ON THE EXPECTED RESULT FOR THE SECOND MOMENT OF TWISTED L-FUNCTIONS

The fourth moment of Dirichlet L-functions is

$$M_4(q) := \sum_{\chi\in X^*(q)} \left|L(\chi,1/2)\right|^4$$

where $X^*(q)$ stands for the set of primitive Dirichlet characters of modulus q whereas the second moment of twisted L-functions is

$$M_{2,f}(q) := \sum_{\chi\in X^*(q)} \left|L(f\times\chi,1/2)\right|^2$$

where f is a *fixed* holomorphic primitive cusp form of level $D_f \geqslant 1$ coprime with q, nebentypus ψ_f of modulus D_f satisfying $\psi_f(-1) = (-1)^{k_f}$ and integer weight $k_f \geqslant 1$ (see appendix [RiRo] for the automorphic background). Note that

$$\operatorname{card}\left(X^*(q)\right) := \varphi^*(q) = q\prod_{p\|q}\left(1-\frac{2}{p}\right)\prod_{p^2|q}\left(1-\frac{1}{p}\right)^2$$

according to [IwKo, Equation (3.7) Page 46]. Finding an asymptotic formula for $M_4(q)$ as $q\to+\infty$ with a power saving in the error term should be philosophically as difficult as the corresponding question for $M_{2,f}(q)$ since

$$M_4(q) = M_{2,\left(\frac{\partial E(1/2,.)}{\partial s}\right)(1/2)}(q)$$

where $E(z,s)$ is the real-analytic Eisenstein series on the modular curve $X_0(1)$ associated to the cusp ∞. Note that $M_4(q)$ is itself the q-analog of the fourth moment of the Riemann zeta function in the t-aspect (see [In] and [Hb2]) but this analogy will not be developed here. D. Heath-Brown ([HB]) for $M_4(q)$ and T. Stefanicki ([St]) for $M_{2,f}(q)$ proved the following analogous results.

Theorem. *If q goes to infinity then*

$$M_4(q) = \frac{1}{2\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}\log^4(q)\varphi^*(q) + O\left(2^{\omega(q)}q\log^3 q\right)$$

where $\omega(q)$ stands for the number of prime divisors of q and

$$M_{2,f}(q) = \frac{6(4\pi)^{k_f}}{\pi\Gamma(k)}||f||^2\prod_{p|q}\frac{(1-p^{-1})^2}{(1-p^{-2})}\left(1-\frac{\lambda_f(p^2)-2}{p}+\frac{1}{p^2}\right)\log(q)\varphi^*(q) + O\left(2^{\omega(q)}q\log^\delta q\right)$$

where $||f||$ stands for Petersson's norm of f and $0<\delta<1$ is any real number satisfying

$$\forall\varepsilon>0,\quad \sum_{p\leqslant x}\left|\lambda_f(p)\right| \leqslant (\delta+\varepsilon)\frac{x}{\log x}.$$

Remark 0.1. It is possible to choose $\delta = (\sqrt{2}+3\sqrt{3})(5\sqrt{2})^{-1}$ according to [Ra].

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Remark 0.2. The formula for $M_4(q)$ is an asymptotic formula for almost all integers but it turns out that if $\omega(q) > \log^{-1} 2 \log_2 q$ then the error term is not smaller than the main term. Similarly, if $\omega(q) > (1-\delta)\log^{-1} q \log_2 q$ then the error term in $M_{2,f}(q)$ is not smaller than the main term such that the formula for $M_{2,f}(q)$ is an asymptotic formula only for a set of integers of zero density according to [HR].

These results have been improved by K. Soundararajan ([So]) for $M_4(q)$ and in [GKR] for $M_{2,f}(q)$ as follows.

Theorem. *If q goes to infinity then*

$$M_4(q) = \frac{1}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} \log^4(q) \varphi^*(q) \left(1 + O\left(\log_2^{-1/2}(q)\right)\right)$$

and if $\omega(q) \ll \exp\left(300^{-1} \log_2 q \log_3^{-1} q\right)$ then

$$M_{2,f}(q) = \frac{6(4\pi)^{k_f}}{\pi\Gamma(k)} ||f||^2 \prod_{p|q} \frac{(1-p^{-1})^2}{(1-p^{-2})} \left(1 - \frac{\lambda_f(p^2)-2}{p} + \frac{1}{p^2}\right) \log(q)\varphi^*(q) \times \left(1 + O\left(\log_2^{-1}(q)\right)\right).$$

M. Young ([Yo]) got an asymptotic formula with a power saving in the error term for $M_4(q)$ in the prime modulus case.

Theorem. *If q goes to infinity among the prime numbers then*

$$M_4(q) = P(\log q)\varphi^*(q) + O\left(\varphi^*(q) q^{-\frac{1}{80}+\frac{\theta}{40}}\right)$$

where P is an explicit polynomial of degree 4 *and* $\theta = 7/64$.

Remark 0.3. Note that $-1/80 + \theta/40 < 0$. The same result should hold without the restriction q prime but it remains an open question so far. It should be very difficult to extend this result to any integer following the proof given in [Yo] since many technical difficulties would occur if q is composite.

Remark 0.4. This particular value of θ is the best currently known approximation towards Ramanujan-Petersson-Selberg's conjecture in $GL(2)$ according to [Ki] and [KH].

It turns out that the analogous result for $M_{2,f}(q)$ is still an open question and the purpose of this note is to identify the underlying analytic issue, which occurs in the second question. Let us state the expected result.

Expected Result. There exists some absolute constant $\alpha > 0$ such that if q goes to infinity then

$$M_{2,f}(q) = P(\log q)\varphi^*(q) + O\left(\varphi^*(q) q^{-\alpha}\right)$$

where P is an explicit polynomial of degree 1.

Proving this requires some new input in order to solve unbalanced shifted convolution problems. Such new input would have many other interesting applications.

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