Are these lattices isomorphic? A quest for finer invariants

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The Lattice Isomorphism Problem (LIP)

Let $\mathbf{b_1}, \ldots, \mathbf{b_n} \in \mathbb{R}^n$ which are $\mathbb{R}$-linearly independant.

$\mathcal{L} := \{\sum_i x_i \mathbf{b_i} \,|\, x_i \in \mathbb{Z}\}$ is a **lattice** in $\mathbb{R}^n$ and $\mathbf{B} = (\mathbf{b_1}| \cdots |\mathbf{b_n})$ is a **basis** of $\mathcal{L}$.

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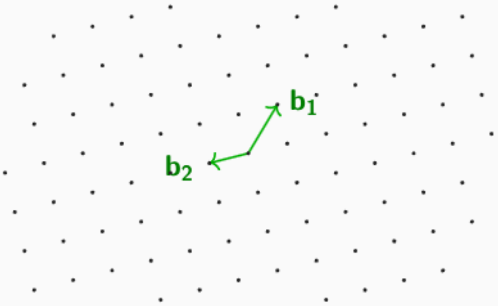

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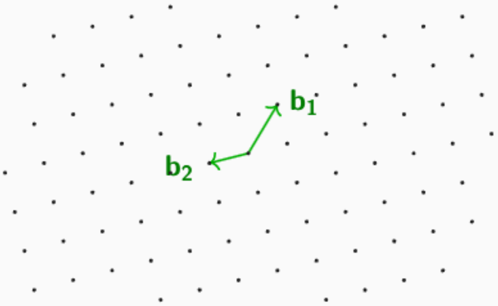

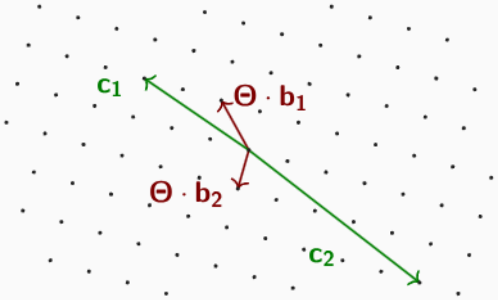

Lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ are **isomorphic** if $\mathcal{L}_2 = \mathbf{\Theta}(\mathcal{L}_1)$ for some $\mathbf{\Theta} \in \mathcal{O}_n(\mathbb{R})$ orthogonal.

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Use in cryptography :

1. Ducas & van Woerden (2021): primitives based on decision LIP.
2. Ducas *et. al.* (2022): signature scheme **HAWK**, security related to **module**-LIP.

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van Gent & van Woerden (2025): For a certain family of module lattices, search module-LIP reduces to **decision** module-LIP.
$\rightsquigarrow$ HAWK reduces to several instances of decision module-LIP.

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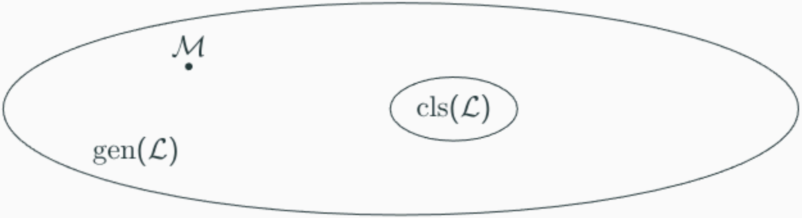

Genus: Can test efficiently if $\mathcal{L}$, $\mathcal{M}$ are in the same **genus** (necessary but not sufficient!). Are there finer and efficiently computable invariants?

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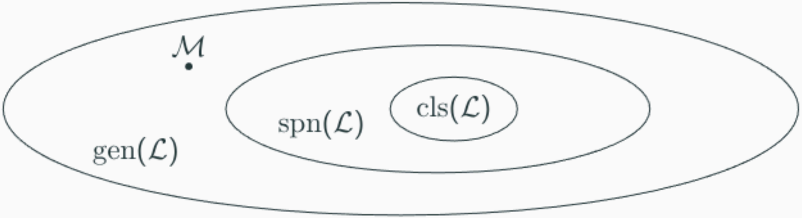

Spinor genus: Ling, Liu and Mendelsohn (2024) gave a poly-time algorithm to determine if $\mathcal{L}$, $\mathcal{M}$ are in the same **spinor genus**. **Limitations:** very strong conditions on the parameters + only defined for quadratic lattices $\Rightarrow$ No impact on HAWK.

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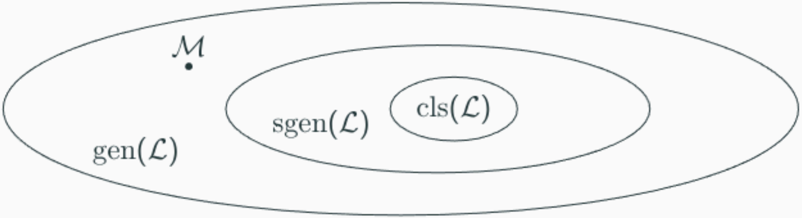

My contribution: An algorithm to determine if $\mathcal{L}, \mathcal{M}$ are in the same **special genus**. Works for all parameters + poly-time for many module lattices, including those in the reduction.

Background on number theory (1)

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K **embeds** into larger fields. Two types of embeddings:

1. **Complex**: $K \hookrightarrow \mathbb{C}$, by sending X to a root of $X^n + 1$ in $\mathbb{C}$.
2. **Local**: $K \hookrightarrow K_\mathfrak{p}$, for any prime ideal $\mathfrak{p}$.

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$$\text{For } \mathcal{L} = \mathbb{Z}_K + \mathbb{Z}_K \text{ and } \mathcal{M} = \mathbb{Z}_K + 2\mathbb{Z}_K, \quad [\mathcal{L} : \mathcal{M}] = 2\mathbb{Z}_K.$$

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For any prime ideal $\mathfrak{p}$, a module lattice $\mathcal{L} \subset K^\ell$ extends to $\mathcal{L}_\mathfrak{p} \subset K_\mathfrak{p}^\ell$ via $K \hookrightarrow K_\mathfrak{p}$.
It is **easy** to check if $\mathcal{L}_\mathfrak{p}$ and $\mathcal{M}_\mathfrak{p}$ are **locally** isomorphic at $\mathfrak{p}$, *i.e.*, if there exists

$$\mathbf{\Theta}_\mathfrak{p} \in \mathcal{U}_\ell(K_\mathfrak{p}) \text{ s.t. } \mathcal{M}_\mathfrak{p} = \mathbf{\Theta}_\mathfrak{p}(\mathcal{L}_\mathfrak{p}), \quad \text{where } \mathcal{U}_\ell \text{ means unitary} : \overline{\mathbf{\Theta}}_\mathfrak{p}^T \cdot \mathbf{\Theta}_\mathfrak{p} = \mathbf{Id}.$$

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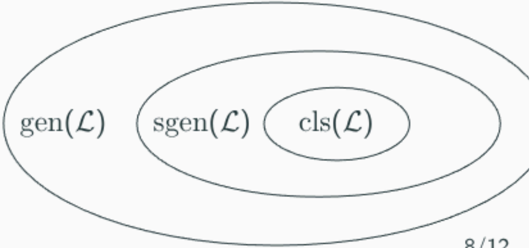
The genus of $\mathcal{L}$ contains the special genus of $\mathcal{L}$, itself containing the isomorphism class of $\mathcal{L}$.

There are **finitely many** classes in a genus.



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We obtain a **polynomial-time** algorithm for testing if $\mathcal{M} \in \mathrm{sgen}(\mathcal{L}_0)$.

More generally it works for any $\mathcal{L}$ and $\mathcal{M} \in \mathrm{gen}(\mathcal{L})$ but it is harder to analyze.

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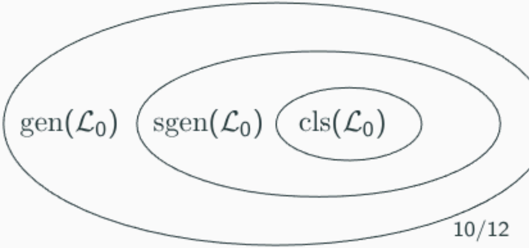
$\#\{\text{isomorphism classes in } \mathrm{gen}(\mathcal{L}_0)\} \approx 2^{1000}$.
$\#\{\text{special genera in } \mathrm{gen}(\mathcal{L}_0)\} \approx 2^{200}$.

$\rightsquigarrow$ Still about $\approx 2^{800}$ classes in $\mathrm{sgen}(\mathcal{L}_0)$!



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- No practical impact on HAWK.
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Thank you for your attention!

References

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