Chapter 4

Symmetric systems. The $L^2$ linear theory

4.1 Symmetric systems, preliminaries

4.1.1 Definitions

Consider the

\[ L = \sum_{j=0}^d \tilde{A}_j(\tilde{x}) \partial_{x_j} + \tilde{B}, \quad \tilde{x} = (x_0, \ldots, x_d) = (t, x) \]

Our goal is to solve the Cauchy problem

\[ \begin{cases} Lu = f, & t \in [0, T], \ x \in \mathbb{R}^d, \\ u|_{t=0} = h, \end{cases} \]

assuming that the system is symmetric in the following sense:

**Definition 4.1.1.** $L$ is symmetric hyperbolic if the $A_j$ are symmetric and $\tilde{A}_0$ is positive definite.

\[ \tilde{A}_0^{-1} = \partial_t + \sum_{j=1}^d \tilde{A}_j(\tilde{x}) \partial_{x_j} + B, \quad \tilde{x} = (x_0, \ldots, x_d) = (t, x) \]

**Lemma 4.1.2.** For all $\tilde{x}$, $\tilde{L}(\tilde{x}, \tilde{\xi})$ is strongly hyperbolic in the direction $dt = (1, 0, \ldots, 0)$ and the cone of hyperbolic directions $\Gamma_{\tilde{x}}$ is the set of $\tilde{\xi}$ such that $\tilde{L}(\tilde{x}, \tilde{\xi})$ is positive definite.

**Assumption 4.1.3.** The coefficients $\tilde{A}_j$ are Lipschitz continuous.
4.1.2 Adoints and weak solutions

Lemma 4.1.4. Let $a \in W^{1,\infty}(\Omega)$. For $u \in H^1(\Omega)$ [resp. $L^2(\Omega)$], $a \partial_{x_j} u$ is well defined in $L^2(\Omega)$ [resp. $H^{-1}(\Omega)$]. In particular, for $u \in L^2(\Omega)$ and $v \in H^1_0(\Omega),$

$$\langle a \partial_{x_j} u, v \rangle_{H^{-1} \times H^1_0} = - \int u \partial_{x_j} (au) dx.$$

The adjoint of $L$ is

$$L^* = \sum_{j=0}^d -\partial_{x_j} \tilde{A}^*_j + \tilde{B}^*.$$  \hspace{1cm} (4.1.4)

Corollary 4.1.5. For $u \in H^1(\tilde{\Omega})$ [resp. $L^2(\tilde{\Omega})$], $Lu$ is well defined in $L^2(\tilde{\Omega})$ [resp. $H^{-1}(\tilde{\Omega})$. There is a similar result for $L^*$ and for $u \in L^2(\tilde{\Omega})$ and $v \in H^1_0(\tilde{\Omega})$,

$$\langle Lu, v \rangle_{H^{-1} \times H^1_0} = \int u(\tilde{x}) L^* v(\tilde{x}) d\tilde{x}.$$

In particular, for $u \in L^2(\tilde{\Omega})$ and $f \in L^2(\tilde{\Omega})$, the equation $Lu = f$ is satisfied in the weak sense, that is in $H^{-1}(\tilde{\Omega})$, if and only if

$$\forall v \in H^1_0(\tilde{\Omega}), \quad \int u(\tilde{x}) L^* v(\tilde{x}) d\tilde{x} = \int f(\tilde{x}) v(\tilde{x}) d\tilde{x}. \hspace{1cm} (4.1.5)$$

4.1.3 Weak and strong solutions of the Cauchy problem

Lemma 4.1.6. If $u \in L^2([0,T] \times \mathbb{R}^d)$ and $\partial_t u \in L^2([0,T]; H^{-1} \mathbb{R}^d)$, then $u \in C^0([0,T]; H^{-\frac{1}{2}}(\mathbb{R}^d))$ and for all $v \in H^1([0,T] \times \mathbb{R}^d),$

$$- \int u(\tilde{x}) \partial_{t} v(\tilde{x}) d\tilde{x} = \int_0^T \langle \partial_t u(t), v(t) \rangle_{H^{-1} \times H^1} dt + \langle u(0), v(0) \rangle_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}} - \langle u(T), v(T) \rangle_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}. \hspace{1cm} (4.1.6)$$

Also recall that $H^1([0,T] \times \mathbb{R}^d) \subset C^0([0,T]; H^{\frac{1}{2}}(\mathbb{R}^d)).$

Corollary 4.1.7. If $u \in L^2([0,T] \times \mathbb{R}^d)$ and $Lu \in L^2([0,T] \times \mathbb{R}^d)$, then $u \in C^0([0,T]; H^{-\frac{1}{2}}(\mathbb{R}^d))$ and for all $v \in H^1([0,T] \times \mathbb{R}^d),$

$$\int u(\tilde{x}) L^* v(\tilde{x}) d\tilde{x} = \int f(\tilde{x}) v(\tilde{x}) d\tilde{x} + \langle u(0), v(0) \rangle_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}} - \langle u(T), v(T) \rangle_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}. \hspace{1cm} (4.1.7)$$
Definition 4.1.8 (Weak $L^2$ solutions of the Cauchy problem). It makes sense.

Corollary 4.1.9. For $f \in L^2([0,T] \times \mathbb{R}^n)$ and $h \in L^2(\mathbb{R}^n)$, $u \in L^2([0,T] \times \mathbb{R}^n)$ is a weak solution of (4.1.2) if and only if, for all $v \in H^1$ such that $v|_{t=T} = 0$, one has

\[
(4.1.8) \quad \int_{[0,T] \times \mathbb{R}^n} f \cdot v \, dt \, dx + \int_{\mathbb{R}^n} h v|_{t=0} \, dx = \int_{[0,T] \times \mathbb{R}^n} u \cdot \nabla v \, dt \, dx.
\]

Definition 4.1.10 (Strong $L^2$ solutions of the Cauchy problem). For $f \in L^2([0,T] \times \mathbb{R}^n)$ and $h \in L^2(\mathbb{R}^n)$, $u \in L^2([0,T] \times \mathbb{R}^n)$ is said to be a strong solution of (4.1.2) if there is sequences $u_k \in H^1([0,T] \times \mathbb{R})$ such that in the limit $k \to +\infty$:

\[
i) \quad \|u - u_k\|_{L^2([0,T] \times \mathbb{R}^n)} \to 0,
\]

\[
ii) \quad \|h - u_k|_{t=0}\|_{L^2(\mathbb{R}^n)} \to 0,
\]

\[
iii) \quad \|f - Lu_k\|_{L^2([0,T] \times \mathbb{R}^n)} \to 0.
\]

Lemma 4.1.11. Strong solutions are weak solutions.

4.2 The $L^2$ energy estimate.

4.2.1 The energy balance

Lemma 4.2.1. If the matrices $A_j$ are symmetric, and $u \in H^1(\bar{\Omega})$ then

\[
2 \text{Re} Lu.\bar{u} = \sum_{j=0}^d \partial_{x_j} (A_j u.\bar{u}) + Ku.\bar{u} \in L^1(\bar{\Omega}).
\]

with $K = 2 \text{Re} B - \sum_{j=0}^d \partial_{x_j} A_j$.

Corollary 4.2.2. If the matrices $A_j$ are symmetric, and $u \in H^1([0,T] \times \mathbb{R}^d)$

\[
2 \text{Re} \int_{[0,T] \times \mathbb{R}^d} Lu.\bar{u} \, d\bar{x} = \int_{[0,T] \times \mathbb{R}^d} Ku.\bar{u} \, d\bar{x}
\]

\[
(4.2.1) \quad + \int_{\mathbb{R}^d} A_0 u \bar{u}(T, x) \, dx - \int_{\mathbb{R}^d} A_0 u \bar{u}(0, x) \, dx.
\]

Proposition 4.2.3. If $L$ is hyperbolic symmetric with Lipschitz coefficients, then there are constants $C$ and $\gamma$ such that for all $u \in H^1([0,T] \times \mathbb{R}^d)$

\[
\|u(t)\|_{L^2} \leq Ce^{\gamma t}\|u(0)\|_{L^2} + C \int_0^t e^{\gamma(t-t')}\|Lu(t')\|_{L^2} \, dt'.
\]

Remark 4.2.4. On $C$ and $\gamma$. 37
4.2.2 Uniqueness of strong solutions

**Theorem 4.2.5.** If the system is hyperbolic symmetric, then any strong solution belongs to $C^0([0,T]; L^2)$ and satisfies the energy estimate (4.2.2). In particular, strong solutions are unique.

*Proof.* Let $u$ be a strong solution and $u_k$ an approximating sequence. The estimate (4.2.2) can be applied to $u_k$ and also to $u_k - u_l$, proving that the $u_k$ are bounded and form a Cauchy sequence in $C^0([0,T]; L^2)$. Therefore the limit $u$ is also in this space, and passing to the limit in the estimates for the $u_k$ we get the estimate for $u$. \qed

4.3 Existence of weak solution

4.3.1 The duality method

The system $L^*$ is hyperbolic symmetric. Therefore there are energy estimates for $L^*$ and changing $t$ to $T - t$, we obtain that for $v \in H^1([0,T] \times \mathbb{R}^d)$ et $t \in [0,T]$ on a

$$\|v(t)\|_{L^2} \leq C \int_t^T \|L^*v(t')\|_{L^2} dt' + C\|v(T)\|_{L^2}.$$ 

Introduce the space $H^1$ of functions $v \in H^1([0,T] \times \mathbb{R}^n)$ such that $v|_{t=T} = 0$. The estimate above implies the following lemma.

**Lemma 4.3.1.** There is a constant $C$ such that for all $v \in H^1$ on $a$ :

$$\|v(0)\|_{L^2(\mathbb{R}^d)} + \|v\|_{L^2([0,T] \times \mathbb{R}^d)} \leq C\|L^*v\|_{L^2([0,T] \times \mathbb{R}^d)}.$$ 

**Theorem 4.3.2.** For all $f \in L^2([0,T] \times \mathbb{R}^d)$ and $h \in L^2(\mathbb{R}^d)$, the problem (4.1.2) has a solution $u \in L^2([0,T] \times \mathbb{R}^d)$.

*Proof.* Consider the space $\mathcal{F} = \{L^*v; v \in H^1\}$ which is a subspace of $L^2([0,T] \times \mathbb{R}^d)$. The mapping $L$ from $H^1$ $L^2$ is injective by (4.3.1). Thus there is a linear inverse mapping $J : \mathcal{F} \mapsto H^1$. For all $g \in \mathcal{F}$ one has $L^*Jg = g$ and by (4.3.1)

$$\|L^*Jg\|_{L^2(\mathbb{R}^d)} + \|g\|_{L^2([0,T] \times \mathbb{R}^d)} \leq C\|g\|_{L^2([0,T] \times \mathbb{R}^d)}.$$ 

Consider the anti-linear form on $H^1$ :

$$\Phi(v) = \int_{[0,T] \times \mathbb{R}^d} f \cdot v \, dt \, dx + \int_{\mathbb{R}^d} h \cdot v|_{t=0} \, dx$$

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and the antilinear form $\Psi$ on $\mathcal{F}$

\[ (4.3.4) \quad \Psi(g) = \Phi(Jg). \]

By (4.3.2) que

\[ (4.3.5) \quad |\Psi(g)| \leq M \|g\|_{L^2([0,T] \times \mathbb{R}^d)} \]

with $M = C(\|f\|_{L^2} + \|h\|_{L^2})$. Hence $\Psi$ can be continuously extended to the closure of $\mathcal{F}$ in $(L^2([0,T] \times \mathbb{R}^d)$, and next on $(L^2([0,T] \times \mathbb{R}^d)$ as an antilinear form with norm less than or equal to $M$. By Riesz Theorem, there is $u \in L^2([0,T] \times \mathbb{R}^d)$ such that for all $g \in L^2$:

\[
\Psi(g) = \int_{[0,T] \times \mathbb{R}^d} u \cdot \nabla g \, dt \, dx.
\]

Therefore, for all $v \in H^1$,

\[
\Phi(v) = \int_{[0,T] \times \mathbb{R}^d} u \cdot \nabla v \, dt \, dx.
\]

This is precisely (4.1.8) and thus $u$ is a solution of (4.1.2). \hfill \Box

### 4.3.2 The approximation method

Let us explain the principle first. The idea is to replace the spatial derivatives $\partial_{x_j}$ by approximations $\partial_{x_j}^\varepsilon$ such that for all $\varepsilon > 0$ the $\partial_{x_j}^\varepsilon$ are bounded operators in $L^2(\mathbb{R}^d)$. Of course, their norm in $L^2$ tends to $+\infty$ as $\varepsilon$ goes to 0, but we assume that they are uniformly bounded from $L^2$ to $H^{-1}$: there is a constant $C$ such that

\[ (4.3.6) \quad \|\partial_{x_j}^\varepsilon u\|_{H^{-1}} \leq C\|u\|_{L^2}. \]

The adjoint operators in $L^2$, $\partial_{x_j}^{\varepsilon*}$, which need not be exactly $-\partial_{x_j}^\varepsilon$, are bounded from $H^1$ to $L^2$:

\[ (4.3.7) \quad \|\partial_{x_j}^{\varepsilon*} v\|_{L^2} \leq C\|v\|_{H^1}. \]

Moreover, $\partial_{x_j}^\varepsilon$ approximates $\partial_{x_j}$ in the distribution sense, that is

\[ (4.3.8) \quad \forall u \in L^2(\mathbb{R}^d), \quad \partial_{x_j}^\varepsilon u \to \partial_{x_j} u \text{ in } H^{-1}, \]

\[ \forall v \in H^1(\mathbb{R}^d), \quad \partial_{x_j}^{\varepsilon*} v \to -\partial_{x_j} v \text{ in } L^2. \]
Consider

\begin{equation}
L^\varepsilon = A_0 \partial_t + \sum_{j=1}^d A_j \partial_j^\varepsilon + B = A_0(\partial_t + K^\varepsilon).
\end{equation}

For all \( \varepsilon > 0 \), \( K^\varepsilon \) is bounded in \( L^2 \) and thus the Cauchy Lipschitz theorem implies that

**Lemma 4.3.3.** For all \( \varepsilon \in ]0, 1] \), \( h \in L^2(\mathbb{R}^d) \), \( f \in L^1([0, T]; L^2(\mathbb{R}^d)) \) the problem

\begin{equation}
L^\varepsilon u^\varepsilon = f, \quad u^\varepsilon_{|t=0} = h
\end{equation}

has a unique solution \( u^\varepsilon \in C^0([0, T]; L^2(\mathbb{R}^d)) \).

**Theorem 4.3.4.** Suppose that the family \( u^\varepsilon \) is bounded in \( C^0([0, T]; L^2) \). Then the Cauchy problem (4.1.2) has a weak solution \( u \in L^2([0, T] \times \mathbb{R}^d) \).

**Proof.** Using (4.3.6), and the we see that \( \partial_t^\varepsilon \) is bounded in \( L^\infty([0, T]; H^{-1}) \) and more precisely that there is \( C \) such that for all \( \varepsilon \in ]0, 1] \):

\[ \|u^\varepsilon(t) - u^\varepsilon(t')\|_{H^{-1}} \leq C|t - t'|. \]

Hence, by Ascoli’s theorem there is a subsequence, still denoted by \( u^\varepsilon \), which converges in \( C^0([0, T]; L^2_{weak}) \) where \( L^2_{weak} \) is the \( L^2 \) space equipped with the weak topology. The convergence in \( C^0([0, T]; L^2_{weak}) \) means that for all \( \varphi \in L^2(\mathbb{R}^d) \), the function \( (u^\varepsilon(t), \varphi)_{L^2} \) converges to \( (u(t), \varphi)_{L^2} \) uniformly in time. In particular, \( u \in L^2([0, T] \times \mathbb{R}^d) \).

For \( v \in H^1([0, T] \times \mathbb{R}^d) \) with \( v(T) = 0 \), one has

\[ \int f \cdot \nu dt dx + \int h \cdot \nu_{|t=0} dx = \int u^\varepsilon \cdot L^{\varepsilon^*} v dt dx \]

where

\[ L^{\varepsilon^*} = -\partial_t A_0 - \sum \partial_j A_j^* + B^* \]

Passing to the limit in \( \varepsilon \) implies that \( u \) is a weak solution of (4.1.2). \( \square \)

**Example 1.**

Let \( J_\varepsilon = (1 - \varepsilon \Delta_x)^{-\frac{1}{2}} \) and \( \partial_j^\varepsilon = \partial_j J_\varepsilon \).

**Proposition 4.3.5.** With this choice, the assumption of Theorem 4.3.4 is satisfied.
Sketch of the proof. We repeat the proof of the energy estimate for $L^\varepsilon$. Because of the boundedness in $L^2$, we can write

$$2\text{Re} \left( A_j \partial_j^\varepsilon u^\varepsilon, u^\varepsilon \right)_{L^2} = \left( (A_j \partial_j^\varepsilon - \partial_j^\varepsilon A_j) u^\varepsilon, u^\varepsilon \right)_{L^2}.$$ 

Using a result of Coifman and Meyer, one can show that the $(A_j \partial_j^\varepsilon - \partial_j^\varepsilon A_j)$ are uniformly bounded in $L^2$. From here the proof continues as for Proposition 4.2.3.

Example 2. We use finite differences: for $j = 1, \ldots, d$, and $\varepsilon \in ]0,1]$, let

$$(4.3.11) \quad \partial_j^\varepsilon u(x) = \frac{1}{2\varepsilon} \left( u(x + \varepsilon e_j) - u(x - \varepsilon e_j) \right)$$

where $\{e_1, \ldots, e_d\}$ is the canonical basis of $\mathbb{R}^d$.

Proposition 4.3.6. With this choice, the assumption of Theorem 4.3.4 is satisfied.

We start with a preliminary estimate.

Lemma 4.3.7. Suppose that $A(x)$ is symmetric and Lipschitz, and $u \in L^2(\mathbb{R}^d)$. Then

$$(4.3.12) \quad \left| \text{Re} \int A_j(x) \partial_j^\varepsilon u(x) \overline{u(x)} dx \right| \leq \left\| \partial_j A \right\|_{L^\infty} \left\| u \right\|^2_{L^2}.$$ 

Proof. Let

$$w(x, y) := 2\text{Re} \ A(x)(u(x + y) - u(x - y))\overline{\pi(x)}$$

$$= A(x)u(x + y)\overline{\pi(x)} - A(x)u(x - y)\overline{\pi(x)}$$

$$+ A(x)u(x)\overline{\pi(x + y)} - A(x)u(x)\overline{\pi(x - y)}.$$ 

Hence

$$\int w(x, y) dx = \int (A(x) - A(x + y))u(x + y)\overline{\pi(x)} dx$$

$$+ \int (A(x) - A(x))u(x)\overline{\pi(x - y)} dx$$

$$\leq 2\|y\|\left\| \partial A \right\|_{L^\infty} \left\| u \right\|^2_{L^2}.$$ 

which implies the lemma. □

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Proof of Proposition 4.3.6. Consider the energy

$$E^\varepsilon = \int_{\mathbb{R}^d} A_0 u^\varepsilon \, dx = \left( A_0 u(t), u(t) \right)_{L^2} \approx \left\| u^\varepsilon(t) \right\|^2_{L^2}.$$  

We have

$$\frac{d}{dt} E^\varepsilon = (\partial_t A_0 u^\varepsilon(t), u^\varepsilon(t))_{L^2} + 2 \text{Re} \left( f(t), u^\varepsilon(t) \right)_{L^2} + \sum_{j=1}^d \int w_j(t, x) \, dx$$

with

$$w_j = 2 \text{Re} A_j \partial_j u^\varepsilon \overline{u^\varepsilon}.$$  

The Lemma implies that

$$\frac{d}{dt} E^\varepsilon \leq C_0 \left\| f(t) \right\|_{L^2} \sqrt{E^\varepsilon} + C_1 E^\varepsilon +$$

and the proposition follows. \(\square\)

4.4 Strong solutions of the Cauchy problem

4.4.1 Weak = strong

We are given a weak solution \(u\) and we want to exhibit a sequence \(u_k\) satisfying the properties listed in the Definition 4.1.10. The principle of the proof is as follows. We look for mollifiers \(J_\varepsilon\) which satisfy the following properties:

1. For all \(\varepsilon > 0\), \(J_\varepsilon\) is a bounded operator from \(L^2(\mathbb{R}^d)\) to \(H^1(\mathbb{R}^d)\) and from to \(H^{-1}(\mathbb{R}^d)\) to \(L^2(\mathbb{R}^d)\);

2. The family \(\{J_\varepsilon, \varepsilon \in [0, 1]\}\) is bounded in the space of operators from \(L^2\) to \(L^2\) and for all \(u \in L^2\) [resp. \(H^1\)], \(J_\varepsilon u \to u \) in \(L^2\) [resp \(H^1\)] as \(\varepsilon \to 0\);

3. For all \(j\), the family of operators \(\{A_0^{-1} A_j(t, x) \partial_{x_j}, J_\varepsilon, \varepsilon \in [0, 1], t \in [0, T]\}\) is bounded in the space of operators from \(L^2\) to \(L^2\).

Proposition 4.4.1. If there exist operators \(J_\varepsilon\) satisfying the properties above, then for all \(f \in L^2([0, T] \times \mathbb{R}^d)\) and \(h \in L^2(\mathbb{R}^d)\), any weak solution \(u \in L^2([0, T] \times \mathbb{R}^d)\) of the problem (4.1.2) is a strong solution.
Proof. Consider the commutators $C^\varepsilon_j = [A_0^{-1} A_j(t, x)\partial_{x_j}, J_\varepsilon]$ acting in $L^2([0, T] \times \mathbb{R}^d)$. By the property 3, they are uniformly bounded, and by property 2, $C^\varepsilon_j v \to 0$ in $L^2$ when $v \in H^1([0, T] \times \mathbb{R}^d)$. By density of $H^1$ in $L^2$ we conclude that

$$\|C^\varepsilon_j u\|_{L^2} \to 0.$$  

Write $L = A_0(\partial_t + K)$. What we have proved is that $[K, J_\varepsilon]u \to 0$ in $L^2([0, T] \times \mathbb{R}^d)$.

Because $u \in L^2([0, T] \times \mathbb{R}^d)$ and $\partial_t u \in L^2([0, T]; H^{-1}(\mathbb{R}^d))$, one easily shows that

1) $\partial_t J_\varepsilon u = J_\varepsilon \partial_t u$, \hspace{1cm} in $L^2([0, T]; H^{-1}(\mathbb{R}^d))$,

2) $(J_\varepsilon u)|_{t=0} = J_\varepsilon(u|_{t=0})$, \hspace{1cm} in $H^{-\frac{1}{2}}(\mathbb{R}^d)$.

Hence we have

1) $J_\varepsilon u \to u$ \hspace{1cm} in $L^2([0, T]; H^{-1}(\mathbb{R}^d))$

2) $L J_\varepsilon u = A_0(J_\varepsilon A_0^{-1} f + [K, J_\varepsilon]u \to f$ in $L^2([0, T]; H^{-1}(\mathbb{R}^d))$

3) $(J_\varepsilon u)|_{t=0} = J_\varepsilon h \to h$ \hspace{1cm} in $L^2(\mathbb{R}^d)$.

proving that $u$ is a strong solution. \hfill \square

4.4.2 Friedrichs Lemma

Consider a function $j \in C_0^\infty(\mathbb{R}^d)$, $j \geq 0$, with

$$\int j(x)dx = 1.$$  

Let

$$j_\varepsilon(x) = \varepsilon^{-d} j(x/\varepsilon), \hspace{1cm} J_\varepsilon u = j_\varepsilon \ast u.$$  

Lemma 4.4.2. The operators $J_\varepsilon$ have the properties 1, 2, 3 listed above.

Proof. Consider a function Lipschitz function $a$ and $u \in H^1$. Let $K_\varepsilon u = J_\varepsilon(a\partial_{x_j} u) - a\partial_{x_j} J_\varepsilon u$. Then

$$K_\varepsilon u(x) = \int j_\varepsilon(y)(a(x-y) - a(x))\partial_{x_j} u(x-y)dy$$

$$K_\varepsilon u(x) = \int K_\varepsilon(x,y)u(x-y)dy.$$
with

\[ K_\varepsilon(x, y) = \partial_y \left( \hat{\jmath}_\varepsilon(y)(a(x - \varepsilon y) - a(x)) \right) \]

One has

\[ |K_\varepsilon(x, y)| \leq 2\|\nabla a\|_{L^\infty} \bar{\jmath}_\varepsilon(y) \]

with

\[ \bar{\jmath}_\varepsilon(y) = \varepsilon^{-d} \tilde{\jmath}(y/\varepsilon), \quad \tilde{\jmath}(y) = j(y) + |y| |\partial_y j(y)|. \]

Hence

\[ |K_\varepsilon u(x)| \leq \int C \bar{\jmath}_\varepsilon(y) |u(x - y)| dy \]

and

\[ (4.4.3) \quad \|K_\varepsilon u\|_{L^2} \leq C \|\tilde{\jmath}\|_{L^1} \|u\|_{L^2}. \]

By density of smooth functions in \( L^2 \), the estimate implies the \( K_\varepsilon \) are uniformly bounded functions from \( L^2 \) into \( L^2 \). Because \( K_\varepsilon u \to 0 \) in \( L^2 \) when \( H^1 \), the uniform bound also implies that

\[ \forall u \in L^2, \lim_{\varepsilon \to 0} \|K_\varepsilon u\|_{L^2} = 0. \]

The proof is similar when \( a \) and \( u \) also depend on \( t \), and for matrices and vectors.

\[ \square \]

### 4.5 The local theory

#### 4.5.1 The cone of hyperbolic directions

**Proposition 4.5.1.** The cone \( \Gamma(t, x) \) of hyperbolic directions at \( (t, x) \) is the set of \( \nu = (\nu_0, \nu_1, \ldots, \nu_d) \) such that the matrix \( \sum \nu_j A_j(t, x) \) is definite positive.

**Proof.** \[ \square \]

**Lemma 4.5.2.** Let \( \lambda_k(t, x, \xi) \) denote the eigenvalues of \( \sum_{j=1}^d \xi_j A_0^{-1} A_j(t, x, \xi) \) and

\[ (4.5.1) \quad c = \max_{[0,T] \times \mathbb{R}^d \times S^{d-1}} \max_k |\lambda_k(t, x, \xi)| < +\infty. \]

Then

\[ (4.5.2) \quad \Gamma = \{ \nu_0 > c|\nu'| \} \subseteq \cap_{t,x} \Gamma(t, x). \]

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Proof. This is clear when \( \nu' = 0 \). When \( \nu' \neq 0 \), we can assume that \( |\nu'| = 1 \) and the assumption is that thus \( \nu_0 > c \). Thus the eigenvalues of \( A := \nu_0 \text{Id} + \sum \nu_j A_0^{-1} A_j \) are positive, as well as the eigenvalues of the conjugate matrix \( A_0^{1/2} A A_0^{-1/2} = A_0^{-1/2} \nu_0 A_0 + \sum \nu_j A_j A_0^{-1/2} \).

Thus this symmetric matrix is definite positive, implying that \( \nu_0 A_0 + \sum \nu_j A_j \) is also positive. \( \square \)

4.5.2 Local energy estimates

Integrate the energy balance on \( \Omega \subset [0,T] \times \mathbb{R}^d \):

\[
2 \text{Re} \int_{\Omega} (Lu, u) dt dx - \int ( Ku, u) dt dx = \sum_{j=0}^d \int_{\partial\Omega} \nu_j (A_j u, u) d\sigma
\]

where \( (\nu_0, \ldots, \nu_d) \) is the outward normal to \( \partial\Omega \).

Consider the polar cone of \( \Gamma \):

\[
\Gamma^\circ = \{(t,x) \in \mathbb{R}^{1+d} : |x| \leq ct \},
\]

and a backward cone

\[
\Omega = \{(t,x), t \in [0,T], |x - \bar{x}| \leq c(t-t) \}.
\]

The lateral boundary of \( \Omega \) is

\[
\partial_l \Omega = \{(t,x), t \in [0,T], |x - \bar{x}| = c(t-t) \}.
\]

Lemma 4.5.3. On \( \partial l\Omega \), the boundary matrix \( \sum \nu_j A_j \) is nonnegative.

Proof. Take for simplicity \( \bar{x} = 0 \). The outer normal at \( (t, x) \in \partial l\Omega \) is \( \delta(c, x/|x|) \) with \( \delta = (1 + c^2)^{1/2} \). Thus the matrix boundary matrix is \( \delta(c A_0 + \sum \nu_j A_j) \) with \( \nu_j = x_j/|x| \) for \( j = 1, \ldots, d \). By the lemma above, it is non negative. \( \square \)

Consider \( \bar{t} \leq T \) and \( \bar{x} \in \mathbb{R}^d \) and \( \Omega \) as above. For \( t \in [0,\bar{t}] \), let \( \omega_t = \{x : |x - \bar{x}| \leq c(t - t)\} \). One has the local energy estimate

Proposition 4.5.4. There are constants \( G \) and \( \gamma \), such that for \( u \in H^1(\Omega) \),

\[
\|u(t)\|_{\omega_t} \leq C e^{\gamma t} \|u(0)\|_{L^2(\omega_0)} + C \int_0^t e^{\gamma(t-t')} \|Lu(t')\|_{L^2(\omega_{t'})} dt'.
\]
Proof. The energy balance applied on $\Omega_t = \Omega \cap \{t' < t\}$ and the lemma imply that

$$
\int_{\omega_t} (A_0 u(t,x),u(t,x))dx \leq \int_{\omega_t} (A_0 u(0,x),u(0,x))dx
+ 2\text{Re} \int_{\Omega_t} (Lu,u)dt'dx + \int_{\Omega_t} |(Ku,u)|dt'dx.
$$

We conclude by Gronwall’s argument. \qed

**Corollary 4.5.5.** If $u$ is a strong solution of the Cauchy problem with source term which vanishes on $\Omega$ and initial data which vanishes on $\omega_0$, then $u = 0$ on $\Omega$.

**Theorem 4.5.6.** For $u_0 \in L^2(\omega_0)$ and $f \in L^2(\Omega)$, the Cauchy problem has a unique strong solution in $L^2(\Omega)$, which in addition is continuous in times with values in $L^2$ and satisfies (4.5.6).