Special Genera of Hermitian Lattices and Applications to HAWK

Lattice Coding & Crypto Meeting, London King's College

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The Lattice Isomorphism Problem (LIP)

Let $\mathbf{b_1}, \ldots, \mathbf{b_n} \in \mathbb{R}^n$ which are $\mathbb{R}$-linearly independant.
$\mathcal{L} := \{\sum_i x_i \mathbf{b_i} \,|\, x_i \in \mathbb{Z}\}$ is a **lattice** in $\mathbb{R}^n$ and $\mathbf{B} = (\mathbf{b_1}| \cdots |\mathbf{b_n})$ is a **basis** of $\mathcal{L}$.

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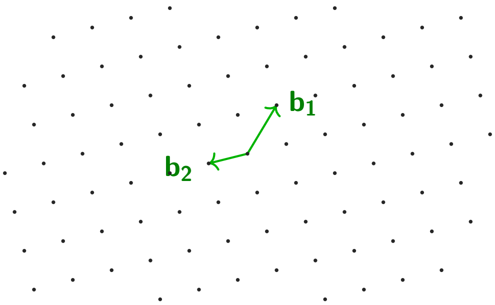

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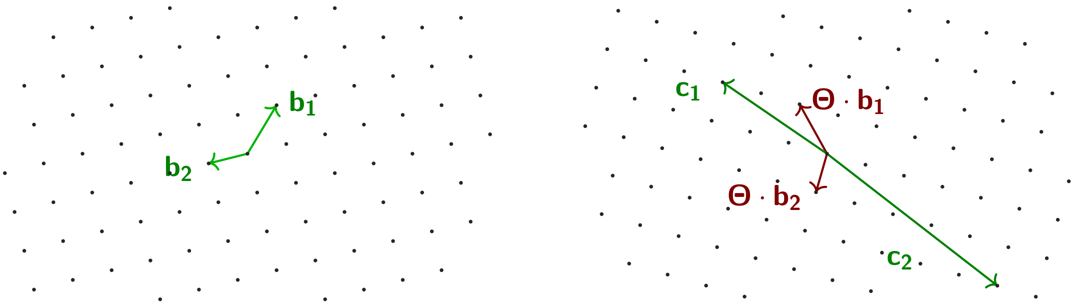

Lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ are **isomorphic** if $\mathcal{L}_2 = \mathbf{\Theta}(\mathcal{L}_1)$ for some $\mathbf{\Theta} \in \mathcal{O}_n(\mathbb{R})$ orthogonal.

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Remark: Can state LIP with **quadratic forms** instead.

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Then the **Gram matrices** $\mathbf{G} := \mathbf{B}^T\mathbf{B}$ and $\mathbf{H} := \mathbf{C}^T\mathbf{C}$ are **congruent**:

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Two point of views, truly equivalent (Cholesky factorization over $\mathbb{R}$).
In practice we prefer quadratic forms. In this talk: keep lattice bases.

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In cryptography :

1. Ducas & van Woerden (2021): primitives based on decision LIP.
2. Ducas *et. al.* (2022): signature scheme **HAWK**, related to search **module**-LIP.

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van Gent & van Woerden (2025): reduce search module-LIP to **decision** module-LIP.
$\rightsquigarrow$ HAWK reduces to several instances of decision module-LIP.

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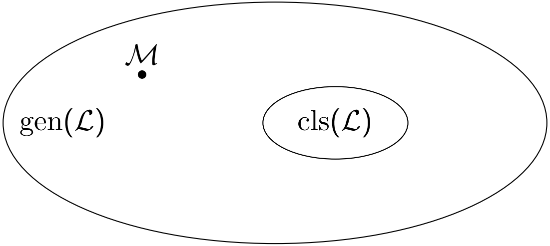

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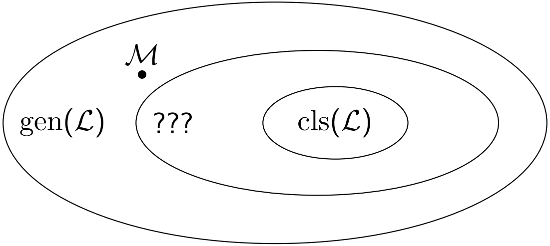

Our goal: Find an equivalence relation $\sim$ on gen($\mathcal{L}$) s.t. the equivalence class of $\mathcal{L}$ is between gen($\mathcal{L}$) and cls($\mathcal{L}$). Also we want $\mathcal{M} \sim \mathcal{L}$ to be **efficiently testable**.

This one is actually important too

Ling, Liu and Mendelsohn (Asiacrypt 24') considered the **spinor genus**. It is defined for a lattice in a **quadratic** space over $\mathbb{Q}$ (*e.g*, $(\mathbb{Q}^\ell, \Phi)$, where $\Phi(\mathbf{v}, \mathbf{w}) = \sum_i v_i w_i$).

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This talk: Shimura (1964) introduced the **special genus**. An avatar of the spinor genus for **Hermitian** lattices. It is efficiently computable for several module lattices. We discuss the impact on HAWK.

Background on Hermitian lattices

Background on number theory (1)

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Its **ring of integers** is $\mathbb{Z}_K = \mathbb{Z}[X]/(X^n + 1)$. $\exists$ a **complex conjugation** $a \mapsto \overline{a}$ on K.

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K **embeds** into larger fields. Two types of embeddings:

1. **Complex**: $K \hookrightarrow \mathbb{C}$, by sending X to a root of $X^n + 1$ in $\mathbb{C}$.
2. **Local**: $K \hookrightarrow K_{\mathfrak{p}}$, for any prime ideal $\mathfrak{p}$.

Background on number theory (2)

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A (full rank) **Hermitian lattice** in V is of the form:

$$\mathcal{L} = \mathfrak{a}_1 \mathbf{b_1} + \cdots + \mathfrak{a}_\ell \mathbf{b}_\ell,$$

where $B = (\mathbf{b_1} | \cdots | \mathbf{b}_\ell) \in \mathbf{GL}_\ell(K)$, and $\mathfrak{a}_1, \ldots, \mathfrak{a}_\ell \subseteq \mathbb{Z}_K$ are ideals.

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1. $C = BU$
2. $u_{i,j} \in \mathfrak{a}_i \mathfrak{b}_j^{-1}$
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Example: $\mathcal{L} = \mathbb{Z}_K \binom{1}{0} + \mathbb{Z}_K \binom{0}{1}$ as in **HAWK**, $\{(\text{pseudo-})\text{bases of } \mathcal{L}\} \supseteq \mathbf{GL}_2(\mathbb{Z}_K)$.

Background on number theory (3)

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The **module index** of $\mathcal{M}$ in $\mathcal{L}$ is:

$$[\mathcal{L} : \mathcal{M}] := \det(B^{-1}C) \cdot \mathfrak{a}_1\mathfrak{b}_1^{-1} \cdots \mathfrak{a}_\ell\mathfrak{b}_\ell^{-1}.$$

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Example: if $K = \mathbb{Q}(i)$ and $\mathcal{L} = \mathbb{Z}[i] + \mathbb{Z}[i]$, $\mathcal{M} = \mathbb{Z}[i] + 2\mathbb{Z}[i]$, then $[\mathcal{L} : \mathcal{M}] = 2\mathbb{Z}[i]$.

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Remark: $[\mathcal{L} : \mathcal{M}]$ well defined even if there is no inclusion.

Genus and special genus

(Locally) isometric Hermitian lattices

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Images of $\mathcal{L}$, $\mathcal{M}$ are denoted by $\mathcal{L}_\mathfrak{p}$, $\mathcal{M}_\mathfrak{p}$. They are "local" Hermitian lattices.

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It is **computationally easy** to test if $\mathcal{L}$, $\mathcal{M}$ are locally isomorphic at $\mathfrak{p}$, *i.e.*, if

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$$\forall \mathfrak{p}, \exists \Theta_\mathfrak{p} \in \mathcal{U}(V_\mathfrak{p}) \text{ s.t. } \mathcal{M}_\mathfrak{p} = \Theta_\mathfrak{p}(\mathcal{L}_\mathfrak{p}).$$

Fact: $\mathcal{L}$ and $\mathcal{M}$ are trivially locally isomorphic at $\mathfrak{p}$, for all but finitely many $\mathfrak{p}$.

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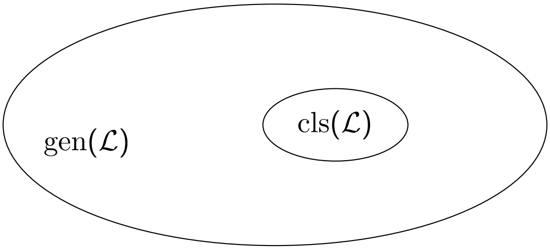
$\to$ poly-time algo to test if $\mathcal{L}$, $\mathcal{M}$ are in the same genus (we write $\mathcal{M} \in \text{gen}(\mathcal{L})$).

The genus of $\mathcal{L}$ contains its isomorphism class, $\text{cls}(\mathcal{L})$.

Fact: $\text{gen}(\mathcal{L})$ is the disjoint union of **finitely many** isomorphism classes.



Special genus of a Hermitian lattice

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$$(\exists\, \Sigma \in \mathcal{U}(V), \forall\, \mathfrak{p}, \exists\, \Theta_{\mathfrak{p}} \in \mathcal{U}(V_{\mathfrak{p}}) \text{ with } \det \Theta_{\mathfrak{p}} = 1) \text{ s.t. } \mathcal{M}_{\mathfrak{p}} = \Sigma \circ \Theta_{\mathfrak{p}}(\mathcal{L}_{\mathfrak{p}}).$$

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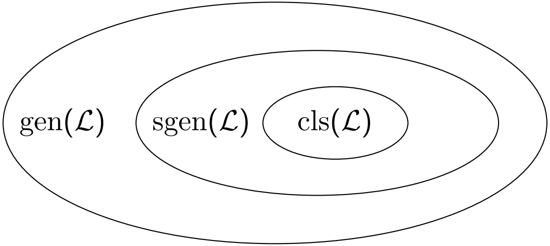
Equivalence relation $\sim$ on $\mathrm{gen}(\mathcal{L})$.
Denote the class of $\mathcal{L}$ by **sgen**$(\mathcal{L})$.

Gives an **intermediate classification** between $\mathrm{gen}(\mathcal{L})$ and $\mathrm{cls}(\mathcal{L})$.



How to distinguish special genera?

Main theoretic result: Shimura's theorem

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Corollary.

Let $\mathcal{M} \in \text{gen}(\mathcal{L}_0)$. $\mathcal{M} \in \text{sgen}(\mathcal{L}_0)$ **iff** $[\mathcal{L}_0 : \mathcal{M}]$ has the form $g \cdot \mathbb{Z}_K$ with $g\overline{g} = 1$.

Main algorithmic tool: Lenstra-Silverberg's algorithm

Testing if an ideal $\mathfrak{a}$ is principal is a **hard problem** for (classical) computers. Moreover if $\mathfrak{a}$ is principal, two generators g, h have $g\overline{g} \neq h\overline{h}$ in general.

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$\Longrightarrow$ We obtain a (classical) poly-time algo to test if $\mathcal{M} \in \text{sgen}(\mathcal{L}_0)$.

Pseudo-code for $\mathcal{L}_0 = \mathbb{Z}_K^2$

Algorithm: Test if $\mathcal{M} \in \mathrm{sgen}(\mathcal{L}_0)$

Input: A pseudo-basis $\mathbf{C} = (C, \mathfrak{b}_1, \mathfrak{b}_2)$ of $\mathcal{M} \in \mathrm{gen}(\mathcal{L}_0)$
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Can't compute a Cholesky factorization **over** $K \rightsquigarrow$ don't have a basis for $\mathfrak{a} = [\mathcal{L}_0 : \mathcal{M}]$.
But we can still check if $\mathfrak{a}$ has a generator with $g\overline{g} = 1$.

Remarks on the general algorithm

Previous algorithm is extended to any $\mathcal{L} \subset K^\ell$ and $\mathcal{M} \in \text{gen}(\mathcal{L})$.

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Remark: For several Hermitian lattices (including HAWK), there are no local conditions to check! :)

Impact on HAWK

Impact on HAWK (1)

Recall: $\mathcal{L}_0 = \mathbb{Z}_K^2$, and fix $m = 512$. We are able to distinguish $\mathrm{sgen}(\mathcal{L}_0) \subseteq \mathrm{gen}(\mathcal{L}_0)$.

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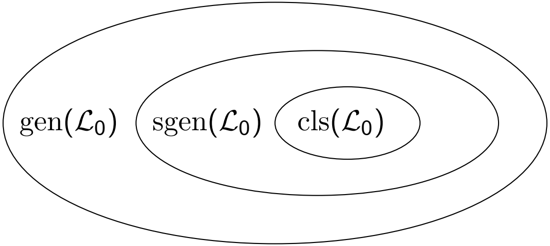
We will approximate:

$$\#\{\text{iso. classes in } \mathrm{sgen}(\mathcal{L}_0)\}$$

$$\approx \frac{\#\{\text{iso. classes in } \mathrm{gen}(\mathcal{L}_0)\}}{\#\{\text{special genera in } \mathrm{gen}(\mathcal{L}_0)\}}$$



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Siegel's mass formula gives a way to compute the mass (van Gent):

$$\text{Mass}(\mathcal{L}_0) = \frac{1}{2^{\frac{m}{2}-1}} \cdot \prod_{\mathfrak{p}} \lambda(\mathcal{L}_{0\mathfrak{p}}) \cdot \left| \frac{\zeta_K}{\zeta_F}(0) \right| \cdot |\zeta_F(-1)|$$

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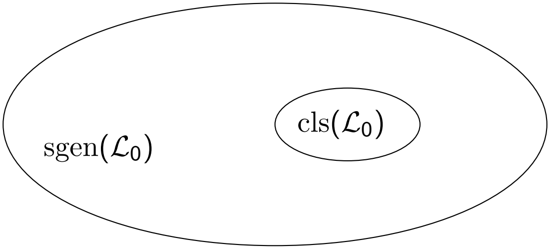
Moreover for $m = 512$, and under GRH, $h_F = 1$. Overall:

$$sh(\mathcal{L}_0) = h_K \approx 2^{200}.$$

Impact on HAWK (4)

For $\mathcal{L}_0$ and $m = 512$ as in HAWK,

$\#\{\text{iso. classes in sgen}(\mathcal{L}_0)\} \approx 2^{800}$



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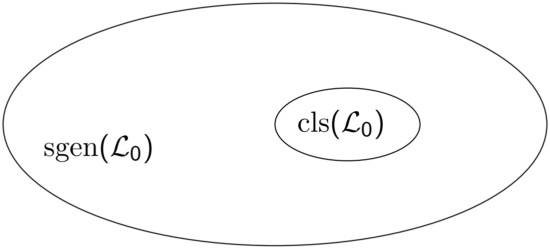

Recall: Have a reduction from HAWK to (several instances) of decision module-LIP.
(Unfortunately) lattices involved are all in sgen($\mathcal{L}_0$) $\rightsquigarrow$ **No impact on HAWK!**

Takeaway and perspectives

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- Despite exponential gain, still exponential number of classes in $\mathrm{sgen}(\mathcal{L}_0)$.
- No concrete impact on the security of HAWK.
- **Open:** $\mathcal{L}_0$ can be seen as a **quadratic** lattices (of rank 4!). Spinor genus? How does it compare with the special genus?

Takeaway and perspectives

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- Despite exponential gain, still exponential number of classes in $\mathrm{sgen}(\mathcal{L}_0)$.
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Thank you for your attention!

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3. The first basis vector is $g^{p-1} \cdot v$ with v **short**. Reduce it modulo p

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$\rightarrow$ We have an implementation in `SageMath`!

References

Léo Ducas, Eamonn W Postlethwaite, Ludo N Pulles, and Wessel van Woerden. Hawk: Module lip makes lattice signatures fast, compact and simple. 2022.

Léo Ducas and Wessel van Woerden. On the lattice isomorphism problem, quadratic forms, remarkable lattices, and cryptography. 2022.

Hendrik W Lenstra Jr and Alice Silverberg. Testing isomorphism of lattices over cmorders. 2019.

Cong Ling, Jingbo Liu, and Andrew Mendelsohn. On the Spinor Genus and the Distinguishing Lattice Isomorphism Problem. 2024.

Goro Shimura. Arithmetic of unitary groups. 1964.

Daniel M. H. van Gent. A note on the genus of the HAWK lattice. 2025.