Boundary of Hurwitz spaces and explicit patching

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Abstract

We describe a general method for the study and computation of Hurwitz spaces of curves of any genus. It is based on a careful combinatorial study of the associated jacobian. The key tool is an adapted cell decomposition of the cohomology of a graph (used here for the intersection graphs of special curves). We illustrate this method in the context of modular curves to produce modular units. We also give a detailed simple example and show how the algebraic difficulty of Hurwitz spaces computation can be reduced to its minimum.

1. Introduction

It is a difficult computational problem, given an n-uple $(\sigma_i)_{1\leq i\leq n}$ of permutations, to explicitly describe the corresponding family of covers of $\mathbb{P}^1$ (Hurwitz space and sometimes the universal curve). An introduction to these questions together with many examples can be found in recent monographs like [14, 21, 15]. See the paper by Klüners and Malle in this volume for examples of realizations. A nice (and successful, see [10] for example) approach starts from a degenerate cover and rebuilds the entire family by deformation. Indeed, if A is a complete discrete valuation ring with valuation ν, uniformizing parameter π and field of fractions K and if $\mathcal{C} \rightarrow \mathrm{Spec}(A)$ is the curve we are trying to compute (assuming that two or more ramification points coalesce modulo π) then the special fiber $\mathcal{C}_\epsilon$ is a covering of a nodal genus zero curve and can be described and often efficiently computed from the σ_i's. For each component γ of $\mathcal{C}_\epsilon$ we denote by ν_γ the corresponding valuation, extending ν to the field of functions of $\mathcal{C}$. The computation of $\mathcal{C}$ from $\mathcal{C}_\epsilon$ can be achieved in four steps

1 — Define two functions x and y on $\mathcal{C}$;

2 — For every component γ of $\mathcal{C}_\epsilon$ compute $\nu_\gamma(x) = a$ and $\nu_\gamma(y) = b$ and define $X_\gamma = x/\pi^a$ and $Y_\gamma = y/\pi^b$;

3 — For each γ, using the monodromy $(\sigma_i)_i$, compute an equation $\mathcal{E}_\gamma(X_\gamma, Y_\gamma) = 0 \pmod{\pi}$ of the component γ;

4 — Patch (in the sense of [11]) all the $\mathcal{E}_\gamma$ together using Hensel's lemma and find an equation $\mathcal{E}(x, y) = 0$ with coefficients in A for the universal curve.

We have detailed and illustrated this method for genus zero covers in [7]. For arbitrary genus covers, steps 3 and 4 are unchanged. We need however to introduce new ideas for steps 1 and 2. Indeed, the functions x and y for genus zero covers can be defined using cross ratios with three branch points and we showed how the multiplicities a and b could be computed as distance functions in the graph of $\mathcal{C}_\epsilon$.

The main goal of this paper is to generalize this to curves of arbitrary genus. We denote by $\bar{K}$ the algebraic closure of K and by k the residue field of A which is assumed to be algebraically closed. The generic curve of $\mathcal{C}$ is denoted by $\mathcal{C}_K$. We assume that $\mathcal{C}$ is regular and $\mathcal{C}_K \otimes_K \bar{K}$ is a smooth curve of genus g. We want to define functions on $\mathcal{C}$ and control their divisor. A method which always works is to form a divisor D on $\mathcal{C}_K$ of degree g, the genus of $\mathcal{C}_K$, with support in the set of ramification points. We then define x to be a function in $\mathcal{L}(D)$ taking value 1 at some extra ramification point P. There remains to compute the valuations $\nu_\gamma(x)$ for every γ. This amounts to computing the vertical part of the divisor of x seen as a function on $\mathcal{C}$. Using intersection theory, the problem boils down to the following one.

PROBLEM 1: *Given a divisor D of degree g on $\mathcal{C}_K$, let E be an effective divisor of degree g equivalent to D. Such a divisor exists because of the surjectivity of the map*

$$\Phi : S^g\mathcal{C}_K \to J_K \tag{1}$$

where J_K is the jacobian of $\mathcal{C}_K$ (we assume some origin $O \in \mathcal{C}_K(K)$ has been given).

The Zariski closure of D (resp. E) is a divisor on $\mathcal{C}$ that we shall denote by D (resp. E) also. Knowing how D intersects the special fiber $\mathcal{C}_\epsilon$ can we deduce how E intersects $\mathcal{C}_\epsilon$?

To answer this question we consider the intersection graph associated to the nodal curve $\mathcal{C}_\epsilon$. In general, we define a graph as in [19]. It is a 5-uple $\mathcal{G} = (V, F, o, e, \rho)$ where V is a set (the vertices) and F another set (the oriented edges) and o and e are two maps from F to V associating to any oriented edge its origin and its end and $\rho : F \to F$ is an involution without fixed points such that $o \circ \rho = e$. If $E \subset F$ is a set of representatives for the orbites of ρ in F then $F = E \cup \rho(E)$ and choosing E is just choosing an orientation for any edge.

Associated to $\mathcal{G}$ there is a CW-complex of dimention 1 (sometimes called the realization of $\mathcal{G}$) which we shall denote by $\mathcal{G}$ also since there is no risk of confusion. In the same spirit, vertices in V and edges in E are seen as subspaces (cells) of the realization of $\mathcal{G}$. We notice however that not every graph is a simplicial complex (there might be several edges between two given vertices). However we define for every positive integer e the e-th division $\mathcal{G}_e$ of $\mathcal{G}$ to be the graph obtained by cutting every edge into e pieces and for $e \geq 3$ we have a simplicial complex.

See [6] for definitions and classical properties of complexes. For any complex $\mathcal{H}$ one denotes by $\mathcal{H}^k$ the set of k-dimensional cells and in particular for a graph $\mathcal{G} = (V, F, o, e, \rho)$ we have $\mathcal{G}^0 = V$ and $\mathcal{G}^1 = E = F/\rho$.

To any finite connected graph $\mathcal{G}$ we shall associate in section 2 a finite CW-complex $\mathcal{K}$ which we call the Kirchhoff complex of $\mathcal{G}$ whose underlying topological space is a torus T of dimension h where $h = \dim H_1(\mathcal{G}, \mathbb{R})$ is the genus of $\mathcal{G}$, and with set of vertices $\mathcal{K}^0$ a finite subgroup of the torus. There is also a surjective continuous "integration" map

$$\upsilon : S^h\mathcal{G} \to T$$

where $S^h\mathcal{G}$ is the h-th symmetric product of $\mathcal{G}$.

The map υ is the combinatorial analogue of the map Φ in equation (1). Contrary to the holomorphic case, there exists a continuous section σ that turns $\sigma \circ \upsilon$ into a retraction. In the simplest non trivial case one has $h = 1$ and $\mathcal{G}$ is made of a loop with trees rooted at vertices in the loop. The retraction $\sigma \circ \upsilon$ just retracts each tree onto its root and fixes points in the cycle.

In case $\mathcal{G}$ is the intersection graph of the special curve $\mathcal{C}_\epsilon$, the map υ is a sort of combinatorial model of Φ. For example, the vertices of $\mathcal{K}$ correspond to connected components of the Néron model of J_K. The Kirchhoff complex is a purely combinatorial object and can be used to predict how the divisor E intersects the special fiber using results in [16, 18, 9, 4, 3]. This is the purpose of section 3.

Apart from its computational interest, a good reason to introduce the Kirchhoff complex is the uniformization method in [3] (e.g. the homotopy theorem 3.5). It would be interesting to compare this complex to the ones constructed in [17].

Section 4 is an interlude and an illustration of the efficiency of the method in the more theoretical context of modular curves. We very easily construct a great deal of modular units (more than in [12] for example). This section has been motivated by a conversation with Bas Edhixoven and the observations in [8]. I thank him for his help. Section 5 extends the methods of sections 2 and 3 in the context of generalized jacobians. It is a useful generalization since it provides a connection with the genus zero situation studied in [7]. A more practical reason to introduce it is that in somes cases (e.g. when there are few or even no rational sections given) only this more general construction succeeds.

Section 6 treats a detailed simple example.

Although our method is quite general, one may notice that in the two examples treated in sections 4 and 6 we develop a little bit more and improve on the general method. Instead of looking at a single degenerate cover we collect information at every special cover in the family. In many cases, using the knowledge we have on differentials, we can construct units on the Hurwitz space in that way, replacing the patching computation of step 4 by mere interpolation. This makes the method more efficient but might not be possible for any family of covers. Since we already illustrated the general patching step in [7], we prefer to give richer and theoretically more interesting examples here.

2. The Kirchhoff complex of a graph

In this section we construct the Kirchhoff complex $\mathcal{K}$ associated to any finite connected graph $\mathcal{G}$. We denote by $V = \mathcal{G}^0$ and $E = \mathcal{G}^1$ the sets of vertices and edges (we have chosen an orientation for any edge). The basic idea is to mimic integration theory on Riemann surfaces. We recall that the genus of $\mathcal{G}$ is $h = \dim H_1(\mathcal{G}, \mathbb{R})$ the dimension of the first homology group.

Let $C_1(\mathcal{G}, \mathbb{R})$ be the space of 1-chains over $\mathbb{R}$. It's the free vector space over $\mathbb{R}$ generated by edges of $\mathcal{G}$ and it is of dimension $|E|$. There is a canonical positive definite pairing $(,)$ on it such that E is an orthonormal family.

For any closed edge $e \in E$ we denote by $d\mu_e$ the uniform measure of total mass 1 on e and set

$$d\mu = \sum_E e d\mu_e.$$

The meaning of this expression is that if X is a subset of the topological space $\mathcal{G}$ such that $X_e = X \cap e$ is a measurable subset of the edge e for every e, then the measure $\mu(X)$ is a vector in $C_1(\mathcal{G}, \mathbb{R})$ defined to be the sum over $e \in E$ of the products $\mu_e(X_e)$ (scalar measure of X_e) times the vector $e \in C_1(\mathcal{G}, \mathbb{R})$:

$$\mu(X) = \sum_{e \in E} \mu_e(X \cap e)e.$$

If a and b are points on $\mathcal{G}$ (not necessarily vertices) and γ is a continuous path from a to b (continuous map from $[0, 1]$ to $\mathcal{G}$) we may define

$$\int_\gamma d\mu \in C_1(\mathcal{G}, \mathbb{R})$$

which only depends on the homology class $[\![\gamma]\!] \in H_1(\mathcal{G}, \{a, b\})$ (the group of relative homology) of γ. What this integral should be is rather straightforward. We give a definition however. Let A and B be closed edges of $\mathcal{G}$ such that $a \in A$ and $b \in B$ and let $\alpha = o(A)$ (resp. $\beta = o(B)$) be the origin of A (resp. B). Let $[\alpha, a] \subset A$ be the interval in A bounded by α and a and let $u = [\![\alpha, a]\!]$

be the corresponding cycle in $H_1(\mathcal{G}, \{a, \alpha\})$. We similarly define $v = [\![\beta, b]\!]$. The boundary $\delta([\![\alpha, a]\!])$ is $a-\alpha$ and similarly $\delta([\![\beta, b]\!]) = b-\beta$. We define a linear map $\lambda : H_1(\mathcal{G}, \{a, b\}) \to C_1(\mathcal{G})$ in the following way. For any cycle $c \in H_1(\mathcal{G}, \{a, b\})$ there is an integer k such that $\delta(c) = k(b-a)$. We set $\lambda(c) = c + k(u-v) + k(\mu_B([\beta, b])B - \mu_A([\alpha, a])A)$. Note that $c+k(u-v)$ is in $H_1(\mathcal{G}, \{\alpha, \beta\}) \subset C_1(\mathcal{G})$.

We define $\int_\gamma d\mu$ to be $\lambda([\![\gamma]\!]) \in C_1(\mathcal{G})$.

We show an example on figure (1).

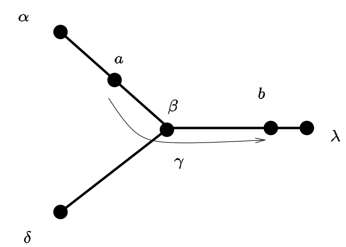

Figure 1: An example of integrating along a path

Here the graph has four vertices α, β, λ, δ. The point a is the middle of $[\alpha, \beta]$ and $\overline{\beta b} = \frac{3}{4}\overline{\beta\lambda}$. The path γ is represented by the arrow. We have $\int_\gamma d\mu = 1/2[\alpha, \beta] + 3/4[\beta, \lambda]$.

If we pick an origin $O \in V$ and take $\mathcal{U}$ to be the universal covering of $\mathcal{G}$ constructed as the space of pathes from O up to homotopy, we can define a map

$$\phi : \mathcal{U} \to H^1(\mathcal{G}, \mathbb{R}) \text{ by } \gamma \mapsto (\int_\gamma d\mu,).$$

The image of $\pi_1(\mathcal{G}, O) \subset \mathcal{U}$ by ϕ is a lattice $\mathcal{T}$ which is contained in $H^1(\mathcal{G}, \mathbb{Z})$. We denote by

$$\varphi : \mathcal{G} \to T = H^1(\mathcal{G}, \mathbb{R})/\mathcal{T}$$

the induced map on quotients.

For any positive integer k we denote by $\phi^k : \mathcal{U}^k \to H^1(\mathcal{G}, \mathbb{R})$ the sum of ϕ with itself k times and similarly for φ^k.

The map φ^h is invariant under permutation of the terms and thus gives rise to a map $\upsilon : S^h\mathcal{G} \to T$. The map φ^h is continuous and since both $S^h\mathcal{G}$ and T are compact, it is a closed map.

Lemma 2.1: *The maps ϕ^h, φ^h and υ are surjective.*

To prove this we take h loops $(\gamma_i)_{1\le i\le h}$ in $\mathcal{G}$ generating the homology $H_1(\mathcal{G}, \mathbb{Z})$ (as in [1] page 26) and show that the restriction of φ^h to $\prod_i \gamma_i$ is surjective. This restriction is a continuous map between two tori of the same dimension and is non singular (and even unimodular) on the homology. The surjectivity follows from the following lemma.

Lemma 2.2: *If T is a torus of dimension $n \geq 1$ and $\phi : T \to T$ is a continuous map such that $\phi_* : H_1(T) \to H_1(T)$ is non singular (as an endomorphism of the free $\mathbb{Z}$-module $H_1(T)$) then ϕ is surjective.*

Let $f_1,...,f_n$ be a basis of the cohomology group $H^1(T)$ and $g_1,...,g_n$ a dual basis of $H_1(T)$. The n-th cohomology $H^n(T)$ is generated by $D = f_1 \wedge ... \wedge f_n$. Assume that ϕ is not surjective. By [6, VI.8.5] we deduce that the restriction of D to the image L of ϕ is zero. This implies that the determinant of $[f_i(\phi_*(g_j))]$ is zero and thus ϕ_* is singular. □

Being surjective and closed, the map φ^h is an identification map (the natural topology on T is the quotient one).

We now study the map v more in detail. We start with the

Definition 1: *A point $x = \{x_1, ..., x_h\}$ in $S^h\mathcal{G}$ is said to be stable if and only if there exist h (closed) edges $(e_i)_{1\leq i\leq h}$ such that $x_i \in e_i$ and $\mathcal{G} - \cup_i e_i$ is a connected tree. Under the same conditions we shall say that $(x_1, ..., x_h) \in \mathcal{G}^h$ is stable. We denote by $\mathcal{J}$ (resp. $\mathcal{K}$) the set of stable points in $\mathcal{G}^h$ (resp. $S^h\mathcal{G}$).*

The space $S^h\mathcal{G}$ is defined to be the quotient of $\mathcal{G}^h$ by the symmetric group $\mathcal{S}_h$ and is easily seen to be Hausdorff and compact. Since $\mathcal{J}$ is closed in $\mathcal{G}^h$, the restriction to $\mathcal{J}$ of the quotient map $\mathcal{G}^h \to S^h\mathcal{G}$ is closed and an identification map also. See [6, I.13] for properties of quotients.

We first prove the following

Lemma 2.3: *$\mathcal{J}$ is a subcomplex of $\mathcal{G}^h$ and $\mathcal{K}$ is a quotient complex of $\mathcal{J}$ by the symmetric group $\mathcal{S}_h$.*

From its definition it is clear that $\mathcal{J}$ is a union of closed cells in $\mathcal{G}^h$ (which we shall call *stable cells*). Further, the boundary of a stable cell is a union of stable cells. Thus $\mathcal{J}$ is a subcomplex of $\mathcal{G}^h$.

The group $\mathcal{S}_h$ is an automorphism group of $\mathcal{J}$. If c is a closed cell of $\mathcal{J}$ and σ an element of $\mathcal{S}_h$ such that $\sigma(c) = c$ then σ *fixes* c. Indeed if $x = (x_1, ..., x_h)$ is in c and $\sigma(x) \neq x$ then there is a $k \in \{1, ..., h\}$ such that $x_{\sigma(k)} \neq x_k$ and both x_k and $x_{\sigma(k)}$ belong to the same edge of $\mathcal{G}$. This is incompatible with the stability of x.

The quotient map $\mathcal{J} \to \mathcal{K}$ is thus a bijection when restricted to an open cell of $\mathcal{J}$. It thus provides $\mathcal{K}$ with a structure of CW-complex and becomes a cellular map. □

We now study the restriction of the map v to $\mathcal{K}$ (which we shall denote by $v_{|\mathcal{K}}$ or just v). It is a continuous map (even an identification map). We denote by $\mathring{\mathcal{K}}$ the union of the relative interiors of all cells of maximum dimension in $\mathcal{K}$. We prove the following

LEMMA 2.4: *The map $v_{|\mathcal{K}}$ is surjective and its restriction to $\mathring{\mathcal{K}}$ is injective.*

By Kirchhoff-Trent theorem, the number of stable cells of dimension h in $S^h\mathcal{G}$ is the index of $\mathcal{T}$ in $H^1(\mathcal{G},\mathbb{Z})$ i.e. the volume of T (see [2, II.3] and [17, 4.5]). On the other hand, every open stable cell of dimension h is mapped bijectively by v onto a parallelogram of volume 1 in T (because the h corresponding edges of $\mathcal{G}$ form a basis of *elementary cocycles* in $H^1(\mathcal{G},\mathbb{Z})$). Because the image by φ^h of non-stable points has zero measure and φ^h is surjective, we deduce that the (open) h-dimensional stable cells in $\mathcal{K}$ map to pairwise disjoint parallelograms (with volume 1) in T. So the image of $\mathring{\mathcal{K}}$ is dense in T and since $\mathcal{K}$ and T are compact we deduce that $v_{|\mathcal{K}}$ is surjective. □

We now study cells of dimension $h-1$ in $\mathcal{K}$ and $\mathcal{J}$ (we call them stable *faces*).

LEMMA 2.5: *If c is a face in $\mathcal{J}$ (resp. $\mathcal{K}$) there exist at most two h-dimensional (closed) cells in $\mathcal{J}$ (resp. $\mathcal{K}$) that contain c (i.e have c as a face).*

A face c in $\mathcal{K}$ is given by a set of $h-1$ edges $\{e_1,...,e_{h-1}\}$ and a vertex v in $\mathcal{G}$ such that $\mathcal{G}-\cup_i e_i$ is a connected graph of genus one and v belongs to the unique elementary cycle in it. Cells in $S^h\mathcal{G}$ of dimension h containing c correspond to edges e in $\mathcal{G}$ containing v. To any such e we associate the face built from $\{e,e_1,...,e_{h-1}\}$. This face is stable if and only if e belongs to the elementary cycle ζ of $\mathcal{G}-\cup_i e_i$ which leaves at most two possibilities. There is a single such face if and only if ζ consists of a single self crossing edge. In that later case, the face c is twice the face of the same cell. □

We now proceed to prove that $v_{\mathcal{K}}$ is an homeomorphism.

We start with

LEMMA 2.6: *The sets $v(\mathring{\mathcal{K}})$ and $v(\mathcal{K}-\mathring{\mathcal{K}})$ are complementary subsets of the torus $T=H^1(\mathcal{G},\mathbb{R})/\mathcal{T}$.*

The union of these two sets is $H^1(\mathcal{G},\mathbb{R})/\mathcal{T}$ since $v_{\mathcal{K}}$ is surjective. Let us now prove that they are disjoint. Let $z\in H^1(\mathcal{G},\mathbb{R})/\mathcal{T}$ and $y\in\mathring{\mathcal{K}}$ and $x\in\mathcal{K}-\mathring{\mathcal{K}}$ with $v(x)=v(y)=z$. There is an open cell A of $\mathcal{K}$ such that x is in $\bar{A}$ the closure of A. There is also an open stable cell B such that $y\in B$. Let $\tilde{A}$ (resp. $\tilde{B}$) be the image of A (resp. B) by v. The point $z=v(x)$ is in the closure of $\tilde{A}$ and since $z=v(y)$ it is in $\tilde{B}$. Since $\tilde{B}$ is open and v is bijective when restricted to $\mathring{\mathcal{K}}$ we deduce that $\tilde{A}=\tilde{B}$ and $A=B$. So we have $x\in\bar{A}-A$ and $y\in A$ with $v(x)=v(y)$. This is impossible because $v(A)$ lifts to a fundamental parallelogram for $H^1(\mathcal{G},\mathbb{Z})$ in $H^1(\mathcal{G},\mathbb{R})$ and cannot contain an inner point congruent to a boundary point modulo $H^1(\mathcal{G},\mathbb{Z})$ (and even less modulo $\mathcal{T}$). □

We now prove

LEMMA 2.7: *The restriction of v to $\mathcal{K}-\mathring{\mathcal{K}}$ is injective.*

Let $z \in H^1(\mathcal{G}, \mathbb{R})$ and $x \in \mathcal{K} - \mathring{\mathcal{K}}$ such that $z = v(x)$. For ϵ a small positive real we define the ball $B(z,\epsilon)$ as the reduction modulo $\mathcal{T}$ of a ball in $H^1(\mathcal{G}, \mathbb{R})$ with radius ϵ whose center maps to z. There is a positive integer I such that for any small enough ϵ the intersection $B(z,\epsilon) \cap v(\mathring{\mathcal{K}})$ is a union of I connected components c_1, c_2,..., c_I, each the intersection of an open h-dimensional cell in $v(\mathring{K})$ having z in its boundary, with the ball $B(z,\epsilon)$. We denote by d_1, ..., d_I the inverse image of c_1, ..., c_I by $v_{|\mathcal{K}}$. These are also intersections of stable cells with a ball having its center in the boundary. For ϵ small enough, the restriction of v to the closure $\bar{d_i}$ of d_i is injective (because it factors through $H^1(\mathcal{G}, \mathbb{R})$ where it gives the intersection of a fundamental parallelogram for $H^1(\mathcal{G}, \mathbb{Z})$ with a small ball). On the other hand, any point in $\mathcal{K}$ above z has to be in the closure of some d_i. So there are finitely many points in $\mathcal{K}$ above z and they define a partition of $\{1, ..., I\}$. We may assume that x belongs only to $\bar{d}_1$, $\bar{d}_2$, ..., $\bar{d}_J$, with $1 \leq J \leq I$. The restriction of v to $\bar{d}_j$ defines an h-dimensional cycle κ_j in $B(z,\epsilon)$. By lemma 2.5 the sum $\kappa = \sum_{1 \leq i \leq J} \kappa_i$ has its boundary contained in the boundary $S(z,\epsilon)$ of $B(z,\epsilon)$ because the components coming from faces of parallelograms cancel by pairs. From [6, IV.6.6] $H_h(B(z,\epsilon), S(z,\epsilon)) = \mathbb{Z}$ is generated by the orientation class. This implies that the union $\cup_{1 \leq i \leq J} v(\bar{d}_i) = \overline{\cup_{1 \leq i \leq J} c_i}$ is either of measure zero or the whole ball. Since the first possibility is excluded (J is positive) we deduce that $I = J$. Thus x is the only point in $\mathcal{K}$ above z. □

We deduce the

THEOREM 2.1: *The restriction of the integration map v to the set $\mathcal{K}$ of stable points is an homeomorphism that turns $\mathcal{K}$ into a cell decomposition of the torus $T = H^1(\mathcal{G}, \mathbb{R})/\mathcal{T}$. The h-dimensional cells in $\mathcal{K}$ are the stable h-dimensional cells in $S^h\mathcal{G}$. They are parallelograms and correspond to spanning trees in $\mathcal{G}$. The set of vertices of $\mathcal{K}$ is the group $H^1(\mathcal{G}, \mathbb{Z})/\mathcal{T}$.*

REMARK 1: *We denote by $\sigma = {v_{|\mathcal{K}}}^{(-1)}$ the inverse map of $v_{|\mathcal{K}}$. This map solves the discrete version of the Jacobi inversion problem. It will be an essential tool for solving problem (1). For practical computations we shall only apply v to vertices in $S^h\mathcal{G}^0$ and σ to points in $H^1(\mathcal{G}, \mathbb{Z})/\mathcal{T}$. We notice that while the computation of v (the integration map) reduces to linear algebra and can be achieved in time polynomial in $\log(|\mathcal{K}^0|)$, on the contrary the evaluation of σ may require exhaustive search and time $|\mathcal{G}^0|^h$. This will not be a problem for us since for the graphs we shall consider in section 3, this $|\mathcal{G}^0|^h$ is small compared to the difficulty of computing families of covers. For example it is trivially bounded by $[(r+1)d]^g$ where d is the degree of the cover and g its genus and r the number of branched points.*

We finish with a natural criterion for stability.

LEMMA 2.8: *A point $x = \{x_1, ..., x_h\}$ in $S^h\mathcal{G}$ is stable if and only if v is locally surjective at x (the image of any neighbourhood of x is a neighbourhood of $v(x)$).*

Indeed, if x is stable the map υ and even $\upsilon_{|\mathcal{K}}$ is locally surjective at x from the above theorem. If x is not stable, then all h-dimensional cells close to x in $S^h\mathcal{G}$ are unstable so their image by υ has measure zero. □

The structure of the cell complex $\mathcal{K}$ is independent of the choice of the origin O.

For any integer $e \geq 1$ let $\mathcal{G}_e$ be the e-th division of $\mathcal{G}$ obtained by cutting any edge into e smaller edges of equal length. The associated Kirchhoff cell complex $\mathcal{K}_e$ to $\mathcal{G}_e$ is the e-th division of $\mathcal{K}$ (every k-dimensional parallelogram is cut into e^k smaller ones).

Definition 2: *A point $x \in \mathcal{G}$ is said to be rational if $\int_O^x d\mu$ is in $C_1(\mathcal{G}, \mathbb{Q})$ (no matter the path). The denominator of x is the smallest positive integer D such that $D\int_O^x d\mu \in C_1(\mathcal{G}, \mathbb{Z})$ or equivalently $x \in \mathcal{G}_D^0$ (the set of vertices of $\mathcal{G}_D$). We denote by $\mathcal{G}(\mathbb{Q})$ the set of rational points of $\mathcal{G}$.*

A point in $T = H^1(\mathcal{G}, \mathbb{R})/\mathcal{T}$ is said to be rigid if it has a unique preimage by υ. Rigid points are dense in T.

The complex $\mathcal{K}$ and its group $\mathcal{K}^0$ of vertices may be seen as discrete analogues of the Picard varieties of degree g and 0 for algebraic curves. They actually interact when dealing with curves over local fields.

We notice that there are many classical cell decompositions of the cohomology of a graph (see [17]). The one we just described has interesting properties with respect to the map υ. We don't know if the Kirchhoff complex is always one of the complexes constructed in [17]. We believe this is not always the case since the Kirchhoff complex is in some sense "diagonal" with respect to the ones in [17]. It borrows cells to all these complexes. At least the Kirchhoff complex has the advantage of beeing unique.

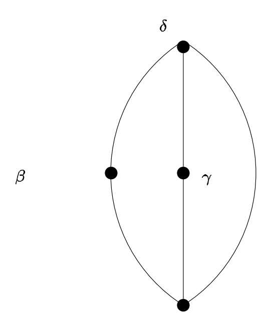

Figure 2: A graph $\mathcal{G}$ with genus $h = 2$

We give an example. The graph we consider is the one on figure (2) coming from [17, 8]. The associated Kirchhoff complex is given on figure (3). Since it is

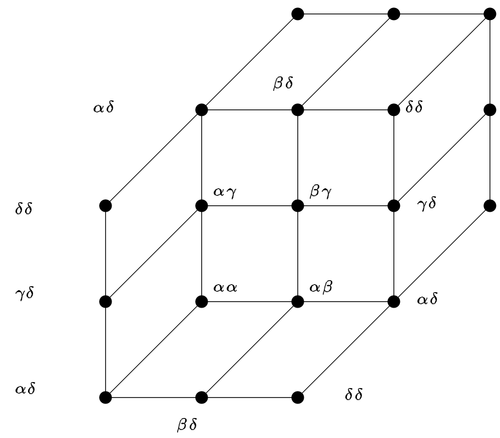

Figure 3: The Kirchhoff complex of $\mathcal{G}$

not easy to write on a torus we represent the periodic complex decomposition of $H^1(\mathcal{G},\mathbb{R})$ associated to it. The letters at vertices give the corresponding stable point in $S^2\mathcal{G}$. We notice that in this particular case, the Kirchhoff complex is one of the two complexes defined by Oda and Seshadri. Note that $\{\beta,\beta\}$ and $\{\gamma,\gamma\}$ are not stable and map to the vertex $\{\alpha,\delta\}$.

3. The Kirchhoff complex of a curve

Le K be a complete local field with discrete valuation ν, ring of integers $\mathcal{O}$, residue field k and algebraic closure $\bar{K}$. The residue field k is assumed to be algebraically closed. Let Spec($\mathcal{O}$) be the spectrum of $\mathcal{O}$ and let $\mathcal{C} \to \mathrm{Spec}(\mathcal{O})$ be a curve over Spec($\mathcal{O}$) with generic fiber $\mathcal{C}_K$ and special fiber $\mathcal{C}_k$. We assume that $\mathcal{C}_K \otimes_K \bar{K}$ is smooth complete and irreducible of genus g and $\mathcal{C}_k$ is a reduced curve with ordinary double points and all components of $\mathcal{C}_k$ are smooth. We assume also that $\mathcal{C}$ is regular of finite type. We denote by $\mathcal{G}$ the graph of the curve $\mathcal{C}_k$ whose vertices correspond to irreducible components and edges to crossings. We assume there is a section o that intersects $\mathcal{C}_k$ at a non singular point o_k. We call O the vertex of $\mathcal{G}$ corresponding to the component of $\mathcal{C}_k$ that crosses o. We call $\mathcal{K}$, υ,..., the associated complex and maps.

DEFINITION 3: *If $L \supset K$ is an extension of local fields with ramification index e then the curve $\mathcal{C} \otimes_{\mathcal{O}} \mathcal{O}_L$ is not regular but after blowing up e times at every crossing, we obtain a regular surface $\mathcal{C}_{\mathcal{O}_L}$ whose intersection graph is the e-th subdivision of $\mathcal{G}$. The surface $\mathcal{C}_{\mathcal{O}_L}$ is called the regular model of $\mathcal{C}$ over L.*

Let $P_K \in \mathcal{C}(K)$ be a point on $\mathcal{C}_K$ and P its Zariski closure and P_k the intersection with $\mathcal{C}_k$. Since $\mathcal{C}$ is regular and P is defined over K, the point P_k is a smooth point of $\mathcal{C}_k$ by [20, IV.4.3]. We thus can associate to P_K a vertex $x(P) \in \mathcal{G}^0$.

More generally, if $P_{\bar{K}} \in \mathcal{C}(\bar{K})$, we can associate to it a point $x(P) \in \mathcal{G}(\mathbb{Q})$ with denominator bounded by the index of ramification of $K(P)/K$.

We may say that $\mathcal{C}_{\mathcal{O}_L}$ is the minimal blow up of $\mathcal{C}$ over L although it is not quite the minimal model for $\mathcal{C}$ over L since $\mathcal{C}$ itself might not be minimal over K.

LEMMA 3.1: *Let L be a finite extension of K with residue field $l \supset k$ (we actually have $l = k$ because k is algebraically closed but we keep the two letters for the sake of clarity) and $D = \sum_i e_i[P_{i,L}] - (\sum_i e_i)[o_K]$ a divisor of degree 0 with all the $P_{i,L} \in \mathcal{C}(L)$. We denote by $\delta_L \in J_K(L)$ the corresponding point of the jacobian J_K of $\mathcal{C}_K$.*

If $J_{\mathcal{O}_L} \to \mathrm{Spec}(\mathcal{O}_L)$ is the Néron model of $J_K \otimes_K L$ then δ_L extends to a section δ of $J_{\mathcal{O}_L} \to \mathrm{Spec}(\mathcal{O}_L)$. The irreducible components of the special fiber J_l of $J_{\mathcal{O}_L}$ are parametrized by the vertices of $\mathcal{K}_e$ where e is the ramification index of L/K. The section δ crosses the component of J_l associated to

$$\sum_i e_i x(P_i) - (\sum_i e_i)O \in \mathcal{K}_e^0 = H^1(\mathcal{G}, \frac{1}{e}\mathbb{Z})/\mathcal{T}.$$

This results from sections 9.5 and 9.6 of [5] and especially theorem 4 and lemma 8 in section 9.5 and theorem 1 in section 9.6. □

When computing families of covers, the points P_i are chosen to be ramification points of the covering. The $x(P_i)$ are then given by the monodromy of the covering (see [7]). Now, because the map Φ in formula (1) is an epimorphism, there exists a possibly ramified extension $M \supset L$ and g points $Q_{i,M} \in \mathcal{C}(M)$ for $1 \leq i \leq g$ such that D is linearly equivalent to $\sum_{1 \leq i \leq g} Q_{i,M} - g.o_K$. We would like to know how the sections Q_i intersect the special fiber $\mathcal{C}_l$ of $\mathcal{C}_{\mathcal{O}_L}$. Note that since $M \supset L$ is possibly ramified, the Q_i may cross $\mathcal{C}_l$ at singular points. This is the kind of phenomenon we would like to avoid. We would even like to compute the $x(Q_i)$ in terms of the $x(P_i)$. We define what a generic situation is and give sufficient conditions for this to hold.

Let L be an extension of K with ramification index e. The set $\mathcal{G}_e^0$ of vertices of the e-th subdivision of $\mathcal{G}$ is in bijection with the set of irreducible components of $\mathcal{C}_l$. For $v \in \mathcal{G}_e^0$ we denote by β_v the associated component and g_v its genus. We have

$$g = h + \sum g_v.$$

DEFINITION 4: *Let L be an extension of K with ramification index e.*

Let $\{Q_1, ..., Q_g\}$ be a family of g points in $\mathcal{C}(\bar{K})$, globally defined over L (i.e. they are permuted by $\mathrm{Gal}(\bar{K}/L)$). We say that the $(Q_i)_{1 \leq i \leq g}$ are L-residually generic if

1 — none of them cross the special fiber $\mathcal{C}_l$ at a singular point of it and thus the $x(Q_i)$ are in $\mathcal{G}_e^0$,

2 — for any component β_v of $\mathcal{C}_l$, there are at least g_v sections crossing β_v at a smooth point of $\mathcal{C}_l$ among the Q_i, *3 — if for all v we remove, from the Q_i, g_v sections crossing β_v at a smooth point, there remains a family of h sections which is stable with respect to the graph $\mathcal{G}_e$ in the sense of the previous paragraph.*

We notice that if the three conditions above are met, then we know where are the sections Q_i crossing the special fiber as explained in remark (1) of section 2.

If θ is a function with horizontal divisor

$$\sum_i e_i[P_{i,L}] - \sum_{1\leq i\leq g}[Q_{i,M}] + (g - \sum_i e_i)[o_K]$$

then we can deduce from the genericity property what is the divisor of θ on the surface $\mathcal{C} \to \mathrm{Spec}(\mathcal{O}_L)$ up to a multiple of the whole special fiber. Indeed, the vertical part of this divisor is equivalent to the opposite of the horizontal part. Since the Kirchhoff complex and the map σ tell us how the horizontal part intersects the special fiber, we deduce how the vertical part intersects vertical divisors. This is enough to determine it up to a multiple of the special fiber. See [13, 5.2] for an account of these classical things. This last computational step is just linear algebra and requires no effort.

REMARK 2: *Once we know the divisor of θ we can evaluate the valuation of $\theta(R_1)/\theta(R_2)$ for two sections $R_i \to \mathrm{Spec}(\mathcal{O}_L)$ provide these sections satisfy the two supplementary conditions below*

4 — The sections R_1 and R_2 cross the special fiber $\mathcal{C}_l$ at regular points $R_{1,l}$ and $R_{2,l}$,

5 — the points $R_{1,l}$ and $R_{2,l}$ are distinct from any of the $P_{i,l}$ and $Q_{i,l}$ and o_k.

Indeed, in such a situation, the valuation of $\theta(R_1)/\theta(R_2)$ is the difference between the multiplicities of θ along the components of $\mathcal{C}_l$ crossed by R_1 and R_2.

We now state the main result in this section which is a sufficient condition for the $(Q_i)_{1\leq i\leq g}$ to be residually generic.

THEOREM 3.1: *Let $D = \sum_i e_i[P_{i,L}] - (\sum_i e_i)[o_K]$ be a divisor of degree zero on $\mathcal{C}_L$ and let $\{Q_1, ..., Q_g\}$ be a family of g points in $\mathcal{C}(\bar{K})$ globally defined over L and such that D is equivalent to $\sum_i [Q_i] - g[o]$. For any component β_v of $\mathcal{C}_l$ (with $v \in \mathcal{G}_e^0$) we denote by χ_v the set of intersection points of β_v either with the other components of $\mathcal{C}_l$ or with the sections P_i and o.*

A sufficient set of conditions for the (Q_i) to be L-residually generic is

6 — for all $v \in \mathcal{G}^0$ and for any divisor F with support in χ_v and degree lower than g_v we have $\ell(F) = 0$ where $\ell(F)$ is the dimension of the linear space $\mathcal{L}(F)$.

7 — *the point*

$$\sum_i e_i x(P_i) - \sum_v g_v v + (g - h - \sum_i e_i).O \in \mathcal{K}$$

is rigid (here the vertices v that appear in the second summation are seen as points of the space $\mathcal{G}$).

Assume condition (2) of definition (4) is not fulfilled. Let v be such that the smooth part of β_v crosses less than g_v sections among the Q_i. Let a be the multiplicity of θ along β_v and set $\Theta = \theta/\pi^a$. Let Θ_v be the restriction of the divisor of Θ to β_v. This principal divisor is the difference between an effective divisor of degree smaller than g_v and a divisor with support in χ_v. This contradicts condition (6). Thus condition (2) follows from condition (6). Assuming condition (6) we thus have condition (2) and we call $\tilde{Q}_1$, $\tilde{Q}_2$, ..., $\tilde{Q}_h$, the remaining Q's after removing g_v intersecting every β_v. From condition (7) we deduce that the family $\{x(\tilde{Q}_1), x(\tilde{Q}_2), ..., x(\tilde{Q}_h)\}$ is stable since it is the unique point in $S^h(\mathcal{G})$ above $\sum_i e_i x(P_i) - \sum_v g_v v + (g - h - \sum_i e_i).O$. So condition (3) holds. Further the $x(P_i)$ are in $\mathcal{G}_e^0$ so $\sum_i e_i x(P_i) - \sum_v g_v v + (g-h-\sum_i e_i).O$ is in $H^1(\mathcal{G}, \frac{1}{e}\mathbb{Z}) = \mathcal{K}_e^0$. This implies that the $x(\tilde{Q}_i)$ for $1 \leq i \leq h$ are in $\mathcal{G}_e^0$. Since the other $x(Q)$'s are in $\mathcal{G}^0$ condition (1) follows. □

Condition (6) says that special points on β_v are as independent as possible. This will be the case in general. In particular when the genus g_v is zero or when there are enough generic points in χ_v, condition (6) holds.

Condition (7) is purely combinatorial and can easily be checked.

Conditions (4) and (5) are sometimes easily checked. For example, if the R_i and the P_i or Q_i lie on distinct components of $\mathcal{C}_l$ or if their fields of definitions are clearly different (e.g. if the $R_{i,l}$ are generic points on $\mathcal{C}_l$ while the $P_{i,l}$ and $Q_{i,l}$ are not).

REMARK 3: *Although the family $\{Q_1, ..., Q_g\}$ might not be unique, the conclusions of theorem (3.1) apply to all of them.*

When computing families of coverings, one can always make the heuristic assumption that the $(Q_i)_{1 \leq i \leq g}$ can be taken to be residually generic and see if the computation is succesful under this assumption.

4. The case of modular curves

In this section N will be an odd integer. We shall study the modular curves of level $2N$. The curve $X(2N)$ is the Hurwitz space of the following family of coverings.

Let $A_N = (\mathbb{Z}/N\mathbb{Z})^2 \rtimes (\mathbb{Z}/2\mathbb{Z})$ be the group of translations and reflections of the affine plane modulo N. For $w \in \mathbb{Z}/N\mathbb{Z}$ we denote by a_w the translation by w and by b_w the map

$$b_w : (\mathbb{Z}/N\mathbb{Z})^2 \to (\mathbb{Z}/N\mathbb{Z})^2 \text{ defined by } z \mapsto -z + w.$$

All the reflections b_w are conjugated and the center of A_N is trivial.

We consider Galois G-coverings of $\mathbb{P}^1$ ramified over four points with monodromy given by four reflections. These coverings are parametrized by quadruplets $(\sigma_4, \sigma_3, \sigma_2, \sigma_1)$ of reflections such that $\sigma_4\sigma_3\sigma_2\sigma_1 = 1$, up to an inner automorphism of A_N.

Such an equivalent class of quadruplets is characterized by the couple (u, v) up to sign where u and v are the products $u = \sigma_2\sigma_1$ and $v = \sigma_3\sigma_2$. The couple (u, v) generates $(\mathbb{Z}/N\mathbb{Z})^2$. We thus have $|GL_2(N)|/2$ isomorphism classes of coverings with given ordered ramification.

We now compute the action of braids on these quadruplets. We shall represent a quadruplet by the associated couple $\pm(u, v)$. We call $t_{1,2}$ the coloured elementary braid that twists the first two strands and $t_{2,3}$ the one twisting the second and third strands.

We know that $t_{1,2}(\sigma_4, \sigma_3, \sigma_2, \sigma_1) = (\sigma_4, \sigma_3, \sigma_2^{\sigma_2\sigma_1}, \sigma_1^{\sigma_2\sigma_1})$ thus

$$t_{1,2}(u, v) = (u, v + 2u)$$

and similarly

$$t_{2,3}(u, v) = (u - 2v, v).$$

We notice that braid action preserves the determinant $[u, v]$ of u and v. Therefore, our Hurwitz space $\mathcal{H}$ is the disjoint union of $\varphi(N)$ connected components, each a covering of $\mathcal{M}_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}$ of degree $|PSL_2(N)|$.

We shall of course restrict to one such component which we call $\tilde{\mathcal{H}}$, corresponding to couples $\pm(u, v)$ with $[u, v] = 1$. This is nothing but $X(2N)$.

We call $\eta : \tilde{\mathcal{H}} \to \mathbb{P}^1 - \{0, 1, \infty\}$ the Hurwitz map that associates to any point in $\tilde{\mathcal{H}}$ the cross ratio $[x_1, x_2, x_3, x_4]$ of the ramification locus of the associated cover. It extends to $\bar{\eta} : \bar{\mathcal{H}} = Y(2N) \to \mathbb{P}^1$. There is a universal elliptic curve with $2N$ torsion

$$\begin{array}{ccc} \bar{\mathcal{H}} & \longleftarrow & E_{2N} \\ {\scriptstyle \bar{\eta}}\downarrow & & \downarrow{\scriptstyle A_N} \\ \bar{\mathcal{M}}_{0,4} & \longleftarrow & \bar{\mathcal{M}}_{0,5} \end{array}$$

where fibers above 0, 1, ∞ are covers of the special curves $\mathcal{S}_{0,\epsilon}$, $\mathcal{S}_{1,\epsilon}$, $\mathcal{S}_{\infty,\epsilon}$ represented below.

We call $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_\infty$ the localizations at 0, 1 and ∞ of $\bar{\mathcal{M}}_{0,5} \to \bar{\mathcal{M}}_{0,4}$.

Ramified points of $\bar{\eta}$ over 0 correspond to cycles under the action of $t_{1,2}$ and

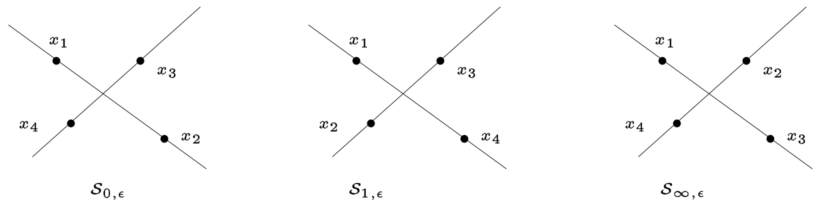

Figure 4: The three basic special curves

thus are parametrized by the $\varphi(N)\psi(N)/2$ possible values of $\pm u$ where $\psi(N) = |\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})|$. The ramification index is N.

For any u of maximal order in $(\mathbb{Z}/N\mathbb{Z})^2$ we call ρ_u the associated ramified point of $\bar{\eta}$ above 0 and $\mathcal{O}_u$ the local ring of $Y(2N)$ at ρ_u. We choose a local parameter π at ρ_u such that $\pi^N = [x_1, x_2, x_3, x_4]$ the local parameter at $0 \in \bar{\mathcal{M}}_{0,4}$. We call $\mathcal{C}_u \to \mathrm{Spec}(\mathcal{O}_u)$ the localization of E_{2N} at ρ_u. The curve $\mathcal{C}_u$ is an admissible G-cover of the curve $\mathcal{S}_0$.

Let us take some $b \in \mathbb{P}^1(\mathbb{C})$ not in $\mathbb{P}^1(\mathbb{R})$. For every value of π in a small real interval $]0, \kappa[$, we choose a basis for $\pi_1(\mathbb{P}^1 - \{0, \pi^N, 1, \infty\}, b)$ depending continuously on π and consider the monodromy $(\sigma_4, \sigma_3, \sigma_2, \sigma_1)$ of the corresponding covering in the family.

We may always assume that $\sigma_1 = b_{-u}$, $\sigma_2 = b_0$, $\sigma_3 = b_v$ with v such that $[u, v] = 1$ and $\sigma_4 = b_{v-u}$.

The special curve $\mathcal{C}_{u,\epsilon}$ is the limit covering when $\pi \to 0$.

We call $\mathfrak{a}$ (resp. $\mathfrak{b}$) the component of $\mathcal{S}_{0,\epsilon}$ containing x_1 and x_2 (resp. x_3 and x_4).

The components of $\mathcal{C}_{u,\epsilon}$ above $\mathfrak{a}$ (resp. $\mathfrak{b}$) are in bijection with the right cosets of A_N modulo the subgroup generated by σ_1 and σ_2 (resp. σ_3 and σ_4) and the singular points of $\mathcal{C}_{u,\epsilon}$ correspond to right cosets of $< \sigma_2\sigma_1 > = < \sigma_4\sigma_3 >$.

For any $W \in (\mathbb{Z}/N\mathbb{Z})^2/ < u >$ we denote by $\mathfrak{a}_W$ (resp. $\mathfrak{b}_W$) the component of $\mathcal{C}_u$ above $\mathfrak{a}$ (resp. $\mathfrak{b}$) associated to the coset $\mathcal{A}_W = \{a_w, b_{-w} | w \in W\} = < b_0, a_u > a_w$ (resp. $\mathcal{B}_W = \{a_w, b_{v-w} | w \in W\} = < b_v, a_u > a_w$).

We see that the components $\mathfrak{a}_W$ and $\mathfrak{b}_W$ follow in this way

$$... \leftrightarrow \mathfrak{b}_{W+v} \leftrightarrow \mathfrak{a}_W \leftrightarrow \mathfrak{b}_W \leftrightarrow \mathfrak{a}_{W-v} \leftrightarrow \mathfrak{b}_{W-v} \leftrightarrow ... \quad (2)$$

where the symbol $\leftrightarrow$ stands for “crosses”.

We see that $\mathcal{C}_{u,\epsilon}$ is a cycle of $2N$ genus zero components mapping alternatively onto $\mathfrak{a}$ and $\mathfrak{b}$.

The curve $\mathcal{C}_u \to \mathrm{Spec}(\mathcal{O}_u)$ is regular at the intersection of two components of its special fiber because the order of any cycle of a_u is N, the order of braid action.

In that case, the intersection graph $\mathcal{G}$ is a cycle with $2N$ vertices and the

Kirchhoff complex $\mathcal{K}$ is just $\mathcal{G}$ itself. So the computations of remark (1) of section 2 are transparent.

Each pair $\{a_w, b_{-w-u}\}$ for $w \in (\mathbb{Z}/N\mathbb{Z})^2$ corresponds to a ramified point ξ_w (with index 2) over x_1 on $\mathcal{C}_u$. The points ξ_w are $2N$-torsion points and define global sections $\xi_w \to Y(2N)$ of the universal elliptic curve with torsion $E_{2N} \to Y(2N)$.

We notice that according to equation (2), the component of $\mathcal{C}_u$ that intersects ξ_w is the $[u,w]$-th above $\mathfrak{a}$ left from the component that intersects ξ_0.

Take $x, y, z, u \in (\mathbb{Z}/N\mathbb{Z})^2$ such that $y, z \notin \{0, x, -x\}$ and let $k = [u,x] \in \mathbb{Z}/N\mathbb{Z}$. We see that the divisor $[\xi_x] + [\xi_{-x}] - 2[\xi_0]$ is principal and we take f_x to be a function with this horizontal divisor. Let $r = [u,y]$ and $s = [u,z]$. For any $t \in \mathbb{Z}/N\mathbb{Z}$ we denote by $||t||$ the absolute value of the residue in $[-N/2, N/2]$.

We want to compute the valuation at ρ_u of the quotient $f_x(\xi_y)/f_x(\xi_z)$. To this end we compute the divisor (f_x) of f_x seen as a function on $\mathcal{C}_u \to \mathrm{Spec}(\mathcal{O}_u)$ up to a multiple of the special fiber $\mathcal{C}_{u,\epsilon}$.

The intersection number of the horizontal part of (f_x) with the components of the special fiber is

- -2 for $\mathfrak{a}_0$,
- $+1$ for $\mathfrak{a}_x$ and $\mathfrak{a}_{-x}$,
- 0 everywhere else.

The same intersection indices are obtained with the vertical divisor

$$\Delta_k = 2k\mathfrak{a}_0 + (2k-1)(\mathfrak{b}_0 + \mathfrak{b}_v) + (2k-2)(\mathfrak{a}_{-v} + \mathfrak{a}_v) + (2k-3)(\mathfrak{b}_{-v} + \mathfrak{b}_{2v}) + \ldots + (\mathfrak{b}_{-(k-1)v} + \mathfrak{b}_{kv}).$$

Indeed, every component of the special fiber crosses itself with multiplicity -2 and its two neighbours with multiplicity 1. The coefficients in Δ_k are thus deduced from the matricial identity below (for $k = 2$).

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{3}$$

Therefore the valuation of $f_x(\xi_y)/f_x(\xi_z)$ at ρ_u is

$$2\min(||r||, ||k||) - 2\min(||s||, ||k||).$$

We call $\theta_{x,y,z}$ the function $f_x(\xi_y)/f_x(\xi_z)$ on $Y(2N)$. It's a unit on $X(2N)$ since torsion points never coalesce on an elliptic curve.

There is a changing level map $Y(2N) \to Y(N)$ which is Galois with group $\mathcal{S}_3$ and corresponds to forgetting the order on the four branch points $\{x_1, x_2, x_3, x_4\}$. It is ramified with index 2 at all points in $\bar{\eta}^{-1}(\{0, 1, \infty\})$ and it maps these points onto the cusps of $Y(N)$. The cusps of $Y(N)$ are in bijection with the points in $\bar{\eta}^{-1}(0)$. We thus shall call $\tilde{\rho}_u$ the cusps of $Y(N)$.

The functions $\theta_{x,y,z}$ are invariant under this $\mathcal{S}_3$ action and thus can be seen as unit functions on $X(N)$. To avoid any confusion we call $\tilde{\theta}_{x,y,z}$ the function $\theta_{x,y,z}$ seen as a function on $Y(N)$.

The valuation of $\tilde{\theta}_{x,y,z}$ at $\tilde{\rho}_u$ is $\min(|||[y,u]|||, |||[x,u]|||) - \min(|||[z,u]|||, |||[x,u]|||)$.

5. The Kirchhoff complex of a marked graph

In this section we extend the method of sections 1, (2) and (3) replacing ordinary jacobians by generalized ones. From a theoretical point of view we believe it is of some interest to define a combinatorial counterpart of generalized jacobians and thus make a link with naive considerations in [7] about distances in trees. In practice this generalization is relevant when due to the lack of rational sections we cannot normalize the functions x and y in the introduction unless we ask them to take value 1 at a point $\mathfrak{m}$ of degree greater than 1. This leads us to the consideration of the generalized jacobian $J_{\mathfrak{m}}$.

Since the methods in this section are very similar to the ones in sections 1, 2 and 3, we shall just stress the differences. The main difficulty is the construction of the Kirchhoff complex of a "non-compact" graph. We again mimic classical integration theory.

We define an *infinite thread* to be a connected graph whose set of vertices is $\mathbb{N}$ and with one edge from m to $m+1$ for any $m \geq 0$. A *thread* of length N is a finite graph whose set of vertices is $[0, N]$ and with one edge from m to $m+1$ for any $0 \leq m \leq N-1$.

Definition 5: *Let $\mathcal{G}$ be a finite graph with genus h, vertices in V and edges in E, $n \geq 2$ an integer and X_1, X_2, ..., X_n a family of (not necessarily distinct) vertices.*

The decorated graph of depth m associated to $\mathcal{G}$ and $X = (X_i)_i$, denoted by ${}^m\mathcal{G}$, is the finite graph obtained by branching a thread of length m on $\mathcal{G}$ at every X_i.

The hung graph associated to $\mathcal{G}$ and $(X_i)_i$, denoted by ${}^\infty\mathcal{G}$, is the limit of ${}^m\mathcal{G}$ when $m \to \infty$ obtained by branching an infinite thread at every X_i.

We denote by ${}^X\mathcal{G}$ the compactification of ${}^\infty\mathcal{G}$ obtained by connecting all the threads in ${}^\infty\mathcal{G}$ at a single extra vertex ω.

We denote by ${}^X_m\mathcal{G}$ the graph obtained by connecting all the threads in ${}^m\mathcal{G}$ at a single extra vertex ω.

Although the topological space ${}^X\mathcal{G}$ is not a graph, it makes sense to speak about edges (these are edges in ${}^\infty\mathcal{G}$) or vertices (vertices in ${}^\infty\mathcal{G}$ plus ω) in it. Let

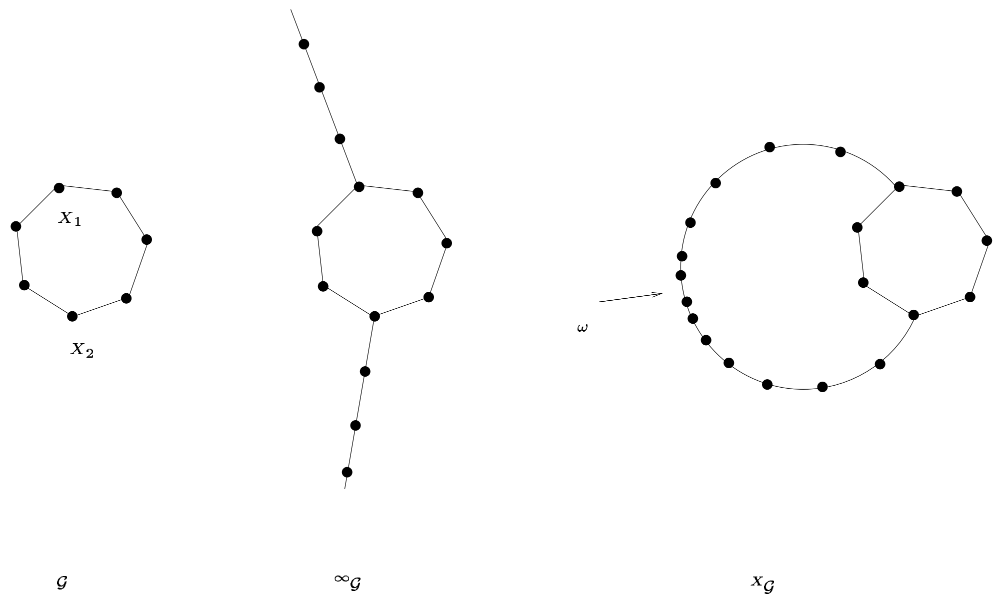

Figure 5: An example of hanging a graph

$C_1({}^X\mathcal{G},\mathbb{R}) = \mathbb{R}^{{}^X\mathcal{G}^1}$ be the space of functions from the set ${}^X\mathcal{G}^1$ of edges in ${}^X\mathcal{G}$ to $\mathbb{R}$. This is not a finite dimensional space. For any edge e of ${}^X\mathcal{G}$ we denote by $d\mu_e$ the uniform measure of total mass 1 on e and set

$$d\mu = \sum_{e\in {}^X\mathcal{G}^1} e d\mu_e.$$

For any positive integer m we call $d\mu_{|m}$ the restriction of $d\mu$ to ${}^m\mathcal{G}$.

Let $H_1({}^X\mathcal{G},\mathbb{R})$ be the first singular homology group of the topological space ${}^X\mathcal{G}$. This can be seen as a finite dimensional subspace of $C_1({}^X\mathcal{G},\mathbb{R})$ with dimension $p = h + n - 1$. Let $H^1({}^X\mathcal{G},\mathbb{R})$ be the space of cocycles (defined to be the dual of $H_1({}^X\mathcal{G},\mathbb{R})$).

If a and b are points on ${}^\infty\mathcal{G}$ (not necessarily vertices) and γ is a continuous piecewise affine path from a to b we define

$$\int_\gamma d\mu \in C_1({}^\infty\mathcal{G},\mathbb{R})$$

as before and if we pick an origin $O \in \mathcal{G}^0$ and take $\mathcal{U}$ to be the universal covering of ${}^\infty\mathcal{G}$ constructed as the space of pathes from O up to homotopy, we can define a map

$$\phi : \mathcal{U} \to H^1({}^X\mathcal{G},\mathbb{R}) \text{ by } \gamma \mapsto (\int_\gamma d\mu,)$$

which makes sense because $\int_\gamma d\mu$ has finite support. The image of $\pi_1({}^\infty\mathcal{G}, O) \subset \mathcal{U}$ by ϕ is a lattice $\mathcal{T}$ of dimension h which is contained in $H^1({}^X\mathcal{G}, \mathbb{Z})$. We denote by

$$\varphi : {}^\infty\mathcal{G} \to T = H^1({}^X\mathcal{G}, \mathbb{R})/\mathcal{T}$$

the induced map on quotients.

It is important to notice that $H^1({}^X\mathcal{G}, \mathbb{R})/\mathcal{T}$ is not compact since the dimension of $H^1({}^X\mathcal{G}, \mathbb{R})$ is $p = h + n - 1$.

For any positive integer k we denote by $\phi^k : \mathcal{U}^k \to H^1({}^X\mathcal{G}, \mathbb{R})$ the sum of ϕ with itself k times and similarly for φ^k.

The map φ^p is invariant under permutation of the terms and thus gives rise to a map $\upsilon : S^p({}^\infty\mathcal{G}) \to T$.

LEMMA 5.1: *The maps φ^p, ϕ^p and υ are surjective.*

We take ${}^X\mathcal{U}$ to be the universal covering of ${}^X\mathcal{G}$ and define for each positive integer m the map

$$\phi_m : {}^X\mathcal{U} \to H^1({}^X\mathcal{G}, \mathbb{R}) \text{ by } \gamma \mapsto (\int_\gamma d\mu_{|m},).$$

We notice that this map is well defined because $d\mu_{|m}$ has finite total mass.

The image of $\pi_1({}^X\mathcal{G}, O) \subset {}^X\mathcal{U}$ by ϕ is the direct sum $\mathcal{T} \oplus \mathcal{T}_m$ of $\mathcal{T}$ and a lattice $\mathcal{T}_m$ of dimension $n-1$ and it is contained in $H^1({}^X\mathcal{G}, \mathbb{Z})$. We denote by

$$\varphi_m : {}^X_m\mathcal{G} \to H^1({}^X\mathcal{G}, \mathbb{R})/\mathcal{T} \oplus \mathcal{T}_m$$

the induced map on quotients. This time $H^1({}^X\mathcal{G}, \mathbb{R})/\mathcal{T} \oplus \mathcal{T}_m$ is a compact torus.

For any positive integer k we denote by $\phi_m^k : {}^X\mathcal{U}^k \to H^1({}^X\mathcal{G}, \mathbb{R})$ the sum of ϕ_m with itself k times and similarly for φ_m^k.

The map φ_m^p is invariant under permutation of the terms and thus gives rise to a map $\upsilon_m : S^p({}^X_m\mathcal{G}) \to H^1({}^X\mathcal{G}, \mathbb{R})/\mathcal{T} \oplus \mathcal{T}_m$.

The maps φ_m^p, ϕ_m^p and υ_m are surjective. This is proved like in section 3 (a continuous map between two tori of the same dimension which is non-singular on the homology is surjective).

Now let y be in $H^1({}^X\mathcal{G}, \mathbb{R})/\mathcal{T}$. For any integer m there is a unique stable point $x_m \in ({}^X_m\mathcal{G})^p$ such that $\upsilon_m(x_m) = y \bmod \mathcal{T} + \mathcal{T}_m$.

There exist two integers M and N such that for any $m \geq M$ the point x_m is in $({}^N\mathcal{G})^p$ and is independent of m. This limit point is mapped onto y by υ. □

DEFINITION 6: *A point $x = \{x_1, ..., x_p\}$ in $S^p({}^\infty\mathcal{G})$ is said to be stable if and only if there exist p (closed) edges $(e_i)_{1 \leq i \leq p}$ in ${}^\infty\mathcal{G}$ such that $x_i \in e_i$ and ${}^X\mathcal{G} - \cup_i e_i$ is connected and simply connected. Under the same conditions we shall say that $(x_1, ..., x_p) \in ({}^\infty\mathcal{G})^p$ is stable. We denote by $\mathcal{J}$ (resp. $\mathcal{K}$) the set of stable points in $({}^\infty\mathcal{G})^p$ (resp. $S^p({}^\infty\mathcal{G})$).*

Lemma 5.2: *A point $x = \{x_1, ..., x_p\}$ is stable if and only if υ is locally surjective at x (the image of any neighbourhood of x is a neighbourhood of $\upsilon(x)$).*

This results from lemma (2.8) because a point is stable if and only if it is stable with respect to the map υ_m for m big enough. □

Theorem 5.1: *The restriction of the integration map υ to the set $\mathcal{K}$ of stable points in $S^p({}^\infty\mathcal{G})$ is an homeomorphism that turns $\mathcal{K}$ into a locally finite cell decomposition of the cylinder $T = H^1({}^X\mathcal{G}, \mathbb{R})/\mathcal{T}$. The p-dimensional cells in $\mathcal{K}$ are the stable p-dimensional cells in $S^p({}^\infty\mathcal{G})$. They are parallelograms and correspond to "spanning trees" in ${}^X\mathcal{G}$. The set of vertices of $\mathcal{K}$ is the group $H^1({}^X\mathcal{G}, \mathbb{Z})/\mathcal{T}$.*

For any integer $e \geq 1$ let ${}^\infty\mathcal{G}_e$ be the e-th division of ${}^\infty\mathcal{G}$ obtained by cutting any edge into e smaller edges of equal length. The associated Kirchhoff cell complex $\mathcal{K}_e$ is the e-th division of $\mathcal{K}$.

A point in T is said to be *rigid* if it has a unique preimage by υ. Rigid points are dense in T.

Let K, $\bar{K}$, $k = \bar{k}$, $\mathcal{O}$, ν, π, $\mathcal{C}$ be as in section 3. Let $n \geq 2$ be an integer and $y_{1,K}$, $y_{2,K}$, ..., $y_{n,K}$, o points in $\mathcal{C}_K(K)$ with y_1, ..., y_n, o the corresponding sections that we assume to cross the special fiber $\mathcal{C}_k$ at pairwise distinct smooth points $y_{1,k}$, $y_{2,k}$, ..., $y_{n,k}$, o_k. Let $\mathcal{G}$ be the graph of $\mathcal{C}_k$ and $X_1 = x(y_1)$, ..., $X_n = x(y_n)$ and $O = x(o)$ the vertices of $\mathcal{G}$ associated to the components that cross y_1, ..., y_n and o. We call ${}^\infty\mathcal{G}$, ${}^X\mathcal{G}$, $\mathcal{K}$, υ,..., the associated complexes and maps. We set $\mathfrak{m} = [y_1] + [y_2] + ... + [y_n]$ (the "module"). Let $\dot{\mathcal{C}}$ be the model of $\dot{\mathcal{C}}_K = \mathcal{C}_K - \{y_1, ..., y_n\}$ with special curve $\dot{\mathcal{C}}_k$ having infinitely many genus zero components corresponding to all concentric circles with radius a multiple of π around the y_i's. To any point P in $\dot{\mathcal{C}}_K(\bar{K})$ one can associate an element $x(P)$ in ${}^\infty\mathcal{G}(\mathbb{Q})$ and to any divisor one can associate a point of the Kirchhoff complex $\mathcal{K}$ of ${}^X\mathcal{G}$. In the same spirit as in section 3 we thus can solve the following

Problem 2: *Given a divisor D of degree $g + n - 1$ on $\dot{\mathcal{C}}_K$, we assume there exists an effective divisor E of degree $g + n - 1$ and $\mathfrak{m}$-equivalent to D (it may not be unique).*

The Zariski closure of D (resp. E) is a divisor on $\dot{\mathcal{C}}$ that we shall denote by D (resp. E) also. Knowing how D intersects the special fiber $\dot{\mathcal{C}}_\epsilon$ can we deduce how E intersects $\dot{\mathcal{C}}_\epsilon$?

6. A simple computational example

In this section we treat a simple example. We shall again take advantage of several special fibers at a time. This way we avoid the heavy computations with series like in [7] and take this opportunity to study a greater variety of degenerate covers. Although these shortcuts may not be possible for all covers, we believe they have to be illustrated once.

We consider degree 3 coverings of $\mathbb{P}^1$ ramified over four points and with ramification type $(3, 3, 2.1, 2.1)$. These are genus 1 coverings. Once given the four ramification points $x_1 = 0$, $x_2 = \infty$, $x_3 = \eta$, $x_4 = 1$ and a basis for the fundamental group, the two possible monodromies are

$$\sigma_1 = [1,2,3],\ \sigma_2 = [1,3,2],\ \sigma_3 = [1,2],\ \sigma_4 = [1,2]$$

and

$$\sigma_1 = [1,2,3],\ \sigma_2 = [1,2,3],\ \sigma_3 = [3,2],\ \sigma_4 = [1,3].$$

The (complete) Hurwitz curve $\mathcal{H}$ parametrizing these covers is a genus zero curve. We see η as a map from $\mathcal{H}$ to $\mathbb{P}^1$ that associates to any point in $\mathcal{H}$ the cross-ratio of the ramification locus in the corresponding cover. The map η is a degree two map ramified above $\eta = 0$ (corresponding to $x_1 = x_3$) and $\eta = \infty$ (corresponding to $x_2 = x_3$).

We call $T \in \mathcal{H}$ the unique point above 0, U the unique point above ∞ and V, W the points above 1.

There is a universal curve

$$\begin{array}{ccc} \mathcal{H} & \longleftarrow & \mathcal{C} \\ \eta \downarrow & & \downarrow \phi \\ \bar{\mathcal{M}}_{0,4} & \longleftarrow & \bar{\mathcal{M}}_{0,5} \end{array}$$

We call $A \to \mathcal{M}_{0,4}$ the unique point on $\mathcal{C}$ mapped onto x_1 by ϕ. We call B the unique point above x_2 and C (resp. E) the unique ramified point above x_3 (resp. x_4). We call D (resp. F) the unique non ramified point above x_3 (resp. x_4). We call $\mathcal{C}_{T,\epsilon}$, $\mathcal{C}_{U,\epsilon}$, $\mathcal{C}_{V,\epsilon}$, $\mathcal{C}_{W,\epsilon}$, the special curves at the corresponding points.

We draw these special curves on figures 6 and 7 (It must be noted that the fiber at W we have represented is obtained after base change of degree 3. The actual fiber would be a quotient of it by a group of order 3. In particular, the thickness at the only intersection point in $\mathcal{C}_{W,\epsilon}$ is $1/3$).

The surface $\mathcal{C} \to \mathcal{H}$ is regular except at two points. One intersection point on each of the special curves $\mathcal{C}_{U,\epsilon}$ and $\mathcal{C}_{T,\epsilon}$ is not smooth.

Indeed the product $\sigma_2\sigma_3 = [1][2,3]$ has two cycles of unequal length and the corresponding nodes on the special fiber $\mathcal{C}_{T,\epsilon}$ have thickness 1 and 2. Thus one of them is singular.

We blow up $\mathcal{C}$ at these two singular points and obtain new special curves $\hat{\mathcal{C}}_{U,\epsilon}$ and $\hat{\mathcal{C}}_{T,\epsilon}$.

We consider two functions on $\mathcal{C}$. The first one is just ϕ. The second one is obtained from the holomorphic differential ω on the genus one curve $\mathcal{H}$. The divisor of ω is 0. The divisor of the differential $d\phi$ is

$$d\phi = 2[A] - 4[B] + [C] + [E]$$

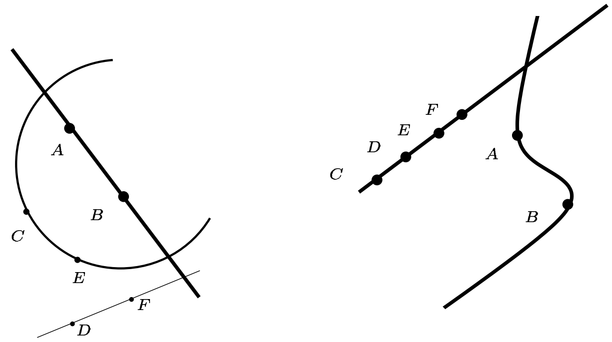

Figure 6: The special fibers $\mathcal{C}_{V,\epsilon}$ and $\mathcal{C}_{W,\epsilon}$

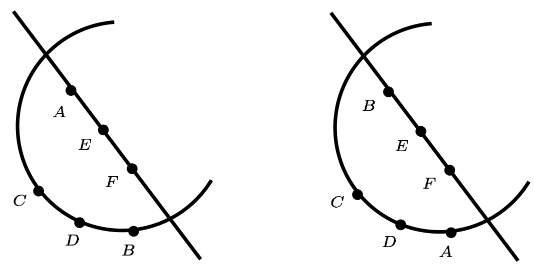

Figure 7: The special fibers $\mathcal{C}_{U,\epsilon}$ and $\mathcal{C}_{T,\epsilon}$

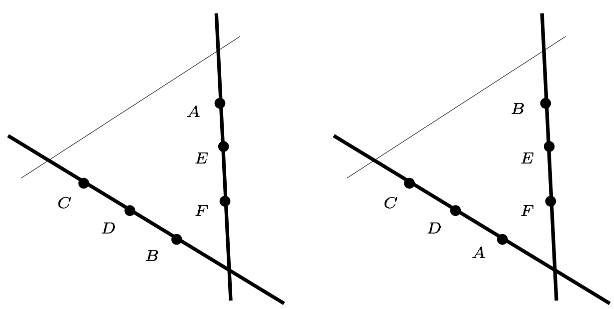

Figure 8: The special fibers $\hat{\mathcal{C}}_{U,\epsilon}$ and $\hat{\mathcal{C}}_{T,\epsilon}$

thus the function $\gamma = d\phi/\omega$ has the same divisor. We set $\theta = \gamma/\phi$ and check that

$$(\theta) = [C] + [E] - [B] - [A].$$

Since θ is defined only up to a constant factor we ask that $\theta(D) = 1$.

Since $\mathcal{H}$ has genus 0, we call f the function on $\mathcal{H}$ taking values 0 at T, ∞ at U and 1 at W.

Because θ has two poles it is of degree two i.e. the field of functions $\mathbb{C}(f)(\mathcal{C})$ of $\mathcal{C}$ is a degree two extension of $\mathbb{C}(f)(\theta)$. Similarly, the function ϕ has degree

3. Thus there is an equation $\mathcal{E}(\theta,\phi)=0$ of degree 3 in θ and 2 in ϕ and with coefficients in $\mathbb{C}(f)$.

We write

$$\mathcal{E}(\theta,\phi)=a_0+a_1\phi+a_2\theta+a_3\phi^2+a_4\phi\theta+a_5\theta^2+a_6\phi^2\theta+a_7\phi\theta^2+a_8\theta^3+a_9\phi^2\theta^2+a_{10}\phi\theta^3+a_{11}\phi^2\theta^3.$$

Local multiplicities of poles at B imply that $a_{11}=a_9=a_6=0$. Indeed the term $a_{11}\phi^2\theta^3$ is the only term with minimal order (-9) in the equation. It thus must be zero. And similarly for $a_9\phi^2\theta^2$ which is the only term of order -8 and $a_6\phi^2\theta$ of order -7.

Then the local multiplicities at A similarly show that $a_8=a_5=a_2=0$ in this order. Further, the cancellation of $d\phi$ at C implies $a_4=0$. Indeed we have

$$d\phi(a_1+2a_3\phi+a_4\theta+a_7\theta^2+a_{10}\theta^3)=-d\theta(a_4\phi+2a_7\theta\phi+3a_{10}\theta^2\phi)$$

and since $d\theta$, $d\phi$, θ and ϕ are of order 0, 1, 1 and 0 we deduce that $a_4=0$.

Thus
$$\mathcal{E}(\theta,\phi)=a_0+a_1\phi+a_3\phi^2+a_7\phi\theta^2+a_{10}\phi\theta^3.$$

Let us now call $\{\phi\}_V$ the vertical part of the divisor of ϕ at $\mathcal{C}_{V,\epsilon}$. Since all intersection indices are zero we have $\{\phi\}_V=0$. We find in a similar way $\{\theta\}_V=(CE)$ where (CE) stands for the component carrying C and E.

Similarly, $\{\phi\}_W=0$ and $\{\theta\}_W=-2/3(AB)$.

It makes sense to compute the divisors of θ and ϕ restricted to the blown up fibers $\hat{\mathcal{C}}_{U,\epsilon}$ and $\hat{\mathcal{C}}_{T,\epsilon}$. We denote by $(\emptyset)$ the extra genus zero component.

Thus $\{\phi\}_U=-(\emptyset)-2(BCD)$ and $\{\theta\}_U=0$.

And similarly $\{\phi\}_T=(\emptyset)+2(CDA)$ and $\{\theta\}_T=0$.

We observe that $\phi(C)$ and $\phi(E)$ are finite when the curve is regular. Further, the study of special curves shows that $\phi(C)$ has a pole of order two at T and a zero of order two at U and is finite at V and W. Similarly, $\phi(E)$ is shown to have no pole and no zero. We normalize taking $a_0=1$ and since $\phi(C)$ and $\phi(E)$ are the two solutions of $\mathcal{E}(0,\phi)=0$ we deduce that there exist two constants k_1 and k_2 such that

$$a_3=k_1k_2f^{-2} \text{ and } a_1=-k_2-k_1f^{-2}.$$

We now study the value of $\phi\theta^3$ at A. This is nothing but $-a_0/a_{10}$. Examination of special fibers shows that this function has a double pole at W and a double zero at T. Therefore

$$a_{10}=k_3(f-1)^2f^{-2}$$

where k_3 is a constant.

Similarly we show that $\theta(F)$ and $\phi(F)$ are constants k_4 and k_5 so

$$\mathcal{E}(k_4,k_5)=0.$$

We deduce that $a_7 = k_6 + k_7 f^{-1} + k_8 f^{-2}$ where k_6, k_7 and k_8 are constants depending on k_1,...,k_5.

There remains to determine the constants k_1, ..., k_8. This is done by specializing. All the constants can be obtained from a single specialization. For example, if we set $f = 0$ in $\mathcal{E}$ and ask that the corresponding cover be ramified above $\phi = 0$, $\phi = 1$, $\phi = \infty$ we find that $k_2 = 1$ and $4k_6^2 = 27k_1k_3^2$.

If we substitute $\theta = 0$ in $\mathcal{E}$ we find that $\eta = f^2/k_1$. Since both f and η take value 1 at W we deduce that $k_1 = 1$.

Since point $D \in \mathcal{C}$ has coordinates $(1, \eta)$ we now set $\theta = 1$ and $\phi = f^2$ in $\mathcal{E}$ and find

$$k_3 = -k_6 \text{ and } k_7 = -2k_6 \text{ and } k_8 = k_6.$$

We deduce that $k_6 = 27/4$.

In the end, the equation of the universal curve $\mathcal{C}$ is

$$4\phi^2 - 4\phi(f^2 + 1) + 4f^2 + 27(f-1)^2\phi\theta^2 - 27\phi\theta^3(f-1)^2 = 0.$$

References

[1] C. Berge. *Graphs and hypergraphs.* North Holland, 1970.

[2] Béla Bollobás. *Modern Graph Theory.* Graduate texts in Mathematics. Springer, 1998.

[3] S. Bosch and W. Lütkebohmert. Stable reduction and uniformization of abelian varieties ii. *Invent. math.*, 1984:257–297, 1984.

[4] S. Bosch and W. Lütkebohmert. Stable reduction and uniformization of abelian varieties i. *Math. Ann.*, 270:349–379, 1985.

[5] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models.* Ergebnisse der Math. und ihrer Grenzgebiete. Springer, 1990.

[6] Glen E. Bredon. *Topology and geometry*, volume 139 of *GTM.* Springer Verlag, 1993.

[7] J.-M. Couveignes. Tools for the computation of families of covers. In H. Völklein, P. Müller, D. Harbater, and J. G. Thompson, editors, *Aspects of Galois Theory*, volume 256 of *London Mathematical Society Lecture Note Series*, pages 38–65. Cambridge University Press, 1999.

[8] Bas Edhixoven. On Néron models, divisors and modular curves. *Prépublication de l'Université de Rennes I*, pages 1–26, 1999.

[9] J. Fresnel and M. van der Put. Uniformisation des variétés abéliennes. *Ann. Fac. Sc. de Toulouse*, 6:7–42, 1994.

[10] Louis Granboulan. Construction d'une extension régulière de $\mathbb{Q}(t)$. *Experimental Math.*, 5:1–13, 1996.

[11] D. Harbater. Galois coverings of the arithmetic line. *Lecture Notes in Math.*, 1240:165–195, 1987.

[12] D. Kubert and S. Lang. Units in the modular function field ii. A full set of units. *Math. Ann.*, 218:175–189, 1975.

[13] Serge Lang. *Fundamentals of diophantine geometry.* Springer, 1983.

[14] G. Malle and B.H. Matzat. *Inverse Galois theory.* Springer Monographs in Mathematics. Springer, 1999.

[15] B.H. Matzat. *Konstruktive Galoistheorie.* Springer, 1987.

[16] D. Mumford. An analytic construction of degenerating abelian varieties over complete rings. *Compositio Math.*, 24:239–272, 1972.

[17] T. Oda and C.S. Seshadri. Compactifications of the generalized jacobian variety. *Trans. Amer. Math. Soc.*, 253:1–90, 1979.

[18] M. Raynaud. Spécialisation du foncteur de Picard. *Publ. Math. Inst. Hautes Études Sci.*, 38:27–76, 1970.

[19] Jean-Pierre Serre. *Arbres, amalgames,* SL_2, volume 46 of *astérisque.* Société Mathématique de France, 1983.

[20] Joseph H. Silverman. *Advanced topics in the arithmetic of elliptic curves.* Graduate Texts in Mathematics. Springer, 1994.

[21] Helmut Völklein. *Groups as Galois groups – an Introduction.* Cambridge Studies in Advanced Math. Cambridge Univ. Press, 1996.