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An explicit density estimate

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Abstract. We prove an explicit upper bound for the number of zeroes of L-functions that are below T in imaginary part and whose real part is larger than some $\sigma > 1/2$.

1. Introduction

Dirichlet L-series $L(s,\chi) = \sum_{n\geq 1} \chi(n)n^{-s}$ associated to primitive Dirichlet characters χ are one of the keys to the distribution of primes. Even the simple case $\chi = 1$ which corresponds to the Riemann zeta-function contains many informations on primes and on the Farey dissection. There have

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been many generalizations of these notions, and they all have arithmetical properties and/or applications, see [5], [62], [46], [50] or [42] for instance. Investigations concerning these functions range over many directions, see [60] on their real zeroes, [25], [43] on their moments, [16] on their values or [15] on their relation with Stickelberger elements. We present in the last section elements of the theory of automatic Dirichlet series. We note furthermore that Dirichlet characters have been the subject of numerous studies, see [7], [66], [11]; Dirichlet series in themselves are still mysterious, see [9] and [12] ; they share many properties with the factorial series, see [10],or with Taylor series, see [13], and are closely related to the theory of quasianalytic classes of functions, for a contemporary snapshot of which we refer the reader to [67]. Dirichlet series may be put in a more general motivic setting; among their important relatives, one finds L-series associated with (classical) modular forms. The special values of those have applications to the arithmetic of abelian varieties; an illustration can be found in [55].

One of the main problem concerns the location of the zeroes of these functions in the strip $0 < \Re s < 1$; the Generalized Riemann Hypothesis asserts that all of those are on the line $\Re s = 1/2$. Assuming finiteness of the set of prime numbers, this conjecture holds, but this assumption is unfortunately too restrictive, cf [30, Proposition 20].

We concentrate here on a special case, as in [61], namely the one of Dirichlet L-series, since the theory can be pushed farther in this case. We are even seeking explicit results, namely results where all implied contants are computed as in [24] and, if possible, are small enough. We concentrate in this paper on estimating

$$N(\sigma, T, \chi) = \sum_{\substack{\rho=\beta+i\gamma,\\ L(\rho,\chi)=0,\\ \sigma\le\beta,|\gamma|\le T}} 1.$$

On the generalised Riemann hypothesis, this quantity vanishes when $\sigma > 1/2$ and we want to bound it from above. An upper bound is however often very powerful, one of the more striking uses of such an estimate being surely Hoheisel Theorem. In [48, Theorem 7], the authors already prove an explicit density estimates for L-functions, namely

$$\sum_{\chi \mod q} N(\sigma, T, \chi) \le \left(\frac{254\,231}{\operatorname{Log} qT} + 17\,102\right)(q^3T^4)^{1-\sigma}(\operatorname{Log} qT)^{6\sigma} + 16\,541(\operatorname{Log} T)^6$$

under some size conditions on T and q we do not reproduce. [20] had in fact proved most of this result, but his bound had the restriction $\chi \neq \chi_0$, the principal character. This result is used in [49] to prove to show that every odd integer $\ge \exp(3\,100)$ is a sum of at most three primes. For problems on asymptotic bases, see [41], and for related problems with the primes, see [45]. Note that it would be interesting to extend other additive problems, e.g. [36], by restricting the summands to be primes. The sequence of primes has some interesting uses in the theory of Fourier series, see [54].

The correct generalization of Dirichlet L-series to $\mathbb{F}_q[T]$ is not obvious, though it is clear that the primes should be the irreducible polynomials. However, see [38], [37], there are many results in the special case where these primes appear as factors of fixed points of the σ function appropriately defined over polynomials instead of integers. See also the remark after Lemma 33.

Here is our main Theorem:

Theorem 11 *For $T \geq 2\,000$, $Q \geq 10$ and $Q \leq T$, as well as $\sigma \geq 1/2$, we have*

$$\sum_{q \leq Q} \sum_{\chi \bmod^* q} N(\sigma, T, \chi) \leq 157 (Q^5 T^3)^{1-\sigma} \operatorname{Log}^{4-\sigma}(Q^2 T) + 6 Q^2 \operatorname{Log}^2(Q^2 T)$$

where $\chi \bmod^* q$ denotes a sum over all primitive Dirichlet character χ to the modulus q.

We can reduce the 157 to $6.4 \cdot 603^{1-\sigma}$. Our result is asymptotically better in case $Q = 1$ than Ingham's, from which we borrow most of the proof, by the power of logarithm: we get the exponent $4 - \sigma$ instead of the classical 5. See [69, Theorem 9.19]. Finding an equivalent result in the case of ζ-functions steaming from more combinatorial approaches, as in [68] or [18], is an open problem.

In case $k = Q = 1$ and $\sigma = 3/4$, our estimate (we take $Q = 10$ in the Theorem above but restrict the LHS to $q = 1$) yields

$$N(3/4, T, \mathbb{1}) \leq 157 \cdot 10^{5/4} T^{3/4} \operatorname{Log}^{11/4}(100T) \leq 4517\, T^{3/4} \operatorname{Log}^{11/4} T$$

when $T \geq 2.9 \cdot 10^{10}$. The bound is also valid when $T \geq 1$ since the Riemann Hypothesis has been verified up to height $T_0 = 2.9 \cdot 10^{10}$, see [70]. For comparison, Chen/ Liu & Wang's result is useless here because of the exponent of T. Note that we can improve on our estimate when the summation is restricted to the trivial character, but we keep such an improvement for a later paper. We should however mention that, when comparing this estimate to the total number of zeroes, see Lemma 61, the above bound is not more than 1/2 this total number (and this is required because of the symetry of the zeroes with respect to $\rho \mapsto 1 - \overline{\rho}$) only when $T \geq 10^{32}$.

In passing we will prove some explicit results of independent interest, like Theorems 41 and 42.

Notations and some definitions We follow closely Ingham's proof as given in [69], paragraph 9.16 through 9.19. We extend it to cover the case of Dirichlet characters.

We consider a real parameter $X \geq 100$ and the following kernel that we use to "mollify" $L(s, \chi)$ (see [23] for instance)

$$M_X(s, \chi) = \sum_{n \leq X} \chi(n)/n^s. \tag{1}$$

We consider

$$\begin{cases} f_X(s,\chi) = M_X(s,\chi)L(s,\chi) - 1, \\ h_X(s,\chi) = 1 - f_X(s,\chi)^2 = L(s,\chi)M_X(s,\chi)(1 - L(s,\chi)M_X(s,\chi)), \\ g_X(s,\chi) = h_X(s,\chi)h_X(s,\overline{\chi}). \end{cases} \quad (2)$$

We observe that zeroes of $L(s,\chi)$ are zeroes of $h_X(s,\chi)$. We use here the fact that $M_X(s,\chi)$ is expected to be a partial inverse of $L(s,\chi)$, due to combinatorial properties of the Moebius function. We in fact needed to extract a subset of the divisors where the total weight is zero, or at least small. In the case of non-negative weights on graphs, this is the subject of [8]. See also [34], but note that the notion of harmonic Dirichlet functions defined therein is far from our Dirichlet series.

We denote by $N_1(\sigma,T,\chi)$ the zeroes ρ of $h_X(s,\chi)$ in the rectangle

$$\Re\rho \ge \sigma, \quad T \ge |\Im\rho| \quad (3)$$

to the exception of those with $\Im\rho = 0$. They are also the zeroes of $g_X(s,\chi)$ with $T \ge \Im\rho \ge 0$ and $\Re s \ge \sigma$. We define furthermore $N_1(\sigma,T_1,T_2,\chi) = N_1(\sigma,T_2,\chi) - N_1(\sigma,T_1,\chi)$ as well as

$$N_1(\sigma,T_1,T_2,Q) = \sum_{q\le Q}\sum_{\chi \bmod^* q} N_1(\sigma,T_1,T_2,\chi).$$

In the course of the proof, we shall also require

$$F(\sigma,T) = \int_{-T}^{T}\sum_{q\le Q}\sum_{\chi \bmod^* q} |f_X(\sigma+it,\chi)|^2 dt. \quad (4)$$

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2. On the size of L-functions

Lemma 21 *Let* χ *be a primitive character of conductor* $q > 1$. *For* $-\frac{1}{2} \le -\eta \le \sigma \le 1+\eta \le \frac{3}{2}$, *we have*

$$|L(s,\chi)| \le \left(\frac{q|1+s|}{2\pi}\right)^{\frac{1}{2}(1+\eta-\sigma)} \zeta(1+\eta)$$

See [58, Theorem 3]. In the same paper, Theorem 4 treats in passing the case $q = 1$, where the above bound for $q = 1$ simply has to be multiplied by $3|\frac{1+s}{1-s}|$. We can treat the term $\zeta(1+\eta)$ by using the inequality

$$\zeta(1+\eta) \le \frac{1+\eta}{\eta} \quad (5)$$

valid for $\eta > 0$. Our main application will be for $\sigma = \Re s = \frac{1}{2}$, for which we can invoke the following recent result of [21]:

Lemma 22 *For $0 \le t \le e$, we have $|\zeta(\frac{1}{2}+it)| \le 2.657$. For $t \ge e$, we have $|\zeta(\frac{1}{2}+it)| \le 3t^{1/6} \operatorname{Log} t$.*

The problem of computing Dedekind ζ-functions and Hecke L-functions is adressed inter alia in [14].

Lemma 23 *If χ is a primitive character of conductor $q \ge 1$, we have (for $T \ge 4$)*

$$\max\bigl\{|L(s,\chi)|, \Re s \ge 0, |\Im s| \le T\bigr\} \le 4.42(qT)^{5/8}.$$

PROOF : We use Lemma 21 with $\eta = 1/4$ in case $q > 1$, to get the upper bound

$$\left(\frac{q(1+\sigma+T)}{2\pi}\right)^{\frac{1}{2}(\frac{5}{4}-\sigma)} \zeta(5/4)$$

In the quotient, worst case is $\sigma = 0$. The quantity $\zeta(5/4) \le 4.6$ is trivially an upper bound in case $\Re s \ge 5/4$. In case $q = 1$, we multiply this bound by 3.001. ⋄ ⋄ ⋄

Lemma 24 *For $\sigma \ge 0$ and $|t| \le T$ where $T \ge 1000$, we have*

$$\operatorname{Log} |h_X(\sigma+it,\chi)| \le 4 \operatorname{Log}\bigl((qT)^{5/8} X\bigr) + 6.$$

PROOF : We use the preceding Lemma and get

$$|h_X(\sigma+it,\chi)| \le \Bigl(\bigl(4.42(qT)^{5/8} X\bigr)^2 + 1\Bigr)^2.$$

⋄ ⋄ ⋄

Lemma 25 *We have for $Q \ge 10$*

$$\max\bigl\{|L(\tfrac{1}{2}+it)|, \chi \bmod^* q \le Q, |t| \le T\bigr\} \le 2\,(QT)^{1/4} \operatorname{Log}(QT) + 3Q^{1/4} \operatorname{Log} Q.$$

PROOF : We use lemma 21 with $\eta = 1/\operatorname{Log}(QT)$ in case $q > 1$ and get the upper bound

$$e^{1/2} \left(\frac{q(\frac{3}{2}+T)}{2\pi}\right)^{1/4} (\operatorname{Log}(QT)+1) \le 2\,(QT)^{1/4} \operatorname{Log}(QT)$$

for $QT \ge 5$. When $QT \le 5$, then we take $\eta = 1/\operatorname{Log} Q$ and numerically check that

$$\left(1+\frac{1}{\operatorname{Log} 2}\right) e^{1/2} \left(\frac{\frac{3}{2}+T}{2\pi}\right)^{\frac{1}{4}} Q^{1/4} \operatorname{Log} Q - 2\,(QT)^{1/4} \operatorname{Log} Q \le 1.7 Q^{1/4} \operatorname{Log} Q$$

when $T \ge 0$. As for the remaining case $QT \le 5$ and $T \le 1$, we add the maximum of $-2T^{1/4} \operatorname{Log} T$ divided by $\operatorname{Log} 10$ to the coefficient of $Q^{1/4} \operatorname{Log} Q$. This readily extends to encompass case $q = 1$. ⋄ ⋄ ⋄

3. Some arithmetical lemmas

Here is a lemma from [22]:

Lemma 31 *We have, for $D \geq 1\,664$*

$$\sum_{d\leq D}\mu^2(d) = \frac{6D}{\pi^2} + \mathcal{O}^*\big(0.1333\sqrt{D}\big).$$

In particular, this is not more than $0.62D$ when $D \geq 1700$.

We shall require explicit computations that involve sums over primes (we convert products in sums via the logarithm). We shall truncate these sums and here is a handy lemma to control the error term.

Lemma 32 *Let f be a C^1 non-negative, non-increasing function over $[P,\infty[$, where $P \geq 3\,600\,000$ is a real number. We have*

$$\sum_{p\geq P} f(p)\operatorname{Log} p \leq (1+\epsilon)\int_P^\infty f(t)dt + \epsilon f(P) + Pf(P)/(5\operatorname{Log}^2 P)$$

with $\epsilon = 1/36260$. When we can only ensure $P \geq 2$, then a similar inequality holds, simply replacing the last $1/5$ by a 4.

Proof : Indeed, a summation by parts tells us that

$$\sum_{p\geq P} f(p)\operatorname{Log} p = -\int_P^\infty f'(t)\vartheta(t)dt - \vartheta(P)f(P)$$

where $\vartheta(x) = \sum_{p\leq x}\operatorname{Log} p$. At this level, we recall two results from [29, Proposition 5.1]

$$\vartheta(x) - x \leq x/36260 \quad (x > 0)$$

and Theorem 5.2 therein (these results may also be found in [28]):

$$|\vartheta(x) - x| \leq 0.2x/(\operatorname{Log}^2 x) \quad (x \geq 3\,600\,000).$$

The Lemma follows readily on applying these estimates. ◇◇◇

Lemma 33 *We have*

$$\sum_{d\leq D}\mu^2(d)\frac{\phi(d)}{d^2} = a\operatorname{Log} D + b + \mathcal{O}^*(0.174)$$

with $a = \prod_{p\geq 2}(p^3 - 2p + 1)/p^3 = 0.4282 + \mathcal{O}^*(10^{-4})$ and

$$b/a = \gamma + \sum_{p\geq 2}\frac{3p-2}{p^3-2p+1}\operatorname{Log} p = 2.046 + \mathcal{O}^*(10^{-4}).$$

See also [1, Lemme 4] to adapt this Lemma to $\mathbb{F}_q[T]$.

Proof : We appeal to [59, Lemma 3.2]. First note that

$$D(s) = \sum_{d\geq 1} \frac{\mu^2(d)\phi(d)}{d^{2+s}} = \prod_{p\geq 2}\Bigl(1+\frac{p-1}{p^{2+s}}\Bigr)$$
$$= \zeta(s)\prod_{p\geq 2}\Bigl(1-\frac{1}{p^{2+s}}-\frac{1}{p^{2+2s}}+\frac{1}{p^{3+2s}}\Bigr) = \zeta(s)H(s)$$

say. We thus get, for $D \geq 1$:

$$\sum_{d\leq D} \mu^2(d)\frac{\phi(d)}{d^2} = H(0)\,\mathrm{Log}\,D + H'(0) + \gamma H(0) + \mathcal{O}^*\bigl(c/D^{1/3}\bigr)$$

with

$$c = \prod_{p\geq 2}\Bigl(1+\frac{1}{p^{5/3}}+\frac{1}{p^{4/3}}+\frac{1}{p^{7/3}}\Bigr) \leq 6$$

and

$$a = H(0) = \prod_{p\geq 2}\frac{p^3-2p+1}{p^3} = 0.4282 + \mathcal{O}^*(10^{-4}).$$

Furthermore

$$\frac{H'(0)}{H(0)} = \sum_{p\geq 2}\frac{3p-2}{p^3-2p+1}\,\mathrm{Log}\,p = 1.4695 + \mathcal{O}^*(10^{-4})$$

We use the following Sage program, see [64], since it implements interval arithmetic from [33]:

```
R = RealIntervalField(64)

def g(n):
    res = 1
    l = factor(n)
    for p in l:
        if p[1] > 1:
            return R(0)
        else:
            res *= (p[0]-1)/p[0]^2
    return R(res)

P = 10000
aaa = R(1)
p = 2
while p <= P:
    aaa *= R(1-2/p^2+1/p^3)
    p = next_prime (p)
eps = 1/R(36260)
x = 3*(1+eps)/R(P)/log(R(P))+3*eps/R(P)^2/log(R(P))+3/4/R(P)/log(R(P))^3
```

```
x = exp(-x)
aaa = aaa * x.union(R(1))

P = 100000
bbb = R(0)
p = 2
while p <= P:
    bbb += R((3*p-2)/(p^3-2*p+1))*log(R(p))
    p = next_prime (p)
x = (log(R(P))+1)/R(P)
bbb = bbb + x.union(R(0)) + R(euler_gamma)

ccc = R(6)

def model(z):
    return aaa * (log(R(z)) + bbb)

def getbounds (zmin, zmax):
    zmin = max (0, floor (zmin))
    zmax = ceil (zmax)
    res = R(0)
    for n in range (1, zmin + 1):
        res += g(n)
    maxi = abs(res - model (zmin)).upper()
    maxiall = maxi
    for n in xrange (zmin + 1, zmax + 1):
        m = model (n)
        maxi = max (maxi, abs(res - m).upper())
        res += g(n)
        maxi = max (maxi, abs(res - m).upper())
        if n % 100000 == 0:
            print "Upto ", n, " : ", maxi, cputime()
            maxiall = max (maxiall, maxi)
            maxi = R(-1000).upper()
    maxi = max (maxi, abs (res - model (zmax)).upper())
    maxiall = max (maxiall, maxi)
    print "La borne pour z >= ", zmax, " : "
    bound = ccc/R(zmax)^(1/3)
    print bound.upper()
    return [maxiall, maxi]

sage: getbounds(10, 3000000)
La borne pour z >=  3000000  :
...
0.0416016764610380824
[0.0532695418028642758, 0.000185953301713212994]
```

to show that

$$\Big|\sum_{d\le D}\mu^2(d)\frac{\phi(d)}{d^2} - a\,\mathrm{Log}\,D - b\Big| \le 0.0533$$

when $10 \le D \le 3\,000\,000$. The conclusion is easy. ⋄ ⋄ ⋄

Lemma 34 *For $N \ge 1$, we have*

$$\frac{6}{\pi^2}\,\mathrm{Log}\,N + 0.578 \le \sum_{n\le N}\mu^2(n)/n \le \frac{6}{\pi^2}\,\mathrm{Log}\,N + 1.166$$

A similar lemma occurs in [63], but with worst constants.

Proof : We proceed as above and get

$$\sum_{n\le N}\mu^2(n)/n = \frac{6}{\pi^2}\big(\mathrm{Log}\,N + 2\sum_{p\ge 2}\frac{\log p}{p^2-1} + \gamma\big) + \mathcal{O}^*(3/N^{1/3}).$$

A similar script as in the previous Lemma yields

$$\Big|\sum_{d\le D}\frac{\mu^2(d)}{d^2} - \frac{6}{\pi^2}\,\mathrm{Log}\,D - b'\Big| \le 0.0695$$

when $10 \le D \le 200\,000$. We present here an easier GP script, see [65], to extend it. Though such a script is usually enough (by which we mean, its result can in most examples be certified by Sage as in the previous Lemma), only the program using MPFR handles correctly the error term.

```
{g(n) =
   my(res = 1.0, dec = factor(n), P = dec[,1], E = dec[,2]);
   for(i = 1, #P,
      my(p = P[i]);
      if(E[i] != 1, return(0));
      res *= 1/p);
   return(res);}

aaa = 6/Pi^2;
bbb = 1.7171176851;
ccc = 3;

{model(z)=aaa*(log(z)+bbb)}

{getsidedbounds(zmin,zmax)=
   my(res = 0.0, m, maxiplus, maximinus, maxiplusall, maximinusall);
   zmin = max( 0, floor(zmin));
   zmax = ceil(zmax);
   for(n=1, zmin, res += g(n));
   m = model(zmin);
```

```
   maxiplus = res - m;
   maxiplusall = maxiplus;
   maximinus = res - m;
   maximinusall = maximinus;
   for(n = zmin+1, zmax,
      m = model(n);
      maxiplus = max(maxiplus, res-m);
      maximinus = min(maximinus, res-m);
      res += g(n);
      maxiplus = max(maxiplus, res-m);
      maximinus = min(maximinus, res-m);

      if(n%100000==0,
         print("Upto ",n," : ", maximinus, " / ", maxiplus);
         maxiplusall = max(maxiplusall, maxiplus);
         maximinusall = min(maximinusall, maximinus);
         maxiplus = -1000;
         maximinus = 1000));
   m = model(zmax);
   maxiplus = max(maxiplus, res - m);
   maxiplusall = max(maxiplusall, maxiplus);
   maximinus = min(maximinus, res - m);
   maximinusall = min(maximinusall, maximinus);
   print("La borne pour z >= ", zmax, " : ", ccc/zmax^(1/3));
   return( [maximinusall, maximinus, maxiplusall, maxiplus]);
}
```

which ensures us that

$$-0.466 \leq \sum_{d\leq D} \mu^2(d)\frac{\phi(d)}{d^2} - \frac{6}{\pi^2}\,\mathrm{Log}\,D - b' \leq 0.122.$$

The conclusion is easy. ⋄ ⋄ ⋄

Here is a handy lemma taken from [40].

Lemma 35 *We have uniformly for real $N \geq 1$ and integer d*

$$\Big|\sum_{\substack{n\leq N,\\ (n,d)=1}} \mu(n)/n\Big| \leq 1.$$

Lemma 36 *We have, for $X \geq 1700$,*

$$\sum_{1<n\leq N}\Big(\sum_{\substack{d|n,\\ d\leq X}} \mu(d)\Big)^2 \leq 2N\Big(\frac{12}{\pi^2}\,\mathrm{Log}\,\frac{N}{X} + 0.6\Big)\Big(\frac{6}{\pi^2}\,\mathrm{Log}\,\frac{N}{X} + 0.6\Big)$$

and

$$\sum_{1<n\le N}\Bigl(\sum_{\substack{d|n,\\ d\le X}}\mu(d)\Bigr)^2 \le 0.43N\operatorname{Log}X + 0.88N + 0.39X^2.$$

It is shown in [27] that this sum is in fact of size N.

Proof : Call $S(N)$ the sum to be studied. For $N\le X^2$, we proceed as follows:

$$\begin{aligned} S(N) &= \sum_{1<n\le N}\Bigl(\sum_{\substack{d|n,\\ d>X}}\mu(d)\Bigr)^2 \\ &= \sum_{\substack{X<d_1,d_2\le N,\\ [d_1,d_2]\le N}}\frac{\mu(d_1)\mu(d_2)N}{[d_1,d_2]} + \mathcal{O}^*\Bigl(\sum_{\substack{X<d_1,d_2\le N\\ [d_1,d_2]\le N}}\mu^2(d_1)\mu^2(d_2)\Bigr). \end{aligned}$$

Let us denote by R the error term above. We have

$$\begin{aligned} R &= \sum_{X^2/N<d\le N}\ \sum_{\substack{X<d_1,d_2\le N,\\ (d_1,d_2)=d,\\ d_1d_2\le dN}}\mu^2(d_1)\mu^2(d_2) \\ &\le \sum_{X^2/N<d\le N}\mu^2(d)\sum_{\substack{X/d<\ell_1\le N/d,\\ (\ell_1,d)=1}}\mu^2(\ell_1)\sum_{\substack{X/d<\ell_2\le N/(d\ell_1),\\ (\ell_2,\ell_1 d)=1}}\mu^2(\ell_2) \\ &\le N\sum_{X^2/N<d\le N}\mu^2(d)\sum_{\substack{X/d<\ell_1\le N/d,\\ (\ell_1,d)=1}}\frac{\mu^2(\ell_1)}{d\ell_1} \\ &\le N\Bigl(\frac{12}{\pi^2}\operatorname{Log}\frac{N}{X}+0.6\Bigr)\Bigl(\frac{6}{\pi^2}\operatorname{Log}\frac{N}{X}+0.6\Bigr). \end{aligned}$$

The last estimate comes from Lemma 34. As for the main term TP, we proceed in a slightly different fashion

$$\begin{aligned} TP &= N\sum_{X^2/N<d\le N}\frac{\mu^2(d)\phi(d)}{d^2}\Bigl(\sum_{\substack{X/d<\ell\le N/d,\\ (\ell,d)=1}}\frac{\mu(\ell)}{\ell}\Bigr)^2 \\ &\le N\sum_{X^2/N<d\le N}\frac{\mu^2(d)\phi(d)}{d^2} \le N(0.85\operatorname{Log}(N/X)+0.35) \end{aligned}$$

by Lemma 33 and 35. Hence

$$S(N)\le 2N\Bigl(\frac{12}{\pi^2}\operatorname{Log}\frac{N}{X}+0.6\Bigr)\Bigl(\frac{6}{\pi^2}\operatorname{Log}\frac{N}{X}+0.6\Bigr).$$

For large N, it would be better to open up and write

$$\begin{aligned} S(N) &= \sum_{d_1,d_2\le X} \frac{\mu(d_1)\mu(d_2)(N-X)}{[d_1,d_2]} + \mathcal{O}^*((0.62X)^2) \\ &= (N-X)\sum_{d\le X}\frac{\mu^2(d)\phi(d)}{d^2}\biggl(\sum_{\substack{n\le X/d,\\(n,d)=1}}\mu(n)/n\biggr)^2 + \mathcal{O}^*((0.62X)^2) \\ &\le (N-X)\sum_{d\le X}\frac{\mu^2(d)\phi(d)}{d^2} + (0.62X)^2 \\ &\le 0.43N\operatorname{Log} X + 0.88N + 0.39X^2 \end{aligned}$$

by invoking Lemma 31, using Selberg's diagonalization process and Lemma 35. This is better than the above when $N \ge X^2$. ⋄⋄⋄

Lemma 37 *For $\sigma > 1$ and $X \ge 10^5$, we have*

$$\sum_{X<n}\frac{\Bigl(\sum_{\substack{d|n,\\d\le X}}\mu(d)\Bigr)^2}{\sigma n^\sigma} \le \frac{1.2\operatorname{Log}^3 X}{X^{\sigma-1}} + \frac{0.51\operatorname{Log} X}{(\sigma-1)X^{2\sigma-2}} + \frac{0.4}{\sigma X^{2\sigma-2}}.$$

Proof : Let $G(\sigma)$ be our sum. We first use an integration by parts:

$$G(\sigma) = \sigma\int_X^\infty \sum_{X<n\le y}\biggl(\sum_{\substack{d|n,\\d\le X}}\mu(d)\biggr)^2\frac{dy}{y^{1+\sigma}}$$

and appeal to Lemma 36 to write

$$\begin{aligned} \frac{G(\sigma)}{\sigma} &\le 2\int_X^Y\Bigl(\frac{12}{\pi^2}\operatorname{Log}\frac{y}{X}+0.6\Bigr)\Bigl(\frac{6}{\pi^2}\operatorname{Log}\frac{y}{X}+0.6\Bigr)\frac{dy}{y^\sigma} \\ &\quad+\int_Y^\infty\Bigl(0.43\operatorname{Log} X + 0.88 + 0.39X^2y^{-1}\Bigr)\frac{dy}{y^\sigma} \\ &\le 2X^{1-\sigma}\int_1^{Y/X}\Bigl(1.48\operatorname{Log}^2 u + 2.19\operatorname{Log} u + 0.36\Bigr)\frac{du}{u^\sigma} \\ &\quad+\frac{0.43\operatorname{Log} X + 0.88}{(\sigma-1)Y^{\sigma-1}} + \frac{0.39X^2}{\sigma Y^\sigma}. \end{aligned}$$

We set $v = u^{1-\sigma}$ and get

$$\begin{aligned} &\int_1^{Y/X}\Bigl(1.48\operatorname{Log}^2 u + 2.19\operatorname{Log} u + 0.36\Bigr)\frac{du}{u^\sigma} \\ &\quad\le \int_{(X/Y)^{\sigma-1}}^1\Bigl(1.48\frac{\operatorname{Log}^2 v}{(\sigma-1)^2} - 2.19\frac{\operatorname{Log} v}{\sigma-1} + 0.36\Bigr)\frac{dv}{(\sigma-1)v} \\ &\qquad\le \frac{1.48}{3}\operatorname{Log}^3(Y/X) + \frac{2.19}{2}\operatorname{Log}^2(Y/X) + 0.36\operatorname{Log}(Y/X). \end{aligned}$$

We choose $Y = X^2$ and get the Lemma. ⋄⋄⋄

4. Large sieve estimates and the like

We first need an explicit version of a theorem of Gallagher (this is [35, Lemma 1], see also [19, Theorem 9]).

Theorem 41 *Let $c > 1$ be a real parameter. With $\tau = e^{2\pi/(cT)}$, we have*

$$\int_{-T}^{T} \Big|\sum_n a_n n^{it}\Big|^2 dt \le \frac{\pi^2}{\sin(\pi/c)^2} T^2 \int_0^\infty \Big|\sum_{y<n\le \tau y} a_n\Big|^2 dy/y$$

Proof : We use Parseval identity to derive and get

$$\int_{-\infty}^{\infty} \Big|\sum_n a_n e^{2i\pi t(\operatorname{Log} n)/(2\pi)} \frac{\sin \pi\delta t}{\pi\delta t}\Big|^2 dt = \int_{-\infty}^{\infty} \Big|\sum_{|\operatorname{Log} n - 2\pi x|\le \pi\delta} a_n \delta^{-1}\Big|^2 dx.$$

We recall that $(\sin \pi\delta t)/(\pi\delta t)$ is non-increasing for $|t| \le 1$ from which we infer

$$\left(\frac{\sin(\pi/c)}{\pi/c}\right)^2 \int_{-(c\delta)^{-1}}^{(c\delta)^{-1}} \Big|\sum_n a_n n^{it}\Big|^2 \le \int_0^\infty \Big|\delta^{-1} \sum_{ze^{-\pi\delta}<n\le e^{\pi\delta}z} a_n\Big|^2 dz/z$$
$$\le \int_0^\infty \Big|\delta^{-1} \sum_{y<n\le e^{2\pi\delta}y} a_n\Big|^2 dy/y$$

with $2\pi x = \operatorname{Log} z$. We take $\delta = 1/(cT)$ and get the result. ◇◇◇

Here is the classical large sieve inequality for primitive characters (see [53]):

Lemma 41 *We have*

$$\sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi \bmod^* q} \Big|\sum_{1\le n\le N} b_n\chi(n)\Big|^2 \le (N-1+Q^2)\sum_n |b_n|^2.$$

We note here that this inequality relies on bounding the largest eigenvalues

Theorem 42 *We have, for $T \ge 2\pi/c$ and any $c > 1$:*

$$\sum_{q\le Q}\sum_{\chi \bmod^* q} \int_0^T \Big|\sum_n a_n\chi(n)n^{it}\Big|^2 dt \le \pi\left(\frac{\pi/c}{\sin(\pi/c)}\right)^2 \sum_n |a_n|^2(7n + cQ^2T).$$

Proof : We use Theorem 41 together with Lemma 41:

$$\sum_{q\le Q}\sum_{\chi \bmod^* q}\int_{-T}^{T}\left|\sum_n a_n\chi(n)n^{it}\right|^2 dt$$
$$\le \frac{\pi^2T^2}{\sin^2(\pi/c)}\int_0^\infty \sum_{q\le Q}\sum_{\chi \bmod^* q}\left|\sum_{y<n\le\tau y} a_n\chi(n)\right|^2 dy/y$$
$$\le \frac{\pi^2T^2}{\sin^2(\pi/c)}\int_0^\infty \sum_{y<n\le\tau y}|a_n|^2[(\tau-1)y+Q^2]dy/y$$
$$\le \frac{\pi^2T^2}{\sin^2(\pi/c)}\sum_{n\ge1}|a_n|^2\int_{n/\tau}^{n}[(\tau-1)y+Q^2]dy/y$$

which reads

$$\frac{\pi^2T^2}{\sin^2(\pi/c)}\sum_{n\ge1}|a_n|^2\big((\tau+\tau^{-1}-2)n+Q^2\operatorname{Log}\tau\big).$$

We conclude the proof by noticing that $(e^x+e^{-x}-2)\le\frac{11}{10}x^2$ when $|x|\le1$. ⋄ ⋄ ⋄

Lemma 42 *We have, for $X\ge 2\,000$ and $T\ge0$,*

$$\sum_{q\le Q}\sum_{\chi \bmod^* q}\int_0^T|M_X(\tfrac12+it,\chi)|^2dt\le(10.4Q^2T+1.8X)\operatorname{Log}X.$$

Proof : From Theorem 42 and one using Lemma 34 and 31, we readily get the upper bound

$$\pi\left(\frac{\pi/c}{\sin(\pi/c)}\right)^2\sum_{n\le X}\frac{\mu^2(n)}{n}(7n+cQ^2T)$$
$$\le\pi\left(\frac{\pi/c}{\sin(\pi/c)}\right)^2 cQ^2T\Big(\frac{6}{\pi^2}\operatorname{Log}X+1.17\Big)+\pi\left(\frac{\pi/c}{\sin(\pi/c)}\right)^2 0.62X.$$

We take for π/c the root of $\tan t-2t$ in $]1,1.5[$, namely $c=2.6953+\mathcal{O}^*(10^{-4})$. This leads to the bound

$$13.63Q^2T\Big(\frac{6}{\pi^2}\operatorname{Log}X+1.17\Big)+3.14X\le10.4Q^2T\operatorname{Log}X+3.14X.$$

This bound is valid for $T\ge2\pi/c$. For smaller T, we use directly

$$\sum_{q\le Q}\sum_{\chi \bmod^* q}\int_0^T|M_X(\tfrac12+it,\chi)|^2dt\le T\sum_{n\le X}\frac{\mu^2(n)}{n}(X+Q^2).$$

⋄ ⋄ ⋄

Lemma 43 *We have, for $X \geq 2\,000$, $Q \geq 10$ and $T \geq 0$,*

$$\tfrac12 F(1/2,T) \leq 20.9 Q^{1/2}(Q^2T + 0.18X)(2T^{1/4}\operatorname{Log}(QT) + 3\operatorname{Log} Q)^2 \operatorname{Log} X.$$

Note that it is important that this Lemma should hold for small T's as well. The method developped here is of course very elementary since we want to be able to compute all the involved constants, and has nothing in common with the technology developped for instance in [26].

Proof : On using (4) and the inequality $|z_1+z_2|^2 \leq 2(|z_1|^2+|z_2|^2)$, we readily see that

$$\tfrac12 F(1/2,T) \leq 2\sum_{q\leq Q}\sum_{\chi \bmod^* q}\int_0^T |M_X(\tfrac12+it,\chi)|^2 dt \max_{\substack{q\leq Q,\\ \Re s=\frac12}} |L(s,\chi)|^2 + 2Q^2T.$$

so that, on appealing to Lemma 25 and 42, we reach the upper bound

$$20.9Q^{1/2}(Q^2T + 0.18X)(2T^{1/4}\operatorname{Log}(QT) + 3\operatorname{Log} Q)^2 \operatorname{Log} X.$$

⋄ ⋄ ⋄

Lemma 44 *We have, for $\delta = 1/\operatorname{Log} X$, $Q^2 \leq X/2000$, $X \geq 10^5$ and $T \geq 0$,*

$$\tfrac12 F(1+\delta,T) \leq 5.4(1+Q^2TX^{-1})\operatorname{Log}^3 X.$$

Proof : We readily get from (4) and Theorem 42 the upper bound

$$\alpha_c \sum_{X<n}\Bigg(\sum_{\substack{d|n,\\ d\leq X}}\mu(d)\Bigg)^2 n^{-2-2\delta}(7n + cQ^2T)$$

with $\alpha_c = \pi\,(\pi/(c\sin(\pi/c)))^2$ and for $T \geq 2\pi/c$. This is not more by Lemma 37 than

$$c\alpha_c(2+2\delta)\frac{Q^2T}{X^{1+2\delta}}\Big(1.2\operatorname{Log}^3 X + \frac{0.51\operatorname{Log} X}{(1+2\delta)X^{1+2\delta}} + \frac{0.4}{(2+2\delta)X^{1+2\delta}}\Big)$$
$$+ \frac{7\alpha_c}{X^{2\delta}}(1+2\delta)\Big(1.2\operatorname{Log}^3 X + \frac{0.51\operatorname{Log} X}{2\delta X^{2\delta}} + \frac{0.4}{(1+2\delta)X^{2\delta}}\Big)$$

which is in turn bounded above by

$$\frac{Q^2Tc\alpha_c}{e^2X}\operatorname{Log}^3 X\Big(2.4 + 2.4\delta + \frac{(2+2\delta)0.51\delta^2}{(1+2\delta)e^2X} + \frac{0.4\delta^3}{e^2X}\Big)$$
$$+ \frac{7\alpha_c}{e^2}\operatorname{Log}^3 X\Big(1.2 + 2.4\delta + \frac{0.51(1+2\delta)\delta^2}{2e^2} + \frac{0.4\delta^3}{e^2}\Big)$$

which is not more than

$$\Big(0.354\frac{Q^2T}{X}c + 1.34\Big)\alpha_c \operatorname{Log}^3 X$$

We select $c = 3.731$. We now should extend this estimate to cover the case of smaller T's. The quantity $\frac{1}{2}F(1+\delta, T)$ is simply bounded above by $\frac{1}{2}F(1+\delta, 2\pi/c)$ which is thus not more than

$$5.39(1 + Q^2(2\pi/c)X^{-1})\operatorname{Log}^3 X \le 5.4\operatorname{Log}^3 X \le 5.4(1 + Q^2 T X^{-1})\operatorname{Log}^3 X.$$

The proof is complete. ◇◇◇

5. Computing some values of Γ'

We shall require values of Γ and Γ' at special points. Most of them are tabulated in [2], but the value of $\Gamma'(5/4)$ is missing. We computed Γ' via $\Gamma'(s) = \psi(s)\Gamma(s)$, where ψ is the Digamma function. It is also given by

$$\psi(s+1) = -\gamma + \int_0^1 \frac{1-x^s}{1-x}dx.$$

When $s = k/n$, we introduce $x = u^n$, so that

$$\begin{aligned}\psi((k+n)/n) &= -\gamma + n\int_0^1 \frac{1-u^k}{1-u^n}u^{n-1}du \\ &= -\gamma + n\int_0^1 \frac{1+u+\cdots+u^{k-1}}{1+u+\cdots+u^{n-1}}u^{n-1}du\end{aligned}$$

where the integrand has no singularity left in the considered range. See [17] on the evolution of this singularity and [57] as well as [39] on the complexity of this computation. We can also get a closed formula by using a partial fraction decomposition. By using [65], we got

$$\psi(5/4) = -0.4897 + \mathcal{O}^*(10^{-4}). \tag{6}$$

6. On the total number of zeroes

Here is a lemma we took from [51].

Lemma 61 *If χ is a Dirichlet character of conductor k, if $T \ge 1$ is a real number, and if $N(T,\chi)$ denotes the number of zeros $\beta + i\gamma$ of $L(s,\chi)$ in the rectangle $0 < \beta < 1$, $|\gamma| \le T$, then*

$$\left|N(T,\chi) - \frac{T}{\pi}\operatorname{Log}\left(\frac{qT}{2\pi e}\right)\right| \le C_2 \operatorname{Log}(qT) \ + C_3$$

with $C_2 = 0.9185$ and $C_3 = 5.512$.

In particular, we have when $Q \ge 10$

$$\begin{aligned}\sum_{q\le Q}\sum_{\chi \bmod^* q} N(6,\chi) &\le \frac{6Q^2}{\pi}\operatorname{Log}\frac{6Q}{2\pi e} + Q^2(0.92\operatorname{Log}(6Q) + 5.6) \\ &\le 4.81 Q^2 \operatorname{Log} Q.\end{aligned} \tag{7}$$

7. A convexity argument

General principle To evaluate $\int_{T_1}^{T_2}\sum_{q\le Q}\sum_{\chi \bmod^* q}|f_X(\sigma_0+it,\chi)|^2dt$, we use a slight extension convexity argument due to [44]. We first are to evaluate this integral in $\frac{1}{2}$ and in $1+\delta$. We set

$$\Phi(s)=\frac{s-1}{s(\cos s)^{1/(2\tau)}}\quad \Re s\in[\tfrac{1}{2},1+\delta] \tag{8}$$

for some parameter $\tau\geq 1\,000$ that we will at the end take to be T_2. Here $\delta=1/(Q^2T_2)$. Of course $\cos s$ does not vanish in the strip we consider. We readily find that $\Phi(s)f_X(s,\chi)=o(1)$ uniformly in $\Re s$ and as $|\Im s|$ goes to infinity. Let us set

$$a=\frac{1+\delta-\sigma}{1+\delta-\frac{1}{2}}\quad,\quad b=\frac{\sigma-\frac{1}{2}}{1+\delta-\frac{1}{2}}. \tag{9}$$

A slight extension of the Hardy-Ingham-Pólya inequality reads

$$\mathfrak{M}(\sigma)\leq \mathfrak{M}(1/2)^a\mathfrak{M}(1+\delta)^b \tag{10}$$

with

$$\mathfrak{M}(\sigma)=\int_{-\infty}^{\infty}\sum_{q\le Q}\sum_{\chi \bmod^* q}|\Phi(\sigma+it)f_X(\sigma+it,\chi)|^2dt. \tag{11}$$

The extension comes from the fact that we have added a summation over characters instead of considering a single function.

Proof : Indeed we follow closely [69, section 7.8] and set

$$\phi(z,\chi)=\frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi(z)f_X(z,\chi)z^{-z}dz \qquad (\sigma\geq 1/2, |\arg z|<\pi/2). \tag{12}$$

Setting $z=ixe^{-i\delta}$ with $0<\delta<\pi/2$, we readily see that

$$\Phi(\sigma+it)f_X(\sigma+it,\chi)e^{-i(\sigma+it)(\frac{1}{2}\pi-\delta)}\quad\text{and}\quad \phi(ixe^{-i\delta},\chi)$$

are Mellin transforms. Using Parseval's formula and Hölder's inequality, we obtain:

$$\begin{aligned}\mathfrak{M}(\sigma)&=2\pi\int_0^\infty\sum_{q\le Q}\sum_{\chi \bmod^* q}|\phi(ixe^{-i\delta},\chi)|^2x^{2\sigma-1}dx\\ &\leq 2\pi\Big(\int_0^\infty\sum_{\substack{q\le Q,\\ \chi \bmod^* q}}|\phi(ixe^{-i\delta},\chi)|^2dx\Big)^a\Big(\int_0^\infty\sum_{\substack{q\le Q,\\ \chi \bmod^* q}}|\phi(ixe^{-i\delta},\chi)|^2x^{1+2\delta}dx\Big)^b\\ &\leq \mathfrak{M}(1/2)^a\mathfrak{M}(1+\delta)^b.\end{aligned}$$

◇◇◇

We now exploit inequality (10) of [69, section 7.8]. We bound above the RHS of (10) via

$$\mathfrak{M}(\sigma) \leq \left(\frac{2}{\cos\sigma}\right)^{1/\tau} \int_{-\infty}^{\infty} e^{-|t|/\tau} \sum_{q\leq Q} \sum_{\chi \bmod^* q} |f_X(\sigma+it,\chi)|^2 dt.$$

On recalling (4), we see that an integration by parts give us

$$\begin{aligned}\mathfrak{M}(\sigma) &\leq \left(\frac{2}{\cos\sigma}\right)^{1/\tau} \int_0^{\infty} e^{-T/\tau} F(\sigma,T)dT/\tau \\ &\leq \left(\frac{2}{\cos\sigma}\right)^{1/\tau} \int_0^{\infty} e^{-t} F(\sigma,t\tau)dt\end{aligned}$$

Lemma 71 *We have, when $T \geq 2\,000$, $Q \geq 10$ and $Q \leq T$, and on selecting $X = Q^2T$ and $\tau = T$,*

$$\mathfrak{M}(1/2) \leq 270(Q^5T^3)^{1/2}\operatorname{Log}^2(Q^2T).$$

Assuming T to be larger would not save much here, the best constant achievable via the proof below being 145.83 instead of 270.

Proof : We appeal to Lemma 43 to infer that $\mathfrak{M}(1/2)/(21\sqrt{Q}\operatorname{Log} X)$ is bounded above by

$$\int_0^{\infty} \mathfrak{N}(t)e^{-t}dt$$

where

$$\begin{aligned}\mathfrak{N}(t) = (4\tau^{3/2}Q^2)t^{3/2}\operatorname{Log} t + (4Q^2\tau^{3/2}\operatorname{Log}(Q\tau))t^{3/2} \\ + (6Q^2\tau^{5/4}\operatorname{Log} Q)t^{5/4}\operatorname{Log} t + (6Q^2\tau^{5/4}\operatorname{Log} Q\operatorname{Log}(Q\tau))t^{5/4} \\ + (9Q^2\tau\operatorname{Log} Q)t + (4\cdot 0.18\cdot X\tau^{1/2})t^{1/2}\operatorname{Log} t \\ + (4\cdot 0.18\cdot X\tau^{1/2}\operatorname{Log}(Q\tau))t^{1/2} + (12\cdot 0.18\cdot X\tau^{1/4}\operatorname{Log} Q)t^{1/4}\operatorname{Log} t \\ + (12\cdot 0.18\cdot X\tau^{1/4}\operatorname{Log} Q\operatorname{Log}(Q\tau))t^{1/4} + 9\cdot 0.18\cdot X\operatorname{Log}^2 Q.\end{aligned}$$

Note that in this proof, we keep X and τ independant of T and Q until the integration has been done. On using values of Γ or of Γ' (see section 5), we get the bound

$$\begin{array}{ll} 2.71(4\tau^{3/2}Q^2) & +1.33(4Q^2\tau^{3/2}\operatorname{Log}(Q\tau)) \\ +0.650(6Q^2\tau^{5/4}\operatorname{Log} Q) & +1.14(6Q^2\tau^{5/4}\operatorname{Log} Q\operatorname{Log}(Q\tau)) \\ +(9Q^2\tau\operatorname{Log} Q) & +0.327(0.72X\tau^{1/2}) \\ +0.887(0.72X\tau^{1/2}\operatorname{Log}(Q\tau)) & -0.443(2.16X\tau^{1/4}\operatorname{Log} Q) \\ +0.907(2.16X\tau^{1/4}\operatorname{Log} Q\operatorname{Log}(Q\tau)) & +1.62X\operatorname{Log}^2 Q. \end{array}$$

We now take $X = Q^2 T$ and $\tau = T$ to get

$$\mathfrak{M}(1/2)/\operatorname{Log}^2(Q^2T) \leq \begin{array}{ll} 19.9 Q^{5/2}T^{3/2} & +112 Q^{5/2}T^{3/2} \\ +82 Q^{5/2}T^{5/4} & +142 Q^{5/2}T^{5/4}\operatorname{Log} Q \\ +189 Q^{5/2}T & +0.43 Q^{5/2}T^{3/2} \\ +13.5 Q^{5/2}T^{3/2} & \\ +41.2 Q^{5/2}T^{5/4}\operatorname{Log} Q & +17.1 Q^2 T \operatorname{Log} Q \end{array}$$

which simplifies into (with $Q \leq T$) the claimed quantity. ⋄⋄⋄

An upper bound for $\mathfrak{M}(1+\delta)$ We appeal to Lemma 44 to infer that

$$\begin{aligned}\mathfrak{M}(1+\delta) &\leq \int_0^\infty 5.4\left(1 + Q^2 t\tau X^{-1}\right) e^{-t} dt \operatorname{Log}^3 X \\ &\leq 5.4\big(1 + \frac{Q^2\tau}{X}\big)\operatorname{Log}^3 X \leq 11 \operatorname{Log}(Q^2T).\end{aligned}$$

An upper bound for $\mathfrak{M}(\sigma)$ We thus conclude that (note that $b = 1 - a$)

$$\begin{aligned}\mathfrak{M}(\sigma) &\leq \Big(270 (Q^5T^3)^{1/2}\operatorname{Log}^2(Q^2T)\Big)^a \left(11 \operatorname{Log}^3(Q^2T)\right)^b \\ &\leq 11\,(270/11)^a\,(Q^5T^3)^{a/2}\operatorname{Log}^{3-a}(Q^2T).\end{aligned}$$

We note that $\sqrt{Q^5T^3}/\operatorname{Log}(Q^2T) = \sqrt{QT}Q^2T/\operatorname{Log}(Q^2T)$ where $Q^2T/\operatorname{Log}(Q^2T) \geq 1$. The exponent a is maximal when $\delta = 0$ and thus

$$\boxed{\mathfrak{M}(\sigma) \leq 11\big(603 Q^5T^3\big)^{1-\sigma}\operatorname{Log}^{1+2\sigma}(Q^2T).}$$

An upper bound for $\int_{T_1 \leq |t| \leq T_2}\sum_q\sum_\chi |f_X(\sigma+it,\chi)|^2 dt$ We simply note that

$$\mathfrak{M}(\sigma) \geq \frac{\big(1 - \frac{1}{1000}\big)^2}{(\cosh\tau)^{1/\tau}} \int_{T_1 \leq |t| \leq T_2} \sum_{q \leq Q} \sum_{\chi \bmod^* q} |f_X(\sigma+it,\chi)|^2 dt.$$

The coefficient is ≥ 0.367 when we choose $\tau = T \geq 2000$.

8. The zero detection Lemma and proof of Theorem 11

For $\sigma_0 \in [\frac{1}{2}, 1]$, a Lemma of [47], reproduced in [69, section 9.9], gives us

$$\begin{aligned}2\pi \int_{\sigma_0}^2 N_1(\sigma, T_1, T_2, \chi) d\sigma &= \int_{T_1}^{T_2} \big(\operatorname{Log}|g_X(\sigma_0 + it, \chi)| - \operatorname{Log}|g_X(2 + it, \chi)|\big) dt \\ &+ \int_{\sigma_0}^2 \big(\arg g_X(\sigma + iT_2, \chi) - \arg g_X(\sigma + iT_1, \chi)\big) d\sigma \qquad (13)\end{aligned}$$

where $\arg g_X(s,\chi)$ is taken to be 0 on the line $\Re s = 2$.

We study the first integral by noticing that

$$\operatorname{Log}|h_X(s,\chi)| \leq \operatorname{Log}(1+|f_X(s,\chi)|^2) \leq |f_X(s,\chi)|^2. \tag{14}$$

On another hand, we have

$$-\operatorname{Log}|h_X(2+it,\chi)| \leq -\operatorname{Log}(1-|f_X(2+it,\chi)|^2 \leq 2|f_X(2+it,\chi)|^2 \tag{15}$$

provided $|f_X(2+it,\chi)|^2 \leq 1/2$ which we prove now:

$$\begin{aligned}|f_X(2+it,\chi)| &\leq \sum_{n\geq X} \frac{|\sum_{d|n}\mu(d)|}{n^2} \leq \sum_{n\geq X}\frac{2^{\omega(n)}}{n^2}\\ &\leq \sqrt{8/3}\sum_{n\geq X}\frac{1}{n^{3/2}} \leq \frac{2\sqrt{8/3}}{(X-1)^{1/2}} \leq 0.462 \leq 1/\sqrt{2}\end{aligned}$$

since $X \geq 100$ and $2^{\omega(n)} \leq \sqrt{8/3}\sqrt{n}$ (use multiplicativity).

Getting an upper bound for the argument is more tricky and relies on the following Lemma from [69, section 9.4]:

Lemma 81 *Let $0 \leq \alpha < \beta \leq 2$ and F be an analytical function, real for real s, holomorphic for $\sigma \geq \alpha$ except maybe at $s = 1$. Let us assume that $|\Re F(2+it)| \geq m > 0$ and that $|F(\sigma'+it')| \leq M$ for $\sigma' \geq \sigma$ and $T \geq t' \geq T_0 - 2$. Then, if $T-2 \geq T_0$ is not the ordinate of a zero of $F(s)$, we have*

$$|\arg F(\sigma+iT)| \leq \frac{\pi}{\operatorname{Log}\frac{2-\alpha}{2-\beta}}\operatorname{Log}\big(M/m\big) + \frac{3\pi}{2}$$

valid for $\sigma \geq \beta$.

The condition concerning the *ordinate* comes from the way we define the logarithm, and hence the argument. It is usually harmless since one can otherwise argue by continuity.

We use this lemma with $\alpha = 0$, $\beta = 1/2$ and $F = g_X(s,\chi)$ which is indeed real on the real axis. We already showed that

$$|\Re g_X(2+it,\chi)| \geq (1-|f_X(2+it,\chi)^2|)(1-|f_X(2+it,\overline{\chi})^2|) \geq (1-0.214^2)^2 \geq 0.91.$$

Hence, for $j = 1, 2$ and using Lemma 23

$$|\arg g_X(\sigma+iT_j)| \leq 14\operatorname{Log}((qT)^{5/8}X) + 26.$$

The use of this lemma asks for $T_1 = 4 + 2$ (the smallest value available). Since we fix this value, we can dispense with the index in T_2 and denote it by T.

Since $|f_X(2+it)| \leq 1/(X-1)$, we get for $\sigma_0 \geq 1/2$

$$\begin{aligned}2\pi\int_{\sigma_0}^{2} N_1(\sigma,T_1,T_2,\chi)d\sigma \leq &\int_{T_1}^{T_2}\big(|f_X(\sigma_0+it,\chi)|^2 + |f_X(\sigma_0+it,\overline{\chi})|^2\big)dt\\ &+\frac{4(T_2-T_1)}{X-1} + 42\operatorname{Log}((qT_2)^{5/8}X) + 78.\end{aligned} \tag{16}$$

We use $\sigma_0 = \sigma_1 - 3/\operatorname{Log}(Q^2T_2)$ and write

$$N_1(\sigma_1, T_1, T_2, \chi) \leq \int_{\sigma_0}^{\sigma_1} N_1(\sigma, T_1, T_2, \chi) d\sigma/(\sigma_1 - \sigma_0)$$

and hence

$$N_1(\sigma_1, T_1, T_2, Q) \leq \int_{T_1}^{T_2} \sum_{q \leq Q} \sum_{\chi \bmod^* q} |f_X(\sigma_0 + it, \chi)|^2 dt \frac{2\operatorname{Log}(Q^2T)}{3\pi} + \frac{4Q^2T_2\operatorname{Log}(Q^2T_2)}{3(X-1)\pi} + \frac{Q^2\operatorname{Log}(Q^2T_2)}{6\pi}\big(42\operatorname{Log}((QT_2)^{5/8}X) + 78\big).$$

We finally use $X = Q^2T$ (forget the subscript: $T_2 = T$), $Q \geq 10$ and $T \geq 2\,000$ to infer that $N(\sigma_1, 6, T, Q)$ is not more than

$$\frac{22}{0.367 \cdot 3\pi}\big(603\operatorname{Log}(Q^2T)\big)^{1-\sigma_1}(Q^5T^3)^{1-\sigma_1}\operatorname{Log}^3(Q^2T) + 5Q^2\operatorname{Log}^2(Q^2T).$$

We simplify and use (7) to get the stated result.

9. Automatic Dirichlet series

Dirichlet L-series are generalizations of the Riemann zeta function $\zeta(s) = \sum 1/n^s$ where the constant sequence 1 in the numerator is replaced by the sequence $(\xi(n))_{n\geq 0}$, with ξ a primitive Dirichlet character. Another possible generalization consists of replacing the constant sequence with sequences having some sort of regularity. In particular it is tempting to look at *automatic sequences* (for definitions and properties of automatic sequences, see, e.g., [6] and [56]).

Definition 91 *Let $d > 1$ be an integer. A sequence $(a_n)_{n\geq 0}$ is said to be* d-automatic *if and only if the set of subsequences $\{(a_{d^k n+r})_{n\geq 0},\ k \geq 0,\ 0 \leq r \leq d^k - 1\}$ is finite.*

Remark 1. – The definition clearly implies that a d-automatic sequence takes only finitely many values.

– Any eventually periodic sequence (in particular the constant sequence 1) is d-automatic for all $d > 1$.

Given an aperiodic automatic sequence $(a_n)_{n\geq 0}$ with values in $\mathbb{N}$, the transcendental analytic function $\sum_{n\geq 0} a_n z^n$ is essentially the inverse Melin transform of the Dirichlet series $\sum_{n\geq 0} a_n/(n+1)^s$. In transcendental number theory, it is a long-standing open problem to determine whether such a function takes transcendental values at every non-zero algebraic point that belongs to the open unit disc. Partial results in this direction were recently obtained in [3] by mean of the Schmidt Subspace Theorem.

Automatic Dirichlet series were studied in [4], [5]. We give more precisely two theorems that can be extracted from results in [4], resp. [5].

Theorem 91 (Allouche-Cohen) *Let $s(n)$ be the sum of the binary digits of the integer n. Then the two Dirichlet series $\sum_{n\geq 0}(-1)^{s(n)}/(n+1)^s$ and $\sum_{n\geq 1}(-1)^{s(n)}/n^s$ can be analytically continued to entire functions.*

Theorem 92 (Allouche-Mendès France-Peyrière) *Let $d > 1$ be an integer. Let $(a(n))_{n\geq 0}$ be a d-automatic sequence with values in $\mathbb{C}$. Then the series $\sum_{n\geq 0} a(n)/(n+1)^s$ can be analytically continued to a meromorphic function on the whole complex plane, whose poles, if any, belong to finitely many semi-lattices on the left.*

Now, analogously to what precedes, one may study the zeroes of, say the function $f(s) := \sum_{n\geq 0}(-1)^{s(n)}/(n+1)^s$. As proven in [4] (see Theorem 1.1 page 532, and the remark on top of Page 534), the function f admits "trivial" zeroes (namely the integers $0, -1, -2, -3, \ldots$) and nontrivial zeroes (namely the complex numbers $\frac{2ik\pi}{\log 2}$ for $k \in \mathbb{Z}$). A "Riemann hypothesis" for f would assert that these nontrivial zeroes and the trivial zeroes are the only zeroes of the function f. This Riemann-like conjecture might be very difficult to prove, and contrarily to the Riemann hypothesis, the proof would probably not make its author(s) famous... Celebrity is certainly not the main motivation of mathematicians. So that it may well be that some of them (including the authors of this paper) get interested in studying in more details the zeroes of f and related series. Here are a few preliminary remarks.

Remark 2. – The zeroes of an automatic Dirichlet series are the poles of its inverse: there is no reason in general that the inverse is also an automatic Dirichlet series. But it might be "so close" to an automatic Dirichlet series that the study of [4] might apply *mutatis mutandis* for studying its poles.

– A classical meta-result in number theory is the difficulty of mixing "additive" and "multiplicative" properties. Alternatively mixing "multiplicative" (in the sense of "multiplicative functions") and "'q-additive" properties (not that far from automaticity) is not easy either.

– Analytic number theory was developed in particular around the zeta function and the distribution of primes. Nothing similar was done for automatic Dirichlet series, which might paradoxally be a positive point, in that many things (including easy ones) remain to be done.

– Results in this area can also be interesting for (theoretical) computer scientists. To cite but one direction, analysis of algorithms and its asymptotics are both close to analytic number theory, and to theoretical computer science: see, e.g., [31, Lemma 2, p. 192] where the Dirichlet series $\sum_{n\geq 1}(-1)^{s(n)}/n^s$ above enters the picture, see also the nice book [32]. Actually properties of automatic Dirichlet series might give new insights on combinatorics on words, in particular in the fine study of the free monoid generated by a finite set, or even in linguistics (a first step is [52]).

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