = (t, y) \in \mathbb{R} \times \mathbb{R}^d$, so that $\nu = (1, 0, \ldots, 0)$. In this case:

$$P(\partial_x) = P(\partial_t, \partial_y) = \sum_{j=0}^{m} A_j(\partial_y)\partial_t^{m-j} \tag{3.0.1}$$

where $A_j$ is a differential operator in $\partial_y$ of degree $j$. In particular $A_0$ is a constant matrix. The hyperbolicity assumption means

$$\det A_0 \neq 0; \tag{3.0.2}$$

$$(\tau, \eta) \in \mathbb{C} \times \mathbb{R}^d, \ |\mathrm{Im}\, \tau| > \gamma_0, \quad \Rightarrow \quad \det P(i\tau, i\eta) \neq 0 \tag{3.0.3}$$

for some $\gamma_0 \geq 0$.

The Cauchy problem reads

$$\begin{cases} P(\partial)u = f & for\ t > 0, \\ \partial_t^j u_{|t=0} = g_j & for\ j = 0, \ldots, m-1. \end{cases} \tag{3.0.4}$$

## 3.1 Fourier synthesis

Our main tool in this chapter is the partial Fourier transform with respect to the space variables $y$. With little risk of confusion, we denote it by $\hat{}$ or $\mathscr{F}$, and specify $\mathscr{F}_{space}$ when necessary.

Assuming that $u$ and $f$ are temperate distributions in $y$, the equation $Pu = f$ reads (at least formally)

$$\sum_{j=0}^{m} A_j(i\eta)\partial_t^{m-j}\hat{u}(t,\eta) = \hat{f}(t,\eta) \tag{3.1.1}$$

and the initial conditions become

$$\partial_t^j \hat{u}(0,\eta) = \hat{g}_j(\eta). \tag{3.1.2}$$

Introduce

$$U(t,\eta) = \begin{pmatrix} \langle\eta\rangle^{m-1}\hat{u} \\ \langle\eta\rangle^{m-2}\partial_t\hat{u} \\ \vdots \\ \partial_t^{m-1}\hat{u} \end{pmatrix}, \quad F(t,\eta) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hat{f} \end{pmatrix}, \quad G(\eta) = \begin{pmatrix} \langle\eta\rangle^{m-1}g_0 \\ \langle\eta\rangle^{m-2}g_1 \\ \vdots \\ g_{m-1} \end{pmatrix}, \tag{3.1.3}$$

$$\mathcal{A}(i\eta) = \begin{pmatrix} 0 & -\langle\eta\rangle & 0 & \dots & 0 \\ 0 & 0 & -\langle\eta\rangle & \dots & 0 \\ \vdots & & \ddots & \ddots & \\ \vdots & & & 0 & -\langle\eta\rangle \\ \tilde{A}_m(i\eta) & & \dots & \dots & \tilde{A}_1(i\eta) \end{pmatrix} \tag{3.1.4}$$

with

$$\langle\eta\rangle = (1+|\eta|^2)^{\frac{1}{2}}, \qquad \tilde{A}_j(i\eta) = \langle\eta\rangle^{1-j}A_0^{-1}A_j(i\eta). \tag{3.1.5}$$

The factors $\langle\eta\rangle^k$ have been introduced so that all the entries of $\mathcal{A}$ have the same order and $\mathcal{A} = O(\langle\eta\rangle)$. Then, the Cauchy problem can be written

$$\partial_t U + \mathcal{A}(i\eta)U = F, \qquad U_{|t=0} = G. \tag{3.1.6}$$

Hence, assuming integrability in time for $\hat{f}$,

$$U(t,\eta) = e^{-t\mathcal{A}(i\eta)}G(\eta) + \int_0^t e^{(s-t)\mathcal{A}(i\eta)}F(s,\eta)ds. \tag{3.1.7}$$

This method is successful if one can perform the inverse Fourier transform, that is if the mutliplicator $e^{-t\mathcal{A}(i\eta)}$ acts in $\mathscr{S}'(\mathbb{R}^d)$. The next proposition answers this question.

**Proposition 3.1.1.** *i) If for some $t_0 \neq 0$ there are $C$ and $M$ such that for all $\eta \in \mathbb{R}^d$*

$$\big|e^{-t_0\mathcal{A}(i\eta)}\big| \le C\langle\eta\rangle^M \tag{3.1.8}$$

*then $P(\partial)$ is hyperbolic in the time direction.*

*ii) Conversely, if $P(\partial)$ is hyperbolic in the time direction, there are $C$ and $\gamma$ such that for all $t$ and $\eta$*

$$\big|e^{-t\mathcal{A}(i\eta)}\big| \le C\langle\eta\rangle^{mN} e^{t\gamma_1}. \tag{3.1.9}$$

Before starting the proof, we collect several elementary remarks on matrices of the form (3.1.4).

**Lemma 3.1.2.** *Consider a matrix of the form*

$$\mathcal{A} = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \\ \vdots & & & 0 & -1 \\ A_m & & \dots & \dots & A_1 \end{pmatrix}, \tag{3.1.10}$$

*with $m$ blocks of dimension $N$ and $P(\tau) = \tau^m \mathrm{Id} + \sum_{j=0}^{m-1} \tau^j A_{m-j}$. Then,*

$$\det(\lambda\mathrm{Id} - \mathcal{A}) = (-1)^{mN} \det P(-\lambda). \tag{3.1.11}$$

*and given $C_0$, there is $C$ such that if $\sup_j |A_j| \le C_0$, then for all $\lambda$ not an eigenvalue of $\mathcal{A}$:*

$$C^{-1}\big|(\lambda\mathrm{Id} - \mathcal{A})^{-1}\big| \le \big(|\lambda| + 1\big)^{m-1}\big|P(-\lambda)^{-1}\big| \le C\big|(\lambda\mathrm{Id} - \mathcal{A})^{-1}\big|. \tag{3.1.12}$$

*Moreover, given $C_0$, there are $\delta_0$ and $C$ such that if $\sup_j |A_j| \le C_0$ and if $\mathcal{A}'$ is $mN \times mN$ matrix such that $|\mathcal{A}'| \le \delta_0$, $\mathcal{A} + \mathcal{A}'$ is conjugated to a matrix of the form* (3.1.10) *with entries $A_j + A'_j$ on the lower row such that $|A'_j| \le C|\mathcal{A}'|$.*

*Proof of Proposition* 3.1.1. **a)** By the lemma above, the roots of $P(-\lambda, i\eta)$ are eigenvalues of $\mathcal{A}(i\eta)$. The estimate (3.1.8) implies that they satisfy

$$e^{-t_0 \mathrm{Re}\,\lambda} = |e^{-t_0\lambda}| \le C\langle\eta\rangle^M.$$

When $t_0 > 0$ we deduce from Lemma 2.3.3 that there is $\gamma$ such that the roots of $\det P(-\lambda, i\eta) = 0$ satisfy $\mathrm{Re}\,\lambda \ge -\gamma$, thus the roots of $\det P(i\tau, i\eta) = 0$

satisfy $\operatorname{Im}\tau \geq -\gamma$ which means that $P$ is hyperbolic in the time direction $dt$.

When $t_0 < 0$ we conclude that the roots of $\det P(i\tau, i\eta) = 0$ satisfy $\operatorname{Im}\tau \leq \gamma$ , which means that $P$ is hyperbolic in the direction $-dt$, thus also in the direction $dt$ by Theorem 2.4.2.

**b)** To prove *ii)*, we use the representation

$$e^{-t\mathcal{A}} = \frac{1}{2i\pi} \int_{\mathcal{C}} e^{-t\lambda} \big(\lambda \mathrm{Id} - \mathcal{A}\big)^{-1} d\lambda \tag{3.1.13}$$

where $\mathcal{C}$ is a contour in $\mathbb{C}$ surrounding the spectrum of $A$. By assumption, it is located in a strip $|\operatorname{Re}\lambda| \leq \frac{1}{2}\gamma_1$. Moreover, there is a constant $K$ such that

$$|\mathcal{A}(i\eta)| \leq K\langle\eta\rangle.$$

If $t > 0$, we choose $\mathcal{C}$ to be the union of the half circle $\mathcal{C}_1 = \{|\lambda + \gamma_1| = R, \operatorname{Re}\lambda \geq -\gamma_1\}$ and of the diameter $\mathcal{C}_2 = \{|\operatorname{Im}\lambda| \leq R\ ,\ \operatorname{Re}\lambda = -\gamma_1\}$, where $R = 2\gamma_1 + 2K\langle\eta\rangle$. In particular, on $\mathcal{C}$, $\operatorname{Re}\lambda \geq -\gamma_1$ and $e^{-t\lambda} \leq e^{t\gamma_1}$.

On $\mathcal{C}_1$, $|\lambda| \geq \gamma_1 + 2K\langle\eta\rangle \geq \frac{1}{2}|\mathcal{A}(i\eta)|$ and thus $\big|(\lambda \mathrm{Id} - A)^{-1}\big| \leq 2|\lambda|^{-1} \leq 4/R$. This shows that the contribution of $\mathcal{C}_1$ to the integral (3.1.13) is less that $2e^{t\gamma_1}$.

The estimate (2.5.2) implies that that on $\mathcal{C}_2$

$$\big|P(-\lambda, i\eta)^{-1}\big| \leq C\langle\eta\rangle^{m(N-1)}$$

and thus by Lemma 3.1.2

$$\big|(\lambda \mathrm{Id} - A)^{-1}\big| \leq C\langle\eta\rangle^{mN-1}. \tag{3.1.14}$$

This implies that the contribution of $\mathcal{C}_2$ is bounded by the right hand side of (3.1.9).

If $t < 0$, we argue in a similar way, integrating over $-\mathcal{C}$. □

The estimate (3.1.9) allows us to apply the inverse Fourier transform to (3.1.7) when the data are temperate in $x$. For instance, in the scale of Sobolev spaces, one can state:

**Theorem 3.1.3.** *If the system is hyperbolic in time, then the Cauchy problem is well posed in Sobolev spaces in the following sense : if $\gamma$, $M$ and $C$ are chosen so that* (3.1.8) *is satisfied, then for all $T > 0$, $\sigma \in \mathbb{R}$, for all $g_j \in H^{\sigma+m-1-j}$ and $f \in L^1([0,T], H^\sigma)$ the Cauchy problem* (3.0.4) *has a unique* *solution* $u \in C^0([0,T]; H^{\sigma-M+m-1})$ *such that* $\partial_t^j u \in C^0([0,T]; H^{\sigma-M+m-1-j})$ *for* $j \leq m-1$ *and*

$$\begin{aligned}\sum_{j=1}^{m-1} \left\|\partial_t^j u(t)\right\|_{H^{\sigma-M+m-j-1}} \leq & Ce^{\gamma t} \sum_{j=1}^{m-1} \left\|g_j\right\|_{H^{\sigma+m-j-1}} \\ & + C \int_9^t e^{\gamma(t-s)} \left\|f(s)\right\|_{H^\sigma} ds.\end{aligned} \tag{3.1.15}$$

## 3.2 Maximal estimates and strong hyperbolicity

The best estimate one can expect in (3.1.9) is

$$\forall \eta \in \mathbb{R}^d, \forall t \geq 0 : \qquad \left|e^{-t\mathcal{A}(i\eta)}\right| \leq Ce^{\gamma t} \tag{3.2.1}$$

in which case the Theorem 3.1.3 holds with $M = 0$. It turns out that the condition above only depends on the principal part of $P$

$$P^{\mathrm{pr}}(\partial_x) = \sum A_{m-j}^{\mathrm{pr}}(\partial_y)\partial_t^j \tag{3.2.2}$$

where $A_k^{\mathrm{pr}}$ is the homogeneous part of degree $k$ of $A_k$. The principal part of $\mathcal{A}$ is defined as

$$\mathcal{A}^{\mathrm{pr}}(i\eta) = \lim_{\rho \to +\infty} \frac{1}{\rho} \mathcal{A}(i\rho\eta) = \begin{pmatrix} 0 & -|\eta| & 0 & \dots & 0 \\ 0 & 0 & -|\eta| & \dots & 0 \\ \vdots & & \ddots & \ddots & \\ \vdots & & & 0 & -|\eta| \\ \tilde{A}_m^{\mathrm{pr}}(i\eta) & & \dots & \dots & \tilde{A}_1^{\mathrm{pr}}(i\eta) \end{pmatrix} \tag{3.2.3}$$

with $\tilde{A}_k^{\mathrm{pr}} = |\eta|^{1-k} A_k^{\mathrm{pr}}(i\eta)$, so that $\mathcal{A}^{\mathrm{pr}}$ is homogeneous of degree one in $\eta$. Note also that $\mathcal{A}^{\mathrm{pr}}$ is odd in $\eta$ in the sense that $\mathcal{A}^{\mathrm{pr}}(-i\eta)$ is conjugated to $-\mathcal{A}^{\mathrm{pr}}(i\eta)$:

$$\mathcal{A}^{\mathrm{pr}}(-\eta) = -\mathcal{E}^1 \mathcal{A}^{\mathrm{pr}}(i\eta)\mathcal{E}, \qquad \mathcal{E} = \mathrm{diag}(\mathrm{Id}_{\mathbb{E}}, -\mathrm{Id}_{\mathbb{E}}, \mathrm{Id}_{\mathbb{E}} \dots). \tag{3.2.4}$$

**Proposition 3.2.1.** *There are* $C$ *and* $\gamma$ *such that the condition* (3.2.1) *is satisfied if and only if there is a constant* $C$ *such that*

$$\forall \eta \in \mathbb{R}^d, \forall t : \qquad \left|e^{t\mathcal{A}^{\mathrm{pr}}(i\eta)}\right| \leq C. \tag{3.2.5}$$

*Proof.* **a)** Suppose that we have (3.2.1). Then,

$$\left|e^{-t\frac{1}{\rho}\mathcal{A}(i\rho\eta)}\right| \le Ce^{\gamma t/\rho}.$$

Together with (3.2.3), this implies (3.2.5).

**b)** Conversely, we use the following remark which concerns exponential of matrices. Considering the ordinary differential equation $\partial_t u + (A+B)u = 0$, we see that

$$e^{-t(A+B)} = e^{-tA} + \int_0^t e^{(s-t)A} e^{-s(A+B)} ds$$

and therefore Gronwall's lemma implies that, for $t \ge 0$,

$$\left|e^{-tA}\right| \le Ce^{\gamma t} \quad \Rightarrow \quad \left|e^{-t(A+B)}\right| \le Ce^{(\gamma + C|B|)t}. \tag{3.2.6}$$

For $|\eta| \ge 1$, we have $A_k(i\eta) - A_k^{\mathrm{pr}}(i\eta) = O(|\eta|^{k-1})$ and here is a constant $K$ such that

$$|\mathcal{A}^{\mathrm{pr}}(i\eta) - \mathcal{A}(i\eta)| \le K \qquad for\ |\eta| \ge 1. \tag{3.2.7}$$

Therefore, (3.2.5) and (3.2.6) imply that

$$\left|e^{-t\mathcal{A}(i\eta)}\right| \le C^{Kt} \qquad for\ |\eta| \ge 1\ and\ t \ge 0$$

estimate (3.2.1) is satisfied for $|\eta| \ge 1$. The estimate (3.2.1) is clear for $|\eta| \le 1$ since there $\mathcal{A}(i\eta)$ is bounded. $\square$

The condition (3.2.5) has several equivalent formulations, as explained in the next proposition.

**Proposition 3.2.2.** *Given matrices $A(a)$ which depend on parameters $a \in \mathscr{A}$, The following conditions are equivalent*

*i) There is a real $C_1$ such that*

$$\forall t \in \mathbb{R}, \forall a \in \mathscr{A} \ : \qquad \left|e^{tA(a)}\right| \le C_1. \tag{3.2.8}$$

*ii) All the the eigenvalues $\lambda$ of $A(a)$ are purely imaginary and semi-simple and there is a real $C_2$ such that all the eigen-projectors $\Pi_\lambda(a)$ satisfy*

$$\forall a \in \mathscr{A} \ : \qquad \left|\Pi_\lambda(a)\right| \le C_2. \tag{3.2.9}$$

*iii) $A(a) - \lambda\mathrm{Id}$ is invertible when $\mathrm{Re}\,\lambda \ne 0$ and there is a real $C_3$ such that*

$$\forall \lambda \notin i\mathbb{R}\ \forall a \in \mathscr{A} \ : \qquad \left|\big(A(a) - \lambda\mathrm{Id}\big)^{-1}\right| \le C_3\left|\mathrm{Re}\,\lambda\right|^{-1}. \tag{3.2.10}$$

*iv) There are definite positive matrices $S(a)$ and there are constants $C_4$ and $c_4 > 0$ such that for all $a \in \mathscr{A}$, $S(a)A(a)$ is skew adjoint and*

$$|S(a)| \leq C_4, \qquad S(a) \geq c_4 \mathrm{Id}. \tag{3.2.11}$$

*v) There is a real $C_5$ such that for all matrix $B$, all $a \in \mathscr{A}$ and all $\rho \in \mathbb{R}$, the eigenvalues of $\rho A(a) + B$ are located in $\{|\mathrm{Re}\,\lambda| < C_5 |B|\}$.*

*Proof.* **a)** *ii)* implies that $A(a)$ has the spectral decomposition $A = \sum \lambda_j \Pi_j$ with $\lambda_j \in i\mathbb{R}$. Thus (3.2.9) implies that $|e^{tA)}| = |\sum e^{t\lambda_j}\Pi_j| \leq NC_2$.

Conversely, *i)* implies that the eigenvalues $\lambda_j$ of $A(a)$ are purely imaginary and semi-simple and thus that $A(a) = \sum \lambda_j \Pi_j$. Moreover,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{t(A(a)-\lambda_j \mathrm{Id})} dt = \sum_k \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{t(\lambda_k - \lambda_j \mathrm{Id})} \Pi_k dt = \Pi_j .$$

Thus, $|\Pi_j| \leq C_1$ if (3.2.8) is true.

**b)** Suppose that *ii)* is satisfied so that $A = \sum \lambda_j \Pi_j$ and $\mathrm{Id} = \sum \Pi_j$. Then

$$S(a) = \sum \Pi_j^* \Pi_j \tag{3.2.12}$$

is definite positive, satisfies $S \geq N^{-1}\mathrm{Id}$, $|S| \leq NC_2^2$, and $SA = \sum \lambda_j \Pi_j^* \Pi_j$ is skew adjoint.

If *iv)* holds then, with $\epsilon = -\mathrm{signRe}\,\lambda$ ,

$$c_4 |\mathrm{Re}\,\lambda| |u|^2 \leq \mathrm{Re}\,\epsilon\big(S(A - \lambda \mathrm{Id})u, u\big) \leq C_4 |(A-\lambda)u| \; |u|$$

implying *iii)* with $C_3 = C_4/c_4$.

If *iii)* is satisfied, then the eigenvalues of $A(a)$ are purely imaginary and semi-simple, for if there were a nondiagonal block in the Jordan's decomposition of $A - \lambda_j \mathrm{Id}$, the norm of $(A - (\lambda_j + \gamma)\mathrm{Id})^{-1}$ would be at least of order $\gamma^{-2}$ when $\gamma \to 0$. Thus $A = \sum \lambda_j \Pi_j$ and

$$\lim_{\gamma\to 0} \gamma\big(A - (\lambda_j + \gamma)\mathrm{Id}\big)^{-1} = \sum_k \lim_{\gamma \to 0} \frac{\gamma}{(\lambda_k - \lambda_j + \gamma)} \Pi_k = \Pi_j ,$$

hence $|\Pi_j| \leq \; C_3$.

**c)** By homogeneity, *iii)* is equivalent to the condition

$$\forall a \in \mathscr{A}, \forall \rho \in \mathbb{R}, \qquad |\mathrm{Re}\,\lambda| \geq C_3 \quad \Rightarrow \quad \big|\big(\rho A(a) - \lambda \mathrm{Id}\big)^{-1}\big| \leq 1.$$

By Lemma 3.2.3 below, this is equivalent to the condition that for all matrix $B$ such that $|B| < 1$, $\rho A - \lambda \mathrm{Id} + B$ is invertible when $|\mathrm{Re}\,\lambda| \geq C_3$, meaning that the spectrum of $\rho A + B$ is contained in $\{|\mathrm{Re}\,\lambda| < C_3$. By homogeneity, this is equivalent to $v)$ with $C_5 = C_3$.

To complete the proof of the proposition, it only remains to prove the next lemma. □

**Lemma 3.2.3.** *The matrix $A$ is invertible with $|A^{-1}| \leq \kappa$ if and only if $A + B$ is invertible for all $B$ such that $|B| < \kappa^{-1}$.*

*Proof.* If $|A^{-1}| \leq \kappa$, then $A + B = A^{-1}(\mathrm{Id} + A^{-1}B)$ is invertible for all $B$ such that $|A^{-1}B| \leq \kappa |B| < 1$.

Conversely, if $A$ is not invertible or if $|A^{-1}| > \kappa$, there is $\underline{u}$ such that $|\underline{u}| = 1$ and $|A\underline{u}| < \kappa^{-1}$. Pick a linear form $\ell$ such that $\ell(\underline{u}) = 1$ and $|\ell| = 1$. Then the matrix $B$ defined by $Bu = \ell(u)A\underline{u}$ satisfies $|B| = |A\underline{u}| < \kappa^{-1}$ but $A - B$ is not invertible since $\underline{u}$ is in its kernel. □

**Corollary 3.2.4.** *The estimate* (3.2.5) *is satisfied if and only if there is a constant $C$ such that for all $(\tau, \eta) \in \mathbb{C} \times \mathbb{R}^d$ and all $u \in \mathbb{E}$:*

$$|\mathrm{Im}\,\tau|(|\tau| + |\eta|)^{m-1}|u| \leq C\big|P^{\mathrm{pr}}(i\tau, i\eta)u\big|. \tag{3.2.13}$$

*Proof.* The proposition above implies that (3.2.5) is equivalent to the estimate

$$\big|\big(\lambda - \mathcal{A}^{\mathrm{pr}}(i\eta)\big)^{-1}\big| \leq C|\mathrm{Re}\,\lambda|^{-1}.$$

When $|\lambda|^2 + |\eta|^2 = 1$, this is equivalent to

$$\big|P^{\mathrm{pr}}(-\lambda, i\eta)\big)^{-1}\big| \leq C|\mathrm{Re}\,\lambda|^{-1}.$$

as shown in Lemma 3.1.2. By homogeneity, this condition is equivalent to (3.2.13). □

This motivates the following definition. We say that $P$ is strongly hyperbolic in the time direction when (3.2.13) is satisfied. Extended to general direction, the definition reads:

**Definition 3.2.5.** *Consider a differential system $P(\partial_x)$ of order $m$ with principal part $P^{\mathrm{pr}}$. It is said to be strongly hyperbolic in the direction $\nu$ if there is a constant $C$ such that for all $\xi \in \mathbb{R}^n$, $\gamma$ real and $u \in \mathbb{E}$:*

$$|\gamma|(|\gamma| + |\xi|)^{m-1}|u| \leq C\big|P^{\mathrm{pr}}(i\xi + \gamma\nu)u\big|. \tag{3.2.14}$$

Note that for $\xi = 0$, this implies that $P^{\mathrm{pr}}(\nu) \neq 0$.

**Theorem 3.2.6.** *Suppose that $P(\partial_t, \partial_y)$ is strongly hyperbolic in the time direction $dt$. Let $\Gamma$ denote de cone of hyperbolic directions which contain $dt$. Then is is strongly hyperbolic in all directions $\theta \in \Gamma$*