TRIGONOMETRIC SERIES AND SELF-SIMILAR SETS

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ABSTRACT. Let F be a self-similar set on $\mathbb{R}$ associated to contractions $f_j(x) = r_j x + b_j$, $j \in \mathcal{A}$, for some finite $\mathcal{A}$, such that F is not a singleton. We prove that if $\log r_i / \log r_j$ is irrational for some $i \neq j$, then F is a set of multiplicity, that is, trigonometric series are not in general unique in the complement of F. No separation conditions are assumed on F. We establish our result by showing that every self-similar measure μ on F is a Rajchman measure: the Fourier transform $\widehat{\mu}(\xi) \to 0$ as $|\xi| \to \infty$. The rate of $\widehat{\mu}(\xi) \to 0$ is also shown to be logarithmic if $\log r_i / \log r_j$ is diophantine for some $i \neq j$. The proof is based on quantitative renewal theorems for random walks on $\mathbb{R}$.

1. Introduction and the main result

The *uniqueness problem* in Fourier analysis that goes back to Riemann [33] in 1868 concerns the following question: suppose we have two converging trigonometric series $\sum a_n e^{2\pi i n x}$ and $\sum b_n e^{2\pi i n x}$ with coefficients $a_n, b_n \in \mathbb{C}$ such that for "many" $x \in [0,1]$ they agree:

$$\sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x} = \sum_{n \in \mathbb{Z}} b_n e^{2\pi i n x}, \tag{1.1}$$

then are the coefficients $a_n = b_n$ for all $n \in \mathbb{Z}$? For how "many" $x \in [0,1]$ do we need to have (1.1) so that $a_n = b_n$ holds for all $n \in \mathbb{Z}$? If we assume (1.1) holds *for all* $x \in [0,1]$, then using Toeplitz operators Riemann [33] proved that indeed $a_n = b_n$ for all $n \in \mathbb{Z}$. However, it would be interesting to see how small the set of $x \in [0,1]$, where (1.1) holds can be so that we have $a_n = b_n$ for all $n \in \mathbb{Z}$. Motivated by this one defines that a subset $F \subset [0,1]$ is a *set of uniqueness* if whenever we have coefficients $a_n, b_n \in \mathbb{C}$, $n \in \mathbb{Z}$, such that (1.1) holds for all $x \in [0,1] \setminus F$, then $a_n = b_n$ for all $n \in \mathbb{Z}$. Here one defines also that if F is not a set of uniqueness, then it is called a *set of multiplicity*. In particular by Riemann's result this shows that the empty set $\varnothing$ is a set of uniqueness and so $[0,1]$ is a set of multiplicity.

Cantor [8] proved that that every closed countable set is a set of uniqueness, and later Young [45] generalised to every countable set. In the uncountable case, however, even if assuming F is very small, uniqueness of F may fail: Menshov [30] constructed a set F of Lebesgue measure 0, which is a set of multiplicity, that is, the uniqueness problem fails if we only assume (1.1) for all $x \in [0,1] \setminus F$. This can be proved using the following criteria, which goes back to Salem [35] that if a set F supports a Borel probability measure μ such

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that the *Fourier transform*

$$\widehat{\mu}(\xi) := \int e^{-2\pi i \xi x}\, d\mu(x), \quad \xi \in \mathbb{R},$$

satisfies $\widehat{\mu}(n) \to 0$ as $|n| \to \infty$, $n \in \mathbb{Z}$, then F is a set of multiplicity. Such measures μ are called *Rajchman measures* in the literature. Hence constructing measures μ with decaying Fourier coefficients provides a way to check whether F is of multiplicity. It remains an open problem to classify which uncountable sets F are of multiplicity and which F are of uniqueness and much work has been done in many examples of F on trying to establish their uniqueness or multiplicity.

In the series of works Salem [35] proved that the middle third Cantor set $C_{1/3}$ is a set of uniqueness. More generally, Salem established that if C_λ is the middle λ-Cantor set with $0 < \lambda < 1/2$, that is, interval of length $1 - 2\lambda$ is removed from the center of $[0, 1]$ at every construction stage, then C_λ is a set of uniqueness when λ^{-1} is a Pisot number. In the opposite case, if λ^{-1} is not a Pisot number, by constructing a Rajchman measure on C_λ, Piatetski-Shapiro [31], Salem and Zygmund [36] established that C_λ is a set of multiplicity.

The Cantor set C_λ is an example of a *self-similar set*. Recall that a subset $F \subset [0, 1]$ is self-similar if there exists similitudes $f_j : [0, 1] \to [0, 1]$, that is, $f_j(x) = r_j x + b_j$, $j \in \mathcal{A}$, for some finite set $\mathcal{A}$, translations $b_j \in \mathbb{R}$ and contractions $0 < r_j < 1$ such that

$$F = \bigcup_{j \in \mathcal{A}} f_j(F).$$

As far as we know nothing is known about the uniqueness or multiplicity of self-similar sets beyond beyond the case of C_λ or if adding finitely many more similitudes with the same contraction ratio λ to the definition, which was done by Salem [35]. For example if we have two different contractions $r_0 = 1/2$ and $r_1 = 1/3$ for the iterated function system, do we expect F to be of multiplicity or of uniqueness? Due to having same contraction ratio λ the case C_λ has a convolution structure, which is helpful when connecting to the algebraic properties of the number λ. In the general case, however, we would need to find a way out of this.

It turns out that the algebraic properties of the additive subgroup Γ generated by the log-contraction ratios $\{-\log r_j : j \in \mathcal{A}\}$ in $\mathbb{R}$ is important in the study of the multiplicity of a self-similar set F with contraction ratios r_j. In particular if this subgroup Γ is dense, which happens when $\log r_j / \log r_\ell$ is irrational for some $j \neq \ell$ (e.g. $r_j = 1/2$ and $r_\ell = 1/3$), we can establish F is a set of multiplicity.

Theorem 1.1. *Let $F \subset [0, 1]$ be a self-similar set associated to contractions $f_j(x) = r_j x + b_j$, $j \in \mathcal{A}$, such that F is not a singleton. If $\log r_j / \log r_\ell$ is irrational for some $i \neq j$, then F is a set of multiplicity.*

Notice that by assuming $\log r_j / \log r_\ell$ is irrational we exclude the case of C_λ as in that case every ratio of logarithms of the contractions is just 1. It remains an open problem to study the case when $\log r_j / \log r_\ell \in \mathbb{Q}$ for all $j \neq \ell$. We predict that here typically F should be a set of uniqueness unless all the contraction ratios are equal, like the case C_λ, and then an algebraic number theoretic condition like λ^{-1} being Pisot needs to be imposed.

In order to prove the multiplicity of a self-similar set F, it is enough by Salem's criterion [35] for multiplicity to find a Rajchman measure supported on F. Hence Theorem 1.1 follows by establishing that all positive dimensional self-similar measures on F are Rajchman measures. Recall that a probability measure μ on $\mathbb{R}$ is called *self-similar* if there exists a finite collection $\{f_j : j \in \mathcal{A}\}$ of similitudes of $\mathbb{R}$ with at least two maps and weights $0 < p_j < 1$, $j \in \mathcal{A}$, with $\sum_{j\in\mathcal{A}} p_j = 1$ such that $\mu = \sum_{j\in\mathcal{A}} p_j f_j \mu$.

Theorem 1.2. *Let $F \subset [0,1]$ be a self-similar set associated to contractions $f_j(x) = r_j x + b_j$, $j \in \mathcal{A}$, such that F is not a singleton. If $\log r_i / \log r_j$ is irrational for some $i \neq j$, then the Fourier transform $\widehat{\mu}(\xi) \to 0$ as $|\xi| \to \infty$ for every self-similar measure μ on F.*

Theorem 1.2 is closely related to another problem in a currently active problem in the community of fractal geometry, where we would like to understand the Fourier transforms of fractal measures, see the book [29] by Mattila for an history and overview. In particular there are various past and recent works on random fractals by Kahane [21, 22], Shmerkin and Suomala [39] and other people [14, 15], connections to Diophantine approximation by Kaufman *et al.* [23, 24], dynamical systems [19, 34] and additive combinatorics [3, 25]. Analysing the spectrum of fractal measures has been particularly important in finding normal numbers from the support of fractals [18, 32, 11] and the study of harmonic analysis defined by fractal measures, see for example applications to the spectrum of convolution operators defined by fractal measures in the work of Sarnak [37] and later by Sidorov and Solomyak [40], and more recently applications to quantum resonances in quantum chaos by Bourgain and Dyatlov [4].

The study of Fourier transforms of self-similar measures in general goes back to the works of Strichartz [41, 42], where an average decay of Fourier transform $\widehat{\mu}(\xi)$ of self-similar measures μ are obtained, where proportions of frequencies $\xi \in \mathbb{R}$ are excluded. More recently a large deviation estimate for these average decays was proved by Tsujii [43]. However, the methods here cannot be used to obtain a full decay over all $|\xi| \to \infty$. Before Theorem 1.2 the only cases of self-similar measures μ where $\widehat{\mu}(\xi) \to 0$ as $|\xi| \to \infty$ was known were self-similar measures on the middle λ-Cantor sets C_λ, $0 < \lambda < 1/2$, by Salem [35], Piatetski-Shapiro [31], Salem and Zygmund [36], and in the overlapping case for the *Bernoulli convolutions* μ_β, $1 < \beta < 2$, which are the distribution of the random sum $\sum \pm\beta^{-k}$ with i.i.d. chosen signs. For Bernoulli convolutions Fourier transforms play an important role as proving that $\widehat{\mu}_\beta(\xi)$ has power decay as $|\xi| \to \infty$ implies μ_β is absolutely continuous, which is a well-known open problem in the field, see Shmerkin [38]. It is known by the results of Erdös [16] and Kahane [20] that the set of $1 < \beta < 2$ such that μ_β does not have a power decay has Hausdorff dimension zero. Moreover, Erdös [16] proved that $\widehat{\mu}_\beta(\xi) \to 0$ as $|\xi| \to \infty$ if and only if β is a Pisot number. In the non-Pisot case the rate of convergence was later shown to be logarithmic for rational number β or some algebraic numbers β by Dai [9] and Bufetov and Solomyak [7], and some power decay for algebraic numbers β has been obtained by Dai, Feng and Wang [10].

Notice that in Theorem 1.1 and Theorem 1.2 there can be any types of overlaps for the maps f_j and no separation conditions are assumed. Typically in the overlapping case the analysis of self-similar sets and measures can be notoriously difficult to understand, say, their Hausdorff dimension has required some deep connections to additive combinatorics, see for example the recent works of Hochman [17], Breuillard-Varju [6] and Varju [44]. The reason overlaps do not cause us any issues is the fact that the main contribution to the Fourier decay comes from controlling the distribution of lengths of the construction intervals, and not their relative positions. Understanding the distribution of the lengths of the construction intervals then can be reduced as a problem of studying the renewal theory for random walk random walk $X_1, X_2, \dots$ on $\mathbb{R}$ driven by $\lambda = \sum_{j\in\mathcal{A}} p_j \delta_{-\log r_j}$. This strategy to establish Fourier decay is similar to what was done in the case of the stationary measure for group actions by the first author in [26]. In the self-similar case case we consider, however, the proof is much more straightforward and we can see the idea governing the Fourier decay more clearly. The irrationality of $\log r_i/\log r_j$ is key to prove the random walk becomes *non-lattice*, that is, not concentrated on an arithmetic progression, which is a key assumption for the renewal theorem we use.

If we want a rate of convergence in Theorem 1.2 using the strategy we present in this paper, one needs to go into the rate of convergence for the renewal theorems we use. Here it is well-known that the diophantine properties of the random walk become an essential property, in particular, how well $\log r_i/\log r_j$ is approximated by rationals. In Diophantine approximation, it is defined that an irrational real number $a \in \mathbb{R}$ is called *diophantine* if for some $c > 0$ and $l > 2$ we have

$$\Big|a - \frac{p}{q}\Big| \geq \frac{c}{q^l}$$

for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$. This happens for example when $a = \log 2/\log 3$ or in general for $a = \log p/\log q$ for coprime p, q, see Baker [1]. Having some diophantine $\log r_i/\log r_j$ in the iterated function system imposes the random walk generated by the contractions to quantitatively avoid lattices and then gives quantitative rates for the renewal theorem. Under this condition, we can improve Theorem 1.2 in the following way:

Theorem 1.3. *Let $F \subset [0,1]$ be a self-similar set associated to contractions $f_j(x) = r_j x + b_j$, $j \in \mathcal{A}$, such that F is not a singleton. If $\log r_i/\log r_j$ is diophantine for some $i \neq j$, for some $\alpha > 0$ we have*

$$|\widehat{\mu}(\xi)| = O\Big(\frac{1}{|\log|\xi||^{\alpha}}\Big), \quad |\xi| \to \infty$$

for every self-similar measure μ on F.

Removing the irrationality of ratios of log-contractions ratios makes the random walk $X_1, X_2, \dots$ on $\mathbb{R}$ driven by $\lambda = \sum_{j\in\mathcal{A}} p_j \delta_{-\log r_j}$ *lattice*, that is, concentrated on arithmetic progressions. Then the renewal theorems do not hold anymore in the same form, and in fact the Fourier transform no longer may not even decay at infinity as given by the middle 1/3 Cantor measure. However, in the case β is not Pisot, the Bernoulli convolution μ_β associated to β provides examples of a measure where the Fourier transform does decay at infinity, even with polynomial rate for some algebraic β, but the additive random walk on $\mathbb{R}$ generated by $\log \beta$ is a lattice. Hence it would be interesting to develop the connection to renewal theory further and find a full classification of self-similar sets F which are of uniqueness and which are of multiplicity.

In this paper we considered the self-similar case, but if we impose the maps f_j to be suitably nonlinear, such as the inverse branches of the Gauss map $x \mapsto 1/x \bmod 1$ and study the Fourier transforms of self-conformal measures μ, then the rates of Fourier decay in Theorem 1.3 for Fourier decay can be improved to power decay, see for example the works [19, 4, 34, 27]. Here the non-lattice condition of contractions $-\log r_j$ is replaced by a non-concentration condition of the log-derivatives of the iterates $-\log(f_{j_1} \circ \dots \circ f_{j_n})'(x)$ as $n \to \infty$ and these types of conditions appear in the Fourier decay properties of multiplicative convolutions in the discretised sum-product theory developed by Bourgain [3].

What about the higher dimensional case? Here the analogue to Theorem 1.2 and Theorem 1.3 would be to understand Fourier transforms $\widehat{\mu}$ of *self-affine measures* μ on $\mathbb{R}^d$. They are measures on $\mathbb{R}^d$ associated to affine contractions $f_j = A_j + b_j$ of $\mathbb{R}^d$, $j \in \mathcal{A}$, for some finite set $\mathcal{A}$, where $b_j \in \mathbb{R}^d$ and $A_j \in \mathrm{GL}(d, \mathbb{R})$ such that

$$\mu = \sum_{j \in \mathcal{A}} p_j f_j \mu$$

for some weights $0 < p_j < 1$, $j \in \mathcal{A}$, with $\sum_{j \in \mathcal{A}} p_j = 1$. In a follow-up paper [28], we apply a similar strategy as we do in this paper by considering renewal theory for random walks on the group $\mathrm{GL}(d, \mathbb{R})$ coming from $\{A_j : i \in \mathcal{A}\}$ to establish a Fourier decay for self-affine measures. The renewal theory we need has been done recently by the first author in [27]. Here the non-lattice condition can be replaced by an irreducibility and proximality assumption of the subgroup Γ generated by $\{A_j : i \in \mathcal{A}\}$ as Bárány, Hochman and Rapaport did recently in their work [2] for the computation of Hausdorff dimension of self-affine measures on $\mathbb{R}^2$. Moreover, due to the better rates for quantitative renewal theorems for random walks in real split groups [27], that is, when the Zariski closure of Γ is $\mathbb{R}$-splitting, we can improve the rates for the Fourier decay of μ to power decay.

Organisation of the paper. The article is organised as follows. In Section 2 we give the quantitative renewal theorems we need for our results and then prove them in Section 4. Then in Section 3 we give the proof of Theorem 1.2, which implies Theorem 1.1 on the multiplicity of self-similar sets, and also in Section 3 we prove the quantiative Theorem 1.3 using the quantitative estimates for the renewal theorem established in Section 2.

2. Quantitative renewal theorems for random walks in $\mathbb{R}$

The proof of Theorem 1.2 and the quantitative version Theorem 1.3 rely on quantitative renewal theorems for random walks on $\mathbb{R}$, which we will give in this section. We will first fix some notation: for two real functions f and g, we write $f = O(g)$, $f \ll g$ or $g \gg f$ if there exists a constant $C > 0$ such that $|f| \leq Cg$, where C only depends on the measure μ. We write $f = O_\varepsilon(g)$ or $f \ll_\varepsilon g$ if the constant C depends on an extra parameter ε.

Let λ be a probability measure on $\mathbb{R}^+$ with finite support and let $|\operatorname{supp} \lambda|$ be the maximal of the support of λ. Let σ be the expectation of λ. We call λ *non-lattice* if the support of λ generates a dense additive subgroup of $\mathbb{R}$. In our case of self-similar measures associated to an iterated function system $f_j(x) = r_j x + b_j$ and weights $\sum_{j \in \mathcal{A}} p_j = 1$, we will apply this in the case of

$$\lambda = \sum_{j \in \mathcal{A}} p_j \delta_{-\log r_j}$$

which is non-lattice as long as the the additive subgroup generated by $-\log r_j$, $j \in \mathcal{A}$, is dense in $\mathbb{R}$.

Let now $X_1, X_2, \dots$ be i.i.d. sequence of random variables with λ. Write for $n \in \mathbb{N}$ the sum

$$S_n := X_1 + X_2 + \cdots + X_n.$$

Thus S_n has the distribution λ^{*n}, where $\lambda^{*n} = \lambda^{*(n-1)} * \lambda$, $n \geq 2$, is the iterated convolution with $\lambda^{*1} := \lambda$. We defined for $t > 0$ the *stopping time*

$$n_t := \inf\{n \in \mathbb{N} : S_n \geq t\},$$

and we define $S_t := S_{n_t}$. The main intuition is that if λ is non-lattice, then the residue distribution $S_t - t$ will converge to a distribution absolutely continuous with respect to the Lebesgue measure when t tends to infinite.

Proposition 2.1. *If λ is non-lattice, then we have for $t > |\operatorname{supp}\lambda| + 1 > 0$ and C^1 function g on $\mathbb{R}$,*

$$\mathbb{E}_t(g(S_t - t)) = \frac{1}{\sigma}\int_{\mathbb{R}^+} g(x)p(x)dx + o_t|g|_{C^1}, \tag{2.1}$$

where o_t tends to zero as t going to ∞ and where $p(x) = \int_{y>x} d\lambda(y)$ is a piecewise constant function and vanishes when x passes the support of λ.

Proof. Let ϱ be a smooth cutoff such that $\varrho_{[-|\operatorname{supp}\lambda|,|\operatorname{supp}\lambda|]} = 1$ and becomes 0 outside of $[-|\operatorname{supp}\lambda| - 1, |\operatorname{supp}\lambda| + 1]$. Take $f(v, u) = g(v + u)\varrho(v)\varrho(u)$. Then $f(v, u) = g(v + u)$ when v, u are in the interval $[-|\operatorname{supp}\lambda|, |\operatorname{supp}\lambda|]$. By definition, we have

$$\mathbb{E}_t(g(S_t - t)) = E_C f(t).$$

This function f satisfies the conditions in Proposition 4.9, and the proof is complete by using Proposition 4.9. ☐

If we want to apply the Diophantine condition on the ratios of the logarithms, we need a rate. This can be obtained in the following:

Definition 2.2. For a probability measure λ on $\mathbb{R}$ and l in $\mathbb{R}^+$, we call it *l-weakly diophantine* if

$$\liminf_{|b|\to\infty} |b|^l |1 - \mathcal{L}\lambda(ib)| > 0,$$

where $\mathcal{L}\lambda$ is the *Laplace transform* of λ, defined for $z \in \mathbb{C}$ by the formula

$$\mathcal{L}\lambda(z) = \int e^{zx} d\lambda(x).$$

More generally, we say that λ is weakly-diophantine if it is l-weakly diophantine.

This definition can be find in [5].

Lemma 2.3. *If there exist r_j, r_k for $j, k \in \mathcal{A}$ such that $\log r_j / \log r_k$ is diophantine, then the measure λ is weakly diophantine.*

Proof. We have that

$$|1 - \mathcal{L}\lambda(ib)| \geq |\operatorname{Re}(p_j(1 - e^{-ib\log r_j}) + p_k(1 - e^{-ib\log r_k}))|$$
$$\gg \max\{d(b\log r_j, 2\pi\mathbb{Z})^2, d(b\log r_k, 2\pi\mathbb{Z})^2\} \gg \max\{d(b_1, \mathbb{Z})^2, d(b_1\frac{\log r_k}{\log r_j}, \mathbb{Z})^2\},$$

with $b_1 = b\log r_j/2\pi$. By the definition of diophantine number, we obtain that for some $l \in \mathbb{N}$

$$\max\{d(b_1, \mathbb{Z})^2, d(b_1\frac{\log r_k}{\log r_j}, \mathbb{Z})^2\} \gg |b_1|^{-2l}.$$

Combing the above two inequalities, we know that the measure λ is weakly diophantine. ☐

Proposition 2.4. *If the measure λ is l-weakly diophantine, then for $t > |\operatorname{supp}\lambda| + 1$ we have*

$$\mathbb{E}_t(g(S_t - t)) = \frac{1}{\sigma}\int_{\mathbb{R}^+} g(x)p(x)dx + O(t^{-1/(4l+1)})|g|_{C^1}. \tag{2.2}$$

Proof. We need to use the weakly diophantine condition to give an estimate of the error term in Proposition 4.9. For the supremum of the norm of $\frac{1}{1-\mathcal{L}\lambda(i\xi)}+\frac{1}{\sigma}\frac{1}{i\xi}$ and its derivative

$$\partial_\xi\left(\frac{1}{1-\mathcal{L}\lambda(i\xi)}+\frac{1}{\sigma}\frac{1}{i\xi}\right)=\frac{-\partial_\xi\mathcal{L}\lambda(i\xi)}{(1-\mathcal{L}\lambda(i\xi))^2}-\frac{1}{\sigma}\frac{1}{i\xi^2},$$

on the interval $[-C_\psi\delta^{-2},C_\psi\delta^{-2}]$, by the definition l-weakly diophantine, we obtain that it is less than $C\delta^{-4l}$. Then by Proposition 4.9

$$O_\delta\le C\delta^{-4l}.$$

Then take $\delta=t^{-1/(4l+1)}$. The proof is complete. □

3. Proof of the Fourier decay

3.1. **Dimension theory and symbolic notations.** Let us write $\mathcal{A}^*$ as the space of all words w with entries in $\mathcal{A}$ of finite length. Moreover, $\mathcal{A}^n$ is the space of all words of length n with entries in $\mathcal{A}$. If $w=w_1w_2\dots w_n\in\mathcal{A}^n$, define the composition

$$f_w:=f_{w_1}\circ\dots\circ f_{w_n}.$$

Then f_w is again a similitude with a contraction

$$r_w:=r_{w_1}\dots r_{w_n}.$$

Using this notation the self-similarity of μ implies that

$$\mu=\sum_{w\in\mathcal{A}^n}p_wf_w\mu,$$

where

$$p_w:=p_{w_1}\dots p_{w_n}>0$$

as the product of weights p_j, $j\in\mathcal{A}$, according to the entries of the word w. See the book by Falconer [12] for more details, notations and history on self-similar sets and measures.

3.2. **Reduction to exponential sums.** Given $\xi\in\mathbb{R}$ and $t>0$, the first step is to reduce the Fourier transform of μ to double μ integrals over exponential sums determined by the stopping time n_t. Recall that we defined in Section 2 for $t>0$ the stopping time

$$n_t:=\inf\{n\in\mathbb{N}:S_n\ge t\},$$

where $S_n=X_1+\dots+X_n$ and X_j are i.i.d. distributed according to

$$\lambda=\sum_{j\in\mathcal{A}}p_j\delta_{\log(1/r_j)}.$$

Let $\mathbb{P}_t$ be the probability distribution on $\mathcal{A}^*$ associated to the stopping time and write

$$\mathcal{W}_t:=\operatorname{spt}\mathbb{P}_t,$$

for the support of $\mathbb{P}_t$. The reason to use the stopping time here is that we want to use the equidistribution phenomenon of the renewal theorem (Proposition 2.1), which combined with high-oscillation can give decay of exponential sums.

Later we will make t depend on ξ and let $|\xi|\to\infty$, but for now we keep everything fixed.

Lemma 3.1. *For every $\xi \in \mathbb{R}$ and $t > 0$ we have*

$$|\widehat{\mu}(\xi)|^2 \leq \iint \sum_{w \in \mathcal{W}_t} p_w e^{-2\pi i \xi (f_w(x) - f_w(y))} \, d\mu(x) \, d\mu(y).$$

Proof. Firstly, by

$$\mu = \sum_{j \in \mathcal{A}} p_j f_j \mu$$

we see that for any $t > 0$ we can write

$$\mu = \sum_{w \in \mathcal{W}_t} p_w f_w \mu.$$

The proof of this is similar to [26, Proposition 3.5]. Hence we obtain

$$\widehat{\mu}(\xi) = \sum_{w \in \mathcal{W}_t} p_w \int e^{-2\pi i \xi f_w(x)} \, d\mu(x).$$

Thus by Cauchy-Schwartz, we have

$$|\widehat{\mu}(\xi)|^2 \leq \sum_{w \in \mathcal{W}_t} p_w \Big| \int e^{-2\pi i \xi f_w(x)} \, d\mu(x) \Big|^2.$$

Opening up we see that

$$\sum_{w \in \mathcal{W}_t} p_w \Big| \int e^{-2\pi i \xi f_w(x)} \, d\mu(x) \Big|^2 = \iint \sum_{w \in \mathcal{W}_t} p_w e^{-2\pi i \xi (f_w(x) - f_w(y))} \, d\mu(x) \, d\mu(y).$$

□

Thus to prove Fourier decay, we would need to prove

$$\iint \sum_{w \in \mathcal{W}_t} p_w e^{-2\pi i \xi (f_w(x) - f_w(y))} \, d\mu(x) \, d\mu(y) \to 0 \tag{3.1}$$

as $|\xi| \to \infty$ for a suitable $t = t(\xi) \to \infty$, and if we want a rate for the Fourier decay, we need to control the speed of convergence in (3.1). In order to do this, we first write $|\xi|$ in the separate form

$$\xi = se^t$$

with $s \in \mathbb{R}$ and $t > 0$ and later we will first take $|s|$ large, then take $t > 0$ large enough depending on s. Using these parameters, write

$$\delta = 1/\sqrt{|s|} > 0.$$

Then define the tube

$$A_\delta = \{(x, y) \in \mathbb{R} : |x - y| \leq \delta\}.$$

We will split (3.1) into two cases depending on how close $|x - y|$ are in terms of the $\delta > 0$ defined above. We will have the following two propositions given in Proposition 3.2 and Proposition 3.3, which together imply Theorem 1.2. For the quantitative part, we also need Proposition 3.4 to control the rate in (3.1).

3.3. **Controlling nearby points.** The first one is on the nearby points $x, y \in \mathbb{R}$, that is, those with $|x-y| \leq \delta$, and here is where we use the fact that F is not a singleton. By [13, Proposition 2.2], due to F is not a singleton, there exist $r_0 > 0$, $\alpha > 0$ and $C > 0$ such that for all $0 < r < r_0$ and $x \in F$ we have

$$\mu(B(x,r)) \leq Cr^{\alpha}. \tag{3.2}$$

A measure which satisfies this condition is sometimes called a Frostman measure. Using the decay (3.2) of the μ measure on balls, we can control the nearby points in the following lemma:

Proposition 3.2. *For any $|\xi| = se^t$, we have*

$$\Big| \iint_{A_\delta} \sum_{w \in \mathcal{W}_t} p_w e^{-2\pi i \xi (f_w(x) - f_w(y))} \, d\mu(x) \, d\mu(y) \Big| \to 0$$

as $|\xi| \to \infty$.

Proof. First of all, since for all $t > 0$ we have that

$$\sum_{w \in \mathcal{W}_t} p_w = 1,$$

we can directly bound using triangle inequality that

$$\Big| \iint_{A_\delta} \sum_{w \in \mathcal{W}_t} p_w e^{-2\pi i \xi (f_w(x) - f_w(y))} \, d\mu(x) \, d\mu(y) \Big| \leq (\mu \times \mu)(A_\delta)$$

as $|e^{i\theta}| = 1$ for all $\theta \in \mathbb{R}$. Using Fubini's theorem we see that

$$(\mu \times \mu)(A_\delta) = \int \mu(B(x,\delta)) \, d\mu(x), \tag{3.3}$$

thus by (3.2) the right-hand side converges to 0 as $\delta \to 0$. □

3.4. **Application of the renewal theorem and high-oscillations.** In the case when $x, y \in \mathbb{R}$ are chosen such that $|x - y| > \delta$, we will use the renewal theory to prove the following convergence.

Proposition 3.3. *Suppose $\log r_j / \log r_\ell$ is irrational for some $j \neq \ell$. Then*

$$\iint_{\mathbb{R}^2 \setminus A_\delta} \sum_{w \in \mathcal{W}_t} p_w e^{-2\pi i \xi (f_w(x) - f_w(y))} \, d\mu(x) \, d\mu(y) \to 0$$

as $|\xi| \to \infty$.

This rate is not quantitative, so in the later section, by adding an extra assumption ($\log r_j / \log r_\ell$ is diophantine) to the renewal theory, gives us a quantitative version (Proposition 3.4).

Proof of Proposition 3.3. By definition of f_j we have that for all $x, y \in [0,1]$ and $w \in \mathcal{W}_t$ the difference

$$f_w(x) - f_w(y) = r_w(x-y).$$

Therefore we can write

$$e^{-2\pi i \xi (f_w(x) - f_w(y))} = e^{-2\pi i \xi (x-y) r_w}$$

Recall that we have fixed $s \in \mathbb{R}$ and $t > 0$ such that ξ has the form $\xi = se^t$. With this $s \in \mathbb{R}$, we can define a function $g_s : \mathbb{R} \to \mathbb{C}$ by

$$g_s(r) := \exp(-2\pi i s e^{-r}), \quad r \in \mathbb{R}.$$

Using g_s we can write for any pair $x, y \in \mathbb{R}$ that

$$\sum_{w \in \mathcal{W}_t} p_w e^{-2\pi i \xi (f_w(x) - f_w(y))} = \mathbb{E}_t(g_{s(x-y)}(S_t - t)),$$

where expectation is with respect to the probability measure $\mathbb{P}_t$ determined by the stopping time n_t.

We will apply the function $g_{s(x-y)}$ in the renewal theorem (see Proposition 2.1) as the renewal function h in Proposition 2.1. Here is where we need to invoke the condition that $\log r_j / \log r_\ell$ is irrational for some $j \neq \ell$. It implies that the i.i.d. random walk $X_1, X_2, \dots$ on $\mathbb{R}$ with the distribution

$$\lambda = \sum_{j \in \mathcal{A}} p_j \delta_{\log(1/r_j)},$$

is non-lattice. Hence we can apply the renewal theorem (Proposition 2.1) to obtain some piecewise continuous function $r \mapsto p(r)$ on $\mathbb{R}$ such that for any $h : \mathbb{R} \to \mathbb{C}$ we have the following convergence of the expectations:

$$\lim_{t \to \infty} \mathbb{E}_t(h(S_t - t)) = \int_{\mathbb{R}} h(r) p(r) \, dr.$$

Applying this with $h = g_{s_1}$ gives us when $|s_1|$ tends to infinite

$$\lim_{t \to \infty} \mathbb{E}_t(g_{s_1}(S_t - t)) = \int_{\mathbb{R}} g_{s_1}(r) p(r) \, dr.$$

If we now look at the right-hand side, since $p(r)$ is a piecewise continuous function, or just integrable, Riemann-Lebesgue lemma implies that

$$\int_{\mathbb{R}} g_{s_1}(r) p(r) \, dr \to 0. \tag{3.4}$$

However, in our case we only know that $x, y \in \mathbb{R}^2 \setminus A_\delta$, so $|x - y| > \delta$ and $|s_1| = |s(x-y)| \in [|s|^{1/2}, C|s|]$, where C depends on the support of λ. Thus to be able to use the above convergence (3.4), we need uniformity for $|s_1|$ in the interval $[|s|^{1/2}, C|s|]$ and to make it more effective using the error term in the renewal theorem Proposition 2.1.

Let us fix $\varepsilon > 0$ small enough. Then first use (3.4) choose $s_0 \in \mathbb{R}$ such that for all $s_1 \in \mathbb{R}$ with $|s_1| \geq |s_0|$ we have

$$\Big| \int_{\mathbb{R}} g_{s_1}(r) p(r) \, dr \Big| \leq \frac{\varepsilon}{2}.$$

Then we take t_0 large enough such that for all $|s_1| \in [|s_0|, C|s_0|^2]$ and $t > t_0$ the error term

$$o_t |g_{s_1}|_{C^1}$$

in Proposition 2.1 is also less than $\varepsilon/2$.

Then for all $g_{s_1} = g_{s(x-y)}$ with s equal to $|s_0|^2$ and $|x - y| > |s|^{-1/2} = |s_0|^{-1}$, we have $|s_1| = |s(x-y)| \in [|s_0|, C|s_0|^2]$. Therefore for all $|\xi| = |s_0|^2 e^t > |s_0|^2 e^{t_0}$, we will have that

$$|\mathbb{E}_t(g_{s(x-y)}(S_t - t))| \leq \varepsilon$$

for all $x, y \in A_\delta$. $\square$

3.5. **Quantitative rate for Fourier decay.** In order to prove a quantitative rate (Theorem 1.3), we need a rate in Proposition 3.3 and for this we give the following

Proposition 3.4. *Suppose* $\log r_j / \log r_\ell$ *is diophantine for some* $j \neq \ell$. *Then there exists* $\alpha > 0$ *such that*

$$\Big| \iint_{\mathbb{R}^2 \setminus A_\delta} \sum_{w \in \mathcal{W}_t} p_w e^{-2\pi i \xi (f_w(x) - f_w(y))} \, d\mu(x) \, d\mu(y) \Big| = O\Big(\frac{1}{|\log|\xi||^\alpha} \Big),$$

as $|\xi| \to \infty$.

Proof. By the quantitative renewal theorem (see Proposition 2.4), we obtain for some $\alpha > 0$ that

$$\Big| \mathbb{E}_t(g_{s_1}(S_t - t)) - \int_{\mathbb{R}} g_{s_1}(r) p(r) \, dr \Big| = O\Big(\frac{s_1}{t^\alpha} \Big).$$

Because the function $p(r) = \int_{x > r} d\lambda(x)$ $(r \geq 0)$ is piecewise constant with a finite number of disconnected points, the decay rate in the main term, the oscillation integral, is given by the oscillation (See [26, Lemma 3.8] for more details)

$$| \int_{\mathbb{R}^+} g_{s_1}(r) p(r) \, dr | = O(\frac{1}{s_1}).$$

Then we take $s = t^{\alpha/2}$, which implies $|s_1| = |s(x-y)| \in [t^{\alpha/4}, C t^{\alpha/2}]$ for $(x, y) \in \mathbb{R}^2 \setminus A_\delta$. Due to $|\xi| = t^{\alpha/2} e^t$, thus after taking logarithms the rate is $O(\frac{1}{|\log|\xi||^{\alpha/4}})$. □

4. Proofs of the renewal theorems

Let us now finish the paper by giving the proofs of the renewal theorems Proposition 2.1 and Proposition 2.4. This follows the similar proofs in [26], but we give the proofs for completeness. Recall that λ is a finite supported probability measure $\mathbb{R}^+$. We define a renewal operator R as follows. For a positive bounded Borel function f on $\mathbb{R}$ and a real number t, we set

$$Rf(t) = \sum_{n=0}^{+\infty} \int f(x - t) \mathrm{d}\lambda^{*n}(x).$$

Because of the positivity of f, this sum is well defined. The classical theory of Blackwell gives us a limit. But a uniform speed of convergence is needed in our application. We will give a proof using the Laplace transform, which fulfills our demands. The renewal theorem will give us an equidistribution phenomenon, where the key input is non-lattice.

First we give a proof of renewal theorem for good functions. Then we prove some regularity properties. These will imply a version of residue process.

4.1. **Laplace transform.** The Laplace transform of a compactly supported probability measure on $\mathbb{R}$ is defined by

$$\mathcal{L}\lambda(z) = \int e^{-zx} \mathrm{d}\lambda(x).$$

By the definition of non-lattice, we have

Proposition 4.1. *If λ is non-lattice, then for any pure imaginary number $i\xi$ not 0, the Laplace transform of λ is different from 1 and*

$$u(i\xi) := \frac{1}{1-\mathcal{L}\lambda(i\xi)} - \frac{1}{\sigma i\xi} \tag{4.1}$$

is holomorphic.

4.2. **Renewal theory for regular functions.** We start to compute the renewal operator. A result for the renewal operator for "good" functions will be proved. Let f be a function on $\mathbb{R}$. Define a norm by $|f|_{L^\infty} = \sup_{\xi\in\mathbb{R}} |f(\xi)|$. Define another norm $|f|_{W^{1,\infty}} = |f|_{L^\infty} + |\partial_\xi f|_{L^\infty}$. Write the Fourier transform $\widehat{f}(\xi) = \int e^{i\xi u} f(u)\mathrm{d}u$.

Proposition 4.2. *Let f be a positive bounded continuous function in $L^1(\mathbb{R}, Leb)$ such that its Fourier transform satisfies $\widehat{f} \in L^\infty$ and $\partial_\xi \widehat{f} \in L^\infty$. Assume that the projection of* $\operatorname{supp}\widehat{f}$ *onto $\mathbb{R}$ is in a compact set K. Then for all $t > 0$, we have*

$$Rf(t) = \frac{1}{\sigma}\int_{-t}^{\infty} f(u)\mathrm{d}u + \frac{1}{t} O_K(|\widehat{f}|_{W^{1,\infty}}),$$

where O_K is the the norm of $u(i\xi)$ and $\partial_\xi u(i\xi)$ on K.

Proof. Combine the following two lemmas. □

Lemma 4.3. *Under the same assumption as in Proposition 4.2, we have*

$$Rf(t) = \frac{1}{\sigma}\int_{t}^{\infty} f(u)\mathrm{d}u + \frac{1}{2\pi}\int e^{it\xi} u(i\xi)\widehat{f}(\xi)\mathrm{d}\xi,$$

where u is defined in Proposition 4.1.

This is a classical computation, for more details please see Lemma 4.6 in [26].

Lemma 4.4. *Under the same assumption as in Proposition 4.2, we have*

$$|\int e^{-it\xi} u(i\xi)\widehat{f}(\xi)\mathrm{d}\xi| \le \frac{1}{t} O_K\left(|\widehat{f}|_{L^\infty} + |\partial_\xi \widehat{f}|_{L^\infty}\right).$$

Proof. Use the fact that $\widehat{f}(\xi)$ is compactly supported and $|\widehat{f}(\xi)|$, $|\partial_\xi \widehat{f}(\xi)| < \infty$. Then applying integration by parts, we have

$$\begin{aligned}\int e^{-it\xi} u(i\xi)\widehat{f}(\xi)\mathrm{d}\xi &= \frac{1}{it}\int e^{-it\xi}\partial_\xi(u(i\xi)\widehat{f}(\xi))\mathrm{d}\xi \\ &= \frac{1}{it}\int e^{-it\xi}\left(\partial_\xi(u(i\xi))\widehat{f}(\xi) + u(i\xi)\partial_\xi\widehat{f}(\xi)\right)\mathrm{d}\xi.\end{aligned}$$

Since the operator norms of $u(i\xi)$ and $\partial_\xi u(i\xi)$ are uniformly bounded on compact regions, the result follows. □

4.3. **Regularity properties of renewal measures.** We want to use convolution to smooth out the target function. There exists an even function ψ such that it is a probability density, and the Fourier transform $\widehat{\psi}$ is compactly supported. Let $\psi_\delta(t) = \frac{1}{\delta^2}\psi(\frac{t}{\delta^2})$. Then $\int_{-\delta}^{\delta}\psi_\delta(t)\mathrm{d}t = \int_{-1/\delta}^{1/\delta}\psi(t)\mathrm{d}t > 1 - C\delta$.

Proposition 4.5. *Let $\delta \leq 1/3$ and $b_2 \geq b_1$. If $b_2 - b_1 \geq 2\delta$, then $t > 0$, we have*

$$R(\mathbf{1}_{[b_1,b_2]})(t) \ll_\psi (b_2 - b_1)(1/\sigma + O_\delta(1 + |b_2| + |b_1|)/t), \tag{4.2}$$

where $O_\delta \leq \sup_{\xi \in [-C_\psi \delta^{-2}, C_\psi \delta^{-2}]}(|u(i\xi)| + |\partial_\xi u(i\xi)|)$.

Proof. If u is in $[b_1, b_2]$, then $[u - b_2, u - b_1]$ contains at least one of $[0, \delta]$ or $[-\delta, 0]$. Therefore

$$\psi_\delta * \mathbf{1}_{[b_1,b_2]}(u) = \int_{b_1}^{b_2} \psi_\delta(u - v)\mathrm{d}v \geq \int_0^\delta \psi(v)\mathrm{d}v \geq (1 - \delta)/2.$$

Then

$$\mathbf{1}_{[b_1,b_2]} \leq 3\psi_\delta * \mathbf{1}_{[b_1,b_2]}. \tag{4.3}$$

It is sufficient to bound $R(\psi_\delta * \mathbf{1}_{[b_1,b_2]})$. Proposition 4.2 implies that

$$R(\psi_\delta * \mathbf{1}_{[b_1,b_2]}) = \frac{1}{\sigma_\lambda} \int_{-t}^{\infty} \psi_\delta * \mathbf{1}_{[b_1,b_2]} + \frac{O_\delta}{t} |\widehat{\psi_\delta} \widehat{\mathbf{1}}_{[b_1,b_2]}|_{W^{1,\infty}}.$$

The first term is less than $\int \psi_\delta * \mathbf{1}_{[b_1,b_2]} = (b_2 - b_1)$. For the second term, we have

$$\begin{aligned} |\widehat{\psi_\delta} \widehat{\mathbf{1}}_{[b_1,b_2]}|_{W^{1,\infty}} &= |\widehat{\psi_\delta} \widehat{\mathbf{1}}_{[b_1,b_2]}|_{L^\infty} + |\partial_\xi \widehat{\psi_\delta} \widehat{\mathbf{1}}_{[b_1,b_2]}|_{L^\infty} \\ &\leq C_\psi(|\mathbf{1}_{[b_1,b_2]}(u)|_{L^1} + |u\mathbf{1}_{[b_1,b_2]}(u)|_{L^1}) \leq C_\psi(b_2 - b_1)(1 + |b_1| + |b_2|). \end{aligned}$$

□

Because every step of the random walk is positive, every trajectory can only stay at most Cs times in the interval $[t, t + s]$, with C depending on λ.

Lemma 4.6. *For real numbers s, t and a point x in X, we have*

$$R(\mathbf{1}_{[0,s]})(t) \ll \max\{1, s\}. \tag{4.4}$$

4.4. **Residue process.** We introduce the residue process, which not only deals with $X_1 + \cdots + X_n$ but also takes into account the next step X_{n+1}. Let f be a positive bounded Borel function on $\mathbb{R}^2$. For $t \in \mathbb{R}$, we define the residue operator by

$$Ef(t) = \sum_{n \geq 0} \int f(y, x - t)\mathrm{d}\lambda^{*n}(x)\mathrm{d}\lambda(y). \tag{4.5}$$

Let $\mathcal{F}_u f(v, \xi) = \int f(v, u)e^{iu\xi}\mathrm{d}u$ be the Fourier transform on $\mathbb{R}_u$. Let F be a function on $\mathbb{R}_v \times \mathbb{R}_\xi$,. Define the infinite norm by

$$|F|_{L^\infty} = \sup_{v,\xi \in \mathbb{R}} |F(v, \xi)|.$$

Proposition 4.7 (Residue process)**.** *If f is a positive bounded continuous function on $\mathbb{R}^2$. Assume that the projection of $\operatorname{supp} \mathcal{F}_u(f)$ onto $\mathbb{R}_\xi$ is contained in a compact set K, and $|\mathcal{F}_u(f)|_{L^\infty}, |\partial_\xi \mathcal{F}_u(f)|_{L^\infty}$ are finite. Then for $t > 0$ and $x \in X$, we have*

$$Ef(t) = \frac{1}{\sigma} \int_{-t}^{\infty} \int_{\mathbb{R}^+} f(y, u)\mathrm{d}\lambda(y)\mathrm{d}u + \frac{1}{t} O_K \left(|\mathcal{F}_u(f)|_{L^\infty} + |\partial_\xi \mathcal{F}_u(f)|_{L^\infty}\right). \tag{4.6}$$

Proof. For a bounded continuous function f on $\mathbb{R}^2$ and $u \in \mathbb{R}$, we define an operator Q by

$$Qf(u) = \int f(y, u)\mathrm{d}\lambda(y).$$

Then

$$Ef(t) = \sum_{n\geq 0} \int Qf(x-t)\mathrm{d}\lambda^{*n}(x) = R(Qf)(t).$$

We want to use Proposition 4.2, so we need to verify the hypotheses. The function Qf is bounded and integrable by the hypotheses on f. Then

$$\widehat{Qf}(\xi) = \int Qf(u)e^{iu\xi}\mathrm{d}u = \int f(y,u)e^{iu\xi}\mathrm{d}u\mathrm{d}\lambda(y) = \int \mathcal{F}_u f(y,\xi)\mathrm{d}\lambda(y).$$

Thus $\widehat{Qf}$ is also compactly supported on ξ.

Lemma 4.8 (Change of norm)**.** *Under the assumptions of Proposition 4.7, we have*

$$|\widehat{Qf}|_{L^\infty_\xi} \ll |\mathcal{F}_u(f)|_{L^\infty},\ |\partial_\xi \widehat{Qf}|_{L^\infty_\xi} \ll |\partial_\xi \mathcal{F}_u f|_{L^\infty}.$$

Proof. The second inequality follows by the same computation as $\widehat{Qf}$. □

By Proposition 4.2, we have

$$\begin{aligned} R(Qf)(t) &= \frac{1}{\sigma}\int_X \int_{-t}^{\infty} Qf(u)\mathrm{d}u + \frac{1}{t}O_K\left(|\widehat{Qf}|_{L^\infty_\xi} + |\partial_\xi \widehat{Qf}|_{L^\infty_\xi}\right) \\ &= \frac{1}{\sigma}\int_X \int_{-t}^{\infty} Qf(u)\mathrm{d}u + \frac{1}{t}O_K\left(|\mathcal{F}_u(f)|_{L^\infty} + |\partial_\xi \mathcal{F}_u(f)|_{L^\infty}\right). \end{aligned}$$

The proof is complete. □

4.5. **Residue process with cutoff.** In this section, we restrict the residue process to the sequences $(X_{n+1}, X_n, \dots, X_1)$ such that $X_n + \dots + X_1 < t \leq X_{n+1} + \dots + X_1$. Let f be a function on $\mathbb{R}^2$. For a C^1 function on $\mathbb{R}_v \times \mathbb{R}_u$, define a norm by

$$|f|_1 = |f|_\infty + |\partial_u f|_\infty. \tag{4.7}$$

Define an operator from bounded Borel functions on $\mathbb{R}^2$ to functions on $\mathbb{R}$ by

$$E_C f(t) = \sum_{n\geq 0} \int_{x<t\leq y+x} f(y, x-t)\mathrm{d}\lambda(y)\mathrm{d}\lambda^{*n}(x).$$

By Lemma 4.12, which will be proved later, this operator is well defined. Let K be a compact set in $\mathbb{R}$. We denote $|K|$ by the supremum of the distance between a point in K and 0.

Proposition 4.9. *Let f be a continuous function on $\mathbb{R}^2$ with $|f|_1$ finite. Assume that the projection of* $\operatorname{supp} f$ *on $\mathbb{R}_v$ is contained in a compact set K. For all $\delta > 0$ and $t > |K| + \delta$, we have*

$$E_C f(t) = \int_{\mathbb{R}^+} \int_{-y}^{0} f(y,u)\mathrm{d}u\mathrm{d}\lambda(y) + O_K(\delta + O_\delta/t)|f|_1, \tag{4.8}$$

where O_δ is the same as in Proposition 4.5.

Remark 4.10. We decompose f into real and imaginary parts, then decompose these two parts into positive and negative parts. Each part satisfies the hypotheses of Proposition 4.9, with the support and the Lipschitz norm bounded by the original one. Thus, it is sufficient to prove this proposition for f positive.

The following lemma connects the operator E_c with E.

Lemma 4.11. *Under the assumptions of Proposition 4.9, let* $f_o(x, v, u) = \mathbf{1}_{-v \le u < 0} f(x, v, u)$. *Then*

$$E_C f(t) = E f_o(t).$$

Before proving this proposition, we describe some regularity properties. They are corollaries of analogous properties for the renewal process. The idea is to decompose the integral according to the last letter.

Lemma 4.12. *There exists* $C > 0$ *such that for all* $t \in \mathbb{R}$ *and* $x \in X$, *we have*

$$E_C(\mathbf{1})(t) = E(\mathbf{1}_{-v \le u < 0})(t) \le C. \tag{4.9}$$

Proof. By Lemma 4.6, we have

$$\sum_{n \ge 0} \lambda \otimes \lambda^{*n}\{(y, x) | x - t \in [-y, 0], y \ge 0\} = \int R(\mathbf{1}_{[-y,0]})(t) \mathrm{d}\lambda(y) \ll \int \max\{1, y\} \mathrm{d}\lambda(y).$$

The proof is complete. □

Using ψ_δ to regularize these functions, we write $f_\delta(v, u) = \int f_o(v, u - u_1)\psi_\delta(u_1)\mathrm{d}u_1 = \psi_\delta * f_o(v, u)$.

Lemma 4.13. *Under the same hypotheses as in Proposition 4.9, we have*

$$E(f_\delta)(t) = \int_{\mathbb{R}^+} \int_{-y}^0 f(y, u) \mathrm{d}u \mathrm{d}\lambda(y) + O(\delta + \frac{O_\delta}{t}(|K| + |K|^2))|f|_\infty.$$

Proof. We want to verify the conditions in Proposition 4.7 and then use this proposition. For the Fourier transform, we have

$$\mathcal{F}_u f_\delta = \mathcal{F}_u(\psi_\delta * f_o) = \widehat{\psi}_\delta \mathcal{F}_u f_o.$$

We need to estimate the infinite norm of $\mathcal{F}_u f_o$. This function equals

$$\int f_o(v, u) e^{i\xi u} \mathrm{d}u = \int_{-v}^0 f(v, u) e^{i\xi u} \mathrm{d}u.$$

Lemma 4.14 (Change of norm)**.** *Under the same hypotheses as in Proposition 4.9, we have*

$$|\mathcal{F}_u f_\delta|_{L^\infty} \le |K||f|_\infty, \ |\partial_\xi \mathcal{F} f_\delta|_{L^\infty} \le |K|^2 |f|_\infty.$$

Proof. Noting that in the integration $|u| \le |v|$, we get the second inequality by the same computation. □

The projection of the support of $\mathcal{F}_u f$ onto $\mathbb{R}_\xi$ is contained in $[-C_\psi \delta^{-2}, C_\psi \delta^{-2}]$. Therefore by Proposition 4.7, we have

$$E(f_\delta)(t) = \frac{1}{\sigma} \int_{-t}^\infty \int_{\mathbb{R}^+} f_\delta(y, u) \mathrm{d}\lambda(y) \mathrm{d}u + \frac{O_\delta}{t} \left(|f|_\infty (|K| + |K|^2) \right).$$

Then

$$\begin{aligned} \int_{-t}^\infty f_\delta(v, u) \mathrm{d}u &= \int_{-t}^\infty \int_{-v}^0 f(v, u_1) \psi_\delta(u - u_1) \mathrm{d}u_1 \mathrm{d}u = \int_{-v}^0 f(v, u_1) \int_{-t}^\infty \psi_\delta(u - u_1) \mathrm{d}u \mathrm{d}u_1 \\ &= \int_{-v}^0 f(v, u_1) \mathrm{d}u_1 - \int_{-v}^0 f(v, u_1) \int_{-\infty}^{-t-u_1} \psi_\delta(u) \mathrm{d}u \mathrm{d}u_1. \end{aligned}$$

Since $t-\delta \geq |K|$, we have $-t-u_1 \leq -t+v \leq -\delta$. By $\int_{-\infty}^{-\delta}\psi_\delta \leq C_\psi \delta$, this implies that $\int_{-t}^{\infty} f_\delta(v,u)\mathrm{d}u = \int_{-v}^{0} f_\delta(v,u)\mathrm{d}u(1+O(\delta))$. Using Lemma 4.12, we have

$$|\int_{\mathbb{R}^+}\int_{-y}^{0} f(y,u)\mathrm{d}u\mathrm{d}\lambda(y)| \leq |f|_\infty E_C(\mathbf{1}) = O(|f|_\infty).$$

Therefore

$$\int_{-t}^{\infty}\int_{\mathbb{R}^+} f_\delta(y,u)\mathrm{d}\lambda(y)\mathrm{d}u = \int_{\mathbb{R}^+}\int_{-y}^{0} f(y,u)\mathrm{d}u\mathrm{d}\lambda(y) + O(\delta|f|_\infty).$$

The proof is complete. □

Proof of Proposition 4.9. To simplifier the notation, we normalize f in such a way that $|f|_\infty = 1$. By Lemma 4.13, we only need to give an estimate of $E(|f_\delta - f_o|)(t)$.

Due to $f_o(v,u) = \mathbf{1}_{-v\leq u<0}(u)f(v,u)$, elementary computation (Lemma 4.26 in [26]) implies that (For simplifying notation, we omit the variable v in the following computation)

$$|f_\delta - f_o|(u) \leq \begin{cases} (|\partial_u f|_\infty + 2)\delta & u \in [-v+\delta, -\delta], \\ 2 & u \in [-v-\delta, -v+\delta] \cup [-\delta,\delta], \\ \psi_\delta * \mathbf{1}_{[-v,0]}(u) & u \in [-v-\delta,\delta]^c. \end{cases}$$

By definition of $|K|$, the first term is less than $(|\partial_u f|_\infty + 2)\delta\mathbf{1}_{[-|K|+\delta,-\delta]}$. The third term equals

$$\begin{aligned}\mathbf{1}_{[-\infty,-v-\delta]\cup[\delta,\infty]}\psi_\delta * \mathbf{1}_{[-v,0]}(u) &= \mathbf{1}_{[-\infty,-v-\delta]\cup[\delta,\infty]}(u)\int_{-v}^{0}\psi_\delta(u-u_1)\mathrm{d}u_1 \\ &= \mathbf{1}_{[-\infty,-v-\delta]\cup[\delta,\infty]}(u)\int_{u}^{u+v}\psi_\delta(u_1)\mathrm{d}u_1.\end{aligned}$$

By definition and the above arguments, we have

$$\begin{aligned}E(|f_\delta - f_o|)(t) &= \sum_{n\geq 0}\int |f_\delta - f_o|(y, x-t)\mathrm{d}\lambda^{*n}(x)\mathrm{d}\lambda(y) \\ &\leq \sum_{n\geq 0}\int \Big((|\partial_u f|_\infty + 2)\delta\mathbf{1}_{[-|K|,-\delta]}(x-t) + 2\mathbf{1}_{[-y-\delta,-y+\delta]\cup[-\delta,\delta]}(x-t) \\ &\quad + \mathbf{1}_{[-\infty,-y-\delta]\cup[\delta,\infty]}(x-t)\int_{x-t}^{x+y-t}\psi_\delta(u_1)\mathrm{d}u_1\Big)\mathrm{d}\lambda^{*n}(x)\mathrm{d}\lambda(y).\end{aligned}$$

By Lemma 4.6, the first term is controlled by $(|\partial_u f|_\infty + 2)\delta|K|$. The second term is less than $R(\mathbf{1}_{[-\delta,\delta]})(t)$. Due to Proposition 4.5, it is controlled by $C_\psi\delta(1/\sigma + O_\delta(1+2\delta)/t)$.

For the third term, we need to change the order of integration. Since $x-t>\delta$ or $x-t<-y-\delta$, we have $u_1 \geq x-t>\delta$ or $u_1 \leq x+y-t \leq -\delta$. We integrate first with respect to u_1, then the third term is less than

$$\begin{aligned}&\int_{[-\infty,-\delta]\cup[\delta,\infty]}\psi_\delta(u_1)\sum_{n\geq 0}\lambda\otimes\lambda^{*n}\{(y,x)|x+y\geq u_1+t\geq x\}\mathrm{d}u_1 \\ &= \int_{[-\infty,-\delta]\cup[\delta,\infty]}\psi_\delta(u_1)E_C(\mathbf{1})(u_1+t)\mathrm{d}u_1.\end{aligned}$$

By Lemma 4.12, the above quantity is less than $C\int_{[-\infty,-\delta]\cup[\delta,\infty]}\psi_\delta(u_1)\mathrm{d}u_1 \ll_\psi \delta$.

Therefore, we have

$$E(|f_\delta - f|)(t) = O(\delta|K| + C_\delta/t)|f|_1.$$

The proof is complete. ∎

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