Nonlinear Hyperbolic Smoothing at a Focal Point *

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Outline.

Abstract. The nonlinear dissipative wave equation $u_{tt} - \Delta u + |u_t|^{h-1} u_t = 0$ in dimension $d > 1$ has strong solutions with the following structure. In $0 \le t < 1$ the solutions have a focusing wave of singularity on the incoming light cone $|x| = 1 - t$. In $\{t \ge 1\}$ that is after the focusing time, they are smoother than they were in $\{0 \le t < 1\}$. The examples are radial and piecewise smooth in $\{0 \le t < 1\}$

§1. Introduction.

We construct real valued finite energy solutions of the dissipative nonlinear wave equation

$$\Box u + \left|u_t\right|^{h-1} u_t = 0\,, \qquad \Box := \partial_t^2 - \Delta_x\,, \qquad 1 < h \in \mathbb{R}, \tag{1.1}$$

which have singularities which are partially smoothed after a focus. Here $\{t, x\} \in \mathbb{R}^{1+d}$, with spatial dimension $d \ge 2$.

A striking classical result of Lions-Strauss [LS] shows that (1.1) is a well behaved evolution equation in $t \ge 0$ in all dimensions. There are two underlying estimates in establishing this result. The first is that solutions have nonincreaing energy. With

$$E(u,t) := \int_{\mathbb{R}^d} \frac{u_t^2}{2} + \frac{|\nabla_x u|^2}{2}\, dx\,, \tag{1.2}$$

one has

$$E(u,t) = E(u,0) - \int_0^T \int_{\mathbb{R}^d} \frac{\left|u_t\right|^{h+1}}{h+1}\, dx\, dt \le E(u,0)\,. \tag{1.3}$$

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* Research partially supported by the U.S. National Science Foundation, U.S. Office of Naval Research, and the NSF-CNRS cooperation program under grant numbers NSF-DMS-9203413, NSF-DMS-9803296, OD-G-N0014-92-J-1245, and NSF-INT-9314095 respectively, and the CNRS through the Groupe de Recherche G1180 POAN.

More generally one has a contractivity estimate which relies on the monotonicity of the nonlinear function

$$F_h(s) := \left|s^{h-1}\right| s\,.$$

Precisely,

$$E(u-v,t) = E(u-v,0) - \int_0^T \int_{\mathbb{R}^d} \big(u_t - v_t\big)\, \big(F_h(u_t) - F_h(v_t)\big)\, dx\, dt \le E(u-v,0)\,. \tag{1.4}$$

The energy dissipation identity is the case $v=0$ of the contractivity identity. These estimates lead to the following fundamental results of J.-L. Lions and W. Strauss.

Theorem 1.1. **[LS].** *If $\{f,g\} \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ then there is a unique soltion u to (1) with*

$$u \in C\big([0,\infty[\,;\, H^1(\mathbb{R}^d)\big)\,, \qquad u_t \in C\big([0,\infty[\,;\, L^2(\mathbb{R}^d)\big) \ \cap\ L^{h+1}([0,\infty[\times\mathbb{R}^d)$$

with Cauchy data

$$u\big|_{t=0} = f \qquad u_t\big|_{t=0} = g\,. \tag{1.5}$$

In addition, the energy laws (1.3) and (1.4) are satisfied by pairs of such solutions as well as the local versions in the truncated cones , $(\underline{x}, R, T) := \big\{|x-\underline{x}|| < R - t\,,\ 0 < t < T \le R\big\}$.

The energy law in , $(\underline{x}, R, R)$ shows that two solutions whose Cauchy data agree on $|x-\underline{x}| \le R$, must agree on cone $|x-\underline{x}| \le R-t$.

Regularity results follow from this by applying the contractivity estimate (1.4) to the solutions $u(t,x)$ and $v = u(t,x+\xi)$. The H^1 modulus of continuity is defined by

$$\omega(u,t,h)^2 := \sup_{0<|\xi|\le h} \|\nabla_x u(t,x) - \nabla_x u(t,x+\xi)\|^2_{L^2(\mathbb{R}^d)} + \|u_t(t,x) - u_t(t,x+\xi)\|^2_{L^2(\mathbb{R}^d)}\,. \tag{1.6}$$

Corollary 1.2. **[LS].** *If u is one of the solutions from Theorem 1, then the H^1 modulus of continuity $\omega(u,t,h)$ is a decreasing function of t. It follows that if $f,g \in H^{\sigma+1} \times H^\sigma$ with $\sigma \in]0,1]$ then*

$$u \in L^\infty\big([0,\infty[\,;\, H^{\sigma+1}(\mathbb{R}^d)\big)\,, \qquad \text{and} \qquad u_t \in L^\infty\big([0,\infty[\,;\, H^{\sigma}(\mathbb{R}^d)\big)\,.$$

For $\sigma \in]0,1[$ one has continuity in time, that is

$$u \in C\big([0,\infty[\,;\, H^{\sigma+1}(\mathbb{R}^d)\big)\,, \qquad \text{and} \qquad u_t \in C\big([0,\infty[\,;\, H^{\sigma}(\mathbb{R}^d)\big)\,.$$

This shows that H^s regularity for $1 \le s \le 2$ propagates forward in time.

The major interest of these results is that they define a strongly nonlinear evolution. By any measure known to man these problems are supercritical when d is large. These problems can not be attacked by using the basic estimates and then treating the nonlinear term as a perturbation writing $u = -\Box^{-1}\big(u_t^h\big)$. In particular for d large and $h \in \mathbb{Z}_{\rm odd}$ the nonlinearity is polynomial and it is not known whether the solutions with data in C_0^∞ are C^∞. Equivalently it is not known if such solutions are locally lipshitzean.

Our main result is the construction of compactly supported solutions which are smoother in $\{t \ge 1\}$ than they are in $\{0 \le t < 1\}$. This includes an explicit solution in closed form computed in §5.

The examples cannot be locally lipshitzean since the result of [GR] shows that if a solution has $\nabla_{t,x}u \in L^\infty_{loc}$, then its H^s_{loc} regularity does not change with time. In particular, in the one dimensional case, if the Cauchy data satisfies $\nabla_{t,x}u(0,x) \in L^\infty_{\rm loc}$, then the solution is lipshizean and therefore the H^s regularity is independent of $t \ge 0$. This lipshitz bound is proved by an argument needed later, so we recall the estimates. Introduce the characteristic combinations

$$u_\pm := \partial_\mp u := \big(\partial_t \mp \partial_x\big)u\,.$$

When $d = 1$, the differential equation (1.1) takes the characteristic form

$$\big(\partial_t \pm \partial_x\big)\, u_\pm + \frac{F_h(u_+ + u_-)}{2^h} = 0\,.$$

Multiplying by $p\, u_\pm^{p-1}$ with even integer p, adding and then integrating dx shows that

$$\partial_t \int_{\mathbb{R}} u_+^p + u_-^p\ dx = -\frac{p}{2^h}\int_{\mathbb{R}} (u_+^{p-1} + u_-^{p-1})\ F_h(u_+ + u_-)\ dx\,.$$

Since for abitrary real a, b one has $(a^{p-1} + b^{p-1})\, F_h(a+b) \ge 0$, it follows that $\int u_+^p + u_-^p\ dx$ is a nonincreasing function of t. Passing to the limit $p \to \infty$ shows that $\sup_{\mathbb{R}} \max\{|u_+|, |u_-|\}$ is a nonincreasing function of t. Thus, if $\nabla_{t,x}u(0,.)$ is initially L^∞ it remains so in $t \ge 0$.

Assumption 1.3. *Suppose that the initial data f, g are piecewise C^2, radial, compactly supported, vanish for $|x| \le 1$, and have singularities only on $|x| = 1$. In addition, f is assumed to be continuous and $g + \partial_r f$ is* **not** *continuous.*

When this assumption is satisfied, $\partial_r f$ and g are radial piecewise smooth and the locus of singularities is $r = 1$. Since $g + \partial_r f$ is not continuous at least one of g and $\partial_r f$ must jump at $r = 1$. This implies that

$$\{f, g\} \in H^\sigma(\mathbb{R}^d) \times H^{\sigma-1}(\mathbb{R}^d) \qquad \Longleftrightarrow \qquad \sigma < 3/2\,. \tag{1.7}$$

Assumption 1.4. *In addition to Assumption 1.3, suppose that $(\partial_t - \partial_r)u(0,r) = g - \partial_r f$ is continuous at $r = 1$.*

Since $g - \partial_r f$ does not jump it follows that the jumps of g and $\partial_r f$ are equal and nonzero. Assumption 1.4 insures that a jump discontinuity in ∇u propagates along the focussing cone $\{|x| = 1 - t\}$ and that the first derivatives are continuous across the outgoing cone

Main Theorem 1.5. *Assume that Assumtions 1.3 and 1.4 are satisfied and that u is the solution from Theorem 1.1. Then*

$$u\,,\, u_t\ \in\ L^\infty\big([0,1[\,;\, H^\sigma(\mathbb{R}^d) \times H^{\sigma-1}(\mathbb{R}^d)\,\big) \qquad \Longleftrightarrow \qquad \sigma < 3/2$$

and u is more regular for $t \ge 1$ in the following senses.

I. *If $d > 2h/(h-1)$, then*

$$u \in L^\infty\big(\,[1,\infty[\,;\, H^2(\mathbb{R}^d)\big) \quad \text{and} \quad u_t \in L^\infty\big(\,[1,\infty[\,;\, H^1(\mathbb{R}^d)\big)\,. \tag{1.8}$$

II. *If $2h/(h-1) - 1 < d \le 2h/(h-1)$, let $\alpha := \big(2h/(h-1)\ - d\big)/2 \in [0, 1/2[$ then for all $\epsilon > 0$*

$$u \in C\big(\,[1,\infty[\,;\, H^{2-\alpha-\epsilon}(\mathbb{R}^d)\big)\ \cap\ C^1\big(\,[1,\infty[\,;\, H^{1-\alpha-\epsilon}(\mathbb{R}^d)\big)\,, \tag{1.9}$$

Remarks. 1. For h fixed, the regularity of the solution for $t \geq 1$ increases linearly from $H^{3/2}$ to H^2 as the dimension increases from $d_1(h) := 2h/(h-1) - 1$ to $d_2(h) = 2h/(h-1)$. For dimensions higher than $d_2(h)$ the wave is H^2 in $t \geq 1$.

2. Theorem 1 of [GR] shows that in order for this smoothing to take place the solutions must not be lipshitzean. For any $\underline{t} \in]0,1[$, the solution is uniformly lipshitzean on $[0, \underline{t}] \times \mathbb{R}^d$, but that the sup norm of the derivatives diverges to infinity as $\underline{t} \to 1$.

3. What is happening is that an incoming spherical wave focusses at $t = 1, x = 0$. Approaching the focus, the amplitudes of u_t and u_t^h diverge to infinity. The nonlinear term acts in a dissipative manner. For $d > 2h/(h-1) - 1$ the nonlinearity is sufficiently large that the effect of the dissipation is so strong that the solution grows more slowly than it would have in the linear case. The idea of the proof in case I is to use the classical energy estimate for the second derivates of u in the domain outside the incoming light cone that is $\big\{\{t,x\} : |x| \geq 1-t\big\}$. The energy identity involves a boundary term on the incoming light cone $|x| = 1-t$. This term is finite for the nonlinear problem and would have been infinite for the linear problem. In this way one shows that the second derivative at time $t = 1$ are square integrable. In case II one shows that they are square integrable with weight r^α, $1 > \alpha \geq 0$. Then an inequality of Hardy type finishes the proof.

4. There are at least two other circumstances where supercritical damping for the same family of equations has been shown to have a regularizing effect on solutions. The first involves families of oscillatory solutions u^ϵ whose angular derivatives $\partial_\omega \nabla_{t,x} u^\epsilon$ are uniformly bounded in L^2 at the same time as $\nabla_{t,x} u^\epsilon$ is bounded in L^2 ([JMR2], [JMR3], [JMR4]). If the initial data is supported in $|x| < 1$, is not compact in energy, and has prinicipal oscillations which initially move toward the origin, in the sense that $(\partial_t - \partial_r)u^\epsilon$ is compact in L^2, then for $t > 1$ the family $\nabla_{t,x} u^\epsilon$ is compact in $L^2(\mathbb{R}^d)$. The noncompactness has been absorbed at the focus.

5. A similar phenomenon was described in [RR3] for families u^ϵ of unifromly disipative first order systems when $d = 1$ and the intial data are the regularizations $j_\epsilon * \mu$ of finite measures. The $L^1(\mathbb{R})$ norm of $u^\epsilon(t)$ decreases in time. It is proved that for $t > 0$ the solutions converge to the solution with initial data given by the nonsingular parts (in the sense of the Lebesgue decomposition) of the measures μ_j. The singular part is absorbed. In particular if the singular part is nonzero, $u^\epsilon(t,x)$ is compact in $L^1(\mathbb{R}_x)$ for $t > 0$ even though the initial data are not.

6. The explicit example of §5, shows that the result of the Main Theorem is sharp when $h = 2$ and $d = 4$.

2. Analysis of the singularities.

The most important step in the proof of the Main Theorem is to analyse the jump discontinuities in the derivatives of the solution for times $0 \leq t \leq 1$. The singularities come from the initial jump discontinuities on the sphere $|x| = 1$.

The finite speed of propagation implies that the solution u in the Main Theorem satisfies $u = 0$ in the truncated cone $|x| < 1 - t$. Uniqueness implies that u is radial. With the usual abuse of notation we write $u = u(t,r)$, and the differential equation in $\{r > 0\}$ becomes

$$u_{tt} - u_{rr} - \frac{d-1}{r}\, u_r + F_h(u_t) = 0\,. \tag{2.1}$$

This is a hyperbolic equation and the coefficient $(d-1)/r$ is smooth in $\{r > 0\}$. The solution we are looking at vanishes in $\{r < 1-t\}$ so is supported in the smooth coefficient region for $0 \leq t < 1$.

Lemma 2.1. i. Piecewise continuity for t < 1. *If Assumption 1.3 is satisfied, then for $0 \leq t < 1$, u is continuous and piecewise C^2 with jumps in the first derivatives restricted to the* *cones* $\{|x| = 1 \pm t\}$. **ii. Piecewise continuity up to t = 1 away from x = 0.** *For any* $\delta > 0$, *u is continous and piecewise C^2 in the regions* $\{r \geq \delta + t\,,\ 0 \leq t \leq 1\}$ *with jumps in the first derivatives restricted to the cones* $\{|x| = 1 \pm t\}$.

Proof of Lemma 2.1. i. Fix $0 < T < 1$. Finite speed shows that for $0 \leq t \leq T$, u is supported in $r \geq 1 - T > 0$ where the coefficient $(d-1)/2r$ in (2.1) is smooth.

The first step is to show that the solution is uniformly lipsitzean on $[0, T] \times \mathbb{R}^d$. Write (2.1) in the characteristic form

$$\big(\partial_t \pm \partial_r\big) u_\pm + \frac{d-1}{2r}(u_+ - u_-) + \frac{F_h(u_+ + u_-)}{2^h} = 0\,, \qquad u_\pm := \big(\partial_t \mp \partial_r\big) u\,. \tag{2.2}$$

The standard local existence theorem for hyperbolic equations in space dimension $d = 1$ shows that u is uniformly lipshitzean on $[0, T_1] \times [1 - T, \infty[$ with T_1 small positive. The same result show that in order to prove that u is lipshitzean up to time T it suffices to prove an *a priori* estimate for $\|\nabla_{t,r} u(t)\|_{L^\infty(\mathbb{R})}$. Precisely, it suffices to show that there is an $M < \infty$ depending only on f, g, so that if $0 \leq t \leq T_2 \leq T$ and u is a lipshitzean solution on $[0, T_2] \times [1 - T, \infty[$, then $\|\nabla_{t,r} u\|_{L^\infty([0,T_2]\times[1-T,\infty[)} \leq M$.

Multiply (2.2) by $p u_\pm^{p-1}$ with even integer p and add the resulting identities to find that

$$\partial_t \big(u_+^p + u_-^p\big) + \partial_r \big(u_+^p - u_-^p\big) + \frac{p(d-1)}{2r}\,\big(u_- - u_+\big)\,\big(u_+^{p-1} + u_-^{p-1}\big) = -\frac{p}{2^h}\big(u_+^{p-1} + u_-^{p-1}\big)\, F_h(u_+ + u_-) \leq 0\,. \tag{2.3}$$

Define

$$\psi(t, p) := \int_{1-T}^{\infty} u_+^p + u_-^p \ dr\,. \tag{2.4}$$

Integrate (2.3) over $[0, t] \times [1 - T, \infty]$. Integrating by parts and using the fact that $r \geq 1 - T > 0$ so no boundary terms arise yields

$$\begin{aligned} &\psi(t) - \psi(0) \\ &\qquad \leq \int_0^t \int_{1-T}^{\infty} \frac{p(1-d)}{2r}\,\big(u_+ - u_-\big)\,\big(u_+^{p-1} + u_-^{p-1}\big)\ dr\,dt \leq c\,p \int_0^t \psi(t)\,dt\,, \end{aligned} \tag{2.5}$$

where

$$c = c(d, T) := \frac{d-1}{1-T} \max_{p \in \mathbb{N}_{even}} \max_{\{u_+, u_-\} \in \mathbb{R}^2 \setminus 0} \frac{(u_- - u_+)(u_+^{p-1} + u_-^{p-1})}{u_+^p + u_-^p} < \infty\,. \tag{2.6}$$

Therefore

$$\psi(t, p) \leq \psi(0, p) + c(d, T)\, p \int_0^t \psi(t, p)\, dt \tag{2.7}$$

and Gronwall's inequality yields

$$\psi(t, p) \leq \psi(0, p)\, e^{c\,p\,t} \tag{2.8}$$

with c independent of p. Taking the p^{th} root gives bound on the L^p norm of $u_\pm$ independent of $t \leq T_2$ and p. Passing to the limit $p \to \infty$ bounds $\|u_\pm(t)\|_{L^\infty([0,T_2]\times[0,\infty[)} \leq M(f, g)$. This estimate completes the proof that u is a uniformly lipshitzean solution of (1.1) on $[0, T]$ supported in $\{r \geq 1 - t\}$.

When F_h is a amooth function that is when $h \in \mathbb{Z}_{odd}$, Theorem 1 of [RR1] applied to the first order system (2.2) implies that u is piecewise smooth with singularities restricted to the two cones.

If one is only interested in showing that Lipshitz continuous solutions are piecewise C^2 then the argument of [RR2] requires only that the nonlinear function F be C^1 which is the case in our problem. More generally if $F \in C^k$ then the argument of [RR] can be carried out to study discontinuities in derivatives of order $k+1$. The details for completeing this part of the proof of part **i.** of the Lemma are left to the reader.

The proof of **ii.** is similar. It suffices to prove an *apriori* estimate

$$\|\nabla_{t,r} u\|_{L^\infty([0,T_2]\times\{r\geq\delta+t\})} \leq M(f,g)$$

with M independent of $T_2 \leq 1$. Introduce

$$\Psi(t,p) := \int_{\delta+t}^{\infty} u_+^p + u_-^p \ dr\,.$$

Integrating (2.3) over the region $\big\{r \geq \delta + t\big\} \cap \big\{0 \leq t \leq \underline{t}\big\}$ yields

$$\Psi(\underline{t},p) - \Psi(0,p) + 2\int_0^{\underline{t}} u_-^p\big|_{r=\delta+t}\, dt$$
$$\leq \int_0^{\underline{t}}\int_{\delta+t}^{\infty} \frac{p(1-d)}{2r}\,\big(u_+ - u_-\big)\,\big(u_+^{p-1} + u_-^{p-1}\big)\ dr\, dt \leq c(d,\delta)\,p\int_0^{\underline{t}} \Psi(t,p)\, dt\,.$$

There is now a boundary term on $r = \delta + t$ which is nonnegative, so improves the estimate. As before this yields an estimate

$$\Psi(t,p) \leq \Psi(0,p)\, e^{c(d,\delta)\,p\,t} \tag{2.9}$$

with c independent of p. Taking the the $p^{\rm th}$ root and then the limit $p \to \infty$ yields the desired Lipshitz estimate. ∎

The next three lemmas prepare for the application of an energy estimate. That estimate is applied to $w := \partial u$ and one needs to control the growth of the boundary values of $(\partial_t - \partial_r)\partial u$ on the shrinking sphere $r = 1 - t$ as t increases to 1. To do this we take advantage of the piecewise smoothness.

The main estimate (2.10) of the next lemma is very important. If the problem had been linear, one would have found that the energy density $r^{(d-1)}(u_-^2 + u_+^2)$ was constant on the incoming characteristic. If $(d-1)(h-1) > 2$ then, the energy density tends to zero as $r \to 0$ which is a result of the nonlinear dissipative mechanism. For linear dissipation, the energy density would converge to a strictly positive quantity.

In comparing the condition of this lemma with those of the Main Theorem it is useful to keep in mind the relation

$$\frac{2h}{h-1} - 1 = \frac{h+1}{h-1}\,.$$

Lemma 2.2. Analysis of the incoming jump. *On the incoming characteristic $r = 1 - t$ one has $u_+ = 0$. If $d > (h+1)/(h-1)$, then as t increases to 1, one has*

$$u_-(t, 1-t) = \frac{c}{r^{1/(h-1)}}\Big(1 + o(1)\Big)\,, \tag{2.10}$$

where the values of u_- are the limits from above, that is from $t > 1-r$ and the constant $c = c(d,h)$ is given in (2.13). In addition the tangential derivative satisfies

$$\big(\partial_t - \partial_r\big)\, u_-(t,1-t) = \frac{C}{r^{h/(h-1)}}\Big(1 + o(1)\Big)\,, \qquad C = C(d,h) = \frac{c(d,h)\,(d-1)}{2} - \frac{c(d,h)^h}{2^h}\,. \tag{2.11}$$

Proof of Lemma 2.2. By finite speed of popagation, both u_+ and u_- vanish in $r < 1-t$. Also by the plus equation in (2.2), u_+ is continuous across $r = 1-t$ which proves that u_+ vanishes on the incoming characteristic.

Next estimate the boundary values b of u_-,

$$b(t, 1-t) := \lim_{\delta \to 0+} u_-(t, 1-t+\delta)\,.$$

Note that b is defined only on the characteristic line $\{r = 1-t\}$.

Since $u_- = 0$ below the characteristic, the jump in u_- from under to over the characteristic is equal to b.

Since $u_- = 0$ below $r = 1-t$ and u_+ vanishes on both sides, the minus equation of (2.2) reads

$$\big(\partial_t - \partial_r\big) b - \frac{d-1}{2\,r}\, b + F_h(b) = 0\,. \tag{2.12}$$

Let $\gamma := r^{(d-1)/2} b = (1-t)^{(d-1)/2} b(t, 1-t)$. Then

$$\big(\partial_t - \partial_r\big)\gamma = r^{(d-1)/2}\left(\big(\partial_t - \partial_r\big) b - \frac{d-1}{2\,r}\, b\right) = -r^{(d-1)/2}\,\frac{F_h(g)}{2^h} = \frac{-F_h(\gamma)}{2^h\, r^{(d-1)(h-1)/2}}\,.$$

Then since $(d-1)(h-1)/2 \neq 1$,

$$\big(\partial_t - \partial_r\big)\,\frac{|\gamma|^{-h+1}}{-h+1} = \frac{-1}{2^h\, r^{(d-1)(h-1)/2}} = \big(\partial_t - \partial_r\big)\,\frac{r^{1-(d-1)(h-1)/2}}{2^h\,\big(1 - (d-1)(h-1)/2\big)}\,.$$

Thus along $r = 1-t$ the quantity

$$\frac{1}{(h-1)|\gamma|^{h-1}} - \frac{1}{2^{h-1}\,\big((d-1)(h-1) - 2\big)\; r^{(d-1)(h-1)/2-1}}$$

is constant.

The hypothesis $(d-1)(h-1) > 2$ guarantees that the exponent of r in the second term is positive, so as as t increases to 1, the radius r shrinks to zero and the second term grows without bound. To compensate this the first term tends to infinity. Thus,

$$\gamma = c\, r^{\frac{d-1}{2} - \frac{1}{(h-1)}}\Big(1 + o(1)\Big)\,, \qquad c = c(d,h) := 2\left(\frac{(d-1)(h-1) - 2}{h-1}\right)^{1/(h-1)}\,. \tag{2.13}$$

In terms of the original variable b, estimate (2.13) is equivalent to (2.10).

To prove (2.11), insert the estimate (2.10) into the identity (2.12). ∎

The energy density along the characteristic is then

$$r^{d-1}\, |u_-|^2 = O\big(r^{d-1}\, r^{-2/(h-1)}\big) = O\big(r^{d-1-2/(h-1)}\big)$$

which is $o(1)$ precisely when the hypothesis of Lemma 2.2 is satisfied.

Lemma 2.3 *If d satisfies the condition $d > (h+1)/(h-1)$ from Lemma 2.2 and α satisfies $\alpha + d > 2h/(h-1)$, then for each $\partial \in \{\partial/\partial_t, \partial/\partial x_1, \dots \partial/\partial x_d\}$ the limits of the derivatives $(\partial_t - \partial_r)\partial u$ from above the incoming light cone satisfy*

$$\int_{\substack{|x|=1-t\\0<t<1}} r^\alpha \Big(\Big|(\partial_t - \partial_r)\partial u\Big|^2 + |\nabla_\omega \partial u|^2 \Big)\, d\sigma \ < \ \infty\,, \tag{2.14}$$

where $|\nabla_\omega w|$ is the length of the angular derivative given by $|\nabla_\omega w|^2 := |\nabla_x w|^2 - |\partial_r w|^2$ and $d\sigma$ the element of surface area.

Proof of Lemma 2.3. In $|x| \geq 1-t$, write

$$\frac{\partial u}{\partial x_j} = \frac{\partial u}{\partial r}\,\frac{\partial r}{\partial x_j} = \frac{x_j}{r}\,\frac{\partial u}{\partial r}\,.$$

Since $\partial_r u$ is spherically symmetric, its angular gradient vanishes so

$$\nabla_\omega\,\frac{\partial u}{\partial x_j} = \big(\nabla_\omega \frac{x_j}{r}\big)\,\frac{\partial u}{\partial r}\,.$$

Use that $2\partial_r = u_- - u_+$ and that on the incoming light cone the second summand vanishes to show that on $|x| = 1-t$

$$= \big(\nabla_\omega \frac{x_j}{r}\big)\,\frac{u_-}{2} = \big(\nabla_\omega \frac{x_j}{r}\big)\,\frac{u_- - u_+}{2} = O(r^{-1}\,r^{-1/(h-1)}) = O(r^{\frac{-h}{h-1}})\,.$$

Similarly, the product rule,

$$(\partial_t - \partial_r)\,\frac{\partial u}{\partial x_j} = \Big((\partial_t - \partial_r)\frac{x_j}{r}\Big)\,\frac{\partial u}{\partial r}\ +\ \frac{x_j}{r}\,(\partial_t - \partial_r)\frac{\partial u}{\partial r}\,. = \Big((\partial_t - \partial_r)\frac{x_j}{2r}\Big)\,u_- + \frac{x_j}{2r}\,(\partial_t - \partial_r)\,u_-\ .$$

Equations (2.10) and (2.11) show that each summand is $O(r^{-h/(h-1)})$.
Thus

$$\int_{\substack{|x|=1-t\\0<t<1}} r^\alpha \Big(\Big|(\partial_t - \partial_r)\,\frac{\partial u}{\partial x_j}\Big|^2\ + |\nabla_\omega \partial u|^2 \Big)\, d\sigma \leq C\,\int_0^1 r^\alpha\,\frac{1}{r^{2h/(h-1)}}\,r^{d-1}\,dr\,.$$

This integral is finite if and only if $\alpha - 2h/(h-1) + d - 1 > -1$ recovering the condition in the Lemma.

The remaining derivative $(\partial_t - \partial_r)\partial_t u = (\partial_t - \partial_r)\partial_r u$ on the incoming cone, so the square integrability follows from the previous estimates. ∎

Lemma 2.4. Analysis of the outgoing jump. *If Assumption 1.4 is satisfied then u is continuously differentiable on a neighborhood of the outgoing cone $\{|x| = 1+t\,,\ t>0\}$. In particular u is locally H^2 on the complement of the incoming light cone, $u \in H^2_{\rm loc}\big((]0,1[\times\mathbb{R}^d) \setminus \{|x| = 1-t\}\big)$.*

Proof of Lemma 2.4. From Lemma 2.1.**ii.**, u is continuous, and, on a neighborhood of $\{r = t\,,\ t > 0\}$, u is piecewise smooth with singularities on $r = t$. It is sufficient to prove that $u_\pm$ are continuous across $\{r = 1 + t\,,\ t > 0\}$.

For u_- this follows from equation $(2.2)_-$ and the facts that $u_\pm$ are locally bounded.

Next we show that the continuity of $u_+(0, r)$ at $r = 1$ from Assumption 1.4 implies the continuity of u_+ across the outgoing characteristic $r = 1 + t$.

The jump in u_+ defined as

$$[u_+](t, 1 + t) := u_+(t+, 1 + t) - u_+(t-, 1 + t)\,, \qquad u_+(t\pm, 1 + t) := \lim_{\delta\to 0} u(t \pm \delta, 1 + t)\,.$$

The first step is to show that the $\lim_{t\to 0}[u_+](t, 1 + t) = 0$.

Equation $(2.2)_+$ shows that $(\partial_t + \partial_r)u_+$ is locally bounded. Thus with $0 < \delta < t << 1$ integrating this equation shows that

$$u_+(t + \delta, 1 + t) - u(0, 1 - \delta) = O(t)\,, \qquad u_+(t - \delta, 1 + t) - u(0, 1 + \delta) = O(t)\,.$$

Letting $\delta \to 0$ and subtracting shows that

$$[u_+](t, 1 + t) = O(t) + \lim_{\delta\to 0} \big(u_+(0, 1 - \delta) - u_+(0, 1 + \delta)\big)\,.$$

The limit on the right is equal to zero thanks to Assumption 1.4 and therefore

$$\lim_{t\to 0+}\ [u_+(t, t)] \ = \ 0\,. \tag{2.15}$$

Define a smooth function $k(t)$ by

$$k(t) := u_+(t - 0, t) + u_-(t, t) = 2\,u_t(t - 0, t)\,.$$

The transport equation satisfied by the jump $[u_+]$ along the outgoing characteristic is derived by taking the difference between the equation $(2.2)_+$ on the upper and lower sides of the characteristic to find

$$(\partial_t + \partial_r)[u_+] + \frac{d-1}{r}\,[u_+] + \frac{F_h\big([u_+] + k(t)\big) - F_h\big(k(t)\big)}{2^h} = 0\,.$$

Define a C^1 function

$$G_h(t, s) := \frac{d-1}{r}\,s + \frac{F_h\big(s + k(t)\big) - F_h\big(k(t)^h\big)}{2^h} \qquad \text{with} \quad G_h(t, 0) = 0$$

to find the nonlinear transport equation

$$(\partial_t + \partial_r)[u_+] + G_h(t, [u_+]) = 0\,. \tag{2.16}$$

The initial value problem defined by (2.15) and (2.16) has the unique solution $[u_+] = 0$ which proves the desired continuous differentiability. ∎

§3. Proof of part I of the Main Theorem.

The next step in the proof is an energy estimate which begins with the energy identity

$$w_t\,\Box w = \partial_t\Big(\frac{w_t^2}{2}+\frac{|\nabla_x w|^2}{2}\Big) - \sum_{j=1}^{d}\partial_j\big(w_t\,\partial_j w\big) := \partial_t e(t,x) - \sum_{j=1}^{d}\partial_j\big(w_t\,\partial_j w\big)\,. \tag{3.1}$$

Lemma 3.1. Energy estimate. *For $0<T<1$ define $\Omega_T := \{\,(t,x)\,:\,1+t>|x|>1-t,\ \ 0<t<T\}$ and suppose that $w\in C^2(\overline{\Omega_T})$ and satisfies $w_t\Box w\le 0$ in Ω_T. Define $2\,e(w,t,x) := w_t^2+|\nabla_x w|^2$. Then*

$$\begin{aligned}\int_{|x|\ge 1-T} e(w,T,x)\,dx - \int_{|x|\ge 1} e(w,0,x)\,dx \ \le \\ \int_{\substack{|x|=1-t\\0<t<T}} \Big((w_t-w_r)^2+|\nabla_\omega w|^2\Big)\frac{d\sigma}{2\sqrt2} + \int_{\substack{|x|=1+t\\0<t<T}} \Big((w_t+w_r)^2+|\nabla_\omega w|^2\Big)\frac{d\sigma}{2\sqrt2}\,.\end{aligned} \tag{3.2}$$

Proof of Lemma 3.1. This identity follows from integrating (3.1) over Ω_T and then integrating by parts to find

$$\begin{aligned}\int_{|x|\ge 1-t} e(t,x)\ dx\,\Big|_{t=0}^{t=T} \le \\ \int_{\substack{|x|=1-t\\0<t<T}} \big(e(t,x)-w_t(t,x)\,w_r(t,x)\big)\frac{d\sigma}{\sqrt2} + \int_{\substack{|x|=1+t\\0<t<T}} \big(e(t,x)+w_t(t,x)\,w_r(t,x)\big)\frac{d\sigma}{\sqrt2}\,.\end{aligned}$$

Simplifying the boundary terms using the identities $2(e\mp w_tw_r) = (w_t\mp w_r)^2+|\nabla_\omega w|^2$ yields (3.2). ∎

End of proof of Main Theorem 1.5.I. For $\partial\in\{\partial_t,\partial/\partial x_1,\dots\partial/\partial x_d\}$ let $w:=\partial u$. Lemma 2.1.**i** proves that $w\in C^2(\overline{\Omega_T})$ for any $T<1$. Applying ∂ to equation (1.1) shows that $\Box w = -3u_t^2\,w_t$ so $w_t\Box w = -2(u_t\,w_t)^2\le 0$. Thus Lemma 3.1 can be applied to this w.

Next consider the terms on the right hand side of (3.2) in the limit $T\to1$. Part **ii.** of Lemma 2.1. implies that the second term is bounded independent of T. Similary, Lemma 2.3 with $\alpha=0$ shows that the integral of the first summand in the first integral on the right of (3.2) is bounded independent of T. This is where the hypothesis $d>2h/(h-1)$ is used.

Taking the limit $T\to1$ in (3.2) implies that

$$\limsup_{T\to1}\int_{1+T>|x|>1-T} e(w,T,x)\,dx \ <\ \infty\,.$$

Inserting the definition $w=\partial u$, this reads

$$\limsup_{T\to1}\int_{1+T>|x|>1-T} \big(\partial_t\partial u(T,x)\big)^2+\big|\nabla_x\partial u(T,x)\big|^2\ dx<\infty$$

Lemma 2.1.**ii** together with the continuous differentiability from Lemma 2.4 imply that

$$\limsup_{T\to1}\int_{|x|>\delta+T} \big(\partial_t\partial u(T,x)\big)^2+\big|\nabla_x\partial u(T,x)\big|^2\ dx<\infty\,.$$

Combining the the last two estimates shows that

$$\limsup_{T\to 1} \int_{|x|>1-T} \big(\partial_t \partial u(T,x)\big)^2 + \big|\nabla_x \partial u(T,x)\big|^2 \; dx < \infty\,. \tag{3.3}$$

Corollary 1.2 with (1.7) imply that

$$\partial u(1,x) \in C\big([0,\infty[\,:\ H^{1/2-\epsilon}(\mathbb{R}^d)\big)\,, \qquad \text{and} \qquad \partial_t \partial u(1,x) \in C\big([0,\infty[\,:\ H^{-1/2-\epsilon}(\mathbb{R}^d)\big)\,. \tag{3.4}$$

Estimate (3.3) together with the continuity (3.4) implies that the restriction of $\partial_{t,x}\partial u(1,x)$ to $\{\mathbb{R}^d \setminus 0\}$ is a square integrable function, that is

$$\int_{\mathbb{R}^d\setminus 0} \big(\partial_t \partial u(1,x)\big)^2 + \big|\nabla_x \partial u(1,x)\big|^2 \; dx < \infty\,. \tag{3.5}$$

Define $G(x)$ to be the square integrable function which is the restriction of $\partial_{t,x}\partial u(1,x)$ to $\mathbb{R}^d \setminus 0$, and let

$$R(x) := G(x) - \partial_{t,x}\partial u(1,x)\,, \qquad \text{so} \qquad \operatorname{supp} R \subset \{0\}\,. \tag{3.6}$$

The regularity (3.4) implies that

$$R \in H^{-1/2-\epsilon}(\mathbb{R}^d)\,. \tag{3.7}$$

Since there are no nonzero elements of this space with support at the orgin it follows that $R = 0$ and therefore that $\partial_{t,x}\partial u(1,x) \in L^2(\mathbb{R}^d)$. Corollary 1.2 with $\sigma = 1$ implies that (1.8) is satisfied so the proof of Main Theorem 1.5.I is complete. ∎

§4. Proof of part II of the Main Theorem.

The difference in the analysis comes from the square integrability near the focus at $t = 1, r = 0$. For the second part of the Main Theorem one needs the weights r^α from Lemma 2.3. To take advantage of the weighted estimates from Lemma 2.3 we use the following weighted energy estimate which reduces to Lemma 3.1 when $\alpha = 0$.

Lemma 4.1. Weighted energy estimate. *For $0 < T < 1$ define $\Omega_T := \{\, (t,x)\ :\ 1+t > |x| > 1-t,\ \ 0 < t < T\}$ and suppose that $w \in C^2\big(\overline{\Omega_T}\big)$, vanishes for $|x| \ge R$ and satisfies $w_t \Box w \le 0$ in Ω_T. Define $2\,e(t,x) := w_t^2 + |\nabla_x w|^2$. Then for all $\alpha \ge 0$,*

$$\begin{aligned}\int_{|x|\ge 1-T} (|x|+1-T)^\alpha\, e(T,x)\, dx - \int_{|x|\ge 1} (|x|+1)^\alpha\, e(0,x)\, dx \le \\ \int_{|x|=1-t} (|x|+1-t)^\alpha \Big((w_t - w_r)^2 + |\nabla_\omega w|^2\Big) \frac{d\sigma}{2\sqrt{2}} \\ + \int_{|x|=1+t} (|x|+1-t)^\alpha \Big((w_t + w_r)^2 + |\nabla_\omega w|^2\Big) \frac{d\sigma}{2\sqrt{2}}\,.\end{aligned} \tag{4.1}$$

Proof. Multiplying (3.1) by a continuous function $\phi(t,x)$ with integrable first derivates yields

$$\begin{aligned}\partial_t\big(\phi\, e(t,x)\big) - \sum_{j=1}^d \partial_j\big(\phi\, w_t\, \partial_j w\big) &= \phi_t e - \sum_{j=1}^d (\partial_j \phi)\,\big(w_t\, \partial_j w\big) \\ &\le \phi_t\, e + |\nabla_x \phi|\,|w_t|\,|\nabla_x w| \le \big(\phi_t + |\nabla_x\phi|\big)\, e\,.\end{aligned}$$

If ϕ satisfies $\phi_t + |\nabla_x \phi| \le 0$, then an integration by parts in Ω_T yields

$$\begin{aligned}0 \ge \int_{|x|\ge 1-t} &\phi(t,x)\, e(t,x)\ dx\Big|_{t=0}^{t=T} \\ &- \int_{|x|=1-t} \phi(t,x)\,\big(e(t,x) - w_t(t,x)\, w_r(t,x)\big)\ \frac{d\sigma}{\sqrt{2}} \\ &- \int_{|x|=1+t} \phi(t,x)\,\big(e(t,x) + w_t(t,x)\, w_r(t,x)\big)\ \frac{d\sigma}{\sqrt{2}}\,.\end{aligned} \tag{4.2}$$

Taking $\phi(t,x) := (|x|+1-t)^\alpha \ge 0$ and using the identities $2(e \mp w_t w_r) = (w_t \mp w_r)^2 + |\nabla_\omega w|^2$ yields (4.1). ∎

Proof of Main Theorem 1.5.II. Using Lemma 2.1.**ii**, Lemma 2.4, estimates (2.14) and (4.1), and reasoning as in the proof of Theorem 1.5.I, yields the following weighted estimates on $\mathbb{R}^d \setminus 0$. Define $\alpha \in]0,1/2]$ by

$$\alpha \ := \ \frac{1}{2}\Big(\frac{2h}{h-1} - d\Big)\,. \tag{4.3}$$

Then,

$$\forall \epsilon > 0\,,\ \partial \in \Big\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x_j}\Big\}\,, \qquad \int_{\mathbb{R}^d\setminus 0} |x|^{2\alpha+2\epsilon}\,\Big(\big(\partial_t \partial u(1,x)\big)^2 + \big|\nabla_x \partial u(1,x)\big|^2\Big)\ dx\ <\ \infty\,. \tag{4.4}$$

An application of Hölder's inequality shows that the function defined for $x \ne 0$ by $\partial_x \partial u(1,x)$ is absolutely integrable on compact subsets of $\mathbb{R}^d$ so defines a distribution. As in the sentence before equation (3.6), this distribution is called $G(x)$. Define R as in (3.6). Then

$$R \in L^1_{\rm loc}(\mathbb{R}^d) + H^{-1/2-\epsilon}(\mathbb{R}^d)\,, \qquad \text{and} \qquad \operatorname{supp} R \subset \{0\}\,.$$

It follows that $R = 0$. Thus, G is equal to the $\partial_x\partial u(1,x)$ where the derivatives are taken in the sense of distributions.

Thus

$$\Phi := \partial u \quad \Rightarrow \quad |x|^{\alpha-\epsilon}\,\nabla_x \Phi \in L^2(\mathbb{R}^d)\,. \tag{4.5}$$

Lemma 4.2. Hardy inequality. *If $\beta \in]0, d/2[$ then there is a $c = c(d,\beta)$ so that for all $\Phi \in \mathcal{S}\,(\mathbb{R}^d)$,*

$$\big\|\, |D|^{1-\beta}\, \Phi \,\big\|_{L^2(\mathbb{R}^d)} \le c\, \big\|\, |x|^\beta\, \nabla_x \Phi\, \big\|_{L^2(\mathbb{R}^d)}\,. \tag{4.6}$$

Proof. Inequality (4.6) follows from the inequality

$$\big\| \, |D|^{-\beta} \, \psi \, \big\|_{L^2(\mathbb{R}^d)} \le c \big\| \, |x|^{\beta} \, \psi \, \big\|_{L^2(\mathbb{R}^d)} \,, \qquad 0 < \beta < \frac{d}{2} \tag{4.7}$$

applied to the first derivatives of Φ.

Inequality (4.7) in turn is a consequence of the boundedness on L^2 of the integral operator with kernel

$$\frac{1}{|x-y|^{d-\beta}} \, \frac{1}{|y|^{\beta}} \,, \qquad 0 < \beta < \frac{d}{2} \,. \tag{4.8}$$

A proof of this boundedness can be found in [SW]. This completes the proof of Lemma 4.2. ∎

Applying (4.6) to the regularizations $\Phi^{\epsilon} := j_{\epsilon} * \partial u$ and passing to the limit $\epsilon \to 0$ yields

$$|D|^{1-\alpha-\epsilon} \, \partial u(1, \, . \,) \in L^2(\mathbb{R}^d) \,. \tag{4.9}$$

An application of Corollary 1.2 completes the proof of the Main Theorem. ∎

§5. An explicit example.

In this section we compute an explicit example exhibiting smoothing of a singularity. The example is self similar so the partial differential equation in t, r becomes a nonlinear equation with singularities of Fuchs type. When $h = 2$ this equation is explicitly solvable.

If v is a solution of (1.1) and $\lambda > 0$, then

$$u_{\lambda} = u_{\lambda}(t, x) := \lambda^{\alpha} \, v(\lambda t, \lambda x) \tag{5.1}$$

is also a solution provided that α and h satisfy the equivalent conditions

$$\alpha = \frac{2-h}{h-1} \,, \qquad h = \frac{\alpha+2}{\alpha+1} \,. \tag{5.2}$$

For the case of quadratic nonlinearity

$$h = 2 \,, \qquad \text{and} \qquad \alpha = 0 \,, \tag{5.3}$$

seek radial self similar solutions, that is solutions satisfying

$$u(t, r) := u(\lambda t, \lambda r) \,. \tag{5.1}$$

Setting $\lambda = 1/r$ shows that

$$u(t, r) = u(t/r, 1) := U(t/r) \,. \tag{5.4}$$

Then

$$u_t = \frac{1}{r} \, U'\Big(\frac{t}{r}\Big) \,, \qquad u_{tt} = \frac{1}{r^2} \, U''\Big(\frac{t}{r}\Big) \,, \qquad u_r = \frac{-t}{r^2} \, U'\Big(\frac{t}{r}\Big) \,, \qquad u_{rr} = \frac{t^2}{r^4} \, U''\Big(\frac{t}{r}\Big) + \frac{2t}{r^3} \, U'\Big(\frac{t}{r}\Big) \,.$$

Therefore equation (2.1) reads

$$0 = \Box u + F_h(u_t) = \Big[\frac{1}{r^2} - \frac{t^2}{r^4}\Big] \, U'' + \Big[\frac{(d-1)t}{r^3} - \frac{2t}{r^3}\Big] \, U' + \frac{1}{r^2} \, U'|U'| \,. \tag{5.5}$$

Multiply by r^2 and set

$$s := t/r\,, \qquad V := U' \tag{5.6}$$

to find

$$(1 - s^2)\, V' + (d-3)s\, V + V|V| = 0\,. \tag{5.7}$$

Consider solutions with

$$U = V = 0 \qquad \text{for} \qquad -\infty < s < -1\,,$$

which corresponds to solutions u which vanish on the incoming cone $\{t < -r\}$.
For $-1 < s < 1$, change variable to

$$V := (1-s^2)^{(d-3)/2}\, W$$

to find that (5.7) is transformed to

$$(1-s^2)\, W' + (1-s^2)^{(d-3)/2}\, W|W| = 0\,. \tag{5.8}$$

Therefore, W never changes sign in $\{-1 < s < 1\}$ and $-W$ is a solution whenever W is a solution. Separating variables in (5.8) yields the positive solution

$$W(s) = \frac{1}{F(s)}\,, \qquad \text{where} \qquad F(s) := \int_{-1}^{s} (1-t^2)^{(d-5)/2}\, dt\,.$$

This integral is finite for $d > 3$ and approaching $s = -1$ from above one has

$$F(s) \;=\; \frac{2}{(d-3)(1-s^2)^{(d-3)/2}}\, \Big(1 + o(1)\Big)\,.$$

Thus, the right hand limit of $V(s)$ at $s = -1$ is given by

$$\lim_{s \searrow -1} V(s) = (d-3)/2\,.$$

Therefore $U' = V$ has a jump discontinuity at $s = -1$, so the first derivatives of the selfsimilar solution has a jump discontinuity on the incoming light cone.
When s increases to $+1$ from below one has

$$V(s) \sim c\,(1-s^2)^{(d-3)/2}\,.$$

Extend V to vanish for $s > 1$,

$$V := 0 \qquad \text{for} \qquad s \geq 1\,. \tag{5.8}$$

The resulting self similar solution is constant inside the outgoing light cone $\{r = t > 0\}$. In addition the first derivatives of u are continuous across this cone.
Near the outgoing cone, one has

$$\nabla_{t,r}\, u \;\sim\; (r-t)^{(d-3)/2}$$

so for all $\epsilon > 0$,

$$\nabla\, u \in H^{(1+(d-3)/2+1/2-\epsilon)}_{\text{loc}} \tag{5.9}$$

For $d = 4$ this example shows that the result of the Main Theorem is sharp. For $d > 4$, the regularity on the outgoing cone increases linearly with d as if the result of the first part of the Main Theorem were true for all $d > 2h/(h-1)$.

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