Computation of 2-groups of narrow logarithmic divisor classes of number fields *

Jean-François JAULENT, Sebastian PAULI,
Michael E. POHST & Florence SORIANO–GAFIUK,

Abstract. We present an algorithm for computing the 2-group $\widetilde{\mathcal{C}\ell}_F^{res}$ of narrow logarithmic divisor classes of degree 0 for number fields F. As an application, we compute in some cases the 2-rank of the wild kernel $WK_2(F)$ and the 2-rank of its subgroup $K_2^\infty(F) := \cap_{n\geq 1} K_2^n(F)$ of infinite height elements in $K_2(F)$.

Résumé. Nous présentons un algorithme de calcul du 2-groupe des classes logarithmiques de degréé nul au sens restreint $\widetilde{\mathcal{C}\ell}_F^{res}$ pour tout corps de nombres F. Nous en déduisons sous certaines hypothèses les 2-rangs du noyau sauvage $WK_2(F)$ ainsi que du sous-groupe $K_2^\infty(F) := \cap_{n\geq 1} K_2^n(F)$ des éléments de hauteur infinie dans $K_2(F)$.

1 Introduction

In [8] J.-F. Jaulent pointed out that the wild kernel $WK_2(F)$ of a number field F can also be studied via logarithmic class groups, the arithmetic of which he therefore developed in [9].

More precisely, if F contains a primitive 2ℓ-th root of unity, the ℓ-rank of $WK_2(F)$ coincides with the ℓ-rank of the logarithmic class group $\widetilde{\mathcal{C}\ell}_F$. So an algorithm for computing $\widetilde{\mathcal{C}\ell}_F$ for Galois extensions F was developed first in [4] and later generalized and improved for arbitrary number field in [3].

In case the prime ℓ is odd and the field F does not contain a primitive ℓ-th root of unity one considers the cyclotomic extension $F' := F(\zeta_\ell)$, uses the isomorphism

$$\mu_\ell \otimes \widetilde{\mathcal{C}\ell}_{F'} \ \simeq\ WK_2(F')/WK_2(F')^\ell \ ,$$

and gets back to F via the so-called transfer (see [11] and [16]). The algorithmic aspect of this is treated in [14].

In case $\ell = 2$, whenever the condition $\zeta_{2\ell} \in F$ is not fulfilled, the relationship between logarithmic classes and exotic symbols is more intricate. For instance, when the number field F has a real embedding, F. Soriano-Gafiuk observed in [15] that one may then define a narrow version of the logarithmic class group

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by analogy with the classical ideal class groups; and she used this in [16] for approximating the wild kernel more closely. But, unexpectedly, the 2-rank of this restricted logarithmic class group $\widetilde{\mathcal{C}\ell}_F^{res}$ sometimes differs from the 2-rank of the group $WK_2(F)$. Moreover, in this case the wild kernel $WK_2(F)$ may differ from its subgroup $K_2^\infty(F) := \cap_{n\geq 1} K_2^n(F)$ of infinite height elements in $K_2(F)$. This was observed by J. Tate and then made more explicit by K. Hutchinson (cf [6, 7]).

That last difficulty was finally solved in [12],where the authors constructed a positive class group *ad hoc* $\mathcal{C}\ell_F^{pos}$ which has the same 2-rank as the wild kernel $WK_2(F)$. Nevertheless, in case the set PE_F of dyadic exceptional primes of the number field F is empty, that group $\mathcal{C}\ell_F^{pos}$ appears as a factor of the full narrow logarithmic class group $\mathcal{C}\ell_F^{res}$ (without any assumption on the degree), so one may still use narrow logarithmic classes in order to compute the 2-rank of the wild kernel.

In the present paper we use the results from [3] on (ordinary) logarithmic class groups and develop an algorithm for computing the narrow groups $\widetilde{\mathcal{C}\ell}_F^{res}$ in arbitrary number fields F. As a consequence, this algorithm calculates the 2-rank of the wild kernel $WK_2(F)$ whenever the field F has no dyadic exceptional places.

The computation of the 2-rank of $WK_2(F)$ in the remaining case ($PE_F \neq \emptyset$) will be solved in a forthcoming article where we compute the finite positive classgroup $\mathcal{C}\ell_F^{pos}$ and its subgroup $\widetilde{\mathcal{C}\ell}_F^{pos}$ of positive classes of degree 0.

2 The group of narrow logarithmic classes $\widetilde{\mathcal{C}\ell}_F^{res}$

In this preliminary section we recall the definition and the main properties of the arithmetic of restricted (or narrow) logarithmic classes. We refer to [9] and [16] for a more detailed account.

Throughout this paper the prime number ℓ equals 2 and F is a number field of degree $n = r + 2c$ with r real places, c complex places and d dyadic places.

According to [8], for every finite place $\mathfrak{p}$ of F there exists a 2-adic $\mathfrak{p}$-valuation $\tilde{v}_\mathfrak{p}$ which is related to the wild $\mathfrak{p}$-symbol in case the cyclotomic $\mathbb{Z}_2$-extension of $F_\mathfrak{p}$ contains i. The degree $\deg_F \mathfrak{p}$ of the place $\mathfrak{p}$ is a 2-adic integer such that the image of $\mathcal{R}_F := \mathbb{Z}_2 \otimes_\mathbb{Z} F^\times$ under the map $\mathrm{Log}\,|\;|_\mathfrak{p}$ is the $\mathbb{Z}_2$-module $\deg_F \mathfrak{p}\, \mathbb{Z}_2$ (see [9]), where Log denotes the usual 2-adic logarithm and $|\;|_\mathfrak{p}$ is the 2-adic absolute value at the place $\mathfrak{p}$.

The construction of the 2-adic logarithmic valuations $\tilde{v}_\mathfrak{p}$ yields:

$$\forall \alpha \in \mathcal{R}_F := \mathbb{Z}_2 \otimes_\mathbb{Z} F^\times \;:\; \sum_{\mathfrak{p}\in Pl_F^0} \tilde{v}_\mathfrak{p}(\alpha) \deg_F \mathfrak{p} \;=\; 0 \;, \tag{1}$$

where Pl_F^0 is the set of finite places of the number field F. Setting

$$\widetilde{\mathrm{div}}_F(\alpha) \;:=\; \sum_{\mathfrak{p}\in Pl_F^0} \tilde{v}_\mathfrak{p}(\alpha)\, \mathfrak{p}$$

with values in $\mathcal{D}\ell_F := \bigoplus_{\mathfrak{p}\in Pl_F^0} \mathbb{Z}_2\,\mathfrak{p}$, we obtain by $\mathbb{Z}_2$-linearity:

$$\deg_F(\widetilde{\operatorname{div}}_F(\alpha)) \;=\; 0 \;\; . \tag{2}$$

We then define the subgroup of logarithmic divisors of degree 0 by:

$$\widetilde{\mathcal{D}\ell}_F \;:=\; \Big\{\mathfrak{a} = \textstyle\sum_{\mathfrak{p}\in Pl_F^0} a_\mathfrak{p}\mathfrak{p} \in \mathcal{D}\ell_F \mid \deg_F \mathfrak{a} := \sum_{\mathfrak{p}\in Pl_F^0} a_\mathfrak{p} \deg_F \mathfrak{p} = 0\Big\}\,;$$

and the group of principal logarithmic divisors as the image of $\mathcal{R}_F$ by $\widetilde{\operatorname{div}}_F$:

$$\widetilde{\mathcal{P}\ell}_F \;:=\; \Big\{\widetilde{\operatorname{div}}_F(\alpha) \mid \alpha \in \mathcal{R}_F\Big\} \;\; .$$

Because of (2), $\widetilde{\mathcal{P}\ell}_F$ is a subgroup of $\widetilde{\mathcal{D}\ell}_F$. And by the so-called extended Gross conjecture the factor group

$$\widetilde{\mathcal{C}\ell}_F \;:=\; \widetilde{\mathcal{D}\ell}_F/\widetilde{\mathcal{P}\ell}_F$$

is a finite group, the *2-group of logarithmic divisor classes (of degree 0)* of the field F introduced in [9].

Now let $PR_F := \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ be the (non empty) set of real places of the field F and F^+ be the subgroup of all totally positive elements in $F^\times$, i.e. the kernel of the sign map

$$\operatorname{sign}_F^\infty : F^\times \to \{\pm 1\}^r$$

which maps $x \in F$ onto the vector of the signs of the real conjugates of x. For

$$\widetilde{\mathcal{P}\ell}_F^{+} \;:=\; \{\widetilde{\operatorname{div}}_F(\alpha) \mid \alpha \in \mathcal{R}_F^+ := \mathbb{Z}_2 \otimes_\mathbb{Z} F^+\}$$

the factor group

$$\widetilde{\mathcal{C}\ell}_F^{res} \;:=\; \widetilde{\mathcal{D}\ell}_F/\widetilde{\mathcal{P}\ell}_F^{+}$$

is the *2-group of narrow (or restricted) logarithmic divisor classes (of degree 0)* introduced in [15]; and it is also finite under the extended Gross conjecture.

In order to make it more suitable for actual computations, we may define it in a slighty different way by introducing real signed divisors of degree 0:

Definition 1. *With the notations above, the 2-group of real signed logarithmic divisors (of degree 0) is the direct sum:*

$$\widetilde{\mathcal{D}\ell}_F^{res} := \widetilde{\mathcal{D}\ell}_F \oplus \{\pm 1\}^r;$$

and the subgroup of principal real signed logarithmic divisors is the image:

$$\widetilde{\mathcal{P}\ell}_F^{res} \;:=\; \{(\widetilde{\operatorname{div}}_F(\alpha), \operatorname{sign}_F^\infty(\alpha)) \mid \alpha \in \mathcal{R}_F\}$$

of $\mathcal{R}_F := \mathbb{Z}_2 \otimes_\mathbb{Z} F^\times$ under the $(\widetilde{\operatorname{div}}_F, \operatorname{sign}_F^\infty)$ map. The factor group:

$$\widetilde{\mathcal{C}\ell}_F^{res} \;:=\; \widetilde{\mathcal{D}\ell}_F^{res}/\widetilde{\mathcal{P}\ell}_F^{res}$$

is the 2-group of narrow logarithmic divisor classes (of degree 0).

Because of the weak approximation theorem, every class in $\widetilde{\mathcal{C}\ell}_F^{res}$ can be represented by a pair $(\mathfrak{a}, \mathbf{1})$ where the vector $\mathbf{1}$ has all entries 1. So the canonical map $\mathfrak{a} \mapsto (\mathfrak{a}, \mathbf{1})$ induces a morphism from $\widetilde{\mathcal{D}\ell}_F$ onto $\widetilde{\mathcal{C}\ell}_F^{res}$, the kernel of which is $\widetilde{\mathcal{P}\ell}_F^{+}$. We conclude as expected:

$$\widetilde{\mathcal{C}\ell}_F^{res} \;=\; \widetilde{\mathcal{D}\ell}_F^{res}/\widetilde{\mathcal{P}\ell}_F^{res} \;\simeq\; \widetilde{\mathcal{D}\ell}_F/\widetilde{\mathcal{P}\ell}_F^{+}.$$

We are now in a situation to present an algorithm for computing narrow logarithmic classes. It uses our previous results of [3] on (ordinary) logarithmic classes and mimics the classical feature concerning narrow and ordinary ideal classes. We note that this algorithm is a bit more intricate in the logarithmic context since the logarithmic units *are not* algebraic numbers and are therefore not exactely known from a numerical point of view.

3 The algorithm for computing $\widetilde{\mathcal{C}\ell}_F^{res}$

We assume in this section that the number field F has at least one real place and that the logarithmic 2-class group $\widetilde{\mathcal{C}\ell}_F$ is isomorphic to the sum

$$\widetilde{\mathcal{C}\ell}_F \;\simeq\; \bigoplus_{j=1}^{\nu} \mathbb{Z}/2^{n_j}\mathbb{Z}$$

subject to $1 \leq n_1 \leq ... \leq n_\nu$. Let $\mathfrak{a}_j \;\; (1 \leq j \leq \nu)$ be fixed representatives of the ν generating divisor classes (of degree 0). We let $(\epsilon_i)_{i=1,\dots,r}$ denote the canonical basis of the multiplicative $\mathbb{F}_2$-space $\{\pm 1\}^r$.

Thus any real signed divisor $(\mathfrak{a}, \epsilon)$ in $\widetilde{\mathcal{D}\ell}_F^{res}$ can be uniquely written:

$$(\mathfrak{a}, \epsilon) \;=\; \left(\sum_{j=1}^{\nu} a_j \mathfrak{a}_j + \widetilde{\operatorname{div}}_F(\alpha), \prod_{i=1}^{r} \epsilon_i^{b_i} \; \operatorname{sign}_F^{\infty}(\alpha) \right)$$

with suitable integers $a_j \in \mathbb{Z}$, $b_i \in \{0, 1\}$ and $\alpha \in \mathcal{R}_F$.

Then the $(\mathfrak{a}_j, \mathbf{1})_{j=1,\dots,\nu}$ together with the $(0, \epsilon_i)_{i=1,\dots,r}$ are a finite set of generators of the narrow class group $\widetilde{\mathcal{C}\ell}_F^{res}$. And we just need to detect the relations among those.

From the description of the logarithmic class group $\widetilde{\mathcal{C}\ell}_F$ above we get:

$$2^{n_j} \mathfrak{a}_j \;=\; \widetilde{\operatorname{div}}_F(\alpha_j),$$

with $\alpha_j \in \mathcal{R}_F$ for $j = 1, \dots, \nu$. So we can define coefficients $c_{\nu+i,j}$ in $\{0, 1\}$ by:

$$\operatorname{sign}_F^{\infty}(\alpha_j) \;=\; ((-1)^{c_{\nu+1,j}}, \dots, (-1)^{c_{\nu+r,j}})$$

Consequently, a first set of relations is given by the columns of the following matrix $A \in \mathbb{Z}_2^{(\nu+r)\times\nu}$:

$$A \;=\; \begin{pmatrix} 2^{n_1} & 0 & \cdots & 0 & 0 \\ 0 & 2^{n_2} & \cdots & 0 & 0 \\ .. & . & \cdots & . & . \\ .. & . & \cdots & . & . \\ 0 & 0 & \cdots & 2^{n_{\nu-1}} & 0 \\ 0 & 0 & \cdots & 0 & 2^{n_\nu} \\ -- & -- & --- & -- & -- \\ & & & & \\ & & c_{i,j} & & \\ & & & & \end{pmatrix} .$$

Now, the ν elements α_j are only given up to logarithmic units. Hence, we must additionally consider the sign-function on the 2-group $\widetilde{\mathcal{E}}_F$ of logarithmic units of F (see [8]). More precisely, in case

$$\widetilde{\mathcal{E}}_F = \{\pm 1\} \times < \tilde{\varepsilon}_1, ..., \tilde{\varepsilon}_{r+c} > \quad , \tag{3}$$

we define exponents $b_{i,j}$ via

$$\mathrm{sign}_F^{\infty}(\tilde{\varepsilon}_j) = \prod_{i=1}^{r} \epsilon_i^{b_{i,j}} \tag{4}$$

and we have, of course:

$$\mathrm{sign}_F^{\infty}(-1) = \prod_{i=1}^{r} \epsilon_i \quad .$$

From a computational point of view things are a bit more complicated. We just know that

$$\widetilde{\mathcal{E}}_F = \{x \in \mathcal{R}_F \mid \forall \mathfrak{p} : \tilde{v}_{\mathfrak{p}}(x) = 0\} \tag{5}$$

is a subgroup of the 2-group of 2-units $\mathcal{E}'_F$. If we assume that there are exactly d places $\mathfrak{p}_1, ..., \mathfrak{p}_d$ containing 2 in F, we have, say,

$$\mathcal{E}'_F \;=\; \{\pm 1\} \times \langle \varepsilon_1, ..., \varepsilon_{r+c+d-1} \rangle \quad .$$

In the same way that in [3], for the calculation of $\widetilde{\mathcal{E}}_F$ we fix a precision e for our 2-adic approximations by requiring for elements ε of $\mathcal{E}'_F$ the relation

$$\tilde{v}_{\mathfrak{p}_i}(\varepsilon) \equiv 0 \bmod 2^e \quad (1 \leq i \leq d) \quad .$$

We obtain a system of generators of $\widetilde{\mathcal{E}}_F$ by computing the nullspace of the matrix

$$M \;=\; \begin{pmatrix} & \mid & 2^e & \cdots & 0 \\ \tilde{v}_{\mathfrak{p}_i}(\varepsilon_j) & \mid & \cdot & \cdots & \cdot \\ & \mid & 0 & \cdots & 2^e \end{pmatrix}$$

with $r + c + 2d - 1$ columns and d rows. We assume that the nullspace is generated by the columns of the matrix

$$M' \;=\; \begin{pmatrix} & C & \\ & & \\ - & - & - \\ & & \\ & D & \end{pmatrix}$$

where C has $r+c+d-1$ and D exactly d rows. It suffices to consider C. Each column $(n_1, ..., n_{r+c+d-1})^{tr}$ of the matrix C corresponds to a unit

$$\prod_{i=1}^{r+c+d-1} \varepsilon_i^{n_i} \in \widetilde{\mathcal{E}}_F \mathcal{R}_F^{2^e}$$

so that we can choose

$$\tilde{\varepsilon} := \prod_{i=1}^{r+c+d-1} \varepsilon_i^{n_i}$$

as an approximation for a logarithmic unit. This procedure yields $k \geq r+c$ logarithmic units. Of course, by the generalized Gross conjecture we would have exactly $r+c$ such units.

If the integer k which we get in our calculations is not much larger than $r+c$ then we will proceed with the k generating elements of $\widetilde{\mathcal{E}}_F$ obtained. Otherwise, we reduce the number of generators by computing a basis of the submodule of $\mathbb{Z}^{r+c+d-1}$ which is the span of the columns of C. Hence, from now on we may assume that we have determined exactly $r+c$ generators $\tilde{\varepsilon}_1, ..., \tilde{\varepsilon}_{r+c}$ of $\widetilde{\mathcal{E}}_F$.

To conclude, with the notations of (4) the columns of the following matrix $R \in \mathbb{Z}_2^{(\nu+r)\times(\nu+2r+c+1)}$ generate all relations for the $(\mathfrak{a}_j, 1)$ and the $(0, \epsilon_j)$:

$$R = \left(\begin{array}{ccc|ccc|cccc} 2^{n_1} & & & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ & \ddots & & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ & & 2^{n_\nu} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ -- & -- & --- & -- & -- & -- & -- & -- & -- & -- \\ & & & 2 & & & b_{1,1} & \cdots & b_{1,r+c} & 1 \\ & c_{i,j} & & & \ddots & & \vdots & & \vdots & \vdots \\ & & & & & 2 & b_{\nu,1} & \cdots & b_{\nu,r+c} & 1 \end{array} \right)$$

4 Applications in K-Theory

We adopt the notations and definitions in this section from [12]. In particular i denotes a primitive fourth root of unity; and we say that the number field F is *exceptional* when i is not contained in the cyclotomic $\mathbb{Z}_2$-extension F^c of F, i.e. whenever the cyclotomic extension $F^c[i]/F$ is not procyclic.

We say that a non-complex place $\mathfrak{p}$ of a number field F is *signed* whenever the local field $F_\mathfrak{p}$ does not contain the fourth root of unity i. These are the places which do not decompose in the extension $F[i]/F$. For such a place $\mathfrak{p}$, there exists a non trivial sign-map

$$\mathrm{sign}_\mathfrak{p} \ : \ F_\mathfrak{p}^\times \to \{\pm 1\},$$

given by the Artin reciprocity map $F_\mathfrak{p}^\times \to \mathrm{Gal}(F_\mathfrak{p}[i]/F_\mathfrak{p})$ of class field theory.

We say that a non-complex place $\mathfrak{p}$ of F is *logarithmically signed* if and only if one has $i \notin F_\mathfrak{p}^c$. These are the places which do not decompose in $F^c[i]/F^c$. So the finite set PLS_F of logarithmic signed places of the field F only contains:

(i) the subset PR_F of infinite real places and

(ii) the subset PE_F of exceptional dyadic places, i.e. the set of logarithmic signed places above the prime 2.

We say that a non-complex place $\mathfrak{p}$ of F is *logarithmically primitive* if and only if $\mathfrak{p}$ does not decompose in the first step E/F of the cyclotomic $\mathbb{Z}_2$-extension F^c/F. Finally we say that an exceptional number field F is *primitive* whenever there exists an exceptional dyadic place which is logarithmically primitive.

Naturally, the task arises to determine logarithmically signed places, i.e. those non complex places of F for which i is not contained in $F^c_{\mathfrak{p}}$:

Proposition 2. *Let $E_{\mathfrak{p}}$ be the first quadratic extension of $F_{\mathfrak{p}}$ in the tower of field extension from $F_{\mathfrak{p}}$ to $F^c_{\mathfrak{p}}$. Then $i \in F^c_{\mathfrak{p}}$ holds precisely for $i \in E_{\mathfrak{p}}$.*

Proof. Since $F^c_{\mathfrak{p}}/F_{\mathfrak{p}}$ is a $\mathbb{Z}_2$-extension, it contains exactly one quadratic extension $E_{\mathfrak{p}}$ of $F_{\mathfrak{p}}$. So we immediately obtain:

$$i \in F^c_{\mathfrak{p}} \;\Leftrightarrow\; F_{\mathfrak{p}}(i) \subseteq E_{\mathfrak{p}} \;\Leftrightarrow\; i \in E_{\mathfrak{p}} \quad .$$

Remark. The extension $E_{\mathfrak{p}}$ is $F_{\mathfrak{p}}(\alpha_k)$ where k is the smallest integer such that α_k does not belong to $F_{\mathfrak{p}}$ with $\alpha_0 = 0$ and $\alpha_{k+1} = \sqrt{2+\alpha_k}$.

We assume in the following that the number field F has no exceptional dyadic place. Let us introduce the group $\mathcal{C}\ell^{res}_F$ of narrow logarithmic classes without any assumption of degree:

$$\mathcal{C}\ell^{res}_F = \mathcal{D}\ell^{res}_F / \widetilde{\mathcal{P}\ell}^{res}_F .$$

Via the degree map, we obtain the direct decomposition:

$$\mathcal{C}\ell^{res}_F \;\simeq\; \mathbb{Z}_2 \oplus \widetilde{\mathcal{C}\ell}^{res}_F ,$$

where the torsion subgroup $\widetilde{\mathcal{C}\ell}^{res}_F$ was yet computed in the previous section. So the quotient of exponent 2

$$^2\mathcal{C}\ell^{res}_F := \mathcal{C}\ell^{res}_F / (\mathcal{C}\ell^{res}_F)^2$$

contains both as hyperplanes the two quotients $^2\widetilde{\mathcal{C}\ell}^{res}_F$ relative to $\widetilde{\mathcal{C}\ell}^{res}_F$ and $^2\mathcal{C}\ell^{pos}_F$ relative to the positive class group $\mathcal{C}\ell^{pos}_F$ introduced in [12].

Since, according to [12], this later gives the 2-rank of the wild kernel $WK_2(F)$. we can extend the results of K. Hutchinson [6, 7] as follows:

Theorem 3. *Let F be a number field which has no exceptional dyadic places.*

(i) If F is not exceptional (i.e. in case $i \in F^c$) the wild kernel $WK_2(F)$ coincide with the subgroup $K_2^\infty(F) = \cap_{n\geq 1} K_2^n(F)$ of infinite height elements in $K_2(F)$; the group $\widetilde{\mathcal{C}\ell}^{res}_F$ of narrow logarithmic classes coincide with the group $\widetilde{\mathcal{C}\ell}_F$ of (ordinary) logarithmic classes; and one has immediately:

$$\mathrm{rk}_2\, WK_2(F) \;=\; \mathrm{rk}_2\, K_2^\infty(F) \;=\; \mathrm{rk}_2\, \widetilde{\mathcal{C}\ell}^{res}_F \;=\; \mathrm{rk}_2\, \widetilde{\mathcal{C}\ell}_F$$

(ii) If F is exceptional (i.e. in case $i \notin F^c$) the subgroup $K_2^\infty(F)$ has index 2 in the wild kernel $WK_2(F)$ and one still has:

$$\mathrm{rk}_2\, WK_2(F) \;=\; \mathrm{rk}_2\, \widetilde{\mathcal{C}\ell}^{res}_F \;\geq 1$$

(ii,a) In case $WK_2(F)$ and $K_2^\infty(F)$ have the same 2-rank, this gives:

$$\mathrm{rk}_2\, K_2^\infty(F) \;=\; \mathrm{rk}_2\, \widetilde{\mathcal{C}\ell}^{res}_F \geq 1.$$

(ii,b) And in case $K_2^\infty(F)$ is a direct summand in $WK_2(F)$, one has:

$$\mathrm{rk}_2\, K_2^\infty(F) \;=\; \mathrm{rk}_2\, \widetilde{\mathcal{C}\ell}^{res}_F - 1.$$

Proof. In the non exceptional case, the number field F is not locally exceptional, i.e. has no logarithmic signed places: $PE_F = PR_F = \emptyset$. In particular, narrow logarithmic classes coincide with ordinariry logarithmic classes and the result follows from [12].

In the exceptional case, the number field F may have real places, so the narrow logarithmic class group $\widetilde{\mathcal{C}\ell}_F^{res}$ may differ from the ordinary logarithmic class group. Moreover, because the assumption $PE_F = \emptyset$ and accordingly to Hutchinson [6, 7], the subgroup $K_2^\infty(F)$ has index 2 in the wild kernel $WK_2(F)$.

Remark. It remains to determine whether a number field F is not exceptional, *i.e.* whether the cyclotomic $\mathbb{Z}_2$ extension F^c contains the fourth root of unity i. Of course if $i \in F^c$ then i is contained in the quadratic subfield E/F in F^c. Now the finite subfields of $\mathbb{Q}^c$ are the real cyclic fields $\mathbb{Q}^{(s)} = \mathbb{Q}[\zeta_{2^{s+2}} + \zeta_{2^{s+2}}^{-1}]$ and the finite extensions of F^c are of the form $F\mathbb{Q}^{(s)}$. So we only need to check whether i is contained in $F[\zeta_{2^{s+2}} + \zeta_{2^{s+2}}^{-1}]$ where s is minimal with $\zeta_{2^{s+2}} + \zeta_{2^{s+2}}^{-1} \notin F$.

5 Examples

The methods described here are implemented in the computer algebra system Magma [2]. Many of the fields used in the examples were results of queries to the QaoS number field database [13, section 6].

The wild kernel $WK_2(F)$ is contained in the tame kernel $K_2(\mathcal{O}_F)$. Let $\mu(F)$ be the order of the torsion subgroup of $F^\times$ and for a prime $\mathfrak{p}$ of F over p denote by $\mu^1(F_\mathfrak{p})$ the p-Sylow subgroup of the torsion subgroup of $F_\mathfrak{p}^\times$. By coupling Moore's exact sequence and the localization sequence [5, section 1] one obtains the index formula [1, equation (6)]:

$$(K_2(\mathcal{O}_F) : WK_2(F)) = \frac{2^r}{|\mu(F)|} \prod_\mathfrak{p} |\mu^1(F_\mathfrak{p})|,$$

where $\mathfrak{p}$ runs through all finite places and r is the number of real places of F.

We apply this in the determination of the structure of $WK_2(F)$ in the cases where the structure of $K_2(\mathcal{O}_F)$ is known.

Abelian groups are given as a list of the orders of their cyclic factors;

[:] denotes the index $(K_2(O_F) : WK_2(F))$;

d_F denotes the discriminant for a number field F;

$\mathcal{C}\ell_F$ denotes the class group, P the set of dyadic places;

$\mathcal{C}\ell'_F$ denotes the 2-part of $Cl/\langle P\rangle$;

$\widetilde{\mathcal{C}\ell}_F$ denotes the logarithmic classgroup;

$\mathcal{C}\ell_F^{res}$ denotes the group of narrow logarithmic classes;

rk_2 denotes the 2-rank of the wild kernel WK_2;

WK_2 denotes the wild kernel in $K_2(F)$;

K_2^∞ denotes the subgroup of infinite height elements in $K_2(F)$.

5.1 Imaginary quadratic fields

K. Belabas and H. Gangl have developed an algorithm for the computation of the tame kernel $K_2\mathcal{O}_F$ [1]. The following table contains the structure of $K_2\mathcal{O}_F$ as computed by Belabas and Gangl and the 2-rank of the wild kernel $WK_2(F)$ calculated with our methods. We also give the structure of the wild kernel if it can be deduced from the structure of $K_2\mathcal{O}_F$ and of the rank of the wild kernel computed here or in [14]. The structure of the tame kernel $K_2(\mathcal{O}_F)$ of all fields except for the starred entries has been proven by Belabas and Gangl.

The table gives the structure of the wild kernel of all imaginary quadratic fields F with no exceptional places and discriminant $|d_F| < 1000$.

d_F	$\mathcal{C}\ell_F$	$K_2(O_F)$	[:]	$\mathcal{C}\ell'_F$	$\widetilde{\mathcal{C}\ell}_F$	$\widetilde{\mathcal{C}\ell}_F^{res}$	rk_2	WK_2	K_2^∞
-68	[4]	[8]	2	[2]	[2]	[2]	1	[4]	[2]
-132	[2,2]	[4]	2	[2]	[2]	[2]	1	[2]	[ ]
-136	[4]	[4]	1	[2]	[2]	[2]	1	[4]	[2]
-164	[8]	[4]	2	[4]	[4]	[4]	1	[2]	[ ]
-228	[2,2]	[12]	6	[2]	[2]	[2]	1	[2]	[ ]
-260	[2,4]	[4]	2	[4]	[4]	[4]	1	[2]	[ ]
-264	[2,4]	[6]	3	[4]	[4]	[4]	1	[2]	[ ]
-292	[4]	[4]	2	[2]	[2]	[2]	1	[2]	[ ]
-328	[4]	[2]	1	[2]	[2]	[2]	1	[2]	[ ]
-356	[12]	[4]	2	[2]	[2]	[2]	1	[2]	[ ]
-388	[4]	[8]	2	[2]	[2]	[2]	1	[4]	[2]
-420	[2,2,2]	[2,4]	2	[2,2]	[2,2]	[2,2]	2	[2,2]	[2]
-452	[8]	[8]	2	[4]	[4]	[4]	1	[4]	[2]
-456	[2,4]	[2]	1	[4]	[4]	[4]	1	[2]	[ ]
-516	[2,6]	[12]	6	[2]	[2]	[2]	1	[2]	[ ]
-520	[2,2]	[2]	1	[2]	[2]	[2]	1	[2]	[ ]
-548	[8]	[4]	2	[4]	[4]	[4]	1	[2]	[ ]
-580	[2,4]	[4]	2	[4]	[4]	[4]	1	[2]	[ ]
-584	[16]	[2]	1	[8]	[8]	[8]	1	[2]	[ ]
-644	[2,8]	[2,16]	2	[2,4]	[2,4]	[2,4]	2	[2,8]	[?]
-708	[2,2]	[4]	2	[2]	[2]	[2]	1	[2]	[ ]
-712	[8]	[2]	1	[4]	[4]	[4]	1	[2]	[ ]
-740	[2,8]	[4]	2	[8]	[8]	[8]	1	[2]	[ ]
-772	[4]	[8]	2	[2]	[2]	[2]	1	[4]	[2]
-776	[20]	[4]	1	[2]	[2]	[2]	1	[4]	[2]
-804	[2,6]	[36]	6	[2]	[2]	[2]	1	[6]	[3]
-836	[2,10]	[4]	2	[2]	[2]	[2]	1	[2]	[ ]
-840	[2,2,2]	[2,6]	3	[2,2]	[2,2]	[2,2]	2	[2,2]	[2]
-868	[2,4]	[2,4]	2	[2,2]	[2,2]	[2,2]	2	[2,2]	[2]
-904	[8]	[4]	1	[4]	[4]	[4]	2	[4]	[2]
-964	[12]	[8]	2	[2]	[2]	[2]	1	[4]	[2]
-996	[2,6]	[4]	2	[2]	[2]	[2]	1	[2]	[ ]

5.2 Real Quadratic Fields

The following table contains all real quadratic fields F with no exceptional places and discriminant $|d_F| < 1000$. All these quadratic fields are exceptional.

d_F	$\mathcal{C}\ell_F$	[:]	$\|P\|$	$\|PE\|$	$\mathcal{C}\ell'_F$	$\widetilde{\mathcal{C}\ell}_F$	$\widetilde{\mathcal{C}\ell}_F^{res}$	rk_2
28	[ ]	8	1	0	[ ]	[ ]	[2]	1
56	[ ]	4	1	0	[ ]	[ ]	[2]	1
60	[2]	24	1	0	[ ]	[ ]	[2]	1
92	[ ]	8	1	0	[ ]	[ ]	[2]	1
120	[2]	4	1	0	[ ]	[ ]	[2]	1
124	[ ]	8	1	0	[ ]	[ ]	[2]	1
156	[2]	8	1	0	[ ]	[ ]	[2]	1
184	[ ]	4	1	0	[ ]	[ ]	[2]	1
188	[ ]	8	1	0	[ ]	[ ]	[2]	1
220	[2]	8	1	0	[ ]	[ ]	[2]	1
248	[ ]	4	1	0	[ ]	[ ]	[2]	1
284	[ ]	8	1	0	[ ]	[ ]	[2]	1
312	[2]	12	1	0	[ ]	[ ]	[2]	1
316	[3]	8	1	0	[ ]	[ ]	[2]	1
348	[2]	24	1	0	[ ]	[ ]	[2]	1
376	[ ]	4	1	0	[ ]	[ ]	[2]	1
380	[2]	8	1	0	[ ]	[ ]	[2]	1
412	[ ]	8	1	0	[ ]	[ ]	[2]	1
440	[2]	4	1	0	[ ]	[ ]	[2]	1
444	[2]	8	1	0	[ ]	[ ]	[2]	1
476	[2]	8	1	0	[2]	[2]	[2, 2]	2
604	[ ]	8	1	0	[ ]	[ ]	[2]	1
632	[ ]	4	1	0	[ ]	[ ]	[2]	1
636	[2]	24	1	0	[ ]	[ ]	[2]	1
668	[ ]	8	1	0	[ ]	[ ]	[2]	1
696	[2]	4	1	0	[ ]	[ ]	[2]	1
732	[2]	8	1	0	[ ]	[ ]	[2]	1
760	[2]	4	1	0	[ ]	[ ]	[2]	1
764	[ ]	8	1	0	[ ]	[ ]	[2]	1
796	[ ]	8	1	0	[ ]	[ ]	[2]	1
824	[ ]	4	1	0	[ ]	[ ]	[2]	1
860	[2]	8	1	0	[ ]	[ ]	[2]	1
888	[2]	12	1	0	[ ]	[ ]	[2]	1
892	[3]	8	1	0	[ ]	[ ]	[2]	1
924	[2, 2]	24	1	0	[2]	[2]	[2, 2]	2
952	[2]	4	1	0	[2]	[2]	[2, 2]	2
956	[ ]	8	1	0	[ ]	[ ]	[2]	1
988	[2]	8	1	0	[ ]	[ ]	[2]	1

The following table contains extensions with class number 32 up to discriminant 222780 and extensions with class number 64 up tp discriminant 805596.

d_F	$\mathcal{C}\ell_F$	[:]	$\lvert P\rvert$	$\lvert PE\rvert$	$\mathcal{C}\ell'_F$	$\widetilde{\mathcal{C}\ell}_F$	$\widetilde{\mathcal{C}\ell}_F^{res}$	rk_2
112924	[2,16]	8	1	0	[16]	[16]	[2,16]	2
120796	[2,16]	8	1	0	[16]	[16]	[2,16]	2
136120	[2,16]	4	1	0	[16]	[16]	[2,16]	2
153660	[2,2,8]	8	1	0	[2,8]	[2,8]	[2,2,8]	3
158844	[2,2,8]	8	1	0	[2,8]	[2,8]	[2,2,8]	3
163576	[2,16]	4	1	0	[2,8]	[2,8]	[2,2,8]	3
170872	[2,16]	4	1	0	[16]	[16]	[2,16]	2
176316	[2,16]	24	1	0	[16]	[16]	[2,16]	2
176440	[2,16]	4	1	0	[16]	[16]	[2,16]	2
196540	[2,16]	8	1	0	[16]	[16]	[2,16]	2
202524	[2,16]	24	1	0	[16]	[16]	[2,16]	2
207480	[2,2,2,4]	4	1	0	[2,2,4]	[2,2,4]	[2,2,2,4]	4
213180	[2,2,2,4]	24	1	0	[2,2,4]	[2,2,4]	[2,2,2,4]	4
221276	[2,16]	8	1	0	[16]	[16]	[2,16]	2
222780	[2,2,8]	8	1	0	[2,8]	[2,8]	[2,2,8]	3
374136	[2,2,16]	12	1	0	[2,16]	[2,16]	[2,2,16]	3
382204	[2,32]	8	1	0	[32]	[32]	[2,32]	2
449436	[2,2,16]	8	1	0	[2,16]	[2,16]	[2,2,16]	3
484764	[2,2,16]	24	1	0	[2,16]	[2,16]	[2,2,16]	3
506940	[2,2,2,8]	24	1	0	[2,2,8]	[2,2,8]	[2,2,2,8]	4
805596	[2,2,16]	24	1	0	[2,16]	[2,16]	[2,2,16]	3

5.3 Biquadratic Extensions

The following table contains quadratic and biquadratic number fields. The biquadratic fields are the compositum of the the first quadratic extensions and one of the other quadratic extensions. All fields are exceptional.

F	d_F	r	$\mathcal{C}\ell_F$	[:]	$\lvert P\rvert$	$\mathcal{C}\ell'_F$	$\widetilde{\mathcal{C}\ell}_F$	$\widetilde{\mathcal{C}\ell}_F^{res}$	rk_2
K	9660	2	[2,2,2]	8	1	[2,2]	[2,2]	[2,2,2]	3
L_1	9340	2	[10]	8	1	[ ]	[ ]	[2]	1
KL_1		4	[2,2,2,10]	128	2	[2,2,2]	[2,2,2]	[2,2,2,2,2]	5
L_2	13020	2	[2,2,2]	24	1	[2,2]	[2,2]	[2,2,2]	3
KL_2		4	[2,2,4]	384	2	[2,2]	[2,2,2]	[2,2,2,2,2,2]	6
L_3	15708	2	[2,2,2]	8	1	[2,2]	[2,2]	[2,2,2]	3
KL_3		4	[2,2,4,28]	128	2	[2,2,4]	[2,2,4]	[2,2,2,2,2,4]	6

5.4 Examples of higher degrees

f	d_F	r	Gal	$\mathcal{C}\ell_F$	[:]	$\|P\|$	$\mathcal{C}\ell'_F$	$\widetilde{\mathcal{C}\ell}_F$	$\widetilde{\mathcal{C}\ell}_F^{res}$	rk_2
x^4+x^2-6x+1	-3312	2	D(4)	[ ]	8	1	[ ]	[ ]	[2]	1
$x^4-2x^3-2x^2+18x+21$	3600	0	E(4)	[2]	6	1	[2]	[2]	[2]	1
x^4+25x^2+400	608400	0	E(4)	[4,8]	72	2	[4]	[4]	[4]	1
$x^4+56x^2-32x+713$	700672	0	D(4)	[2,2,4]	4	1	[2,4]	[2,4]	[2,4]	2
$x^4+3x^2-30x+66$	723600	0	D(4)	[4,8]	24	2	[2]	[2]	[2]	1
$x^4-2x^3+29x^2-28x+417$	781456	0	E(4)	[4,8]	2	1	[2,8]	[2,8]	[2,8]	2
$x^4+10x^2-28x+18$	815360	0	D(4)	[4,8]	4	1	[2,4]	[4]	[4]	1
$x^4+12x^2-40x+81$	825600	0	D(4)	[4,8]	1	1	[8]	[8]	[8]	1
$x^6+2x^5-4x^4-16x^3+6x^2+44x+308$	-6832605533873152	0	S(6)	[2,2]	8	2	[2]	[2]	[2]	1
$x^6-2x^4+10x^2+12x+260$	-3797563908766976	0	S(6)	[2,2]	8	2	[2]	[2]	[2]	1
$x^6+2x^5+4x^4-2x^2-4x+260$	-382132112360448	0	S(6)	[4]	48	2	[4]	[4]	[4]	1
$x^6-26x^4-16x^3+90x^2-52x+68$	-212547578875136	4	S(6)	[2]	128	2	[ ]	[ ]	[2,2]	2
$x^6-7x^4+14x^2-7$	1075648	6	C(6)	[ ]	128	1	[ ]	[ ]	[2]	1
$x^8+4x^7-8x^6-42x^5+11x^4$ $+130x^3+15x^2-106x+11$	8090338299904	8	[A(4)^2]2	[ ]	512	2	[ ]	[ ]	[2,2]	2
$x^8-2x^7-27x^6+62x^5+185x^4$ $-520x^3-40x^2+832x-496$	9082363580416	8	E(8)	[ ]	512	2	[ ]	[ ]	[2]	2
$x^8-4x^7-20x^6+30x^5+105x^4$ $-30x^3-168x^2-78x-3$	9299377062144	8	[A(4)^2]2	[ ]	2048	2	[ ]	[ ]	[2]	1
$x^8+2x^7-22x^6-8x^5+159x^4$ $-160x^3-110x^2+186x-47$	9451049953536	8	[A(4)^2]2	[ ]	2048	2	[ ]	[ ]	[2,2]	2

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Jean-François Jaulent
Université Bordeaux I
Institut de Mathématiques
351, Cours de la Libération
33405 Talence Cedex, France
jaulent@math.u-bordeaux1.fr

Sebastian Pauli
University of North Carolina
Department of Mathematics and Statistics
Greensboro, NC 27402, USA
s_pauli@uncg.edu

Michael E. Pohst
Technische Universität Berlin
Institut für Mathematik MA 8-1
Straße des 17. Juni 136
10623 Berlin, Germany
pohst@math.tu-berlin.de

Florence Soriano-Gafiuk
Université de Metz
Département de Mathématiques
Ile du Saulcy
57000 Metz, France
soriano@poncelet.univ-metz.fr