DIFFRACTIVE NONLINEAR GEOMETRIC OPTICS *

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Contents.

This is the first of a series of articles analysing high frequency solutions of of hyperbolic partial differential equations over time scales beyond those for which the geometric optics approximation is valid. Here and in [DR 2], we treat problems for which approximate solutions with infinitely small residual can be constructed. Key hypotheses are that there is one fundamental linear phase, the nonlinearities are odd, and the spectrum of the profiles or envelopes are contained in the odd integers. In [JMR 6] problems not satisfying these hypotheses are discussed. In those more general problems rectification effects are analysed.

§1. The origin of Schrödinger type approximations.

The standard approach to the Maxwell Equations when applied to laser propagation is to make approximations which lead to equations of Schrödinger type. This simple fact raises at least the following three questions.

- How is it that models with finite speed lead to approximations with infinite speed?
- Why is it that the classic model of nondispersive wave propagation, the Maxwell Equations in vacuum, are approximated by the classic model of dispersive wave propagation?
- How come these approximations are not common within the subject of partial differential equations where such high frequency problems are treated?

Note that any high order implicit difference approximation is dispersive and has infinite speed.

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In addition to curiosity about such questions there are physical problems where the appropriate approximations are at present not clear. Among these we mention, Raman and Brillouin scattering by lasers, continuum generation, and optical parametric oscillators.

Our goal is to clarify the nature and range of validity of the approximations leading to Schrödinger like equations. The present paper addresses the simplest mathematical situations leading to such approximations.

The basic fact is that for solutions of constant coefficient problems with linear phase and wavelength of order ε, the behavior for times $t \sim 1/\varepsilon$ have an asymptotic expansion involving a slow time $T = \varepsilon t$ and for which the evolution is described by a Schrödinger type equation. The equation arises in an approximation in which there are three distinct scales, the wavelength ε and two longer scales 1 and $1/\varepsilon$.

This is most clear in simple explicitly solvable models. Consider the initial value problem for $u(t, y)$ with $x := (t, y) \in \mathbb{R}^{1+d}$,

$$\Box u := u_{tt} - \Delta\, u := \frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{d} \frac{\partial^2 u}{\partial y_j^2} = 0\,. \tag{1.1}$$

The partial Fourier Transform of the solution is given by

$$\hat{u}(t, \eta) = \hat{u}(0, \eta)\, \cos t|\eta| + \hat{u}_t(0, \eta)\, \frac{\sin t|\eta|}{|\eta|}\,. \tag{1.2}$$

Take initial data oscillating with wavelength of order ε and linear phase equal to y_1/ε,

$$u^\varepsilon(0, y) := f(y)\, e^{iy_1/\varepsilon}\,, \qquad u^\varepsilon_t(0, y) = 0\,, \qquad f \in \mathcal{S}(\mathbb{R}^d)\,. \tag{1.3}$$

Then

$$\hat{u}(0, \eta) = \hat{f}(\eta - \mathbf{e}_1/\varepsilon)\,, \qquad \hat{u}_t(0, \eta) = 0\,, \tag{1.4}$$

and the solution is the sum of two terms

$$u^\varepsilon_\pm(t, y) := \frac{1}{2\,(2\pi)^{d/2}} \int \hat{f}(\eta - \mathbf{e}_1/\varepsilon)\, e^{i(y\eta \mp t|\eta|)}\, d\eta\,. \tag{1.5}$$

Initial data with $u_t \neq 0$ lead to similar expressions with an additional complication of a factor $1/|\eta|$. The contributions from η near 0, is $O(\varepsilon^\infty)$ because of the rapid decay of factors analogous to $f(\eta - \mathbf{e}_1/\varepsilon)$.

We analyse u^ε_+, the other being entirely analogous. For ease of reading the subscript plus is omitted. Introduce $\zeta := \eta - \mathbf{e}_1/\varepsilon$ to find

$$u^\varepsilon(t, y) = \frac{1}{2\,(2\pi)^{d/2}} \int \hat{f}(\zeta)\, e^{iy(\mathbf{e}_1 + \varepsilon\zeta)/\varepsilon}\, e^{-it|\mathbf{e}_1 + \varepsilon\zeta|/\varepsilon}\, d\zeta\,. \tag{1.6}$$

Expanding the exponent to first order in ε yields

$$|\,\mathbf{e}_1 + \varepsilon\zeta\,| = (1 + \varepsilon\zeta_1) + O(\,\varepsilon^2\,|\zeta|^2\,)\,. \tag{1.7}$$

Define

$$B(t, y) := \frac{1}{2\,(2\pi)^{d/2}} \int \hat{f}(\zeta)\, e^{i(y\zeta - t\zeta_1)}\, d\zeta \tag{1.8}$$

to find

$$u^\varepsilon(t,y) = e^{i(y_1-t)/\varepsilon}\, B(t,y) + O(\varepsilon\, t)\,. \tag{1.9}$$

Estimating as for (1.15) in the sequel yields

$$\big\|\, e^{-i(y_1-t)/\varepsilon}\, u^\varepsilon(t,y) \;-\; B(t,y)\,\big\|_{H^s(\mathbb{R}^d)} \le C\,\varepsilon t\,\big\|\, f\,\big\|_{H^{s+2}(\mathbb{R}^d)}\,. \tag{1.10}$$

The amplitude B satisfies the simple transport equation

$$\frac{\partial B}{\partial t} + \frac{\partial B}{\partial y_1} = 0\,. \tag{1.11}$$

The error estimates (1.9) and (1.10) show that the approximation is useful as long as $t = o(1/\varepsilon)$. (1.9) is the standard approximation of geometric optics.

This approximation has the following geometric interpretation. One has a superposition plane waves $e^{i(x\omega+t|\omega|)}$ with $\omega \sim (1/\varepsilon,0,\ldots,0) + O(1)$. Replacing ω by $(1/\varepsilon,0,\ldots,0)$ and $|\omega|$ by $1/\varepsilon$ yields the approximation (1.10). The wave vectors make an angle of order ε with $\mathbf{e}_1$ so they remain close for times small compared with $1/\varepsilon$. For longer times the fact that the rays are not parallel is important. The wave begins to spread out. Parallel rays is a reasonable approximation for times $t = o(1/\varepsilon)$.

The analysis just performed can be carried out without fundamental change for initial oscillations with nonlinear phase $\psi(y)$ and for variable coefficient operators (see [L],[R]).

The approximation (1.9) cannot remain valid for large time. The approximate solution consists of waves rigidly moving with velocities $\pm\mathbf{e}_1$. The amplitudes are constant along the rays $y_1 - t = c^{st}$. However for ε fixed and time tending to infinity the solution of the initial value problem decays like $t^{-(d-1)/2}$. In fact for large time the solution resembles an outgoing spherical wave $a(y/|y|, |y|-t)/|y|^{(d-1)/2}$. Thus eventually the collumnation of the solution degrades and the wave spreads over regions which grow linearly in time. These waves spread beyond the regions reached by the rays. The penetration of waves into regions not reached by the rays of geometric optics is called *diffraction*. Finding approximations for times beyond the validity of geometric optics amounts to studying the onset of diffractive effects.

To study times and distances of order $1/\varepsilon$, insert the second order Taylor expansion

$$\Big|\, |e_1 + \varepsilon\zeta| - \big(1 + \varepsilon\zeta_1 + \frac{\varepsilon^2}{2}(\zeta_2^2 + \ldots + \zeta_d^2)\big)\,\Big| \le C\,|\varepsilon\zeta|^3 \tag{1.12}$$

into the integral (1.6) to find that

$$u^\varepsilon(t,y) = e^{i(y_1-t)/\varepsilon}\,\frac{1}{2\,(2\pi)^{d/2}}\int \hat f(\zeta)\, e^{iy\zeta - t\zeta_1}\, e^{i\epsilon t(\zeta_2^2+\ldots+\zeta_d^2)/2}\, d\zeta + O(\varepsilon^2 t) \tag{1.13}$$

Introduce the slow variable $T := \varepsilon t$ and

$$B(T,t,y) := \frac{1}{2\,(2\pi)^{d/2}}\int \hat f(\zeta)\, e^{iy\zeta}\, e^{it\zeta_1}\, e^{iT(\zeta_2^2+\ldots+\zeta_d^2)/2}\, d\zeta\,.$$

Then

$$u^\varepsilon(t,y) = e^{i(y_1-t)/\varepsilon}\, B(\varepsilon t, t, y) + O(\varepsilon^2 t) \tag{1.14}$$

uniformly on $\mathbb{R}^{1+d}$. In fact,

$$e^{-i(y_1-t)/\varepsilon}\, u^\varepsilon(t,y) - B(\varepsilon t,t,y) = \frac{1}{2\,(2\pi)^{d/2}} \int \hat f(\zeta)\,\big(\, e^{it|\mathbf{e}_1+\varepsilon\zeta|/\varepsilon} - e^{it(1+\varepsilon\zeta_1+\varepsilon^2|\zeta_\perp|^2/2)/\varepsilon}\,\big)\; d\zeta\,.$$

Use $|e^{iz}-e^{iw}| \le |z-w|$ together with (1.12) to show that

$$\Big|\, e^{it|\mathbf{e}_1+\varepsilon\zeta|/\varepsilon} - e^{it(1+\varepsilon\zeta_1+\varepsilon^2|\zeta_\perp|^2/2)/\varepsilon}\,\Big| \le C\,t\,\varepsilon^2\,|\zeta|^3$$

and therefore

$$\big\|\, e^{-i(y_1-t)/\varepsilon}\, u^\varepsilon(t,y) - B(\varepsilon t,t,y)\,\big\|_{H^s(\mathbb{R}^d)} \le C\,t\,\varepsilon^2\,\|f\|_{H^{s+3}(\mathbb{R}^d)}\,, \tag{1.15}$$

which is a quantitative version of (1.14).

The profile B satisfies a pair of partial differential equations.

$$(\partial_t + \partial_{y_1})B = 0, \qquad \text{and} \qquad \big(2i\partial_T - \Delta_{y_2,\dots y_d}\big)B = 0\,. \tag{1.16}$$

The first is the transport equation of geometric optics and the second is the Schrödinger equation which we were looking for. These together with the initial condition

$$B(0,0,y) = f(y)$$

suffice to uniquely determine B. The first equation in (1.14) is solved by writing

$$B = A(T, y_1 - t, y_2, \cdots, y_d) \tag{1.17}$$

in which case $A(T, y_1, y_2, \cdots, y_d)$ is determined from the Schrödinger equation

$$\big(2i\partial_T - \Delta_{y_2,\dots y_d}\big)A = 0\,, \qquad A(0,y) = f(y)\,. \tag{1.18}$$

The variable y_1 enters only as a parameter.

For $t = o(1/\varepsilon)$ one has $T \to 0$, and setting $T = 0$ in (1.14) recovers the approximation of geometric optics. Thus (1.14) matches the asymptotics for $t = o(1/\varepsilon)$ and those for $t = O(1/\varepsilon)$.

A typical solution of (1.18) has spatial width which grows linearly in T. Thus the width of our solution u^ε grows linearly in εt which is consistent with the geometric observation that the wave vectors comprising u make an angle $O(\varepsilon)$ with $\mathbf{e}_1$.

In contrast to the case of the geometric optics expansion, the last results do not extend to nonlinear phases. Note that the rays associated to nonlinear phases diverge linearly in time and the geometric optics approximation decays correspondingly. The formulas valid for times of order 1, remain valid for large times. The approximations of Schrödinger type describe the interaction over long periods of times of parallel rays. In the same way, one does not find such Schrödinger approximations for linear phases when the geometric approximations are not governed by transport equations. The classic example is conical refraction.

The approximation (1.15) clearly presents three scales; the wavelength ε, the lengths of order 1 on which f varies, and, the lengths of order $1/\varepsilon$ traveled by the wave on the time scales of the variations of B with respect to the slow time T.

Summary. *The Schrödinger approximations are intimately related to linear phases for which the rays are parallel. They provide diffractive corrections for times $t \sim 1/\varepsilon$ to solutions of wavelength ε which are adequately described by geometric optics for times $t \sim 1$.*

These key features, parallel rays and three scales are commonly satisfied by laser beams. The beam is comprised of virtually parallel rays. A typical example with three scales would have wavelength $\sim 10^{-6} m$, the width of the beam $\sim 10^{-3} m$, and the propagation distance $\sim 1m$.

For nonlinear phases the long time behavior is different. With suitable convexity hypotheses on the wave fronts, nonlinear transport equations along rays yield nonlinear geometric optics optics approximations valid gloablly in time (see [Go]).

2. Formulating the ansatz.

We study solutions of semilinear symmetric hyperbolic systems with constant coefficients and nonlinearity which is or order J near $u = 0$. The quasilinear case is discussed in §6.

Symmetric hyperbolicity hypothesis. *Suppose that*

$$L^\epsilon(\partial_x) := \sum_{\mu=0}^{d} A_\mu \frac{\partial}{\partial x_\mu} + \epsilon\, L_0 = A_0\partial_0 + A_1\partial_1 + \cdots + A_d\partial_d + \epsilon\, L_0 := L_1(\partial_x) + \varepsilon\, L_0\,. \tag{2.1}$$

is a constant coefficient symmetric hyperbolic system of order one with timelike variable $t := x_0$, that is, the coefficients A_μ are $N \times N$ hermitian symmetric matrices with A_0 strictly positive.

The fact that the lower order term appers with a factor ε is crucial. In §2.1 this is discussed in terms of interaction times. A remark in §3.3 shows that the construction of approximate solutions can fail if one has L_0 instead of εL_0. In §5, the factor ε is crucial in the proof of the approximation theorem if $L_0 + L_0^*$ has a negative eigenvalue.

The nonlinear differential differential equation to solve is

$$L^\varepsilon(\partial_x)\, u^\varepsilon + F(u^\varepsilon) = 0\,, \tag{2.2}$$

where u^ε is a family of $\mathbb{C}^N$ valued functions.

Order J hypothesis. *The nonlinear function F is smooth on a neighborhood of $0 \in \mathbb{C}^N$, and the nonlinear terms are of order $J \geq 2$ in the sense that*

$$|\beta| \leq J - 1 \quad \Longrightarrow \quad \partial^\beta_{u,\overline{u}} F(0) = 0\,.$$

The Taylor expansion at the origin is then

$$F(u) = \Phi(u) + O(|u|^{J+1})\,, \tag{2.3}$$

where Φ is a homogeneous polynomial of degree J in $u, \overline{u}$.

§2.1. Time of nonlinear interaction.

The amplitude of nonlinear waves in crucially important. Our solutions have amplitude $u^\varepsilon = O(\varepsilon^p)$ where the exponent $p > 0$ is chosen that the nonlinear term $F(u) = O(\varepsilon^{pJ})$ affects the principal term in the asymptotic expansion for times of order $1/\varepsilon$. The time of nonlinear interaction is comparable to the times for the onset of diffractive effects.

The time of nonlinear interaction is estimated as follows. Denote by $S(t)$ the propagator for the linear operator L^ε. Then in $L^2(\mathbb{R}^d)$, $||S(t)|| \leq Ce^{\varepsilon|t|}$. The Duhamel representation

$$u(t) = S(t)\, u(0) - \int_0^t S(t-\sigma)\, F(u(\sigma))\, d\sigma$$

suggests that the contribution of the nonlinear term at time t is of order $t\varepsilon^{pJ} e^{\varepsilon t}$. For the onset of diffraction, $t \sim 1/\varepsilon$ so the accumulated effect is expected to be $O(\varepsilon^{pJ-1})$. For this to be comparable to the size of the solution we chose p so that $pJ - 1 = p$.

Definition. *For nonlinearities satisfying the the order J hypothesis, the* **standard normalization** *is to choose p so that*

$$p := \frac{1}{J-1}\,. \tag{2.4}$$

A similar estimate shows that the contribution of the εL_0 is no larger than $C\varepsilon t$ so the natural interaction time is no shorter than $t \sim 1/\varepsilon$. In particular this term does not influence the principal term in the linear geometric optics description of the solution for $t \sim 1$.

§2.2. Harmonics and rectification.

Introduce a general linear phase $\beta.x = \tau t + \eta.y$ with $\beta = (\tau, \eta) \in \mathbb{R}^{1+d}$. The oscillating factor is then $e^{i\beta.x/\varepsilon}$. The remarks of the previous section suggest replacing B by $\varepsilon^p B$.

Nonlinearity will normally create harmonics $e^{im\beta.x/\varepsilon}$ with $m \in \mathbb{Z}$. The waves with these phases will then interact with each other. With this in mind, the leading term in (1.15) is replaced by

$$\varepsilon^p\, B(\varepsilon x, x, x.\beta/\varepsilon)\,, \qquad \text{with} \qquad B(X, x, \theta) \in C^\infty(\mathbb{R}^{1+d} \times \mathbb{R}^{1+d} \times \mathbb{T})$$

periodic in θ.

A special role is played by the harmonic with $m = 0$ which is nonoscillatory. Such a term occurs from a J-linear interaction of harmonics $e^{im_j\beta.x/\varepsilon}$ with

$$m_1 + m_2 + \cdots + m_J = 0\,. \tag{2.5}$$

If the oscillatory waves propagate with speed v, then one expects such rectification to produce source terms of the form $f(y - vt)$. This is expected to create a term like $L^{-1}\big(f(y - vt)\big)$ which in general is a wave dispersing in all direction of space. For times $t \sim 1/\varepsilon$ which interest us, such wave will be small compared to the waves which propagate nearly sharply in the direction v. This suggests the following facts proved in [JMR 6],

- *The rectified waves are correctors to the principal term in the asymptotics.*
- *The rectified waves are not described by expressions analogous to the principal term.*

In the analysis which follows these rectification effects are not present. Then, correctors to all orders can be constructed having the same form as the principal term. The rectification is avoided by assuming that the nonlinear function F is odd, and that the profiles B in the asymptotic expansion are periodic functions of θ whose spectrum is contained in the odd integers $\mathbb{Z}_{\text{odd}}$. Note that for J and m_j odd the sum on the left hand side of (2.5) is odd and therefore never equal to zero. More generally, if $B(\theta)$ had odd spectrum and F is odd, then $F(B(\theta))$ has odd spectrum.

Oddness hypothesis. *The Taylor expansion of F at $u = 0$ contains only monomials of odd degree.*

This hypothesis is satisfied if and only if the even part $F(u) + F(-u)$ vanishes to infinite order at $u = 0$. In particular it is satisfied for odd functions F. Note that quadratic nonlinearities are excluded by the oddness hypothesis.

The leading term in the *ansatz* is $O(\varepsilon^p)$ where p from (2.4) is fractional for $J \geq 3$. The terms in the Taylor expansion of F generate terms in ε^{np}. These consideration suggest the *ansatz*

$$u^\varepsilon = \varepsilon^p\, a(\varepsilon, \varepsilon x, x, \beta.x/\varepsilon)\,, \qquad a(\varepsilon, X, x, \theta) \sim \sum_{j \in p\mathbb{N}} \varepsilon^j\, a_j(X, x, \theta)\,, \tag{2.6}$$

where the smooth profiles a_j are periodic in θ and satisfy

$$\mathrm{spec}\, a_j \in \mathbb{Z}_{\mathrm{odd}} \,. \tag{2.7}$$

¿From the example in §1, we expect the leading profile a_0 to satisfy a homogeneous transport equation $\partial_t a_0 + v.\partial_y a_0 = 0$ with respect to x and to be as rapidly decreasing in X, x as this permits. In order for $\varepsilon^{j+p} a_{j+p}$ to be small compared to the preceding $\varepsilon^j a_j$ term for times $t \sim 1/\varepsilon$ it is necessary that $\lim \varepsilon a(X, x/\varepsilon, \theta) = 0$ as $\varepsilon \to 0$. Our profiles satisfy the stronger condition that for all α

$$\partial^\alpha_{X,x,\theta}\, a_j \in L^\infty([0,T] \times \mathbb{R}^d \times \mathbb{R}^{1+d} \times \mathbb{T}) \,. \tag{2.8}$$

§3. Equations for the profiles.

The goal is to find u^ε as in the *ansatz* (2.6)-(2.8) which is an approximate solution of (2.2). Compute

$$L^\varepsilon(\partial_x)\, u^\varepsilon = \varepsilon^{p-1}\, b(\varepsilon, \varepsilon x, x, x.\beta/\varepsilon) \,,$$

where

$$b(\varepsilon, X, x, \theta) = L_1(\beta)\, \partial_\theta a + \varepsilon\, L_1(\partial_x)\, a + \varepsilon^2\, (\, L_1(\partial_X) + L_0\,)\, a \,. \tag{3.1}$$

Then (2.6) and (2.8) imply that

$$b(\varepsilon, X, x, \theta) \sim \sum_{j \in p\mathbb{N}} \varepsilon^j\, b_j(X, x, \theta) \,. \tag{3.2}$$

The Taylor expansion of F at 0, yields

$$F(\varepsilon^p a(\varepsilon, X, x, \theta)) = \varepsilon^{pJ}\, c(\varepsilon, X, x, \theta) \,, \qquad c(\varepsilon, X, x, \theta) \sim \sum_{j \in p\mathbb{N}} \varepsilon^j\, c_j(X, x, \theta) \,. \tag{3.3}$$

with

$$c_0(X, x, \theta) = \Phi(\, a_0(X, x, \theta)\,) \,. \tag{3.4}$$

Note that in computing (3.4) the order J hypothesis guarantees that the leading contribution of the nonlinear is this $O(\varepsilon^{p+1})$ term. More generally, a_j contributes terms which are no larger than $O(\varepsilon^{(p+j)J}) = O(\varepsilon^{p+jJ+1}) = o(\varepsilon^{p+j})$ to the nonlinear term. Thus for $j > 0$, c_j is determined by $\{a_k \,:\, k < j\}$.

Adding the expressions for $L^\varepsilon u^\varepsilon$ and $F(u^\varepsilon)$ shows that

$$L^\epsilon(\partial) u^\varepsilon + F(u^\varepsilon) = \varepsilon^{p-1}\, r(\varepsilon, \varepsilon x, x, \beta.x/\varepsilon) \,, \qquad r(\varepsilon, X, x, \theta) \sim \sum_{j \in p\mathbb{N}} \varepsilon^j\, r_j(X, x, \theta) \,. \tag{3.5}$$

The strategy is to choose the profiles a_j so that the profiles r_j of the residual are identically equal to zero.

Setting $r_j = 0$ for $j = 0, 1, 2$ yields the equations

$$L_1(\beta)\, \partial_\theta a_0 = 0 \,, \tag{3.6}$$

$$L_1(\beta)\, \partial_\theta a_1 + L_1(\partial_x) a_0 = 0 \,, \tag{3.7}$$

$$L_1(\beta)\,\partial_\theta a_2 + L_1(\partial_x)a_1 + (L_1(\partial_X) + L_0)\,a_0 + \Phi(a_0) = 0\,. \tag{3.8}$$

Note that since j assumes fractional values, these are not the three leading terms in (3.5). However, they are the terms which lead to the determination of a_0.

With the convention that $a_j = 0$ whenever $j < 0$ the equation $r_j = 0$ reads

$$L_1(\beta)\,\partial_\theta a_{j+1} + L_1(\partial_x)a_j + (L_1(\partial_X) + L_0)\,a_{j-1} + c_{j-1} = 0\,. \tag{3.9}$$

§3.1. Analysis of equation 3.6.

Expand in a Fourier series

$$a_0(X,x,\theta) = \sum_{m\in\mathbb{Z}_{\rm odd}} \mathrm{a}_m(X,x)\,e^{im\theta}\,.$$

Then

$$L_1(\beta)\,\partial_\theta a_0 = \sum_{m\in\mathbb{Z}_{\rm odd}} im\,L_1(\beta)\,\mathrm{a}_m(X,x)\,e^{im\theta}\,.$$

In order for there to be nontrivial solutions of (3.6) one must have

$$\det\,L_1(\beta) = 0\,. \tag{3.10}$$

Equivalently β must belong to the characteristic variety of L. There are two naturally defined matrices which play a central role in the sequel.

Definition. *For $\beta \in \mathbb{R}^{1+d}$ let $\pi(\beta)$ be the linear projection on the kernel of $L_1(\beta)$ along the range of $L_1(\beta)$. Denote by $Q(\beta)$ the partial inverse defined by*

$$Q(\beta)\,\pi(\beta) = 0\,, \qquad \text{and} \qquad Q(\beta)\,L_1(\beta) = I - \pi(\beta)\,. \tag{3.11}$$

Symmetric hyperbolicity implies that both $\pi(\beta)$ and $Q(\beta)$ are hermitian symmetric. In particular $\pi(\beta)$ is an orthogonal projector. With this notation, equation (3.6) is equivalent to

$$\pi(\beta)\,a_0 = a_0\,. \tag{3.12}$$

This asserts that the principal profile is polarized along the kernel of $L_1(\beta)$.

§3.2. Analysis of equation 3.7.

Equation (3.7) involves both a_0 and a_1. Multiplying by $\pi(\beta)$ annihilates $L_1(\beta)$ so eliminates the a_1 term to give

$$\pi(\beta)\,L_1(\partial_x)\,\pi(\beta)\,a_0 = 0\,. \tag{3.13}$$

A vector $w \in \mathbb{C}^N$ vanishes if and only if

$$\pi(\beta)\,w = 0\,, \qquad \text{and} \qquad Q(\beta)\,w = 0\,.$$

Thus the information in (3.7) complementary to (3.13) is obtained by multiplying (3.13) by $Q(\beta)$. This yields

$$(I - \pi(\beta))\,a_1 = -Q(\beta)\,L_1(\partial_x)\,\partial_\theta^{-1} a_0\,. \tag{3.14}$$

Equations (3.12) and (3.13) are the fundamental equations of linear geometric optics (see [R]). As such they determine the dynamics of $\pi(\beta)a_0$ with respect to the time t. The following hypothesis guarantees that the linear geometric optics is simple. It excludes for example β along the optic axis of conical refraction.

Simple characteristic variety hypothesis. *$\beta = (\underline{\tau}, \underline{\eta})$ and there are neighborhoods ω of $\underline{\eta}$ in $\mathbb{R}^d$ (resp. $\mathcal{O}$ of β in $\mathbb{R}^{1+d}$) so that for each $\eta \in \omega$ there is exactly one point $(\tau(\eta), \eta) \in \mathcal{O} \cap \mathrm{char}\, L_1$.*

Proposition 3.1. *If the simple characteristic variety hypothesis is satisfied then the functions, $\tau(\eta)$, $\pi(\tau(\eta), \eta)$ and $Q(\tau(\eta), \eta)$ are real analytic on ω. In addition*

$$\pi(\beta)\, L_1(\partial_x)\, \pi(\beta) = \pi(\beta)\, A_0\, \pi(\beta) \Big(\partial_t - \sum_{j=1}^{d} \frac{\partial \tau(\eta)}{\partial \eta_j} \frac{\partial}{\partial y_j} \Big) . \tag{3.15}$$

Proof. Since

$$L_1(\tau, \eta) = A_0^{1/2} \Big(\tau I + \sum_{j=1}^{d} \eta_j A_0^{-1/2}\, A_j\, A_0^{-1/2} \Big) A_0^{1/2} ,$$

the solutions τ are the eigenvalues of the hermitian matrix

$$H(\eta) := -A_0^{-1/2} \,(\, \sum_{j=0}^{d} \eta_j\, A_j \,)\, A_0^{-1/2} .$$

Choose $r > 0$ so that for η near $\underline{\eta}$ there is exactly one eigenvalue $\tau(\eta)$ in the disk of center $\underline{\tau}$ and radius r. Then the real analyticity of $\pi(\eta)$ follows from the contour integral representation

$$\pi(\eta) = \frac{1}{2\pi i} \oint_{|z|=r} \big(zI - H(\eta)\,\big)^{-1} dz .$$

The analyticity of τ and Q then follow from the formulas

$$\tau(\eta) = \frac{\mathrm{trace}\, H(\eta)\, \pi(\tau(\eta), \eta)}{\mathrm{trace}\, \pi(\tau(\eta), \eta)} , \quad Q(\tau(\eta), \eta) = (I - \pi(\tau(\eta), \eta)) \big(\pi(\tau(\eta), \eta) + L_1(\tau(\eta), \eta) \big)^{-1} .$$

Differentiate the identity

$$L_1\big(\tau(\eta), \eta\big)\ \pi(\tau(\eta), \eta) = 0$$

with respect to η_j to find

$$L_1(\tau(\eta), \eta)\, \frac{\partial}{\partial \eta_j}\, \pi(\tau(\eta), \eta) + \big(\frac{\partial \tau(\eta)}{\partial \eta_j}\, A_0 + A_j \big)\, \pi(\tau(\eta), \eta) = 0 .$$

Multiplying by $\pi(\tau(\eta), \eta)$ eliminates the first term to give

$$\pi(\tau(\eta), \eta) \big(\frac{\partial \tau(\eta)}{\partial \eta_j}\, A_0 + A_j \big)\, \pi(\tau(\eta), \eta) = 0 . \tag{3.16}$$

Using this for the summands on the right hand side of the identity

$$\pi(\beta)\, L_1(\partial_x)\, \pi(\beta) = \pi(\beta) A_0 \pi(\beta)\, \frac{\partial}{\partial t} + \sum_{j=1}^{d} \pi(\beta) A_j \pi(\beta)\, \frac{\partial}{\partial x_j}$$

yields (3.15). The proof is complete. ∎

Definitions. *If $\underline{\tau}, \underline{\eta}$ belongs to the characteristic variety and satisfies the simplicity assumption, define the transport operator V and group velocity v by*

$$V(\underline{\tau}, \underline{\eta}; \partial_x) := \partial_t - \sum_{j=1}^{d} \frac{\partial \tau(\underline{\eta})}{\partial \eta_j} \frac{\partial}{\partial x_j} := \partial_t + v.\partial_y \,. \tag{3.17}$$

Also define $\gamma(\underline{\tau}, \underline{\eta}) \in \mathrm{Hom}\big(\ker L_1(\underline{\tau}, \underline{\eta})\big)$ by

$$\pi(\underline{\tau}, \underline{\eta})\, L_0\, \pi(\underline{\tau}, \underline{\eta}) = \gamma(\underline{\tau}, \underline{\eta})\, \pi(\underline{\tau}, \underline{\eta})\,. \tag{3.18}$$

The $\underline{\tau}, \underline{\eta}$ dependence of π, Q, V and γ will often be omitted when there is little risk of confusion. Since V is scalar and πA_0 is an invertible map from the image of π to itself, it follows that (3.13) is equivalent to

$$V(\partial_x)\, \pi(\beta)\, a_0 = 0\,.$$

§3.3. Analysis of equation 3.8.

The information in (3.8) is split in two by multiplying by $Q(\beta)$ and by $\pi(\beta)$ which yield

$$(I - \pi(\beta))a_2 = -Q(\beta)\, \partial_\theta^{-1} \Big(L_1(\partial_x)a_1 + (L_1(\partial_X) + L_0)\, a_0 + \Phi(a_0) \Big)\,, \tag{3.19}$$

and

$$\pi\, (L_1(\partial_X) + L_0)\, \pi\, a_0 + \pi\, L_1(\partial_x)a_1 + \pi\, \Phi(a_0) = 0\,. \tag{3.20}$$

Equation (3.19) determines a part of a_2 in terms of earlier profiles. To interpret (3.20), write

$$a_1 = \pi\, a_1 + (I - \pi)\, a_1$$

and use (3.15) and (3.18) for the first summand and (3.14) for the second terms to find

$$V(\partial_X)\, \pi\, A_0\, a_0 + \gamma\, a_0 - \pi\, L_1(\partial_x)\, Q(\beta)\, L_1(\partial_x)\, \partial_\theta^{-1}\, a_0 + \pi\, \Phi(a_0) = -\pi\, L_1(\partial_x)\, \pi\, a_1\,. \tag{3.21}$$

The scalar operator $V(\partial_x)$ commutes with all the linear operators in (3.21) and therefore (3.12) and the last equation of §3.2 imply that it annihilates the left hand side of (3.20). Using Proposition 3.1 this shows that $V(\partial_x)\, \pi\, A_0\, \pi\, V(\partial_x)\, \pi\, a_1 = 0$. Since $V(\partial_x)$ is scalar it commutes with $\pi\, A_0\, \pi$ which is an invertible linear map on the range of π. It follows that $V(\partial_x)^2 \pi a_1 = 0$. This together with the condition (2.8) that a_1 is bounded implies that

$$V(\partial_x)\, a_1 = 0\,. \tag{3.22}$$

Thus the last equation in §3.2 together with (3.21) yield the following pair of equations for $a_0 = \pi(\beta)\, a_0$,

$$V(\partial_x)\, a_0 = 0\,, \qquad V(\partial_X)\, \pi\, A_0\, a_0 + \gamma\, a_0 - \pi\, L_1(\partial_x)\, Q(\beta)\, L_1(\partial_x)\, \partial_\theta^{-1}\, a_0 + \pi\, \Phi(a_0) = 0\,. \tag{3.23}$$

This is analogous to (1.16).

Remark. If $\pi\, L_1(\partial)\,\pi$ were not a simple transport operator, for example in the case of conical refraction, it would not necessarily annihilate $\Phi(a_0)$ or $\pi\, L_1\, Q\, L_1\, \partial_\theta^{-1} a_0$. A similar difficulty arises if one studies $L_1 + L_0$ instead of $L_1 + \varepsilon L_0$. In that case one finds $(V+\gamma)a_0 = 0$ and it is not necessarily true that $V+\gamma$ annihilates $\Phi(a_0)$. ∎

Just as the operator $\pi\, L_1\, \pi$ is a scalar transport operator when the simple characteristic variety hypothesis is satisfied, the next result shows that the operator $\pi\, L_1\, Q\, L_1\, \pi$ appearing in (3.23) is a scalar second order operator.

Proposition 3.2. *If the simple characteristic variety hypothesis is satisfied then*

$$\begin{aligned}\pi(\beta)\, L_1(\partial_x)\, Q(\beta)\, L_1(\partial_x)\, \pi(\beta) = -\frac{1}{2}\pi A_0 \pi \left(\sum_{j,k} \frac{\partial^2 \tau}{\partial \eta_j \partial \eta_k} \frac{\partial^2}{\partial y_j \partial y_k} \right) \\ - \pi\, A_0\, Q\, A_0\, \pi\, V(\partial)^2 + \pi \big(A_0\, Q\, L_1(\partial) + L_1(\partial)\, Q\, A_0 \big) \pi V(\partial)\end{aligned} \tag{3.24}$$

where $\pi = \pi(\tau(\eta), \eta)$, $Q = Q(\tau(\eta), \eta)$, *and* $V(\partial)$ *is defined in (3.17).*

Proof. Differentiating (3.16) with respect to η_k yields

$$\begin{aligned}\left(\frac{\partial}{\partial \eta_k} \pi(\tau(\eta), \eta) \right) \left(\frac{\partial \tau}{\partial \eta_j} A_0 + A_j \right) \pi(\tau(\eta), \eta) + \pi(\tau(\eta), \eta) \frac{\partial^2 \tau}{\partial \eta_j \partial \eta_k} A_0 \pi(\tau(\eta), \eta) + \\ \pi(\tau(\eta), \eta) \left(\frac{\partial \tau}{\partial \eta_j} A_0 + A_\mu \right) \left(\frac{\partial}{\partial \eta_k} \pi(\tau(\eta), \eta) \right) = 0\,.\end{aligned} \tag{3.25}$$

Differentiate $I - \pi(\tau(\eta), \eta) = L(\tau(\eta), \eta)\, Q(\tau(\eta), \eta)$ with respect to η_k to find

$$-\frac{\partial}{\partial \eta_k}\, \pi(\tau(\eta), \eta) = \left(\frac{\partial \tau}{\partial \eta_k} A_0 + A_k \right) Q + L \frac{\partial}{\partial \eta_k}\, Q(\tau(\eta), \eta)\,.$$

Multiplying on the left by π yields

$$\pi(\tau(\eta), \eta) \left(\frac{\partial}{\partial \eta_k}\, \pi(\tau(\eta), \eta) \right) = -\pi(\tau(\eta), \eta) \left(\frac{\partial \tau}{\partial \eta_k} A_0 + A_k \right) Q(\tau(\eta), \eta).$$

The adjoint of this identity reads

$$\left(\frac{\partial}{\partial \eta_k} \pi(\tau(\eta), \eta) \right) \pi(\tau(\eta), \eta) = -Q(\tau(\eta), \eta) \left(\frac{\partial \tau}{\partial_{\eta_k}} A_0 + A_k \right) \pi(\tau(\eta), \eta).$$

Multiply (3.25) on the left and the right by $\pi((\tau(\eta), \eta)$ and use the last two identities to eliminate the $\partial \pi$ terms to find

$$\begin{aligned}-\pi \left(\frac{\partial \tau}{\partial \eta_k} A_0 + A_k \right) Q \left(\frac{\partial \tau}{\partial \eta_j} A_0 + A_j \right) \pi + \pi\, \frac{\partial^2 \tau}{\partial \eta_j \partial \eta_k}\, A_0\, \pi \\ - \pi \left(\frac{\partial \tau}{\partial \eta_j} A_0 + A_j \right) Q \left(\frac{\partial \tau}{\partial \eta_k} A_0 + A_k \right) \pi = 0\,.\end{aligned}$$

Multiplying by $\partial^2/\partial y_j \partial y_k$ and summing over j,k yields

$$\pi \sum_{k=1}^{d} \left(\frac{\partial \tau}{\partial \eta_k} A_0 + A_k \right) \frac{\partial}{\partial y_k} \; Q \sum_{j=1}^{d} \left(\frac{\partial \tau}{\partial \eta_j} A_0 + A_j \right) \frac{\partial}{\partial y_j} \, \pi$$
$$= -\, \frac{1}{2} \, \pi \, A_0 \, \pi \, \left(\sum_{j,k} \frac{\partial^2 \tau}{\partial \eta_j \partial \eta_k} \frac{\partial^2}{\partial y_j \partial y_k} \right) .$$

In this identity use the sum of the two expressions

$$A_0 \sum_k \frac{\partial \tau}{\partial \eta_k} \frac{\partial}{\partial y_k} = -A_0 \, V(\partial) + A_0 \partial_t \, , \qquad \sum_k A_k \, \frac{\partial}{\partial y_k} = L_1(\partial) - A_0 \, \partial_t \, .$$

to show that

$$\pi \, (-A_0 \, V(\partial) + L_1(\partial)) \, Q \, (-A_0 \, V(\partial) + L_1(\partial)) \, \pi = -\frac{1}{2} \, \pi A_0 \pi \left(\sum_{j,k=1}^{d} \frac{\partial^2 \tau}{\partial \eta_j \partial \eta_k} \, \frac{\partial^2}{\partial y_j \partial y_k} \right) .$$

The proposition follows. ∎

When (3.24) is used in the second equation in (3.23), the last two terms in (3.24) annihilate a_0 thanks to the first equation in (3.23). Let R denote the real homogeneous scalar second order differential operator

$$R(\partial_y) := \frac{1}{2} \sum_{j,k=1}^{d} \frac{\partial^2 \tau}{\partial \eta_j \partial \eta_k} \, \frac{\partial^2}{\partial y_j \partial y_k} \, . \tag{3.27}$$

The equations satisfied by $a_0 = \pi \, a_0$ are then

$$V(\partial_x) \, a_0 = 0 \, , \qquad V(\partial_X) \, \pi \, A_0 \, a_0 + \gamma \, a_0 + \pi \, A_0 \, R(\partial_y) \, \partial_\theta^{-1} \, a_0 + \pi \, \Phi(a_0) = 0 \, . \tag{3.28}$$

The equations for the higher profiles are derived in similar fashion. $(I - \pi) a_j$ is determined by setting Q times the the case $j-1$ of (3.9) equal to zero to find

$$\begin{aligned} (I - \pi) \, a_j &= -Q \, \partial_\theta^{-1} \Big(L_1(\partial_x) a_{j-1} + (L_1(\partial_X) + L_0) \, a_{j-2} + c_{j-2} \Big) \\ &= -Q \, \partial_\theta^{-1} \, L_1(\partial_x) a_{j-1} + H_j(a_{k \le j-2}) \, . \end{aligned} \tag{3.29}$$

In all cases the right hand side is is a function of the profiles $\{a_k \, : \, k \le j-1\}$ and their derivatives. A special case is those $j < 1$ for which one finds $(I - \pi) a_j = 0$.
The dynamics for $\pi \, a_j$ is determined by setting π times the case j of (3.9) equal to zero to find

$$\pi \, (L_1(\partial_X) + L_0) \, \pi \, a_j + \pi \, L_1(\partial_x) \, (I - \pi) \, a_{j+1} + \pi c_j = -\pi \, L_1(\partial_x) \, \pi \, a_{j+1} \, .$$

Simplifying using (3.15), (3.18), and (3.30) yields

$$V(\partial_X) \, \pi \, A_0 \, a_j + \gamma \, a_j - \pi \, L_1(\partial_x) \, Q \, L_1(\partial_x) \, \partial_\theta^{-1} \, \pi \, a_j + \pi c_j = -\pi \, A_0 \, \pi \, V(\partial_x) \, \pi \, a_{j+1} \, . \tag{3.30}$$

By induction one shows that $V(\partial_x)$ annihilates the left hand side of this equation. Then exactly as in the derivation of (3.22) one finds $V(\partial_x)\, \pi\, a_{j+1} = 0$. The expression (3.28) for the complementary projection then shows by induction that

$$V(\partial_x)\, a_j = 0\,, \qquad j \in p\mathbb{N}\,. \tag{3.31}$$

For $j > 0$ the term c_j is a function of the profiles a_k with $k < j$. Then using (3.24) equation (3.30) takes the form

$$V(\partial_X)\, \pi\, A_0\, a_j + \gamma\, \pi\, a_j + \pi\, A_0\, R(\partial_y)\, \partial_\theta^{-1}\, \pi\, a_j + \pi K_j(a_{k<j}) = 0\,, \qquad 0 < j \in p\mathbb{N}\,. \tag{3.32}$$

Theorem 3.3. *If u^ε is given by (2.6)-(2.8) then in order that $L(\partial)u^\varepsilon + F(u^\varepsilon) \sim 0$ in C^∞ it is sufficient that the principal profile a_0 satisfies (3.12) and (3.28), and that the profiles a_j with $j > 0$ satisfy the equations (3.29), (3.31), and (3.32).*

§4. Solvability of the profile equations.

The first equation in (3.28) holds if and only if

$$a_0(T, Y, t, y, \theta) = \mathrm{a}(T, Y, y - vt, \theta)\,, \qquad \mathrm{a}(T, Y, y, \theta) := a_0(T, Y, 0, y, \theta) \tag{4.1}$$

The second equation in (3.28) then holds if and only if

$$V(\partial_X)\, \pi\, A_0\, \mathrm{a} + \gamma\, \mathrm{a} + \pi\, A_0\, R(\partial_y)\, \partial_\theta^{-1}\, \mathrm{a} + \pi\Phi(\mathrm{a}) = 0\,. \tag{4.2}$$

In the linear case with the spectrum of a_0 equal to the single point k, the operator ∂_θ^{-1} is simply multiplication by $1/\imath k$ and the principal part of (4.2) is a scalar Schrödinger type operator whose second order part, $R(\partial_y)$, is determined by the the second order Taylor polynomial τ at η. The $A_0 R\, \partial_\theta^{-1}$ term is antisymmetric thanks to the factor ∂_θ^{-1}.

Equation (3.31) suggests writing

$$a_j(T, Y, t, y, \theta) := \mathrm{a}_j(T, Y, y - vt, \theta)\,, \tag{4.3}$$

in which case (3.32) takes the form

$$V(\partial_X)\, \pi\, A_0\, \pi\, \mathrm{a}_j + \gamma\, \pi\, \mathrm{a}_j + \pi\, A_0\, R(\partial_y)\, \partial_\theta^{-1}\, \pi\, \mathrm{a}_j + \pi K_j(a_{k<j}) = 0\,. \tag{4.4}$$

Similarly (3.19) becomes

$$(I - \pi)\, \mathrm{a}_j = H_j(\mathrm{a}_{k \le j-1})\,. \tag{4.5}$$

The idea is to construct smooth a_j tending to zero as $Y, y \to \infty$,

$$\mathrm{a}_j \in C^\infty\big([0, T_*[\ ; \cap_s H^s(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T})\big)\,, \qquad \mathrm{spec}\, \mathrm{a}_j \subset \mathbb{Z}_{\mathrm{odd}}\,. \tag{4.6}$$

If the profiles tend rapidly to zero as $Y \to \infty$ then they can be replaced by profiles independent of Y as explained in §7.2 of [JMR 6]. Profiles satisfying (4.6) take into account nonnegligible modulations on the long scale Y. The published version in the Polytéchnique seminar produced rapidly decaying profiles which was not a good decision.

Theorem 4.1. *Suppose that for all $j \in p\mathbb{N}$, initial data $g_j \in \cap_s H^s(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T})$ are given satisfying $\pi g_j = g_j$ and $\operatorname{spec} g_j \subset \mathbb{Z}_{\rm odd}$. Then there is a $T_* \in]0, \infty]$ and a unique a_0 satisfying (4.6), (3.12), (3.28), and $\mathrm{a}_0|_{T=0} = g_0$. If $T_* < \infty$ then*

$$\lim_{T \nearrow T_*} ||\mathrm{a}_0||_{L^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{T})} = \infty \tag{4.7}$$

For $j > 0$ there are unique a_j satisfying (4.6), (4.5), and (4.4) where T_* is that from a_0.

Proof. Denote by $z := (Y, y) \in \mathbb{R}^{2d}$ with $\zeta := (\Xi, \eta) \in \mathbb{R}^{2d}$ the dual variable. The partial Fourier transform of $\mathrm{a}(T, Y, \theta)$ is a function of $(T, \zeta, n) = (T, \Xi, \eta, n)$.
Define the positive definite matrix $E := \pi\, A_0\, \pi + (I - \pi)$, and $\tilde{\mathrm{a}} := E\mathrm{a}$. Then equation (4.2) is equivalent to

$$\pi\tilde{\mathrm{a}} = \tilde{\mathrm{a}}\,, \qquad \frac{\partial \tilde{\mathrm{a}}}{\partial T} + P(\partial_Y, \partial_y, \partial_\theta)\, \tilde{\mathrm{a}} + \pi\, \Phi(E^{-1}\tilde{\mathrm{a}}) = 0\,. \tag{4.8}$$

where acting on functions with mean zero in θ, P is the Fourier multiplier with symbol

$$P(\Xi, \eta, n) := \gamma - iv.\Xi + i\,\pi(\beta)\ R(\eta)\, \frac{1}{n}\, E^{-1}\, \pi(\beta)\,. \tag{4.9}$$

Extend the operator to all functions by setting the symbol equal to zero for $n = 0$. Then P annihilates functions which are independent of θ. The key to the analysis is that the matrices P satisfy

$$\big\|\, P + P^* \,\big\|_{\mathbb{C}^N \to \mathbb{C}^N} \le C_1\,. \tag{4.10}$$

Definition. *For $s \in \mathbb{N}$, Γ^s denotes the Hilbert space of functions $f(z, \theta) \in L^2(\mathbb{R}^{2d} \times \mathbb{T})$ such that*

$$\alpha \in \mathbb{N}^{2d+1}, \quad |\alpha| \le s \quad \Longrightarrow \quad \partial^\alpha_{z,\theta} f \in L^2(\mathbb{R}^{2d} \times \mathbb{T})\,.$$

The space Γ^s_π is the subset of Γ^s consisting of functions whose spectrum is contained in $\mathbb{Z}_{\rm odd}$, and which satisfy the polarization $\pi f = f$.

The solutions of (4.2) are constructed by Picard iteration. The key facts are that the associated linear evolutions are bounded on Γ^s and that Γ^s is mapped to itself by smooth functions. These well known facts are summarized in a lemma.

Lemma 4.2. i. Linear estimate. *With C_1 from (4.10),*

$$\big\|e^{TP(\partial_z,\partial_\theta)}\big\|_{\Gamma^s \to \Gamma^s} \ \le\ C\, e^{C_1|T|}\,. \tag{4.11}$$

ii. Gagliardo-Nirenberg Inequalities. *If $u \in L^\infty(\mathbb{R}^{2d}) \cap \Gamma^s$ then for any $\nu \in \mathbb{N}^{4d}$ with $0 < |\nu| < s$, $\partial^\nu_{z,\theta}\, u \in L^{2s/|\nu|}$ and with a constant independent of u*

$$\|\, \partial^\nu_{z,\theta}\, u\, \|_{L^{2s/|\nu|}} \le\ C\ \|\, u\, \|^{1-|\nu|/s}_{L^\infty}\ \|\, u\, \|^{|\nu|/s}_{\Gamma^s}\,. \tag{4.13}$$

iii. Schauder's Lemma. *If G is a smooth function such that $G(0) = 0$, $s > (2d+1)/2$, and $u \in \Gamma^s$, then $G(u) \in \Gamma^s$. If in addition G is odd and satisfies $\pi\, G = G$, then G maps Γ^s_π to itself.*

iv. Moser Inequality. *For all $M > 0$, there is a constant $C(s, M, G)$ such that if $u, w \in \Gamma^s$ satisfy*

$$\|\, w\, \|_{L^\infty(\mathbb{R}^{2d}\times\mathbb{T})} \le M, \qquad \text{and} \qquad ||\, \partial^\alpha_z\, u\, ||_{L^\infty(\mathbb{R}^{2d}\times\mathbb{T})} \le M \quad \forall\ |\alpha| \le s,$$

then

$$\big\|\, G(u + w) - G(u)\, \big\|_{\Gamma^s}\ \le\ C\, \big\|\, w\, \big\|_{\Gamma^s}\,. \tag{4.14}$$

Using the results of the Lemma, standard Picard iteration constructs for each $s > (2d+1)/2$ a unique solution $u \in C([0, T_*(s)[\,; \Gamma^s)$ with

$$\liminf_{t \nearrow T_*(s)} \|u(t)\|_{\Gamma^s} = \infty\,. \tag{4.15}$$

In fact, inequality (4.14) with $w = 0$ shows that $||G(u)||_{\Gamma^s} \le C(||u||_{L^\infty})\,||G(u)||_{\Gamma^s}$. This estimate shows that so long as the L^∞ norm of u stays bounded, the Γ^s norm can grow at most exponentially, and therefore cannot explode in finite time. This proves (4.7) and at the same time shows that the time of explosion T_* is independent of $s > 2d+1$.

It follows that the unique solution belongs to $C([0, T_*[\,; \Gamma^s)$ for all s. Using the differential equation to express time derivatives in terms of spatial derivatives, it follows that $\partial_t^j u \in C([0, T_*[\,; \Gamma^s)$ for all s. This is equivalent to the desired regularity (4.7). The construction of a_0 is complete.

The construction of the higher profiles requires only the solution of linear equations using (4.12). ∎

§5. Convergence toward exact solutions.

Given a_j satisfying (4.6), Borel's theorem constructs

$$\mathrm{a}(\varepsilon, T, Y, y, \theta) \sim \sum_{j \in p\mathbb{N}} \varepsilon^j\, \mathrm{a}_j(T, Y, y, \theta) \quad \text{in} \quad C^\infty\big([0, T_*[\,; \cap_s H^s(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T})\big)\,, \tag{5.1}$$

with $\mathrm{spec}\,\mathrm{a} \subset \mathbb{Z}_{\mathrm{odd}}$. The $\sim$ in (5.1) means that for all $T \in]0, T_*[$, $\alpha \in \mathbb{N}^{2d+2}$ and $m \in \mathbb{N}$

$$\Big\| \partial^\alpha_{T,z,\theta}\big(\mathrm{a}(\varepsilon, T, Y, y, \theta) - \sum_{|j| \le m} \varepsilon^j\, \mathrm{a}_j(T, Y, y, \theta)\big)\Big\|_{L^2([0,T]\times\mathbb{R}^{2d}\times\mathbb{T})} \le C\,\varepsilon^{m+1}\,. \tag{5.2}$$

The profile a is defined by

$$a(\varepsilon, T, Y, t, y, \theta) \;:=\; \mathrm{a}(\varepsilon, T, Y, y - vt, \theta)\,. \tag{5.3}$$

When a_j, or equivalently a_j, are solutions of the appropriate profile equations, approximate solutions u^ε are defined by

$$u^\varepsilon(t, y) := \varepsilon^p\, a(\varepsilon, \varepsilon x, x, x.\beta/\varepsilon) = \varepsilon^p\, \mathrm{a}(\varepsilon, \varepsilon t, \varepsilon y, y - vt, x.\beta/\varepsilon)\,. \tag{5.4}$$

The profile equations are computed exactly so that in this case

$$L^\varepsilon(u^\varepsilon) + F(u^\varepsilon) = \mathrm{r}^\varepsilon(\varepsilon t, \varepsilon y, y - vt, x.\beta/\varepsilon) \tag{5.5}$$

with

$$\mathrm{r}^\varepsilon(T, Y, y, \theta) \;\sim\; 0 \quad \text{in} \quad C^\infty\big([0, T_*[\,; \cap_s H^s(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T})\big)\,. \tag{5.6}$$

In particular, for any $c < T_*$, u^ε satisfies the hypotheses of the following stability theorem which then implies that for Cauchy data very close to those of u^ε the exact solution of the initial value problem exists for $0 \le t \le c/\varepsilon$ and is very close to u^ε. The theorem is a variant of the stability theorem of Gues, [G1]. The modifications are to assume L^∞ control of the approximate solution to simplify the demonstration, and that the time scale is longer corresponding to smaller amplitudes. These ideas were first introduced in [D].

Approximation Theorem 5.1 *Suppose the derivatives of F of order less than or equal to J vanish at the origin, that $p = 1/(J-1)$, and with $c > 0$*

$$u^\varepsilon = \varepsilon^p\, U^\varepsilon(t,y) \in C^\infty([0,c/\varepsilon] \times \mathbb{R}^d) \qquad \text{for} \quad 0 < \varepsilon \leq 1 \tag{5.7}$$

has $\varepsilon\partial$ derivatives which are $O(\varepsilon^p)$ in the sense that for all $\alpha \in \mathbb{R}^{1+d}$,

$$\big\|\, (\varepsilon\partial_x)^\alpha\, U^\varepsilon\, \big\|_{L^\infty([0,c/\varepsilon]\times\mathbb{R}^d)} \leq C_\alpha\,. \tag{5.8}$$

Suppose in addition that u^ε is an infinitely accurate approximate solution in the sense that

$$L^\varepsilon(u^\varepsilon) + F(u^\varepsilon) \sim 0 \quad \text{in} \quad C^\infty\big([0,c/\varepsilon]\ ;\, \cap_s H^s(\mathbb{R}^d)\big)\,, \tag{5.9}$$

that is for any α and M

$$\big\|\, \partial_x^\alpha\, \big(L^\varepsilon(u^\varepsilon) + F(u^\varepsilon)\big)\, \big\|_{L^2([0,c/\varepsilon]\times\mathbb{R}^d)} \leq C(\alpha, M)\, \varepsilon^M\,. \tag{5.10}$$

Define $v^\varepsilon \in C^\infty([0, T_*(\varepsilon)[\times\mathbb{R}^d)$ to be the maximal solution of the initial value problem

$$L^\varepsilon\, v^\varepsilon + F(v^\varepsilon) = h^\varepsilon\,, \qquad \text{and} \qquad v^\varepsilon(0,y) = u^\varepsilon(0,y) + g^\varepsilon(y)\,. \tag{5.11}$$

If

$$h^\varepsilon \sim 0 \quad \text{in} \quad C^\infty\big([0,c/\varepsilon]\ ;\, \cap_s H^s(\mathbb{R}^d)\big) \qquad \text{and} \quad g^\varepsilon \sim 0 \quad \text{in} \quad \cap_s\, H^s(\mathbb{R}^d)\,, \tag{5.12}$$

then there is an $\varepsilon_0 > 0$ so that for $\varepsilon < \varepsilon_0$, $T_*(\varepsilon) > c/\varepsilon$ and

$$u^\varepsilon - v^\varepsilon \sim 0 \quad \text{in} \quad C^\infty\big(\, [0,c/\varepsilon]\ ;\, \cap_s H^s(\mathbb{R}^d)\big)\,. \tag{5.13}$$

Remarks. In this result, the oddness hypothesis is not needed. The result and the proof are valid for operators of the form $L^\varepsilon = L_1(\partial) + L_0(\varepsilon)$ with

$$L_0(\varepsilon) + L_0(\varepsilon) \geq -C\, I\,, \qquad ||L_0(\varepsilon)||_{\mathbb{C}^N} \leq C\, \varepsilon^{-M}\,.$$

The positive part of $L_0(\varepsilon)$ can be polynomially large. The ϵ in the zero order term of (2.1) is needed for the construction of the approximate solution. ∎

Proof. Step 1. Taylor's Theorem absorbs the ε^p's. Define $w^\varepsilon, W^\varepsilon$, and V^ε by

$$v^\varepsilon := u^\varepsilon + w^\varepsilon := \varepsilon^p U^\varepsilon + \varepsilon^p W^\varepsilon := \varepsilon^p V^\varepsilon\,.$$

Then the equation for v^ε holds if and only if the perturbation w^ε satisfies an equation

$$L^\varepsilon\, w^\varepsilon + F(u^\varepsilon + w^\varepsilon) - F(u^\varepsilon) \sim 0\,, \qquad w^\varepsilon(0) = g^\varepsilon \sim 0\,. \tag{5.14}$$

The strategy is to show that this problem has a solution $w^\varepsilon \sim 0$ on $0 \leq t \leq c/\varepsilon$.
Taylor's Theorem expresses $F(u+w) - F(w) = H(u,w)\, w$ where the smooth function H has derivatives of order less that or equal to $J-2$ vanishing at the $(0,0)$. Therefore

$$H(\varepsilon^p U^\varepsilon, \varepsilon^p W^\varepsilon) = \varepsilon^{(J-1)p}\, G(\varepsilon, U^\varepsilon, W^\varepsilon) = \varepsilon\, G(\varepsilon, U^\varepsilon, W^\varepsilon)\,.$$

where G is a smooth function. Equation (5.14) then reads

$$L^\varepsilon\, \varepsilon^p\, W^\varepsilon + \varepsilon\, G(\varepsilon, U^\varepsilon, W^\varepsilon)\, \varepsilon^p\, W^\varepsilon \sim 0\,.$$

Cancelling the ε^p factors yields

$$L^\varepsilon\, W^\varepsilon + \varepsilon\, G(U^\varepsilon, W^\varepsilon)\, W^\varepsilon \sim 0\,, \qquad W^\varepsilon(0) \sim 0 \quad \text{in} \quad \cap_s\, H^s(\mathbb{R}^d)\,. \tag{5.15}$$

where the key is the factor ε in front of the $G\,W$ term. Roughly one expects growth like $e^{\varepsilon t}$ with sources $O(\varepsilon^\infty)$ which yields $W^\varepsilon = O(\varepsilon^\infty)$ for times up to c/ε.

Step 2. $\varepsilon\partial$ estimates for W^ε. Introduce the H^s_ε norms, each equivalent to the norm in $H^s(\mathbb{R}^d)$ by

$$\|f\|^2_{H^s_\varepsilon} := \sum_{|\alpha|\le s} \|\,(\varepsilon\partial_y)^\alpha f\,\|^2_{L^2(\mathbb{R}^d)}\,. \tag{5.16}$$

This norm is a scaling of that in H^s in the sense that if $g(y) := f(\varepsilon y)$ then

$$\|\,(\,\varepsilon\partial_y\,)^\alpha\, g\,\|^2_{L^2} = \varepsilon^{d/2}\|\,\partial^\alpha_y f\,\|^2_{L^2}\,, \qquad \text{so} \qquad \|\,g\,\|^2_{H^s_\varepsilon} \;=\; \varepsilon^{d/2}\,\|\,f\,\|^2_{H^s}\,. \tag{5.17}$$

This scaling property immediately implies the Sobolev and Moser inequalities

$$\|f\|_{L^\infty} \le \frac{C_s}{\varepsilon^{d/2}}\, \|f\|_{H^s_\varepsilon}\,, \qquad \text{and} \qquad \|G(u)\|_{H^s_\varepsilon} \le C_s(\|u\|_{L^\infty})\,\|u\|_{H^s_\varepsilon}\,. \tag{5.18}$$

In the same way, the the propagation estimate for L^ε

$$\|u(t)\|_{H^s_\varepsilon} \le C(s)\left(\|u(0)\|_{H^s_\varepsilon} + \int_0^t \|L^\varepsilon u(\sigma)\|_{H^s_\varepsilon}\, d\sigma\right) \tag{5.19}$$

follows from the case $\varepsilon = 1$. Inequality (5.19) for $\varepsilon = 1$ can be proved directly by the standard energy method or by using the Fourier Transform as in the proof of (4.12).

Choose $\varepsilon_1 > 0$ so that for $\varepsilon \le \varepsilon_1$, $\|W(0)\|_{L^\infty} \le 1/2$. Since U^ε is bounded in L^∞, it follows that so long as

$$\|W^\varepsilon(t)\|_{L^\infty} \le 1 \tag{5.20}$$

one has

$$\|G(\varepsilon, U^\varepsilon, W^\varepsilon)\, W^\varepsilon\|_{H^s_\varepsilon} \;\le\; C(s)\left(1 + \|W^\varepsilon\|_{H^s_\varepsilon}\right). \tag{5.21}$$

Applying inequality (5.19) to W^ε and using (5.15) and (5.21) yields for all $n \in \mathbb{N}$ and $s > d/2$

$$\|W^\varepsilon(t)\|_{H^s_\varepsilon} \;\le\; C(s,n)\left(\varepsilon^n + \varepsilon \int_0^1 \left(1 + \|W^\varepsilon(\sigma)\|_{H^s_\varepsilon}\right)\, d\sigma\right). \tag{5.22}$$

Gronwall's inequality yields

$$\|W^\varepsilon(t)\|_{H^s_\varepsilon} \;\le\; C(s,n)\,\varepsilon^n\, e^{\varepsilon C(s,n) t}\,. \tag{5.23}$$

Step 3. Endgame. First choose $s > d/2$ and $n > d/2$. Then with the constants C_s from (5.18), $C(s,n)$ from (5.23) and c from (5.7), choose $\varepsilon_0 \le \varepsilon_1$ so that

$$\varepsilon \le \varepsilon_0 \quad \Longrightarrow \quad \frac{C_s}{\varepsilon^{d/2}}\, C(s,n)\,\varepsilon^n\, e^{C(s,n)\,c} \le 1/2\,.$$

Then (5.23) together with (5.18) show that for as long as W^ε exists in $0 \le t \le c/\varepsilon$, one has $\|W^\varepsilon\|_{L^\infty} \le 1/2$.

The first consequence of this conclusion is that for $\varepsilon \le \varepsilon_0$, the maximal solution of the initial value problem (5.15) defining W^ε exists for $0 \le t \le c/\varepsilon$ and satisfies (5.20) throughout that interval. Since v^ε is expressed in terms of W^ε it follows that v^ε exists on this interval.

Once this is known it follows that inequality (5.23) is valid for all s, n and $0 \le t \le c/\varepsilon$. This implies in that for all α and m, there is a $C(\alpha, m)$ so that for all $\varepsilon \le \varepsilon_0$

$$\|\partial_y^\alpha \, w^\varepsilon\|_{L^2} \le C(\alpha, m)\, \varepsilon^m \,. \tag{5.24}$$

Thus to prove (5.13) it suffices to prove estimates analogous to (5.24) for time derivatives of w^ε. These follow from (5.24) by using the differential equation (5.14) to express time derivatives in terms of spatial derivatives. ∎

§6. The quasilinear case.

A few ideas are needed to extend the analysis to the case of quasilinear equations

$$L(u, \partial_x)\, u + F(u) := \sum_{\mu=0}^{d} A_\mu(u)\, \partial_\mu u + F(u) = 0 \,. \tag{6.1}$$

For simplicity of reading, the εL_0 term from (2.1) will be omitted in this section. The system (6.1) is assumed to be symmetric hyperbolic in the sense that the coefficients A_μ are smooth hermitian symmetric valued functions of u, and, for each u, $A_0(u)$ is positive definite.

6.1. Order of nonlinearity and amplitudes.

Suppose that the quasilinear terms are of order $2 \le K \in \mathbb{N}$ in the sense that

$$|\alpha| \le K - 2 \quad \Longrightarrow \quad \partial_{u,\overline{u}}^\alpha \big(A_\mu(u) - A_\mu(0) \big)_{u=0} = 0 \,. \tag{6.2}$$

Then $A_\mu(u) - A_\mu(0)$ is of order $K-1$ and so its product with $\partial_\mu u$ is of order K.

If the solutions have amplitude of order ε^p with derivatives of order ε^{p-1}, then the size of the quasilinear terms is $\varepsilon^{p(K-1)}\,\varepsilon^{p-1}$. Accumulating for time T one obtains $T\,\varepsilon^{p(K-1)}\,\varepsilon^{p-1}$ Setting this equal to the order of magnitude of the solutions, ε^p, yields the following estimate for the time of quasilinear interaction

$$T_{\text{quasilinear}} \sim \frac{1}{\varepsilon^{pK-p-1}} \,.$$

The **standard normalization** for diffractive nonlinear geometric optics is obtained by taking the time of interaction to be of order ε^{-1}, that is

$$p = \frac{2}{K-1} \,. \tag{6.3}$$

To insure that the the semilinear term does not have a time of interaction shorter than ε^{-1}, supposes that F is of order J satisfying (2.4), that is

$$\frac{2}{K-1} = \frac{1}{J-1} \qquad \text{equivalently} \qquad J = \frac{K+1}{2} \,. \tag{6.4}$$

Since K will be assumed odd, (6.4) determines an integer J.

It is possible to consider quasilinear problems for which the left hand sides in (6.4) are smaller than the right hand sides. In such cases, the quasilinear terms do not affect the leading term in the approximation (see [DR 2]).

6.2. Profile equations.

Oddness hypothesis. *In addition to the oddness hypothesis from §2, assume that K is odd and that the Taylor expansion of $A_\mu(u) - A_\mu(0)$ at $u = 0$ contains only monomials of odd order.*

With the normalizations of section 6.1, the ansatz (2.6)-(2.8) is then appropriate. The degree $K-1$ Taylor polynomial of $A_\mu(u) - A_\mu(0)$ is denoted Λ_μ, so for $u \approx 0$,

$$A_\mu(u) - A_\mu(0) = \Lambda_\mu(u) + O(|u|^K)\,.$$

Expanding $L(u^\varepsilon, \partial)u^\varepsilon + F(u^\varepsilon)$ as in §3, and setting the terms of order ε^{p-1}, ε^p and ε^{p+1} equal to zero yields the equations

$$L(0,\beta)\, a_0 = 0\,, \tag{6.5}$$

$$L(0,\beta)\, a_1 + L(0,\partial_x)\, a_0 = 0\,, \tag{6.6}$$

$$L(0,\beta)\, a_2 + L(0,\partial_x)\, a_1 + L(0,\partial_X)\, a_0 + \sum \beta_\mu\, \Lambda_\mu(a_0)\partial_\theta a_0 + \Phi(a_0) = 0\,. \tag{6.7}$$

Equation (6.5) requires that β belong to the characteristic variety of the linearized operator $L(0,\partial)$. Construct the associated projector $\pi(\beta)$ and partial inverse $Q(\beta)$. Equations (6.5) and (6.6) are then equivalent to (3.12), (3.13), and (3.14).

Assume that the characteristic variety of $L(0,\partial)$ satisfies the simplicity assumption at β. Proposition 3.1 then shows that (3.13) is a transport equation.

Setting $\pi(\beta)$ times (6.7) equal to zero yields the analogue of (3.21),

$$\begin{aligned} V(\partial_X)\,\pi\, A_0\, a_0 - \pi\, L(0,\partial_x)\, Q(\beta)\, L(0,\partial_x)\, \partial_\theta^{-1}\, a_0 + \\ \pi \sum \beta_\mu \Lambda_\mu(a_0)\, \partial_\theta a_0 + \pi\, \Phi(a_0) = \pi\, L(0,\partial_x)\, \pi\, a_1\,. \end{aligned} \tag{6.8}$$

The key change is the appearance of the quasilinear terms $\Lambda_\mu(a_0)\partial_\theta a_0$ on the left. They are qualitatively of the form $u^{K-1}\,\partial_\theta u$. The analysis which leads to (3.22) and therefore the elimination of the right hand side is exactly as before. The analysis of Proposition 3.2 applies to $L(0,\partial)$ and one finds the quasilinear equations for the principal profile a_0,

$$V(\partial_x)\, a_0 = 0\,, \qquad V(\partial_X)\,\pi\, A_0\, a_0 + \pi\, A_0\, R(\partial_y)\partial_\theta^{-1}\, a_0 + \pi \sum \beta_\mu \Lambda_\mu(a_0)\, \partial_\theta a_0 + \pi\, \Phi(a_0) = 0\,. \tag{6.9}$$

The analogous equations for the correctors a_j with $j \in p\mathbb{N} \setminus 0$ are linear. The solvability of (6.9) is proved as in §4 with a little more work because of the quasilinear term. The L^∞ norm in the explosion criterion (4.8) is correspondingly replaced by the Lipshitz norm.

§**6.3. Convergence.**

The proof of convergence uses the following quasilinear approximation theorem.

Approximation Theorem 6.1. *Suppose that (6.2) and (6.4) are satisfied, that p is defined by (6.3), and that and u^ε, h^ε and g^ε satisfy (5.7), (5.8), and (5.12). Suppose that u^ε is an infinitely accurate approximate solution in the sense that*

$$L(u^\varepsilon, \partial)\, u^\varepsilon + F(u^\varepsilon) \sim 0 \quad \text{in} \quad C^\infty\big([0, c/\varepsilon]\,;\, \cap_s H^s(\mathbb{R}^d)\big)\,, \tag{6.10}$$

Define $v^\varepsilon \in C^\infty([0, T_*(\varepsilon)[\times \mathbb{R}^d)$ to be the maximal solution of the initial value problem

$$L(v^\varepsilon, \partial)\, v^\varepsilon + F(v^\varepsilon) = h^\varepsilon\,, \qquad \text{and} \qquad v^\varepsilon(0,y) = u^\varepsilon(0,y) + g^\varepsilon(y)\,. \tag{6.11}$$

Then there is an $\varepsilon_0 > 0$ so that for $\varepsilon < \varepsilon_0$, $T_*(\varepsilon) > c/\varepsilon$ and

$$u^\varepsilon - v^\varepsilon \sim 0 \quad \text{in} \quad C^\infty\big(\,[0, c/\varepsilon]\,;\, \cap_s H^s(\mathbb{R}^d)\big)\,. \tag{6.12}$$

Outline of Proof. Define $v^\varepsilon, w^\varepsilon$ and W^ε as in the proof of Theorem 5.1, except that the value of p is now given by (6.3). Taylor's Theorem yields the analogue of equation (5.15),

$$\Big(L(0,\partial_x) + \sum_{\mu=0}^{d} \varepsilon^2 H_\mu(\varepsilon, U^\varepsilon, W^\varepsilon)\,\partial_\mu + \varepsilon\, G(\varepsilon, U^\varepsilon, \partial U^\varepsilon, W^\varepsilon) \Big) W^\varepsilon \sim 0\,. \tag{6.13}$$

The key observation is that so long as

$$\| \, W^\varepsilon, \ \varepsilon \partial W^\varepsilon \, \|_{L^\infty([0,t]\times\mathbb{R}^d)} \leq 1 \tag{6.14}$$

the derivatives of the coefficients $\varepsilon^2 H_\mu$ are $O(\varepsilon)$. Then the natural growth rates are $O(e^{\varepsilon t})$ which means that one has stability estimates for times as long as c/ε. That is, for $s > d+1$, there is a constant $C(s)$ so that so long as $0 \leq t \leq c/\varepsilon$ and (6.14) holds one has

$$\|W(t)\|_{H_\varepsilon} \leq C \Big(\|W(0)\|_{H^s_\varepsilon} + \int_0^t \| \, \{ L(0,\partial_x) + \sum_{\mu=0}^{d} \varepsilon^2 H_\mu(\varepsilon, U^\varepsilon, W^\varepsilon)\,\partial_\mu \ + \varepsilon\, G(\varepsilon, U^\varepsilon, \partial U^\varepsilon, W^\varepsilon) \,\}\, W^\varepsilon(\sigma) \, \|_{H^s_\varepsilon} \, d\sigma \Big)\,.$$

The endgame is as in Theorem 5.1. ∎

References

[D] P. Donnat, Quelques contributions mathématiques en optique non linéaire, thèse, École Polytechnique, 1994.

[DR 1] P. Donnat and J. Rauch, Dispersive nonlinear geometric optics, J, Math. Phys. 1997, to appear

[DR 2] P. Donnat and J. Rauch, Diffractive nonlinear geometric optics for dispersive hyperbolic systems, in preparation.

[Go] P. Godin, Global oscillatory waves for second order quasilinear wave equations, Trans. Amer. Math. Soc., 346(2)(1994), 523-547.

[G 1] O. Gues, Développements asymptotiques de solutions exactes de systèmes hyperboliques quasilinéaires, Asymptotic Anal., 6(1993), 241-269.

[G 2] O. Gues, Ondes multidimensionnelles ε-stratifiées et oscillations, Duke Math. J., 68(1992), 401-446.

[HMR] J. Hunter, A. Majda, and R. Rosales, Resonantly interacting weakly nonlinear hyperbolic waves II: several space variables, Stud. Appl. Math., 75(1986), 187-226.

[JMR 1] J.-L. Joly, G. Metivier, and J. Rauch, Resonant one dimensional nonlinear geometric optics J. Funct. Anal., 114(1993), 106-231.

[JMR 2] J.-L. Joly, G. Metivier, and J. Rauch, Coherent and focusing multidimensional nonlinear geometric optics, Annales de L'École Normale Supérieure de Paris, 28(1995), 51-113.

[JMR 3] J.-L. Joly, G. Metivier, and J. Rauch, Generic rigorous asymptotic expansions for weakly nonlinear multidimensional oscillatory waves, Duke Math. J., 70(1993), 373-404.

[JMR 4] J.-L. Joly, G. Metivier, and J. Rauch, Coherent nonlinear waves and the Wiener algebra, Ann. Inst. Fourier, 44(1994), 167-196.

[JMR 5] J.-L. Joly, G. Metivier, and J. Rauch, Nonlinear oscillations beyond caustics, Comm. Pure and Appl. Math.,XLIX(1996), 443-529.

[JMR 6] J.-L. Joly, G. Metivier, and J. Rauch, Diffractive nonlinear geometric optics with rectification, in preparation.

[JR] J.L. Joly and J. Rauch, Justification of multidimensional single phase semilinear geometric optics, Trans. Amer. Math. Soc., 330(1992), 599-625.

[Lax] P.D. Lax, Oscillatory solutions of oscillatory initial value problems, Duke Math. J., 24(1957), 627-645.

[MRS] A. Majda, R. Rosales and M. Schonbeck, A canonical system of integrodifferential equations arising in resonant nonlinear acoustics, Stud. Appl. Math., 79(1988), 205-262.

[R] J. Rauch, Lectures on geometric optics, IAS-Park City Summer School on Nonlinear Waves, July, 1995.

[Se] D. Serre, Oscillations non linéaires des systèmes hyperboliques : méthodes et résultats qualitatifs, Ann. Inst. Henri Poincaré, 8(1991) 351-417.

[Sch] S. Schochet, Fast singular limits of hyperbolic partial differential equations, J. Diff. Eq., 114(1994), 474-512.

[T] L. Tartar, Solutions oscillantes des équations de Carleman, Séminaire Goulaouic-Schwartz, École Polytechnique, Paris, année 1980-0981.

[Wa] T. Wagenmaker, Analytic solutions and resonant solutions of hyperbolic partial differential equations, PhD Thesis, University of Michigan, 1993.

[Wi] M. Williams, Resonant reflection of multidimensional semilinear oscillations, Comm. Partial Diff. Eq., 18(1993), 1901-1959.