Existence of quasilinear relaxation shock profiles

Guy Métivier[*], Benjamin Texier [†], Kevin Zumbrun[‡]

July 9, 2009

Abstract

We establish existence with sharp rates of decay and distance from the Chapman–Enskog approximation of small-amplitude quasilinear relaxation shocks in the general case that the profile ODE may become degenerate. Our method of analysis follows the general approach used by Métivier and Zumbrun in the semilinear case, based on Chapman–Enskog expansion and the macro–micro decomposition of Liu and Yu. In the quasilinear case, however, we find it necessary to apply a parameter-dependent Nash-Moser iteration to close the analysis, whereas, in the semilinear case, a simple contraction-mapping argument sufficed.

Contents

[*]MAB Université de Bordeaux I, 33405 Talence Cedex , France; metivier@math.u-bordeaux.fr., partially supported by European network HYKE, HPRN-CT-2002-00282.
[†]TODO
[‡]Indiana University, Bloomington, IN 47405; kzumbrun@indiana.edu: K.Z. thanks the University of Bordeaux I for its hospitality during the visit in which this work was carried out. Research of K.Z. was partially supported under NSF grants number DMS-0070765 and DMS-0300487.

1 Introduction

We consider the problem of existence of relaxation profiles

$$U(x,t) = \bar{U}(x-st), \quad \lim_{z\to\pm\infty} \bar{U}(z) = U_\pm \tag{1.1}$$

of a general relaxation system

$$U_t + A(U)U_x = Q(U), \tag{1.2}$$

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ q \end{pmatrix}, \tag{1.3}$$

in one spatial dimension, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^r$, where, for some smooth v_* and f,

$$q(u, v_*(u)) \equiv 0, \quad \Re\sigma(\partial_v q(u, v_*(u))) \le -\theta,\ \theta > 0, \tag{1.4}$$

$\sigma(\cdot)$ denoting spectrum, and

$$\begin{pmatrix} A_{11} & A_{12} \end{pmatrix} = \begin{pmatrix} \partial_u f & \partial_v f \end{pmatrix}. \tag{1.5}$$

Here, we are thinking particularly of the case n bounded and $r \gg 1$ arising through discretization or moment closure approximation of the Boltzmann equation or other kinetic models; that is, we seek estimates and proof independent of the dimension of v.

For fixed n, r, the existence problem has been treated in [YZ, MaZ1] under the additional assumption

$$\det(A - sI) \neq 0 \tag{1.6}$$

corresponding to nondegeneracy of the traveling-wave ODE. However, as pointed out in [MaZ2, MaZ3], this assumption is unrealistic for large models, and in particular is not satisfied for the Boltzmann equations, for which the eigenvalues of A are constant particle speeds of all values, hence cannot be uniformly satisfied for discrete velocity or moment closure approximations. Our goal here, therefore, is to revisit the existence problem without the assumption (1.6).

The latter problem was treated in [MZ2] for the semilinear case, which includes discrete velocity approximations of Boltzmann's equations, and for Boltzmann's equation (semilinear but infinite-dimensional) in [MZ3]. We mention also the proof, by similar methods, of positivity of Boltzmann shock profiles in [LY] and the original proof, by different methods, of existence of Boltzmann profiles in [CN]. The new application here is to moment closure approximations of Boltzmann's and other kinetic equations, which are in general quasilinear.

Our main result is to show existence with sharp rates of decay and distance from the Chapman–Enskog approximation of small-amplitude quasilinear relaxation shocks in the general case that the profile ODE may become degenerate. See Sections 2 and 3 for model assumptions and description of the Chapman–Enskog approximation, and Section 4 for a statement of the main theorem. Our method of analysis, as in [MZ2, MZ3] is based on Chapman–Enskog expansion and the macro-micro decomposition of [LY]. The main difference in this analysis from those of the previous works is that, due to a subtle loss of derivatives, *in the quasilinear case, we find it necessary to apply Nash-Moser iteration to close the analysis*, whereas in the semilinear case a simple contraction-mapping argument sufficed. Indeed, we require a nonstandard, parameter-dependent, Nash–Moser iteration scheme, indexed by amplitude $\varepsilon \to 0$, for which the linear solution operator loses not only derivatives but powers of ε. In this, we make convenient use of a general scheme developed in [TZ] for the treatment of such problems, which also arise in certain weakly nonlinear optics problems involving oscillatory solutions with large amplitudes or times of existence.

We note that spectral stability has been shown for general small-amplitude quasilinear relaxation profiles in [MaZ3], without the assumption (1.6), under the assumption that the profile exist and satisfy exponential bounds like those of the viscous case. The results obtained here verify that assumption, completing the analysis of [MaZ3]. Existence results in the absence of condition (1.6) have been obtained in special cases in [MaZ4, DY] by quite different methods (for example, center-manifold expansion near an assumed single degenerate point [DY]). However, the decay bounds as stated, though exponential, are not sufficiently sharp with respect to ε for the needs of [MaZ3].

2 Model, assumptions, and the reduced system

Taking without loss of generality $s = 0$, we study the traveling-wave ODE

$$A(U)U' = Q(U), \tag{2.1}$$

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} \partial_u f(u,v) & \partial_v f(u,v) \\ A_{21}(u,v) & A_{22}(u,v) \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ q(u,v) \end{pmatrix} \tag{2.2}$$

governing solutions (1.1), where

$$q(u, v_*(u)) \equiv 0, \quad \Re\sigma(\partial_v q(u, v_*(u))) \leq -\theta, \ \theta > 0. \tag{2.3}$$

We make the standard assumption of *symmetric–dissipativity* [Y]:

Assumption 2.1. *(SD) There exists a smooth, symmetric and uniformly positive definite matrix $S(U)$ such that*

i) for all U, $S(U)A$ is symmetric,

ii) for all equilibria $U_* = (u, v_*(u))$, $\Re S\, dQ(U_*)$ is nonpositive with

$$\dim\ker \Re S dQ = \dim\ker dQ \equiv n. \tag{2.4}$$

In (2.4) and below, $\Re M$ denotes symmetric part of the matrix M, i.e. $\frac{1}{2}(M + M^*)$.

By the change of coordinates $v \to v - v_*(u, v)$, we may take without loss of generality

$$v_*(u, v) \equiv 0, \quad dQ = \begin{pmatrix} 0 & 0 \\ 0 & \partial_v q \end{pmatrix} \tag{2.5}$$

without changing either the assumed structure (1.2), (2.1) or (since it is coordinate-independent) the property of symmetrizability. Note that symmetry of SdQ, together with (2.4), then implies both block-diagonal structure

$$S = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix} \tag{2.6}$$

and definiteness and proper rank of $\Re S_{22}\partial_v q$. Likewise, symmetry of SA together with (2.6) yields symmetry of $S_{11}A_{11}$ and $S_{22}A_{22}$ as well as

$$(S_{11}A_{12})^T = S_{22}A_{21}. \tag{2.7}$$

We make the simplifying assumption (2.5) throughout the paper.

We make also the Kawashima assumption of *genuine coupling* [K]:

Assumption 2.2. *(GC) For all equilibria $U_* = (u, v_*(u))$, there exists no eigenvector of A in the kernel of $dQ(U_*)$. Equivalently, given Assumption 2.1 (see [K]), there exists in a neighborhood $\mathcal{N}$ of the equilibrium manifold a skew symmetric $K = K(U)$ such that*

$$\Re(KA - SdQ)(U) \geq \theta > 0 \tag{2.8}$$

for all $U \in \mathcal{N}$.

Recall [Y] that the reduced, Navier–Stokes type equations obtained by Chapman–Enskog expansions are

$$f_*(u)' = (b_*(u)u')', \tag{2.9}$$

where, under the simplifying assumption (2.5),

$$\begin{aligned} f_*(u) &:= f(u, 0), \\ b_*(u)u' &:= -A_{12}\partial_v q^{-1} A_{21}(u, 0). \end{aligned} \tag{2.10}$$

For the reduced system (2.9), symmetric–dissipativity becomes:

(sd) There exists $s(u)$ symmetric positive definite such that $s\,df_*$ is symmetric and sb_* is symmetric positive semidefinite, with $\dim\ker\Re sb_* = \dim\ker b_*$.

We have likewise a notion of genuine coupling [K]:

(gc) There is no eigenvector of df_* in $\ker b_*$.

We note first the following important observation of [Y].

Proposition 2.3 ([Y])**.** *Let* (2.1) *as described above be a symmetric–dissipative system satisfying the genuine coupling condition (GC). Then, the reduced system* (2.9) *is a symmetric–dissipative system satisfying genuine coupling condition (gc).*

Proof. Assuming without loss of generality (2.5), we find that $s = S_{11}$ is a symmetrizer, since $sdf_* = S_{11}A_{11}$ is symmetric as already observed, and $sb_* = -S_{11}A_{12}(S_{22}\partial_v q)^{-1}S_{22}A_{21}$ is definite with proper rank by the corresponding properties of $S_{22}\partial_v q$ together with (2.7). Computing that (gc) is the condition that no eigenvector of A_{11} lie in $\ker A_{21}$, we see that (GC) and (gc) are equivalent. □

Besides the basic properties guaranteed by Lemma 2.3, we assume that the reduced system satisfy the following important additional conditions.

Assumption 2.4. (i) The matrix $b_*(u)$ has constant left kernel, with associated eigenprojector π_* onto $\ker b_*$, and (ii) The matrix $a_* := \pi_* df_* \pi_*(u)|_{\ker b_*}$ is uniformly invertible.

Assumption 2.4 ensures that the zero-speed profile problem for the reduced system,

$$f_*(u)' = (b_*(u)u')', \quad \lim_{z\to\pm\infty} u(z) = u_\pm \tag{2.11}$$

or, after integration from $-\infty$ to x,

$$b_*(u)u' = f_*(u) - f_*(u_\pm), \tag{2.12}$$

may be expressed as a nondegenerate ODE in u_2, coordinatizing $u = (u_1, u_2)$ with $u_1 = \pi_* u$ and $u_2 = (I - \pi_*)u$ [MaZ3, Z1, GMWZ]. Next, we assume that the classical theory of weak shocks can be applied to (2.11), assuming that the flux f_* has a genuinely nonlinear eigenvalue near 0:

Assumption 2.5. *In a neighborhood $\mathcal{U}_*$ of a given base state u_0, df_* has a simple eigenvalue α near zero, with $\alpha(u_0) = 0$, and such that the associated hyperbolic characteristic field is genuinely nonlinear, i.e., after a choice of orientation, $\nabla\alpha \cdot r(u_0) < 0$, where r denotes the eigendirection associated with α.*

Remark 2.6. Assumption 2.5 is standard, and is satisfied in particular for the compressible Navier–Stokes equations resulting from Chapman–Enskog approximation of the Boltzmann equation. Assumptions 2.1 and 2.2 are verified in [Y] for a wide variety of discrete kinetic models.[1] Assumptions 2.4 and 2.5 on the reduced equations must be checked in individual cases.

[1]For example, both discrete kinetic models [PI] used to approximate the Boltzmann equation [PI] and BGK models [JX, N] used to approximate general hyperbolic conservation laws; see pp. 289–294 [Y]. Note for each of these examples that the symmetrizer S is not constant, but depends nontrivially on U.

3 Chapman–Enskog approximation

Integrating the first equation of (2.1) and noting that $f(u,v)_\pm = f_*(u_\pm)$, we obtain

$$
\begin{aligned}
f(u,v) &= f_*(u\pm),\\
A_{21}(u,v)u' + A_{22}(u,v)v' &= q(u,v).
\end{aligned}
\tag{3.1}
$$

Taylor expanding the first equation, we obtain

$$f(u,0) + f_v(u,0)v + O(v^2) = f_*(u_\pm),$$

or

$$f_*(u) + f_v(u,0)v + O(v^2) = f_*(u_\pm). \tag{3.2}$$

Taylor expanding the second equation, we obtain

$$A_{21}(u,0)u' + O(|v||u'|) + O(|v'|) = \partial_v q(u,0)v + O(|v|^2),$$

or, inverting $\partial_v q$,

$$v = \partial_v q(u,0)^{-1} A_{21}(u,0)u' + O(|v|^2) + O(|v||u'|) + O(|v'|). \tag{3.3}$$

Substituting (3.3) into (3.2) and rearranging, we thus obtain the approximate viscous profile ODE

$$b_*(u)u' = f_*(u) - f_*(u_\pm) + O(v^2) + O(|v||u'|) + O(|v'|). \tag{3.4}$$

Motivated by (3.3)–(3.4), we define an approximate solution $(\bar u_{CE}, \bar v_{CE})$ of (3.1) by choosing $\bar u_{CE}$ as a solution of

$$b_*(\bar u_{CE})\bar u'_{CE} = f_*(\bar u_{CE}) - f_*(u_\pm), \tag{3.5}$$

and $\bar v_{CE}$ as the first approximation given by (3.3)

$$\bar v_{CE} = c_*(\bar u_{CE})\bar u'_{CE}. \tag{3.6}$$

3.0.1 Higher-order correctors

Further expanding the second equation as

$$A_{21}(u,0)u' + A_{22}(u,0)v' + O(|v||u'| + |v||v'|) = \partial_v q(u,0)v + O(|v|^2)$$

and setting $v = \bar v_{CE} + \tilde v$, $u = \bar u_{CE}$, we obtain

$$A_{22}(u,0)(\bar v_{CE})' + O(|v||u'| + |v||v'|) = \partial_v q(u,0)\tilde v + O(|v|^2)$$

or, inverting $\partial_v q$,

$$\tilde{v} = \partial_v q(u, 0)^{-1} A_{22}(u, 0) u' + O(|v|^2 + |v||u'| + |v||v'|). \tag{3.7}$$

Accordingly, we define

$$\bar{v}_{CE,2} = \bar{v}_{CE} + \partial_v q(\bar{u}_{CE}, 0)^{-1} A_{22}(u_{CE}, 0) u'_{CE} \tag{3.8}$$

as a second-order corrector for v. Substituting $\bar{v}_{CE,2}$ into the first equation and discarding the Taylor remainder as before, we obtain a second-order corrector $\bar{u}_{CE,2}$ for u. We can continue this process of Chapman–Enskog expansion to all orders to obtain an approximation

$$\bar{U}^N_{CE} := \bar{U}_{CE,1} + \bar{U}_{CE,2} + \ldots, \bar{U}_{CE,N} \tag{3.9}$$

to order N, where $\bar{U}_{CE,1} := \bar{U}_{CE}$ is the basic approximant at the first step.

3.0.2 Existence and decay bounds

Small amplitude shock profiles solutions of (3.5) are constructed using the center manifold analysis of [Pe] under conditions (i)-(ii) of Assumption 2.4; see discussion in [MaZ4].

Proposition 3.1. *Under Assumptions 2.5 and 2.4, in a neighborhood of (u_0, u_0) in $\mathbb{R}^n \times \mathbb{R}^n$, there is a smooth manifold $\mathcal{S}$ of dimension n passing through (u_0, u_0), such that for $(u_-, u_+) \in \mathcal{S}$ with amplitude $\varepsilon := |u_+ - u_-| > 0$ sufficiently small, and direction $(u_+ - u_-)/\varepsilon$ sufficiently close to $r(u_0)$, the zero speed shock profile equation (3.5) has a unique (up to translation) solution $\bar{u}_{CE}$ in $\mathcal{U}_*$. The shock profile is necessarily of* Lax type*: i.e., with dimensions of the unstable subspace of $df_*(u_-)$ and the stable subspace of $df_*(u_+)$ summing to one plus the dimension of u, that is $n+1$.*

Moreover, there is $\theta > 0$ and for all k there is C_k independent of (u_-, u_+) and ε, such that

$$|\partial_x^k(\bar{u}_{CE} - u_\pm)| \le C_k \varepsilon^{k+1} e^{-\theta\varepsilon|x|}, \quad x \gtrless 0. \tag{3.10}$$

and, more generally,

$$|\partial_x^k(\bar{u}_{CE,j}| \le C_k \varepsilon^{j+k+1} e^{-\theta\varepsilon|x|}, \quad x \gtrless 0. \tag{3.11}$$

We denote by $\mathcal{S}_+$ the set of $(u_-, u_+) \in \mathcal{S}$ with amplitude $\varepsilon := |u_+ - u_-| > 0$ sufficiently small and direction $(u_+ - u_-)/\varepsilon$ sufficiently close to $r(u_0)$ such that the profile $\bar{u}_{CE}$ exists. Given $(u_-, u_+) \in \mathcal{S}_+$ with associated profile $\bar{u}_{CE}$, we define $\bar{v}_{CE}$ by (3.6) and

$$\bar{U}^N_{CE} := (\bar{u}^N_{CE}, \bar{v}^N_{CE}). \tag{3.12}$$

It is an approximate solution of (3.1) in the following sense:

Corollary 3.2. *For fixed u_- and amplitude $\varepsilon := |u_+ - u_-|$ sufficiently small,*

$$\begin{aligned} \mathcal{R}_u^N &:= f(\bar u_{CE}^N, \bar v_{CE}^N) - f_*(u\pm), \\ \mathcal{R}_v^N &:= g(\bar u_{CE}^N, \bar v_{CE}^N)' - q(\bar u_{CE}^N, \bar v_{CE}^N) \end{aligned} \tag{3.13}$$

satisfy

$$\begin{aligned} |\partial_x^k \mathcal{R}_u^N(x)| &\le C_{k,N}\varepsilon^{N+k+4}e^{-\theta\varepsilon|x|}, \\ |\partial_x^k \mathcal{R}_v^N(x)| &\le C_{k,N}\varepsilon^{N+k+3}e^{-\theta\varepsilon|x|}, \quad x \gtrless 0, \end{aligned} \tag{3.14}$$

where $C_{k,N}$ is independent of (u_-, u_+) and $\varepsilon = |u_+ - u_-|$.

Proof. For $N = 0$, $k = 0$, bounds (3.14) follow by expansions (3.2) and (3.3), definitions (3.5) and (3.6), and bounds (3.10). Bounds for N, $k > 0$ follow similarly. □

4 Statement of the main theorem

We are now ready to state the main result. Define a base state $U_0 = (u_0, 0)$ and a neighborhood $\mathcal{U} = \mathcal{U}_* \times \mathcal{V}$.

Theorem 4.1. *Let Assumptions 2.1, 2.2, and 2.4 hold on the neighborhood $\mathcal{U}$ of U_0, with $f, A, Q \in C^\infty$. Then, there are $\varepsilon_0 > 0$ and $\delta > 0$ such that for $(u_-, u_+) \in \mathcal{S}+$ with amplitude $\varepsilon := |u_+ - u_-| \le \varepsilon_0$, the standing-wave equation* (2.1) *has a solution $\bar U$ in $\mathcal{U}$, with associated Lax-type equilibrium shock (u_-, u_+), satisfying for all k, N:*

$$\begin{aligned} \left|\partial_x^k(\bar U - \bar U_{CE}^N)\right| &\le C_{k,N}\varepsilon^{k+N+2}e^{-\delta\varepsilon|x|}, \\ |\partial_x^k(\bar u - u_\pm)| &\le C_k\varepsilon^{k+1}e^{-\delta\varepsilon|x|}, \quad x \gtrless 0, \\ \left|\partial_x^k(\bar v - v_*(\bar u)\right| &\le C_k\varepsilon^{k+2}e^{-\delta\varepsilon|x|}, \end{aligned} \tag{4.1}$$

where $\bar U_{CE} = (\bar u_{CE}, \bar v_{CE})$ is the approximating Chapman–Enskog profile defined in (3.12), *and C_k, $C_{k,N}$ are independent of ε. Moreover, up to translation, this solution is unique within a ball of radius $c\varepsilon$ about $\bar U_{CE}$ in norm $\varepsilon^{-1/2}\|\cdot\|_{L^2} + \varepsilon^{-3/2}\|\partial_x \cdot\|_{L^2} + \cdots + \varepsilon^{-11/2}\|\partial_x^5 \cdot\|_{L^2}$, for $c > 0$ sufficiently small and K sufficiently large. (For comparison, $\bar U_{CE} - U_\pm$ is order ε in this norm, by* (4.1)*(ii)–(iii).)*

Bounds (4.1) show that (i) the behavior of profiles is indeed well-described by the Navier–Stokes approximation, and (ii) profiles indeed satisfy the exponential decay rates required for the proof of spectral stability in [MaZ3]. From the second observation, we obtain immediately from the results of [MaZ3] the following stability result.

Corollary 4.2 ([MaZ3])**.** *Under the assumptions of Theorem 4.1, the resulting profiles $\bar U$ are spectrally stable for amplitude ε sufficiently small, in the sense that the linearized operator $L := \partial_x A(\bar U) - dQ(\bar U)$ about $\bar U$ has no L^2 eigenvalues λ with $\Re\lambda \ge 0$ and $\lambda \ne 0$.*

Proof. In [MaZ3], under the same structural conditions assumed here, it was shown that small-amplitude profiles of general quasilinear relaxation systems are spectrally stable, provided that $|\bar U'|_{L^\infty} \le C|U_+ - U_-|^2$, $|\bar U''(x)| \le C|U_+ - U_-|\,|\bar U'(x)|$, and

$$\Big|\frac{\bar U'}{|\bar U'|} + \mathrm{sgn}(\eta) R_0\Big| \le C\,|U_+ - U_-|, \qquad R_0 := \begin{pmatrix} r(u_0) \\ dv_*(U_0) r(u_0) \end{pmatrix}, \tag{4.2}$$

where $r(u_0)$ as defined in Theorem 4.1 is the eigenvector of df_* at base point U_0 in the principal direction of the shock. These conditions are readily verified using (4.1). □

The remainder of the paper is devoted to the proof of Theorem 4.1.

5 Outline of the proof

5.1 Linear and nonlinear perturbation equations

Defining the perturbation variable $U := \bar U - \bar U^N_{CE}$, where $\bar U^N_{CE}$ is as in (3.9), we obtain from (3.1) the nonlinear perturbation equations $\Phi^\varepsilon(U) = 0$, where

$$\Phi^\varepsilon(U) := \begin{pmatrix} f_1(\bar U^\varepsilon_{CE} + U) - f_*(U_-) \\ (A_{21}(\bar U^\varepsilon_{CE} + U)(\bar u^\varepsilon_{CE} + u)' + (A_{22}(\bar U^\varepsilon_{CE} + U)(\bar v^\varepsilon_{CE} + v)' - q(\bar U^\varepsilon_{CE} + U) \end{pmatrix}. \tag{5.1}$$

Formally linearizing Φ^ε about an approximate solution $\tilde U$, we obtain

$$(\Phi^\varepsilon)'(\tilde U) U = \begin{pmatrix} A_{11} u + A_{12} v \\ A_{21} u' + A_{22} v' - Q_{22} v - bU \end{pmatrix}, \tag{5.2}$$

where

$$A = df(\bar U^\varepsilon_{CE} + \tilde U), \quad Q_{22} = \partial_v q(\bar U^\varepsilon_{CE} + \tilde U), \tag{5.3}$$

and

$$bU = \big(d(A_{21}, A_{22})(\bar U^\varepsilon_{CE} + \tilde U) U\big)(\bar U^\varepsilon_{CE} + \tilde U)'. \tag{5.4}$$

The associated linearized equation for a given forcing term F is

$$(\Phi^\varepsilon)'(\tilde U) U = F = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{5.5}$$

We have also

$$(\Phi^\varepsilon)''(\tilde U)(U, \hat U) = \begin{pmatrix} N_1(\tilde U)(U, \hat U) \\ N_2(\tilde U)(U, \hat U)' + N_3(\tilde U)(U, \hat U) \end{pmatrix}, \tag{5.6}$$

where $N_j(\tilde U)$ are quadratic forms depending smoothly on $\tilde U$.

5.2 Functional analytic setting

The coefficients and the error term $\mathcal{R}$ are smooth functions of $\bar{U}_{CE'}$ and its derivative, so behave like smooth functions of εx. Thus, it is natural to solve the equations in spaces which reflect this scaling. We do not introduce explicitly the change of variables $\tilde{x} = \varepsilon x$, but introduce norms which correspond to the usual H^s norms in the $\tilde{x}$ variable :

$$\|f\|_{H^s_\varepsilon} = \varepsilon^{\frac{1}{2}}\|f\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|\partial_x f\|_{L^2} + \cdots + \varepsilon^{\frac{1}{2}-s}\|\partial_x^s f\|_{L^2}. \tag{5.7}$$

We also introduce weighted spaces and norms, which encounter for the exponential decay of the source and solution: introduce the notations.

$$< x > := (x^2+1)^{1/2} \tag{5.8}$$

For $\delta \geq 0$ (sufficiently small), we denote by $H^s_{\varepsilon,\delta}$ the space of functions f such that $e^{\delta\varepsilon<x>}f \in H^s$ equipped with the norm

$$\|f\|_{H^s_{\varepsilon,\delta}} = \varepsilon^{\frac{1}{2}}\sum_{k\leq s}\varepsilon^{-k}\|e^{\delta\varepsilon<x>}\partial_x^k f\|_{L^2}. \tag{5.9}$$

Note that for $\delta \leq 1$, this norm is equivalent, with constants independent of ε and δ, to the norm

$$\|e^{\delta\varepsilon<x>}f\|_{H^s_\varepsilon}.$$

For fixed δ, introduce spaces $E_s := H^s_{\varepsilon,\delta}$ with norm $\|\cdot\|_s = \|\cdot\|_{H^s_{\varepsilon,\delta}}$ and $F_s := \begin{pmatrix} H^{s+1}_{\varepsilon,\delta} \\ H^s_{\varepsilon,\delta}\end{pmatrix}$ with norm $|\begin{pmatrix} f \\ g\end{pmatrix}|_s = \|f\|_{H^{s+1}_{\varepsilon,\delta}} + \|g\|_{H^s_{\varepsilon,\delta}}$.

5.3 Nash Moser iteration scheme

Lemma 5.1. *$|\Phi(0)|_{H^s_{\delta,\varepsilon}} \leq C\varepsilon^{N+2}$ for all $0 \leq s \leq \bar{s}$, some $C > 0$.*

Proof. Immediate from (3.14) and (5.7). □

Lemma 5.2. *Φ^ε is Frechet differentiable from $H^{s+1}_{\varepsilon,\delta} \to H^s_{\varepsilon,\delta}$, for all $s \geq 0$, $\varepsilon > 0$, $\delta \geq 0$, and, for $s_0 \geq 1$, all s such that $s_0 + 1 \leq s + 1 \leq \bar{s}$, and all $U, V, W \in H^{s+1}_{\varepsilon,\delta}$,*

$$|\Phi^\epsilon(U)|_s \leq C_0(1 + |U|_{H^{s+1}_{\varepsilon,\delta}} + |U|_{H^{s_0+1}_{\varepsilon,\delta}}|U|_{H^s_{\varepsilon,\delta}}), \tag{5.10}$$

$$|(\Phi^\epsilon)'(U)\cdot V|_s \leq C_0(|V|_{H^{s+1}_{\varepsilon,\delta}} + |V|_{H^{s_0+1}_{\varepsilon,\delta}}|U|_{H^{s+1}_{\varepsilon,\delta}}), \tag{5.11}$$

and

$$\begin{aligned}|(\Phi^\epsilon)''(U)\cdot(V,W)|_s \leq C_0\big(&|V|_{H^{s_0+1}_{\varepsilon,\delta}}|W|_{H^{s+1}_{\varepsilon,\delta}} + |V|_{H^{s+1}_{\varepsilon,\delta}}|W|_{H^{s_0+1}_{\varepsilon,\delta}} \\ &+ |U|_{H^{s+1}_{\varepsilon,\delta}}|V|_{H^{s_0+1}_{\varepsilon,\delta}}|W|_{H^{s_0+1}_{\varepsilon,\delta}}\big),\end{aligned} \tag{5.12}$$

where C is uniformly bounded for $|U|_{H^{s_0+1}_{\varepsilon,\delta}} \leq C$, for any fixed value of δ.

Proof. Standard, using Moser's inequality, definition (5.2), the fact that $|\cdot|_{H^s_{\varepsilon,\delta}}$ is a fixed weighted norm in coordinates $\tilde{x} = \varepsilon x$, and working in $\tilde{x}$ coordinates, with $\partial_x = \varepsilon \partial_{\tilde{x}}$. $\square$

Proposition 5.3. *Under the assumptions of Theorem 4.1, there are $\varepsilon_0 > 0$ and $\delta > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$, for $\varepsilon \in]0, \varepsilon_0]$, $\delta \in [0, \delta_0]$, equation (5.5) has a solution operator $\Psi^\varepsilon(\tilde{U})$ (i.e., there exists a formal right inverse for $(\Phi^\varepsilon)'(\tilde{U})$), such that, for all s such that $s_0 + 2 \le s + 1 \le \bar{s}$, $s_0 = 3$, $F = \begin{pmatrix} f \\ g \end{pmatrix} \in F_s$, and $U \in H^{s+r}_{\varepsilon,\delta}$ such that*

$$|\tilde{U}|_{H^{s_0+2}_{\varepsilon,\delta}} \le C\epsilon, \tag{5.13}$$

there holds the estimate

$$\begin{aligned} \|\Psi^\varepsilon(\tilde{U})F\|_{H^s_{\varepsilon,\delta}} &\le C\varepsilon^{-1}\big(\|\tilde{U}\|_{H^{s+1}_{\varepsilon,\delta}}|F|_{s_0+2} + \|F\|_{s+1}\big) \\ &= C\varepsilon^{-1}\big(\|\tilde{U}\|_{H^{s+1}_{\varepsilon,\delta}}(\|f\|_{H^{s_0+3}_{\varepsilon,\delta}} + \|g\|_{H^{s_0+2}_{\varepsilon,\delta}}) + (\|F\|_{H^{s+2}_{\varepsilon,\delta}} + \|g\|_{H^{s+1}_{\varepsilon,\delta}})\big), \end{aligned} \tag{5.14}$$

where $C = C(|\tilde{U}|_{H^{s_0+2}_{\varepsilon,\delta}})$ is a non-decreasing function of $|\tilde{U}|_{H^{s_0+2}_{\varepsilon,\delta}}$.

The proof of this proposition, carried out in Sections 6–8 is essentially identical to that of the corresponding proposition (Prop. 5.2) of [MZ2] in the semilinear case. Once it is established, existence and uniqueness follow by the abstract Nash–Moser theorems developed in [TZ], reproduced for completeness in Appendix A.

Proof of Theorem 4.1 (Existence). The profiles $\bar{U}^N_{CE}$ exist if ε is small enough. Comparing, we find that Lemma 5.2, Proposition 5.3, and Lemma 5.1 verify, respectively, Assumptions A.1, A.2, and A.3 of Appendix A, with $s_0 = 3$, $\gamma_0 = 0$, $\gamma = 1$, $k = N+2$, $m = r = 1$, $r' = 0$, and arbitrary $\bar{s}$. Taking $\bar{s}$ sufficiently large, and applying the Nash Moser Theorem A.4 of Appendix A, we thus obtain existence of a solution U^ε of (5.1) with $|U^\varepsilon|_{H^{s+1}_{\varepsilon,\delta}} \le C\varepsilon^{N+1}$. Defining $\bar{U}^\varepsilon := \bar{U}^N_{CE} + U^\varepsilon$, and noting by Sobelev embedding that $|h|_{H^{s+1}_{\varepsilon,\delta}}$ controls $|e^{\delta\varepsilon|x|}h|_{L^\infty}$, we obtain the result. $\square$

Proof of Theorem 4.1 (Uniqueness). Applying Theorem A.5 for $s_0 = 3$, $\gamma_0 = 0$, $\gamma = 1$, $k = 3$, $m = r = 1$, $r' = 0$, we obtain uniqueness in a ball of radius $c\varepsilon$ in $H^4_{\varepsilon,0}$, $c > 0$ sufficiently small, under the additional phase condition (A.19). We obtain unconditional uniqueness from this weaker version by the observation that phase condition (A.19) may be achieved for any solution $\bar{U} = \bar{U}_{CE} + U$ with

$$\|U'\|_{L^\infty} \le c\varepsilon^2 << \bar{U}'_{CE}(0) \sim \varepsilon^2$$

by translation in x, yielding $\bar{U}_a(x) := \bar{U}(x+a) = \bar{U}_{CE}(x) + U_a(x)$ with

$$U_a(x) := \bar{U}_{CE}(x+a) - \bar{U}_{CE}(x) + U(x+a)$$

so that, defining $\phi := \bar U'/|\bar U'|$, we have $\partial_a\langle \phi, U_a\rangle \sim \langle \phi, \bar U'_{CE} + U'\rangle = \langle \phi, (1+o(1))\bar U' + U'\rangle = (1+o(1))|\bar U'| \sim \varepsilon^2$ and so (by the Implicit Function Theorem applied to $h(a) := \varepsilon^{-2}\langle \phi, U_a\rangle$, together with the fact that $\langle \phi, U_0\rangle = o(\varepsilon)$ and that $\langle \phi, \bar U'_{NS}\rangle \sim |\bar U'_{NS}| \sim \varepsilon^2$) the inner product $\langle \phi, U_a\rangle$, hence also ΠU_a may be set to zero by appropriate choice of $a = o(\varepsilon^{-1})$ leaving U_a in the same $o(\varepsilon)$ neighborhood, by the computation $U_a - U_0 \sim \partial_a U \cdot a \sim o(\varepsilon^{-1})\varepsilon^2$. □

It remains to prove existence of the linearized solution operator and the linearized bounds (5.14), which tasks will be the work of the rest of the paper. We concentrate first on estimates, and prove the existence next, using a viscosity method.

6 Internal and high frequency estimates

We begin by establishing a priori estimates on solutions of the equation (5.5) This will be done in two stages. In the first stage, carried out in this section, we establish energy estimates showing that "microscopic", or "internal", variables consisting of v and derivatives of (u, v) are controlled by and small with respect to the "macroscopic", or "fluid" variable, u. In the second stage, carried out in Section 7, we estimate the macroscopic variable u by Chapman–Enskog approximation combined with finite-dimensional ODE techniques such as have been used in the study of fluid-dynamical shocks [MZ1, MaZ5, PZ, Z1].

6.1 The basic H^1 estimate

We consider the equation

$$\begin{pmatrix} A_{11}u + A_{12}v \\ A_{21}u' + A_{22}v' + bU - Q_{22}v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{6.1}$$

and its differentiated form:

$$(AU' - Q + b)U = \begin{pmatrix} f' \\ g \end{pmatrix}, \tag{6.2}$$

where $b = \tilde b(\bar U^N_{CE})'$, and A, Q, $\tilde b$ are smooth functions of $\bar U_{CE} + \tilde U$, with $\|\tilde U\|_4$, $\|\bar U^N_{CE}\|_{s+1}$ both order ε (the first by assumption, the second by estimates (3.11)). We shall freely use below the resulting coefficient bounds

$$|\partial_x^{k+1}A|, |\partial_x^{k+1}Q|, |\partial_x^{k+1}K|, \le C\varepsilon^{2+k}, \quad |\partial_x^k b| \le C\varepsilon^{2+k} \tag{6.3}$$

for $0 \le k \le 3$ and

$$|\partial_x^{j+1}A|_{L^2}, |\partial_x^{j+1}Q|_{L^2}, |\partial_x^{j+1}K|_{L^2} \le C\varepsilon^{j+1/2}(\varepsilon + \|\tilde U\|_{s+1}), \quad |\partial_x^j b| \le C\varepsilon^{j+1/2}(\varepsilon + \|\tilde U\|_{s+1}) \tag{6.4}$$

for $0 \le j \le s$, where K is the Kawashima multiplier (a smooth function of A). The internal variables are $U' = (u', v')$ and v.

Proposition 6.1. *Under the assumptions of Theorem 4.1, there are constants C, $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$ and $0 \le \delta \le \delta_0$, $f \in H^2_{\varepsilon,\delta}$, $g \in H^1_{\varepsilon,\delta}$ and $U = (u, v) \in H^1_{\varepsilon,\delta}$ satisfying* (6.1)*, one has*

$$\|U'\|_{L^2_{\varepsilon,\delta}} + \|v\|_{L^2_{\varepsilon,\delta}} \le C\big(\|(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|u\|_{L^2_{\varepsilon,\delta}}\big). \tag{6.5}$$

Multiplying by symmetrizer S (block-diagonal, by assumption (2.5)), we obtain an ODE

$$\widetilde{A}\widetilde{U}' - \widetilde{Q}\,\widetilde{U} + \widetilde{C}\widetilde{U} = \widetilde{F}, \tag{6.6}$$

where

$$\widetilde{A} = SA, \quad \widetilde{Q} = SQ = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{Q}_{22} \end{pmatrix}, \tag{6.7}$$

with $\Re\widetilde{Q}_{22}$ negative definite, $\widetilde{F} = SF$, and

$$\widetilde{C} = O(\bar{u}'_{CE})\widehat{C} = O(\varepsilon^2)\widehat{C} \tag{6.8}$$

comprising commutator terms and $-S_{22}bU$.

We first prove the estimate (6.5) for $\delta = 0$. Dropping hats and tildes, the ODE reads

$$AU' - QU + \varepsilon^2 CU = F, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & Q_{22} \end{pmatrix}, \tag{6.9}$$

A symmetric and $\Re Q_{22}$ negative definite. Likewise, the genuine coupling condition still holds, which, by the results of [K], is equivalent to the *Kawashima condition*, and there is a smooth $K = \widetilde{K}(\bar{u}_{CE}) = -\widetilde{K}^*$ such that $\Re(KA - SQ)$ is definite positive. Therefore, there is $c > 0$ such that for all $\varepsilon \le \varepsilon_0$ and $x \in \mathbb{R}$:

$$\tilde{q} \le -c\mathrm{Id}, \qquad \Re(KA - SQ) \ge c\mathrm{Id}. \tag{6.10}$$

Lemma 6.2. *There is a constant C such that for ε sufficiently small, $f \in H^2$, $\tilde{U} \in H^2$, $g \in H^1$, and $U \in H^1$ satisfying* (6.9)*, with $\|\tilde{U}\|_2 \le C\varepsilon$, one has*

$$\|U'\|_{L^2} + \|v\|_{L^2} \le C\big(\|f\|_{H^2} + \|g\|_{H^1} + \varepsilon\|u\|_{L^2}\big). \tag{6.11}$$

Proof. Introduce the symmetrizer

$$\mathcal{S} = \partial_x^2 + \partial_x \circ K - \lambda. \tag{6.12}$$

One has

$$\begin{aligned}
&\Re\partial_x^2 \circ (A\partial_x - Q) = \frac{1}{2}\partial_x \circ A' \circ \partial_x - \partial_x \circ Q \circ \partial_x - \Re\partial_x \circ Q' \\
&\Re\partial_x \circ K(A\partial_x - Q) = \partial_x \circ \Re KA \circ \partial_x - \mathrm{Re}\,\partial_x \circ KQ \\
&\Re(A\partial_x - Q) = \frac{1}{2}A' - Q.
\end{aligned}$$

Thus

$$\begin{aligned}\Re\mathcal{S}\circ(A\partial_x - Q) =&\partial_x\circ(\Re AK - Q)\circ\partial_x + \lambda Q\\ &+\frac{1}{2}\partial_x\circ A'\circ\partial_x - \frac{1}{2}\lambda A' - \Re\partial_x\circ Q' - \Re\partial_x\circ KQ.\end{aligned}$$

Therefore, for $U\in H^2(\mathbb{R})$, (6.10) implies that

$$\begin{aligned}\Re(\mathcal{S}F, U)_{L^2} \geq\; & c\|\partial_x U\|^2_{L^2} + \lambda c\|v\|^2_{L^2}\\ &-\frac{1}{2}\|(A)'\|_{L^\infty}\big(\|\partial_x U\|^2_{L^2} + \lambda\|U\|^2_{L^2}\big)\\ &-\|(Q)'\|_{L^\infty}\|U\|_{L^2}\|\partial_x U\|_{L^2} - \|K\|_{L^\infty}\|\partial_x U\|_{L^2}\|qv\|_{L^2}\\ &-\varepsilon^2(|C|_{L^\infty}|U|^2_{H^1} + |C'|_{L^2}|U|^\infty_L).\end{aligned}$$

Taking

$$\lambda = \frac{2}{c}\|K\|^2_{L^\infty}\|q\|_{L^\infty},$$

and using that

$$\|(A)'\|_{L^\infty} + \|(Q)'\|_{L^\infty} = O(\varepsilon^2), \quad \|(C)'\|_{L^2} \sim \|(A)''\|_{L^2} \sim \varepsilon^{3/2}(\varepsilon + \|\tilde{U}\|_2) = O(\varepsilon^{5/2}) \tag{6.13}$$

and $|U|_{L^\infty} \leq \|U\|_1 = \varepsilon^{-1/2}\|U\|_{L^2} + \varepsilon^{1/2}\|U\|_{H^1}$, yields

$$\|U'\|^2_{L^2} + \|v\|^2_{L^2} \lesssim \Re(\mathcal{S}F, U)_{L^2} + \varepsilon^2\big(\|U\|^2_{L^2} + \|U'\|^2_{L^2}\big).$$

In the opposite direction,

$$\begin{aligned}\Re(\mathcal{S}F, U)_{L^2} \leq&\|\partial_x U\|_{L^2}\big(\|\partial_x(F)\|_{L^2} + \|K\|_{L^\infty}\|F\|_{L^2}\big)\\ &+\lambda\big(\|(u)'\|_{L^2}\|f\|_{L^2} + \|v\|_{L^2}\|g\|_{L^2}\big).\end{aligned}$$

Using again that the derivatives of the coefficients are $O(\varepsilon^2)$, this implies that

$$\begin{aligned}\Re(\mathcal{S}F, U)_{L^2} \lesssim&\big(\|f\|_{H^2} + \|g\|_{H^1}\big)\|U'\|_{L^2}\\ &+\varepsilon^2\|f\|_{L^2}\|u\|_{L^2} + \|g\|_{L^2}\|v\|_{L^2},\end{aligned}$$

The estimate (6.11) follows provided that ε is small enough.

This proves the lemma under the additional assumption that $U\in H^2$. When $U\in H^1$, the estimates follows using Friedrichs mollifiers. □

Proof of Proposition 6.1. This follows similarly as in the proof of Lemma 6.2, making the change of variables $U\to e^{\delta\varepsilon|x|}U$ and absorbing commutators. See the proof of Proposition 6.1, [MZ2]. □

6.2 Higher order estimates

Proposition 6.3. *There are constants C, $\varepsilon_0 > 0$, $\delta_0 > 0$ and for all $k \geq 2$, there is C_k, such that $0 < \varepsilon \leq \varepsilon_0$, $\delta \leq \delta_0$, $U \in H^s_{\varepsilon,\delta}$, $\tilde U \in H^{s+1}_{\varepsilon,\delta}$, $f \in H^{s+1}_{\varepsilon,\delta}$ and $g \in H^s_{\varepsilon,\delta}$ satisfying (6.9), with $\|\tilde U\|_{H^2_{\varepsilon,\delta}} \leq C\varepsilon$, there holds*

$$
\begin{aligned}
\|\partial_x^k U'\|_{L^2_{\varepsilon,\delta}} + \|\partial_x^k v\|_{L^2_{\varepsilon,\delta}} &\leq C\|\partial_x^k(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} \\
&\quad + \varepsilon^k C_k\big(\|U'\|_{H^{k-1}_{\varepsilon,\delta}} + \varepsilon\|v\|_{H^{k-1}_{\varepsilon,\delta}} + \varepsilon\|u\|_{L^2_{\varepsilon,\delta}}\big) \\
&\quad + C_k\varepsilon^{k+1}\|\tilde U\|_{H^{k+2}_{\varepsilon,\delta}}(\|v\|_{H^1_{\varepsilon,\delta}} + \varepsilon\|U\|_{H^2_{\varepsilon,\delta}}).
\end{aligned} \tag{6.14}
$$

Proof. Differentiating (6.1) k times, yields

$$
A\partial_x^k U - Q\partial_x^k U = \begin{pmatrix} \partial_x^k f' \\ \partial_x^k g + r_k \end{pmatrix}, \tag{6.15}
$$

where

$$
r_k = -\partial_x^{k-1}\big((\partial_x Q_{22})v\big) - \partial_x^{k-1}\big((\partial_x A)\,\partial_x U\big) - \partial_x^{k-1}\big((\partial_x C)\,U\big).
$$

The H^1 estimate yields

$$
\begin{aligned}
\|\partial_x^k U'\|_{L^2_{\varepsilon,\delta}} + \|\partial_x^k v\|_{L^2_{\varepsilon,\delta}} \leq C\big(&\|\partial_x^k(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} \\
&+\varepsilon\|\partial_x^k u\|_{L^2_{\varepsilon,\delta}} + \|\partial_x r_k\|_{L^2_{\varepsilon,\delta}} + \|r_k\|_{L^2_{\varepsilon,\delta}}\big),
\end{aligned}
$$

for $0 \leq k \leq s$, with $r_0 = 0$ when $k = 0$.

Using Moser's inequality together with (6.3) and (6.4), we may estimate

$$
\begin{aligned}
\|r_k\|_{L^2_{\varepsilon,\delta}} \leq C_k(&|\partial_x Q|_{L^\infty}\|\partial_x^{k-1} v\|_{L^2} + |\partial_x^k Q|_{L^2}\|v\|_{L^\infty} \\
&+ |\partial_x A|_{L^\infty}\|\partial_x^k U\|_{L^2} + |\partial_x^k A|_{L^2}\|\partial_x U\|_{L^\infty} \\
&+ |\partial_x C|_{L^\infty}\|\partial_x^{k-1} U\|_{L^2} + |\partial_x^k C|_{L^2}\|U\|_{L^\infty}) \\
\leq C_k(&\varepsilon^{k+1}\|v\|_{H^{k-1}_{\varepsilon,\delta}} + \varepsilon^{k+2}\|U\|_{H^{k-1}_{\varepsilon,\delta}} + \varepsilon^2\|\partial_x^k U\|_{L^2_{\varepsilon,\delta}}) \\
&+ C_k(\varepsilon^{k+1}\|\tilde U\|_{H^k_{\varepsilon,\delta}}\|v\|_{H^1_{\varepsilon,\delta}} + \varepsilon^{k+2}\|\tilde U\|_{H^k_{\varepsilon,\delta}}\|U\|_{H^2_{\varepsilon,\delta}} + \varepsilon^{k+2}\|\tilde U\|_{H^{k+1}_{\varepsilon,\delta}}\|U\|_{H^1_{\varepsilon,\delta}}),
\end{aligned}
$$

obtaining the result by absorbing (smaller) highest-order terms from $\|\partial_x r_k\|_{L^2_{\varepsilon,\delta}}$ on the left-hand side.

□

7 Linearized Chapman–Enskog estimate

7.1 The approximate equations

It remains only to estimate $\|u\|_{L^2_{\varepsilon,\delta}}$ in order to close the estimates and establish (6.5). To this end, we work with the first equation in (6.1) and estimate it by comparison with the Chapman-Enskog approximation (see the computations Section 3).

From the second equation

$$A_{21}u' + A_{22}v' - g = dq_v v,$$

we find

$$v = \partial_v q^{-1}\Big((A_{21} + A_{22}\partial_v dv_*(\bar u_{CE}))u' + A_{22}v' - g\Big). \tag{7.1}$$

Introducing v in the first equation, yields

$$(A_{11} + A_{12}dv_*(\bar u_{CE}))u + A_{12}v = f,$$

thus

$$(A_{11} + A_{12}dv_*(\bar u_{CE}))u' = f' - A_{12}v' - d^2v_*(\bar u_{CE})(\bar u'_{CE}, u).$$

Therefore, (7.1) can be modified to

$$v = c_*(\bar u_{CE})u' + r \tag{7.2}$$

with

$$\begin{aligned} r = d_v^{-1}q(\bar u_{CE}, &v_*(\bar u_{CE}))\Big(A_{22}(v)' - g \\ &+ dv_*(\bar u_{CE})\big(f' - A_{12}v' - d^2v_*(\bar u_{CE})(\bar u'_{CE}, u)\big)\Big). \end{aligned}$$

This implies that u satisfies the linearized profile equation

$$\bar b_* u' - \bar{df}_* u = A_{12}r - f \tag{7.3}$$

where $\bar b_* = b_*(\bar u_{CE})$ and $\bar{df}_* := df_*(\bar u_{CE}) = A_{11} + A_{12}dv_*(\bar u_{CE})$.

7.2 L^2 estimates and proof of the main estimates

Proposition 7.1. *For $\|\tilde U\|_4 \le C\varepsilon$, the operator $(\bar b_*\partial_x - \bar{df}_*)(\tilde U)$ has a right inverse $(b_*\partial_x - df^*)^\dagger$*

$$\|(\bar b_*\partial_x - \bar{df}_*)^\dagger h\|_{L^2_{\varepsilon,\delta}} \le C\varepsilon^{-1}\|h\|_{L^2_{\varepsilon,\delta}}, \tag{7.4}$$

uniquely specified by the property that the solution $u = (b_*\partial_x - df^*)^\dagger h$ satisfies

$$\ell_\varepsilon \cdot u(0) = 0. \tag{7.5}$$

for certain unit vector ℓ_ε.

Proof. Standard asymptotic ODE techniques, using the gap and reduction lemmas of [MZ1, MaZ3, PZ], where the assumption $\|\tilde U\|_{H^4_{\varepsilon,\delta}} \le C\varepsilon$ gives the needed control on coefficients; see the proof of Proposition 7.1, [MZ2]. □

Proposition 7.2. *There are constants C, $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for $\varepsilon \in]0, \varepsilon_0]$, $\delta \in [0, \delta_0]$, $f \in H^3_{\varepsilon,\delta}$, $g \in H^2_{\varepsilon,\delta}$ and $U \in H^2_{\varepsilon,\delta}$ satisfying* (5.5) *and* (7.5)

$$\big\|U\big\|_{H^2_{\varepsilon,\delta}} \le C\varepsilon^{-1}\big(\|f\|_{H^3_{\varepsilon,\delta}} + \big\|g\big\|_{H^2_{\varepsilon,\delta}}\big). \tag{7.6}$$

Proof. Going back now to (7.3), u satisfies

$$\bar b_* u' - \bar d f_* u = O(|v'| + |g| + |f'| + \varepsilon^2 |u|) - f,$$

If in addition u satisfies the condition (7.5) then

$$\|u\|_{L^2_{\varepsilon,\delta}} \le C\varepsilon^{-1}(\|v'\|_{L^2_{\varepsilon,\delta}} + \|(f,f',g)\|_{L^2_{\varepsilon,\delta}} + \varepsilon^2\|u\|_{L^2_{\varepsilon,\delta}}). \tag{7.7}$$

By Proposition 6.1 and Proposition 6.3 for $k=1$, we have

$$\big\|U'\big\|_{L^2_{\varepsilon,\delta}} + \big\|v\big\|_{L^2_{\varepsilon,\delta}} \le C\big(\big\|(f,f',f'',g,g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\big\|u\big\|_{L^2_{\varepsilon,\delta}}\big). \tag{7.8}$$

$$\begin{aligned} \|U''\|_{L^2_{\varepsilon,\delta}} + \big\|v'\big\|_{L^2_{\varepsilon,\delta}} &\le \\ &C\big(\big\|(f',f'',f''',g',g'')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\big\|U'\big\|_{L^2_{\varepsilon,\delta}} + \varepsilon^2\big\|u\big\|_{L^2_{\varepsilon,\delta}}\big). \end{aligned} \tag{7.9}$$

Combining these estimates, this implies

$$\begin{aligned} \big\|v'\big\|_{L^2_{\varepsilon,\delta}} &\le C\big(\big\|(f',f'',f''',g',g'')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\big\|(f,f',f'',g,g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon^2\big\|u\big\|_{L^2_{\varepsilon,\delta}}\big) \\ &\le C\big(\varepsilon\big\|(f,f',f'',g,g')\|_{H^1_{\varepsilon,\delta}} + \varepsilon^2\big\|u\big\|_{L^2_{\varepsilon,\delta}}\big). \end{aligned}$$

Substituting in (7.7), yields

$$\varepsilon\|u\|_{L^2_{\varepsilon,\delta}} \le C\big(\|(f,f',g)\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|(f,f',f'',g,g')\|_{H^1_{\varepsilon,\delta}} + \varepsilon^2\|u\|_{L^2_{\varepsilon,\delta}}\big).$$

Hence for ε small,

$$\varepsilon\|u\|_{L^2_{\varepsilon,\delta}} \le C\big(\|(f,f',g)\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|(f,f',f'',g,g')\|_{H^1_{\varepsilon,\delta}}\big). \tag{7.10}$$

Plugging this estimate in (7.8)

$$\big\|U'\big\|_{L^2_{\varepsilon,\delta}} + \big\|v\big\|_{L^2_{\varepsilon,\delta}} + \varepsilon\big\|u\big\|_{L^2_{\varepsilon,\delta}} \le C\big\|(f,f',f'',g,g')\|_{H^1_{\varepsilon,\delta}}. \tag{7.11}$$

Hence, with (7.9), one has

$$\begin{aligned} \|U''\|_{L^2_{\varepsilon,\delta}} + \big\|v'\big\|_{L^2_{\varepsilon,\delta}} &\le \\ &C\big(\big\|(f',f'',f''',g',g'')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\big\|(f,f',f'',g,g')\|_{H^1_{\varepsilon,\delta}}\big). \end{aligned} \tag{7.12}$$

Therefore,

$$\|U'\|_{H^1_{\varepsilon,\delta}} + \|v\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|u\|_{L^2_{\varepsilon,\delta}} \le C\|f, f', f'', g, g'\|_{H^1_{\varepsilon,\delta}} \tag{7.13}$$

The left hand side dominates

$$\|U'\|_{H^1_{\varepsilon,\delta}} + \varepsilon\|U'\|_{L^2_{\varepsilon,\delta}} = \varepsilon\|U'\|_{H^2_{\varepsilon,\delta}}$$

and the right hand side is smaller than or equal to $\|f\|_{H^2_{\varepsilon,\delta}} + \|g\|_{H^1_{\varepsilon,\delta}}$. The estimate (7.6) follows. □

Knowing a bound for $\|u\|_{L^2_{\varepsilon,\delta}}$, Proposition 6.3 implies by induction the following final result.

Proposition 7.3. *There are constants C, $\varepsilon_0 > 0$ and $\delta_0 > 0$ and for $s \ge 3$ there is a constant C_s such that for $\varepsilon \in]0, \varepsilon_0]$, $\delta \in [0, \delta_0]$, $f \in H^{s+1}_{\varepsilon,\delta}$, $g \in H^s_{\varepsilon,\delta}$, $\tilde U \in H^{s+1}_{\varepsilon,\delta}$, and $U \in H^s_{\varepsilon,\delta}$ satisfying* (5.5), (A.19), *and* (5.13), *one has*

$$\begin{aligned}\|U\|_{H^s_{\varepsilon,\delta}} &\le C\varepsilon^{-1}\big(\|\tilde U\|_{H^{s+1}_{\varepsilon,\delta}}|F|_{s_0+2} + \|F\|_{s+1}\big) \\ &= C\varepsilon^{-1}\big(\|\tilde U\|_{H^{s+1}_{\varepsilon,\delta}}(\|f\|_{H^{s_0+3}_{\varepsilon,\delta}} + \|g\|_{H^{s_0+2}_{\varepsilon,\delta}}) + (\|F\|_{H^{s+2}_{\varepsilon,\delta}} + \|g\|_{H^{s+1}_{\varepsilon,\delta}})\big),\end{aligned} \tag{7.14}$$

8 Existence for the linearized problem

To complete the proof of Proposition 5.3, it remains to demonstrate existence for the linearized problem. This can be carried out as in [MZ2] by the vanishing viscosity method, with viscosity coefficient $\eta > 0$, obtaining existence for each positive η by standard boundary-value theory, and noting that our previous A Priori bounds (7.14) persist under regularization for sufficiently small viscosity $\eta > 0$, so that we can obtain a weak solution in the limit by extracting a weakly convergent subsequence. We omit these details, referring the reader to Section 8, [MZ2]. The asserted estimates then follow in the limit by continuity.

A A Nash–Moser Theorem with losses

For completeness, we give in this appendix the parameter-dependent Nash–Moser theory developed in [TZ], specialized for clarity to the present, Hilbert space, setting. The main novelty of this treatment is to allow losses of powers of the parameter $\varepsilon \to 0$ in the linearized solution operator. For a proof of this result, see [TZ]; for a more general discussion of Nash–Moser iteration methods, see [H, AG, XSR], and references therein.

Consider two families of Banach spaces $\{E_s\}_{s\in\Re}$, $\{F_s\}_{s\in\Re}$, and a family of equations

$$\Phi^\epsilon(u^\epsilon) = 0, \qquad u^\epsilon \in E_s, \tag{A.1}$$

indexed by $\epsilon \in (0,1)$, where for all ϵ,

$$\Phi^\epsilon \in C^2(E_s, F_{s-m}), \qquad \text{for all } s \leq \bar{s}, \tag{A.2}$$

for some $m \geq 0$ and some $\bar{s} \in \Re$.

Let $|\cdot|_s$ denote the norm in E_s and $\|\cdot\|_s$ denote the norm in F_s. The norms $|\cdot|_s$ and $\|\cdot\|_s$ may be ϵ-dependent (as in our application here). We assume that the embeddings

$$E_{s'} \hookrightarrow E_s, \qquad F_{s'} \hookrightarrow F_s, \qquad s \leq s', \tag{A.3}$$

hold, and have norms less than one:

$$|\cdot|_s \leq |\cdot|_{s'}, \quad \|\cdot\|_s \leq \|\cdot\|_{s'}, \qquad s \leq s'. \tag{A.4}$$

We assume the interpolation property[2]:

$$|\cdot|_{s+\sigma} \lesssim |\cdot|_s^{\frac{\sigma'-\sigma}{\sigma'}} |\cdot|_{s+\sigma'}^{\frac{\sigma}{\sigma'}}, \qquad 0 < \sigma < \sigma'. \tag{A.5}$$

We assume in addition the existence of a family of regularizing operators

$$S_\theta : \quad E_s \to E_s, \qquad \theta > 0,$$

such that for all $s \leq s,'$,

$$|S_\theta u - u|_s \lesssim \theta^{s-s'} |u|_{s'}. \tag{A.6}$$

$$|S_\theta u|_{s'} \lesssim \theta^{s'-s} |u|_s. \tag{A.7}$$

Assumption A.1. *For some $s_0 \in \Re$, some $\gamma_0 \geq 0$, for all s such that*

$$s_0 + m \leq s + m \leq \bar{s},$$

for all $u, v, w \in E_{s+m}$,

$$\|\Phi^\epsilon(u)\|_s \leq C_0(1 + |u|_{s+m} + |u|_{s_0+m}|u|_s); \tag{A.8}$$

$$\|(\Phi^\epsilon)'(u) \cdot v\|_s \leq C_0(|v|_{s+m} + |v|_{s_0+m}|u|_{s+m}), \tag{A.9}$$

and

$$\begin{aligned} \|(\Phi^\epsilon)''(u) \cdot (v,w)\|_s \leq C_0\big(&|v|_{s_0+m}|w|_{s+m} + |v|_{s+m}|w|_{s_0+m} \\ &+ |u|_{s+m}|v|_{s_0+m}|w|_{s_0+m}\big) \end{aligned} \tag{A.10}$$

where $C_0 = C_0(\epsilon, |u|_{s_0+m})$ satisfies

$$\sup_\epsilon \sup_{|u|_{s_0+m} \lesssim \epsilon^{\gamma_0}} C_0 < +\infty. \tag{A.11}$$

[2] In (A.5) and below, $|u|_s \lesssim |v|_{s'}$ stands for $|u|_s \leq C|v|_{s'}$, for some $C > 0$ depending on s and s' but not on ϵ, nor on u and v.

Assumption A.2. *For some $\gamma \geq 0, r \geq 0, r' \geq 0$, for all s such that*

$$s_0 + m + \max(r, r') \leq s + \max(r, r') \leq \bar{s}, \tag{A.12}$$

for all $u \in E_{s+r}$ such that

$$|u|_{s_0+m} \lesssim \epsilon^{\gamma}, \tag{A.13}$$

the map $(\Phi^\epsilon)'(u) : E_{s+m} \to F_s$ has a right inverse $\Psi^\epsilon(u)$:

$$(\Phi^\epsilon)'(u)\Psi^\epsilon(u) = \mathrm{Id} : \quad F_s \to F_s,$$

satisfying, for all $\phi \in F_{s+r'}$,

$$|\Psi^\epsilon(u)\phi|_s \leq \epsilon^{-1} C(\|\phi\|_{s_0+m+r'} |u|_{s+r} + \|\phi\|_{s+r'}), \tag{A.14}$$

where C is a non-decreasing function of its arguments s and $|u|_{s_0+m+r}$.

Assumption A.3. *There holds the bound*

$$\|\Phi^\epsilon(0)\|_s \lesssim \epsilon^k, \tag{A.15}$$

for some k and s satisfying

$$\max(2, 1 + \gamma_0, 1 + \gamma) < k, \tag{A.16}$$

$$C(k) \leq \bar{s} - s_0 - m, \tag{A.17}$$

where $C(k)$ is a certain positive function (see [TZ]) and $s \in [s_0 + m, \bar{s} - C(k)]$.

Theorem A.4 (Existence)**.** *Under Assumptions* A.1, A.2 *and* A.3*, for ϵ small enough, there exists a real sequence θ_j^ϵ, satisfying $\theta_j^\epsilon \to +\infty$ as $j \to +\infty$ and ϵ is held fixed, such that the sequence*

$$u_0^\epsilon := 0, \qquad u_{j+1}^\epsilon := u_j^\epsilon + S_{\theta_j^\epsilon} v_j^\epsilon, \qquad v_j^\epsilon := -\Psi^\epsilon(u_j^\epsilon)\Phi^\epsilon(u_j^\epsilon),$$

is well defined and converges, as $j \to \infty$ and ϵ is held fixed, to a solution u^ϵ of (A.1) *in $s + m$ norm, which satisfies the bound*

$$|u^\epsilon|_s \lesssim \epsilon^{k-1}. \tag{A.18}$$

Theorem A.5 (Uniqueness)**.** *Under Assumptions* A.1, A.2 *and* A.3*, for ϵ small enough, if $(\Phi^\epsilon)'$ is invertible, i.e., Ψ^ϵ is also a left inverse, then the solution described in Thm A.4 is unique in a ball of radius $o(\varepsilon^{\max(1,\gamma_0,\gamma)})$ in $s_0 + 2m + r'$ norm. More generally, if $\hat{u}^\epsilon$ is a second solution within this ball, then $(\hat{u}^\epsilon - u^\epsilon)$ is approximately tangent to* $\mathrm{Ker}(\Phi^\epsilon)'(u^\epsilon)$*, in the sense that its distance in s_0 norm from* $\mathrm{Ker}(\Phi^\epsilon)'(u^\epsilon)$ *is $o(|\hat{u}^\epsilon - u^\epsilon|_{s_0})$. In particular, if* $\mathrm{Ker}(\Phi^\epsilon)'(u^\epsilon)$ *is finite-dimensional, then u is the unique solution in the ball satisfying the additional "phase condition"*

$$\Pi_{\mathrm{Ker}(\Phi^\epsilon)'(u^\epsilon)}(\hat{u}^\epsilon - u^\epsilon) = 0, \tag{A.19}$$

where $\Pi_{\mathrm{Ker}(\Phi^\epsilon)'(u^\epsilon)}$ is any uniformly bounded projection onto $\mathrm{Ker}(\Phi^\epsilon)'(u^\epsilon)$ *(in a Hilbert space, any orthogonal projection onto* $\mathrm{Ker}(\Phi^\epsilon)'(u^\epsilon)$*).*

References

[AG] S. Alinhac and P. Gerard, *Opérateurs pseudo-différentiels et théorème de Nash-Moser.* Savoirs Actuels. InterEditions, Paris; Editions du Centre National de la Recherche Scientifique (CNRS), Meudon, 1991. 190 pp. *Pseudo-differential operators and the Nash-Moser theorem.* Translated from the 1991 French original by Stephen S. Wilson. Graduate Studies in Mathematics, 82. American Mathematical Society, Providence, RI, 2007. viii+168 pp.

[CN] R. Caflisch and B. Nicolaenko, *Shock profile solutions of the Boltzmann equation,* Comm. Math. Phys. 86 (1982), no. 2, 161–194.

[DY] A. Dressel and W.-A. Yong, *Existence of traveling-wave solutions for hyperbolic systems of balance laws,* Arch. Ration. Mech. Anal. 182 (2006), no. 1, 49–75.

[GMWZ] Gues, O., Metivier, G., Williams, M., and Zumbrun, K., *Paper 4*, *Navier-Stokes regularization of multidimensional Euler shocks*, Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 1, 75–175.

[H] R. S. Hamilton, *The inverse function theorem of Nash and Moser.* Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 1, 65–222.

[JX] S. Jin and Z. Xin, *The relaxation schemes for systems of conservation laws in arbitrary space dimensions,* Comm. Pure Appl. Math. 48 (1995), no. 3, 235–276.

[K] S. Kawashima, *Systems of a hyperbolic–parabolic composite type, with applications to the equations of magnetohydrodynamics*, thesis, Kyoto University (1983).

[LY] T.-P. Liu and S.-H. Yu, *Boltzmann equation: micro-macro decompositions and positivity of shock profiles*, Comm. Math. Phys. 246 (2004), no. 1, 133–179.

[MP] A. Majda and R. Pego, *Stable viscosity matrices for systems of conservation laws*, J. Diff. Eqs. 56 (1985) 229–262.

[MaZ1] C. Mascia and K. Zumbrun, *Pointwise Green's function bounds and stability of relaxation shocks.* Indiana Univ. Math. J. 51 (2002), no. 4, 773–904.

[MaZ2] C. Mascia and K. Zumbrun, *Stability of large-amplitude shock profiles of general relaxation systems*, SIAM J. Math. Anal. 37 (2005), no. 3, 889–913.

[MaZ3] C. Mascia and K. Zumbrun, *Spectral stability of weak relaxation shock profiles*, Preprint (2008).

[MaZ4] C. Mascia and K. Zumbrun, *Stability of small–amplitude shock profiles of symmetric hyperbolic–parabolic systems,* Comm. Pure Appl. Math. 57 (2004), no.7, 841–876.

[MaZ5] C. Mascia and K. Zumbrun, *Pointwise Green function bounds for shock profiles of systems with real viscosity,* Arch. Rational Mech. Anal. 169 (2003), no.3, 177–263.

[MZ1] Métivier, G. and Zumbrun, K., *Viscous Boundary Layers for Noncharacteristic Nonlinear Hyperbolic Problems*, Memoirs AMS, 826 (2005).

[MZ2] Métivier, G. and Zumbrun, K., *Existence of semilinear relaxation shocks*, to appear, JMPA.

[MZ3] Métivier, G. and Zumbrun, K., *Existence and sharp localization in velocity of small-amplitude Boltzmann shocks*, in preparation.

[N] R. Natalini, *Recent mathematical results on hyperbolic relaxation problems,* Analysis of systems of conservation laws (Aachen, 1997), 128–198, Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math., 99, Chapman & Hall/CRC, Boca Raton, FL, 1999.

[Pe] R.L. Pego, *Stable viscosities and shock profiles for systems of conservation laws,* Trans. Amer. Math. Soc. 282 (1984) 749–763.

[PI] T. Platkowski and R. Illner, *Discrete velocity models of the Boltzmann equation: a survey on the mathematical aspects of the theory,* SIAM Rev. 30 (1988), no. 2, 213–255.

[PZ] Plaza, R. and Zumbrun, K., *An Evans function approach to spectral stability of small-amplitude shock profiles*, Discrete Contin. Dyn. Syst. 10 (2004) 885–924.

[TZ] B. Texier and K. Zumbrun, *Nash–Moser iterates and singular perturbations*, preprint (2008).

[XSR] X. Saint-Raymond, *A simple Nash-Moser implicit function theorem.* Enseign. Math. (2) 35 (1989), no. 3-4, 217–226.

[Y] W.-A. Yong *Basic structures of hyperbolic relaxation systems*, Proc. Roy. Soc. Edinburgh Sect. A 132 (2002), no. 5, 1259–1274.

[YZ] W.-A. Yong and K. Zumbrun *Existence of relaxation shock profiles for hyperbolic conservation laws*, SIAM J. Appl. Math. 60 (2000) no.5, 1565–1575.

[Z1] K. Zumbrun, *Multidimensional stability of planar viscous shock waves*, "Advances in the theory of shock waves", 307–516, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, 2001.