Recent Results in Non Linear Geometric Optics

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1. Introduction.

Many recent works are devoted to the study of high frequency oscillatory nonlinear waves and to nonlinear geometric optics. In this talk we review several basic problems and results in this field.

High frequency linear waves are modelled by phase-amplitude expansions

$$(1.1) \qquad u^\varepsilon(t,x) \;=\; a(t,x,\varepsilon)\; e^{i\,\varphi(t,x)/\varepsilon} \quad \text{with} \quad a(t,x,\varepsilon) \;\sim\; \sum_{n\geq 0} \varepsilon^n \; a_n(t,x)\,.$$

This leads to linear geometric optics ([La]) : the phase φ satisifies an eikonal equation and the amplitudes a_n are determined by transport equations. For nonlinear waves, the natural extension of (1.1) which includes the phenomenon of generation of harmonics is

$$(1.2) \qquad u^\varepsilon(t,x) \;\sim\; \underline{u}(t,x) \;+\; \varepsilon^p \sum_{n\geq 0} \varepsilon^n \; U_n(t,x,\varphi(t,x)/\varepsilon)\,.$$

The $U_n(t,x,\theta)$ are periodic (or quasi/almost periodic) in the θ variable and φ is a real phase function. In the summation, n runs in a countable set of nonnegative numbers. Typically, n is an integer, a half integer... . $\underline{u}$ is a a background state, ε is a wavelength and ε^p is the order of magnitude of the energy of the oscillations (assuming that $\partial_\theta U_0 \neq 0$). Basic questions are existence, propagation, interaction, reflection of waves (1.2).

In contrast with linear waves, the choice of the parameter p for the amplitude of the oscillations is very important. It is strongly related to the scale of time T under consideration. For small times or small amplitudes (i.e. large p), the propagation is mainly linear. The idea is to increase the scale of energy ε^p or the time of propagation T to reach the regime where *the first* nonlinear effects appear. Examples of relations between p and T are be computed in sections 2 and 4.

The waves u^ε are solutions of partial differential equations. Examples from nonlinear optics are Maxwell equations which describe the propagation of an electromagnetic wave in a nonlinear medium modelled by a polarisation vector P :

$$\begin{cases} \partial_t B + \mathrm{curl} E = 0\,, \\ \partial_t E - \mathrm{curl} B = -\partial_t P \end{cases} \tag{1.3}$$

Different models for the interaction light-matter have been considered (see [Do]). Here we mention two of them :

$$\varepsilon^2 \partial_t^2 P + P = E + f(E, P) \qquad \text{anharmonic oscillator model}$$

$$\begin{cases} \varepsilon^2 \partial_t^2 P + P = (N_0 - N) E \\ \partial_t N = E \cdot \partial_t P \end{cases} \qquad \text{Maxwell-Bloch}$$

Note that the size of several coefficients is related to the wavelength ε.

Other examples of equations for u^ε are nonlinear wave equations

$$(\partial_t^2 - \Delta_x)\, u \;+\; f(t, x, u, \nabla_{t,x}\, u) \;=\; 0\,. \tag{1.4}$$

One can also consider quasilinear equations, such that Euler's equations for gas dynamics and thus study acoustic waves. For the purpose of this talk, we restrict ourselves to first order semilinear systems. Possibly after rescaling the dependent and independent variables u and (t, x), we suppose that the equation reads

$$L(\varepsilon\partial)\, u = \; f(t, x, u, \varepsilon) \tag{1.5}$$

where

$$L(\varepsilon\partial) \;:=\; \varepsilon\partial_t \;+\; \sum_{j=1}^{d} \varepsilon A_j(t, x)\partial_{x_j} \;+\; E(t, x)\,, \tag{1.6}$$

and the A_j are symmetic, E is skew symmetric and $f = O(\varepsilon + |u|^2)$. The case $E = 0$, after disivion by ε in (1.5), covers the usual examples of hyperbolic equations such as the reduction of (1.4) to a first order system. Maxwell equations (1.3) are of the form (1.5) with $u = (B, E, P, \varepsilon\partial_t P)$ or $u = (B, E, P, \varepsilon\partial_t P, N)$. Quasinilear analogues cover the case of Euler's equations.

Note that nonlinear geometric optics technics are not restricted to the specific expansion (1.2). Rescaling both the dependent and independent variable, is always possible. The important feature is that two scales are present. They are (t, x) and $\varphi(t, x)/\varepsilon$ in (1.2), but one can as well consider scales like $(\varepsilon t, \varepsilon x)$ and (t, x). In particular, the analysis applies to *singular* systems

$$\partial_t U^\varepsilon \;+ \sum A_j \partial_{x_j} U^\varepsilon \;+\; \frac{1}{\varepsilon} \sum B_j \partial_{\theta_j} U^\varepsilon = F(t, x, U^\varepsilon, \varepsilon)\,, \tag{1.6}$$

see §3.3 and the references there. One can also consider nonperiodic profiles U_n in θ, such as rapidly decreasing functions. They are used to model solutions which resemble simple waves.

The paper is organized as follows. In §2, we review the basic problem of the propagation of single wave (1.2). The main task is to justify the existence of exact solutions close to the formal solutions given by BKW methods. The interaction of several waves is described by expansions similar to (1.2) but with several phases. It is discussed in §3, where several different results are briefly presented. Three scales expansions are introduced in §4. They are usefull to describe long time propagation and diffractive effects

2. One phase expansions.

2.1. Standard amplitudes

We look for asymptotic solutions

$$u^\varepsilon(t,x) \;\sim\; \varepsilon^p \sum_{n\ge 0} \varepsilon^n\, U_n(t,x,\varphi(t,x)/\varepsilon)\,. \tag{2.1}$$

of equation (1.4) for times $0 \le t \le T$. To determine the natural size for the amplitude ε^p, consider the ordinary differential case where $L(\varepsilon\partial) = \varepsilon\partial_t$. Then

$$u(t) \;=\; u(0) \;+\; \varepsilon^{-1} \int_0^t f(u(s))\, ds\,. \tag{2.2}$$

Suppose that f is of order $J \ge 2$ near $u = 0$. Then,

$$u = O(\varepsilon^p) \quad \Rightarrow \quad \varepsilon^{-1} \int_0^t f(u) = O(T\,\varepsilon^{pJ-1})\,.$$

The balance is for $T\varepsilon^{pJ-1} \approx \varepsilon^p$, thus

$$p = \frac{1}{J-1} \quad \text{for} \quad T = O(1)\,. \tag{2.3}$$

For simplicity, we continue the discussion assuming that $J = 2$ and thus $p = 1$.

2.2. Formal solutions

Plugging the expansion into the equation (2.1) leads to an infinite system of equations (see [CB])

$$L(d\varphi\,\partial_\theta)U_0 \;=\; 0\,, \tag{2.4}$$

and for $n \ge 0$

$$L(d\varphi\,\partial_\theta)U_{n+1} \;+\; L_1(\partial)U_n \;+\; F_n \;=\; 0\,. \tag{2.5}$$

Here $L(d\varphi\,\partial_\theta) = (\partial_t\varphi + \sum \partial_{x_j}\varphi A_j)\partial_\theta + E$, $L(\partial) \;:=\; \partial_t \;+\; \sum\; A_j\,\partial_{x_j}$, and F_n is a function of $(U_0, \dots U_n)$. In particular, $F_0 = -\frac{1}{2}\, f''(0)(U_0, U_0)$.

One analyzes $L(d\varphi\partial_\theta)$ on Fourier series in θ :

$$L(d\varphi\partial_\theta)\Big(\sum_\alpha V_\alpha\, e^{i\alpha\theta}\Big) \;=\; \sum_\alpha L(i\alpha d\varphi)V_{0,\alpha}\, e^{i\alpha\theta}\,.$$

The projector on its kernel is (formally)

$$\mathbb{P}\Big(\sum_\alpha V_\alpha\, e^{i\alpha\theta}\Big) \;:=\; \sum_\alpha \Pi_\alpha V_\alpha\, e^{i\alpha\theta}\,.$$

where Π_α is the orthogonal projector on the kernel of $L(i\alpha d\varphi)$. Because the $L(i\alpha d\varphi)$ are skew adjoint, the projector on the image of $L(d\varphi\partial_\theta)$ is (formally) $1-\mathbb{P}$.

For (2.4) to have a nontrivial solution, one requires that at least one among the projector Π_α is non trivial, and it is natural to assume that this holds for $\alpha=1$. This means that φ satisfies the eikonal equation

$$\det L(id\varphi) = \det L(t,x,id\varphi(t,x)) \;=\; 0\,. \tag{2.6}$$

Then, (2.4) is equivalent to the polarization condition

$$U_0 = \mathbb{P}U_0 \quad\Leftrightarrow\quad \begin{cases} U_{0,\alpha} = 0 & \alpha\notin\mathcal{C} \\ U_{0,\alpha} = \Pi_\alpha U_{0,\alpha} & \alpha\in\mathcal{C}\end{cases} \tag{2.7}$$

where $\mathcal{C}$ is the set of integers α such that $\det L(i\alpha d\varphi)=0$.

In the *nondispersive case* ($E=0$ in (1.5)), $\mathcal{C}=\mathbb{Z}$, $\Pi_0 = Id$ and for $\alpha\neq 0$, $\Pi_\alpha = \Pi$ is the projector on the kernel of $L(id\varphi)$. Introducing the average $\underline{V}$ and the oscillation $V^* = V - \underline{V}$ of the periodic function V, (2.7) is then equivalent to the condition $\Pi U_0^* = U_0^*$.

In the *dispersive case* ($E\neq 0$), $\mathcal{C}$ is limited. In many examples, such as the Maxwell equations of §1, one has $\mathcal{C}=\{-1,0,1\}$, which means that there is no generation of harmonics in the leading term.

The propagation equation for U_0 is obtained by applying $\mathbb{P}$ to the equation (2.5) :

$$\mathbb{P}L_1(\partial)\mathbb{P}U_0 \;+\; \mathbb{P}F_0(U_0) \;=\; 0\,, \tag{2.8}$$

In the nondispersive case, this is a coupled system for the average and oscillations of U_0 :

$$\begin{cases} L_1(\partial)\,\underline{U_0} \;+\; \underline{F_0} &=& 0 \\ (I-\Pi)\,U_0^* &=& 0 \\ \Pi\,L_1(\partial)\,\Pi U_0^* \;+\; \Pi\,F_0^* &=& 0 \end{cases}$$

In the dispersive case, (2.8) is a coupled system of equations for the Fourier coefficients $U_{0,\alpha}$ with $\alpha\in\mathcal{C}$.

Example. Suppose that $\lambda(t,x,\xi)$ is a real eigenvalue of constant multiplicity of the hermitian symmetric matrix $\sum \xi_j\, A_j(t,x) - iE$ and that $\alpha\varphi$ satisfies

$$\alpha\partial_t\varphi(t,x) \;+\; \lambda(t,x,\alpha\partial_x\varphi(t,x)) \;=\; 0\,. \tag{2.9}$$

Then the first order part of the propagator $\Pi_\alpha L_1(\partial)\Pi_\alpha$ is the vector field :

$$\mathbb{X}_\alpha \;:=\; \partial_t \;+\; \sum_j \frac{\partial\lambda}{\partial\xi_j}(t,x,\alpha\partial_x\varphi)\,\partial_{x_j}\,. \tag{2.10}$$

In the nondispersive case, this is independent of $\alpha \neq 0$. In the dispersive case the group velocity $\mathbf{v}_\alpha := \nabla_\xi\lambda(t,x,\alpha d\varphi)$ depends on α. The constant multiplicity assumption is satisfied generically, but variable multiplicities do occur. A classical example is conical refraction.

Given C^∞ Cauchy data for φ and U_0 with $\mathbb{P}U_0 = U_0$, one can solve the eikonal equation (2.6) and the nonlinear profile equation (2.7) (2.8) on a suitable domain of determinacy Ω. The other terms U_n are determined similarly on the same domain Ω, solving *linear* equations. In the most favorable cases, this leads to approximate solutions, that is to functions u^ε_{app} which satsify (2.1) and

$$L(\varepsilon\partial)\,u^\varepsilon_{app} \;-\; f(u^\varepsilon_{app}) \;=\; O(\varepsilon^\infty)\,.$$

2.3. Exact solutions

Theorem ([JR 1], [DR]) *Let u^ε_{app} be a formal solution on a domain of determinacy Ω for $L(\varepsilon\partial)$. Suppose that*

$$u^\varepsilon_0(x) \;-\; u^\varepsilon_{app}(0,x) \;=\; O(\varepsilon^\infty)\,.$$

Then the Cauchy problem initial data u^ε_0, has a unique solution u^ε for $\varepsilon \leq 1$, defined and smooth on a domain $\Omega' \subset \Omega$ independent of ε, and satisfying

$$u^\varepsilon(t,x) \;-\; u^\varepsilon_{app}(t,x) \;=\; O(\varepsilon^\infty)$$

Moreover, Ω' can be taken arbitrarily close to Ω if ε is restricted to be small enough.

The main part of the proof is not to compare u^ε and u^ε_{app}. It is to prove the *existence* of u^ε on a domain independent of ε. Note that the classical existence theorems for smooth solutions do not apply. The Theorem means that the small arbitrary oscillations contained in the remainders $O(\varepsilon^N)$ are kept under control. They interact with the main oscillation u^ε_{app} and also with themselves, but they cannot be organized in a coherent way to affect the leading oscillation.

Further results and problems.

1) Similar results are available for quasilinear equations and perturbations of a background solution $\underline{u}$ ([G 2], [G 3]). The phase φ satisfies the eikonal equation for the linearized operator on $\underline{u}$. The main novelty is that the transport operator contains an additional nonlinear term in $\partial_\theta U_0$. In particular, the transport equation for U_0 looks like a Burger's equation and thus may develop shocks. The other equations for U_n, $n \geq 1$ remain linear. The expansions are justified, as long as the phase φ and the first profile U_0 exist and remain smooth. See also §3.1 for results about weak solutions in 1-D.

2) Another interesting feature of [G 3] is the justification of expansions with *almost periodic profiles*. In the nondispersive case, the Fourier analysis of the inverse of $L(d\varphi\partial_\theta)$ is much more delicate since the Fourier spectrum is now $\mathbb{R}$ in place of $\mathbb{Z}$ and ∂_θ^{-1} does not act in the space of almost periodic functions with vanishing mean value. As a consequence, the second term U_1 is not defined in general. However, the profile equations still make sense and define the U_0. [G 3] provides a construction of solutions which satisfy

$$u^\varepsilon(t,x) \; = \; \varepsilon^p \, U_0(t,x,\varphi(t,x)/\varepsilon) \; + \; o(\varepsilon^p)\,.$$

3) In some cases, the "standard amplitude" computed at the beginning of this section leads to linear geometric optics (the nonlinearity vanishes on the polarization). This happens for example for Maxwell-Bloch equations or for quasilinear waves associated to linearly degenerate eigenvalues. To reach nonlinear phenomena one can consider larger amplitudes or longer times. In some cases, approximate solutions can still be constructed but new *stability conditions* are needed to justify the approximation. This is related to known examples of instabilities such as Raman an Brillouin instabilities in optics. For Maxwell Bloch equation which is is quadratic ($J = 2$), the standard amplitude is $p = 1$. Formal solutions with $p = 1/2$ can be computed ([D]). The justification is in progress. For quasilinear systems, the standard amplitude is $p = 1$. Formal solutions with $p = 0$, associated with linearly degenerate eigenvalues are proposed in [Se 1], [Se 2]. The justification in space dimension one is made in [He].

3. Multiphase expansions

Consider the interaction of oscillatory waves. The interesting phenomenon is the creation of a new outgoing oscillation of the same magnitude, from incoming oscillations. This happens when the phases φ_j of the incoming waves and the phase φ of the outgoing wave satisfy the *resonance relation*

$$\varphi \; = \; \sum_j \varphi_j \,, \tag{3.1}$$

or more generally

$$\alpha\,\varphi \; = \; \sum_j \alpha_j, \varphi_j \,, \quad \alpha, \alpha_j \in \mathbb{Z}\,, \tag{3.2}$$

for the resonances among harmonics. The analysis is made through multiphase expansions

$$u^\varepsilon(t,x) \ \sim\ \varepsilon^p \sum_{n\geq 0} \varepsilon^n\, U_n(t,x,\psi_1(t,x)/\varepsilon,\dots,\psi_m(t,x)/\varepsilon)\,, \tag{3.3}$$

where the $U_j(t,x,\theta_1,\dots\theta_m)$ are periodic in the variables θ. The leading term U_0 is often sought as a superposition of incoming and outgoing waves

$$U_0(t,x,\psi_1(t,x)/\varepsilon,\dots,\psi_m(t,x)/\varepsilon) \ =\ \sum_j\ S_j(t,x,\varphi_j(t,x)/\varepsilon)\,. \tag{3.4}$$

with phases φ_j which are linear combination of the ψ_k satisfying the eikonal equation. However, the number of directions for the phases φ_j can be infinite, even if only three of them are present in the Cauchy data, the other ones being created by interactions (see [JR 2], [JMR 2], [JMR 8]).

A formal derivation of the equations of nonlinear geometric optics, is given in [HK 1], [MR] and [HMR], see also [JMR 1], [JMR 3]. The equations of propagation have the same structure as in §2 :

$$U_0 = \mathbb{P}U_0\,, \quad \mathbb{P}L_1(\partial)\mathbb{P}U_0\ +\ \mathbb{P}F_0(U_0)\ =\ 0\,. \tag{3.5}$$

The operator $\mathbb{P}$ acts on multiperiodic functions of $\theta = (\theta_1,\dots,\theta_m)$

$$\mathbb{P}\Big(\sum_\alpha V_\alpha\, e^{i\alpha\theta}\Big)\ :=\ \sum_\alpha \Pi_\alpha V_\alpha\, e^{i\alpha\theta}\,.$$

where Π_α is the orthogonal projector on the kernel of $L(i\alpha d\psi)$, which is only non trivial when $\alpha\psi := \sum \alpha_j\psi_j$ is eikonal. Thus the structure of the resonances is encoded in $\mathbb{P}$.

The justification of the asymptotic expansions for exact solutions is delicate. Simple examples show that complete expansions (3.3) are not true in general. Instead, one has to study a larger class of solutions which satisfy the weaker estimate

$$u^\varepsilon(t,x)\ =\ U_0(t,x,\psi_1(t,x)/\varepsilon,\dots,\psi_m(t,x)/\varepsilon)\ +\ o(1)\,, \tag{3.6}$$

3.1. Results in one space dimension

Solutions satisfying (3.6) are constructed [JMR 1] in wide class of situations. Partial previous justifications were given in [T], [McLPT], [J], [Ka]. In general (3.6) is proved in L^p for all $p<\infty$. The justification in L^∞ requires more assumptions. This is due to weak resonances that is linear combinations (3.1) or (3.2) which satisfy the eikonal equation only on a small set. They create new waves whose energy is localized on isolated curves. These waves are small in L^1 but $O(1)$ in L^∞. As a consequence (3.6) is not always true in L^∞. This phenomenon is studied in detail in [G 1]. In the absence of weak resonances, L^∞ convergence is proved in [JMR 1].

The justification of nonlinear geometric optics also covers the framewok of weak solutions of quasilinear systems of conservation laws : see [Che 2] which generalizes a result of [Sch 1] and particular cases previously treated in [DiPM], [Liu], [Che 1].

The interaction of (strong) shock waves or contact discontinuities and small amplitude oscillations is described in [Co 1] and [Co 2].

3.2. Focusing and caustics

In space dimension $d \geq 2$, an major phenomenon is the *focusing* of oscillations. This phenomenon is already present for linear equations and corresponds to the breakdown of existence of smooth solutions of the eikonal equation. When rays focus, amplitudes grow and the large amplitudes can be amplified by nonlinearities. For example, it may imply that the domain of existence of the (weak) solutions u^ε, shrinks to the empty set as ε tends to zero (see [JMR 2] [JMR 7]). Focusing and blow up can be created by the principal oscillations themselves. This is called *direct focusing* in [JMR 2]. But nonlinear interactions make the problem much harder. Focusing and blow up can be created by phases not present in the principal term of the expansion, but which are generated after several interactions. This phenomenon is explored in detail in [JMR 2], where it is called *hidden focusing*.

In the extreme opposite direction, when it is combined with strongly dissipative mechanisms, focusing can lead to a complete absorbtion of oscillations, in finite time. The oscillations disappear when they reach the caustic set. An example of such a behaviour is given in [JMR 5].

A general study of caustics for dissipative or sublinear equations is performed in [JMR 7] [JMR 10]. In this case, global existence is known in advance together with suitable energy estimates. As in the linear case, the desciption of the solution involves oscillatory integrals and the transport equations hold on the Lagrangian manifold which is the geometric solution of the eikonal equation. Approximations are shown in L^p for some $p \geq 2$. However, no precise behaviour near the caustic is given. In particular, a challenging open question, is the validity of the formal expansions near the caustic, computed in [HK 2].

3.3 Coherent expansions

A framework where all focusing is excluded, besides the one space dimensionnal case, occurs when one considers a *coherent* set of phases. This assumption means that the dichotomy between characteristic and noncharacteristic phases (3.2) is clear.

Definition ([JMR 2]). *A real vector space $\Phi \subset C^\infty(\Omega)$ is L-coherent when for all $\varphi \in \Phi\backslash\{0\}$, one of the following two condition holds :*

i) $\det L(t,x,d\varphi(t,x)) \;\equiv\; 0$ *and* $d\varphi(t,x) \neq 0$ *at every point* $(t,x) \in \Omega$*,*

ii) $\det L(t,x,d\varphi(t,x)) \;\neq 0$ *at every point* $(t,x) \in \Omega$*.*

A typical and important example is given by linear phases for a constant coefficient background system. Then, expansions like (3.3) or (3.5) can be justified, see [JMR 2] [JMR 3] [JMR 4] [Sch 2] [De].

An important idea is to introduce the *fast variables* $\theta = (\theta_1, \ldots, \theta_m)$ as independent variables and to look for *exact* solutions of the form

$$u^\varepsilon(t,x) = \varepsilon^p\, U^\varepsilon(t,x,\psi_1(t,x)/\varepsilon,\ldots,\psi_m(t,x)/\varepsilon)\,. \tag{3.7}$$

The phases ψ_j belong to the coherent space Φ and are independent over $\mathbb{Q}$, not necessarily over $\mathbb{R}$. This allows the study of quasiperiodic oscillations. The equation for U^ε is a *singular* system

$$\partial_t U^\varepsilon + \sum A_j \partial_{x_j} U^\varepsilon + \frac{1}{\varepsilon}\sum B_j \partial_{\theta_j} U^\varepsilon = F(t,x,U^\varepsilon,\varepsilon) \tag{3.8}$$

The B_j are skew symmetric and thus L^2 estimates are available. The coherence assumption is used to prove H^s estimates.

The principal profile U_0 is determined by (3.5). However, the equations for U_1 leads to small divisors problems for the summation of its Fourier series. *Generically*, the summation of the Fourier series can be performed and complete expansions (3.3) can be constructed ([JMR 3]) (see also [De] for the case of the wave equation). Using a general theorem from [G 2], this implies the existence of exact solutions satisfying (3.3). In general, only U_0 is defined and one can can prove (3.7) : [JMR 2], [Sch 2]. Note that theses results have analogues for quasilinear equations. Almost periodic semilinear oscillations are investigated in [JMR 4].

3.4. Further Results.

1. Another important problem is the reflection of oscillatory waves on boundaries. A formal approach is described in [MA]. The tranverse reflection at a boundary is studied in [Chi] for two speed equations, and in [Wi 1] under more general hypothesis. The possibility of glancing mode for flat boundaries and linear coherent phases, is studied in [Wi 2]. Oscillations near a diffractive point, for semilinear wave equations with globally Lipschitzean nonlinearity, are studied in [Che 3]. The interaction of a multidimensional strong shock and oscillations in the regime of weakly nonlinear geometric optics is discussed in [Wi 3], [Wi 4].

2. The equations of geometric optics expansions are simpler than the original full set of equations, and thus they give detailed descriptions of several mechanisms of nonlinear interaction. Applications to the study of qualitative behavior of solutions of systems of conservation laws, are numerous. See e.g. [Hu 1], [MR], [MRS], [JMR 9].

3. The validity of geometric optics can be extended beyond the category of solutions of the form (1.1), to more general families of "oscillating" solutions u^ε. This is shown in [JMR 6] and [MS] for semilinear systems either of size 3×3 or with quadratic interaction. This illustrates a link between geometric optics and compensated compactness, in the spirit of [DiP 1], [DiP 2], [T], [McLPT], [Wa].

4. Three scales expansions

In geometric optics, the waves are propagated along rays. However, even for parallel rays, diffractive effects appear for long time propagation. Then "paraxial approximations" are valid and the dispersion is due to the curvature of the characteristic variety. The analysis is performed through a three scale expansion

$$u^\varepsilon(t,x) \;\sim\; \varepsilon^p \sum_{n\geq 0} \varepsilon^n\, U_n(\varepsilon t, t, x, \varphi(t,x)/\varepsilon)\,. \tag{4.1}$$

for times $t \leq O(1/\varepsilon)$. Related expansions in other scales are

$$u^\varepsilon(t,x) \;\sim\; \varepsilon^p \sum_{n\geq 0} \varepsilon^n\, U_n(t, x, \psi(t,x)/\varepsilon, \varphi(t,x)/\varepsilon^2)\,. \tag{4.2}$$

for times $t \leq O(1)$ (See [Hu 2])

The initial data for u^ε correspond to "initial data" for the $U_n(T,t,x,\theta)$ on $T = t = 0$.

Assume here that $L(\varepsilon\partial)$ has constant coefficients and is non dispersive ($E = 0$). Consider an eigenvalue $\lambda(\xi)$ of constant multiplicity of $A(\xi) = \sum \xi_j A_j$ and assume that $\varphi(t,x) = \omega t + k\cdot x$ with $\omega = -\lambda(k)$. In particular, φ satisfies the eikonal equation.

4.1. Standard amplitudes As in section 2.1, the balance is $T\varepsilon^{pJ-1} \approx \varepsilon^p$. For $T = O(1/\varepsilon)$ this holds for $p = 2/(J-1)$.

4.2. Formal expansions

Plugging (4.1) into (1.4) and ordering the equations in powers of ε, one obtains

$$L(d\varphi)\,\partial_\theta U_0 = 0\,, \tag{4.3}$$

$$L(d\varphi)\,\partial_\theta U_1 \;+\; L(\partial_{t,x})U_0 \;= 0\,, \tag{4.4}$$

$$L(d\varphi)\,\partial_\theta U_2 \;+\; L(\partial_{t,x})U_1 \;+\; \partial_T U_0 \;=\; f_J(U_0). \tag{4.5}$$

where f_J is the J-th homogeneous part of the Taylor expansion of f at $u = 0$.

a) The first equation (4.3) is equivalent to

$$\Pi U_0^* = U_0^* \quad \text{(polarization)}. \tag{4.6}$$

where $\Pi :=$ is the orthogonal projector on $\ker L(d\varphi)$. We use the notation $\underline{V}$ and $V^* = V - \underline{V}$ for the average and oscillations of V.

b) Split $F := L(d\varphi)\partial_\theta U_1 + L(\partial)U_0$ into $\underline{F} + \Pi F^* + (Id - \Pi)F^*$. The (4.4) is equivalent to

$$L(\partial_{t,x})\,\underline{U_0} = 0 \tag{4.7}$$

$$X(\partial_{t,x})U_0^* = 0 \tag{4.8}$$

$$(Id - \Pi)U_1^* = -QL(\partial_{t,x})\partial_\theta^{-1}U_0^* \tag{4.9}$$

where X is the transport field $X(\partial_{t,x}) = \partial_t + \mathbf{v}\cdot\partial_x := \partial_t + \sum_{j=1}^d \partial_{\xi_j}(k)\,\partial_{x_j}$ which, thanks to the constant multilicity assumption satisfies $\Pi L(\partial_{t,x})\Pi = X(\partial_{t,x})\Pi$. In addition, Q is the partial inverse of $L(d\varphi)$ suchthat $QL(d\varphi) = L(d\varphi)Q = Id - \Pi$ and $Q\Pi = \Pi Q = 0$. Finally, ∂_θ^{-1} is the inverse of ∂_θ acting on functions of vanishing mean value. .

c) Similarly, (4.6) is equivalent to

$$\partial_T\,\underline{U_0} + L(\partial_{t,x})\,\underline{U_1} = \underline{f_J(U_0)} \tag{4.10}$$

$$\partial_T U_0^* + X(\partial_{t,x})\Pi U_1^* - \Pi L(\partial_{t,x})QL(\partial_{t,x})\partial_\theta^{-1}U_0^* = \big(\Pi f_J(U_0)\big)^* \tag{4.11}$$

$$(Id - \Pi)U_2^* = \partial_\theta^{-1}Q(Id - \Pi)\big(\big(\Pi f_J(U_0)\big)^* - \partial_T U_0^* - L(\partial_{t,x})U_1^*\big) \tag{4.12}$$

We have used (4.9) in (4.11). The constant multiplicity also implies that

$$S := \Pi L(\partial_{t,x})QL(\partial_{t,x})\Pi = \frac{1}{2}\sum_{j,l}\partial^2_{\xi_j,\xi_l}\lambda(k)\partial^2_{x_j,x_l}\,\Pi \tag{4.13}$$

4.3. Simplified case

Assume that J is odd and the data have an odd Fourier spectrum. Then one looks for solutions U_n in the space of periodic functions with odd spectrum

$$U(\theta) = \sum_{\alpha\in 2\mathbb{Z}+1} U_\alpha\, e^{i\alpha\theta}\,.$$

This space is preserved by odd polynomials and thus by f_J. In particular $\underline{U} = 0$.

The equations (4.7) and (4.10) are void. (4.6), (4.8) (4.11) become

$$U_0^* = \Pi U_0^*\,, \quad XU_0^* = 0\,, \tag{4.14}$$

$$\partial_T U_0^* - S\partial_\theta^{-1}U_0^* + \Pi f_J(U_0^*) = X(\partial_{t,x})\Pi U_1^* \tag{4.15}$$

X anihilates the left hand side of (4.2). Thus $X^2\Pi U_1 = 0$. The condition ΠU_1 bounded for large t then requires $X\Pi U_1 = 0$. Finally, the equations for U_0 are

$$\begin{cases} U_0^* = \Pi U_0^*\,, \quad XU_0^* = 0\,, \\ \partial_T U_0^* - S\partial_\theta^{-1}U_0^* + \Pi f_J(U_0^*) = 0 \end{cases} \tag{4.16}$$

Theorem ([DJMR]) *i) The equations for U_0 can be solved, and similar equations for the U_n, $n > 0$, giving approximate solutions u^ε_{app} on $[0, \overline{T}/\varepsilon] \times \mathbb{R}^d$.*

ii) If $u_0^\varepsilon(x) \sim u^\varepsilon_{app}(0,x)$, for all $\underline{T} < \overline{T}$, there is $\varepsilon(\underline{T}) > 0$ such that the Cauchy problem with initial data u_0^ε has a solution u^ε on $[0, \underline{T}/\varepsilon] \times \mathbb{R}^d$ and $u^\varepsilon \sim u^\varepsilon_{app}$.

4.4. The general case, rectification

We refer to [JMR 11] for the general case. The new phenomenon is *rectification* which occurs when the oscillations U^* affect the main field $\underline{U}$. The analysis is much more delicate and complete expansions (4.1) are not available in general. This is due to the fact that the rectified waves have no definite polarization and propagate in all directions. However, the leading term U_0 is well defined as a solution of equations which take the rectification effects into account. In this case, the errors are $o(\varepsilon)^p$.

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