ON QUANTUM HUYGENS PRINCIPLE AND RAYLEIGH SCATTERING

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ABSTRACT. We prove several minimal photon/phonon velocity estimates below the ionization threshold for a particle system coupled to the quantized electromagnetic or phonon field. Using some of these results, we prove the asymptotic completeness (for Rayleigh scattering) on the states for which the expectation of either the photon/phonon number operator or an operator testing the photon/phonon infrared behaviour is uniformly bounded on corresponding dense sets. By extending a recent result of De Roeck and Kuipiainen in a straightforward way, we show that the second of these conditions is satisfied for the spin-boson model.

1. Introduction

In this paper we study the long-time dynamics of a non-relativistic particle system coupled to the quantized electromagnetic or phonon field. For energies below the ionization threshold, we prove several lower bounds on the growth of the distance of the escaping photons to the particle system. Using some of these results, we prove asymptotic completeness (for Rayleigh scattering) on the states for which the expectation of the photon number is bounded uniformly in time.

Model. First, we consider the standard model of non-relativistic quantum electrodynamics in which particles are minimally coupled to the quantized electromagnetic field. The state space for this model is given by $Rd\Gamma(\omega)$ (see Supplement I for the definition). The operator $H := H_p \otimes F$, where $H_p$ is the particle state space, $F$ is the bosonic Fock space, $F \equiv \Gamma(h) := C^\infty_0 \otimes h$, based on the one-photon space $h := L^2(\mathbb{R}, C^2)$ ($\otimes_n^\infty$ stands for the symmetrized tensor product of $n$ factors, $C^2$ accounts for the photon polarization). Its dynamics is generated by the Hamiltonian

$$H = \sum_{j=1}^n \frac{1}{2m_j} (-i\nabla_{x_j} - \kappa_j A_\xi(x_j))^2 + U(x) + H_f. \quad (1.1)$$

Here, $m_j$ and $x_j$, $j = 1, \ldots, n$, are the (‘bare’) particle masses and the particle positions, $U(x) = \langle x \rangle$ is the total potential affecting the particles, and $\kappa_j$ are coupling constants related to the particle charges. Moreover, $A_\xi := \xi \ast A$, where $\xi$ is an ultraviolet cut-off satisfying e.g. $|\partial^m \xi(k)| \lesssim \langle k \rangle^{-3}$, $|m| = 0, \ldots, 3$, and $A(y)$ is the quantized vector potential in the Coulomb gauge (div $A(y) = 0$), describing the quantized electromagnetic field and given by

$$A_\xi(y) = \sum_{\lambda=1,2} \int \frac{dk}{\sqrt{2\omega(k)}} \xi(k)\varepsilon_\lambda(k)(e^{ik\cdot y}a_\lambda(k) + e^{-ik\cdot y}a_\lambda^*(k)).$$

Here, $\omega(k) = |k|$ denotes the photon dispersion relation ($k$ is the photon wave vector), $\lambda$ is the polarization, and $a_\lambda(k)$ and $a_\lambda^*(k)$ are photon annihilation and creation operators acting on the Fock space $F$ (see Supplement I for the definition). The operator $H_f$ is the quantum hamiltonian of the quantized electromagnetic field, describing the dynamics of the latter, given by $H_f = d\Gamma(\omega)$, where $d\Gamma(\tau)$ denotes the lifting of a one-photon operator $\tau$ to the photon Fock space, $d\Gamma(\tau)|_C = 0$ for $n = 0$ and, for $n \geq 1$,

$$d\Gamma(\tau)|_C = \sum_{j=1}^n 1 \otimes \cdots 1 \otimes \tau \otimes 1 \otimes \cdots 1. \quad (1.2)$$

(See Supplement II for definitions related to the creation and annihilation operators and for the expression of $d\Gamma(\tau)$ in terms of these operators. Here and in what follows, the integrals without indication of the domain of integration are taken over entire $\mathbb{R}^3$.)
We assume that $U(x) \in L^2_{\text{loc}}(\mathbb{R}^3)$ and is either confining or relatively bounded with relative bound 0 w.r.t. $-\Delta_x$, so that the particle hamiltonian $H_p := -\sum_{j=1}^{n} \frac{1}{2m_j} \Delta_{x_j} + U(x)$, and therefore the total hamiltonian $H$, are self-adjoint.

This model goes back to the early days of quantum mechanics (it appears in the review [20] as a well-known model and is elaborated in an important way in [53]); its rigorous analysis was pioneered in [21, 22] (see [56, 62] for extensive references).

Next we consider the standard phonon model of the solid state physics (see e.g. [45]). The state space for it is given by $H := H_p \otimes F$, where $H_p$ is the particle state space and $F \equiv \Gamma(\mathfrak{h}) = \mathbb{C} \otimes_{m=1}^{\infty} \otimes_{\mathfrak{h}}^{\otimes} \mathfrak{h}$ is the bosonic Fock space based on the one-phonon space $\mathfrak{h} := L^2(\mathbb{R}^3, \mathbb{C})$. Its dynamics is generated by the Hamiltonian

$$H := H_p + H_f + I(g),$$  

(1.3)

acting on $H$, where $H_p$ is a self-adjoint particle system Hamiltonian, acting on $H_p$, and $H_f = d\Gamma(\omega)$ is the phonon hamiltonian acting on $F$, where $\omega = \omega(k)$ is the phonon dispersion law ($k$ is the phonon wave vector). For acoustic phonons, $\omega(k) \approx |k|$ for small $|k|$ and $c \leq \omega(k) \leq c^{-1}$, for some $c > 0$, away from 0, while for optical phonons, $c \leq \omega(k) \leq c^{-1}$, for some $c > 0$, for all $k$. To fix ideas, we consider below only the most difficult case $\omega(k) = |k|$.

The operator $I(g)$ acts on $H$ and represents an interaction energy, labeled by a coupling family $g(k)$ of operators acting on the particle space $H_p$. In the simplest case of linear coupling (the dipole approximation in QED or the phonon models), $I(g)$ is given by

$$I(g) := \int (g^*(k) \otimes a(k) + g(k) \otimes a^*(k)) dk,$$  

(1.4)

where $a^*(k)$ and $a(k)$ are the phonon creation and annihilation operators acting on $F$, and $g(k)$ is a family of operators on $H_p$ (coupling operators), for which we assume the following condition

$$\|\eta_2^{|\mu|} |\partial^\mu g(k)\|_{H_p} \lesssim |k|^{1-|\mu|-2|\mu|}, \quad |\mu| \leq 2,$$  

(1.5)

where $\eta_1$ and $\eta_2$ are bounded, positive operators with unbounded inverses, the specific form of which depends on the models considered and will be given below.

A primary example for the particle system to have in mind is an electron in a vacuum or in a solid in an external potential $V$. In this case, $H_p = \epsilon(p) + V(x)$, with $\epsilon(p)$ being the standard non-relativistic kinetic energy, $\epsilon(p) = \frac{1}{2m} |p|^2 \equiv -\frac{1}{2m} \Delta_x$ (the Nelson model), or the electron dispersion law in a crystal lattice (a standard model in solid state physics), acting on $H_p = L^2(\mathbb{R}^3)$. The coupling family is given by $g(k) = |k|^3 \xi(k) e^{ikx}$, where $\xi(k)$ is the ultraviolet cut-off, satisfying e.g. $|\partial^m \xi(k)| \lesssim (k)^{-2-\mu}$, $m = 0, \ldots, 3$ (and therefore $g(k)$ satisfies (1.5), with $\eta_1 = 1$ and $\eta_2 = (x)^{-1}$ with $\langle x \rangle = (1 + |x|^2)^{1/2}$). For phonons, $\mu = 1/2$, and for the Nelson model, $\mu \geq -1/2$. To have a self-adjoint operator $H$ we assume that $V$ is a Kato potential and that $\mu \geq -1/2$. This can be easily upgraded to an $N$–body system (e.g. an atom or a molecule, see e.g. [37, 56]). Another example – the spin-boson model – will be defined below.

Note that the QED hamiltonian (1.1) can be written in the form (1.3), with $I(g)$ being quadratic in the creation and annihilation operators $a_{\sigma}^\dagger(h,k)$, and the coupling functions satisfying estimates of the form (1.5) with $\mu = -1/2$, $\eta_1 = (p)^{-1}$ or 1, and $\eta_2 = (x)^{-1}$. However, once we performed the generalized Pauli-Fierz transform of [55] (see below), the infrared behaviour of $g$ improves considerably.

A key fact here is that for the particle models discussed above, there is a spectral point $\Sigma \in \sigma(H)$, called the ionization threshold, s.t. below $\Sigma$, the particle system is well localized:

$$\|\langle p \rangle^2 e^{\delta|x|} f(H)\| \lesssim 1,$$  

(1.6)

for any $0 \leq \delta < \text{dist}(\text{supp} f, \Sigma)$ and any $f \in C_0^\infty((-\infty, \Sigma))$. In other words, states decay exponentially in the particle coordinates $x$ ([34, 6, 7]). To guarantee that $\Sigma > \inf \sigma(H_p) \geq \inf \sigma(H)$, we assume that the potentials $U(x)$ or $V(x)$ are such that the particle hamiltonian $H_p$ has discrete eigenvalues below the essential spectrum ([34, 6, 7]). Furthermore, $\Sigma$, for which (1.6) is true, is given by $\Sigma := \lim_{R \to \infty} \inf_{\varphi \in C^\infty(\varphi, H \varphi)} \langle \varphi | H \varphi \rangle$, where the infimum is taken over $D_R = \{ \varphi \in D(H) \mid \varphi(x) = 0 \text{ if } |x| < R, \| \varphi \| = 1 \}$ (see [34]; $\Sigma$ is close to $\inf \sigma_{\text{ess}}(H_p)$).

An abstract version of (1.6) is that there is $\Sigma \in \sigma(H)$ s.t. the following estimate holds

$$\|\eta_2^{-n} \eta_1^{-m} f(H)\| \lesssim 1, \quad 0 \leq n, m \leq 2,$$  

(1.7)
for any $f \in C_0^\infty((\infty, \Sigma))$ and for some $\Sigma > \text{inf } \sigma(H_\rho)$. We can think of the spin-boson model as having $\Sigma = \infty$.

**Problem.** In all above cases, the Hamiltonian $H$ is self-adjoint and generates the dynamics through the Schrödinger equation,

$$i\partial_t \psi_t = H\psi_t.$$  \hspace{1cm} (1.8)

As initial conditions, $\psi_0$, we consider states below the ionization threshold $\Sigma$, i.e. $\psi_0$ in the range of the spectral projection $E_{(-\infty,\Sigma)}(H)$. In other words, we are interested in processes, like emission and absorption of radiation, or scattering of photons on an electron bound by an external potential (created e.g. by an infinitely heavy nucleus or impurity of a crystal lattice), in which the particle system (say, an atom or a molecule) is not being ionized.

Denote by $\Phi_j$ and $E_j$ the eigenfunctions and the corresponding eigenvalues of the Hamiltonian $H$, below $\Sigma$, i.e. $E_j < \Sigma$. The following are the key characteristics of the evolution of a physical system, in progressive order the refined information they provide and in our context:

- **Local decay** stating that some photons are bound to the particle system while others (if any) escape to infinity, i.e. the probability that they occupy any bounded region of the physical space tends to zero, as $t \to \infty$.

- **Minimal photon velocity bound** with speed $c$ stating that, as $t \to \infty$, with probability $\to 1$, the photons are either bound to the particle system or depart from it with the distance $\geq c' t$, for any $c' < c$.

Similarly, if the probability that at least one photon is at the distance $\geq c't$, $c'' > c$, from the particle system vanishes, as $t \to \infty$, we say that the evolution satisfies the **maximal photon velocity bound** with speed $c$.

- **Asymptotic completeness** on the interval $(-\infty, \Sigma)$ stating that, for any $\psi_0 \in \text{Ran } E_{(-\infty,\Sigma)}(H)$, and any $\epsilon > 0$, there are photon wave functions $f_{j\epsilon} \in \mathcal{F}$, with a finite number of photons, s.t. the solution, $\psi_t = e^{-itH}\psi_0$, of the Schrödinger equation, (1.8), satisfies

\begin{equation}
\limsup_{t \to \infty} \|e^{-itH}\psi_0 - \sum_j e^{-\text{i}E_j t} \Phi_j \otimes_s e^{-\text{i}H_{j\epsilon} t} f_{j\epsilon}\| \leq \epsilon.
\end{equation}

(IIIt will be shown in the text that $\Phi_j \otimes_s f_{j\epsilon}$ is well-defined, at least for the ground state $(j = 0)$.) In other words, for any $\epsilon > 0$ and with probability $\geq 1 - \epsilon$, the Schrödinger evolution $\psi_t$ approaches asymptotically a superposition of states in which the particle system with a photon cloud bound to it is in one of its bound states $\Phi_j$, with additional photons (or possibly none) escaping to infinity with the velocity of light.

The reason for $\epsilon > 0$ in (1.9) is that for the state $\Phi_j \otimes_s f$ to be well defined, as one would expect, one would have to have a very tight control on the number of photons in $f$, i.e. the number of photons escaping the particle system. (See the remark at the end of Subsection 5.4 for a more technical explanation.) For massive bosons $\epsilon > 0$ can be dropped (set to zero), as the number of photons can be bound by the energy cut-off.

We define the photon velocity in terms of its space-time (and sometimes phase-space-time) localization. In a quantum theory this is formulated in terms of quantum localization observables and related quantum probabilities. We describe the photon position by the operator $y := i\nabla_k$ on $L^2(\mathbb{R}^3)$, canonically conjugate to the photon momentum $k$. To test the photon localization, we use the observables $d\Gamma(1_\Omega(y))$, where $1_\Omega(y)$ denotes the characteristic function of a subset $\Omega$ of $\mathbb{R}^3$. We also use the localization observables $\Gamma(1_\Omega(y))$, where $\Gamma(\chi)$ is the lifting of a one-photon operator $\chi$ (e.g. a smoothed out characteristic function of $y$) to the photon Fock space, defined by

$$\Gamma(\chi) = \oplus_{n=0}^\infty (\otimes^n \chi),$$  \hspace{1cm} (1.10)

(so that $\Gamma(e^b) = e^{d\Gamma(b)}$), and then to the space of the total system. Let also $T_g = \Gamma(\tau_g)$, with $\tau_g : f(y) \to f(g^{-1}y)$. The observables $d\Gamma(1_\Omega(y))$ and $\Gamma(1_\Omega(y))$ have the following natural properties:

- $d\Gamma(1_{\Omega_2})(y) = d\Gamma(1_{\Omega_1}(y)) + d\Gamma(1_{\Omega_2}(y))$ and $\Gamma(1_{\Omega_1}(y))\Gamma(1_{\Omega_2}(y)) = 0$, for $\Omega_1$ and $\Omega_2$ disjoint,

- $T_g X_\Omega(y)T_g^{-1} = X_{g^{-1}\Omega}(y)$, where $X_\Omega(y)$ stands for either $d\Gamma(1_{\Omega_1}(y))$ or $\Gamma(1_{\Omega_1}(y))$.

The observables $d\Gamma(1_\Omega(y))$ can be interpreted as giving the number of photons in Borel sets $\Omega \subset \mathbb{R}^3$. They are closely related to those used in [24, 32, 47] (and discussed earlier in [49] and [1]) and are consistent with a theoretical description of the detection of photons (usually via the photoelectric effect, see e.g. [50]). The
quantity \( \langle \psi, \Gamma(1\Omega(y))\psi \rangle \) is interpreted as the probability that the photons are in the set \( \Omega \) in the state \( \psi \). This said, we should mention that the subject of photon localization is still far from being settled.\(^1\)

The fact that for photons the observables we use depend on the choice of polarization vector fields, \( \varepsilon_\lambda(k) \), \( \lambda = 1, 2 \), is not an impediment here as our results imply analogous results for e.g., localization observables of Mandel [49] and of Amrein and Jauch and Piron [1, 43]: \( \text{d}\Gamma(f_{\Omega}^{\text{man}}) \) and \( \text{d}\Gamma(f_{\Omega}^{\text{ip}}) \), where \( f_{\Omega}^{\text{man}} := P^\perp 1\Omega(y)P^\perp \) and \( f_{\Omega}^{\text{ip}} := 1\Omega(y) \cap P^\perp \), respectively, acting in the Fock space based on the space \( h = L^2_{\text{trans}}(\mathbb{R}^3; \mathbb{C}) := \{ f \in L^2(\mathbb{R}^3; \mathbb{C}) : k \cdot f(k) = 0 \} \) instead of \( h = L^2(\mathbb{R}^3; \mathbb{C}) \). Here \( P^\perp : f(k) \rightarrow |k|^2 k \cdot f(k) \) is the orthogonal projection on the transverse vector fields and, for two orthogonal projections \( P_1 \) and \( P_2 \), the symbol \( P_1 \cap P_2 \) stand for the orthogonal projection on the largest subspace contained in \( \text{Ran} \, P_1 \) and \( \text{Ran} \, P_2 \).

We say that the system obeys the quantum Huygens principle if the Schrödinger evolution, \( \psi_t = e^{-itH}\psi_0 \), obeys the estimates

\[
\int_1^\infty dt \, t^{-\alpha} \left\| \text{d}\Gamma(\chi_{\Omega(y)}^1) \frac{1}{2} \psi_t \right\|^2 \lesssim \left\| \psi_0 \right\|^2, \tag{1.11}
\]

for some norm \( \| \psi_0 \|_0 \), some 0 < \( \alpha' \leq 1 \), and for any \( \alpha < 0 \) and \( c > 0 \) such that either \( \alpha < 1 \), or \( \alpha = 1 \) and \( c < 1 \). In other words there are no photons which either diffuse or propagate with speed \( < 1 \). Here \( \chi^1(\Omega) \) denotes a smoothed out characteristic function of the set \( \Omega \), which is defined at the end of the introduction.

The maximal velocity estimate, as proven in [10], states that, for any \( c' > 1 \),

\[
\left\| \text{d}\Gamma(\chi_{\Omega(y)}^1) \frac{1}{2} \psi_t \right\| \lesssim t^{-\gamma} \left( \| \text{d}\Gamma(\Omega(y)) \| + 1 \right)^{1/2} \left\| \psi_0 \right\|, \tag{1.12}
\]

with \( \gamma < \min(\frac{1}{2}(1 - \frac{1}{c'}), \frac{1}{10}) \) for (1.1), and \( \gamma < \min(\frac{1}{2} (\frac{c'}{2c'} - 1), \frac{1}{2 + \frac{1}{c'}}) \) for (1.3)–(1.5) with \( \mu > 0 \).

Considerable progress has been made in understanding the asymptotic dynamics of non-relativistic particle systems coupled to quantized electromagnetic or phonon field. The local decay property was proven in [7, 8, 9, 11, 27, 28, 30, 31], by the combination of the renormalization group and positive commutator methods. The maximal velocity estimate was proven in [10].

An important breakthrough was achieved recently in [13], where the authors proved relaxation to the ground state and uniform bounds on the number of emitted massless bosons in the spin-boson model. (Importance of both questions was emphasized earlier by Jürg Fröhlich.)

In quantum field theory, asymptotic completeness was proven for (a small perturbation of) a solvable model involving a harmonic oscillator (see [3, 61]), and for models involving massive boson fields ([17, 24, 25, 26]). Moreover, [32] obtained some important results for massless bosons (the Nelson model) in confined potentials (see below for a more detailed discussion). Motivated by the many-body quantum scattering, [17, 24, 25, 26, 32] defined the main notions of scattering theory on Fock spaces, such as wave operators, asymptotic completeness and propagation estimates.

**Results.** Now we formulate our results. We consider both the minimal coupling model (1.1) and the linear coupling model (1.3) with the linear interaction (1.4) and the coupling operators \( g(k) \) satisfying (1.5) with \( \mu > -1/2 \).

It is known (see [7, 35]) that the operator \( H \) has a unique ground state (denoted here as \( \Phi_{gs} \)) and that generically (e.g. under the Fermi Golden Rule condition) it has no eigenvalues in the interval \((E_{gs}, \Sigma)\), where \( E_{gs} \) is the ground state energy (see [8, 27, 31]). We assume that this is exactly the case:

\[
\text{Fermi’s Golden Rule } ([6, 7]) \text{ holds.} \tag{1.13}
\]

(If the particle system has an infinite number of eigenvalues accumulating to its ionization threshold – which is the bottom of its essential spectrum – then to rule out the eigenvalues in the spectral interval of interest we should replace \( \Sigma \) by \( \Sigma - \epsilon \) for some fixed \( \epsilon \), which is understood from now on.) Treatment of the (exceptional) situation when such eigenvalues do occur requires, within our approach, proving a delicate

\(^1\)The issue of localizability of photons is a tricky one and has been intensely discussed in the literature since the 1930 and 1932 papers by Landau and Peiers [46] and Pauli [52] (see also a review in [44]). A set of axioms for localization observables was proposed by Newton and Wigner [51] and Wightman [63] and further generalized by Jauch and Piron [43]. Observables describing localization of massless particles, satisfying the Jauch-Piron version of the Wightman axioms, were constructed by Amrein in [1].

\(^2\)Since polarization vector fields are not smooth, using them to reduce the results from one set of localization observables to another would limit the possible time decay. However, these vector fields can be avoided by using the approach of [48].
estimate \( \|P_\Omega f(H)\| \lesssim \langle g \rangle \), where \( P_\Omega \) denotes the projection onto \( \mathcal{H}_\rho \otimes \Omega \) (where \( \Omega := 1 + 0 + \ldots \) is the vacuum in \( \mathcal{F} \)) and \( f \in C^\infty_0((E_{gs}, \Sigma) \setminus \sigma_{pp}(H)) \), uniformly in \( d(\text{supp } f, \sigma_{pp}(H)) \).

Let \( N := d(1) \) be the photon (or phonon) number operator and \( N_\rho := d(\Gamma(\omega^{-\rho})) \) be the photon (or phonon) low momentum number operator. In what follows we let \( \psi_t \) denote the Schrödinger evolution, \( \psi_t := e^{-itH}\psi_0 \), i.e. the solution of the Schrödinger equation (1.8), with an initial condition \( \psi_0 \), satisfying \( \psi_0 = f(H)\psi_0 \) with \( f \in C^\infty_0((-\infty, \Sigma)) \). More precisely, we will consider the following sets of initial conditions

\[
\mathcal{Y}_\rho := \{ \psi_0 \in f(H)D(N_\rho)^{\frac{1}{2}}, \text{ for some } f \in C^\infty_0((-\infty, \Sigma)) \},
\]

and

\[
\mathcal{Y}_0 := \{ \psi_0 \in f(H)(D(d\Gamma(g))) \cap D(d\Gamma(b)^2), \text{ for some } f \in C^\infty_0((E_{gs}, \Sigma)) \},
\]

where \( b := \frac{1}{2}(k \cdot y + y \cdot k) \).

For \( A \geq -C \), we denote \( \|\psi_0\|_A := \|(A + C + 1)^{\frac{3}{2}} \psi_0\| \). We define \( \nu_\rho \geq 0 \) as the smallest real number satisfying the inequality

\[
\langle \psi_t, N_\rho \psi_t \rangle \lesssim t^{\nu_\rho} \|\psi_0\|^2,
\]

for any \( \psi_0 \in \text{Ran } E_{(-\infty, \Sigma)}(H) \), where \( \|\psi\|^2 := \|\psi\|^2_{N_\rho} \). With \( \nu_\rho \) defined by (1.14), we prove the following two results.

**Theorem 1.1** (Quantum Huygens principle). Consider the Hamiltonian (1.1), or the Hamiltonian (1.3)–(1.4) satisfying (1.5) with \( \mu > -1/2 \) and (1.7). Let either \( \alpha < 1 \), or \( \alpha = 1 \) and \( c < 1 \). Assume

\[
\alpha > \max\left( \frac{1}{6}(5 + \nu_1 - \nu_0), \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + 2\mu \right),
\]

where for (1.1), \( \mu = 1/2 \). Then for any initial condition \( \psi_0 \in \mathcal{Y}_1 \), the Schrödinger evolution, \( \psi_t \), satisfies, for any \( \alpha > 1 \), the following estimate

\[
\int_1^\infty dt \ t^{-\alpha - \nu_0} \|d\Gamma(x \mid \psi_t \rangle \| \lesssim \|\psi_0\|^2.
\]

For the coupling function \( g \), we introduce the norm \( \langle g \rangle := \sum_{|\alpha| \leq 2} \|\eta_{\rho}^{\alpha} \partial_{g}^{\alpha} g\|_{L^2(\mathbb{R}^3, \mathcal{H}_\rho)} \). We have

**Theorem 1.2** (Weak minimal photon escape velocity estimate). Consider the Hamiltonian (1.1) with the coupling constants \( \kappa_j \) sufficiently small, or the Hamiltonian (1.3)–(1.4) satisfying (1.5) with \( \mu > -1/2 \), (1.7) and \( \langle g \rangle \ll 1 \). Assume (1.13), \( \nu_1 < \alpha < 1 - \nu_0 \) and \( c > 0 \). Then for any initial condition \( \psi_0 \in \mathcal{Y}_\#, \) the Schrödinger evolution, \( \psi_t \), satisfies the estimate

\[
\|d\Gamma(x \mid \psi_t \rangle \| \lesssim t^{-\gamma} (\|\psi_0\|^2_{d(\Gamma(g))} + \|\psi_0\|^2_{d(\Gamma(b)^2)}),
\]

where \( \gamma < \frac{1}{2} \min(1 - \alpha - \nu_0, \frac{1}{2}(\alpha - \nu_0 - 1)) \).

**Remarks.**

1) It was shown in [10] (see (A.1) of Appendix A) that, for any \(-1 \leq \rho \leq 1 \), the inequality (1.14) is satisfied with

\[
\nu_\rho \leq \frac{1 + \rho}{2 + \mu}
\]

(this generalizes an earlier result due to [32]). Also, the bound

\[
\|\psi_t\|_{H^j} \lesssim \|\psi_0\|_{H^j}
\]

shows that (1.14) holds for \( \rho = -1 \) with \( \nu_{-1} = 0 \).

2) The estimate (1.16) is sharp if \( \nu_0 = 0 \). Assuming this and taking \( \nu_1 = (3/2 + \mu)^{-1} \) (see (A.7) of Appendix A), the conditions on \( \alpha \) in Theorems 1.1 and 1.2 become \( \alpha > \frac{5}{6} + \frac{1}{6(3/2 + \mu)} \), and \((3/2 + \mu)^{-1} < \alpha < 1 \), respectively.

3) The estimate (1.17) states that, as \( t \to \infty \), with probability \( \to 1 \), either all photons are attached to the particle system in the combined ground state, or at least one photon departs the particle system with the distance growing at least as \( \mathcal{O}(t^\alpha) \). (1.17) for \( \mu \geq 1/2 \), some \( \alpha > 0 \) and \( \psi_0 \in E_{\Delta}(H) \), with \( \Delta \subset (E_{gs}, \epsilon_1 - \mathcal{O}(\langle g \rangle)) \) and \( \epsilon_1 \) the first excited eigenvalue of \( H_\rho \), can be derived directly from [9, 10].)
4) With some more work, one can remove the assumption (1.13) and relax the condition on \( \psi_0 \) in Theorem 1.2 to the natural one: \( \psi_0 \in P_2D(d\Gamma(y)) \), where \( P_2 \) is the spectral projection onto the orthogonal complement of the eigenfunctions of \( H \) with the eigenvalues in the interval \((-\infty, \Sigma)\).

Our next result is

**Theorem 1.3 (Asymptotic Completeness).** Consider the hamiltonian (1.1) with the coupling constants \( \kappa_j \) sufficiently small, or the hamiltonian (1.3)–(1.4) satisfying (1.5) with \( \mu > 0, (1.7) \) and \( (g) \ll 1 \). Assume (1.13) and suppose that either

\[
\| N^{\frac{1}{2}} \psi_t \| \lesssim \| N^{\frac{1}{2}} \psi_0 \| + \| \psi_0 \|,
\]

for any \( \psi_0 \in f(H)D(N^{1/2}) \), with \( f \in C_0^\infty(E_{gs}, \Sigma) \), uniformly in \( t \in [0, \infty) \), or

\[
\| N^{\frac{1}{2}} \psi_t \| \lesssim 1,
\]

uniformly in \( t \in [0, \infty) \), for any \( \psi_0 \in D \), where \( D \) is such that \( D \cap D(d\Gamma(\omega^{-1/2}(y)\omega^{-1/2} \frac{1}{2})) \) is dense in \( \text{Ran} \ E_{(-\infty, \Sigma)}(H) \). Then the asymptotic completeness holds on \( \text{Ran} \ E_{(-\infty, \Sigma)}(H) \).

Assumption (1.20) can be replaced by the slightly weaker hypothesis that there exist \( 1/2 \leq \delta_1 \leq \delta_2 \) such that for any \( \psi_0 \in f(H)D(N^{1/2}) \), with \( f \in C_0^\infty(E_{gs}, \Sigma) \), \( \| N^{\delta_1} \psi_t \| \lesssim \| N^{\delta_2} \psi_0 \| + \| \psi_0 \| \), uniformly in \( t \in [0, \infty) \).

The advantage of Assumption (1.21) is that the uniform bound on \( N_1 = d\Gamma(\omega^{-1}) \) is required to hold only for an arbitrary dense set of initial states and, as a result, can be verified for the massless spin-boson model by modifying slightly the proof of [13] (see the discussion below). Hence the asymptotic completeness in this case holds with no implicit conditions.

As we see from the results above, the uniform bounds, (1.20) or (1.21), on the number of photons (or phonons) emerge as the remaining stumbling blocks to proving the asymptotic completeness without qualifications. The difficulty in proving these bounds for massless fields is due to the same infrared problem which pervades this field and which was successfully tackled in other central issues, such as the theory of ground states and resonances (see [5, 56] for reviews), the local decay and the maximal velocity bound.

For massive bosons (e.g. optical phonons), the inequality (1.20) (as well as (1.14), with \( \nu_0 = 0 \)) is easily proven and the proof below simplifies considerably as well. In this case, the result is unconditional. It was first proven in [17] for the models with confined particles, and in [24] for the Rayleigh scattering.

**Spin-boson model.** Another example fitting into our framework, and the simplest one, is the spin-boson model describing an idealized two-level atom, with state space \( \mathcal{H}_p = \mathbb{C}^2 \) and hamiltonian \( H_p = \sigma^3 \), where \( \sigma^1, \sigma^2, \sigma^3 \) are the usual \( 2 \times 2 \) Pauli matrices, and \( \varepsilon > 0 \) is an atomic energy, interacting with the massless bosonic field. The total hamiltonian is given by (1.3)–(1.4), with the coupling family given by \( g(k) = \omega^\mu q(k)\sigma^3 + \frac{1}{2} (\sigma^1 + i\sigma^2) \). For the spin-boson model, we can take \( \Sigma = \infty \).

As was mentioned above, for the spin-boson model, a uniform bound, \( \langle \psi_t, e^{\delta N} \psi_t \rangle \leq C(\psi_0) < \infty, \delta > 0 \), on the number of photons, on a dense set of \( \psi_0 \)'s, was recently proven in the remarkable paper [13] (see the discussion below).

To verify (1.21) for the spin-boson model, with \( \mu > 0 \), we proceed precisely in the same way as in [13], but using a stronger condition on the decay of correlation functions,

\[
\int_0^\infty dt (1 + t)^\alpha |h(t)| < \infty, \quad \text{with } h(t) := \int_{\mathbb{R}^3} dk e^{-it|k|} (1 + |k|^{-1})|g(k)|^2,
\]

for some \( \alpha \geq 1 \), instead of Assumption A of [11], and bounding the observable \( (1 + \kappa N_{1/2})^2 \) instead of \( e^{\kappa N} \). Assumption C of [11] on initial states has to be replaced in the same manner. Assuming that our condition (1.10) on the coupling function \( g \) is satisfied with \( \mu > 0 \) (and \( \eta = 1 \)), we see that (1.22) holds with \( \alpha = 1 + 2\mu \).

The form of the observable \( e^{\kappa N} \) enters [13] through the estimate \( \| K_{u,v} \| \lesssim C|h(u - v)| \) of the operator \( K_{u,v} \) defined in [13, (3.4)] and the standard estimate [13, (4.36)]. Both extend readily to our case (the former with \( h(t) \) given in (1.22)). Moreover, [13, (4.36)] is used in the proof that pressure vanishes – Eq. (4.39) in [13] – and the latter also follows from our Proposition A.1 (We can also use the observable \( \Gamma(\omega^{-\lambda}) = d\Gamma(-\lambda \ln \omega) \) and analyticity – rather than perturbation – in \( \lambda \).

**Generalized Pauli–Fierz transformation and a new class of hamiltonians.** We consider for simplicity a single negatively charged particle in an external potential. Then, absorbing the absolute value of
We see that the new hamiltonian (1.26) is of the form of the form (1.30) with \( \eta \).
The coupling operators, \( H \), with \( \eta \), the latter operator is of the following general form

\[
H = (p + A(x))^2 + H_f + U(x).
\] (1.23)

The coupling function \( g_x^{\text{red}}(k, \lambda) := |k|^{-1/2}(\xi(k)\varepsilon(x)k)e^{ikx} \) in (1.23) is more singular in the infrared than is allowed by our techniques \((\mu > 0)\). To go around this problem we use the (unitary) generalized Pauli–Fierz transformation (see [55])

\[
U := e^{i\Phi(g_x)} ,
\] (1.24)
to pass from \( H \), given in (1.23), to the new hamiltonian \( \tilde{H} := UHU^* \), where \( \Phi(h) \) is the operator-valued field, \( \Phi(h) := \frac{1}{\sqrt{2}}(a^*(h) + a(h)) \) and the function \( \eta_x(k, \lambda) \) is defined as follows. Let \( \varphi \in \mathcal{C}\mathcal{C}(\mathbb{R}; \mathbb{R}) \) be a non-decreasing function such that \( \varphi(r) = r \) if \( |r| \leq 1/2 \) and \( |\varphi(r)| = 1 \) if \( |r| \geq 1 \). For \( 0 < \nu < 1/2 \), we define

\[
\eta_x(k, \lambda) := \frac{\xi(k)}{|k|^{1/2} + \nu} |\varphi(|\xi|\varepsilon(k) \cdot x) |.
\] (1.25)

We note that the definition of \( \Phi(h) \) gives \( A(x) = \Phi(g_x^{\text{red}}) \). Using (II.7) and (II.8) of Supplement II, we compute

\[
\tilde{H} = (p + \tilde{A}(x))^2 + E(x) + H_f + V(x),
\] (1.26)

where

\[
\begin{align*}
\tilde{A}(x) & := \Phi(\tilde{g}_x), & \tilde{g}_x(k, \lambda) & := g_x^{\text{red}}(k, \lambda) - \nabla_x \eta_x(k, \lambda), \\
E(x) & := \Phi(e_x), & e_x(k, \lambda) & := i|k|\eta_x(k, \lambda), \\
V(x) & := U(x) + \frac{1}{2} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |k|\eta_x(k, \lambda)^2 dk.
\end{align*}
\] (1.27)

The operator \( \tilde{H} \) is self-adjoint with domain \( \mathcal{D}(\tilde{H}) = \mathcal{D}(H) = \mathcal{D}(p^2 + H_f) \) (see [38, 39]). Now, the coupling functions (form factors) \( \tilde{g}_x(k, \lambda) \) and \( e_x(k, \lambda) \) in the transformed hamiltonian, \( \tilde{H} \), satisfy the estimates that are better behaved in the infrared \((10)\):

\[
\begin{align*}
|\partial^m_k \tilde{g}_x(k, \lambda)| & \lesssim \langle k \rangle^{-3}|k|^{1/2-|m|}\langle x \rangle^{1/2+|m|}, \\
|\partial^m_k e_x(k, \lambda)| & \lesssim \langle k \rangle^{-3}|k|^{1/2-|m|}\langle x \rangle^{1+|m|}.
\end{align*}
\] (1.28)

We see that the new hamiltonian (1.26) is of the form

\[
\tilde{H} = H_p + H_f + \tilde{I}(g),
\] (1.30)

with \( H_p := -\Delta + V(x) \), \( H_f = d\Gamma(\omega) \) and with \( \tilde{I}(g) := p \cdot \tilde{A}(x) + \tilde{A}(x) \cdot p + \tilde{A}(x)^2 + E(x) \). We see that the latter operator is of the following general form

\[
\tilde{I}(g) := \sum_{i,j} \int \int d\nu(g) d\nu(g) g_{ij}(k_{i}(\nu_i), k_{j}(\nu_j)) \otimes a^*(k_{i}(\nu_i)) a(k_{j}(\nu_j)).
\] (1.31)

where the summation in \( i, j \) ranges over the set \( \nu, j \geq 0, 1 \leq i + j \leq 2, \nu_i := (\nu_1, \ldots, \nu_p), \nu_j := (j_k, \lambda_j) \),

\[
\int d\nu(g) := \prod_i \sum_{\lambda_i} \int dk_i a^\nu(k_i) a^\nu(k_i) := \prod_i a^\nu(k_i),
\]

and the coupling operators, \( g_{ij} := g_{ij}(k_{i}(\nu_i), k_{j}(\nu_j)) \) obey

\[
g_{ij}(k_{i}(\nu_i), k_{j}(\nu_j)) = g_{ij}^*(k_{j}(\nu_j), k_{i}(\nu_i)),
\] (1.32)

and satisfy the estimates

\[
\|\eta_1^{-i-j}\eta_2^{[\alpha]} \partial^\alpha g_{ij}(k_{i}(\nu_i), k_{j}(\nu_j))\| \lesssim \sum_{m=1}^{i+j} \prod_{\nu=1}^{i+j} (\nu^\mu (\nu)^{-2-\mu} |k_m|^{-|\alpha|}),
\] (1.33)

with \( \eta_1 = \langle p \rangle^{-1}, \eta_2 = \langle x \rangle^{-1/\nu}, \mu = 1/2, |\alpha| \leq 2, \) and \( 1 \leq i + j \leq 2 \).

The bound (1.6) holds for both (1.23) and (1.30). Actually, we can consider a class of hamiltonians of the form (1.30) with \( H_f \) as above and \( \tilde{I}(g) \) given by (1.31), with the coupling operators, \( g_{ij}(k_{i}(\nu_i), k_{j}(\nu_j)) \), obeying (1.32) and (1.33) with \( \mu > -1/2 \), and \( \eta_1 \) and \( \eta_2 \) estimating operators (unbounded, positive operators with bounded inverses) on the particle space \( \mathcal{H}_p \), satisfying (1.7). We define the norm \( \langle g \rangle := \sum_{1 \leq i + j \leq 2} \sum_{|\alpha| \leq 2} \|\eta_1^{-i-j}\eta_2^{[\alpha]} \partial^\alpha g_{ij}\| \) of the vector coupling operators \( g := (g_{ij}) \), extending the norms of
the scalar coupling operators \( g \), introduced above. It is easy to extend Theorems 1.1–Theorem 1.3 to the hamiltonians of the form (1.30)–(1.33) satisfying (1.7):

**Theorem 1.4.** Theorem 1.1–Theorem 1.3 still hold if we replace hamiltonians of the form (1.3)–(1.5) with hamiltonians of the form (1.30)–(1.33), with (1.7).

Comparison with earlier results. For models involving massive bosons fields, some minimal velocity estimates are proven in [17]. For massless bosons, Theorems 1.1 and 1.2 seem to be new. As was mentioned above, asymptotic completeness was proven for (a small perturbation of) a solvable model involving a harmonic oscillator (see [3, 61]), and, for models involving massive boson fields, in [17] for confined systems, in [24] below the ionization threshold for non-confined systems, and in [25] for Compton scattering.

The paper [32] treats the Nelson model (1.3)–(1.4), with abstract conditions on the coupling function \( g \) (allowing a coupling function of the form \( g(k) = |k|^\alpha \xi(k) e^{ikx} \) where \( \xi(k) \) is the ultraviolet cut-off, with various conditions on \( M \) depending on the results involved), and with \( V(x) \) growing at infinity as \( V(x) \geq c_0|x|^{2\alpha} - c_1 \), \( c_0 > 0 \), \( \alpha > 0 \). In this case, in particular, the ionization threshold \( \Sigma \) is equal to \( \infty \).

We reproduce the main results of [32] (Theorems 12.4, 12.5 and 13.3), which are coached in different terms than ours and present another important view of the subject. Let

\[ f_f \]

\[ \{ | \}

\[ \chi \]

\[ b \]

\[ \{ | \}

\[ \{ | \}

\[ \{ | \}

Then Proposition 12.2 and Theorem 12.3 of [32] state that the operators \( P_+ \) exist provided \( \rho > 1/\mu + 1 \), are independent of the choice of \( f \), and are orthogonal projections commuting with \( \rho \). Furthermore, let \( K^+ := \{ \Phi \in \mathcal{H} : a_+ (a_\Phi) \Phi = 0, \forall h \in \mathcal{H} \} \) (called in [32] the set of asymptotic vacua), where (formally) \( a_\pm (h) := \text{slim}_{\text{t} \to \pm \infty} e^{itH} a(e^{-it\omega}) e^{-itH} \) and \( H_+ := \text{Ran} P^+_c \) (the spaces containing states with only a finite number of photons in the region \( \{ |y| \geq c't \} \) as \( t \to \infty \), for all \( c' > c \). Assuming \( M > 1 \) and \( \mu > 0 \), Theorems 12.4 and 12.5 state that the operator \( \Gamma^+_\rho (f_0) \) exists and is equal to the orthogonal projection on the space \( K^+ := K^+ \cap H_+ \), provided \( 0 < c < c' < 1 \) and \( \rho > 1/\mu + 1 \). Assuming in addition that the Mourre estimate \( Q(H) [H, iB] \Phi = 0 \) holds on an open interval \( \Delta \subset \mathbb{R} \), with the conjugate operator \( B := \text{d}(\Gamma(b), c_0 \geq 0 \) and \( R \in \mathcal{H} \) compact, then for \( 0 < c < c(\Delta, c_0) \), one has \( Q(H) K^+ = Q(H) H_{pp} \), where \( H_{pp} \) is the pure point spectrum eigenspace of \( H \). The latter property is called in [32] geometric asymptotic completeness. Combining results of [7, 8, 27] one can probably prove a Mourre estimate, with \( B \) as conjugate operator, in any spectral interval above \( E_g \) and below \( \Sigma \) and for the coupling function \( g \) given by \( g(k) = |k|^\alpha \xi(k) e^{ikx} \), with \( M > 1/2 \).

Our approach is similar to the one of [32] in as much as it also originates in ideas of the quantum many-body scattering theory. At this the similarities end.

**Approach and organization of the paper.** In this paper, as in earlier works, we use the method of propagation observables, originating in the many body scattering theory ([58, 59, 42, 33, 64, 14], see [16, 41] for a textbook exposition and a more recent review). It was extended to the non-relativistic quantum electrodynamics in [17, 32, 23, 24, 25, 26] and to the \( P(\phi)^2 \) quantum field theory, in [18]. We formalize this method in the next section.

After that we prove key propagation estimates in Sections 3 and 4. Instead of \( |y| \), these estimates involve the operator \( b_e \) defined as \( b_e := \frac{1}{2} (v(k) \cdot y + y \cdot v(k)) \), where \( v(k) := \frac{k}{\omega + t} \), for \( \epsilon = t^{-\kappa} \), with some \( \kappa > 0 \). Since the vector field \( v(k) \) is Lipschitz continuous and therefore generates a global flow, the operator \( b_e \) is self-adjoint. We show in Section 6 that these propagation estimates give the estimates (1.16) and (1.17). (We could have also used the operators \( b_\epsilon \), with \( \epsilon > 0 \) constant, or \( b := \frac{1}{2} (k \cdot y + k \cdot y) \) [32] used the non-self-adjoint operator \( b_0 := \frac{1}{2} (\frac{k}{c^2} \cdot y + \frac{\omega}{c^2} \cdot y) \). Using \( b_0 \) avoids some technicalities, as compared to the other two operators. At the expense of slightly lengthier computations but gaining simpler technicalities, one can also modify \( b_e \) to make it bounded, by multiplying it with the cut-off function \( \chi_{|\omega| < 1} \) with \( c' > 1 \), such that the maximal velocity estimate (1.12) holds, or use the smooth vector field \( v(k) := \frac{k}{\sqrt{\omega^2 + t^2}} \), instead of \( v(k) = \frac{k}{\sqrt{\omega^2 + t^2}} \). In Section 6 we show how to pass from the observable \( b_e \) to \( |y| \).

Once the minimal velocity estimates are proven, the first step in the proof of the asymptotic completeness is to decouple the photons in the expanding ball \( \{ b_e \leq ct^\alpha \} \) from those outside \( \{ b_e \geq ct^\alpha \} \). To this end we use the second quantization, \( \Gamma(j) : \Gamma(\Phi) \to \Gamma(\Phi \otimes \Phi) \) of a partition of unity \( j : h \to j_0 h + j_\infty h \) on the one-photon
space, \( j : \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h} \), with \( j_0 \) localizing a photon to a region \( \{ b_c \leq ct^\alpha \} \), and \( j_\infty \), to \( \{ b_c \geq ct^\alpha \} \), and satisfying \( j_0^2 + j_\infty^2 = 1 \). Defining the adjoint map \( j^* : h_0 \oplus h_\infty \to j_0^* h_0 + j_\infty^* h_\infty \), so that \( j^* j = j_0^2 + j_\infty^2 = 1 \), and using \( \Gamma(j)^* \Gamma(j) = \Gamma(jj^*) \), we see that \( \Gamma(j)^* \Gamma(j) = 1 \).

The partition \( \Gamma(j) \) is further refined as \( \tilde{\Gamma}(j) := U \Gamma(j) : \Gamma(h) \to \Gamma(h) \otimes \Gamma(h) \), where \( U : \Gamma(h) \oplus \Gamma(h) \to \Gamma(h) \otimes \Gamma(h) \) is the unitary map defined through the relations \( U \Omega = \Omega \otimes \Omega \). \( U a^*(h) = [a^*(h_1) \otimes 1 + 1 \otimes a^*(h_2)] U \), for any \( h = (h_1, h_2) \in \mathfrak{h}_0 \oplus \mathfrak{h}_\infty \), and is then lifted from the Fock space \( \mathcal{F} = \Gamma(h) \) to the full state space \( \mathcal{H} = \mathcal{H}_\mathfrak{p} \otimes \mathcal{F} \).

As above, \( \tilde{\Gamma}(j)^* \tilde{\Gamma}(j) = 1 \). Using \( \tilde{\Gamma}(j) \), we define the Deift-Simon wave operators,

\[
W_\pm := s\text{-}\lim_{t \to \infty} e^{i H t} \tilde{\Gamma}(j) e^{-i H t},
\]

where \( H := H \otimes 1 + 1 \otimes H_f \) on the auxiliary space \( \tilde{\mathcal{H}} := \mathcal{H} \otimes \mathcal{F} \). The first minimal velocity estimate for \( b_c \) implies that these operators exist (see Subsection 5.2). The existence of the Deift-Simon wave operators implies that

\[
\psi_t = \tilde{\Gamma}(j)^* e^{-i H t} e^{i H t} \tilde{\Gamma}(j) e^{-i H t} \psi_0 = \tilde{\Gamma}(j)^* e^{-i H t} \phi_0 + o_t(1),
\]

where \( \phi_0 := W_\pm \psi_0 \). Since \( e^{-i H t} = e^{-i H_f t} \otimes e^{-i H_f t} \), we see that the first term on the r.h.s. describes the photons in the expanding ball \( \{ b_c \leq ct^\alpha \} \) decoupled from those outside \( \{ b_c \geq ct^\alpha \} \).

Next, let \( \Delta = [E_p, a] \subset \mathbb{R} \), where \( a < \Sigma \), and \( \Delta' := [0, a - E_p] \). The existence of \( W_+ \) implies the property \( W_+ \chi_\Delta(H) = \chi_\Delta(H) W_+ \), which gives \( \phi_0 = \chi_\Delta(H) \phi_0 \). The latter relation together with \( \chi_\Delta(H') = (\chi_\Delta(H) \otimes \chi_\Delta(H_f)) \chi_\Delta(H) \) imply \( \phi_0 = (\chi_\Delta(H) \otimes \chi_\Delta(H_f)) \phi_0 \). Next, we use that for all \( \epsilon > 0 \), there is \( \delta = \delta(\epsilon) > 0 \), such that

\[
\| (\chi_\Delta(H) \otimes 1) \phi_0 - (\chi_\Delta(H) \otimes 1) \phi_0 - (P_{gs} \otimes 1) \phi_0 \| \leq \epsilon,
\]

where \( \Delta_\epsilon := [E_p + \delta, a] \) and \( P_{gs} \) is the orthogonal projection onto the ground state of \( H \). Applying this equation and the relations \( e^{-i H t} = e^{-i H_f t} \otimes e^{-i H_f t} \) and \( e^{-i H_f t} P_{gs} = e^{-i E_p t} P_{gs} \) to (1.35) gives, after some manipulations with energy cut-offs,

\[
\psi_t = \tilde{\Gamma}(j)^* e^{-i E_p t} P_{gs} \otimes e^{-i H_f t} \chi_\Delta(H_f) \phi_0 + \tilde{\Gamma}(j)^* \phi_t + O(\epsilon) + o_t(1),
\]

where \( \phi_t = (e^{-i H_f t} \chi_\Delta(H) \otimes e^{-i H_f t} \chi_\Delta(H_f)) \phi_0 \). Now, let \( (j_0, j_\infty) \) be localized similarly to \( (j_0, j_\infty) \) and satisfy \( j_0^* j_0 = j_0^2, j_\infty^* j_\infty = j_\infty \). Then, as shown below, the adjoint \( \tilde{\Gamma}(j)^* \) to the operator \( \tilde{\Gamma}(j) \) can be represented as \( \tilde{\Gamma}(j)^* = \tilde{\Gamma}(j)^* (\Gamma(j_0) \otimes \Gamma(j_\infty)) \). Using this equation in (1.35) and using that \( (\Gamma(j_0) \otimes 1) \phi_t \to 0 \), as \( t \to \infty \), by the second minimal velocity estimate for \( b_c \), we see that the second term on the r.h.s. of (1.37) vanishes, as \( t \to \infty \).

To conclude the proof of the asymptotic completeness, we pass from the operator \( \tilde{\Gamma}(j)^* \) to the (scattering) map \( I \), defined first by (see [40, 17, 24])

\[
I(\Phi \otimes f) = \left( \begin{array}{c} p + q \\ p \end{array} \right)^{1/2} \Phi \otimes_s f,
\]

for any \( \Phi \in \mathcal{H}_\mathfrak{p} \otimes (\otimes_2 \mathfrak{h}) \) and \( f \in \otimes_2 \mathfrak{h} \), and then extended to a dense subspace of \( \tilde{\mathcal{H}} \). To this end we use the formula \( \tilde{\Gamma}(j)^* = \Gamma(j_0) \otimes \Gamma(j_\infty) \), for any operator \( j : h \to j_0 h \oplus j_\infty h \), and some elementary estimates in order to remove \( \Gamma(j_0^*) \otimes \Gamma(j_\infty^*) \).

To simplify the exposition, in Sections 2–6, we consider hamiltonians of the form (1.3)–(1.4), with the coupling operators \( g(k) \) satisfying (1.5), where \( \eta_1 \) and \( \eta_2 \) obey (1.7). In Section 7 we extend the results to hamiltonians of the form (1.30)–(1.31) with the coupling operators \( g_{ij} \) satisfying (1.33) and prove Theorem 1.4. In Section 8 we present the extension of the results to the minimal coupling model (1.23).

Finally, a low momentum bound of [10] and some standard technical statements are given in Appendices A, B, C and D. The paper is essentially self-contained. In order to make it more accessible to non-experts, we included Supplement I giving standard definitions, proof of the existence and properties of the wave operators, and Supplement II defining and discussing the creation and annihilation operators (see also [19, 15]).

**Notations.** For functions \( A \) and \( B \), we will use the notation \( A \lesssim B \) signifying that \( A \leq CB \) for some absolute (numerical) constant \( 0 < C < \infty \). The symbol \( E_\Delta \) stands for the characteristic function of a set \( \Delta \), while \( \chi_{\leq 1} \) denotes a smoothed out characteristic function of the interval \( (-\infty, 1) \), that is it is in \( C^\infty(\mathbb{R}) \), is non-decreasing, and \( = 1 \) if \( x \leq 1/2 \) and \( = 0 \) if \( x \geq 1 \). Moreover, \( \chi_{\geq 1} := 1 - \chi_{\leq 1} \) and \( \chi_{=$1} \) stands for the derivative of \( \chi_{\geq 1} \). Given a self-adjoint operator \( a \) and a real number \( \alpha \), we write \( \chi_{\alpha \leq \alpha} := \chi_{\alpha \leq 1} \), and
likewise for $\chi_{n \geq \alpha}$. Finally, $D(A)$ denotes the domain of an operator $A$, $\langle x \rangle := (1 + |x|^2)^{1/2}$, $O(\epsilon)$ denotes an operator bounded by $C \epsilon$, $o_t(1)$ denotes a real number tending to 0 as $t \to \infty$, and $C(\epsilon)o_t(1)$ denotes a real number (depending on $\epsilon$ and $t$) which goes to 0 as $t \to \infty$ for any fixed $\epsilon$.

2. Method of propagation observables

Many steps of our proof use the method of propagation observables which we formalize in what follows. Let $\psi_t = e^{-itH}\psi_0$, where $H$ is a hamiltonian of the form (1.3)–(1.4), with the coupling operator $g(k)$ satisfying (1.5). The method reduces propagation estimates for our system say of the form

$$
\int_0^\infty dt \|G^4_1 \psi_t\|^2 \lesssim \|\psi_0\|^2_\#, \tag{2.1}
$$

for some norm $\| \cdot \|_\# \geq \| \cdot \|$, to differential inequalities for certain families $\phi_t$ of positive, one-photon operators on the one-photon space $L^2(\mathbb{R}^3)$. Let

$$
d\phi_t := \partial_t \phi_t + i[\omega, \phi_t],
$$

and let $\nu_\rho \geq 0$ be determined by the estimate (1.14). We isolate the following useful class of families of positive, one-photon operators:

**Definition 2.1.** A family of positive operators $\phi_t$ on $L^2(\mathbb{R}^3)$ will be called a *one-photon weak propagation observable*, if it has the following properties

- there are $\delta \geq 0$ and a family $p_t$ of non-negative operators, such that
  $$
  \|\omega^{\delta/2} \phi_t \omega^{\delta/2}\| \lesssim \langle t \rangle^{-\nu_\delta} \quad \text{and} \quad d\phi_t \geq p_t + \sum_{\text{finite}} \text{rem}_i,
  $$

  where rem$_i$ are one-photon operators satisfying
  $$
  \|\omega^{\delta/2} \text{rem}_i \omega^{\delta/2}\| \lesssim \langle t \rangle^{-\lambda_i},
  $$

  for some $\rho_i$ and $\lambda_i$, s.t. $\lambda_i > 1 + \nu_{\rho_i}$,

- for some $\lambda' > 1 + \nu_\delta$ and with $\eta_1$, $\eta_2$ satisfying (1.7),

  $$
  \left( \int \|\eta_1 \eta_2' (\phi_t g)(k)\|^2 \|\omega(k)^\delta dk\right)^{1/2} \lesssim \langle t \rangle^{-\lambda'}.
  \tag{2.4}
  $$

(Here $\phi_t$ acts on $g$ as a function of $k$.)

Similarly, a family of operators $\phi_t$ on $L^2(\mathbb{R}^3)$ will be called a *one-photon strong propagation observable*, if

$$
d\phi_t \leq -p_t + \sum_{\text{finite}} \text{rem}_i,
$$

with $p_t \geq 0$, rem$_i$ are one-photon operators satisfying (2.3) for some $\lambda_i > 1 + \nu_{\rho_i}$, and (2.4) holds for some $\lambda' > 1 + \nu_\delta$.

Recall the notations $N_\rho = d\Gamma(\omega^{-\rho})$ and

$$
\Upsilon_\rho = \{ \psi_0 \in f(H)D(N^4_\rho), \text{ for some } f \in C_0^\infty((-\infty, \Sigma)) \}. \tag{2.6}
$$

Notice that, since $N_{-1} f(H) = H f(H)$ is bounded, one easily verifies that $\Upsilon_\rho \subset \Upsilon_{\rho'}$ for $\rho \geq \rho' \geq -1$. The following proposition reduces proving inequalities of the type of (2.1) to showing that $\phi_t$ is a one-photon weak or strong propagation observable, i.e. to *one-photon estimates of $d\phi_t$ and $\phi_t g$*.

**Proposition 2.2.** If $\phi_t$ is a one-photon weak (resp. strong) propagation observable, then we have either the weak propagation estimate, (2.1), or the strong propagation estimate,

$$
\langle \psi_t, \Phi_t \psi_t \rangle + \int_0^\infty dt \|G^4_1 \psi_t\|^2 \lesssim \|\psi_0\|^2_\#, \tag{2.7}
$$

with the norm $\|\psi_0\|^2_\# := \|\psi_0\|^2_3 + \|\psi_0\|^2_0$, where $\Phi_t := d\Gamma(\phi_t)$, $G_t := d\Gamma(p_t)$, $\|\psi_0\|_\delta := \|\psi_0\|_\delta$ and $\|\psi_0\|_\phi := \sum \|\psi_0\|_{\rho_i}$, on the subspace $\Upsilon_{\max(\delta, \rho_1)}$. 

Before proceeding to the proof we present some useful definitions. Consider families $\Phi_t$ of operators on $\mathcal{H}$ and introduce the Heisenberg derivative

$$D\Phi_t := \partial_t \Phi_t + i[H, \Phi_t],$$

with the property

$$\partial_t \langle \psi_t, \Phi_t \psi_t \rangle = \langle \psi_t, D\Phi_t \psi_t \rangle. \quad (2.8)$$

**Definition 2.3.** A family of operators $\Phi_t$ on a subspace $\mathcal{H}_1 \subset \mathcal{H}$ will be called a (second quantized) weak propagation observable, if for all $\psi_0 \in \mathcal{H}_1$, it has the following properties

- $\sup_{t} \langle \psi_t, \Phi_t \psi_t \rangle \lesssim \langle \psi_0 \rangle^2$;
- $D\Phi_t \geq G_t + \text{Rem}$, where $G_t \geq 0$ and $\int_0^\infty dt \langle \psi_t, \text{Rem} \psi_t \rangle \lesssim \langle \psi_0 \rangle^2$, for some norms $\|\psi_0\|$, $\|\cdot\| \geq \|\cdot\|$. Similarly, a family of operators $\Phi_t$ will be called a strong propagation observable, if it has the following properties

- $\Phi_t$ is a family of non-negative operators;
- $D\Phi_t \leq -G_t + \text{Rem}$, where $G_t \geq 0$ and $\int_0^\infty dt \langle \psi_t, \text{Rem} \psi_t \rangle \lesssim \langle \psi_0 \rangle^2$, for some norm $\|\cdot\| \geq \|\cdot\|$. If $\Phi_t$ is a weak propagation observable, then integrating the corresponding differential inequality sandwiched by $\psi_t$’s and using the estimate on $\langle \psi_t, \Phi_t \psi_t \rangle$ and on the remainder Rem, we obtain the (weak propagation) estimate (2.1), with $\|\psi_0\| := \|\psi_0\| + \|\psi_0\|^2$. If $\Phi_t$ is a strong propagation observable, then the same procedure leads to the (strong propagation) estimate (2.7).

**Proof of Proposition 2.2.** Let $\Phi_t := d\Gamma(\phi_t)$. To prove the above statement we use the relations (see Supplement II)

$$D_0 d\Gamma(\phi_t) = d\Gamma(d\phi_t), \quad i[I(g), d\Gamma(\phi_t)] = -I(i\phi_tg), \quad (2.9)$$

where $D_0$ is the free Heisenberg derivative,

$$D_0 \Phi_t := \partial_t \Phi_t + i[H_0, \Phi_t].$$

valid for any family of one-particle operators $\phi_t$, to compute

$$D\Phi_t = d\Gamma(d\phi_t) - I(i\phi_tg). \quad (2.10)$$

Denote $\langle A \rangle_\psi := \langle \psi, A \psi \rangle$. Applying the Cauchy-Schwarz inequality, we find the following version of a standard estimate

$$\|\langle I(g) \rangle_\psi \| \leq 2 \left( \int \|\eta_1 \eta_2 g(k)\|^2 \tilde{\eta}(k) \tilde{\omega}(k) \tilde{\omega}(k) \tilde{\eta}(k) \tilde{\omega}(k) \right)^{1/2} \eta_0^{-1} \eta_2^{-1} \psi \| \psi \| \| \psi \|. \quad (2.11)$$

Using that $\psi_t = f_1(H)\psi_t$, with $f_1 \in C^\infty_\delta(-\infty, \Sigma)$, $f_1 f = f$, and using (1.7), we find $\|\eta_0^{-1} \eta_2^{-2} \| \psi \| \lesssim \| \psi \|$. Taking this into account, we see that the equations (2.11), (2.4) and (1.19) yield

$$\|\langle I(i\phi_t g) \rangle_\psi \| \lesssim \langle t \rangle^{-\lambda + \nu} \| \psi_0 \| \| \psi \|. \quad (2.12)$$

Next, using (2.3), we find $\pm \text{rem}_t \leq \|\omega^\rho_\Sigma \| \psi_0 \| \| \psi_0 \|^2 \| \omega_\Sigma \| \lesssim \langle t \rangle^{-\lambda} \omega_\Sigma$. This gives $\pm d\Gamma(\text{rem}_t) \lesssim \langle t \rangle^{-\lambda} d\Gamma(\omega_\Sigma)$, which, due to the bound (1.14), leads to the estimate

$$\|\langle d\Gamma(\text{rem}_t) \rangle_\psi \| \lesssim \langle t \rangle^{-\lambda + \nu} \| \psi_0 \| \| \psi_0 \|^2. \quad (2.13)$$

Let $G_t := d\Gamma(p_t)$ and Rem := $\sum_{\text{finite}} d\Gamma(\text{rem}_t) - I(i\phi_tg)$. We have $G_t \geq 0$, and, by (2.12) and (2.13),

$$\int_0^\infty dt \| \langle \psi_t, \text{Rem} \psi_t \rangle \| \lesssim \| \psi_0 \| \| \psi \|, \quad (2.14)$$

with $\| \psi_0 \| := \| \psi_0 \|^2 + \| \psi_0 \|^2$, $\| \psi_0 \| := \| \psi_0 \| \| \psi \| \| \psi \| := \sum_i \| \psi_0 \| \| \psi \|$. In the strong case, (2.5) and (2.10) imply

$$D\Phi_t \leq -G_t + \text{Rem}, \quad (2.15)$$

and hence by (2.14), $\Phi_t$ is a strong propagation observable.

In the weak case, (2.2) and (2.10) imply

$$D\Phi_t \geq G_t + \text{Rem}. \quad (2.16)$$
Since \( \phi_t \leq \| \omega^{\delta/2} \phi_t \| \omega^{-\delta} \| \omega^{-\delta} \| \langle t \rangle^{-\nu_s} \omega^{-\delta} \| \), we have \( d\Gamma(\phi_t) \lesssim \langle t \rangle^{-\nu_s} d\Gamma(\omega^{-\delta}) \). Using this estimate and using again the bound (1.14), we obtain

\[
(\psi_t, \Phi_t \psi_t) \lesssim \langle t \rangle^{-\nu_s} (d\Gamma(\omega^{-\delta})) \psi_t \lesssim \| \psi_0 \|_3^2.
\]  

(2.17)

Estimates (2.14) and (2.17) show that \( \Phi_t \) is a weak propagation observable.

\( \square \)

Proposition 2.4. Let \( \phi_t \) be a one-photon family satisfying

- either, for some \( \delta \geq 0 \),
  \[ \| \omega^{\delta/2} \phi_t \| \omega^{\delta/2} \| \lesssim \langle t \rangle^{-\nu_s} \text{ and } d\phi_t \geq p_t - d\tilde{\omega}_t + \text{rem}, \]  
  (2.18)

  or
  \[ d\phi_t \leq -p_t + d\tilde{\omega}_t + \sum_{\text{finite}} \text{rem}_i, \]  
  (2.19)

  where \( p_t \geq 0, \text{rem}_i \) are one-photon operators satisfying (2.3), and \( \tilde{\omega}_t \) is a weak propagation observable,

- (2.4) holds.

Then, depending on whether (2.18) or (2.19) is satisfied, \( \Phi_t := d\Gamma(\phi_t) \) is a weak, or strong, propagation observable, on the subspace \( \Upsilon_{\max(\delta, \rho_t)} \), and therefore we have either the weak or strong propagation estimates, (2.1) or (2.7), on this subspace.

Proof. Given Proposition 2.4 and its proof, the only term we have to control is \( d\Gamma(d\tilde{\omega}_t) \). Using that \( \tilde{\omega}_t \) is a weak propagation observable and using (2.8), (2.10) and (2.12) for \( \tilde{\Phi}_t := \frac{d\Gamma(\tilde{\omega}_t)}{dt} \), we obtain

\[ \left| \int_0^\infty dt \langle d\Gamma(\tilde{\omega}_t) \rangle \psi_t \right| \lesssim \| \psi_0 \|_\#^2, \]  

(2.20)

with \( \| \psi_0 \|_\#^2 := \| \psi_0 \|_\#^2 + \| \psi_0 \|_\infty^2, \| \psi_0 \|_* := \| \psi_0 \|_* \), which leads to the desired estimates. \( \square \)

Remarks.

1) Proposition 2.2 reduces a proof of propagation estimates for the dynamics (1.8) to estimates involving the one-photon datum \((\omega, g)\) (an ‘effective one-photon system’), parameterizing the hamiltonian (1.3). (The remaining datum \( H_p \) does not enter our analysis explicitly, but through the bound states of \( H_p \) which lead to the localization in the particle variables, (1.7)).

2) The condition on the remainder in (2.2) can be weakened to \( \text{rem} = \text{rem}' + \text{rem}'' \), with \( \text{rem}' \) and \( \text{rem}'' \) satisfying (2.3) and

\[ |\text{rem}''| \lesssim \chi_{|y| \geq c't}, \]  

(2.21)

for \( c' \) as in (1.12), respectively. Moreover, (2.3) can be further weakened to

\[ \int_0^\infty dt \left| \langle \psi_t, d\Gamma(\text{rem}_i) \psi_t \rangle \right| < \infty. \]  

(2.22)

3) An iterated form of Proposition 2.4 is used to prove Theorem 1.1.

3. The first propagation estimate

Let \( \nu_t \geq 0 \) be the same as in (1.14) and recall the operator \( b_t \) defined in the introduction. We write it as

\[ b_t := \frac{1}{2} (\theta_t \nabla \omega \cdot y + y \cdot \nabla \omega \theta_t), \quad \text{where } \theta_t := \frac{\omega}{\omega_t}, \quad \omega_t := \omega + \epsilon, \quad \epsilon = t^{-\kappa}. \]  

(3.1)

Theorem 3.1. Consider hamiltonians of the form (1.3)–(1.4) with the coupling operators satisfying (1.5) with \( \mu > -1/2 \) and (1.7). Let \( \nu_1 - \nu_0 < \kappa < 1 \). If either \( \alpha < 1 \), or \( \alpha = 1 \) and \( c < 1 \), and

\[ \alpha > \max((3/2 + \mu)^{-1}, (1 + \kappa)/2, 1 - \kappa + \nu_1 - \nu_0), \]  

(3.2)

then for any initial condition \( \psi_0 \in \Upsilon_1 \), the Schrödinger evolution, \( \psi_t \), satisfies, for any \( a > 1 \), the following estimates

\[ \int_1^\infty dt \ t^{-\alpha - a\nu} \| d\Gamma(\chi_{\mu_{\omega_t} = 1}) \psi_t \|_2 ^2 \lesssim \| \psi_0 \|_1 ^2. \]  

(3.3)
If \( \nu_0 = 0, \mu > 0, \alpha \) satisfies (3.2) and \( \alpha < \frac{1}{\beta} \), with \( \beta > 1 \), then, with the notation \( \chi \equiv X (\frac{\theta^2}{2})^{\frac{3}{2}} \leq 1 \),

\[
\int_1^\infty dt \, t^{-\alpha} \|d\Gamma (\theta^2 \chi) \frac{\partial}{\partial t} \|^2 \lesssim \|\psi_0\|^2.
\]  

(3.4)

**Proof.** We will use the method of propagation observables outlined in Section 2. We consider the one-parameter family of one-photon operators

\[
\phi_t := t^{-\alpha \nu} \chi, \quad v := \frac{b_e}{ct^\alpha},
\]

(3.5)

where \( a > 1 \). To show that \( \phi_t \) is a weak one-photon propagation observable, we obtain differential inequalities for \( \phi_t \). We use the notation

\[
\chi_\alpha \equiv \chi_{\nu \geq 1}.
\]

Recall that \( d\phi_t = \theta_t \phi_t + i[\omega, \phi_t] \). To compute \( d\phi_t \), we use the expansion

\[
d\phi_t = t^{-\alpha \nu} (dv)\chi_\alpha' + \sum_{i=1}^{2 \text{ rem, \ } i},
\]

(3.6)

\[
\text{rem}_1 := t^{-\alpha \nu} [d\chi_\alpha - (dv)\chi_\alpha'], \quad \text{rem}_2 := -a\nu t^{-1} \phi_t.
\]

(3.7)

Using the definitions in (3.1), we compute

\[
dv = \frac{1}{ct^\alpha} (\theta - \frac{\alpha b_e}{t} + \partial_t b_e).
\]

(3.8)

Next, we have \( \partial_t b_e = \frac{\alpha}{\sqrt{2\pi \sigma}} (e^{-1} \theta \nabla \cdot y + \text{h.c}) \) on \( D(b_e) \), which, due to the relation \( \frac{1}{2}(\omega_{\alpha}^{-1} \nabla \cdot y + \text{h.c}) = \omega_{\alpha}^{-1/2} \partial_t b_e \omega_{\alpha}^{-1/2} \), becomes

\[
\partial_t b_e = \frac{\alpha}{\sqrt{2\pi \sigma}} \omega_{\alpha}^{-1/2} \partial_t b_e \omega_{\alpha}^{-1/2}.
\]

(3.9)

Using that (see Lemma B.1 of Appendix B)

\[
\omega_{\alpha}^{-1/2} b_e \omega_{\alpha}^{-1/2} \chi_\alpha = \omega_{\alpha}^{-1/2} b_e \chi_\alpha \omega_{\alpha}^{-1/2} + O(t^{\frac{\beta}{2}}),
\]

and that \( b_e \geq 0 \) on supp \( \chi_\alpha' \), we obtain

\[
\partial_t b_e \chi_\alpha' \geq \frac{-\text{const}}{t^{1-\frac{\beta}{2}}},
\]

(3.10)

The relations (3.6)–(3.10), together with \( \frac{b_e}{ct^\alpha} \chi_\alpha' \leq \chi_\alpha' \), imply

\[
d\phi_t \geq t^{-\alpha \nu} \left( \frac{\theta}{ct^\alpha} - \frac{\alpha}{t} \right) \chi_\alpha' + \sum_{i=1}^{3 \text{ rem, \ } i},
\]

(3.11)

where \( \text{rem}_1 \) and \( \text{rem}_2 \) are given in (3.7) and

\[
\text{rem}_3 = O(t^{-1 - \alpha + \frac{\beta}{2} - \alpha \nu}).
\]

(3.12)

This, together with \( \theta_t = 1 - \frac{t^\alpha}{\omega_{\alpha}} \) and \( \omega_{\alpha}^{-1} \chi_\alpha' = \omega_{\alpha}^{-1/2} \omega_{\alpha}^{-1/2} + O(t^{-\alpha + \frac{\beta}{2}}) \) (see again Lemma B.1 of Appendix B), implies

\[
d\phi_t \geq t^{-\alpha \nu} \left( \frac{1}{ct^\alpha} - \frac{\alpha}{t} \right) \chi_\alpha' + \sum_{i=1}^{5 \text{ rem, \ } i},
\]

(3.13)

\[
\text{rem}_4 := \frac{1}{ct^{\alpha + \alpha \nu}} \omega_{\alpha}^{-1/2} \chi_\alpha \omega_{\alpha}^{-1/2}, \quad \text{rem}_5 = O(t^{-2\alpha + \frac{\beta}{2} - \alpha \nu}).
\]

(3.14)

We have \( \|\phi_t\| \leq t^{-\alpha \nu} \) and therefore the first estimate in (2.2) holds with \( \delta = 0 \). If either \( \alpha < 1 \) (and \( t \) sufficiently large), or \( \alpha = 1 \) and \( c < 1 \), then \( p_t := t^{-\alpha \nu} \left( \frac{1}{ct^\alpha} - \frac{\alpha}{t} \right) \chi_\alpha' \) is non-negative, which implies the second estimate in (2.2). Thus (2.2) holds. By the definition (3.6) and Corollary B.3 of Appendix B for \( i = 1 \), and by an explicit form for \( i = 2, 3, 4, 5 \), we have the estimates

\[
\|\omega^{\nu_1/2} \text{rem}_i \omega^{\nu_1/2} \| \lesssim t^{-\lambda_i},
\]

(3.15)
$i = 1, 2, 3, 4, 5$, with $\rho_1 = \rho_2 = \rho_3 = \rho_5 = 0$, $\rho_4 = 1$, $\lambda_1 = 2\alpha - \kappa + \alpha \nu_0$, $\lambda_2 = 1 + \alpha \nu_0$, $\lambda_3 = 1 + \alpha - \kappa/2 + \alpha \nu_0$, $\lambda_4 = \alpha + \kappa + \alpha \nu_0$, and $\lambda_5 = 2\alpha - \kappa/2 + \alpha \nu_0$. We remark here that the $i = 2$ term is absent if $\nu_0 = 0$. The relation (3.15) implies (2.3) with $\rho = \rho_1$ and $\lambda = \lambda_1$ provided $\lambda_1 > 1 + \nu_0$.

Finally, in the same way as [10, Lemma 3.1], one shows (by replacing $|y|$ with $b_c$ in that lemma) that, under (1.5) for some $-\frac{1}{2} \leq \mu \leq \frac{1}{2}$,

$$\|\eta_n^2 \chi \frac{d}{dx} \geq 1 g(k)\|_{L^2(\mathbb{R}^3; H)} \lesssim t^{-\tau}, \quad \tau < \left(\frac{3}{2} + \mu\right)\alpha,$$

which implies (2.4) with $\lambda' < \nu_0 + \frac{(3/2 + \mu)\alpha}{5}$. Hence $\phi_t$ is a weak one-photon propagation observable, provided $2\alpha > 1 + \kappa + \nu_0 - \alpha \nu_0$, $\alpha - \kappa > \nu_0 - \alpha \nu_0$, $\alpha + \kappa > 1 + \nu_1 - \alpha \nu_0$, and $(\frac{3}{2} + \mu)\alpha > 1$. Therefore, by Proposition 2.2 and under the conditions on the parameters above,

$$\int_{1}^{\infty} dt \ t^{-\alpha - \nu_0} \|d\Gamma(\chi'_\alpha)\|_{\mathcal{L}(H)}^2 \lesssim \|\psi_0\|^2.$$  

This, due to the definition of $\chi'_\alpha$, implies the estimate (3.3).

We now prove (3.4). We use again the notation $\chi_\alpha \equiv \chi_{\alpha \geq 1}$, where $v := \frac{b_c}{ct}$, and we denote $w := (\frac{b_c}{ct})^3$. We consider the one-parameter family of one-photon operators

$$\phi_t := \chi_\alpha \chi,$$

and show that $\phi_t$ is a weak one-photon propagation observable. We have $\|\phi_t\| \leq 1$ and therefore, due to the assumption $\nu_0 = 0$, the first estimate in (2.2) holds with $\delta = 0$. Now, we show the second estimate in (2.2). To compute $d\phi_t$, we use the expansion

$$d\phi_t = \chi(dv)\chi'_\alpha \chi + \chi'(dw)\chi_\alpha + \chi_\alpha (dw)\chi' + \sum_{i=1,2} \text{rem}_i,$$

where

$$\text{rem}_1 := \chi(d\chi_\alpha - (dv)\chi'_\alpha) \chi, \quad \text{rem}_2 := (d\chi - (dw)\chi')\chi_\alpha + \text{h.c.}.$$  

As in (3.8)–(3.10), we have

$$\chi(dv)\chi'_\alpha \chi \geq \frac{1}{ct\alpha} \chi\left(\theta_c - \frac{\alpha b_c}{t}\right)\chi'_\alpha \chi + \text{rem}_3,$$

where $\text{rem}_3 = O(t^{-1 - \alpha - \kappa/2})$. We consider the term $-(\alpha b_c)/(ct^{\alpha + 1})$ in (3.21). Since $b_c = \theta_c^{1/2} \theta_c^{-1/2}$, where, recall, $b_c = \frac{1}{2}(\nabla \cdot y + \text{h.c.})$, we obtain, using in particular Lemma B.1 of Appendix B and Hardy’s inequality, that

$$\chi b_c \chi'_\alpha \chi = \chi(\chi'_\alpha) \theta_c \chi \chi'_\alpha \chi = \theta_c^{1/2} \chi \chi'_\alpha \chi \theta_c^{1/2} + O(t^\epsilon),$$

and the maximal velocity cut-off gives $\chi b_c \chi \leq \epsilon t$. Thus, commuting again $\chi$ through $\theta_c^{1/2}$ and $(\chi'_\alpha)^{1/2}$, we obtain

$$-\chi b_c \chi'_\alpha \chi \geq -\epsilon t \chi \theta_c \chi \chi'_\alpha \chi + O(\frac{1}{t^{1 - \kappa}}).$$

Proceeding in the same way for the term $\theta_c/(ct^\alpha)$ in (3.21) gives

$$\chi(\theta_c - \frac{\alpha b_c}{t})\chi'_\alpha \chi \geq (1 - \alpha \epsilon) \chi \theta_c^{1/2} \chi \chi'_\alpha \chi + O(\frac{1}{t^{1 - \kappa}}).$$

Next, we compute

$$dw = 2\left(\frac{b_c}{(ct)^2} - \frac{w}{t}\right).$$

By Lemma B.1 of Appendix B, we have

$$\chi'(dw)\chi_\alpha + \chi_\alpha (dw)\chi' = -2(\chi_\alpha)^{1/2}(-\chi')^{1/2}(dw)(-\chi')^{1/2}(\chi_\alpha)^{1/2} + O(\frac{1}{t^{1 + \alpha - \kappa}}).$$

Using that $dw \leq (\frac{1}{4} - 1)\frac{1}{t}$ on the support of $\chi'$ and that $\chi' \leq 0$ and $\bar{\epsilon} > 1$, we obtain

$$(-\chi')^{1/2}(dw)(-\chi')^{1/2} \geq (1 - \frac{1}{\bar{\epsilon}})\frac{1}{t}(-\chi').$$
The relations (3.19), (3.21), (3.24) and (3.25) imply

$$d\phi_t \geq p_t + \tilde{p}_t - \sum_{i=1,2,3,4} \text{rem}_i,$$

where \(\text{rem}_4 = O(\frac{1}{t^{\alpha-\kappa}})\) and

$$p_t := \frac{1 - \alpha c}{ct^\alpha} \theta^{1/2}_t \chi^{1/2}_\alpha \theta^{1/2}_t,$$

$$\tilde{p}_t := (1 - \frac{1}{\tilde{c}}) \frac{1}{t} \bar{\chi}^{1/2}_\alpha (-\chi') \chi^{1/2}_\alpha.$$

The terms \(p_t\) and \(\tilde{p}_t\) are non-negative, provided \(\alpha < \frac{1}{\tilde{c}}\) and \(\tilde{c} > 1\). This implies the second estimate in (2.2). Next, we claim the estimates

$$\|\text{rem}_i\| \lesssim t^{-\lambda},$$

\(i = 1, 2, 3, 4, \) with \(\lambda = 2\alpha - \kappa\). Indeed, the definition (3.20) and Corollary B.3 of Appendix B imply (3.29) for \(i = 1\). The estimate for \(i = 3, 4\) are obvious. To estimate \(\text{rem}_2\), we write

$$(d\chi - (dw)\chi')\chi_\alpha = (d\chi - (dw)\chi')v\tilde{\chi}_\alpha \chi,$$

where \(\tilde{\chi}_\alpha = v^{-1}\chi_\alpha\), recall \(v = \frac{b_t}{ct^\alpha}\) and use that \(b_t = \theta_t b + i\bar{\omega}_t^2\). Using that, by Lemma B.4 of Appendix B,

$$\|d\chi - (dw)\chi'\| \lesssim t^{-1},$$

and commuting \(b\) through \(\tilde{\chi}_\alpha\) gives

$$(d\chi - (dw)\chi')\chi_\alpha = \frac{1}{ct^\alpha} (d\chi - (dw)\chi')\theta_t \tilde{\chi}_\alpha b \chi + O(\frac{1}{t^{1+\alpha-\kappa}}).$$

By Lemma B.4, we also have

$$\|(d\chi - (dw)\chi')\omega\| \lesssim t^{-2}.$$ Combining this with (3.30) and the estimates \(\omega^{-1}_t = O(t^\kappa)\) and \(b\chi = O(t)\), we obtain

$$(d\chi - (dw)\chi')\chi_\alpha = O(\frac{1}{t^{1+\alpha-\kappa}}),$$

and hence the estimate for \(i = 2\) follows.

The relation (3.29) implies (2.3) with \(\lambda = 2\alpha - \kappa\), for \(\text{rem} = \text{rem}_i\), provided \(2\alpha - \kappa > 1\). Finally, as above, (2.4) holds with \(\lambda' < a\nu_0 + (\frac{3}{2} + \mu)\alpha\) by (3.16). This yields (3.4). \(\square\)

4. THE SECOND PROPAGATION ESTIMATE

Recall the norm \(\langle g\rangle = \sum_{|\alpha| \leq 2} \|\eta_1 \eta_2^{|\alpha|} \partial^\alpha g\|_{L^2(\mathbb{R}^3, H_\text{reg})}\) for the coupling function \(g\) and the notation \(\langle A \rangle_{\psi} = \langle \psi, A\psi\rangle\).

**Theorem 4.1.** Consider hamiltonians of the form (1.3)–(1.4) with the coupling operators satisfying (1.5) with \(\mu > -1/2\) and (1.7). Assume that (1.13) holds. Let \(\langle g\rangle\) be sufficiently small, \(\nu_1 < \kappa < \nu_2\), and \(0 < \alpha < 1\). Let \(\psi_0 \in \mathcal{Y}_\#\). Then the Schrödinger evolution, \(\psi_t\), satisfies the estimate

$$\|\Gamma(\chi_{\eta_1,\eta_2}^{(\psi_0)})\|_{L^2(\mathbb{R}^3, \mathcal{H}_{\text{reg}})} \lesssim t^{-\delta} \left( \|\psi_0\|_{L^2(\Gamma(\psi_0))}^2 + \|\psi_0\|_{L^2(\Gamma(\psi_0_2))}^2 \right),$$

for \(0 \leq \delta < \frac{1}{2} \min(\kappa - \nu_1, 1 - \kappa, 1 - \nu_1)\) and any \(c > 0\), where, recall, \(b = \frac{1}{2}(k - y - y \cdot k)\).

We define \(B_\alpha := d\Gamma(b_\alpha)\) and \(B_{\alpha, c} := B_\alpha(c c_\alpha)\). As is [10, Proposition B.3 and Remark B.4], one verifies that \(\mathcal{Y}_2 \subset D(d\Gamma(\psi_0)) \subset D(B_\alpha)\). The proof of Theorem 4.1 is based on the following result (cf. [58, 42]).

**Proposition 4.2.** Under the conditions of Theorem 4.1, the evolution \(\psi_t = e^{-it\mathcal{H}}\psi_0\) obeys

$$\|\chi_{B_{\alpha, c}} \psi_t\| \lesssim t^{-\delta'} \left( \|\psi_0\|_{L^2(\Gamma(\psi_0))}^2 + \|\psi_0\|_{L^2(\Gamma(\psi_0_2))}^2 \right),$$

for any \(0 < c < (1 - C(\langle g\rangle))/(1 + \kappa)\), where \(\delta' := \frac{1}{2} \min(\frac{1-C(\langle g\rangle)}{c} - 1 - \kappa, 1 - \kappa, \kappa - \nu_1)\).

**Remark.** The constant \(C\) is independent of \(\gamma_0 := \text{dist}(E_{gs}, \text{supp} f)\) (but the implicit constant appearing in the right hand side of (4.2) does depend on \(\gamma_0\)).
We use the local decay properties established in [28] and [8]. Let Ω := 1
\[\Omega = \int_{\mathbb{R}^d} |\phi|^2 \, dx\]
where η := χ_{B_{c,t}}(x) denotes a smoothed out characteristic function of the interval [c, t]. First, we compute the main term, M, in (4.3). We leave out a standard proof of f(H) ∈ C^1(B_c) (see e.g. [27, Theorem 8]) and standard domain questions such as that Y_2 ⊂ D(B_c). We have
\[DB_{c,t} = \frac{1}{c!} DB_c - \frac{1}{t} B_{c,t}.\]

The computations below are understood in the sense of quadratic forms on D. Since, by (II.3) of Supplement II, i[H_2, B_c] = N_e, where N_e := dΓ(θ_e), we have
\[DB_e = N_e + I_1,\]
where I_1 := i[I(g), B_c] = -i(ib_0 g) (see (II.5) of Supplement II). To estimate the operator N_e from below, we use that θ_e = 1 - \frac{c}{t}, to obtain
\[N_e = N - \epsilon d\Gamma(\omega_e^{-1}).\]

Next, Lemma C.2 of Appendix B and the bound (1.14) show that
\[\langle \phi_1 d\Gamma(\omega_e^{-1}) \phi_1 \rangle_{\psi_t} \leq t^{-\delta} \|\psi_0\|^2_t + t^{-1+\nu_0} \|\psi_0\|^2.\]

Define the first estimating operator E_1 := N + \eta_1^{-1} \eta_2^{-1} + 1. By (1.5), the condition μ > -1/2 and (2.11) (with δ = 0), we have
\[\|\eta_1 \eta_2 I_1 (N + 1)^{-1/2} \| \leq \|\eta_1 \eta_2 b_0 g\| \leq \|g\|,\]
and hence,
\[I_1 \geq -C(g) E_1.\]

Combining this with the definition of M, (1.7), (4.5), (4.6), (4.7) and (4.8), we obtain
\[\langle M \rangle_{\psi_t} \leq -\frac{1}{c!} d\Gamma(\omega_e^{-1}) N - t^{-1} B_e - C(g) \phi_1 + c \rho \phi_1 \rangle_{\psi_t} + C \rho^{-\rho} (t^{-\eta_0} \|\psi_0\|^2_t + t^{-1+\nu_0} \|\psi_0\|^2).\]

Let Ω := 1 ⊕ 0 ⊕ ... be the vacuum in F and P_Ω be the orthogonal projection on the subspace H_0 ⊕ Ω, P_Ω := ⟨Ω, Ψ⟩_F ⊗ Ω. We now use the following

**Lemma 4.3.** Assume (1.5) with μ > -1/2, (1.7), (1.13), (g) sufficiently small and \( f \in C^\infty_0 (\Sigma) \). Then
\[\|P_\Omega e^{-itH} f(H) u\| \lesssim t^{-\delta} \|\langle B \rangle u\|, \quad s < 1/2.\]

**Proof.** We use the local decay properties established in [28] and [8]. Let c_j := (e_j + e_{j+1})/2 and δ_j := e_{j+1} - e_j. We decompose the support of f into different regions, writing
\[f(H) = f(H) \chi_{H < c_0} + \sum_{\text{finite}} f(H) \chi_j(H),\]
where \( \chi_j(H) \) denotes a smoothed out characteristic function of the interval [c_j - δ_j/4, c_j + δ_j/4]. Using P_Ω = P_Ω(B), and [28], we obtain
\[\|P_\Omega e^{-itH} f(H) \chi_{H < c_0} u\| \leq \|\langle B \rangle^{-1} e^{-itH} f(H) \chi_{H < c_0} u\| \lesssim t^{-s} \|\langle B \rangle u\|,\]
for s < 1/2.
To prove (4.21), it suffices to show that
\[ \tilde{\chi}_j(H)[H, i\tilde{B}_j]\tilde{\chi}_j(H) \geq m_0\tilde{\chi}_j(H)^2, \]
holds for some positive constant \( m_0 \). By an abstract result of [42], this implies
\[ \|\langle \tilde{B}_j \rangle^{-s}e^{-itH}\tilde{\chi}_j(H)\langle \tilde{B}_j \rangle^{-s}\| \lesssim t^{-s}, \]
for \( s < 1 \). Since the operator \( \tilde{B}_j \) is of the form \( \tilde{B}_j = B + M_j \), where \( M_j \) is a bounded operator, it then follows that
\[ \|\langle B \rangle^{-s}e^{-itH}\tilde{\chi}_j(H)\langle B \rangle^{-s}\| \lesssim t^{-s}, \]
and hence, using again that \( P_\Omega(B) = P_\Omega \), we obtain
\[ \|P_\Omega e^{-itH}\tilde{\chi}_j(H)u\| = \|\langle B \rangle^{-1}e^{-itH}\tilde{\chi}_j(H)u\| \lesssim t^{-s}\|\langle B \rangle u\|. \] (4.15)
Equations (4.13), (4.14) and (4.15) give (4.12). □

Together with \( \varphi_1 P_\Omega = P_\Omega \), the estimate (4.12) gives
\[ \langle \varphi_1 P_\Omega \varphi_1 \rangle_{\varphi_t} = \|\langle P_\Omega \rangle_{\psi_t} \lesssim t^{-2s}\|\langle B \rangle \psi_0\|^2 \lesssim t^{-2s}\|\psi_0\|^2_{B^2}. \] (4.16)
Combining this with \( N \geq 1 - P_\Omega \) and (4.11), we obtain
\[ \langle M \rangle_{\psi_t} \lesssim -\frac{1}{c \rho C} \langle \varphi_1[1 - t^{-1}B_{tC} - C(g)]\varphi_1 \rangle_{\psi_t} + \frac{C}{t^{1-\rho}} (\epsilon t^{\rho_0}\|\psi_0\|_1^2 + t^{-1+\nu_0}\|\psi_0\|_2^2 + t^{-2s}\|\psi_0\|_{B^2}^2). \] (4.17)
Now, recalling the definition \( \varphi(B_{tC}) := (B_{tC} - 2)\chi_{B_{tC} \leq 1} \), we compute
\[ B_{tC} \varphi' + \rho(-\varphi) = B_{tC}(\chi + (B_{tC} - 2)\chi') - \rho(B_{tC} - 2)\chi = (1 - \rho)(B_{tC} + 2\rho)\chi + B_{tC}(B_{tC} - 2)\chi'. \]
Next, using that \( B_{tC}\chi \leq \chi, B_{tC}(B_{tC} - 2)\chi' \leq (B_{tC} - 2)\chi' \), we find furthermore
\[ B_{tC}\varphi' + \rho(-\varphi) \leq (1 + \rho)\chi + (B_{tC} - 2)\chi' = \rho\chi + \varphi' \leq (1 + \rho)\varphi'. \] (4.19)
This, together with (4.17), with \( \varphi_1^2 = \varphi' \), gives
\[ \langle M \rangle_{\psi_t} \lesssim -\frac{\sigma}{c} - \frac{1}{c - \rho} \frac{1}{t^{1-\rho}} \langle \varphi' \rangle_{\psi_t} + \frac{C}{t^{1-\rho}} (\epsilon t^{\rho_0}\|\psi_0\|_1^2 + t^{-1+\nu_0}\|\psi_0\|_2^2 + t^{-2s}\|\psi_0\|_{B^2}^2), \] (4.20)
where \( \sigma := 1 - C(g) \).

Next, we introduce the second estimating operator \( E_2 := N + \eta^{-2} + 1 \), with \( \eta^2 := \eta_1^2 \eta_2^2 \eta_3^2 \), and show that the remainder, \( R \), defined in (4.4) satisfies
\[ R \leq Ct^{-\epsilon}e^{-1}E_2. \] (4.21)
To prove (4.21), it suffices to show that
\[ \|E_2^{-\frac{1}{2}}RE_2^{-\frac{1}{2}}\| \lesssim t^{-2}\epsilon^{-1}. \] (4.22)
Proceeding as in the proof of Lemma B.2 of Appendix B, using the Helffer-Sjöstrand formula (B.1), one verifies that
\[ \|E_2^{-\frac{1}{2}}RE_2^{-\frac{1}{2}}\| \lesssim t^{-2}\|RE_2^{-\frac{1}{2}}B_2E_2^{-\frac{1}{2}}\|, \] (4.23)
where \( B_2 := [B_{tC}, B_{tC}, H] \). Now, writing \( B_2 = [B_{tC}, B_{tC}, H] + I_2 \), where \( I_2 := [B_{tC}, B_{tC}, I(g)] \), and using the elementary computations (II.3) and (II.5) of Supplement II, we find \( [B_{tC}, B_{tC}, H] = d\Gamma(\epsilon\theta_{\omega_\epsilon}^2) \) and \( I_2 = I(b_\epsilon^2 g) \). The estimate \( e\theta_{\omega_\epsilon} \lesssim \epsilon^{-1} \) implies
\[ \|(1 + N)^{-\frac{1}{2}}d\Gamma(\epsilon\theta_{\omega_\epsilon}^2)(1 + N)^{-\frac{1}{2}}\| \lesssim \epsilon^{-1}. \] (4.24)
Moreover, (1.5), the condition \( \mu > -1/2 \) and (2.11) (with \( \delta = 0 \)) yield
\[
\| \eta_1 v_2^2 I_2 (1 + N)^{-\frac{1}{2}} \| \lesssim \| \eta_1 v_2^2 b^2 g \| \lesssim \epsilon^{-1} \langle g \rangle,
\]
and hence
\[
\| E_2^{-\frac{1}{2}} I_2 E_2^{-\frac{1}{2}} \| \lesssim \epsilon^{-1} \langle g \rangle.
\]
Thus, we obtain
\[
\| E_2^{-\frac{1}{2}} B_2 E_2^{-\frac{1}{2}} \| \lesssim \epsilon^{-1},
\]
which together with (4.23) implies (4.22). Together with Equations (4.3) and (4.20) and the fact that \( \| \eta^{-1}_1 \eta^{-2}_2 f(H) \| \lesssim 1 \), this implies
\[
\langle D \Phi \rangle \psi_t \leq -\left( \frac{\sigma}{c} - 1 - \rho \right) t^{-1+\rho} \langle \varphi' \rangle \psi_t
\]
\[
+ C(\epsilon t^{\nu_1} + \rho^{-1}) \| \psi_0 \|^2 + t^{-2+\nu_0 + \rho} \| \psi_0 \|^2 + t^{-1+\rho-2\gamma} \| \psi_0 \| B^2).
\]
(4.28)
Thus, choosing \( s \) such that \( 2s - \rho > 0 \), (4.28), together with the observation \( \Phi_t \geq t^s \chi_{B,1} \), the conditions \( \frac{\sigma}{c} - 1 > \rho \), \( \rho < 1 \leq 2 - \nu_0 \), Hardy’s inequality \( \| \psi_0 \|_1 \lesssim \| \psi_0 \|_{d \Gamma(y)} \) and the trivial inequality \( \| \psi_0 \|_0 \lesssim \| \psi_0 \|_{d \Gamma(y)} \), implies that
\[
t^\rho \langle \chi \rangle \psi_t \leq \langle D \Phi \rangle \psi_t = \langle D \Phi \rangle \psi_t \big|_{t=0} + \int_0^t \langle D \Phi \rangle \psi_t \, ds
\]
\[
\leq -B_t \chi_{B,1} \| \psi_0 \| + C(\epsilon t^{\nu_1} + \rho^{-1} + 1) \| \psi_0 \|^2 + \| \psi_0 \|^2_{B^2}.
\]
Using \( -B_t \chi_{B,1} \| \psi_0 \| \lesssim \| \psi_0 \|^2_{d \Gamma(y)} \), and choosing \( \epsilon = t^{-\kappa} \), we find
\[
\langle \chi \rangle \psi_t \leq C(t^{-\rho+\kappa} + t^{\nu_1 - \kappa} + t^{-\epsilon}) \| \psi_0 \|^2_{d \Gamma(y)} + \| \psi_0 \|^2_{B^2},
\]
which in turn gives (4.2).

**Proof of Theorem 4.1.** Since \( N = d \Gamma(1) \) and \( B_t = d \Gamma(b_t) \) commute, we have
\[
\Gamma(\chi_{\frac{\mu}{\nu+1}}) \leq \chi_{B_t, \leq N \nu} = \chi_{B_t, \leq N \nu} \chi_{N \leq c' t^\epsilon} + \chi_{N \leq c' t^\epsilon}
\]
\[
\leq \chi_{B_t, \leq c' \nu' + c \nu'} + \chi_{N \leq c' \nu'},
\]
(4.29)
where \( \nu := \alpha + \gamma \) and \( \nu' := cc' \). We choose \( c' \ll 1/c \), so that \( 0 < c' \ll 1 \). Next, we have
\[
\| \chi_{N \leq c' \nu} \psi_t \| \leq (c')^{-\frac{1}{2}} t^{-\frac{1}{2}} \| \chi_{N \leq c' \nu} N^\frac{1}{2} \psi_t \|
\]
\[
\leq (c')^{-\frac{1}{2}} t^{-\frac{1}{2}} \| N^\frac{1}{2} \psi_t \|,
\]
which, together with (1.14) (with \( \rho = 0 \)), implies
\[
\| \chi_{N \leq c' \nu} \psi_t \| \lesssim t^{-\frac{1}{2} + \frac{\nu}{2}} \| \psi_0 \|_0.
\]
(4.30)
The inequality (4.29) with \( \nu = -1 \), Proposition 4.2 and the inequality (4.30) (with \( \gamma = 1 - \alpha \)) imply the estimate (4.1). \( \square \)

5. **Proof of Theorem 1.3**

5.1. **Partition of unity.** We begin with a construction of a partition of unity which separates photons close to the particle system from those departing it. Following [17, 24] (cf. the many-body scattering construction), it is defined by first constructing a partition of unity \( (j_0, j_\infty) \), \( j_0^2 + j_\infty^2 = 1 \), on one-photon space, \( \mathfrak{h} = \mathcal{L}^2(\mathbb{R}^3) \), with \( j_0 \) localizing a photon to a region near the particle system (the origin) and \( j_\infty \) near infinity, and then associating with it the map \( j : h \rightarrow h \otimes \mathfrak{h} \), given by \( j : h \rightarrow j_0 h \otimes j_\infty h \). After that we lift the map \( j \) to the Fock space \( \mathcal{F} = \Gamma(\mathfrak{h}) \) by using \( \Gamma(j) : \Gamma(h) \rightarrow \Gamma(h \otimes \mathfrak{h}) \) (defined in (1.10)). Next, we consider the adjoint map \( j^* : h_0 \otimes h_\infty \rightarrow j_0^* h_0 + j_\infty^* h_\infty \), which we also lift to the Fock space \( \mathcal{F} := \Gamma(\mathfrak{h}) \) by using \( \Gamma(j^* : \Gamma(h \otimes \mathfrak{h}) \rightarrow \Gamma(h) \). By definition, the operator \( \Gamma(j) \) has the following properties
\[
\Gamma(j)^* = \Gamma(j^*), \quad \Gamma(j) \Gamma(j) = \Gamma(j) j_j).
\]
(5.1)
Since \( j^* j = j_0^2 + j_\infty^2 = 1 \), this implies the relation \( \Gamma(j)^* \Gamma(j) = 1 \), which is what we mean by a partition of unity of the Fock space \( \mathcal{F} := \Gamma(\mathfrak{h}) \).
We compute, using the Helffer-Sjöstrand formula (see (B.1) of Appendix B) for
Proof.
Assume Lemma 5.2.
Proof.
\[ \Gamma(j) := U\Gamma(j) : \Gamma(h) \rightarrow \Gamma(h) \otimes \Gamma(h). \] (5.3)
We lift \( \Gamma(j) \), as well as \( \hat{\Gamma}(j) \), from the Fock space \( \mathcal{F} = \Gamma(h) \) to the full state space \( \mathcal{H} = \mathcal{H}_p \otimes \mathcal{F} \), so that e.g. \( \hat{\Gamma}(j) : \mathcal{H} \rightarrow \mathcal{H} \otimes \Gamma(h) \). Now, the partition of unity relation on \( \mathcal{H} \) becomes \( \hat{\Gamma}(j) \Gamma(j) = 1 \) (in particular, \( \hat{\Gamma}(j) \) is an isometry).

Finally, we specify \( j_0 \) to be the operator \( \chi_{\epsilon \leq 1} \) and define \( j_\infty \) by the relation \( j_0^2 + j_\infty^2 = 1 \) (hence \( j_\infty \) is of the form \( \chi_{\epsilon \geq 1} \)), with \( v = \frac{b_\epsilon}{\epsilon^{2\alpha}} \), \( b_\epsilon \) is defined in the introduction, \( \epsilon = t^{-\kappa} \), and the parameters \( \alpha \) and \( \kappa \) satisfy \( 1 - \mu/(6 + 3\mu) < \alpha < 1 \) and \( 1 + \nu_1 - \alpha < \kappa < 1/2(5\alpha - 3) \).

5.2. Deift-Simon wave operators. We define the auxiliary space \( \hat{\mathcal{H}} := \mathcal{H} \otimes \mathcal{F} \), which will serve as our repository of asymptotic dynamics, which is governed by the hamiltonian \( \hat{\mathcal{H}} := H \otimes 1 + 1 \otimes H_f \) on \( \hat{\mathcal{H}} \). With the partition of unity \( \hat{\Gamma}(j) \), we associate the Deift-Simon wave operators,
\[ W_\pm := \text{s-lim}_{t \rightarrow \infty} W(t), \quad \text{where} \quad W(t) := e^{i\hat{\mathcal{H}}t} \hat{\Gamma}(j) e^{-i\hat{\mathcal{H}}t}, \] (5.4)
which map the original dynamics, \( e^{-i\hat{\mathcal{H}}t} \), into auxiliary one, \( e^{-i\mathcal{H}_t} \) (to be further refined later). Our goal is to prove

**Theorem 5.1.** Assume (1.5) with \( \mu > 0 \), (1.7), and that one of the implicit conditions of Theorem 1.3 is satisfied. Then the Deift-Simon wave operators exist on \( \text{Ran} \ E_{(-\infty, \Sigma)}(H) \) and satisfy
\[ W_+ P_{gs} = P_{gs}, \] (5.5)
and, for any smooth, bounded function \( f \),
\[ W_+ f(H) = f(\hat{H})W_+. \] (5.6)

**Proof.** We begin with the following lemma

**Lemma 5.2.** Assume (1.5) with \( \mu > 0 \) and (1.7). For any \( f \in C_0^\infty((-\infty, \Sigma)) \) and \( \psi_0 \in f(H)D(N_1^{1/2}) \),
\[ \| (\hat{\Gamma}(j)f(H) - f(\hat{H}\Gamma(j))\psi_0 \| \leq t^{-\alpha^\gamma} \| \psi_0 \|_1. \] (5.7)

**Proof.** We compute, using the Helffer-Sjöstrand formula (see (B.1) of Appendix B) for \( f(H) \) and \( f(\hat{H}) \),
\[ \hat{\Gamma}(j)f(H)\psi_0 = f(\hat{H})\Gamma(j)\psi_0 = R, \]
where
\[ R := \frac{1}{\pi} \int \partial_z \bar{f}(z)(\hat{H} - z)^{-1}(\hat{H}\hat{\Gamma}(j)^{-1}(H - z)^{-1}\psi_0 \text{ dRe } z \text{ dIm } z. \] (5.8)

Using \( (H_p \otimes 1 \otimes 1)(1 \otimes \hat{\Gamma}(j)) = (1 \otimes \hat{\Gamma}(j))(H_p \otimes 1) \), we decompose \( \hat{H}\hat{\Gamma}(j) - \hat{\Gamma}(j)H = G_0 + G_1 \), where
\[ G_0 = \hat{H}_f\hat{\Gamma}(j) - \hat{\Gamma}(j)H_f, \] (5.9)
with \( \hat{H}_f = H_f \otimes 1 + 1 \otimes H_f \), and
\[ G_1 := (I(g) \otimes 1)\hat{\Gamma}(j) - \hat{\Gamma}(j)I(g). \] (5.10)

We consider \( G_0 \). A straightforward computation gives \( \Gamma(j)d\Gamma(c) = d\Gamma(c)\Gamma(j) + d\Gamma(j, jc - c) \), where \( c = \text{diag}(c, c) : h \otimes h \rightarrow h \otimes h \) and
\[ d\Gamma(a, c)|_{\otimes^*_n h} = \sum_{j=1}^{n} a \otimes \cdots \otimes a \otimes c \otimes a \otimes \cdots \otimes a. \] (5.11)

It follows from this relation and the equalities \( Ud\Gamma(c) = (d\Gamma(c) \otimes 1 + 1 \otimes d\Gamma(c))U \) that ([17, 24])
\[ \hat{\Gamma}(j)d\Gamma(c) = (d\Gamma(c) \otimes 1 + 1 \otimes d\Gamma(c))\hat{\Gamma}(j) + d\Gamma(j, jc - c), \] (5.12)
where \(d\Gamma(a, c) := Ud\Gamma(a, c)\). We have \(\omega j - j\omega = ([\omega, j_0], [\omega, j_\infty])\), and, by Corollary B.3 of Appendix B,

\[
[\omega, j_\#] = \frac{\theta}{e^{\nu t}} j'_\# + r, \tag{5.13}
\]

where \(j_\#\) stands for \(j_0\) or \(j_\infty\), \(j'_\#\) is the derivative of \(j_\#\) as a function of \(v = \frac{h}{m}\), and \(r\) satisfies \(\|r\| \lesssim t^{-2\alpha + \kappa}\). Since \(\theta \leq 1\) and since \(\kappa < \alpha\), we deduce that \([\omega, j_\#] = O(t^{-\alpha})\). This gives \(G_0 = -d\Gamma(j, \omega - j\omega) = d\Gamma(j, O(t^{-\alpha}))\). Let \(\tilde{N} := N \otimes 1 + 1 \otimes N\) be the number operator on \(\mathcal{H}\). (5.12) with \(c = 1\) implies \((\tilde{N} + 1)^{-1/2}G_0 = G_0(N + 1)^{-1/2}\). By (C.6) of Appendix C, we then obtain that

\[
\|G_0(N + 1)^{-1}\| = \|(\tilde{N} + 1)^{-\frac{1}{2}}G_0(N + 1)^{-\frac{1}{2}}\| \lesssim t^{-\alpha}.
\]

Using the easy property that \(H \in C^1(N)\) (see e.g. [10, Lemma A.6]), we have \(\|(N + 1)(H - z)^{-1}(N + 1)^{-1}\| \lesssim |\text{Im} z|^{-2}\), and hence

\[
\|G_0(H - z)^{-1}\psi_t\| \lesssim t^{-\alpha}|\text{Im} z|^{-2}\|(N + 1)\psi_t\|. \tag{5.14}
\]

Applying Corollary A.3 of Appendix A, we obtain

\[
\|G_0(H - z)^{-1}\psi_t\| \lesssim t^{-\alpha + \frac{1}{2\pi}}|\text{Im} z|^{-2}\|\psi_0\|_1. \tag{5.15}
\]

Now, we address \(G_1\). We use the definition \(\tilde{\Gamma}(j) = U\Gamma(j)\) to obtain \(\tilde{\Gamma}(j)a^#(h) = Ua^#(jh)\tilde{\Gamma}(j)\), where \(a^#\) stands for \(a\) or \(a^*\). Then using (5.2), and \(j_0j_0 + j_\infty j_\infty = 1\), we derive

\[
\tilde{\Gamma}(j)a^#(h) = (a^#(jh) \otimes 1 + 1 \otimes a^#(j_hh))\tilde{\Gamma}(j). \tag{5.16}
\]

This implies

\[
\tilde{\Gamma}(j)I(g) = (I(j_0g) \otimes 1 + 1 \otimes I(j_\infty g))\tilde{\Gamma}(j). \tag{5.17}
\]

The equation (5.17) gives

\[
G_1 = (I((1 - j_0)g) \otimes 1 - 1 \otimes I(j_\infty g))\tilde{\Gamma}(j). \tag{5.18}
\]

Due to the inequality (3.16), we have

\[
\|\eta_1 \eta_j^2j_\infty g\|_{L^2} \lesssim t^{-\lambda}, \quad \|\eta_1 \eta_j^2(1 - j_0)g\|_{L^2} \lesssim t^{-\lambda}, \tag{5.19}
\]

with \(\lambda < (\mu + \frac{3}{2})\alpha\). Using this, we have in addition

\[
\|G_1(N + \eta^2 + 1)^{-1}\| \lesssim t^{-\lambda}, \tag{5.20}
\]

where \(\eta^2 := \eta_j^2 \eta_j^2 \eta_j^2\). Hence, using (1.7) and, as above, that \(\|(N + 1)(H - z)^{-1}(N + 1)^{-1}\| \lesssim |\text{Im} z|^{-2}\), we obtain

\[
\|G_1(H - z)^{-1}\psi_t\| \lesssim t^{-\lambda + \frac{1}{2\pi}}|\text{Im} z|^{-2}\|\psi_0\|_1. \tag{5.21}
\]

From (5.8), (5.15), (5.21), the properties of the almost analytic extension \(\tilde{f}\) and the estimate \(\|(H - z)^{-1}\| \lesssim |\text{Im} z|^{-1}\), we conclude that (5.7) holds. \hfill \Box

We want to show that the family \(W(t) := e^{iHt}\tilde{\Gamma}(j)e^{-iHt}\) form a strong Cauchy sequence as \(t \to \infty\). Let \(\psi_0 \in f(H)D(N_{1/2}^2), f \in C_0^\infty((0, \Sigma))\) and \(f_1 \in C_0^\infty((-\infty, \Sigma))\) be such that \(f_1f = f\). Lemma 5.2 implies that

\[
W(t)\psi_0 = \tilde{W}(t)\psi_0 + O(t^{-\alpha + \frac{1}{2\pi}}\|\psi_0\|_1), \tag{5.22}
\]

where and

\[
\tilde{W}(t) := e^{iHt}f_1(H)\tilde{\Gamma}(j)e^{-iHt}f_1(H).
\]

Hence, since our conditions on \(\alpha\) imply \(\alpha > 1/(2 + \mu)\), it suffices to show that \(\tilde{W}(t)\) form a strong Cauchy sequence as \(t \to \infty\).

First suppose Assumption (i) of Theorem 1.3. We define \(\chi_m := \chi_{N < m}\) and \(\overline{\chi}_m := \chi_{N \geq m}\), so that \(\chi_m + \overline{\chi}_m = 1\). First, we show that, for any \(\psi_0 \in D(N_{1/2}^2)\),

\[
\sup_t \|\chi_m \overline{\chi}(t)\psi_0\| \lesssim m^{-\frac{3}{2}}\|\psi_0\|_0. \tag{5.23}
\]
Indeed, by Assumption (1.20),
\[
\| \hat{N} \frac{1}{\Delta} e^{iHt} f_1(H) \hat{\Gamma}(j) \psi_s \| \lesssim \| \hat{N} \frac{1}{\Delta} \hat{\Gamma}(j) \psi_s \| + \| \hat{\Gamma}(j) \psi_s \|. 
\]  
(5.24)

The boundedness of \( \hat{\Gamma}(j) \) and the definition \( \psi_t := e^{-iHt} \psi_0 \) imply \( \| \hat{\Gamma}(j) \psi_s \| \leq \| \psi_0 \| \). Equation (5.12) with \( c = 1 \) implies \( \hat{N} \frac{1}{\Delta} \hat{\Gamma}(j) = \hat{\Gamma}(j) \hat{N} \frac{1}{\Delta} \). The latter relation, boundedness of \( \hat{\Gamma}(j) \) and Assumption (1.20) give
\[
\| \hat{N} \frac{1}{\Delta} \hat{\Gamma}(j) \psi_s \| = \| \hat{\Gamma}(j) \hat{N} \frac{1}{\Delta} \psi_s \| \lesssim \| \psi_0 \|, 
\]  
and therefore, by (5.24), \( \| \hat{N} \frac{1}{\Delta} e^{iHt} f_1(H) \hat{\Gamma}(j) \psi_s \| \lesssim \| \psi_0 \| \). Since this is true uniformly in \( t, s \), it implies \( \| \hat{N} \frac{1}{\Delta} \hat{W}(t) \psi_0 \| \lesssim \| \psi_0 \| \), which yields (5.23). Equation (5.23) implies that
\[
\sup_{t, t'} \| \chi_m(\hat{W}(t') - \hat{W}(t)) \psi_0 \| \lesssim m^{-\frac{1}{2}} \| \psi_0 \|. 
\]  
(5.25)

Now we show that, for any \( m > 0 \) and for any \( \psi_0 \in D(d\Gamma((j)_0)) \cap \text{Ran} E_{(-\infty, \Sigma)}(H) \),
\[
\| \chi_m(\hat{W}(t') - \hat{W}(t)) \psi_0 \| \to 0, 
\]  
(5.26)
as \( t, t' \to \infty \). This together with (5.25) implies that \( \hat{W}(t) \) form a strong Cauchy sequence. We write
\[
\hat{W}(t') - \hat{W}(t) = \int_t^{t'} ds \partial_s \hat{W}(s), 
\]  
(5.27)
and compute \( \partial_s \hat{W}(t) = e^{iHt} f_1(H) G e^{-iHt} f_1(H) \), where \( G := i(\hat{H} \hat{\Gamma}(j) - \hat{\Gamma}(j) H) + \partial_t \hat{\Gamma}(j) \). We write \( G = \hat{G}_0 + iG_1 \), where
\[
\hat{G}_0 := iG_0 + \partial_t \hat{\Gamma}(j), 
\]  
and \( G_0 \) and \( G_1 \) are defined in (5.9)–(5.10). We consider \( \hat{G}_0 \). Using the notation \( \tilde{d}j := i(\omega j - j\omega) + \partial_t j, \) with \( \omega = \text{diag}(\omega, \omega) \), and (5.12), we compute readily
\[
\hat{G}_0 = Ud\Gamma(\tilde{j}, \tilde{d}j) = d\Gamma(\tilde{j}, \tilde{d}j). 
\]  
(5.28)

Write \( \tilde{j}' = (j_0', j_{\infty}') \), where \( j_0', j_{\infty}' \) are the derivatives of \( j_0, j_{\infty} \) as functions of \( v = \frac{\omega}{c\ell} \). We first find a convenient decomposition of \( \tilde{d}j \). Using \( \tilde{d}j f = (dj_0 f, dj_{\infty} f) \), with \( dc = i[\omega, c] + \partial_t c, (3.8) \) and Corollary B.3 of Appendix B, we compute
\[
\tilde{d}j = (j_0', j_{\infty}') (\frac{\theta}{ct^\alpha} - \frac{ab}{ct^{\alpha + 1}}) + O(t^{-2\alpha + \kappa}). 
\]  
(5.29)
We insert the maximal velocity partition of unity \( \chi_{w \leq 1} + \chi_{w > 1} = 1 \), with \( w := (\frac{\omega}{c\ell})^2 \) and \( c > 1 \), into this formula and use the notation \( \chi = \chi_{w \leq 1} \) and the relation \( v j_{\#}' = \Theta(1) \), valid due to the localization of \( j_{\#}' \), to obtain
\[
\tilde{d}j = \frac{1}{ct^\alpha} \theta^{1/2} \chi(j_0', j_{\infty}') \chi \theta^{1/2} + \text{rem}_{\ell}, 
\]  
(5.30)
\[
\text{rem}_{\ell} = O(t^{-1}) \chi(j_0', j_{\infty}') \chi + O(t^{-2\alpha + \kappa}) + O(t^{-\alpha}) \chi_{w > 1}. 
\]  
(5.31)
These relations give
\[
\hat{G}_0 = G_0' + \text{Rem}_{\ell}, 
\]  
(5.32)
where \( G_0' := \frac{1}{ct^\alpha} Ud\Gamma(j, \ell_{\ell}), \) with \( \ell_{\ell} = (c_0, c_{\infty}) := (\theta^{1/2} \chi j_0' \chi \theta^{1/2}, \theta^{1/2} \chi j_{\infty}' \chi \theta^{1/2}), \) and
\[
\text{Rem}_{\ell} := \hat{G}_0 - G_0' = Ud\Gamma(j, \text{rem}_{\ell}). 
\]  
Next, we write
\[
A := \sup_{\| \phi_s \| = 1} \left| \int_t^{t'} ds \partial_s (\phi_s, G_0 \psi_s) \right|, 
\]  
where \( \phi_s := e^{-iHt} f_1(H) \chi_m \phi_0 \). By (C.5) of Appendix C, \( G_0' \) satisfies
\[
|\langle \phi, G_0' \psi \rangle| \leq \frac{1}{ct^\alpha} (\| d\Gamma(|c_0|) \|^{\frac{1}{2}} \otimes 1 \| d\Gamma(|c_0|) \|^{\frac{1}{2}} \| \psi \| + \| 1 \otimes d\Gamma(|c_{\infty}|) \|^{\frac{1}{2}} \| \phi \| \| d\Gamma(|c_{\infty}|) \|^{\frac{1}{2}} \| \psi \|). 
\]  
(5.33)
By the Cauchy-Schwarz inequality, (5.33) implies
\[
\int_t^{t'} ds |\langle \phi_s, G_0' \psi_s \rangle| \lesssim \left( \int_t^{t'} ds \, s^{-\alpha} \|d\Gamma(|c_0|)^{1/2} \otimes 1 \phi_s \|_2^2 \right)^{1/2} \left( \int_t^{t'} ds \, s^{-\alpha} \|d\Gamma(|c_0|)^{1/2} \psi_s \|_2^2 \right)^{1/2}
\]
\[
\quad + \left( \int_t^{t'} ds \, s^{-\alpha}\|1 \otimes d\Gamma(|c_\infty|)^{1/2} \phi_s \|_2^2 \right)^{1/2} \left( \int_t^{t'} ds \, s^{-\alpha}\|d\Gamma(|c_\infty|)^{1/2} \psi_s \|_2^2 \right)^{1/2}.
\]
Since \(|c_0|, |c_\infty|\) are of the form \(\theta^1/2 \chi \frac{1}{\sqrt{\pi m}} = 1 \theta^1/2\), the minimal velocity estimate (3.4) implies
\[
\int_1^\infty ds \, s^{-\alpha}\|d\Gamma(|c|)^{1/2} \phi_s \|_2^2 \lesssim \|\chi_m \phi_0 \|_2^2 \lesssim m\|\phi_0\|_2^2,
\]
where \(d\Gamma(|c|)^{1/2}\) stands for \(d\Gamma(|c_0|)^{1/2}\) or \(d\Gamma(|c_\infty|)^{1/2}\), and
\[
\int_1^\infty ds \, s^{-\alpha}\|d\Gamma(|c|)^{1/2} \psi_s \|_2^2 \lesssim \|\psi_0\|_2^2,
\]
with \(d\Gamma(|c|)^{1/2} = d\Gamma(|c_0|)^{1/2}\) or \(d\Gamma(|c_\infty|)^{1/2}\), provided that \(\alpha < 1/\bar{c}\). The last three relations give
\[
\sup_{\|\phi_0\|_2 = 1} \left| \int_t^{t'} ds \langle \phi_s, G_0' \psi_s \rangle \right| \to 0, \quad t, t' \to \infty. \tag{5.34}
\]
Likewise, applying (C.6) of Appendix C first with \(c_1 = 1\) and \(c_2 = 1\), next with \(c_1 = 1\) and \(c_2 = \chi_{w \geq 1}\),
where recall \(w = \left(\frac{|v|}{ct}\right)^2\), and then applying (C.5) with \(c_0 = \chi_{m_0} x\) and \(c_\infty = \chi_{m_\infty} x\), we see that \(\text{Rem}_t\) satisfies
\[
|\langle \phi_s, \text{Rem}_t \psi_s \rangle| \lesssim \|\hat{N} \frac{1}{2} \phi_s\| \left( t^{-2\alpha + \kappa} \|N \frac{1}{2} \psi_s\| + t^{-1} \|d\Gamma(\chi_{w \geq 1}^{1/2}) \frac{1}{2} \psi\| + t^{-\alpha} \|d\Gamma(\chi_{m \geq 1}^{1/2}) \frac{1}{2} \psi\| \right). \tag{5.35}
\]
Now, using (5.35) and the Cauchy-Schwarz inequality, we obtain
\[
\int_t^{t'} ds |\langle \phi_s, \text{Rem}_t \psi_s \rangle| \leq \left( \int_t^{t'} ds \, s^{-\tau} \|\hat{N} \frac{1}{2} \phi_s\|_2^2 \right)^{1/2} \left\{ \left( \int_t^{t'} ds \, s^{-2(2\alpha - \kappa) + \tau} \|N \frac{1}{2} \psi_s\|_2^2 \right)^{1/2} \right. \\
\quad \left. + \left( \int_t^{t'} ds \, s^{-2\tau} \|d\Gamma(\chi_{w \geq 1}^{1/2}) \frac{1}{2} \psi_s\|_2^2 \right)^{1/2} \right\}. \tag{5.36}
\]
Let \(\tau > 1\) and \(\alpha = 2 - \tau\). Then by the estimate (3.3), we have
\[
\int_1^\infty ds \, s^{-2\gamma + \tau} \|d\Gamma(\chi_{w \geq 1}^{1/2}) \frac{1}{2} \psi_s\|_2^2 \lesssim \|\psi_0\|_2^2,
\]
and by the maximal velocity estimate (1.12), we have
\[
\int_1^\infty ds \, s^{-2(2\alpha - \kappa) + \tau} \|N \frac{1}{2} \psi_s\|_2^2 \lesssim \|\psi_0\|_{d\Gamma(\psi)},
\]
provided that \(\alpha > 1 - 2\gamma/3\), where, recall, \(\gamma < \frac{3}{2} \min\left( \frac{c-1}{2\alpha c-1}, \frac{1}{2\alpha + \kappa} \right)\). One verifies that \(c>1\) can be chosen such that this condition is satisfied and \(\alpha < 1/\bar{c}\). Moreover, Assumption (1.20) implies
\[
\int_1^\infty ds \, s^{-2(2\alpha - \kappa) + \tau} \|N \frac{1}{2} \psi_s\|_2^2 \lesssim \|\psi_0\|_0,
\]
provided that \(5\alpha > 3 + 2\kappa\). This and the fact that, by Assumption (1.20), the first integral on the r.h.s. of (5.36) converge yield
\[
\sup_{\|\phi_0\|_2 = 1} \left| \int_t^{t'} ds \langle \phi_s, \text{Rem}_t \psi_s \rangle \right| \to 0, \quad t, t' \to \infty. \tag{5.37}
\]
Equations (5.34) and (5.37) imply that
\[
A = \left\| \int_t^{t'} ds \chi_m f_1(\hat{H}) e^{i \hat{H} s} \hat{G}_0 \psi_s \right\| \to 0, \quad t, t' \to \infty. \tag{5.38}
\]
Now we turn to \(G_1\). The equations (5.18), (5.19), (2.11) (with \(\delta = 0\), (1.7) and \(\hat{N} \frac{1}{2} \Gamma(j) = \Gamma(j) \hat{N} \frac{1}{2}\) imply that
\[
\|f(\hat{H})G_1(N + 1)^{-1/2}\| \lesssim t^{-\lambda}, \tag{5.39}
\]
\end{quote}
for $\lambda < (\mu + \frac{3}{2})\alpha$. Together with Assumption (1.20), this implies that $\|f(\hat{H})G_1\psi\| \lesssim t^{-\lambda}\|\psi_0\|_0$, and hence

$$\left\| \int_t^{t'} ds f(\hat{H})e^{i\hat{H}s}G_1\psi_s \right\| \to 0, \quad t, t' \to \infty,$$

provided that $\alpha > (\mu + \frac{3}{2})^{-1}$. This together with (5.38) gives (5.26) which, as was mentioned above, together with (5.25) shows that $\hat{W}(t)$ is a Cauchy sequence as $t \to \infty$. Hence by (5.22) $\hat{W}(t)$ is a strong Cauchy sequence. This implies the existence of $W_+$.

The proof of the existence of $W_+$ under the assumption (1.21) of Theorem 1.3 is similar, except that we do not need to introduce the cutoff $\chi_m$. We use instead a variant of the weighted propagation estimates of Theorem 3.1. For reader’s convenience we give this proof in Appendix E.

Finally, the proofs of (5.5) and (5.6) are standard. We present the second one. By (5.4), we have $W_+e^{iHs} = \text{s-lim } e^{iH\Gamma(j)e^{-iH(t+s)}} = \text{s-lim } e^{iH(t-s)}\Gamma(j)e^{-iHt'} = e^{iHs}W_+$, which implies (5.6).

\[\square\]

5.3. Scattering map. We define the space $\mathcal{H}_\text{fin} := \mathcal{H}_\text{p} \otimes \mathcal{F}_\text{fin} \otimes \mathcal{F}_\text{fin}$, where $\mathcal{F}_\text{fin} = \mathcal{F}_\text{fin}(\mathfrak{h})$ is the subspace of $\mathcal{F}$ consisting of vectors $\Psi = (\psi_n)_{n=0}^\infty \in \mathcal{F}$ such that $\psi_n = 0$, for all but finitely many $n$, and the (scattering) map $I : \mathcal{H}_\text{fin} \to \mathcal{H}$ as the extension by linearity of the map (see [40, 17, 24])

$$I : \Phi \otimes \prod_{i=1}^n a^*(h_i)\Omega \to \prod_{i=1}^n a^*(h_i)\Phi, \quad (5.40)$$

for any $\Phi \in \mathcal{H}_\text{p} \otimes \mathcal{F}_\text{fin}$ and for any $h_1, \ldots, h_n \in \mathfrak{h}$. (Another useful representation of $I$ is $I : \Phi \otimes f \to \left(\begin{array}{c} p + q \\ p \end{array}\right) \Phi \otimes f$, for any $\Phi \in \mathcal{H}_\text{p} \otimes (\otimes^n \mathfrak{h})$ and $f \in \otimes^n \mathfrak{h}$.) As already clear from (5.40), the operator $I$ is unbounded. Let

$$\mathfrak{h}_0 := \left\{ h \in L^2(\mathbb{R}^3), \int dk (1 + \omega(k)^{-1})|h(k)|^2 < \infty \right\}. \quad (5.41)$$

Properties of the operator $I$ used below are recorded in the following

**Lemma 5.3** ([17, 24, 32]). For any operator $j : h \to j_0h \oplus j_{\infty}h$ and $n \in \mathbb{N}$, the following relations hold

$$\hat{\Gamma}(j)^* = i\Gamma(j_0^*) \otimes \Gamma(j_{\infty}^*), \quad (5.42)$$

$$D((H + i)^{-n/2}) \otimes (\otimes^n \mathfrak{h}_0) \subset D(I). \quad (5.43)$$

**Proof.** Since the operators involved act only on the photonic degrees of freedom, we ignore the particle one. For $g, h \in \mathfrak{h}$, we define embeddings $i_0g := (g, 0) \in \mathfrak{h} \oplus \mathfrak{h}$ and $i_\infty h := (0, h) \in \mathfrak{h} \oplus \mathfrak{h}$. By the definition of $U$ (see (5.2)), we have the relations $U^*a^*(g)\otimes 1 = a^*(i_0g)U^*$, and $U^*1 \otimes a^*(h) = a^*(i_\infty h)U^*$. Hence, using in addition $U^*\Omega \otimes \Omega = \Omega$, we obtain

$$U^*\prod_{i=1}^n a^*(g_i)\Omega \otimes \prod_{i=1}^n a^*(h_i)\Omega = \prod_{i=1}^n a^*(i_0g_i)\prod_{i=1}^n a^*(i_\infty h_i)\Omega. \quad (5.44)$$

By the definition of $\Gamma(j)$ and the relations $j^*i_0g = j_0^*g$ and $j^*i_\infty h = j_{\infty}^*h$, this gives

$$\Gamma(j)^*U^*\prod_{i=1}^n a^*(g_i)\Omega \otimes \prod_{i=1}^n a^*(h_i)\Omega = \prod_{i=1}^n a^*(j_0^*g_i)\prod_{i=1}^n a^*(j_{\infty}^*h_i)\Omega. \quad (5.44)$$

Now, by the definition of $\hat{\Gamma}(j)$ (see (5.2)), we have $\hat{\Gamma}(j)^* = \Gamma(j)^*U^*$. On the other hand by (5.40), the r.h.s. is $i\Gamma(j_0^*) \otimes i\Gamma(j_{\infty}^*) \prod_{i=1}^n a^*(g_i)\Omega \otimes \prod_{i=1}^n a^*(h_i)\Omega$. This proves (5.42).

To prove (5.43), we use the following elementary properties ([24, 32]):

The operator $H_f^2(H + i)^{-n}$ is bounded \forall $n \in \mathbb{N}$,

$$\|a^*(h_1) \cdots a^*(h_n)(H_f + 1)^{-n/2}\| \leq C_n\|h_1\|_\omega \cdots \|h_n\|_\omega, \quad (5.45)$$

and, for any $h_1, \cdots, h_n \in \mathfrak{h}_0$, where $\mathfrak{h}_0$ is defined in (5.41),

$$\|\phi h_1 \cdots a^*(h_n)(H_f + 1)^{-n/2}\| \leq C_n\|h_1\|_\omega \cdots \|h_n\|_\omega\|H + i\|^{n/2}\Phi < \infty. \quad (5.46)$$

This gives the second statement of the lemma. \[\square\]
5.4. Asymptotic completeness. Recall that \( P_{gs} \) denotes the orthogonal projection onto the ground state subspace of \( H \). Below, the symbol \( C(\epsilon)\omega(1) \) stands for a real function of \( \epsilon \) and \( t \) such that, for any fixed \( \epsilon \), \( |C(\epsilon)\omega(1)| \to 0 \) as \( t \to \infty \), and we denote by \( \chi_\Omega(\lambda) \) a smoothed out characteristic function of a set \( \Omega \). In this section we prove the following result.

**Theorem 5.4.** Assume the conditions of Theorem 1.3 for hamiltonians of the form (1.3)–(1.4), and let \( a < \Sigma \), \( \Delta = [E_{gs}, a] \) and \( \Delta' = [0, a - E_{gs}] \). Then, for every \( \epsilon' > 0 \) and \( \phi_0 \in \text{Ran}_\Delta(H) \), there is \( \phi_{o\epsilon'} \), s.t.

\[
\limsup_{t \to \infty} \|\psi_t - \hat{I}(e^{-iE_{gs}t} P_{gs} \otimes e^{-iHt} \chi_\Delta(H))\phi_{0\epsilon'}\| = O(\epsilon'),
\]

which implies (1.9).

**Proof.** Let \( \alpha \) and \( \kappa \) be fixed such that the conditions of Theorems 3.1, 4.1 and 5.1 hold. Let \( (\tilde{j}_0, \tilde{j}_\infty) = (\chi_{v \leq 1}, \chi_{v \geq 1}) \) be the partition of unity defined in Subsection 5.1, where \( v = \frac{b}{2\kappa} \). Since \( \tilde{j}_0^2 + \tilde{j}_\infty^2 = 1 \), the operator \( \hat{\Gamma}(j) \) is, as mentioned above, an isometry. Using the relation \( \Gamma(j)^* \Gamma(j) = 1 \), the boundedness of \( \hat{\Gamma}(j)^* \), and the existence of \( W_+ \), we obtain

\[
\psi_t = \hat{\Gamma}(j)^* e^{-iHt} \hat{\Gamma}(j) e^{-iHt} \phi_0 = \hat{\Gamma}(j)^* e^{-iHt} \phi_0 + o_1(1),
\]

where \( \phi_0 := W_+ \psi_0 \). Next, using the property \( W_+ \chi_\Delta(H) = \chi_\Delta(H) W_+ \), which gives \( W_+ \phi_0 = \chi_\Delta(H) W_+ \phi_0 \), and \( \chi_\Delta(H) = (\chi_\Delta(H) \otimes \chi_\Delta'(H)) \chi_\Delta(H) \), and again using \( \chi_\Delta(H) W_+ \phi_0 = W_+ \phi_0 = \phi_0 \), we obtain

\[
\phi_0 = (\chi_\Delta(H) \otimes \chi_\Delta'(H)) \phi_0.
\]

For all \( \epsilon' > 0 \), there is \( \delta' = \delta'(\epsilon') > 0 \), such that

\[
\| (\chi_\Delta(H) \otimes 1) \phi_0 - (\chi_\Delta(H) \otimes 1) \phi_0 - (P_{gs} \otimes 1) \phi_0 \| \leq \epsilon',
\]

with \( \Delta_\epsilon = [E_{gs} + \delta', a] \). The last two relations give

\[
\phi_0 = ((\chi_\Delta(H) + P_{gs}) \otimes \chi_\Delta'(H)) \phi_0 + O(\epsilon').
\]

Furthermore, let \( (\tilde{j}_0, \tilde{j}_\infty) \) be of the form \( \tilde{j}_0 = \tilde{\chi}_{v \leq 1}, \tilde{j}_\infty = \tilde{\chi}_{v \geq 1} \) where \( \tilde{\chi} \), has the same properties as \( \chi \), and satisfy \( \tilde{j}_0 \tilde{j}_0 = \tilde{j}_0, \tilde{j}_\infty \tilde{j}_\infty = \tilde{j}_\infty \). Then, by (5.42), the adjoint \( \hat{\Gamma}(j)^* \) to the operator \( \hat{\Gamma}(j) \) can be represented as

\[
\hat{\Gamma}(j)^* = \hat{\Gamma}(j)^* (\Gamma(\tilde{j}_0) \otimes \Gamma(\tilde{j}_\infty)).
\]

Using this equation in (5.52), together with the relations \( e^{-iHt} = e^{-iHt} \otimes e^{-iHt} \) and \( e^{-iHt} P_{gs} = e^{-iE_{gs}t} P_{gs} \), gives

\[
\psi_t = \hat{\Gamma}(j)^* \psi_{t\epsilon'} + A + B + C + O(\epsilon') + o_1(1),
\]

where

\[
\psi_{t\epsilon'} := (e^{-iE_{gs}t} P_{gs} \otimes e^{-iHt} \chi_\Delta'(H)) \phi_{0\epsilon'},
\]

\[
A := \hat{\Gamma}(j)^* (\Gamma(\tilde{j}_0) e^{-iHt} \chi_\Delta(H) \otimes \Gamma(\tilde{j}_\infty) e^{-iHt} \chi_\Delta'(H)) \phi_{0\epsilon'},
\]

\[
B := \hat{\Gamma}(j)^* ((\Gamma(\tilde{j}_0) - 1) e^{-iE_{gs}t} P_{gs} \otimes \Gamma(\tilde{j}_\infty) e^{-iHt} \chi_\Delta'(H)) \phi_{0\epsilon'},
\]

\[
C := \hat{\Gamma}(j)^* (e^{-iE_{gs}t} P_{gs} \otimes (\Gamma(\tilde{j}_\infty) - 1) e^{-iHt} \chi_\Delta'(H)) \phi_{0\epsilon'}.
\]

Since \( \hat{\Gamma}(j)^* \) is bounded, the minimal velocity estimate, (4.1), gives (here we use that the first components of \( \phi_{o\epsilon'} \) are in \( D(\Delta(\langle y \rangle)) \))

\[
\| A \| \leq \| (\Gamma(\tilde{j}_0) e^{-iHt} \chi_\Delta(H) \otimes 1) \phi_{0\epsilon'} \| = C(\epsilon')o(1).
\]

Now we consider the term given by \( B \). We begin with

\[
\| B \| \leq \| (\Gamma(\tilde{j}_0) - 1) P_{gs} \|.
\]
Since $0 \leq \tilde{j}_0 \leq 1$, we have that $0 \leq 1 - \Gamma(\tilde{j}_0) \leq 1$. Using this, the relations $1 - \Gamma(\tilde{j}_0) \leq d\Gamma(\tilde{\chi}_{\nu \geq 1}) \leq t^{-2\alpha}d\Gamma(\tilde{\beta}_0^2)$, we obtain the bound
\begin{equation}
\| (\Gamma(\tilde{j}_0) - 1)u \|^2 \leq \| (1 - \Gamma(\tilde{j}_0))^{1/2}u \|^2 \leq t^{-2\alpha}\| d\Gamma(\tilde{\beta}_0^2)^{1/2}u \|^2.
\end{equation}
Using the pull-through formula, one verifies that $d\Gamma(\tilde{\beta}_0^2) P_{gs}$ is bounded and that $\| d\Gamma(\tilde{\beta}_0^2)^{1/2} P_{gs} \| = O(t^\alpha)$ (see Appendix D, Lemma D.1). Hence, since $\kappa < \alpha$, the above estimates yield
\begin{equation}
\| B \| = o_t(1).
\end{equation}
Next, using $\Gamma(j_\infty)e^{-iH_ft} = e^{-iH_ft}\Gamma(e^{i\omega t}j_\infty e^{-i\omega t})$ and $e^{i\omega t}b_\epsilon e^{-i\omega t} = b_\epsilon + \theta_\epsilon t$, it is not difficult to verify (see Appendix C, Lemma C.4) that
\begin{equation}
\| C \| \leq \| 1 \otimes (\Gamma(e^{i\omega t}j_\infty e^{-i\omega t}) - 1)\phi_0 \| \to 0,
\end{equation}
as $t \to \infty$, and hence we obtain
\begin{equation}
\| C \| = C(\epsilon')o_t(1).
\end{equation}
Inserting the previous estimates into (5.54) shows that
\begin{equation}
\psi_t = \tilde{\Gamma}(j)^*\psi_{te'} + O(\epsilon') + C(\epsilon')o_t(1).
\end{equation}
Next, we want to pass from $\tilde{\Gamma}(j)^*$ to $I$ using the formula (5.42). To this end we use estimates of the type (5.61) and (5.62) in order to remove the term $\Gamma(j_0) \otimes \Gamma(j_\infty)$. Hence, we need to bound $I$, for instance by introducing a cutoff in $N$. Let $\chi_m := \chi_{N < m}$ and $\tilde{\chi}_m := 1 - \chi_m$ and write $\tilde{\Gamma}(j)^*\psi_{te'} = \chi_m\tilde{\Gamma}(j)^*\psi_{te'} + \tilde{\chi}_m\tilde{\Gamma}(j)^*\psi_{te'}$. Using that $N^{1/2}\tilde{\Gamma}(j)^* = \tilde{\Gamma}(j)^*N^{1/2}$ and that by Lemma D.1 of Appendix D (see also [7, 35]), Ran $P_{gs} \subset D(N^{1/2})$, and therefore $\psi_{te'} \in D(N^{1/2})$, we estimate
\begin{equation}
\|\chi_m\tilde{\Gamma}(j)^*\psi_{te'}\| \lesssim m^{-\frac{1}{2}}\|N^{1/2}\psi_{te'}\| = m^{-\frac{1}{2}}C(\epsilon').
\end{equation}
Now, we can use (5.42) to obtain
\begin{equation}
\psi_t = \chi_m I(\tilde{\Gamma}(j_0) \otimes \tilde{\Gamma}(j_\infty))\psi_{te'} + O(\epsilon') + C(\epsilon')o_t(1) + C(\epsilon')o_m(1).
\end{equation}
Using $\|\chi_m I\| \leq 2m^{1/2}$ together with estimates of the type (5.61) and (5.62), we find (here we need the cutoff $\chi_m$)
\begin{equation}
\psi_t = \chi_m I\psi_{te'} + O(\epsilon') + C(\epsilon',m)o_t(1) + C(\epsilon')o_m(1).
\end{equation}
Since $\phi_0 \in \mathcal{H} \otimes \mathcal{F}_{\text{fin}}(b_0)$, we can write $\psi_{te'}$ as $\psi_{te'} = \Phi_{gs} \otimes f_{te'}$, with $f_{te'} \in \mathcal{F}_{\text{fin}}(b_0)$, and therefore $\psi_{te'} \in D(I)$ (here we need that $f_{te'}$ is in $\mathcal{F}_{\text{fin}}(b_0)$). Hence $\chi_m I\psi_{te'} = I\psi_{te'} + C(\epsilon')o_m(1)$. Combining this with (5.65) and remembering (5.55), we obtain
\begin{equation}
\psi_t = I(e^{-iE_\epsilon t}P_{gs} \otimes e^{-iH_f t})\chi_{\Delta}(H_f)\phi_0 + O(\epsilon') + C(\epsilon',m)o_t(1) + C(\epsilon')o_m(1).
\end{equation}
Letting $t \to \infty$, next $m \to \infty$, the equation (5.47) follows. 

**Remark.** The reason for $\epsilon'$ in the statement of the theorem is we do not know whether $(P_{gs} \otimes 1)W_+ \psi_0 \in D(I)$. Indeed, if the latter were true, then the relations (5.66), (5.51) and $\| \phi_0 - \phi_{te'} \| \leq \epsilon'$, where $\phi_0 := W_+ \psi_0$, would give
\begin{equation}
\psi_t = I(e^{-iE_\epsilon t}P_{gs} \otimes e^{-iH_f t})\chi_{\Delta}(H_f)\phi_0 + O(\epsilon') + C(\epsilon',m)o_t(1) + C(\epsilon')o_m(1),
\end{equation}
which, after letting $t \to \infty$, next $m \to \infty$ and then $\epsilon' \to 0$, gives
\begin{equation}
\lim_{t \to \infty} \| \psi_t - I(e^{-iE_\epsilon t}P_{gs} \otimes e^{-iH_f t})\chi_{\Delta}(H_f)W_+ \psi_0 \| = 0.
\end{equation}
6. Proof of Minimal Velocity Estimates

In this section we use Theorems 3.1 and 4.1 to prove the minimal velocity estimates of Theorems 1.1 and 1.2 for Hamiltonians of the form (1.3)–(1.4), with the coupling operators \( y(k) \) satisfying (1.5) and (1.7).

Proof of Theorem 1.1 for Hamiltonians of the form (1.3)–(1.4). To prove (1.16), we use several iterations of Proposition 2.4. We consider the one-parameter family of one-photon operators

\[ \phi_t := t^{-a\nu_0} \chi_{w_{\alpha} \geq 1}, \]

with \( w_{\alpha} := \left( \frac{|\alpha|}{\epsilon t^{\alpha}} \right)^2, \alpha > 1 \), and \( \nu_0 \geq 0 \), the same as in (1.14). We use the notation \( \check{\chi}_{\alpha} \equiv \chi_{w_{\alpha} \geq 1} \). As in (3.6)–(3.7), we use the expansion

\[ d\phi_t = t^{-a\nu_0} (dw_{\alpha}) \check{\chi}_{\alpha} + \sum_{i=1}^{2} \text{rem}_i, \]

(6.1)

We compute

\[ dw_{\alpha} = \frac{2b}{(ct^{\alpha})^2} - \frac{2\alpha w_{\alpha}}{t}, \]

(6.3)

where \( \tilde{b} := \frac{1}{2} (\nabla \cdot y + \text{h.c.}) \). We write \( b = b_\epsilon + \epsilon \left( \frac{1}{\omega_\epsilon} \nabla \cdot y + \text{h.c.} \right) \), where, recall, \( \omega_\epsilon := \omega + \epsilon, \epsilon := t^{-\kappa} \). We choose \( \kappa > 0 \) satisfying

\[ 4\alpha - 3 > \kappa > 2 - 2\alpha + \nu_1 - \nu_0. \]

(6.4)

Using the notation \( v = \frac{b_{\epsilon}}{ct^{\alpha}} \) and the partition of unity \( \chi_{v \geq 1} + \chi_{v \leq 1} = 1 \), we find \( b_\epsilon \geq ct^{\alpha} + (b_{\epsilon} - ct^{\alpha}) \chi_{v \leq 1} \). Commutator estimates of the type considered in Appendix B (see Lemma B.5) give \( \chi_{v \leq 1}(\check{\chi}_{\alpha})^{1/2} = \mathcal{O}(t^{-\alpha + \kappa}) \) for \( \bar{c} > c/2 \), which, together with \( b_{\epsilon}(\check{\chi}_{\alpha})^{1/2} = \mathcal{O}(t^{\alpha}) \), yields

\[ (\check{\chi}_{\alpha})^{1/2} b_{\epsilon} \chi_{v \leq 1} (\check{\chi}_{\alpha})^{1/2} \geq -\bar{c}^{\alpha}(\check{\chi}_{\alpha})^{1/2} \chi_{v \leq 1} (\check{\chi}_{\alpha})^{1/2} - Ct^\kappa \check{\chi}_{\alpha}. \]

The last two estimates, together with \( v \leq 1 \) on \( \text{supp} \check{\chi}_{v \geq 1} \), give \( d\phi_t \geq p_t - \tilde{p}_t + \text{rem} \), where

\[ p_t := \frac{2}{\nu_0} \frac{1}{ct^{\alpha}} \left( \frac{c}{(ct^{\alpha})^2} - \frac{\alpha}{t} \right) \check{\chi}_{\alpha}, \]

\[ \tilde{p}_t := \frac{2(c + \bar{c})}{ct^{\alpha} + a\nu_0} \left( \check{\chi}_{\alpha} \right)^{1/2} \chi_{v \leq 1} (\check{\chi}_{\alpha}^{1/2}), \]

and \( \text{rem} = \sum_{i=1}^{4} \text{rem}_i \), with \( \text{rem}_1 \) and \( \text{rem}_2 \) given by (6.2),

\[ \text{rem}_3 := \frac{c}{(ct^{\alpha})^2} \frac{1}{\omega_\epsilon} \nabla \cdot y + \text{h.c.} \right) \check{\chi}_{\alpha}, \]

and \( \text{rem}_4 = \mathcal{O}(t^{-2\alpha + \kappa - a\nu_0}) \). If \( \alpha = 1 \), then we choose \( c > (c')^2 \) so that \( p_t \geq 0 \).

As in the proof of Theorem 3.1, we deduce that the remainders \( \text{rem}_i \), \( i = 1, 2, 3, 4 \), satisfy the estimates (3.15), \( i = 1, 2, 3, 4 \), with \( \rho_1 = \rho_3 = 1, \rho_2 = \rho_4 = 0, \lambda_1 = 2\alpha + a\nu_0, \lambda_2 = 1 + a\nu_0, \lambda_3 = 1 + a\nu_0 + 2\alpha - \kappa + a\nu_0 \) and \( \lambda_4 = 2\alpha - \kappa + a\nu_0 \). In particular, the estimate for \( i = 1 \) follows from Lemma B.4. Since \( 2\alpha > \alpha + \kappa > 1 + \nu_1 - a\nu_0 \) and \( 2\alpha - \kappa > 1 \), the remainder \( \text{rem} = \sum_{i=1}^{4} \text{rem}_i \) gives an integrable term. (Note \( \text{rem}_2 = 0 \), if \( \nu_0 = 0 \).)

Now, we estimate the contribution of \( \tilde{p}_t \). Let \( \gamma = 2\alpha - 1 \leq \alpha \) and decompose \( \tilde{p}_t = p_{t1} + p_{t2} \), where

\[ p_{t1} := \frac{c'}{\lambda_{c_1} + a\nu_0} \left( \check{\chi}_{\alpha} \right)^{1/2} \chi_{c_1 \gamma \leq b_{\epsilon} \leq ct^{\alpha} \left( \check{\chi}_{\alpha} \right)^{1/2}}, \]

\[ p_{t2} := \frac{c''}{\lambda_{c_1} + a\nu_0} \left( \check{\chi}_{\alpha} \right)^{1/2} \chi_{c_1 \gamma} (\check{\chi}_{\alpha})^{1/2}, \]

with \( \chi_{c_1 \gamma \leq b_{\epsilon} \leq ct^{\alpha}} \equiv \chi_{c_1 \gamma \leq 1}, \chi_{\gamma} \equiv \chi_{\gamma \geq 1}, \chi_{\gamma \geq 1}, \xi_{\gamma} \equiv \chi_{\gamma = 1}, \text{ where } c_1 < 1 \text{ if } \gamma = 1 \text{ and } c_1 < \alpha(c')^2 \text{ if } \gamma < 1, \text{ and } c'' := (c' + \bar{c})/c' \). First, we estimate the contribution of \( p_{t1} \). Since \( \left( \check{\chi}_{\alpha} \right)^{1/2} \left( \chi_{c_1 \gamma \leq b_{\epsilon} \leq ct^{\alpha}} \right)^{1/2} = \mathcal{O}(t^{-\gamma + \kappa}) \) (see Lemma B.1 of Appendix B) and since \( \alpha + \gamma - \kappa > 1 \), it suffices to estimate the contribution of \( c't^{-\alpha - a\nu_0} \chi_{c_1 \gamma \leq b_{\epsilon} \leq ct^{\alpha}} \). To this end, we use the propagation observable

\[ \phi_{t1} := t^{-a\nu_0} h_{\alpha} \chi_{\gamma}, \]

(6.5)
where \( h_\alpha \equiv h(\frac{t}{ct^\gamma}), h(\lambda) := \int_\lambda^\infty ds \chi_s \leq 1 \). As in (3.10), we have
\[
h_\alpha \partial_t b_\alpha \chi_\gamma' \leq \frac{\text{const}}{t^{1-\kappa/2}}, \quad h_\alpha \partial_t b_\alpha h_\gamma \geq -\frac{\text{const}}{t^{1-\kappa/2}}.
\] (6.6)

Using this together with (3.6)–(3.8), we compute
\[
d\phi_{t1} \leq \frac{1}{ct^{\alpha + \alpha \nu_0}} (\theta - \frac{\alpha b_\alpha}{t}) h_\alpha' \chi_\gamma + \frac{1}{ct^{\alpha + \alpha \nu_0}} h_\alpha \chi_\gamma' (\theta - \frac{\gamma b_\gamma}{t}) + \sum_{i=1}^3 \text{rem}_i',
\]
where \( \text{rem}_i' \) is a sum of two terms of the form of \( \text{rem}_1 \) given in (3.6)–(3.7), with \( \chi_\alpha \) replaced by \( h_\alpha \), in one, and by \( \chi_\gamma \), in the other, \( \text{rem}_2' := O(t^{-1-\gamma+\kappa/2-\alpha \nu_0}) \), and \( \text{rem}_3' := -\alpha \nu_0 t^{-1} \phi_{t1} \). We estimate
\[
\theta - \frac{\alpha b_\alpha}{t} \geq 1 - \frac{1}{\omega_a t^\kappa} - \frac{\alpha c}{t^{1-\kappa}}
\]
on supp \( h_\alpha' \) and
\[
\theta - \frac{\gamma b_\gamma}{t} \leq 1 - \frac{1}{\omega_a t^\kappa} - \frac{\gamma c_1}{2t^{1-\gamma}}
\]
on supp \( \chi_\gamma' \). Using \( h_\alpha' \leq 0, \chi_\gamma' \geq 0, h_\alpha \leq 1 - \frac{b_\alpha}{ct^\gamma} \) and \( \frac{b_\gamma}{ct^\gamma} = O(t^{-\alpha+\gamma}) \) on supp \( \chi_\gamma' \), this gives
\[
d\phi_{t1} \leq -p_1' + \tilde{p}_{t1} + \text{rem}_i',
\]
with \( \text{rem}_i := \sum_{i=1}^4 \text{rem}_i' \), \( \text{rem}_4 := \omega^{-1/2} O(t^{-\alpha-\kappa-\alpha \nu_0}) \omega^{-1/2} \), and
\[
p_1' := t^{-\alpha \nu_0} (1 - \frac{\alpha}{t}) h_\alpha' \chi_\gamma, \quad \tilde{p}_{t1} := \frac{1}{ct^{\alpha + \alpha \nu_0}} \chi_\gamma'.
\]
By (3.3), since \( \gamma > \max((3/2+\mu)^{-1}, (1+\kappa)/2, 1 - \kappa - \nu_1 - \nu_0) \), the term \( \tilde{p}_{t1} \) gives an integrable contribution. We deduce as above that the remainders \( \text{rem}_i', i = 1, 2, 3, 4 \), satisfy the estimates (3.15), \( i = 1, 2, 3, 4 \), with \( \rho_1 = \rho_2 = \rho_3 = 0, \rho_4 = 1, \lambda_1 = 2 - \gamma - \kappa + \alpha \nu_0, \lambda_2 = 1 + \gamma - \kappa/2 + \alpha \nu_0, \lambda_3 = 1 + \alpha \nu_0, \lambda_4 = \alpha + \kappa + \alpha \nu_0 \).

Since \( 2 - \gamma > 1, \gamma > \kappa/2, \) and \( \alpha + \kappa > 1 + \nu_1 + \nu_0 \), the remainder \( \text{rem}_i' = \sum_i \text{rem}_i' \) is integrable. Finally, (2.4) with \( \chi' < \alpha \nu_0 + (\frac{3}{2} + \mu) \gamma \) holds by the inequality (3.16). Hence, \( \phi_{t1} \) is a strong one-photon propagation observable and therefore we have the estimate
\[
\int_1^\infty dt \| d\Gamma(p_{t1})(\frac{1}{t}) \psi_t \|^2 \lesssim \int_1^\infty dt \| d\Gamma(p_{t1})(\frac{1}{t}) \psi_t \|^2 \lesssim \| \psi_0 \|^2.
\] (6.7)
(In fact, by multiplying the observable (6.5) by \( t^\delta \) for an appropriate \( \delta > 0 \), we can obtain a stronger estimate.)

Now, we consider \( \phi_{t2} \). Recall the notations \( \tilde{\chi}_\alpha \equiv \chi_{\omega_\alpha \geq 1}, \omega_\alpha = (\frac{\| \chi_\alpha \|_p}{\| \chi_\alpha \|_p})^2, \) and \( h_\gamma \equiv h(\nu_\gamma) \), with \( h(\lambda) = \int_\lambda^\infty ds \chi_s \leq 1 \) and \( \nu_\gamma \equiv \frac{\int \chi_\gamma \partial_t}{{\int \chi_\gamma} \partial t} \). We use the propagation observable
\[
\phi_{t2} := t^{-\alpha \nu_0} (\tilde{\chi}_\alpha \nu_\gamma + h_\gamma \tilde{\chi}_\alpha).
\] (6.8)

Using (3.8), (3.9), (6.3), \( b = b_\gamma + \epsilon \frac{1}{ct^\kappa} (\frac{1}{\omega_a} \nabla \cdot \nu + \text{h.c.}) \), \( b_\gamma \leq ct^\gamma \) on supp \( \chi_{\nu_\gamma} \leq 1, \gamma = 2\alpha - 1 \) and \( ||(\tilde{\chi}_\alpha')^{1/2}, h_\gamma|| = O(t^{-\gamma + \kappa}) \) (see Lemma B.1 of Appendix B), we compute
\[
d\phi_{t2} \leq t^{-\alpha \nu_0} \left( \left( \frac{c_1}{(\epsilon c)^2} - \alpha \right) \frac{1}{t} (\tilde{\chi}_\alpha')^{1/2} h_\gamma (\tilde{\chi}_\alpha')^{1/2} + \tilde{\chi}_\alpha h_\gamma' (dv_\gamma) + (dv_\gamma) h_\gamma' \tilde{\chi}_\alpha + \sum_{i=1}^4 \text{rem}_i'' \right),
\]
where \( \epsilon \gamma_\gamma = \frac{\partial}{\partial t} + \frac{\gamma b_\gamma}{ct^\gamma} \), \( \text{rem}_i'' \) is a term of the form of \( \text{rem}_1 \) given in (3.7), with \( \chi_\alpha \) replaced by \( \tilde{\chi}_\alpha \), likewise, \( \text{rem}_2'' \) is a term of the form of \( \text{rem}_1 \) given in (3.7), with \( \chi_\alpha \) replaced by \( h_\gamma \), \( \text{rem}_3'' = O(t^{-1-\gamma+\kappa/2-\alpha \nu_0}) \) and \( \text{rem}_4'' := -\alpha \nu_0 t^{-1} \phi_{t2} \). To estimate \( dv_\gamma \), we use that \( \tilde{\chi}_\alpha' \geq 0, h_\gamma' \leq 0, \theta = 1 - t^{-\kappa} \omega_\gamma^{-1}, \nu_\gamma h_\gamma' \leq h_\gamma', \) and
\[
\tilde{\chi}_\alpha h_\gamma' (dv_\gamma) + (dv_\gamma) h_\gamma' \tilde{\chi}_\alpha = -\chi_\alpha^{1/2} (h_\gamma')^{1/2} (dv_\gamma) (h_\gamma')^{1/2} \tilde{\chi}_\alpha^{1/2} + O(t^{-\gamma + \kappa})
\]
(see again Lemma B.1 of Appendix B), to obtain
\[
d\phi_{t2} \leq -p_{t2} + \text{rem}'',
\]
with $\text{rem}'' := \sum_{i=1}^{6} \text{rem}_i''$, $\text{rem}_0'' = \mathcal{O}(t^{-2\gamma-\kappa-a\alpha})$, $\text{rem}_6'' = \omega^{-1/2}\mathcal{O}(t-\gamma-\kappa-a\alpha)\omega^{-1/2}$ and (at least for $t$ sufficiently large)

$$p_{t}'' := t^{-a\alpha} \left[ - \left( \frac{2c_1}{(c')^2} - 2\alpha \right) \frac{1}{t} (\chi_0')(1/2)h_1(\chi_0')(1/2) + (1 - \frac{c_1}{ct^\gamma}) \frac{1}{ct^\gamma} \chi_0'(1/2)h_1^2 \right].$$

Since $\frac{c_1}{c't^\gamma} < \alpha$ and either $\gamma < 1$ or $\gamma = 1$ and $c_1 < 1$, and $\chi_0' \geq 0$ and $h_1 \leq 0$, both terms in the square braces on the r.h.s. are non-positive. We deduce as above that the remainders $\text{rem}_i''$, $i = 1, \ldots, 6$, satisfy the estimates (3.15), $i = 1, \ldots, 6$, with $p_1 = p_0 = 1$, $p_2 = p_3 = p_4 = p_5 = 0$, $\lambda_1 = 2\alpha + a\alpha$, $\lambda_2 = \lambda_5 = 2\gamma - \kappa + a\alpha$, $\lambda_3 = 1 + \gamma - \kappa + a\alpha$, $\lambda_4 = 1 + a\alpha$, $\lambda_6 = \gamma + \kappa + a\alpha$. Since $2\alpha > \gamma + \kappa > 1 + \nu_1 - a\alpha$, $2\gamma - \kappa > 1$ and $\gamma > \kappa/2$, the condition (2.3) is satisfied. Moreover, (2.4) with $\lambda' < a\alpha + \frac{2}{c} + \mu$ holds by (3.16). Therefore $\phi_{t'}$ is a strong one-photon propagation observable and we have the estimate

$$\int_1^\infty dt \|d\Gamma(p_{t})\frac{1}{2}\psi\|^{2} \lesssim \int_1^\infty dt \|d\Gamma(p_{t}')\frac{1}{2}\psi\|^{2} \lesssim \|\psi_{0}\|^{2}.$$  \hfill (6.9)

(In fact, by multiplying the observable (6.8) by $t^{\delta}$ for an appropriate $\delta > 0$, we can obtain a stronger estimate.)

Since $p_{t}'' = p_{t1} + p_{t2}$, estimates (6.7) and (6.9) imply the estimate

$$\int_1^\infty dt \|d\Gamma(p_{t})\frac{1}{2}\psi\|^{2} \lesssim \|\psi_{0}\|^{2},$$  \hfill (6.10)

which due to $\chi_0' \approx \chi_{v=1}$, implies the estimate (1.16).

Proof of Theorem 1.2 for hamiltonians of the form (1.3)−(1.4). Recall the notations $v = \frac{k}{c'}$ and $w_{\alpha} = \left(\frac{\gamma}{c'}\right)^{2}$. To prove (1.17), we begin with the following estimate, proven in the localization lemma B.5 of Appendix B:

$$\chi_{v \geq 1} \chi_{w \leq 1} = \mathcal{O}(t^{-(\alpha - \kappa)}),$$  \hfill (6.11)

for $\epsilon = t^{-\kappa}$, $\kappa < \alpha$, and $c' < c/2$. Now, let $\chi_{v \leq 1}^{2} + \chi_{v \geq 1}^{2} = 1$ and write

$$\chi_{v \leq 1}^{2} = \chi_{v \leq 1} \chi_{w \leq 1} \chi_{v \leq 1} + R \leq \chi_{v \leq 1}^{2} + R,$$  \hfill (6.12)

where $R := \chi_{v \leq 1}^{2} \chi_{w \leq 1} \chi_{v \geq 1} + \chi_{v \geq 1}^{2} \chi_{w \leq 1} \chi_{v \leq 1} + \chi_{v \leq 1}^{2} \chi_{w \leq 1} \chi_{v \geq 1}$. The estimates (6.11) and (6.12) give

$$\chi_{w \leq 1}^{2} \leq \chi_{v \leq 1}^{2} + \mathcal{O}(t^{-(\alpha - \kappa)}),$$  \hfill (6.13)

which in turn implies

$$\|\Gamma(\chi_{w \leq 1})^{1/2}\psi\| \lesssim \|\Gamma(\chi_{v \leq 1})^{1/2}\psi\| + C t^{-(\alpha - \kappa)/2} \|(N + 1)^{1/2}\psi\|.$$  \hfill (6.14)

This, together with (4.1), yields (1.17).

\hfill \square

7. PROOF OF THEOREM 1.4: THE MODEL (1.30)−(1.33)

In this section we extend the results of Sections 3−5 to hamiltonians of the form (1.30)−(1.33), with the operators $\eta_j$, $j = 1, 2$, satisfying (1.7), and prove Theorem 1.4. First, to extend the results of Section 2 to the present case, we replace the assumption (2.4) by the assumptions

$$\left\{ \begin{array}{l}
\left( \|\eta_{1} \|_{2}^{2} \left( \tilde{\phi}(g)_{ij}(k) \right) \right)_{\mathcal{H}_{k}} \omega(k) \delta dk \right)^{1/2} \lesssim (t)^{-\lambda'}, \quad i + j = 1, \\
\left( \|\eta_{2} \|_{2}^{2} \left( \tilde{\phi}(g)_{ij}(k) \chi_{k}^{2} \right) \right)_{\mathcal{H}_{k}} \prod_{\ell = 1,2} (1 + \omega(k_{\ell}))^{-1/2} + \omega(k_{\ell})^{1/2} \omega(k_{\ell})^{1/2} dk \right)^{1/2} \lesssim (t)^{-\lambda'}, \quad i + j = 2,
\end{array} \right.$$  \hfill (7.1)

where $\lambda'$ is the same as in (2.4) and, for any one-particle operator $\phi$ acting on $\mathfrak{g}$, we define $\tilde{\phi}(g)_{ij} := \phi g_{ij}$, for $i + j = 1$, and $(\tilde{\phi}g)_{2,0} := (\tilde{\phi}g)_{0,2} := (\phi \otimes 1 + 1 \otimes \phi) g_{2,0}$, $(\tilde{\phi}g)_{1,1} := (\phi \otimes 1 - 1 \otimes \phi) g_{1,1}$. Then we replace the second relation in (2.9) by the relation (see Supplement II)

$$i[\tilde{L}(g), d\Gamma(\phi)] = -\tilde{L}(i\tilde{\phi}g),$$  \hfill (7.2)
which is valid for any one-particle operator \( \phi \), and replace the estimate (2.11) by the estimate
\[
|\langle \hat{I}(g) \rangle| \leq \sum_{i+j=1} \left( \int \eta_1 \eta_2 g_{ij}(k) \frac{d^3 \omega(k)}{4 \pi} \right)^2 \left( \eta_1^{-1} \eta_2^{-2} \psi \right) \| \psi \|_d
\]
\[
+ \sum_{i+j=2} \left( \int \eta_1 \eta_2 g_{ij}(k_1, k_2) \frac{d^3 \omega(k)}{4 \pi} \prod_{\ell=1,2} (1 + \omega(k_\ell)^{-1} + \omega(k_\ell)^{\delta}) \right)^2 \left( || \psi ||^4 + \| \psi \|_{-1} \right) \| \psi \|,  \tag{7.3}
\]
which, as in (2.11), implies, together with (7.1) and (1.7),
\[
|\langle \hat{I}(i\hat{\omega}_t g) \rangle| \lesssim t^{-\lambda + \mu} \| \psi_0 \|_d,  \tag{7.4}
\]
for any \( \psi_0 \in \mathcal{Y}_d \), where \( \mathcal{Y}_d \) is defined in (2.6). This completes the extension of the results of Section 2, and therefore of Section 3, to hamiltonians of the form (1.30)–(1.33).

To extend the results of Section 4, we have to extend the estimates (4.10) and (4.26) for \( I_1 = i[I(g), B] \) and \( I_2 = [B, [B, I(g)]] \) and the estimate (4.22) for the remainder, \( R \), defined in (4.4), to the interactions of the form (1.31)–(1.33). Using that \( I_1 := i[I(g), B] = i[\hat{\omega}_t g] \) and \( I_2 := [B, [B, I(g)]] = \hat{\omega}_t \tilde{g} \), where \( \tilde{g} \) is defined by the same rules as \( \phi \) after (7.1), and using (7.3), we obtain
\[
\hat{I}_1 \geq -C(g) \tilde{E}_1,  \tag{7.5}
\]
and
\[
\| \tilde{E}_2^{-\frac{1}{2}} \tilde{I}_2 \tilde{E}_2^{-\frac{1}{2}} \| \lesssim \epsilon^{-1}(g), \tag{7.6}
\]
where, recall, \( \langle g \rangle := \sum_{i+j=2} \sum_{|\alpha|, |\beta| \leq 2} \| \eta_1^{2-i+j} \eta_2^{\alpha} \partial^\alpha g_{ij} \| \) are the norm of the vector coupling operators \( g := (g_{ij}) \), defined in the introduction after (1.33). Next, \( \tilde{E}_1 := N + \eta_2 \eta_1^{-1} \eta_2^{-1} + \eta_2^{-6} + 1 \), and \( \tilde{E}_2 := N + H + \eta_1 \eta_2^{-2} \eta_1^{-2} \eta_2^{-2} + \eta_2^{-8} + 1 \) are new estimating operators. This extends (4.10) and (4.26). Let \( \tilde{R} \) be defined by (4.4), with \( B_1 \) and \( H \) replaced by \( \tilde{B}_1 := i[H, B] \) and \( \tilde{H} \). By (4.23), with \( R \) and \( B_2 := [B_c, [B_c, H]] \) replaced by \( \tilde{R} \) and \( \tilde{B}_2 := [B_c, [B_c, H]] \), and (7.6), we obtain the extension of (4.22) to the interactions of the form (1.31)–(1.33):
\[
\| \tilde{E}_1^{-\frac{1}{2}} \tilde{R} \tilde{E}_1^{-\frac{1}{2}} \| \lesssim t^{-2} \epsilon^{-1}.  \tag{7.7}
\]

To extend the results of Section 5 to hamiltonians of the form (1.30)–(1.33), we have to prove estimates of the type (5.20) and (5.39) for the operator
\[
\tilde{G}_1 := (\hat{I}(g) \otimes 1) \hat{\Gamma}(j) - \hat{\Gamma}(j) \hat{I}(g),  \tag{7.8}
\]
which replaces \( G_1 \) defined in (5.10). To this end, we first extend the relations (5.17), (5.18) to the interactions of the form (1.31). First, we use
\[
\hat{\Gamma}(j) a^\#(h) = \hat{a}^\#(h) \hat{\Gamma}(j),  \tag{7.9}
\]
where \( \hat{a}^\#(h) := a^\#(j_0 h) \otimes 1 \oplus 1 \otimes a^\#(j_\infty h) \), with \( a^\# \) standing for \( a \) or \( a^* \). This together with (7.8) and the notation \( \hat{a}_\lambda^\#(k) := a_\lambda^\#(k) \otimes 1 - \hat{a}_\lambda^\#(k) = (1 - j_0) a_\lambda^\#(k) \otimes 1 - 1 \otimes j_\infty a_\lambda^\#(k) \) gives
\[
\tilde{G}_1 = I_\#(g) \hat{\Gamma}(j),  \tag{7.10}
\]
where
\[
I_\#(g) = \sum_\lambda \int dk \left( g_{01}(k) \otimes \hat{a}_\lambda(k) + \text{h.c.} \right)  \tag{7.11}
\]
\[
+ \sum_{\lambda_1, \lambda_2} \int dk_1 dk_2 \left( g_{02}(k_1, k_2) \otimes (\hat{a}_{\lambda_1}(k_1) \hat{a}_{\lambda_2}(k_2) + \hat{a}_{\lambda_1}(k_1) \hat{a}_{\lambda_2}(k_2) + \hat{a}_{\lambda_1}(k_1) \hat{a}_{\lambda_2}(k_2)) + \text{h.c.} \right)  \tag{7.12}
\]
\[
+ \sum_{\lambda_1, \lambda_2} \int dk_1 dk_2 g_{11}(k_1, k_2) \otimes (\hat{a}_{\lambda_1}(k_1) \hat{a}_{\lambda_2}(k_2) + \hat{a}_{\lambda_1}(k_1) \hat{a}_{\lambda_2}(k_2) + \hat{a}_{\lambda_1}(k_1) \hat{a}_{\lambda_2}(k_2)).  \tag{7.13}
\]
Here the notation \( g_{01}(k) \otimes \hat{a}_\lambda(k) \) should be read as \((1 - j_0) g_{01}(k)(a_\lambda(k) \otimes 1) - (j_\infty g_{01})(k)(1 \otimes a_\lambda(k))\), and likewise in the second and third lines. Using this and (3.16), we have in addition
\[
|\langle \hat{H}_f + 1 \rangle^{-\frac{1}{2}} \tilde{G}_1 (N + \tilde{\eta}^{-2} + 1)^{-1} \| \lesssim t^{-\lambda},  \tag{7.14}
\]
with \( \tilde{\eta}^2 := \eta_2^2(1 + \eta_1^2)\eta_2^2 \), recall, \( \hat{H}_f = H_f \otimes 1 + 1 \otimes H_f \), and
\[
\| f(\hat{H}) \tilde{G}_1(N + 1)^{-\frac{1}{2}} \| \lesssim t^{-\lambda}. \tag{7.15}
\]
This extends the proof of the existence and properties of the Deift-Simon wave operators (see Theorem 5.1) to the interactions of the form (1.31)–(1.33). The remainder of the proof goes the same way as the proof of Theorem 5.1.

8. Proof of Theorems 1.1–1.3 for the QED Model

8.1. Proof of Theorems 1.1–1.2 for the QED model. We have shown the statements of Theorems 1.1 and 1.2 for hamiltonians of the form (1.30)–(1.33), with the operators \( \eta_j, j = 1, 2 \), satisfying (1.7), and therefore for the operator (1.26). To translate Theorems 1.1 and 1.2 from \( H \), given by (1.26), to the QED hamiltonian (1.23), we use the following estimates ([10])
\[
\left\| d\Gamma(\chi_1(w)) \frac{1}{2} \psi \right\|^2 \lesssim \left\langle U \psi, d\Gamma(\chi_1(w)) U \psi \right\rangle + t^{-\alpha_d} \| \psi \|^2, \tag{8.1}
\]
\[
\left\| \Gamma(\chi_2(w)) \frac{1}{2} \psi \right\|^2 \lesssim \left\langle U \psi, \Gamma(\chi_2(w)) U \psi \right\rangle + t^{-\alpha_d} \| \psi \|^2, \tag{8.2}
\]
where \( w := \frac{\eta}{k} \), valid for any functions \( \chi_1(w) \) and \( \chi_2(w) \) supported in \( \{ |w| \leq \epsilon \} \) and \( \{ |w| \geq \epsilon \} \), respectively, for some \( \epsilon > 0 \), for any \( \psi \in \mathcal{D}(H)D(N^{1/2}) \), with \( f \in C_0^\infty((-\infty, \Sigma)) \), and for \( 0 \leq d < 1/2 \). (8.1) follows from estimates of Section 2 of [10] and (8.2) can be obtained similarly (see (II.8) and (II.9)). Using these estimates for \( \psi_0 = e^{-itH} \psi_0 \), with an initial condition \( \psi_0 \) in either \( \Upsilon_1 \) or \( \Upsilon_2 \), together with \( U e^{-itH} \psi_0 = e^{-it\hat{H}} \psi_0 \), and applying Theorems 1.1 and 1.2 for \( \hat{H} \) to the first terms on the r.h.s., we see that, to obtain Theorems 1.1 and 1.2 for the hamiltonian (1.23), we need, in addition, the estimates
\[
\left\langle \psi, U^* N_1 U \psi \right\rangle \lesssim \left\langle \psi, (N_1 + 1) \psi \right\rangle, \tag{8.3}
\]
\[
\left\langle \psi, U^* d\Gamma(y) U \psi \right\rangle \lesssim \left\langle \psi, \left( d\Gamma(y) + (x)^2 \right) \psi \right\rangle, \tag{8.4}
\]
\[
\left\| U^* d\Gamma(b) U \psi \right\| \lesssim \left\| (d\Gamma(b) + (x)^2) \psi \right\|, \tag{8.5}
\]
where, recall, \( N_1 = d\Gamma(\omega^{-1}) \) and \( b = \frac{1}{2} (k \cdot y + y \cdot k) \).

To prove (8.3), we see that, by (II.8), we have
\[
U^* N_1 U = e^{\Phi(q_x)} d\Gamma(\omega^{-1}) e^{-i\Phi(q_x)} = N_1 \Phi(i\omega^{-1} q_x) + \frac{1}{2} \| \omega^{-1/2} q_x \|_b^2. \tag{8.6}
\]
(Since \( \omega^{-1} q_x \notin \mathfrak{h} \), the field operator \( \Phi(i\omega^{-1} q_x) \) is not well-defined and therefore this formula should be modified by introducing, for instance, an infrared cutoff parameter \( \sigma \) into \( q_x \). One then removes it at the end of the estimates. Since such a procedure is standard, we omit it here.) This relation, together with
\[
| \left\langle \psi, \Phi(i\omega^{-1} q_x) \psi \right\rangle | \lesssim \left( \int \omega^{-3-2\nu+\epsilon} |k|^{-6} dk \right)^{\frac{1}{2}} \left\| d\Gamma(\omega^{-1}) \frac{1}{2} \psi \right\| \| \psi \|, \tag{8.7}
\]
for any \( \epsilon > 0 \), which follows from the bounds of Lemma 1.1 of Supplement I, and
\[
\| \omega^{-\frac{1}{2}} q_x \|_b \lesssim \| \omega^{-1-\nu} |k|^{-3} \|_b, \tag{8.8}
\]
implies (8.3).

To prove (8.4) and (8.5), we proceed similarly, using, instead of (8.7) and (8.8), the estimates
\[
| \left\langle \psi, \Phi(i(y) q_x) \psi \right\rangle | \lesssim \left( \int \omega^{-2-2\nu} |k|^{-6} dk \right)^{\frac{1}{2}} \left\| d\Gamma(\omega^{-1}) \frac{1}{2} \psi \right\| \| \langle x \rangle \psi \| \lesssim \left( \int \omega^{-2-2\nu} |k|^{-6} dk \right)^{\frac{1}{2}} \left\| d\Gamma(y) \frac{1}{2} \psi \right\| \| \langle x \rangle \psi \|, \tag{8.9}
\]
and
\[
\| \langle y \rangle^{\frac{1}{2}} q_x \|_b \lesssim \| x \|^{\frac{1}{2}} \| \omega^{-1-\nu} |k|^{-3} \|_b, \tag{8.10}
\]
and
\[ \| \Phi(i b q_x) \psi \| \lesssim \left( \int \omega^{-2-\nu} (k)^{-6} dk \right)^{\frac{1}{2}} \| (x)(H_f + 1)^{\frac{1}{2}} \psi \|, \] (8.11)
and
\[ \langle q_x, b q_x \rangle_h \lesssim \langle x \rangle \| \omega^{-\frac{1}{2} - \nu} (k)^{-3} \|_h^2. \] (8.12)

8.2. Proof of Theorem 1.3. We present the parts of the proof of Theorem 1.3 for the Hamiltonian (1.23) which differ from that for the Hamiltonian (1.3), with the interaction (1.4). To begin with, the existence and the properties of the Deift–Simon wave operators on \( \text{Ran}_{(-\infty, \Sigma)}(H) \)
\[ W_\pm := \text{s-lim}_{t \to \pm \infty} W(t), \quad \text{with} \quad W(t) := e^{it\bar{H}} \bar{\Gamma}(j)e^{-itH}, \] (8.13)
where \( \bar{H} := H \otimes 1 + 1 \otimes H_f \) and the operators \( \bar{\Gamma} \) and \( j = (j_0, j_\infty) \) are defined in Subsection 5.1, are equivalent to the existence and the properties of the modified Deift–Simon wave operators
\[ W_{\pm}^{(\text{mod})} := \text{s-lim}_{t \to \pm \infty} (e^{-i\Phi(q_x) \otimes 1})e^{it\bar{H}} \bar{\Gamma}(j)e^{-itH} e^{i\Phi(q_x)}, \] (8.14)
on \( \text{Ran}_{(-\infty, \Sigma)}(\bar{H}) \) (where \( \bar{H} = e^{-\Phi(q_x)} H e^{\Phi(q_x)} \) is given in (1.30)).

To prove the existence of \( W_{\pm}^{(\text{mod})} \), we observe that, due to (7.9), we have \( \bar{\Gamma}(j) \Phi(h) = \hat{\Phi}(h) \bar{\Gamma}(j) \), where
\[ \hat{\Phi}(h) := \Phi(j_0 h) \otimes 1 + 1 \otimes \Phi(j_\infty h), \] (8.15)
which in turn implies that
\[ \bar{\Gamma}(j) e^{i\Phi(h)} = e^{i\hat{\Phi}(h)} \bar{\Gamma}(j). \] (8.16)
Therefore
\[ (e^{-i\Phi(q_x) \otimes 1})e^{it\bar{H}} \bar{\Gamma}(j)e^{-itH} e^{i\Phi(q_x)} = (e^{-i\Phi(q_x) \otimes 1})e^{it\bar{H}} e^{i\Phi(q_x)} \bar{\Gamma}(j)e^{-it\bar{H}} = e^{it\bar{H}^{(\text{mod})}} \bar{\Gamma}(j)e^{-it\bar{H}} + \text{Rem}_t, \] (8.17)
where \( \bar{H}^{(\text{mod})} := \bar{H} \otimes 1 + 1 \otimes H_f \) and
\[ \text{Rem}_t := (e^{-i\Phi(q_x) \otimes 1})e^{it\bar{H}} (e^{i\Phi(q_x)} - e^{i\Phi(q_x) \otimes 1}) \bar{\Gamma}(j)e^{-it\bar{H}}. \]

We claim that
\[ \text{s-lim}_{t \to \pm \infty} \text{Rem}_t = 0. \] (8.18)
Indeed, let \( R := \hat{\Phi}(q_x) - \Phi(q_x) \otimes 1 = \Phi((j_0 - 1) q_x) \otimes 1 + 1 \otimes \Phi(j_\infty q_x) \) and \( \hat{N} := N \otimes 1 + 1 \otimes N \). Using (1.25), Lemma II.1 of Supplement II and (3.16), we obtain
\[ \| R(\hat{N} + 1)^{-\frac{1}{2}} \| \lesssim \|(j_0 - 1) q_x \|_h + \| j_\infty q_x \|_h \lesssim t^{-\alpha \tau}(x)^{1+\tau}, \]
for any \( \tau < 1 \). From this estimate and the relation \( e^{i\Phi(q_x)} - e^{i\Phi(q_x) \otimes 1} = -i \int_0^1 dse^{(1-s)i\Phi(q_x)} R(e^{s(i\Phi(q_x) \otimes 1)), \) it is not difficult to deduce that
\[ \| (e^{i\Phi(q_x) \otimes 1})(\hat{N} + \langle x \rangle^{2+2\tau} + 1)^{-1} \| \lesssim t^{-\alpha \tau}. \]
Furthermore, we have \( (\hat{N} + \langle x \rangle^{2+2\tau} + 1) \bar{\Gamma}(j) = \bar{\Gamma}(j)(\hat{N} + \langle x \rangle^{2+2\tau} + 1) \), and, as in Corollary A.3 of Appendix A, with \( \mu = 1/2 \), one can verify that \( \| N e^{-it\bar{H}} \psi_0 \| \lesssim t^{3/2} \| \psi_0 \| \) for any \( \psi_0 \in f(\bar{H})D(N_{1/2}), f \in C_0^\infty((-\infty, \Sigma)) \). Using in addition that \( \| \langle x \rangle^{2+2\tau} f(\bar{H}) \| \) is finite, it follows that \( \text{Rem}_t \) strongly converges to 0 on \( \text{Ran}_{(-\infty, \Sigma)}(\bar{H}) \) provided that \( \alpha \tau > 2/5 \).

The equations (8.13), (8.17) and (8.18) imply
\[ W_{\pm}^{(\text{mod})} := \text{s-lim}_{t \to \pm \infty} e^{it\bar{H}^{(\text{mod})}} \bar{\Gamma}(j)e^{-it\bar{H}}. \] (8.19)

The proof of the existence and properties of the Deift–Simon wave operators (8.19) is a special case of the corresponding proof for the Hamiltonian (1.30)–(1.31) (see Section 7).

Finally, we comment on the proof of Theorem 5.4 for the Hamiltonian (1.23) in the QED case. It goes in the same way as in Section 5, until the point where we have to show that \( \| d\Gamma(h^2 \frac{\partial}{\partial x}) \|_{L^2} = O(t^\epsilon) \) in
Likewise, with a minor modification of the proof above, we obtain the following bound for the present case. This estimate can be proven by using the generalized Pauli-Fierz transformation (1.24) together with (II.9), to obtain
\[ \| d\Gamma(b_c^2)^2 \|_{\hat{\Phi}_{gs}}^2 = \left\langle \hat{\Phi}_{gs}, (d\Gamma(b_c^2) - \Phi(b_c^2 q_x) + \frac{1}{2} (b_c^2 q_x)_b) \hat{\Phi}_{gs} \right\rangle, \]  
(8.20)
where \( \hat{\Phi}_{gs} := U\Phi_{gs} \). Using Lemma I.1 of Supplement I, (1.6) and the fact that \( \hat{\Phi}_{gs} \in D(N^{1/2}) \), we can estimate the second term of the r.h.s. of (8.20) as
\[ \left| \left\langle \hat{\Phi}_{gs}, \Phi(b_c^2 q_x)_b \hat{\Phi}_{gs} \right\rangle \right| \leq \|\langle x\rangle^3 \hat{\Phi}_{gs} \| \|\langle x\rangle^{-3} \Phi(b_c^2 q_x)(N + 1)^{-\frac{1}{2}} \| (N + 1)^{\frac{1}{2}} |\hat{\Phi}_{gs}| \leq t^2. \]
Likewise, \( |\langle \hat{\Phi}_{gs}, (b_c^2 q_x)_b \hat{\Phi}_{gs} \rangle| \leq t^2 \). To estimate the first term of the r.h.s. of (8.20), we write
\[ \| d\Gamma(b_c^2)^2 \|_{\hat{\Phi}_{gs}}^2 = \sum_{\lambda} \| b\alpha(k) \hat{\Phi}_{gs} \|^2 dk. \]
Applying the standard pull-through formula gives
\[ a\lambda(k) \hat{\Phi}_{gs} = -(\hat{H} - E_{gs} + |k|)^{-1} (p + \hat{A}(x) : \hat{g}_s(k) + e_x(k)) \hat{\Phi}_{gs}. \]
We then easily conclude that \( \| b\alpha(k) \hat{\Phi}_{gs} \|_b = O(t^\kappa) \) in the same way as in Lemma D.1 of Appendix D.

**Appendix A. Photon # and Low Momentum Estimate**

For simplicity, consider hamiltonians of the form (1.3)–(1.4), with the coupling operators \( g(k) \) satisfying (1.5) and (1.7) with \( \mu > -1/2 \). The extension to hamiltonians of the form (1.30)–(1.31) is done along the lines of Section 7. Recall the notations \( \langle A \rangle_\psi = \langle \psi, A \psi \rangle \), \( N_\rho = d\Gamma(\omega^{-\rho}) \) and \( \Upsilon_\rho = \{ \psi \in f(H)D(N_\rho^{1/2}) \text{ for some } f \in C_0^\infty((-\infty, \Sigma)) \} \). The idea of the proof of the following estimate follows [32] and [10].

**Proposition A.1.** Let \( \rho \in [-1, 1] \). For any \( \psi \in \Upsilon_\rho \),
\[ \langle N_\rho \rangle_\psi \lesssim t^{\nu_\rho} \| \psi \|_\rho^2, \quad \nu_\rho = \frac{1 + \rho}{2 + \mu}. \]  
(A.1)

**Proof.** Decompose \( N_\rho = K_1 + K_2 \), where
\[ K_1 := d\Gamma(\omega^{-\rho} \chi_{t^\rho \omega \leq 1}) \quad \text{and} \quad K_2 := d\Gamma(\omega^{-\rho} \chi_{t^\rho \omega \leq 1}). \]
Then, by (1.19),
\[ \langle K_2 \rangle_\psi \leq (d\Gamma(t^{(1 + \rho)} \omega \chi_{t^\rho \omega \leq 1}))_\psi \leq t^{(1 + \rho)} \langle H_f \rangle_\psi \lesssim t^{(1 + \rho)} \| \psi \|. \]  
(A.2)

On the other hand, we have by (2.10),
\[ DK_1 = d\Gamma(\alpha \omega^{1 + \rho} \chi_{t^\rho \omega \leq 1}) - I(\omega^{-\rho} \chi_{t^\rho \omega \leq 1} g). \]  
(A.3)

Since \( \| \eta_1 g(k) \|_{\Upsilon_\rho} \lesssim |k|^\alpha (\omega(k))^{-2 - \mu} \) (see (1.5)), we obtain
\[ \int dk \omega(k)^{-2\rho} \chi_{t^\rho \omega \leq 1} g(k) \|_{\Upsilon_\rho}^2 (\omega(k)^{-1} + 1) \lesssim t^{-2(1 + \mu - \rho)\alpha}. \]  
(A.4)

This together with (2.11) and (1.19) gives
\[ |\langle I(\omega^{-\rho} \chi_{t^\rho \omega \leq 1}) \rangle_\psi| \lesssim t^{-(1 + \mu - \rho)\alpha} \| \psi \|^2. \]  
(A.5)

Hence, by (A.3), since \( \partial_t \langle K_1 \rangle_\psi = \langle DK_1 \rangle_\psi, \chi_{t^\rho \omega \leq 1} \leq 0 \), we obtain
\[ \partial_t \langle K_1 \rangle_\psi \lesssim t^{-(1 + \mu - \rho)\alpha} \| \psi \|^2, \]
and therefore
\[ \langle K_1 \rangle_\psi \leq Ct^{\nu'} \| \psi \|^2 + \langle N_\rho \rangle_\psi, \]  
(A.6)

where \( \nu' = 1 - (1 + \mu - \rho)\alpha \), if \( (1 + \mu - \rho)\alpha < 1 \) and \( \nu' = 0 \), if \( (1 + \mu - \rho)\alpha > 1 \). Estimates (A.6) and (A.2) with \( \alpha = \frac{1}{2 + \mu} \), if \( \rho > -1 \), give (A.1). The case \( \rho = -1 \) follows from (1.19). \( \square \)

**Remark.** A minor modification of the proof above give the following bound for \( \rho > 0 \) and \( \nu'_\rho := \frac{\rho}{2 + \mu} \),
\[ \langle N_\rho \rangle_\psi \lesssim t^{\nu'_\rho} (\| \psi \|^2 + \| \psi \|^2) + \langle N_\rho \rangle_\psi. \]  
(A.7)
Corollary A.2. For any \( \psi_0 \in \mathcal{Y}_\rho \), \( \gamma \geq 0 \) and \( c > 0 \),
\[
\| \chi_{K_{n-\epsilon^2}^\gamma} \psi_t \| \lesssim t^{-\frac{\alpha}{2} + \frac{\gamma}{2(2\gamma + 1)}} \| \psi_0 \|^2 + t^{-\frac{\alpha}{2}} (K_{\rho}) \psi_0. \tag{A.8}
\]

Proof. We have
\[
\| \chi_{K_{n-\epsilon^2}^\gamma} \psi_t \| \leq c^{-\frac{\alpha}{2}} t^{-\frac{\alpha}{2}} \| \chi_{K_{n-\epsilon^2}^\gamma} K_{\rho} \frac{1}{2} \psi_t \| \leq c^{-\frac{\alpha}{2}} t^{-\frac{\alpha}{2}} \| K_{\rho} \frac{1}{2} \psi_t \|
\]
Now applying (A.1) we arrive at (A.8).
\[\square\]

Corollary A.3. Let \( \psi_0 \in \mathcal{Y}_1 \). Then \( \psi_0 \in D(N) \) and
\[
\langle N^2 \rangle_{\psi_t} \lesssim t^{-\frac{\alpha}{2}} \| \psi_0 \|^2. \tag{A.9}
\]

Proof. By the Cauchy-Schwarz inequality, we have \( N^2 \leq d\Gamma(\omega) d\Gamma(\omega^{-1}) = H_f N_1 \), and hence
\[
\langle N^2 \rangle_{\psi_t} \leq \langle N_{1}^{\frac{1}{2}} H_f \rangle_{\psi_t} \psi_t \tag{A.10}
\]
Under the assumption (1.5) with \( \mu > 0 \), one verifies that \( H_f [N_1^{\frac{1}{2}}, (H - E_{g^s} + 1)^{-1}] \) is bounded. Since \( H_f (H - E_{g^s} + 1)^{-1} \) is also bounded, we obtain
\[
\langle N^2 \rangle_{\psi_t} \lesssim \| N_{1}^{\frac{1}{2}} \psi_t \| (\| N_{1}^{\frac{1}{2}} (H - E_{g^s} + 1) \psi_t \| + \| (H - E_{g^s} + 1) \psi_t \|). \tag{A.11}
\]
Applying Proposition A.1 gives
\[
\| N_{1}^{\frac{1}{2}} \psi_t \| \lesssim t^{-\frac{\alpha}{2}} \| \psi_0 \| + \| N_{1}^{\frac{1}{2}} \psi_0 \|, \tag{A.12}
\]
and
\[
\| N_{1}^{\frac{1}{2}} (H - E_{g^s} + 1) \psi_t \| \lesssim t^{-\frac{\alpha}{2}} \| \psi_0 \| + \| N_{1}^{\frac{1}{2}} (H - E_{g^s} + 1) \psi_0 \|
\]
where we used in the last inequality that \( N_{1}^{\frac{1}{2}} \hat{f}(H) (N_1 + 1)^{-\frac{1}{2}} \) is bounded for any \( \hat{f} \in C_0^\infty(\mathbb{R}^3) \). Combining (A.10), (A.11) and (A.12), we obtain
\[
\langle N^2 \rangle_{\psi_t} \lesssim t^{-\frac{\alpha}{2}} (\| N_{1}^{\frac{1}{2}} \psi_0 \|^2 + \| \psi_0 \|^2). \tag{A.13}
\]
Hence (A.9) is proven.
\[\square\]

Appendix B. One-particle commutator estimates

In this appendix, we estimate some localization terms and commutators appearing in Section 3. We begin with recalling the Helffer-Sjöstrand formula that will be used several times. Let \( f \) be a smooth function satisfying the estimates \( |\partial_x^n f(s)| \leq C_n(s)^{\rho-n} \) for all \( n \geq 0 \), with \( \rho < 0 \). We consider an almost analytic extension \( \hat{f} \) of \( f \), which means that \( \hat{f} \) is a \( C^\infty \) function on \( \mathbb{C} \) such that \( \hat{f}|_{\mathbb{R}} = f \),
\[
\text{supp } \hat{f} \subset \{ z \in \mathbb{C}, \ |\text{Im } z| \leq C \langle \text{Re } z \rangle \},
\]
\[
|\hat{f}(z)| \leq C \langle \text{Re } z \rangle^\rho \quad \text{and, for all } n \in \mathbb{N},
\]
\[
|\frac{\partial \hat{f}}{\partial z}(z)| \leq C_n \langle \text{Re } z \rangle^{\rho-1-n} |\text{Im } z|^n.
\]
Moreover, if \( f \) is compactly supported, we can assume that this is also the case for \( \hat{f} \). Given a self-adjoint operator \( A \), the Helffer-Sjöstrand formula (see e.g. [16, 41]) allows one to express \( f(A) \) as
\[
f(A) = \frac{1}{\pi} \int \frac{\partial \hat{f}(z)}{\partial z}(A - z)^{-1} \text{dRe } z \text{dIm } z. \tag{B.1}
\]
Now recall that $b_\epsilon := \frac{1}{2}(\theta_\epsilon \nabla \omega \cdot y + \text{h.c.})$, where $\theta_\epsilon = \frac{\omega}{\omega_\epsilon}$, $\omega_\epsilon := \omega + \epsilon = t^{-\kappa}$, with $\kappa \geq 0$. We have the relations
\[ i[\omega, b_\epsilon] = \theta_\epsilon, \quad i[\omega, y^2] = \frac{1}{2} (\nabla \omega \cdot y + y \cdot \nabla \omega), \] and, using in particular Hardy’s inequality, one can verify the estimate
\[ \| y^2 b_\epsilon \langle y \rangle^{-2} \| = O(t^\kappa). \] (B.3)

The following lemma is a straightforward consequence of the Helffer–Sjöstrand formula together with (B.2) and (B.3). We do not detail the proof.

**Lemma B.1.** Let $h, \tilde{h}$ be smooth function satisfying the estimates $|\partial^\alpha h(s)| \leq C_n(s)^{-n}$ for $n \geq 0$ and likewise for $\tilde{h}$. Let $w_\alpha = (|y|/c_1 t^\alpha)^2$, $v_\beta = b_\epsilon/(c_2 t^\beta)$, with $0 < \alpha, \beta \leq 1$. The following estimates hold
\[ [h(w_\alpha), \omega] = O(t^{-\alpha}), \quad [\tilde{h}(v_\beta), \omega] = O(t^{-\beta}), \]
\[ [h(v_\beta), \omega t^{-\frac{1}{2}}] = O(t^{\frac{1}{2} \kappa - \beta}), \quad b_\epsilon [h(v_\beta), \omega t^{-\frac{1}{2}}] = O(t^{\frac{1}{2} \kappa}), \]
\[ [h(w_\alpha), b_\epsilon] = O(t^\kappa), \quad [h(w_\alpha), \tilde{h}(v_\beta)] = O(t^{-\beta + \kappa}), \quad \tilde{b}_\epsilon [h(w_\alpha), \tilde{h}(v_\beta)] = O(t^\kappa). \]

Now we prove the following abstract result.

**Lemma B.2.** Let $h$ be a smooth function satisfying the estimates $|\partial^\alpha h(s)| \leq C_n(s)^{-n}$ for $n \geq 0$. Assume an operator $v$ is s.t. the commutators $[v, \omega]$ and $[v, [v, \omega]]$ are bound, and for some $z$ in $\mathbb{C} \setminus \mathbb{R}$, $(v - z)^{-1}$ preserves $D(\omega)$. Then the operator $r := [h(v), \omega] - [v, \omega] h'(v)$ is bounded as
\[ \|r\| \lesssim \|[v, [v, \omega]]\|. \] (B.4)

**Proof.** We would like to use the Helffer–Sjöstrand formula (B.1) for $h$. Since $h$ might not decay at infinity, we cannot directly express $h(v)$ by this formula. Therefore, we approximate $h(v)$ as follows. Consider $\varphi \in C^\infty_0(\mathbb{R}; [0, 1])$ equal to 1 near 0 and $\varphi_R(\cdot) = \varphi(\cdot/R)$ for $R > 0$. Let $\tilde{h}$ be an almost analytic extensions of $h$ such that $|h|_R = h$,
\[ \supp \tilde{h} \subset \{ z \in \mathbb{C} : |\text{Im} z| \leq C(\text{Re} z) \}, \] (B.5)
\[ |\tilde{h}(z)| \leq C \text{ and, for all } n \in \mathbb{N}, \]
\[ |\partial^\alpha \tilde{h}(z)| \leq C_n(\text{Re} z)^{\rho - 1 - n} |\text{Im} z|^{n}. \] (B.6)

Similarly let $\varphi \in C^\infty(\mathbb{C})$ be an almost analytic extension of $\varphi$ satisfying these estimates. As a quadratic form on $D(\omega)$, we have
\[ [h(v), \omega] = \text{s-lim}_{R \to \infty} \left( (\varphi_R h)(v), \omega \right). \] (B.7)

Since $(v - z)^{-1}$ preserves $D(\omega)$ for some $z$ in the resolvent set of $v$ (and hence for any such $z$, see [2, Lemma 6.2.1]), we can compute, using the Helffer–Sjöstrand representation (see (B.1)) for $(\varphi_R h)(v)$,
\[ [(\varphi_R h)(v), \omega] = \frac{1}{\pi} \int \partial_z (\varphi_R \tilde{h})(z) [(v - z)^{-1}, \omega] \text{dRe} z \text{dIm} z \]
\[ = -\frac{1}{\pi} \int \partial_z (\varphi_R \tilde{h})(z) (v - z)^{-1} [v, \omega] (v - z)^{-1} \text{dRe} z \text{dIm} z \]
\[ = [v, \omega] (\varphi_R h)'(v) + r_R, \] (B.8)
as a quadratic form on $D(\omega)$, where
\[ r_R = -\frac{1}{\pi} \int \partial_z (\varphi_R \tilde{h})(z) [(v - z)^{-1}, [v, \omega]] (v - z)^{-1} \text{dRe} z \text{dIm} z \]
\[ = \frac{1}{\pi} \int \partial_z (\varphi_R \tilde{h})(z) (v - z)^{-1} [v, [v, \omega]] (v - z)^{-2} \text{dRe} z \text{dIm} z. \] (B.9)

Now, using $(v - z)^{-1} = O(|\text{Im} z|^{-1})$, we obtain that
\[ \| (v - z)^{-1} [v, [v, \omega]] (v - z)^{-2} \| \lesssim |\text{Im} z|^{-3} \|[v, [v, \omega]]\|. \] (B.10)

Besides, for all $n \in \mathbb{N}$,
\[ |\partial_z (\varphi_R \tilde{h})(z)| \leq C_n(\text{Re} z)^{\rho - 1 - n} |\text{Im} z|^n, \] (B.11)
where $C_n > 0$ is independent of $R \geq 1$. Using (B.9) together with (B.10), we see that there exists $C > 0$ such that $\|r_R\| \leq C\|v, [v, \omega]\|$, for all $R \geq 1$. Finally, since $(\varphi_R h)'(v)$ converges strongly to $h'(v)$, the lemma follows from (B.8) and the previous estimate. \hfill \square

We want apply the lemma above to the time-dependent self-adjoint operator $v := \frac{h}{ct}$.  

**Corollary B.3.** Let $h$ be a smooth function satisfying the estimates $|\partial_s^n h(s)| \leq C_n \langle s \rangle^{-n}$ for $n \geq 0$ and let $v := \frac{h}{ct}$, where $c > 0$, $\epsilon = t^{-\kappa}$, with $0 \leq \kappa \leq \beta \leq 1$. Then the operator $r := dh(v) - (dv)h'(v)$ is bounded as

$$\|r\| \lesssim t^{-\lambda}, \quad \lambda := 2\alpha - \kappa.$$  

**Proof.** Observe that

$$dh(v) - (dv)h'(v) = [h(v), i\omega] - [v, i\omega]h'(v) + \partial_t h(v) - (\partial_t v)h'(v).$$

It is not difficult to verify that $(v - z)^{-1}$ preserves $D(\omega)$ for any $z \in \mathbb{C} \setminus \mathbb{R}$. Hence it follows from the computations

$$[v, i\omega] = t^{-\alpha}\theta_\epsilon, \quad [v, [v, i\omega]] = t^{-2\alpha}\theta_\epsilon^2\omega^2\epsilon,$$  

that we can apply Lemma B.2. The estimate

$$\|h(v), i\omega] - [v, i\omega]h'(v)\| \lesssim t^{-2\alpha + \kappa}$$

then gives

$$\|h(v), i\omega] - [v, i\omega]h'(v)\| \lesssim t^{-2\alpha + \kappa}.$$  

It remains to estimate $\|\partial_t h(v) - (\partial_t v)h'(v)\|$. It is not difficult to verify that $D(b_\epsilon)$ is independent of $t$. Using the notations of the proof of Lemma B.2 and the fact that $\partial_t h(v) = s\lim_{R \to \infty} \partial_t (\varphi_R h)(v)$, we compute

$$\partial_t (\varphi_R h)(v) = \frac{1}{\pi} \int \partial_z (\varphi_R h)(z)\partial_t (v - z)^{-1} dRe z dIm z$$

$$= \frac{1}{\pi} \int \partial_z (\varphi_R h)(z)(v - z)^{-1}\partial_t (v)(v - z)^{-1} dRe z dIm z$$

$$= (\partial_t v)(\varphi_R h)'(v) + r'_R,$$  

where

$$r'_R = \frac{1}{\pi} \int \partial_z (\varphi_R h)(z)(v - z)^{-1}\partial_t v(v - z)^{-1} dRe z dIm z$$

$$= \frac{1}{\pi} \int \partial_z (\varphi_R h)(z)(v - z)^{-2}\partial_t v(v - z)^{-2} dRe z dIm z.$$  

Now using $\partial_t v = -\frac{\partial b}{ct^3} + \frac{1}{ct\epsilon}\partial_t b_\epsilon$ together with (3.9), we estimate

$$[v, \partial_t v] = \mathcal{O}(t^{-1-2\alpha+\kappa}) b_\epsilon + \mathcal{O}(t^{-1-2\alpha+2\kappa}).$$

From this, the properties of $\varphi$, $\tilde{h}$, and $\kappa \leq \beta$, we deduce that $\|r'_R\| \lesssim t^{-1-\alpha+\kappa} \lesssim t^{-2\alpha+\kappa}$ uniformly in $R \geq 1$. This concludes the proof of the corollary. \hfill \square

The following lemma is taken from [10]. Its proof is similar to the proof of Lemma B.2

**Lemma B.4.** Let $h$ be a smooth function satisfying the estimates $|\partial_s^n h(s)| \leq C_n \langle s \rangle^{-n}$ for $n \geq 0$ and $0 \leq \delta \leq 1$. Let $w_\alpha = (|y|/ct^{\alpha})^2$ with $0 < \alpha \leq 1$. We have

$$[h(w_\alpha), i\omega] = \frac{1}{ct^{\alpha}} h'(w_\alpha) \left( \frac{y}{ct^{\alpha}} \cdot \nabla \omega + \nabla \omega \cdot \frac{y}{ct^{\alpha}} \right) + \text{rem},$$

with

$$\|\omega^\delta \text{ rem} \omega^\delta\| \lesssim t^{-\alpha(1+\delta)}.$$  

Now we prove a localization lemma. Let $v_\alpha := \frac{h}{ct^\alpha}$, $w_\alpha := (|y|/ct^{\alpha})^2$.

**Lemma B.5.** Let $\kappa < \alpha$. We have, for $c < c'/2$,

$$\chi_{v_{\alpha} \geq 1} \chi_{w_{\alpha} \leq 1} = \mathcal{O}(t^{-(\alpha+\kappa)}).$$  

(B.16)
Proof. We omit the subindex $\alpha$ in $w_\alpha$ and $v_\alpha$ write $w \equiv w_\alpha$ and $v \equiv v_\alpha$. Observe that by the definition of $\chi$ (see Introduction) and the condition $c < c'/2$, we have $\chi_{|y| \geq c't^\alpha} \chi_{|y| \leq c t^\alpha} = 0$. Let $c < \bar{c} < c'/2$ and let $\chi_{|y| \leq \bar{c} t^\alpha}$ be such that $\chi_{|y| \geq \bar{c} t^\alpha} \chi_{|y| \leq c t^\alpha} = \chi_{|y| \geq \bar{c} t^\alpha}$ and $\chi_{|y| \geq c t^\alpha} \chi_{|y| \leq \bar{c} t^\alpha} = 0$. Define $b_\epsilon := \bar{c} t^\alpha b \chi_{|y| \leq \bar{c} t^\alpha}$. It follows from the expression of $b_\epsilon$ that $\langle |u, b_\epsilon u| \rangle \leq \|u|||y||y||u||^2$, and hence we deduce that $\langle |u, \bar{b}_\epsilon u| \rangle \leq \bar{c} t^\alpha ||u||^2$. This gives $\chi_{b_\epsilon \geq c't^\alpha} = 0$. Using this, we write

$$
\chi_{b_\epsilon \geq c't^\alpha} \chi_{|y| \leq \bar{c} t^\alpha} = (\chi_{b_\epsilon \geq c't^\alpha} - \chi_{b_\epsilon \geq c't^\alpha}) \chi_{|y| \leq \bar{c} t^\alpha}.
$$

(B.17)

Let $\bar{\nu} := \frac{b_\epsilon}{\bar{c} t^\alpha}$. Denote $g(v) := \chi_{|y| \geq \bar{c} t^\alpha}$ and $g(\bar{\nu}) := \chi_{|y| \geq \bar{c} t^\alpha}$. We will use the construction and notations of the proof of Lemma B.2. Using the Helffer-Sjöstrand formula for $(\varphi_R g)(c)$, we write

$$
(\varphi_R g)(v) - (\varphi_R g)(\bar{\nu}) = \frac{1}{\pi} \int \partial_z (\varphi_R g)(z)[(v - z)^{-1} - (\bar{\nu} - z)^{-1}] \, d\text{Re } z \, d\text{Im } z
$$

$$
= -\frac{1}{\pi} \int \partial_z (\varphi_R g)(z)(v - z)^{-1} - (\bar{\nu} - z)^{-1} \, d\text{Re } z \, d\text{Im } z.
$$

(B.18)

Now we show that $(v - \bar{\nu})(\bar{\nu} - z)^{-1} \chi_{|y| \leq \bar{c} t^\alpha} = O(t^{-\alpha - \kappa}) |\text{Im } z|^2$. We have

$$v - \bar{\nu} = (1 - \bar{c} \chi_{|y| \leq \bar{c} t^\alpha}) \frac{b_\epsilon}{\bar{c} t^\alpha} + \bar{c} \chi_{|y| \leq \bar{c} t^\alpha} \frac{b_\epsilon}{\bar{c} t^\alpha} (1 - \bar{c} \chi_{|y| \leq \bar{c} t^\alpha}),$$

and we observe that, by Lemma B.1,

$$[(1 - \bar{c} \chi_{|y| \leq \bar{c} t^\alpha}), b_\epsilon] = O(t^\kappa).$$

(B.19)

Thus

$$v - \bar{\nu} = (1 + \bar{c} \chi_{|y| \leq \bar{c} t^\alpha}) \frac{b_\epsilon}{\bar{c} t^\alpha} (1 - \bar{c} \chi_{|y| \leq \bar{c} t^\alpha}) + O(t^{-\alpha - \kappa}),$$

Moreover, we can write

$$(1 - \bar{c} \chi_{|y| \leq \bar{c} t^\alpha})(\bar{\nu} - z)^{-1} \chi_{|y| \leq \bar{c} t^\alpha} = [(1 - \bar{c} \chi_{|y| \leq \bar{c} t^\alpha}), (\bar{\nu} - z)^{-1}] \chi_{|y| \leq \bar{c} t^\alpha}
$$

$$= - (\bar{\nu} - z)^{-1} [(1 - \bar{c} \chi_{|y| \leq \bar{c} t^\alpha}), \frac{b_\epsilon}{\bar{c} t^\alpha}] (\bar{\nu} - z)^{-1} \chi_{|y| \leq \bar{c} t^\alpha}
$$

$$= O(t^{-\alpha - \kappa}) |\text{Im } z|^2),$$

where we used (B.19) to obtain the last estimate. This implies the statement of the lemma.

Remark. The estimate (B.16) can be improved to $\chi_{\omega>\bar{c}} \chi_{\omega \leq 1} = O(t^{-m(\alpha - \kappa)})$, for any $m > 0$, if we replace $\omega_\epsilon := \omega + \epsilon$ in the definition of $b_\epsilon$ by the smooth function $\omega_\epsilon := \sqrt{\omega^2 + \epsilon^2}$.

APPENDIX C. ESTIMATES OF $d\Gamma$, $d\bar{\Gamma}$ AND $\Gamma$

In this appendix we prove technical statements about $d\Gamma$, $d\bar{\Gamma}$ and $\Gamma$, used in the main text. Most of the results we present here are close to known ones. We begin with the following standard result, which was used implicitly at several places.

Lemma C.1. Let $a, b$ be two self-adjoint operators on $\mathfrak{h}$ with $b \geq 0$, $D(b) \subset D(a)$ and $\|a\varphi\| \leq \|b\varphi\|$ for all $\varphi \in D(b)$. Then $D(d\Gamma(b)) \subset D(d\Gamma(a))$ and $\|d\Gamma(a)\Phi\| \leq \|d\Gamma(b)\Phi\|$ for all $\Phi \in D(d\Gamma(b))$.

Next, we have the following lemma which was used in the proof of Proposition 4.2. We recall the notations $B_\epsilon = d\Gamma(b_\epsilon)$ and $B_{\epsilon, t} = \frac{B_\epsilon}{\epsilon^2}$.

Lemma C.2. Let $f \in C^0_0(\mathbb{R}^3)$. Then

$$\|d\Gamma(\omega_\epsilon^{-1}) \frac{1}{2} f(B_{\epsilon, t})(1 + d\Gamma(\omega^{-1}) + t^{-1} \epsilon^{-2} N)^{-\frac{1}{2}} \| \lesssim 1,$$

(C.1)

uniformly w.r.t. $\epsilon > 0$ and $t > 0$.

Proof. By interpolation, if suffices to prove that

$$\|d\Gamma(\omega_\epsilon^{-1}) f(B_{\epsilon, t})(1 + d\Gamma(\omega^{-1}) + t^{-1} \epsilon^{-2} N)\| \lesssim 1.$$

(C.2)

To this end, we write

$$d\Gamma(\omega_\epsilon^{-1}) f(B_{\epsilon, t}) = f(B_{\epsilon, t}) d\Gamma(\omega_\epsilon^{-1}) + [d\Gamma(\omega_\epsilon^{-1}), f(B_{\epsilon, t})].$$
Since \( \| f(B_{x,t}) \| \lesssim 1 \) and \( d\Gamma(\omega^{-1})^2 \leq d\Gamma(\omega^{-1})^2 \), the first term is bounded as
\[
\| f(B_{x,t})d\Gamma(\omega^{-1})(1 + d\Gamma(\omega^{-1})) \| \lesssim 1. \tag{C.3}
\]
To estimate the second term, we write as above, using the Helffer-Sjöstrand formula,
\[
f(B_{x,t}) = \frac{1}{\pi} \int \partial_z \bar{f}(z)(B_{x,t} - z)^{-1} d\Re z \, d\Im z,
\]
where \( \bar{f} \) denotes an almost analytic extension of \( f \). This gives
\[
[d\Gamma(\omega^{-1}), f(B_{x,t})] = \frac{1}{\pi} \int \partial_z \bar{f}(z)(B_{x,t} - z)^{-1}[B_{x,t}, d\Gamma(\omega^{-1})](B_{x,t} - z)^{-1} d\Re z \, d\Im z, \tag{C.4}
\]
with
\[
[B_{x,t}, d\Gamma(\omega^{-1})] = (ct)^{-1}d\Gamma(\theta, \omega^{-2}).
\]
Since \( \| d\Gamma(\theta, \omega^{-2})(N)^{-1} \| \lesssim c^{-2} \), and since \( B_{x,t} \) commutes with \( N \), we obtain that
\[
\|([B_{x,t}^{-1}, d\Gamma(\omega^{-1})](B_{x,t} - z)^{-1})(N)^{-1} \| \lesssim t^{-1}c^{-2}|\Im z|^{-2},
\]
Hence the formula (C.4) shows that
\[
\| [d\Gamma(\omega^{-1}), f(B_{x,t})](N)^{-1} \| \lesssim t^{-1}c^{-2},
\]
which, together with (C.3), imples (C.2) and hence (C.1) by interpolation. \( \square \)

We recall that, given two operators \( a, c \) on \( \mathfrak{h} \), the operator \( d\Gamma(a, c) \) was defined in (5.11), and \( d\Gamma(a, c) := Ud\Gamma(a, c)u \).

**Lemma C.3.** Let \( j = (j_0, j_\infty) \) and \( c = (c_0, c_\infty) \), where \( j_0, j_\infty, c_0, c_\infty \) are operators on \( \mathfrak{h} \). Furthermore, assume that \( j_0 j_0 + j_\infty j_\infty \leq 1 \). Then we have the relation
\[
\| \langle \tilde{\phi}, d\Gamma(j, c) \psi \rangle \| \leq \| d\Gamma(\{c_0\}) \dot{\frac{1}{2}} \otimes \mathbf{1} \tilde{\phi} \| \| d\Gamma(\{c_\infty\}) \dot{\frac{1}{2}} \psi \| + \| \mathbf{1} \otimes d\Gamma(\{c_\infty\}) \dot{\frac{1}{2}} \tilde{\phi} \| \| d\Gamma(\{c_0\}) \dot{\frac{1}{2}} \psi \|. \tag{C.5}
\]
Likewise, with \( c_1 : \mathfrak{h} \to \mathfrak{h} \otimes \mathfrak{h} \) and \( c_2 : \mathfrak{h} \to \mathfrak{h} \), we have
\[
\| \langle u, d\Gamma(j, c_1 c_2) \psi \rangle \| \leq \| d\Gamma(c_1 c_2) \dot{\frac{1}{2}} u \| \| d\Gamma(c_1 c_2) \dot{\frac{1}{2}} \psi \|. \tag{C.6}
\]
**Proof.** Let \( \tilde{\phi} = U^* \tilde{\phi} \) and for an operator \( b \) on \( \mathfrak{h} \) define operators \( \tilde{i}_0 b := \text{diag}(b, 0) \) and \( i_\infty b := \text{diag}(0, b) \) on \( \mathfrak{h} \otimes \mathfrak{h} \). Since \( U^* d\Gamma(\{c_0\}) \dot{\frac{1}{2}} \otimes \mathbf{1} U = d\Gamma(\{i_0 c_0\}) \dot{\frac{1}{2}} U \) and \( U^* \mathbf{1} \otimes d\Gamma(\{c_\infty\}) \dot{\frac{1}{2}} U = d\Gamma(\{i_\infty c_\infty\}) \dot{\frac{1}{2}} U \), the statement of the lemma is equivalent to
\[
\| \langle \tilde{\phi}, d\Gamma(j, c) \psi \rangle \| \leq \| d\Gamma(\{i_0 c_0\}) \dot{\frac{1}{2}} \tilde{\phi} \| \| d\Gamma(\{c_0\}) \dot{\frac{1}{2}} \psi \| + \| d\Gamma(\{i_\infty c_\infty\}) \dot{\frac{1}{2}} \tilde{\phi} \| \| d\Gamma(\{c_\infty\}) \dot{\frac{1}{2}} \psi \|. \tag{C.7}
\]
We decompose \( d\Gamma(j, c) = d\Gamma(j, i_0 c_0) + d\Gamma(j, i_\infty c_\infty) \) and estimate each term separately. We have, using that \( \| j \| \leq 1 \),
\[
\| \langle \tilde{\phi}, d\Gamma(j, i_0 c_0) \psi \rangle \| \leq \sum_{l=1}^n \| (i_0 c_0) l \tilde{\phi} \| \| (i_0 c_0) l \dot{\frac{1}{2}} \psi \|,
\]
where \( (i_0 c_0) l := 1 \otimes \cdots \otimes 1 \otimes i_0 c_0 \otimes 1 \otimes \cdots \otimes 1 \), with the operator \( i_0 c_0 \) appearing in the \( l \)th component of the tensor product. By the Cauchy-Schwarz inequality, we obtain
\[
\| \langle \tilde{\phi}, d\Gamma(j, i_0 c_0) \psi \rangle \| \leq \sum_{l=1}^n \| (i_0 c_0) l \tilde{\phi} \| \| (i_0 c_0) l \dot{\frac{1}{2}} \psi \| \leq \left( \sum_{l=1}^n \| (i_0 c_0) l \tilde{\phi} \|^2 \right)^\frac{1}{2} \left( \sum_{l=1}^n \| (i_0 c_0) l \dot{\frac{1}{2}} \psi \|^2 \right)^\frac{1}{2} = \| d\Gamma(\{i_0 c_0\}) \dot{\frac{1}{2}} \tilde{\phi} \| \| d\Gamma(\{i_0 c_0\}) \dot{\frac{1}{2}} \psi \|.
\]
Since \( \| d\Gamma(\{i_0 c_0\}) \dot{\frac{1}{2}} \psi \|_{F(\mathfrak{h} \otimes \mathfrak{h})} = \| d\Gamma(\{c_0\}) \dot{\frac{1}{2}} \psi \|_{F(\mathfrak{h})} \), we obtain the first term in the r.h.s. of (C.7). The second one is obtained exactly in the same way. (C.6) can be proven in a similar manner. \( \square \)

In the following lemma, as in the main text, the operator \( j_\infty \) on \( L^2(\mathbb{R}^3) \) is of the form \( j_\infty = \chi_{\text{ray}} \frac{b_\epsilon}{\epsilon} \geq 1 \), where, recall, \( b_\epsilon = \frac{1}{2}(v_\epsilon(k) \cdot y + \text{h.c.}) \), where \( v_\epsilon(k) = \theta_\epsilon \nabla \omega, \ \theta_\epsilon = \frac{\omega}{\omega + \epsilon} \), and \( \epsilon = t^{-\kappa}, \kappa > 0 \).
Lemma C.4. Assume $\alpha + \kappa > 1$. Let $u \in \mathcal{F}$. Then $\| (\Gamma(j_{\infty}) - 1)e^{-iH_1 t}u \| \to 0$, as $t \to \infty$.

Proof. Assume that $u \in D(d\Gamma((y)))$. Using unitarity of $e^{-iH_1 t}$ and the fact that $e^{-iH_1 t} = \Gamma(e^{-i\omega t})$, we obtain
\[
\left\| (\Gamma(j_{\infty}) - 1)e^{-iH_1 t}u \right\| = \left\| (\Gamma(e^{i\omega t}j_{\infty}e^{-i\omega t}) - 1)u \right\| \leq \left\| d\Gamma(e^{i\omega t}j_{\infty}e^{-i\omega t})u \right\|,
\]
where $j_{\infty} = 1 - j_{\infty}$. Using the identity $e^{i\omega t}b_{\epsilon}e^{-i\omega t} = b_{\epsilon} + \theta t$, and the Helffer-Sjöstrand formula show that
\[
e^{it\omega}(\frac{b_{\epsilon}}{ct^a} \leq 1) e^{-it\omega} = \chi(\frac{b_{\epsilon} + \theta t}{ct^a} \leq 1).
\]
Since $\alpha + \kappa > 1$, we have $\chi_{\frac{b_{\epsilon} + \theta t}{ct^a} \leq 1} = \chi_{\frac{b_{\epsilon}}{ct^a} \leq 1} + O(t^{-(\alpha + \kappa - 1)})$. Due to $\frac{2b_{\epsilon}}{t} \geq 1$ on supp $\chi_{\frac{b_{\epsilon}}{ct^a} \leq 1}$ for $t$ sufficiently large, we have
\[
\left\| \chi_{\frac{b_{\epsilon}}{ct^a} \leq 1} \phi \right\| \leq \frac{-2b_{\epsilon}}{t} \left\| \chi_{\frac{b_{\epsilon}}{ct^a} \leq 1} \phi \right\| \leq \frac{2(y)}{t} \phi,
\]
and therefore
\[
\left\| d\Gamma(\chi_{\frac{b_{\epsilon}}{ct^a} \leq 1})u \right\| \leq \frac{2}{t} \left\| d\Gamma((y))u \right\|.
\]
Together with (C.8), this shows that $\| (\Gamma(j_{\infty}) - 1)e^{-iH_1 t}u \| \to 0$, for $u \in D(d\Gamma((y)))$. Since $D(d\Gamma((y)))$ is dense in $\mathcal{F}$, this concludes the proof.

APPENDIX D. ESTIMATES OF $P_{gs}$

Lemma D.1. Assume (1.5) with $\mu > -1/2$ and (1.7). Then $\text{Ran}(P_{gs}) \subset D(N^{\frac{1}{2}}) \cap D(d\Gamma(b_{\epsilon}^2)^{\frac{1}{2}})$, in other words, the operators $N^\frac{1}{2}P_{gs}$ and $d\Gamma(b_{\epsilon}^2)^{\frac{1}{2}}P_{gs}$ are bounded. Moreover, we have $\|d\Gamma(b_{\epsilon}^2)^{\frac{1}{2}}P_{gs}\| = O(t^\kappa)$.

Proof. Let $\Phi_{gs} \in \text{Ran}(P_{gs})$. The statement of the lemma is equivalent to the properties that
\[
k \mapsto \| a(k)\Phi_{gs} \|, \quad k \mapsto \| b_{\alpha}(k)\Phi_{gs} \| \in L^2(\mathbb{R}^3),(D.1)
\]
and that $\| b_{\alpha}(k)\Phi_{gs} \|_{L^2(\mathbb{R}^3)} = O(t^\kappa)$. The well-known pull-through formula gives
\[
a(k)\Phi_{gs} = -(H - E_{gs} + |k|)^{-1}g(k)\Phi_{gs}.
\]
Since $\|(H - E_{gs} + |k|)^{-1}\| \leq |k|^{-1}$ one easily deduces that $\| a(k)\Phi_{gs} \| \in L^2(\mathbb{R}^3)$ for any $\mu > -1/2$. Likewise, using in addition that $b_{\epsilon} = \omega_{e}^{-1}\frac{1}{2}(k \cdot \nabla_k + \nabla_k \cdot k) - i\omega/(2\omega^2)$, together with
\[
\| (k \cdot \nabla_k + \nabla_k \cdot k)(H - E_{gs} + |k|)^{-1}) \| \leq \| k(H - E_{gs} + |k|)^{-2}) \| \leq |k|^{-1},
\]
and (1.5)–(1.7), one easily deduces that $\| b_{\alpha}(k)\Phi_{gs} \|_{L^2(\mathbb{R}^3)} = O(t^\kappa)$ for any $\mu > -1/2$.

APPENDIX E. THE PROOF OF THE EXISTENCE OF $W_+$ UNDER ASSUMPTION (1.21)

Let $\rho_\nu := \chi_{\frac{\nu}{\omega}}^{1/2} \omega^{\nu/2}$ and recall that $\chi \equiv \chi_{w \leq 1}$, with $w = (\frac{e^{at}}{a})^2$, and $v = \frac{b_{\epsilon}}{ct^a}$. We begin with the following weighted propagation estimates, which are a straightforward extensions of the estimates of Theorem 3.1:
\[
\int_1^\infty dt \ t^{-\beta} \left\| d\Gamma(\rho_1^t \chi_{(\omega^{\nu} = 1)}) \frac{1}{2} \psi_t \right\|^2 \lesssim \| \psi_0 \|^2, \quad (E.1)
\]
for $\mu$ and $\alpha$ as in Theorem 3.1 and any $\psi_0 \in \mathcal{H}$, and, if in addition assumption (1.21) of Theorem 1.3 holds,
\[
\int_1^\infty \| d\Gamma(\omega^{-1/2} \chi_{(w^{\nu} = 1)} \omega^{-1/2}) \frac{1}{2} \psi_t \|^2 \lesssim C(\psi_0), \quad (E.2)
\]
and
\[
\int_1^\infty \| d\Gamma(\rho_{t-1}^\nu \chi_{(\nu^{\nu} = 1}) \rho_{t-1}) \frac{1}{2} \psi_t \|^2 \lesssim C(\psi_0). \quad (E.3)
\]
for any $\psi_0 \in \mathcal{D}$. Likewise, under assumption (1.21) of Theorem 1.3, the proof of the maximal velocity estimate (1.12) of [10] can easily be extended to the following weighted maximal velocity estimate:
\[
\| d\Gamma(\omega^{-1/2} \chi_{(w^{\nu} = 1)} \omega^{-1/2}) \frac{1}{2} \psi_t \| \lesssim \left\| (d\Gamma(\omega^{-1/2}(y)\omega^{-1/2}) + 1) \frac{1}{2} \psi_0 \right\| + C(\psi_0), \quad (E.4)
\]
for any $c > 1$, $\gamma < \min\left(\frac{\delta - \frac{1}{2}}{2c-1}, \frac{1}{2}\right)$ and $\psi_0 \in \mathcal{D} \cap \mathcal{D}(d \Gamma(\omega^{-1/2}(y_0/y_0^{-1/2})^\frac{1}{2})$.

We only mention that to obtain for instance (E.2), we estimate the interaction term using (2.11) with $\delta = -1/2$ together with the inequality (3.16) and the assumption (1.21).

Now, let $\psi_0 \in \mathcal{D} \cap D(d \Gamma(\omega^{-1/2}(y_0/y_0^{-1/2})^\frac{1}{2})$. We decompose $(\hat{W}(t') - \hat{W}(t)) \psi_0$ as in Equations (5.27)–(5.31). Using the commutator estimates of Appendix B and Hardy’s inequality, we verify that

$$\rho^{*-1}(j'_0, j'_\infty) \rho_1 = \theta^1\chi(j'_0, j'_\infty) \chi \theta^1 + O(t^{-\alpha+(1+\kappa)/2}),$$

and likewise for the remainder terms $\text{rem}_t$. Hence Equations (5.30)–(5.31) can be transformed into

$$\frac{d}{dt} \rho^+ = \frac{1}{ct} \rho^1(j'_0, j'_\infty) \rho_{-1} + \omega^{1/2} \text{rem}_t \omega^{-1/2}$$

$$\text{rem}_t' = \text{rem}_t + O(t^{-2n+(1+\kappa)/2}),$$

where $\text{rem}_t$ is given in (5.31). These relations give

$$G_0 = \tilde{G}'_0 + \text{Rem}'_t,$$

where $\tilde{G}'_0 := \frac{1}{ct} U d \Gamma(j, \tilde{c})$, with $\tilde{c}_t = (\tilde{c}_0, \tilde{c}_\infty) := (\rho^+_1 j'_0 \rho_{-1}, \rho^+_1 j'_\infty \rho_{-1})$, and

$$\text{Rem}'_t := G_0 - \tilde{G}'_0 = U d \Gamma(j, \text{rem}_t').$$

Next, we consider, as above, $\tilde{A} = \sup_{\|\psi_0\|=1} |\int t' ds (\hat{\phi}_s, G_0 \psi_s)|$, where $\hat{\phi}_s = e^{-it \hat{H}} \hat{f} \hat{H}_0$. Let $a_0 = \rho^+_1 j'_0 \rho_{-1}$, $b_0 = |j'_0|^{1/2} \rho_{-1}$, $a_\infty = \rho^+_1 j'_\infty \rho_{-1}$, $b_\infty = |j'_\infty|^{1/2} \rho_{-1}$.

We have $\tilde{c}_0 = -a_0 b_0$, $\tilde{c}_\infty = a_\infty b_\infty$. Exactly as for (C.5), one can show that, if $c = (a_0 b_0, a_\infty b_\infty)$, where $a_0, b_0, a_\infty, b_\infty$ are operators on $\mathfrak{h}$, then

$$\langle \hat{\phi}_s, d \Gamma(j, c) \psi_s \rangle \leq \|d \Gamma(a_0 a_\infty^*)^2 \otimes 1 \hat{\phi}_s\|\|d \Gamma(b_0^* b_\infty^*)^2 \hat{\psi}_s\|$$

$$+ \|1 \otimes d \Gamma(\hat{a}_\infty a_\infty^*)^2 \hat{\phi}_s\|\|d \Gamma(b_0^* b_\infty^*)^2 \hat{\psi}_s\|.$$ (E.8)

Hence $\tilde{G}'_0$ satisfies

$$\langle \hat{\phi}_s, \tilde{G}'_0 \psi_s \rangle \leq \frac{1}{ct} \|d \Gamma(a_0 a_\infty^*)^2 \otimes 1 \hat{\phi}_s\|\|d \Gamma(b_0^* b_\infty^*)^2 \hat{\psi}_s\|$$

$$+ \|1 \otimes d \Gamma(\hat{a}_\infty a_\infty^*)^2 \hat{\phi}_s\|\|d \Gamma(b_0^* b_\infty^*)^2 \hat{\psi}_s\|.$$ (E.9)

By the Cauchy-Schwarz inequality, (E.9) implies

$$\int t' ds |\langle \hat{\phi}_s, \tilde{G}'_0 \psi_s \rangle| \lesssim \left( \int t' ds s^{-\alpha} \|d \Gamma(a_0 a_\infty^*)^2 \otimes 1 \hat{\phi}_s\|^2 \right)^\frac{1}{2} \left( \int t' ds s^{-\alpha} \|d \Gamma(b_0^* b_\infty^*)^2 \hat{\psi}_s\|^2 \right)^\frac{1}{2}$$

$$+ \left( \int t' ds \|1 \otimes d \Gamma(\hat{a}_\infty a_\infty^*)^2 \hat{\phi}_s\|^2 \right)^\frac{1}{2} \left( \int t' ds s^{-\alpha} \|d \Gamma(b_0^* b_\infty^*)^2 \hat{\psi}_s\|^2 \right)^\frac{1}{2}.$$}

Since $a_0 a_\infty^*$ and $a_\infty a_\infty^*$ are of the form $\rho^+_1 \chi b_\infty \rho_{-1}$, the weighted minimal velocity estimate (E.3) implies

$$\int_1^\infty ds s^{-\alpha} \|d \Gamma(c_{#1} c_{#1}^*)^2 \hat{\phi}_s\|^2 \lesssim \|\hat{\phi}_0\|^2,$$

where $d \Gamma(c_{#1} c_{#1}^*)^2$ stands for $d \Gamma(a_0 a_0^*)^2 \otimes 1$ or $1 \otimes d \Gamma(a_\infty a_\infty^*)^2$. Likewise, since $b_0^* b_0$ and $b_0^* b_\infty$ are of the form $\rho^+_1 \chi b_\infty \rho_{-1}$, the weighted minimal velocity estimate (E.1) implies

$$\int_1^\infty ds s^{-\alpha} \|d \Gamma(c_{#2} c_{#2}^*)^2 \hat{\psi}_s\|^2 \lesssim C(\psi_0),$$

with $\psi_{#2} = b_0$ or $b_\infty$. The last three relations give

$$\sup_{\|\hat{\phi}_s\|=1} \left|\int t' ds \langle \hat{\phi}_s, \tilde{G}'_0 \psi_s \rangle\right| \to 0, \quad t', t' \to \infty.$$ (E.10)
Applying likewise Lemma C.3 of Appendix C, one verifies that $Rm_{\nu}^1$ satisfies
\[ \|\langle \hat{\phi}, Rm_{\nu}^1 \psi \rangle\| \lesssim \|\hat{\phi}\| \left( t^{-2\alpha+(1+\alpha)/2} \|d\Gamma(\omega^{-1})\|^2 + t^{-1} \|d\Gamma(\omega^{-1/2} \chi_{\omega>1}^{-1/2})\|^2 \right). \]

Using (1.21), the weighted minimal velocity estimate (E.2) and the weighted maximal velocity estimate (E.4), we conclude as above that
\[ \sup_{\|\hat{\phi}_0\|=1} \left| \int_t^{t'} ds \langle \hat{\phi}_s, Rm_{\nu}^1 \psi_s \rangle \right| \to 0, \quad t, t' \to \infty. \] (E.11)

Equations (E.10) and (E.11) then imply
\[ \bar{A} = \| \int_t^{t'} ds f(\hat{H}) e^{itH} \| \to 0, \quad t, t' \to \infty. \] (E.12)

The estimate of $G_1$ is the same as above, which shows that $\hat{W}(t)$, and hence $W(t)$, are strong Cauchy sequences. Thus the limit $W_+$ exists.

**Supplement I. The wave operators**

In this supplement we briefly review the definition and properties of the wave operator $\Omega_+$, and establish its relation with $W_+$ in Theorem 1.2 below. For simplicity we consider again hamiltonians of the form (1.3)–(1.4). Let $\mathcal{H}_b \equiv \mathcal{H}_{pp}(H) \cap \mathcal{I}_{(-\infty, \Sigma)}(H)$ be the space spanned by the eigenfunctions of $H$ with the eigenvalues in the interval $(-\infty, \Sigma)$. Define $h_0 := (h \in L^2(\mathbb{R}), \int |h|^2 (|k|-1 + |k|^2) dk < \infty)$. The wave operator $\Omega_+$ on the space $\mathcal{H}_b \otimes \mathcal{F}_{\text{fin}}(h_0)$, is defined by the formula
\[ \Omega_+ := \Pi^{-1} e^{itH} I(e^{-itH} \otimes e^{-itH}). \] (I.1)

As in [17, 23, 24, 36], it is easy to show

**Theorem 1.1.** Assume (1.5) with $\mu \geq -1/2$ and (1.7). The wave operator $\Omega_+$ exists on $\mathcal{H}_b \otimes \mathcal{F}_{\text{fin}}(h_0)$ and extends to an isometric map, $\Omega_+ : \mathcal{H}_{\text{as}} \to \mathcal{H}$, on the space of asymptotic states, $\mathcal{H}_{\text{as}} := \mathcal{H}_b \otimes \mathcal{F}$.

**Proof.** Let $h_1(k) := e^{-it|k|} h(k)$. For $h \in D(\omega^{-1/2})$, s. t. $\partial^{*} h \in D(\omega^{1/2})$, $|\alpha| \leq 2$, we define the asymptotic creation and annihilation operators by (see [17, 23, 24, 36])
\[ a^\pm_\#(h) := \lim_{t \to \pm \infty} e^{itH} a^\#(h_1) e^{-itH}, \]
for any $\Phi \in D([H]^{1/2}) \cap \text{Ran} E_{(-\infty, \Sigma)}(H)$. Here $a^\#$ stands for $a$ or $a^*$. To show that $a^\pm_\#(h)$ exist (see [23, 36]), we define $a^\pm_\#(h) := e^{itH} a^\#(h_1) e^{-itH}$ and compute $a^\pm_\#(h) - a^\pm_\#(h) = f_t ds \partial_s a^\pm_\#(h)$ and $\partial_s a^\pm_\#(h) = i e^{itH} G e^{-itH}$, where $G := [H, a^\#(h_1)] - a^\#(\omega h_2) = \langle g, h_1 \rangle \times_{(dk)}$ for $a^\# = a^*$ and $\langle h_1, g \rangle \times_{(dk)}$ for $a^\# = a$. Thus the proof of existence reduces to showing that one-photon terms of the form $\langle \eta g, h_1 \rangle$ are integrable in $t$. By (1.5), we have $\|\langle \eta g, h_1 \rangle \times_{(dk)}\|_{\mathcal{H}_b} \leq (1 + t)^{-1-\varepsilon}$, with $0 < \varepsilon < \mu + 1$, which is integrable. Moreover, as in [23, 36] one can show that $a^\pm_\#(h)$ satisfy the canonical commutation relations and relations $a_\pm(h) \Phi = 0$, and
\[ \lim_{t \to \pm \infty} e^{itH} a^\#(h_1, t) \cdots a^\#(h_n, t) e^{-itH} \Phi = a^\#_+(h_1) \cdots a^\#_+(h_n) \Phi, \] (I.2)
for any $\Psi \in \mathcal{H}_b$, $h_1, \cdots, h_n \in h_0$, and any $\Phi \in \mathcal{I}_{(-\infty, \Sigma)}(H)$. We define the wave operator $\Omega_+$ on $\mathcal{H}_{\text{fin}}$ by
\[ \Omega_+(\Phi \otimes a^*(h_1) \cdots a^*(h_n) \Omega) := a^*_+(h_1) \cdots a^*_+(h_n) \Phi. \] (I.3)

Using the canonical commutation relations, one sees that $\Omega_+$ extends to an isometric map $\Omega_+ : \mathcal{H}_{\text{as}} \to \mathcal{H}$. Using the relation $e^{itH}(\Phi \otimes a^\#(h_1) \cdots a^\#(h_n) \Omega) = (e^{itH} \Phi_{\text{fin}}) \otimes (a^\#(h_1, t) \cdots a^\#(h_n, t) \Omega)$, the definition of $I$ and (1.2), we identify the definition (1.3) with (1.1). □
Recall that $P_{gs}$ denotes the orthogonal projection onto the ground state subspace of $H$. Let $\tilde{P}_{gs} := 1 - P_{gs}$ and $\tilde{P}_{\Omega} := 1 - P_{\Omega}$, where, recall, $P_{\Omega}$ is the projection onto the vacuum sector in $F$. Theorem 5.4 and its proof imply the following result.

**Theorem I.2.** Under the conditions of Theorem 5.4, we have on $\text{Ran} \chi_{\Delta}(H)$

$$\Omega_{+}(P_{gs} \otimes \tilde{P}_{\Omega})W_{+}\tilde{P}_{gs} + P_{gs} = 1. \tag{I.4}$$

**Proof.** Let $\psi_{0} \in \text{Ran} \chi_{\Delta}(H)$. For every $\epsilon'' > 0$ there is $\delta'' = \delta(\epsilon'') > 0$, s.t.

$$\|\psi_{0} - \psi_{0\epsilon''} - P_{gs}\psi_{0}\| \leq \epsilon'',$$

where $\psi_{0\epsilon''} = \chi_{\Delta,\epsilon''}(H)\psi_{0}$, with $\Delta,\epsilon'' = [E_{gs} + \delta, a]$. Proceeding as in the proof of Theorem 5.4 with $\psi_{0\epsilon''}$ instead of $\psi_{0}$, we arrive at (see (5.66))

$$\psi_{0\epsilon''} = e^{-iHt}(e^{-iE_{gs}\epsilon''P_{gs}} \otimes e^{-iHt}(\chi_{(0,a-E_{gs})(H_{f}))\phi_{0\epsilon''} + O(\epsilon') + C(\epsilon', m)\phi_{t}(1) + C(\epsilon')\phi_{m}(1), \tag{I.6}$$

where we choose $\phi_{0\epsilon''}$ such that $\phi_{0\epsilon''} \in D(\text{def}(\mathcal{F}_{in}(\delta_{0}))$ and $\|W_{+}\phi_{0\epsilon''} - \phi_{0\epsilon''}\| \leq \epsilon'$. Now using Theorem I.1, we let $t \to \infty$, next $m \to \infty$ to obtain

$$\psi_{0\epsilon''} = \Omega_{+}(P_{gs} \otimes \chi_{(0,a-E_{gs})(H_{f}))\phi_{0\epsilon''} + O(\epsilon'). \tag{I.7}$$

Since $\Omega_{+}$ is isometric, hence bounded, we can let $\epsilon' \to 0$, which gives

$$\psi_{0\epsilon''} = \Omega_{+}(P_{gs} \otimes \chi_{(0,a-E_{gs})(H_{f}))W_{+}\psi_{0\epsilon''} = \Omega_{+}(P_{gs} \otimes \tilde{P}_{\Omega})W_{+}\tilde{P}_{gs}\psi_{0\epsilon''}. \tag{I.8}$$

Here we used that $\chi_{(0,a-E_{gs})(H_{f})} = \tilde{P}_{\Omega}\chi_{(0,a-E_{gs})(H_{f})}$, together with $\chi_{(0,a-E_{gs})(H_{f})}W_{+}\psi_{0\epsilon''} = W_{+}\psi_{0\epsilon''}$ and $\psi_{0\epsilon''} = \tilde{P}_{gs}\psi_{0\epsilon''}$. Introducing (I.8) into (I.5) and letting $\epsilon'' \to 0$, we obtain

$$\psi_{0} = \Omega_{+}(P_{gs} \otimes \tilde{P}_{\Omega})W_{+}\tilde{P}_{gs}\psi_{0} + P_{gs}\psi_{0},$$

which gives (I.4). \qed

**Supplement II. Creation and annihilation operators on Fock spaces**

Recall that the propagation speed of the light and the Planck constant divided by $2\pi$ are set equal to 1. Recall also that the one-particle space is $\mathfrak{h} := L^{2}(\mathbb{R}^{3}; \mathbb{C})$, for phonons, and $\mathfrak{h} := L^{2}(\mathbb{R}^{3}; \mathbb{C}^{2})$, for photons. In both cases we use the momentum representation and write functions from this space as $u(k)$ and $u(k, \lambda)$, respectively, where $k \in \mathbb{R}^{3}$ is the wave vector or momentum of the photon and $\lambda \in \{-1, +1\}$ is its polarization.

With each function $f \in \mathfrak{h}$, one associates creation and annihilation operators $a(f)$ and $a^{*}(f)$ defined, for $u \in \otimes_{\mathfrak{h}}\mathfrak{h}$, as

$$a^{*}(f) : u \rightarrow \sqrt{n + 1}f \otimes_{\mathfrak{h}} u \quad \text{and} \quad a(f) : u \rightarrow \sqrt{n}(f, u)_{\mathfrak{h}}, \tag{II.1}$$

with $(f, u)_{\mathfrak{h}} := \int f(k)\overline{u}(k, k_{1}, \ldots, k_{n-1}) dk$, for phonons, and $(f, u)_{\mathfrak{h}} := \sum_{\lambda=1,2} \int \mathcal{F}(k, \lambda)u_{\lambda}(k, \lambda, k_{1}, \lambda_{1}, \ldots, k_{n-1}, \lambda_{n-1})$ for photons. They are unbounded, densely defined operators of $\Gamma(\mathfrak{h})$, adjoint of each other (with respect to the natural scalar product in $\mathfrak{F}$) and satisfy the canonical commutation relations (CCR):

$$[a^{\#}(f), a^{\#}(g)] = 0, \quad [a(f), a^{*}(g)] = \langle f, g \rangle_{\mathfrak{h}},$$

where $a^{\#} = a$ or $a^{*}$. Since $a(f)$ is anti-linear and $a^{*}(f)$ is linear in $f$, we write formally

$$a(f) = \int \overline{f(k)}a(k) dk, \quad a^{*}(f) = \int f(k)a^{*}(k) dk,$$

for phonons, and

$$a(f) = \sum_{\lambda=1,2} \int \mathcal{F}(k, \lambda)a_{\lambda}(k), \quad a^{*}(f) = \sum_{\lambda=1,2} \int \mathcal{F}(k, \lambda)a^{*}_{\lambda}(k),$$

for photons.
operators on canonical commutation relations for photons. Here \(a(k)\) and \(a^*(k)\) and \(a_\lambda(k)\) and \(a^*_\lambda(k)\) are unbounded, operator-valued distributions, which obey (again formally) the canonical commutation relations (CCR):

\[
\begin{align*}
[a^\#(k), a^\#(k')] &= 0, &\quad [a(k), a^*(k')] &= \delta(k - k'), \\
[a^\#(k), a^*_\lambda(k')] &= 0, &\quad [a_\lambda(k), a^*(k')] &= \delta_\lambda, \delta(k - k'),
\end{align*}
\]

where \(a^\# = a\) or \(a^*\) and \(a^\#_\lambda = a_\lambda\) or \(a^*_\lambda\).

Given an operator \(\tau\) acting on the one-particle space \(\mathfrak{h}\), the operator \(d\Gamma(\tau)\) (the second quantization of \(\tau\)) defined on the Fock space \(\mathcal{F}\) by (1.2), can be written (formally) as \(d\Gamma(\tau) := \int dka^*_\lambda(k)\tau a_\lambda(k)\), for phonons, and \(d\Gamma(\tau) := \sum_{\lambda=1,2} \int dka^*_\lambda(k)\tau a_\lambda(k)\), for photons. Here the operator \(\tau\) acts on the \(k\)-variable. The precise meaning of the latter expression is (1.2). In particular, one can rewrite the quantum Hamiltonian \(H_f\) in terms of the creation and annihilation operators, \(a\) and \(a^*\), as

\[
H_f = \sum_{\lambda=1,2} \int dka^*_\lambda(k)\omega(k)a_\lambda(k)
\]

for photons, and similarly for phonons.

The relations below are valid for both phonon and photon operators. Commutators of two \(d\Gamma\) operators reduces to commutators of the one-particle operators:

\[
[d\Gamma(\tau), d\Gamma(\tau')] = d\Gamma([\tau, \tau']).
\]

Let \(\tau\) be a one-photon self-adjoint operator. The following commutation relations involving the field operator \(\Phi(f) = \frac{1}{\sqrt{2}}(a^*(f) + a(f))\) can be readily derived from the definitions of the operators involved:

\[
\begin{align*}
[\Phi(f), \Phi(g)] &= i\text{Im}(f,g)\mathfrak{h}, \quad (\text{II.4}) \\
[\Phi(f), d\Gamma(\tau)] &= i\Phi(\tau f), \quad (\text{II.5}) \\
[\Gamma(\tau), \Phi(f)] &= \Gamma(\tau)a((1 - \tau)f) - a^*((1 - \tau)f)\Gamma(\tau). \quad (\text{II.6})
\end{align*}
\]

Exponentiating these relations, we obtain

\[
\begin{align*}
e^{i\Phi(f)}\Phi(g)e^{-i\Phi(f)} &= \Phi(g) - \text{Im}(f,g)\mathfrak{h}, \quad (\text{II.7}) \\
e^{i\Phi(f)}d\Gamma(\tau)e^{-i\Phi(f)} &= d\Gamma(\tau) - \Phi(\tau f) + \frac{1}{2}\text{Re}(\omega, f)\mathfrak{h} \quad (\text{II.8}) \\
e^{i\Phi(f)}\Gamma(\tau)e^{-i\Phi(f)} &= \Gamma(\tau) + \int_0^1 ds e^{is\Phi(f)}(\Gamma(\tau)a((1 - \tau)f) - a^*((1 - \tau)f)\Gamma(\tau))e^{-s\Phi(f)}. \quad (\text{II.9})
\end{align*}
\]

Finally, we have the following standard estimates for annihilation and creation operators \(a(f)\) and \(a^*(f)\), whose proof can be found, for instance, in [7], [31, Section 3], [37]:

**Lemma II.1.** For any \(f \in \mathfrak{h}\) such that \(\omega^{-\rho/2}f \in \mathfrak{h}\), the operators \(a^\#(f)(d\Gamma(\omega^\rho) + 1)^{-1/2}\), where \(a^\#(f)\) stands for \(a^*(f)\) or \(a(f)\), extend to bounded operators on \(\mathcal{H}\) satisfying

\[
\begin{align*}
\|a(f)(d\Gamma(\omega^\rho) + 1)^{-1/2}\| &\leq \|\omega^{-\rho/2}f\|_\mathfrak{h}, \\
\|a^*(f)(d\Gamma(\omega^\rho) + 1)^{-1/2}\| &\leq \|\omega^{-\rho/2}f\|_\mathfrak{h} + \|f\|_\mathfrak{h}.
\end{align*}
\]

If, in addition, \(g \in \mathfrak{h}\) is such that \(\omega^{-\rho/2}g \in \mathfrak{h}\), the operators \(a^\#(f)a^\#(g)(d\Gamma(\omega^\rho) + 1)^{-1}\) extend to bounded operators on \(\mathcal{H}\) satisfying

\[
\begin{align*}
\|a(f)a(g)(d\Gamma(\omega^\rho) + 1)^{-1}\| &\leq \|\omega^{-\rho/2}f\|_\mathfrak{h}\|\omega^{-\rho/2}g\|_\mathfrak{h}, \\
\|a^*(f)a(g)(d\Gamma(\omega^\rho) + 1)^{-1}\| &\leq \|\omega^{-\rho/2}f\|_\mathfrak{h} + \|f\|_\mathfrak{h}\|\omega^{-\rho/2}g\|_\mathfrak{h} + \|g\|_\mathfrak{h}. \\
\end{align*}
\]


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