# Cryptanalysis of the module-Lattice Isomorphism Problem

LACS Seminar, Luxembourg University

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# The Lattice Isomorphism Problem (LIP)

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$\mathcal{L} := \{\sum_i x_i \mathbf{b_i} \mid x_i \in \mathbb{Z}\}$ is a **lattice** in $\mathbb{R}^n$ and $\mathbf{B} = (\mathbf{b_1} | \cdots | \mathbf{b_n})$ is a **basis** of $\mathcal{L}$.

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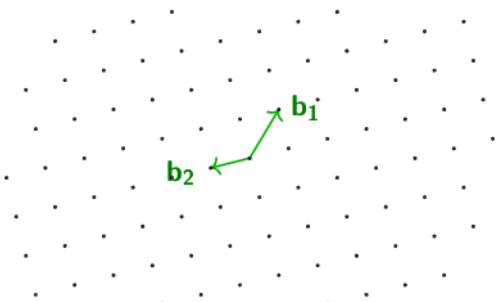

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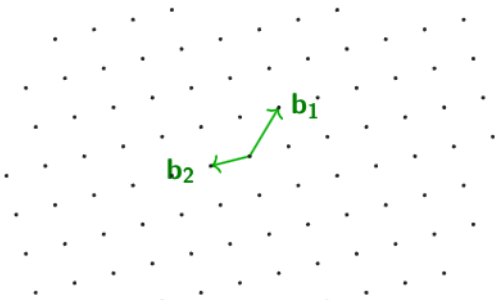


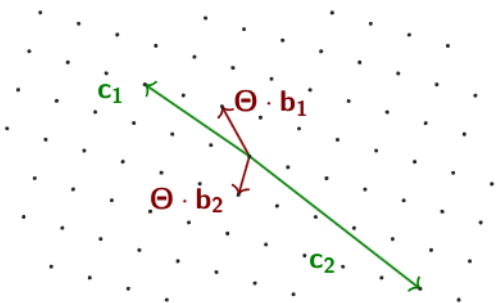


Lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ are **isomorphic** if $\mathcal{L}_2 = \mathbf{\Theta}(\mathcal{L}_1)$ for some $\mathbf{\Theta} \in \mathcal{O}_n(\mathbb{R})$ orthogonal.

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**Remark:** Can state LIP with quadratic forms instead, useful for constructions.

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In cryptography:

- Ducas, van Woerden (2021): primitives based on **decision LIP**;
- Ducas *et. al.* (2022): signature scheme **HAWK**. Its security is related to the hardness of **search module-LIP**.

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*Can we use the extra structure of module-LIP to solve it more efficiently than LIP?*

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2. **Results on search module-LIP**

   We target rank two module lattices, as in HAWK.

3. **Focus on decision module-LIP**

   We introduce computable invariants and discuss the impact on HAWK.

# Background on number theory

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$\longrightarrow$ These fields are among the most common in lattice-based cryptography.

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**Example:** For power-of-two cyclotomic fields, $K \hookrightarrow \mathbb{C}^{d/2} \simeq \mathbb{R}^d$.

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Let $\ell > 0$. An **isometry** of $K^\ell$ is a linear map $\mathbf{\Theta} : K^\ell \to K^\ell$ such that

$$\mathbf{\Theta}^* \mathbf{\Theta} = \mathbf{Id},$$

where $\mathbf{\Theta}^* = \overline{\Theta^T}$ is the transpose-conjugate.

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Moreover, $\{\text{Bases of } \mathcal{L}\} = \mathbf{GL}_2(\mathbb{Z}_K)$.

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**Fact:** Security of HAWK related to module-LIP for $\mathcal{L} = \mathbb{Z}_K\binom{1}{0} + \mathbb{Z}_K\binom{0}{1}$. Precisely:

$$\text{Break module-LIP} \Longrightarrow \text{key recovering in HAWK.}$$

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- Other choices of $K$: quantum polynomial-time attack. Classical poly-time for a large family of fields.

  B. Allombert, A. Pellet-Mary, W. van Woerden (EUROCRYPT 2025).

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All papers follow the same principle:

Reduce rank-2 module-LIP over $K$ to rank-1 module-LIP over an extension $\mathcal{A}/K$.

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**Remark**: rank-1 module-lattices are ideal-lattices.

- When $K$ is totally real, $\mathcal{A}$ is a number field. We use Gentry-Szydlo's algorithm.
- When $K$ is totally complex, $\mathcal{A}$ is a quaternion algebra. No similar tool is known.
- For other fields, $\mathcal{A}$ is a number field and $\exists$ extension of Gentry-Szydlo's algorithm.

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for a polynomially many pairs $(\mathcal{L}, \mathcal{M})$.

$\longrightarrow$ Gives a new direction to target HAWK!

# Focus on decision module-LIP

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**Strategy.** This is very hard in high dimensions. Instead, we check for isomorphisms over **larger spaces**. Only a necessary condition $\to$ gives a **partial distinguisher**.

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Two module-lattices $\mathcal{L}, \mathcal{M} \subset K^{\ell}$ are said to be in the same **genus** if they are locally isomorphic at **every** prime ideal $\mathfrak{p}$. Notation: $\mathcal{M} \in \mathrm{gen}(\mathcal{L})$.

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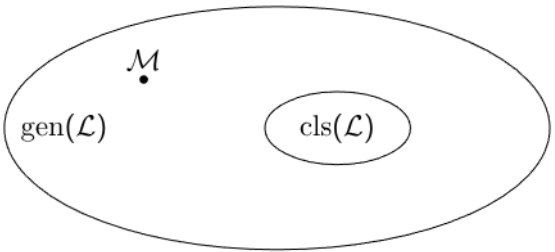


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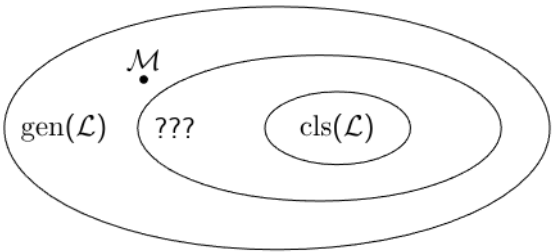


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**Rest of this talk:** Define and distinguish special genera + discuss impact on HAWK.

# Focus on decision module-LIP (5/10)

Module-lattices $\mathcal{L}$, $\mathcal{M} \subset K^\ell$ belong to the same **special genus** if:

$$(\exists\, \mathbf{\Theta}, \forall\, \mathfrak{p}, \exists\, \mathbf{\Theta}_\mathfrak{p} \text{ with } \det \mathbf{\Theta}_\mathfrak{p} = 1) \qquad \text{s.t.} \qquad \mathcal{M}_\mathfrak{p} = \mathbf{\Theta} \circ \mathbf{\Theta}_\mathfrak{p}(\mathcal{L}_\mathfrak{p}).$$

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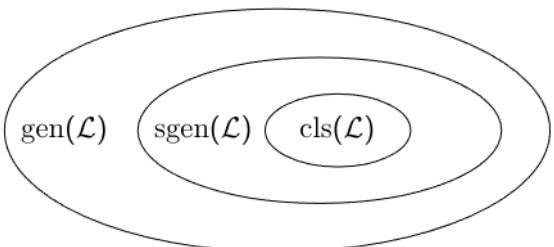
Equivalence relation $\sim$ on $\mathrm{gen}(\mathcal{L})$,
denote the class of $\mathcal{L}$ by $\mathrm{sgen}(\mathcal{L})$.

We have: $\mathrm{cls}(\mathcal{L}) \subseteq \mathrm{sgen}(\mathcal{L}) \subseteq \mathrm{gen}(\mathcal{L})$.

Gives an **intermediate classification** between $\mathrm{gen}(\mathcal{L})$ and $\mathrm{cls}(\mathcal{L})$.



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**Remark:** $[\mathcal{L} : \mathcal{M}]$ is well defined even if there is no inclusion.

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**Consequence:** Assume $\mathcal{M} \in \text{gen}(\mathcal{L})$. Then $\mathcal{M} \in \text{sgen}(\mathcal{L})$ **iff**

$$[\mathcal{L} : \mathcal{M}] \text{ has the form } g \cdot \mathbb{Z}_K \text{ with } g\overline{g} = 1.$$

# Focus on decision module-LIP (8/10)

**Remark:** Testing $\mathfrak{a} = g \cdot \mathbb{Z}_K$ with $g\overline{g} = 1$ is basically module-LIP between $\mathbb{Z}_K$ and $\mathfrak{a}$.

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There is a (classical) **poly-time** algorithm that given an ideal $\mathfrak{a}$ and $q \in K$ s.t. $\mathfrak{a} \cdot \overline{\mathfrak{a}} = q \cdot \mathbb{Z}_K$, computes $g \in K$ such that $\mathfrak{a} = g \cdot \mathbb{Z}_K$ and $g\overline{g} = q$ (if it exists).

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- **Fundamental tool** for the cryptanalysis of (rank-2) search module-LIP.
- A generalization to **quaternionic** ideals would break HAWK.

$\longrightarrow$ We obtain a (classical) poly-time algo to test if $\mathcal{M} \in \mathrm{sgen}(\mathcal{L})$.

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**Remark:** For several module lattices (including HAWK), **no** local conditions to check. $\longrightarrow$ provable poly-time complexity!

# Focus on decision module-LIP (10/10)

Fix $K = \mathbb{Q}[X]/(X^d + 1)$ and $\mathcal{L} = \mathbb{Z}_K\binom{1}{0} + \mathbb{Z}_K\binom{0}{1}$ as in HAWK.

We have an effective distinguisher for $\mathrm{sgen}(\mathcal{L}) \subseteq \mathrm{gen}(\mathcal{L})$.

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**Question 1:** Recall that HAWK reduces to several decision module-LIP instances. Are these module-lattices in the same special genus?

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Are these module-lattices in the same special genus? **YES** $\longrightarrow$ No impact on HAWK...

**Question 2:** Is $\mathrm{sgen}(\mathcal{L})$ really finer than $\mathrm{gen}(\mathcal{L})$?
How many iso. classes in $\mathrm{gen}(\mathcal{L})$? in $\mathrm{sgen}(\mathcal{L})$?

# Focus on decision module-LIP (10/10)

Fix $K = \mathbb{Q}[X]/(X^d + 1)$ and $\mathcal{L} = \mathbb{Z}_K\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + \mathbb{Z}_K\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ as in HAWK.

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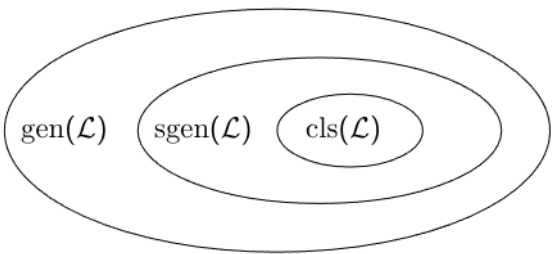
**Question 2:** Is $\mathrm{sgen}(\mathcal{L})$ really finer than $\mathrm{gen}(\mathcal{L})$?
How many iso. classes in $\mathrm{gen}(\mathcal{L})$? in $\mathrm{sgen}(\mathcal{L})$? For $d = 512$, we (heuristically) have:

$\#\{\text{iso. classes in } \mathrm{gen}(\mathcal{L})\} \approx 2^{1000}$

$\#\{\text{iso. classes in } \mathrm{sgen}(\mathcal{L})\} \approx 2^{800}$



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- **Open question:** Other computable invariants?

# Bonus: Gentry-Szydlo's algorithm (1/2)

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3. The first basis vector is $g^{p-1} \cdot v$ with $v$ **short**. Reduce it modulo $p$.

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$\rightarrow$ We have an implementation in `SageMath`!