Principalization of ideals in Abelian extensions of number fields

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With an Appendix by Georges GRAS [1] and Jean-François JAULENT [2]

Abstract

We give a self-contained proof of a general conjecture of G. Gras on principalization of ideals in Abelian extensions of a given field L, which was solved by M. Kurihara in the case of totally real extensions L of the rational field $\mathbb{Q}$.

More precisely, for any given extension L/K of number fields, in which at least one infinite place of K totally splits, and for any ideal class c_L of L, we construct a finite Abelian extension F/K, in which all infinite places totally split, such that c_L become principal in the compositum $M = LF$.

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A – INTRODUCTION

When M/L is an Abelian extension of number fields, the problem of knowing which ideals of L become principal in M is difficult, even if M/L is cyclic. Class field theory gives partial answers. For instance, the Artin-Furtwängler theorem states that when $M = H_L$ is the Hilbert class field of L, all ideals of L become principal in M; but the other cases are more mysterious.

When M/L is cyclic, we still have no general answer but the problem is easier. The kernel of the natural map $j : \mathrm{Cl}_L \to \mathrm{Cl}_M$ is partly known as explained below:

At first, cohomology of cyclic groups says that this kernel is a part of $\hat{H}^1(\mathcal{G}, E_M)$, where $\mathcal{G} = \mathrm{Gal}(M/L)$ and E_M is the group of units of M; $\hat{H}^1(\mathcal{G}, E_M)$ itself is not well known but its order can be deduced from the order of $\hat{H}^0(\mathcal{G}, E_M)$ using the Herbrand quotient; then, the order of $\hat{H}^0(\mathcal{G}, E_M)$ depends partly on the natural map $E_L \to U_S$, where U_S is the subgroup of the unit idèles of L which is the product of the groups of local units at the places ramified in M/L (S is the set of such ramified places); finally, the map $E_L \to U_S$ depends on the Frobenius' of the primes of S in the extension $L[\mu_h, E_L^{1/h}]/L$ (where $h = [M : L]$ and where μ_h is the group of hth roots of unity). Obviously, this makes sense only if the primes of S are prime to the degree of the extension.

However, even in this cyclic case, $\mathrm{Ker}(j)$ is not completely given by these Frobenius', so that knowing such Frobenius', we cannot get really more than a lower bound of the order of $\mathrm{Ker}(j)$.

This article deals with such techniques, which allow us to prove, for instance, this easy fact: if a cyclic extension M/L is ramified at only one finite place $\mathfrak{q}$ prime to $[M : L]$, then, as soon as the ramification index $e_{\mathfrak{q}}$ is large enough (precisely, when $|\mathrm{Cl}_L|$ divides $e_{\mathfrak{q}}$), the capitulation kernel $\mathrm{Ker}\,j$ contains at least the class of $\mathfrak{q}$ in Cl_L. This can be easily established by studying $\hat{H}^1(\mathcal{G}, E_M)$ and may be seen as a particular case of our main theorem.

On the other hand, this article proves a conjecture of Georges Gras and generalizes a result of Masato Kurihara ([1] and [2]), so that the theorem given here is not only an abstract lower bound of $\mathrm{Ker}\,j$, using cohomology of cyclic groups, Dirichlet–Herbrand theorem on units of number fields, class field theory, and Kummer duality. Note also that we make use here of new asymptotic methods ("take n large enough", where n is related to the degree $[F : K]$ in a suitable manner).

Note finally that in the theorem below the hypothesis "at least one infinite place of K totally splits in L/K" is necessary: in [1], Georges Gras gives

examples of extensions L/K with ideals which do not become principal in the compositum LK^{ab}, where K^{ab} is the maximal Abelian extension of K.

B – MAIN THEOREM AND COROLLARIES

Main Theorem. *Let L/K be a finite extension of number fields in which at least one infinite place of K totally splits. There exists a finite Abelian extension F/K, in which all infinite places split totally, such that every ideal of L becomes principal in the compositum $M = LF$.*

Corollary 1 ($K = \mathbb{Q}$). *If L is a totally real number field, any ideal of L principalizes in a real cyclotomic extension of L (i.e., in the compositum of L with a real subfield of a suitable cyclotomic extension of $\mathbb{Q}$).*

Corollary 1 was proved by Masato Kurihara in [2].

Notation: In what follows, K^{ab} denotes the maximal Abelian extension of K and K_+^{ab} its maximal totally real subextension (i.e., the subextension of K^{ab}/K fixed under the decomposition groups of the infinite places of K).

Definition: Let us say that a number field N is *principal* when Cl_N is trivial.

Then:

Corollary 2. *(i) Let K be a number field with at least one complex place. Then any field containing K^{ab} is principal.*
(ii) Let K be a totally real number field. Any field containing K_+^{ab}, the Galois closure of which has at least one real place is principal.

Corollary 3. *(i) Any totally real field containing $\mathbb{Q}_+^{ab}$ is principal.*
(ii) Any field containing $\mathbb{Q}(i)^{ab}$ is principal.

Corollary 3, (i), and (ii) for $\mathbb{Q}(i)$ or any imaginary quadratic field, was proved by Masato Kurihara ([2], theorem 1.1, p. 35 and theorem A.1, p. 46).

Corollary 4. *(i) Let K be a number field with at least one complex place. Then K^{ab} is principal.*
(ii) Let K be a totally real number field. Any field containing K_+^{ab} which is contained in one of the subfields of K^{ab} fixed by a complex conjugation is principal.

Corollary 4 proves a conjecture of Georges Gras (Conjecture (0.5), p. 405 of [1]).

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I. Proof of the main theorem : preliminaries

To prove the theorem, we fix an ideal $\mathfrak{a}_L$ of L, and we shall construct a finite Abelian extension F of K, which is totally split at all infinite places, cyclic in most cases, such that $\mathfrak{a}_L$ principalizes in the compositum LF. Obviously, any ideal $\mathfrak{b}_L$ of L with the same class in Cl_L will become principal in LF as well, so that it is enough to fix the class c_L of $\mathfrak{a}_L$ in Cl_L. Now, if $(c_i)_i$ is a finite generating system of Cl_L, we will obtain a corresponding set $(F_i)_i$ of extensions, and every ideal of L will principalize in LF, where F is the compositum of the $(F_i)_i$; this will prove the theorem.

So, in what follows, we fix c_L in Cl_L, and must find F. As the class group of L is the direct sum of its p-parts, for all prime numbers p, we may assume that the order of c_L in Cl_L is a power of a prime p. So, c_L and p are fixed; Cl_L is now the p-part of the class group of K, and H_L the maximal p-extension contained in the Hilbert class field of L.

(1) One may assume $c_L \in \mathrm{Cl}_L^{p^a}$ for an arbitrary given integer a

Definition: Let us call *Abelian compositum* of the extension L/K any extension $N = LF$, where F is a finite Abelian extension of K, totally split at all infinite places of K.

One fixes an integer a; in case $c_L \notin \mathrm{Cl}_L^{p^a}$, one will build an Abelian compositum L' of L/K, such that the extended class $c_{L'} = j(c_L)$ satisfies $c_{L'} \in \mathrm{Cl}_{L'}^{p^a}$; so, if N' is an Abelian compositum of L'/K in which $c_{L'}$ principalizes, it is as well an Abelian compositum of L/K, so that one can legitimately replace L by L', in which case one has $c_{L'} \in \mathrm{Cl}_{L'}^{p^a}$.

Let's build such a $L' = LF_0$ as follows: let $\mathfrak{q}$ be a prime of L satisfying the following three conditions:

(i) $\mathfrak{q}$ totally splits in $L/\mathbb{Q}$;

(ii) $\mathrm{Frob}(\mathfrak{q}, L[\mu_{2p^a}]/L) = \mathrm{id}$;

(iii) $\mathrm{Frob}(\mathfrak{q}, H_L/L) = c_L$.

If such a $\mathfrak{q}$ exists with say $\mathfrak{q}|q$, the first two conditions imply the existence of a (cyclic) subfield F_0' of $\mathbb{Q}(\mu_q)$, with degree $[F_0' : \mathbb{Q}] = p^a$, which is totally ramified at the prime q; the first condition implies that the compositums $F_0 = KF_0'$ and $L' = LF_0$ have again a degree p^a over K and L, respectively. Now L'/L is totally ramified at $\mathfrak{q}$, say $\mathfrak{q} = \mathfrak{q}'^{p^a}$, for a prime $\mathfrak{q}'$ in L'; so,

according to the third condition, the extended class $c_{L'} = j(c_L)$ satisfies:

$$c_{L'} = \overline{\mathfrak{q}} = \overline{\mathfrak{q}'}^{\,p^a} \in \mathrm{Cl}_{L'}^{p^a}$$

as expected.

Now we only have to verify the existence of such a prime $\mathfrak{q}$. The three conditions defining $\mathfrak{q}$ all depend on $\mathrm{Frob}(q, \widetilde{H}_L[\mu_{2p^a}]/\mathbb{Q})$, where $\widetilde{H}_L$ is the Galois closure of H_L over $\mathbb{Q}$; the first two conditions are equivalent to the fact that this Frobenius is in the subgroup $\mathrm{Gal}(\widetilde{H}_L[\mu_{2p^a}]/L[\mu_{2p^a}])$; so they are compatible with the last condition for any c_L in Cl_L, if and only if one has: $L[\mu_{2p^a}] \cap H_L = L$; if this is right, the Čebotarev theorem states there are infinitely many $\mathfrak{q}$ satisfying the conditions. When this is wrong, we replace L by $L'' = L[\mu_{2p^a}] \cap H_L$ which verifies $L''[\mu_{2p^a}] \cap H_{L''} = L''$.

As explained above, this last change is possible in case $L'' = L[\mu_{2p^a}] \cap H_L$ is an Abelian compositum of L/K. In fact, one has $L'' = LF$ where F is contained in the maximal p-subfield of $K[\mu_{2p^a}]$, which is clearly Abelian and finite but in which infinite places are maybe not totally split for $p = 2$.

So, for $p = 2$, we shall complete the proof, and we take in this particular case $L'' = LK[\mu_{2^b}]_+$, where the symbol $+$ denotes the maximal ∞-split subextension over K, and where b is an integer, which is chosen large enough so that L'' contains $H_L \cap LK[\mu_{2^{a+1}}]_+$ and L''/L has degree at least 2.

Suppose we have found a prime $\mathfrak{q}''$ of L'' satisfying the following conditions:

(i) $\mathfrak{q}''$ totally splits in $L''/\mathbb{Q}$;

(ii) $\mathrm{Frob}(\mathfrak{q}'', L''[\mu_{2^{a+1}}]/L'') = \mathrm{id}$;

(iii) $\mathrm{Frob}(\mathfrak{q}'', H_{L''}/L'') = c_{L''}$,

where $c_{L''}$ is extended from c_L. Then, let F_0' be the totally real subfield of $\mathbb{Q}[\mu_q]$ of degree 2^a over $\mathbb{Q}$, thus $F_0 = K[\mu_{2^b}]_+ F_0'$ and $L''' = LF_0 = L'' F_0'$ (which is an Abelian compositum of L/K). If $\mathfrak{q}'''$ denotes the unique prime of L''' above $\mathfrak{q}''$, the extended class $c_{L'''}'''$ in $\mathrm{Cl}_{L'''}$ satisfies the expected condition:

$$c_{L'''} = \overline{\mathfrak{q}''} = \overline{\mathfrak{q}'''}^{\,2^a} \in \mathrm{Cl}_{L'''}^{2^a}.$$

So to conclude we only have to prove the existence of such a prime $\mathfrak{q}''$ satisfying the three conditions above. But this existence follows from the Čebotarev theorem as soon as the image of $c_L \in \mathrm{Gal}(H_{L''}/L'')$ in $\mathrm{Gal}(H_{L''} \cap L''[\mu_{2^{a+1}}]/L'')$ is trivial.

To check this last point, let us observe that in the class field description the extension of ideal classes j corresponds to the transfer map Ver. Here L'' contains $H_L \cap LK[\mu_{2^{a+1}}]_+$, so $H_{L''} \cap L''[\mu_{2^{a+1}}] = L'''$ is either L'' or $L''[i]$, and

the image of $j_{L''/L}(c_L)$ in $\mathrm{Gal}(L'''/L'')$ is trivial, since one has:

$$\mathrm{Ver}_{B/A \to B'/A'}(\sigma) = \sigma^{[A':A]}$$

when B'/A is Abelian, so:

$$\mathrm{Res}_{L'''}(\mathrm{Ver}_{H_L/L \to H_{L''}/L''}(c_L)) = \mathrm{Ver}_{H_L/L \to L'''/L''}(c_L) = c_L^{[L'':L]} = \mathrm{id}.$$

(2) One may assume that L/K is Galois

Let $\tilde{L}$ denotes the Galois closure of L over K. Assume the theorem is proved for $\tilde{L}/K$ (in which at least one infinite place totally splits as in L/K). Hence, there exists an Abelian compositum $\tilde{L}F$ of $\tilde{L}/K$ such that every ideal of $\tilde{L}$ principalize in $\tilde{L}F$; so, c_L principalizes in $\tilde{L}F$ but maybe does not in LF and we have to study this case.

(a) When c_L is norm in $\tilde{L}/L$, say $c_L = \mathrm{N}_{\tilde{L}/L}(\tilde{c}_{\tilde{L}})$, the class $\tilde{c}_{\tilde{L}} \in \mathrm{Cl}_{\tilde{L}}$ principalizes in $\tilde{L}F$, say $j_{\tilde{L}F/\tilde{L}}(\tilde{c}_{\tilde{L}}) = 1$ in $\mathrm{Cl}_{\tilde{L}F}$; so we obtain: $c_L = \mathrm{N}_{\tilde{L} \cap LF/L}(c'_{\tilde{L} \cap LF})$ with $c'_{\tilde{L} \cap LF} = (\mathrm{N}_{\tilde{L}/(\tilde{L} \cap LF)}(\tilde{c}_{\tilde{L}}))$ and the class $c'_{\tilde{L} \cap LF}$, which satisfies $j_{LF/\tilde{L} \cap LF} \circ \mathrm{N}_{\tilde{L}/(\tilde{L} \cap LF)}(\tilde{c}_{\tilde{L}}) = \mathrm{N}_{\tilde{L}F/LF} \circ j_{\tilde{L}F/\tilde{L}}(\tilde{c}_{\tilde{L}}) = 1$, principalizes in FL; and so does c_L.

(b) When c_L is not a norm in $\tilde{L}/L$, maybe c_L does not principalize in LF. But $\mathrm{N}_{\tilde{L}/L}(\mathrm{Cl}_{\tilde{L}})$ contains $\mathrm{Cl}_L^{[\tilde{L}:L]} = \mathrm{Cl}_L^{p^a}$, where p^a is the largest power of p dividing $[\tilde{L} : L]$. According to Section (1) we replace L by $L' = LF_0$ such that $c_L \in \mathrm{Cl}_{L'}^{p^a}$. Since $[\tilde{L}' : L'] = [\tilde{L}F_0 : LF_0]$ divides $[\tilde{L} : L]$, c_L is norm in $\tilde{L}'/L'$ and (a) applies.

II. Proof of the theorem : constructing the extension F

(3) The method and a first condition about the prime $\mathfrak{q}$

From now on we suppose L/K is Galois. The prime p and the class c_L are fixed and we must build an Abelian compositum LF of L/K such that c_L principalizes in LF. For convenience, we choose F/K as a cyclic p-extension, ramified at only one finite place $\mathfrak{q}$ of K, with ramification index $e_{\mathfrak{q}}(F/K) = p^n$ for some integer n. We will see that under some conditions about $\mathfrak{q}$, when n is large enough, c_L becomes principal in LF, or in $L'F$, where L' is a convenient Abelian compositum of L/K. At the end of the proof, in Section (6), we will study the existence of such a $\mathfrak{q}$ satisfying all conditions.

Now for a given integer n and a given prime $\mathfrak{q}$ of K, we wonder if c_L principalizes in $M = LF$, where F is a cyclic p-extension of K with $e_{\mathfrak{q}}(F/K) = p^n$, unramified but at $\mathfrak{q}$ and in which all infinite places split totally. The first question is the existence of such an extension F/K. Class field theory gives the

answer as follows.

Indeed, N being the maximal Abelian extension of K unramified but at $\mathfrak{q}$ and ∞-split, class field theory describes the Galois group $\mathrm{Gal}(N/K)$ from the quotient

$$J_K \Big/ K^\times . \prod_{v \mid \infty} K_v^\times . \prod_{\mathfrak{q}' \neq \mathfrak{q}} U_{\mathfrak{q}'} \ ,$$

where J_K is the idèle group of K and $U_{\mathfrak{q}'}$ the subgroup of local units of the completion $K_{\mathfrak{q}'}$ of K at the place $\mathfrak{q}'$. The inertia subgroup of $\mathfrak{q}$ in N/K is, according to class field theory, isomorphic to the quotient

$$U_{\mathfrak{q}} \Big/ \overline{U_{\mathfrak{q}} \cap (K^\times . \prod_{v \mid \infty} K_v^\times . \prod_{\mathfrak{q}' \neq \mathfrak{q}} U_{\mathfrak{q}'})} = U_{\mathfrak{q}} \Big/ \overline{E_K} \ ,$$

where E_K is the group of global units in K and the overlining means topological closure in $U_{\mathfrak{q}}$ of the diagonal embedding. Of course if F exists, it is contained in the maximal p-extension of N, whose ramification subgroup is the p-part of $U_{\mathfrak{q}}\big/\overline{E_K}$, denoted $(U_{\mathfrak{q}}\big/\overline{E_K})_p$.

We suppose now:

- $\mathfrak{q} \nmid p$.

So, if F exists, p^n divides $|(U_{\mathfrak{q}}\big/\overline{E_K})_p|$, that is, under the assumption $\mathfrak{q} \nmid p$:

- $\mu_{p^n} \subset K_{\mathfrak{q}}^\times$,

and

- $E_K \subset U_{\mathfrak{q}}^{p^n}$.[3]

These necessary conditions suffice to ensure the existence of F, according to an obvious lemma, which reads: if A is an Abelian finite group and C is a cyclic subgroup of A of order divisible by p^n, then there exists a cyclic quotient of A in which the image of C has order p^n.

Now we suppose that $\mathfrak{q} \nmid p$ satisfies the two conditions above together with the additional assumption:

- $\mathfrak{q}$ is unramified in L/K.[4]

So F exists; all primes $\mathfrak{q}_L \mid \mathfrak{q}$ are ramified in $M/L = LF/L$ with the same index $e_{\mathfrak{q}} = p^n$; and we have $[M : L] = p^{n+d}$ for some positive integer d.

(4) Obtaining a large cohomology group $\hat{H}^0(\mathcal{G}, E_M)$

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[3] The canonical embedding of E_K in $K_{\mathfrak{q}}^\times$ must be contained in $U_{\mathfrak{q}}^{p^n}$ since $(U_{\mathfrak{q}})_p$ is here a cyclic group.

[4] In fact, in the sequel Bosca will suppose that $\mathfrak{q}$ is totally split in L/K.

Let $\mathcal{G}$ denote the Galois group $\mathrm{Gal}(M/L)$, which is cyclic of order p^{n+d}, and $G = \mathrm{Gal}(L/K)$.

According to the Dirichlet–Herbrand theorem, the character of the representation $\mathbb{Q} \otimes_{\mathbb{Z}} E_L$ of $G = \mathrm{Gal}(L/K)$ given by the group of global units E_L is:

$$\chi_{E_L} = \sum_{v|\infty} \mathrm{Ind}_{D_v}^{G} 1_{D_v} - 1_G \ ,$$

where 1_G is the trivial character of G and 1_{D_v} the trivial character of the decomposition subgroup D_v.

Since at least one infinite place splits totally in the extension L/K, the character of $\mathbb{Q} \otimes_{\mathbb{Z}} (E_L/\mu_L E_K)$ satisfies

$$\chi_{E_L/\mu_L E_K} \geq \chi_{\mathbb{Z}[G]} - 1 \ ,$$

and we can deduce from this the existence of a map: [5]

$$\varphi : E_L/\mu_L E_K \longrightarrow \mathbb{Z}$$

such that, in $\mathrm{Hom}(E_L/\mu_L.E_K, \mathbb{Z})$, φ generates a $\mathbb{Z}[G]$-submodule whose character is $\chi_{\mathbb{Z}[G]} - 1$.

On the other hand, since $L[\mu_{p^n}, E_L^{1/p^n}]/L[\mu_{p^n}]$ is a Kummer extension, if $\mathfrak{q}_L$ is one of the primes of L dividing $\mathfrak{q}$, the Frobenius automorphism

$$\sigma = \mathrm{Frob}(\mathfrak{q}_L, L[\mu_{p^n}, E_L^{1/p^n}]/L)$$

corresponds, in the Kummer duality, to the map:

$$u \mapsto (u^{1/p^n})^{(\sigma-1)} \in \mu_{p^n}, \quad \text{for all } u \in E_L,$$

and we impose the new condition:

- this map σ coincides with $\lambda_n : E_L \longrightarrow E_L/\mu_L E_K \overset{\varphi}{\longrightarrow} \mathbb{Z} \longrightarrow \mu_{p^n}$,

where the left map is the natural one and the right one is surjective (we must choose a primitive p^nth root of unity for the right map, but this choice does not change the Frobenius defined up to conjugation: changing the choice of the root of unity is the same as changing the choice of a prime $\mathfrak{q}' \mid \mathfrak{q}$ in $L[\mu_{p^n}]$).

So, the property of φ leads to the following facts:

Let ϕ denotes the map (see the Appendix):

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[5] See the details in the Appendix.

$$E_L \longrightarrow R_G := \{\sum_g \alpha_g \, g \in \mathbb{Z}[G] \mid \sum_g \alpha_g = 0\}$$
$$u \longmapsto \sum_g \varphi(g^{-1}(u)) \, g \; ;$$

$\operatorname{Im}(\phi)$ has finite index, say r. So, p^δ being the maximal power of p dividing r, one has:

$$|\{E_L/\{u \in E_L \mid \forall g \in G, \lambda_n(g(u)) = 1\}| \geq |R_G/R_G^{p^n}|/p^\delta = p^{n(|G|-1)-\delta} \; .$$

On the arithmetical side, the ramification indices in M/L of all primes $\mathfrak{q}_L \mid \mathfrak{q}$ of L are all equal to p^n; so, with $\mathfrak{q}_M|\mathfrak{q}_L|\mathfrak{q}$ in $M/L/K$, one has [6]:

$$\mathrm{N}_{M/L}(E_M) \subset \mathrm{N}_{M/L}\Big(\prod_{\mathfrak{q}_M|\mathfrak{q}} U_{\mathfrak{q}_M} \Big) = \prod_{\mathfrak{q}_L|\mathfrak{q}} U_{\mathfrak{q}_L}^{p^n} \; ,$$

and then

$$\{u \in E_L \mid \forall g \in G, \lambda_n(g(u)) = 1\} = E_L \cap \prod_{\mathfrak{q}_L|\mathfrak{q}} U_{\mathfrak{q}_L}^{p^n} \supset \mathrm{N}_{M/L}(E_M) \; ,$$

so that

$$|H^0(\mathcal{G}, E_M)| = |E_L/\mathrm{N}_{M/L}(E_M)| \geq |E_L/\{u \in E_L \mid \forall g \in G, \lambda_n(g(u)) = 1\}| \; ,$$

and we finally have from the character theory side:

$$|H^0(\mathcal{G}, E_M)| \geq p^{n(|G|-1)-\delta} \; .$$

(5) Study of $\hat{H}^1(\mathcal{G}, E_M)$ and upper bound for $|I_M^{\mathcal{G}}/P_M^{\mathcal{G}}|$

Recall that $M := FL$. According to [4], chapter IX, §1, since the cyclic extension M/L splits totally at all infinite places, the Herbrand quotient $q(\mathcal{G}, E_M)$ of the units is given by:

$$q(\mathcal{G}, E_M) := \frac{|\hat{H}^0(\mathcal{G}, E_M)|}{|\hat{H}^1(\mathcal{G}, E_M)|} = \frac{1}{[M : L]} = \frac{1}{p^{n+d}} \; ,$$

and this gives:

$$|\hat{H}^1(\mathcal{G}, E_M)| = p^{n+d} \cdot |\hat{H}^0(\mathcal{G}, E_M)| \geq p^{n+d} \cdot p^{n(|G|-1)-\delta} = p^{n|G|+d-\delta} \; .$$

On the other hand, one has the canonical isomorphism:

$$\hat{H}^1(\mathcal{G}, E_M) \simeq P_M^{\mathcal{G}}/P_L$$

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[6] Using the fact that the global norm is the product of the corresponding local norms.

where $P_M^{\mathcal{G}}$ is the group of principal ideals of M which are invariant under $\mathcal{G}$.

So one obtains:
$$|P_M^{\mathcal{G}}/P_L| \geq p^{n|G|+d-\delta}.$$

Thus, from the formula:[7]
$$|I_M^{\mathcal{G}}/I_L| = \prod_{\mathfrak{q}_L|\mathfrak{q}} e_{\mathfrak{q}_L} = p^{n|G|},$$

where I_M is the group of fractional ideals of M, one deduces:
$$|I_M^{\mathcal{G}}/P_M^{\mathcal{G}}| = \frac{|I_M^{\mathcal{G}}/P_L|}{|P_M^{\mathcal{G}}/P_L|} = \frac{|I_M^{\mathcal{G}}/I_L| \cdot |I_L/P_L|}{|P_M^{\mathcal{G}}/P_L|} = \frac{p^{n|G|} \cdot |\mathrm{Cl}_L|}{|P_M^{\mathcal{G}}/P_L|} \leq \frac{p^{n|G|} \cdot |\mathrm{Cl}_L|}{p^{n|G|+d-\delta}},$$

that is:
$$|I_M^{\mathcal{G}}/P_M^{\mathcal{G}}| \leq |\mathrm{Cl}_L| \, p^{\delta-d}.$$

Note that the number in the right hand side does not depend on n.

(6) Does the class c_L principalize in M ?

Here, we also suppose:

- $\mathrm{Frob}(\mathfrak{q}_L, H_L/L) = c_L$.

M' being the maximal subfield of M/L in which $\mathfrak{q}_L$ totally splits, one has:
$$\mathfrak{q}_L = \prod_{\mathfrak{q}'_M|\mathfrak{q}_L \text{ in } M'/L} \mathfrak{q}'_M \, ;$$

and for all prime $\mathfrak{q}'_M|\mathfrak{q}_L$ of M'/L, we have $\mathfrak{q}'_M = \mathfrak{q}_M^{p^n}$, where $\mathfrak{q}_M$ is the unique prime of M dividing $\mathfrak{q}'_M$.

Finally in M,
$$\mathfrak{q}_L = \left(\prod_{\mathfrak{q}'_M|\mathfrak{q}_L} \mathfrak{q}_M \right)^{p^n},$$

with $\displaystyle\prod_{\mathfrak{q}'_M|\mathfrak{q}_L} \mathfrak{q}_M \in I_M^{\mathcal{G}}$. According to Section **(5)**, one has:
$$|I_M^{\mathcal{G}}/P_M^{\mathcal{G}}| \leq |\mathrm{Cl}_L| \, p^{\delta-d},$$

and then:
$$\left(\prod_{\mathfrak{q}'_M|\mathfrak{q}_L} \mathfrak{q}_M \right)^{|\mathrm{Cl}_L| \cdot p^{\delta-d}} \in P_M^{\mathcal{G}}.$$

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[7] Since we have supposed that $\mathfrak{q}$ splits totally in L/K.

Hence, in Cl_M, the extended class c_M of c_L satisfies both:

$$c_M = \bar{\mathfrak{q}}_L = \left(\prod_{\mathfrak{q}'_M | \mathfrak{q}_L} \bar{\mathfrak{q}}_M \right)^{p^n} \quad \text{and} \quad \left(\prod_{\mathfrak{q}'_M | \mathfrak{q}_L} \bar{\mathfrak{q}}_M \right)^{|\mathrm{Cl}_L| \, p^{\delta-d}} = 1 \ .$$

So, w being such that $|\mathrm{Cl}_L| = p^w$, c_L becomes principal in M under the assumption:

$$n \geq w + \delta - d.$$

(7) Existence of $\mathfrak{q}$

We just proved that c_L principalizes in M when n is large enough and when $\mathfrak{q}$ (or $\mathfrak{q}_L | \mathfrak{q}$) satisfies the following six conditions:

(1) $\mathfrak{q} \nmid p$,

(2) $\mu_{p^n} \subset K_{\mathfrak{q}}^{\times}$,

(3) $E_K \subset U_{\mathfrak{q}}^{p^n}$,

(4) $\mathfrak{q}$ totally splits in L/K,

(5) $\mathrm{Frob}(\mathfrak{q}_L, L[\mu_{p^n}, E_L^{1/p^n}]/L) = \lambda_n$,

(6) $\mathrm{Frob}(\mathfrak{q}_L, H_L/L) = c_L$.

The definition of $\lambda_n \in \mathrm{Gal}(L[\mu_{p^n}, E_L^{1/p^n}]/L)$ shows it is trivial on $L[\mu_{p^n}, E_K^{1/p^n}]$. Then conditions (4) and (5) imply (2) and (3), so we only study compatibility between conditions (1), (4), (5), (6). This compatibility is possible if and only if c_L and λ_n are equal on the extension $H_L \cap L[\mu_{p^n}, E_L^{1/p^n}]$ of L.

Let m be the integer such that $|(\mu_L)_p| = p^m$; one has, where the exponent ab means Abelian subextension over L:

$$H_L \cap L[\mu_{p^n}, E_L^{1/p^n}] \subset H_L \cap (L[\mu_{p^n}, E_L^{1/p^n}])^{ab} = H_L \cap L[\mu_{p^{n+m}}, E_L^{1/p^m}] \ ;$$

let m' be the integer such that $H_L \cap L[\mu_{p^\infty}] = L[\mu_{p^{m'}}]$, so $m' \geq m$ and

$$H_L \cap L[\mu_{p^n}, E_L^{1/p^n}] = H_L \cap L[\mu_{p^{m'}}, E_L^{1/p^m}] \ .$$

The exponent of the Galois group $\mathrm{Gal}(L[\mu_{p^{m'}}, E_L^{1/p^m}]/L)$ is less than $p^{m'}$, so is that of $\mathrm{Gal}(H_L \cap L[\mu_{p^n}, E_L^{1/p^n}]/L)$. According to Section **(1)**, taking $a = m'$, one can suppose that $c_L \in \mathrm{Cl}_L^{p^{m'}}$ (replacing L by L' as in Section**(1)**; note that $m'(L') = m'(L)$ because L'/L is unramified at all places dividing p, so that $c_L \in \mathrm{Cl}_L^{p^{m'}(L')}$ as expected); in that case, the restriction of c_L is trivial on $H_L \cap L[\mu_{p^n}, E_L^{1/p^n}]$.

About the restriction of λ_n on $H_L \cap L[\mu_{p^n}, E_L^{1/p^n}]$, we can as well suppose it is trivial, by replacing eventually λ_n by $\lambda_n^{p^{m'}}$ (*i.e.* φ by $p^{m'}\varphi$) which has the same properties.[8]

Up to replacing L by L' and choosing a convenient φ, the restrictions of c_L and of λ_n are both trivial on $H_L \cap L[\mu_{p^n}, E_L^{1/p^n}]$: Čebotarev theorem then ensures the existence of infinitely many convenient primes $\mathfrak{q}$ of K satisfying all conditions, and each one gives us an Abelian compositum M in which L principalizes. This proves the Main Theorem.

III. Proofs of the corollaries

Corollary 1 is just the case $K = \mathbb{Q}$, and the Kronecker-Weber theorem which states that Abelian extensions of $\mathbb{Q}$ are cyclotomic.

Corollary 2 is equivalent to the following fact: Let K be a number field and K_+^{ab} its maximal ∞-split Abelian extension. Any field L containing K_+^{ab} and in which at least one infinite place totally splits over K is principal.

To prove this, let $\mathfrak{a}$ be a fractional ideal of finite type of L. Of course $\mathfrak{a}$ is as well a fractional ideal of a subfield $L_\mathfrak{a}$ of L with finite degree over $\mathbb{Q}$. We may assume $L_\mathfrak{a} \supset K$, then $L_\mathfrak{a}/K$ is an extension of number fields in which at least one infinite place is totally split. According to the main theorem, $\mathfrak{a}$ principalizes in an Abelian compositum $L_\mathfrak{a}F$ of $L_\mathfrak{a}/K$; but $L_\mathfrak{a}F$ is contained in $L_\mathfrak{a}K_+^{ab} \subseteq L$ and so $\mathfrak{a}$ becomes principal in L.

Corollary 3 results from corollary 2, taking $K = \mathbb{Q}$ and $K = \mathbb{Q}(i)$, respectively.

Corollary 4 is a consequence of Corollary 2.

$$D - APPENDIX \ ^9$$

In this Appendix we make precise the construction of the map:

$$\varphi: \ E_L/\mu_L E_K \longrightarrow \mathbb{Z},$$

used by Bosca in Subsection (4). This has some importance since the existence and the properties of φ make use in a crucial manner of the main hypothesis of the paper: "at least one infinite place of K splits totally in L/K"; without this assumption the problem of principalization is impossible as explained at the end of the introduction.

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[8] See the Appendix.
[9] Written by G. Gras and J.-F. Jaulent.

The notations are those of the main text.

(8) Definition of φ

Notation: For a group of global units E we denote by $\overline{E}$ the quotient of E by its torsion subgroup μ.

Let N be the norm in L/K and let $_{\mathrm{N}}E$ be the kernel of N in E. From the exact sequence:

$$1 \longrightarrow {_{\mathrm{N}}\overline{E}_L} \longrightarrow \overline{E}_L \longrightarrow \mathrm{N}(\overline{E}_L) \subseteq \overline{E}_K \longrightarrow 1 \quad (\text{finite index})$$

we get:

$$\mathbb{Q} \otimes_{\mathbb{Z}} (\overline{E}_L) = \mathbb{Q} \otimes_{\mathbb{Z}} ({_{\mathrm{N}}\overline{E}_L}) \oplus \mathbb{Q} \otimes_{\mathbb{Z}} (\overline{E}_K) \ i.e. \ \ \mathbb{Q} \otimes_{\mathbb{Z}} (\overline{E}_L/\overline{E}_K) = \mathbb{Q} \otimes_{\mathbb{Z}} ({_{\mathrm{N}}\overline{E}_L}).$$

Since at least one infinite place of K splits totally in the extension L/K, the Dirichlet–Herbrand theorem implies that the character of $\mathbb{Q} \otimes_{\mathbb{Z}} ({_{\mathrm{N}}\overline{E}_L})$ contains $\chi(\mathbb{Q}[G]) - 1$ and that the representation $\mathbb{Q} \oplus \mathbb{Q} \otimes_{\mathbb{Z}} ({_{\mathrm{N}}\overline{E}_L})$ contains at least a representation R isomorphic to $\mathbb{Q}[G]$.

We can put $R = \mathbb{Q} \otimes_{\mathbb{Z}} \langle \theta \rangle$ with $\theta = \rho.\varepsilon_*$, where $\rho \in \mathbb{Q}^{\times}$, $\rho \neq \pm 1$, and where $\varepsilon_* \in {_{\mathrm{N}}\overline{E}_L}$ may be seen as a " relative Minkowski unit ".

Thus any element $u \in R$ is written, in a unique manner, $u = \theta^{\omega}$ with $\omega = \sum_{g \in G} \alpha_g \, g \in \mathbb{Q}[G]$. It follows that u is a unit (in $\langle \varepsilon_* \rangle_{\mathbb{Z}[G]}$) if and only if $\sum_{g \in G} \alpha_g = 0$. We note that in this case the α_g can be taken in $\frac{1}{m}\mathbb{Z}[G]$ for a suitable $m \in \mathbb{Z}$ (for instance $m = |G|$ in particular cases, but if necessary we can adjust the value of m large enough, multiple of $|G|$; at the end of the reasoning in Subsection (7), Bosca uses this possibility); for the same reasons, the choice of ε_* is not crucial and $\langle \varepsilon_* \rangle_{\mathbb{Z}[G]}$ is not necessarily a direct summand in $_{\mathrm{N}}\overline{E}_L$.

In any case m depends only on L/K and not on n.

Noting that

$$(\overline{E}_L/{_{\mathrm{N}}\overline{E}_L}) \, \overline{E}_K = \overline{E}_L/{_{\mathrm{N}}\overline{E}_L} \oplus \overline{E}_K$$

is killed by $|G|$, for $\varepsilon \in \overline{E}_L$ we have:

$$\varepsilon^{|G|} = \eta_* \, \varepsilon_0, \ \ \eta_* \in {_{\mathrm{N}}\overline{E}_L}, \ \ \varepsilon_0 \in \overline{E}_K.$$

Conclusion. *The map φ is defined as follows: Working in $\mathbb{Q} \oplus \mathbb{Q} \otimes_{\mathbb{Z}} ({_{\mathrm{N}}\overline{E}_L})$, in which R is a direct summand, we associate with $\varepsilon \in \overline{E}_L$ the component of ε^m on R, of the form θ^{ω}, with $\omega = \sum_{g \in G} \alpha_g \, g \in \mathbb{Z}[G]$, where $\sum_{g \in G} \alpha_g = 0$,*

then we put:

$$\varphi(\varepsilon) = \alpha_1 \in \mathbb{Z} \,.$$

This map is trivial on $\overline{E}_K$ and defines an element of $\mathrm{Hom}(E_L/\mu_L \,.\, E_K, \mathbb{Z})$ with the G-module action defined as usual by:

$$\psi^h(x) := \psi(x^{h^{-1}}), \quad \text{for all } \psi \in \mathrm{Hom}(E_L/\mu_L \,.\, E_K, \mathbb{Z}) \ \text{ and all } h \in G.$$

It is clear that φ generates a $\mathbb{Z}[G]$-submodule whose character is $\chi(\mathbb{Z}[G]) - 1$. More precisely, a straightforward computation gives $\varphi^g(\varepsilon) = \alpha_g$ for any $g \in G$, thus $\omega = \sum_{g \in G} \alpha_g \, g = \sum_{g \in G} \varphi(\varepsilon^{g^{-1}}) \, g$.

In the sequel we put $\omega =: \phi(\varepsilon)$.

At this step, Bosca introduces (middle of Subsection (4)) the map λ_n:

$$\lambda_n : \; E_L \longrightarrow E_L/\mu_L \, E_K \overset{\varphi}{\longrightarrow} \mathbb{Z} \longrightarrow \mu_{p^n} \longrightarrow 1$$

by a choice of a primitive p^nth root of unity.

This yields an element of $\mathrm{Hom}(E_L, \mu_{p^n})$ which will be by abuse of notation identified, via the Kummer duality between radicals and Galois groups, with the corresponding element σ' of $\mathrm{Gal}(L[\mu_{p^n}, E_L^{1/p^n}]/L[\mu_{p^n}])$.

Then Bosca creates a new condition by saying that σ' coincide with the Frobenius $\sigma = \mathrm{Frob}(\mathfrak{q}_L, L[\mu_{p^n}, E_L^{1/p^n}]/L)$, which is the key idea for the proof of the conjecture. Indeed, the Galois properties of φ give the inequality $|H^0(\mathcal{G}, E_M)| \geq p^{n(|G|-1)-\delta}$ (see the last part of Subsection (4)) which is the main ingredient for Subsection (5).

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