Fourier dimension and spectral gaps for stationary measures on circle

Jialun LI

Contents

1 Introduction

Definition 1.1. *Let $C^\gamma(\mathbb{P}(\mathbb{R}^2))$ be the space of $\gamma-$Hölder function on $\mathbb{P}(\mathbb{R}^2)$. For $z\in\mathbb{C}$, let P_z be an operator on C^γ given by*

$$P_zf(x)=\int e^{z\log\frac{\|gv\|}{\|v\|}}f(gx)\mathrm{d}\mu(g),\ \textit{where } x=\mathbb{R}v\in\mathbb{P}(\mathbb{R}^2).$$

Theorem 1.2 (Spectral gap)**.** *Let μ be a Borel probability measure on $\mathrm{SL}_2(\mathbb{R})$ with finite exponential moment, such that the support of μ generates a Zariski dense subgroup.*

For every $\gamma>0$ small enough, there exists $\delta>0$ such that for all $|b|>1$and $|a|$ small enough the spectral radius of P_{a+ib} acting on $C^\gamma(\mathbb{P}(\mathbb{R}^2))$ satisfies

$$\rho(P_{a+ib})<1-\delta.$$

This should be compared with random walks on $\mathbb{R}$. Let μ be a Borel probability measure on $\mathbb{R}$ with finite support. Then

$$\liminf_{|b|\to\infty}|1-\hat{\mu}(ib)|=0.$$

The proof is direct. Let $\{x_1, \dots, x_l\}$ be the support of μ. Then $\hat{\mu}(ib) = \sum_{1\leq j\leq l} \mu(x_j)e^{ibx_j}$, and we only need to find b such that the terms are uniformly near 1. Using the fact that $\liminf_{b\to\infty} d_{\mathbb{R}^l}(b(x_1, \dots, x_l), 2\pi\mathbb{Z}^l) = 0$, we have the claim.

We can also compare with the counting problem in hyperbolic surfaces. The spectral gap is used to obtain an exponential remainder term as in [LP82], [Nau05] and [Sto11]. An analogue application is given in Section 5.1, that is an exponential remainder term in renewal theorem.

The main ingredient for proving these result is a property about the power decay of the Fourier coefficients of the μ-stationary measure.

Theorem 1.3 (Fourier decay)**.** *Let μ be a Borel probability measure on $\mathrm{SL}_2(\mathbb{R})$ with finite exponential moment, such that the support of μ generates a Zariski dense subgroup. Let $X = \mathbb{P}(\mathbb{R}^2)$ and let ν be the μ-stationary measure on X.*

For every $\gamma > 0$, there exists $\epsilon_0 > 0$ depending on μ such that there exists $\delta_1 > 0$ such that the following holds. For any $\varphi \in C^2(X)$, $r \in C^\gamma(X)$ such that $|\varphi'| \geq |\xi|^{-\epsilon_0}$ on the support of r, $|r| \leq 1$ and

$$\|\varphi\|_{C^2} + c_\gamma(r) \leq |\xi|^{\epsilon_0},$$

then

$$\left|\int e^{i\xi\varphi(x)} r(x) \mathrm{d}\nu(x)\right| \leq |\xi|^{-\delta_1} \quad \text{for all } |\xi| \text{ large enough.} \tag{1.1}$$

Remark 1.4. *As a consequence, the Fourier coefficients of the measure ν converge to zero with a power decay. This is also a generalization of the same theorem for the Patterson-Sullivan measures as in [BD17].*

Inspired by [BD17], we introduce the following definition, which is the key input to using the machine of the discretized sum-product estimate.

Definition 1.5 (Non-concentration hypothesis)**.** *Let μ be a Borel probability measure on $\mathrm{SL}_2(\mathbb{R})$ with finite exponential moment, such that the support of μ generates a Zariski dense subgroup. Let σ_μ be the Lyapunov constant. We say that μ satisfies non-concentration hypothesis if there exist $C_1, \epsilon_1, c_1 > 0$ such that for all $x \in X, h \in G$ and $n \in \mathbb{N}$ we have*

$$\sup_{a\in\mathbb{R}} \mu^{*n}\{g \in G | \frac{\|h\|^2\|v\|^2 e^{2n\sigma_\mu}}{\|hgv\|^2} \in [a \mp e^{-\epsilon_1 n}]\} \leq C_1 e^{-c_1 n} \tag{1.2}$$

Consider a degenerate case that μ is supported on diagonal matrices. counterexample on $\mathbb{R}$, with $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$, where δ_x is the Dirac measure.

We will make use of some classic notation: For two real numbers A and B, we write $A = O(B), A \ll B$ or $B \gg A$ if there exists constant $C > 0$ such that $|A| \leq CB$, where C only depends on the ambient group G and the measure μ. We write $A = O_\epsilon(B), A \ll_\epsilon B$ or $B \gg_\epsilon A$ if the constant C depends on an extra parameter $\epsilon > 0$.

2 Random walks on Lie groups

We will write V for $\mathbb{R}^2$, equipped with the norm $\|v\|^2 = v_1^2 + v_2^2$. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. For g in G, let $\|g\|$ be the operator norm. Let σ_μ be the Lyapunov constant of μ, defined by

$$\sigma_\mu = \lim_{n\to\infty} \frac{1}{n} \log \|g_1 \cdots g_n\|,$$

almost surely for $(g_1, g_2, \dots)$ following the law of $\mu^{\otimes\mathbb{N}}$. For g in G and $x = \mathbb{R}v$ in X, we define the Iwasawa cocycle by

$$\sigma(g, x) = \frac{\|gv\|}{\|v\|}.$$

Let $G = KA^+K$ be a Cartan decomposition, where A^+ is the semigroup of diagonal matrices $\mathrm{diag}\{a, a^{-1}\}$ with $a \geq 1$ and $K = SO(2)$. For an element g in G, under the Cartan decomposition, it can be written as $g = k_g a_g l_g$. We will write $\kappa(g) = \log \|g\|$, called the Cartan projection of g, because we have $a_g = \mathrm{diag}\{e^{\kappa(g)}, e^{-\kappa(g)}\}$. Let $x_g^M = \mathbb{R}k_g e_1$ and $y_g^m = \mathbb{R}^t l_g e_2$, called the density point of g and g^t, respectively.

For $r > 0$, g in $GL(V)$, let

$$b_g^M(r) = \{x \in \mathbb{P}V | d(x, x_g^M) \leq r\},$$
$$B_g^m(r) = \{x \in \mathbb{P}V | \delta(x, y_g^m) \geq r\}$$

2.1 Distance and norm

This section deals with general g in $GL(V)$ acting V, where V is a finite dimensional vector space with euclidean norm. We need some technical control of distance.

Definition 2.1. *Let $x = \mathbb{R}v, x' = \mathbb{R}v'$ be two points in X. We define the projective distance by*

$$d(x, x') = \frac{\|v \wedge v'\|}{\|v\|\|v'\|}$$

This is a lemma about the relation between cocycle, Cartan projection and the distance of points

Lemma 2.2. *[BQ16, Lem 17.11, Lem 13.2, Lem 14.2] For any g in $GL(V)$ and x, x' in $\mathbb{P}V$, we have*

$$|\sigma(g, x) - \sigma(g, x')| \leq \frac{\sqrt{2}d(x, x')}{\min(\delta(x, y_g^m), \delta(x', y_g^m))} \tag{2.1}$$

$$\kappa(g) + \log \delta(x, y_g^m) \leq \sigma(g, x) \tag{2.2}$$

$$d(gx, x_g^M) \leq \|g\|^{-2}/\delta(x, y_g^m) \tag{2.3}$$

As a corollary, we have

Lemma 2.3. *For g in $GL(V)$, let $0 \leq \beta \leq \gamma_{1,2}(g)$. If $\beta < \delta^2$, then*

$$gB_g^m(\delta) \subset b_g^M(\frac{\beta}{\delta}) \subset b_g^M(\delta).$$

This is a lemma of distance, we use Cartan projection.

Lemma 2.4. *For any g, h in $GL(V)$ and $x = \mathbb{R}v, x' = \mathbb{R}v'$ in $\mathbb{P}V$, we have*

$$e^{-2\kappa(g)} \leq \frac{d(gx, gx')}{d(x, x')} \leq \frac{e^{-2\kappa(g)}}{\delta(x, y_g^m)\delta(x', y_g^m)} \tag{2.4}$$

$$\kappa(gh) \geq \kappa(g) + \kappa(h) + \log \delta(x_h^M, y_g^m) \tag{2.5}$$

Proof. By (2.2), we have

$$d(gx, gx') = \frac{\|g(v \wedge v')\|}{\|gv\|\|gv'\|} \leq \frac{\| \wedge^2 g\|}{\|g\|^2 \delta(x, y_g^m)\delta(x', y_g^m)} = \frac{\gamma_{1,2}(g)}{\delta(x, y_g^m)\delta(x', y_g^m)}$$

$$d(gx, gx') = \frac{\|gv \wedge gv'\|}{\|v \wedge v'\|} \frac{\|v \wedge v'\|}{\|v\|\|v'\|} \frac{\|v\|\|v'\|}{\|gv\|\|gv'\|} \geq \gamma_{1,2}(g)\delta(x \wedge x', y_{\wedge^2 g}^m)d(x, x')$$

On the other hand, $\kappa(gh) \geq \sigma(gh, l_h^{-1}e_1) = \sigma(g, x_h^M) + \kappa(h) \geq \kappa(g) + \kappa(h) + \log \delta(x_h^M, y_g^m)$. □

This lemma says that when x is not in the neighborhood of y_h^m, then if h is a general element, this kind of regularity will be inherited to hx.

2.2 Large deviation principles

In the next proposition we summarize the large deviations principle for the cocycle and for the Cartan projection,

Proposition 2.5. *[BQ16, Thm13.11, Thm 13.17] Under the assumptions of Theorem , for every $\epsilon > 0$ there exist $C, c > 0$ such that for all $n \in \mathbb{N}$ and $x \in X$ we have*

$$\mu^{*n}\{g \in G|\ |\sigma(g,x) - n\sigma_\mu| \geq n\epsilon\} \leq Ce^{-c\epsilon n}, \tag{2.6}$$
$$\mu^{*n}\{g \in G|\ |\kappa(g) - n\sigma_\mu| \geq n\epsilon\} \leq Ce^{-c\epsilon n}, \tag{2.7}$$

The following proposition describes regularity properties of μ^{*n}, which is a corollary of the large deviations principle.

Proposition 2.6. *[BQ16, Prop14.3] Under the assumptions of Theorem , for every $\epsilon > 0$ there exist C, c such that for all x, x' in X and $n \geq 1$ we have*

$$\mu^{*n}\{g \in G|\ d(gx, x') \leq e^{-n\epsilon}\} \leq Ce^{-c\epsilon n}, \tag{2.8}$$
$$\mu^{*n}\{g \in G|\ d(x_g^M, x) \leq e^{-n\epsilon}\} \leq Ce^{-c\epsilon n}, \tag{2.9}$$
$$\mu^{*n}\{g \in G|\ d(y_g^m, x) \leq e^{-n\epsilon}\} \leq Ce^{-c\epsilon n}, \tag{2.10}$$

Theorem 2.7 (Hölder regularity)**.** *[Gui90][BL85, Chapter 6,Prop. 4.1] Under the assumptions of Theorem , there exist constants $C > 0$, $c > 0$ such that for every x in X and $r > 0$ we have*

$$\nu(B(x,r)) \leq Cr^c. \tag{2.11}$$

2.3 Derivative

Lemma 2.8. *Let g be in G and x be in X. We fix the unit tangent vector field on X. Then the derivative $g'(x)$ can be viewed as real numbers, and we have*

$$g'(x) = e^{-2\sigma(g,x)}.$$

Lemma 2.9 (uniform non integrability)**.** *If $d(y_g^m, y_h^m)$, $d(x, y_g^m)$, $d(x, y_h^m) > c$ and $\|g\|, \|h\| \geq 1/c^2$ we have*

$$1/c \geq |\partial_x(\sigma(g,x) - \sigma(h,x))| \geq c/2 \tag{2.12}$$

and

$$|\partial_x\sigma(g,x)|,\ |\partial_{xx}\sigma(g,x)| \leq 1/c \tag{2.13}$$

Remark 2.10. *These properties tell us the random walk in a large set behave like uniformly expanding map, and has nonvanish derivative.*

Proof. Suppose $g = \mathrm{diag}\{a, a^{-1}\}$. By definition

$$\begin{aligned}\sigma(g,x) = \log\frac{\|gv\|}{\|v\|} &= \log \|\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\begin{pmatrix}\cos x \\ \sin x\end{pmatrix}\| \\ &= 1/2\log(a^2\cos^2 x + a^{-2}\sin^2 x)\end{aligned}$$

Hence, using $\cos x \geq c$ we have

$$\begin{aligned}\partial_x\sigma(g,x) &= -\frac{(a^2 - a^{-2})\cos x\sin x}{a^2\cos^2 x + a^{-2}\sin^2 x} = -\tan x + \frac{a^{-2}(\cos x\sin x + \sin^3 x\cos^{-1} x)}{a^2\cos^2 x + a^{-2}\sin^2 x} \\ &= -\tan x + O(a^{-4}c^{-3})\end{aligned}$$

The condition $d(y_g^m, y_h^m) > c$ means that $|\sin(\theta_h)| \geq c$.

$$\begin{aligned}|\partial_x\sigma(g,x) - \partial_x\sigma(h,x)| &= |\tan(x) - \tan(x+\theta_h) + c^{-3}O(e^{-4\kappa(g)} + e^{-2\kappa(h)})| \\ &= |\frac{\sin(\theta_h)}{\cos(x+\theta_h)\cos(x} + c^{-3}O(e^{-4\kappa(g)})| \geq c/2\end{aligned}$$

For the estimate of the second order derivative, by definition we have

$$\partial_{xx}\sigma(g,x) = -(a^2 - a^{-2})\frac{a^2\cos^4 x - a^{-2}\sin^4 x}{(a^2\cos^2 x + a^{-2}\sin^2 x)^2} \leq 1/c$$

The proof is complete. □

3 Sum-product estimates

3.1 Non concentration condition

If we want to get the non-concentration directly, then this becomes an effective local limit estimate, which is difficult due to the lack of spectral gap. Hence, we transfer it to the Hölder regularity of stationary measure.

Another technique is to replace tiny interval by a neighborhood of diagonal. This technique is already used in [BD17]. We first get an estimate of both the action and the position. Then we fix the position to get the non-concentration estimate. But if we do this naively, different a in $\mathbb{R}$ will obstruct us getting an estimate uniformly on all a in $\mathbb{R}$. The estimate of the diagonal implies uniform estimate for all a in $\mathbb{R}$.

Let $\psi(g,x) = \exp(-2\sigma(h_0 g, x) + 2\kappa(h_0) + 2\omega(g)\sigma_\mu)$ for x in X and g in $G^{\times n}$.

Definition 3.1. *We say that μ satisfies diagonal non-concentration, if there exist $\epsilon, c, C > 0$ such that for all n in $\mathbb{N}$ and x in X*

$$\mu^{*3n}\{(g,l,h)||\psi(g,hx) - \psi(l,hx)| \leq e^{-\epsilon n}\} \leq Ce^{-c\epsilon n}$$

This is a probabilistic analogue of [BD17, Lem.2.16], which is the main input for the sum-product estimates. Our new observation is that this implies non concentration, a rather pleasant form.

Lemma 3.2. *Diagonal non-concentration implies non concentration condition.*

Proof. Let $\delta = e^{-\epsilon n}$ and $\beta = e^{-n\sigma_\mu}$. Separate $g = g_1 g_2$ such that g_1, g_2 follow the same law of μ^{*n}. Fix g_2 and measure on g_1. Then

$$\begin{aligned}A :=& \sup_a \mu^{*2n}\{g|\psi(g,x) \in [a \mp \delta]\} \\ \leq& \int_G \sup_a \mu^{*n}\{g_1|\psi(g_1, g_2 x) \in e^{2\sigma(g_2,x)}\beta^{-2}[a \mp \delta]\}\mathrm{d}\mu^{*n}(g_2)\end{aligned}$$

By large deviation principle, for g_2 outside an exponential small set, we have $e^{2\sigma(g_2,x)-2n\sigma_\mu} \leq \delta^{-1/2}$. Let $\delta' = \delta^{1/2}$. Therefore

$$A \leq \int_G \sup_a \mu^{*n}\{g_1|\psi(g_1, g_2 x) \in [a \mp \delta']\}\mathrm{d}\mu^{*n}(g_2) + O_\epsilon(\delta^{c_1}),$$

where $c_1 > 0$ depends on ϵ. Replace sup estimate by diagonal estimate

$$\begin{aligned}A \leq& \int_G \mu^{*n}\otimes\mu^{*n}\{(g_1, g_1')||\psi(g_1, g_2 x) - \psi(g_1', g_2 x)| \leq 2\delta'\}^{1/2}\mathrm{d}\mu^{*n}(g_2) + O_\epsilon(\delta^{c_1}) \\ \leq& \mu^{*3n}\{(g_1, g_1', g_2)||\psi(g_1, g_2 x) - \psi(g_1', g_2 x)| \leq 2\delta'\}^{1/2} + O_\epsilon(\delta^{c_1})\end{aligned}$$

The proof ends by Definition 3.1. □

3.2 Hölder regularity

This is the key new ingredient compared with [BD17]. Using other representations, we can get more information on the cocycle. Our method is flexible, which can be generalized to higher rank case. This idea has already been used in [Aou13] for problem concerning transience of algebraic subvariety of split real lie groups.

The key tool is the large deviation principle

Lemma 3.3. *Let V be an irreducible representation of G. There exist $\epsilon, c, C > 0$ such that for v in V and f in V^* we have*

$$\mu^{*n}\{g||f(gv)| \leq \|f\|\|gv\|e^{-\epsilon n}\} \leq Ce^{-c\epsilon n}.$$

Lemma 3.4. *Diagonal non-concentration is a consequence of the following: There exist ϵ_1, c_1 such that for all $x \in X$, $n \in \mathbb{N}$ there exists G_n in G^2 with $\mu^{*n} \otimes \mu^{*n}\{G_n\} \geq 1 - C_1 e^{-c_1 \epsilon n}$, such that for $(g, l) \in G_n$ we have*

$$\mu^{*n}\{h||\psi(g, hx) - \psi(l, hx)| \leq e^{-\epsilon_1 n}\} \leq C_1 e^{-c_1 \epsilon_1 n}.$$

The idea is that we will transform the inequality of cocycle to an inequality of polynomial function on v, with some exponentially small error. Then the polynomial function can be interpreted as an linear form on the symmetric product of the vector space V. In order to using large deviation principle, we need to estimate the norm of this linear form. The norm will satisfy our demand of largeness if g, l are in "general position".

Proof. Let $\beta = e^{-n\sigma_\mu}$ and let $\delta = e^{-\epsilon n}$, which will be determined later. Take n large enough depending on ϵ such that $\delta \leq 1/2$. We will prove that we can take

$$G_n = \{(g, l)||d(y_l^m, y_g^m), \delta(x_g^M, y_{h_0}^m) \geq \delta, \|g\|, \|l\| \in \beta^{-1}[\delta, \delta^{-1}]\}$$

Suppose that $(g, l) \in G_n$. Due to $\psi(g, x) = \frac{\|h_0\|^2\beta^{-2}\|v\|^2}{\|h_0 gv\|^2}$, we have

$$|\psi(g, x) - \psi(l, x)| = \frac{\|h_0\|^2\beta^{-2}\|v\|^2}{\|h_0 gv\|^2\|h_0 lv\|^2}|\|h_0 gv\|^2 - \|h_0 lv\|^2| \tag{3.1}$$

By the definition of G_n, the fraction term satisfies

$$\frac{\|h_0\|^2\beta^{-2}\|v\|^2}{\|h_0 gv\|^2\|h_0 lv\|^2} \geq \frac{\beta^{-2}}{\|h_0\|^2\|g\|^2\|l\|^2\|v\|^2} \geq \frac{1}{\|h_0\|^2\beta^2\delta^{-2}\|v\|^2} \tag{3.2}$$

Now we consider the linear form on $Sym^2 V$

$$f(v) := \|h_0 gv\|^2 - \|h_0 lv\|^2 \tag{3.3}$$

By (3.1), (3.2), (3.3)

$$|\psi(g, x) - \psi(l, x)| \geq \frac{f(v)}{\|h_0\|^2\beta^2\delta^{-2}\|v\|^2}. \tag{3.4}$$

Next we want to use large deviation principle, so we give an estimate of $\|f\|$. We take $x = \mathbb{R}v = y_l^m$, and we get $\|h_0 lv\| \leq \|h_0\|\beta\delta^{-1}$. Due to $d(y_g^m, y_l^m) \geq \delta$, by Lemma 2.3, we have $gx \in b(x_g^M, \|g\|/\delta) \subset b(x_g^M, \delta/2)$. Hence $\delta(y_{h_0}^m, gx) \geq \delta/2$, and by (2.2) we have

$$\|h_0 gv\| \geq \|h_0\|\|g\|\delta(y_{h_0}^m, gx)\delta(y_g^m, x)\|v\| \geq \|h_0\|\beta^{-1}\delta^{O(1)}\|v\|.$$

Therefore when ϵ is small compared to σ_μ, by $\delta < 1/2$ we have

$$\|f\| = \sup_v \frac{|f(v)|}{\|v\|} \geq \|h_0\|^2(\beta^{-2}\delta^{O(1)} - \beta^2\delta^{-2}) \geq \|h_0\|^2\beta^{-2}\delta^C, \tag{3.5}$$

for some constant $C > 0$. Then by (3.4), (3.5)

$$|\psi(g,x) - \psi(l,x)| \geq \frac{f(v)\delta^{C+2}}{\|f\|\|v\|^2}. \tag{3.6}$$

By Lemma, we have

$$\mu^{*n}\{h||f(hv)| \leq \|f\|\|hv\|\delta\} = O_\epsilon(\delta^{c_1}), \tag{3.7}$$

where $c_1 > 0$ depends on ϵ. If we fix ϵ small and take $e^{-\epsilon_1 n} = \delta^{C+3}$, then by (3.6), (3.7) the result holds for n large. For n small, the result follows by replacing C_1 by a large number. □

3.3 Key combinatorial tool

One of the key tools is the sum-product estimates.

Proposition 3.5. *[Bou10, Lemma 8.43] For every $\kappa_1 > 0$, there exist ϵ, ϵ' and $k \in \mathbb{N}$, such that the following holds. Let N be a large integer depending on κ_1. For all probability measures λ supported in $[1/2, 1]$, which satisfies*

$$\sup_x \lambda(B(x,\sigma)) < \sigma^{\kappa_1}$$

for all $\sigma \in [N^{-1}, N^{-\epsilon}]$, we have

$$\left| \int \exp(2i\pi\tau x_1 \cdots x_k) \mathrm{d}\lambda(x_1) \cdots \mathrm{d}\lambda(x_k) \right| \leq N^{-\epsilon'}, \tag{3.8}$$

for all $\tau \in \mathbb{R}, |\tau| \sim N$.

An adaptation to the case of several different measures

Proposition 3.6. *[BD17, Prop. 3.2] Fix $\kappa_1 > 0$. Then there exist $k \in \mathbb{N}, \epsilon > 0$ depending only on κ_1 such that the following holds. Let $C_0 > 0$ and $\lambda_1, \dots \lambda_k$ be Borel measures on $[C_0^{-1}, C_0] \subset \mathbb{R}$ such that $\lambda_j(\mathbb{R}) \leq C_0$. Let $\tau \in \mathbb{R}, |\tau| \geq 1$, and assume that for all $\sigma \in [C_0^{-1}|\tau|^{-1}, C_0|\tau|^{-\epsilon}]$ and $j = 1, \dots, k$*

$$\lambda_j \times \lambda_j(\{(x,y) \in \mathbb{R}^2 : |x - y| \leq \sigma\}) \leq C_0\sigma^{\kappa_1}.$$

Then there exists a constant C_1 depending only on C_0, κ_1 such that

$$|\int \exp(2\pi i\tau x_1 \cdots x_k) \mathrm{d}\lambda_1(x_1) \dots \mathrm{d}\lambda_k(x_k)| \leq C_1|\tau|^{-\epsilon}$$

In our case, the measures are not compactly supported, hence we give another version

Proposition 3.7. *For every $\kappa_0 > 0$, there exist ϵ_2 and $k \in \mathbb{N}$ depending only on κ_0 such that the following holds. Let $C_0 > 0$ and let $\tau \in \mathbb{R}, |\tau| > 1$. Let $\lambda_1, \dots \lambda_k$ be Borel measures supported on $[|\tau|^{-\epsilon_3}, |\tau|^{\epsilon_3}]$, where $\epsilon_3 = \min\{\epsilon_2, \epsilon_2\kappa_0\}/10k$, with total mass less than 1. Assume that for all $\sigma \in [|\tau|^{-2}, |\tau|^{-\epsilon_2}]$ and $j = 1, 2, \dots k$ we have*

$$\sup_x \lambda_j(B(x,\sigma)) < C_0\sigma^{\kappa_0}. \tag{3.9}$$

Then there exists a constant C_2 depending only on C_0, κ_0 such that for all $\varsigma \in \mathbb{R}, |\varsigma| \in [|\tau|^{3/4}, |\tau|^{5/4}]$ we have

$$\left| \int \exp(i\varsigma x_1 \cdots x_k) \mathrm{d}\lambda_1(x_1) \cdots \mathrm{d}\lambda_k(x_k) \right| \leq C_2|\tau|^{-\epsilon_2}. \tag{3.10}$$

Proof. Let ϵ as in Proposition 3.6 with $\kappa_1 = \kappa_0/2$, and let $\epsilon_2 = \epsilon/4$.

Divide $[|\tau|^{-\epsilon_3}, \tau^{\epsilon_3}]$ into $[2^l, 2^{l+1}]$. We rescale the measure in each interval to $[1/2, 1]$. Let $\lambda^l(A) = \lambda|_{[2^{l-1},2^l]}(2^l A)$. For $\sigma \in [\tau^{-3/2}, \tau^{-\epsilon_2/2}]$ we have

$$\begin{aligned}\lambda^l \times \lambda^l\{(x,y) \in \mathbb{R}^2 ||x-y| \le \sigma\} &\le \lambda \times \lambda\{(2^l x, 2^l y) \in \mathbb{R}^2 ||x-y| \le \sigma\} \\ &= \lambda \times \lambda\{(x,y) \in \mathbb{R}^2 ||x-y| \le 2^l\sigma\} \le \sup_{a\in\mathbb{R}} \lambda B(a, 2^l\sigma) \le C_0(2^l\sigma)^{\kappa_0}\end{aligned}$$

where we use $2^l\sigma \in [\tau^{-3/2-\epsilon_3}, \tau^{-\epsilon_2/2+\epsilon_3}] \subset [\tau^{-2}, \tau^{-\epsilon_2/4}]$. Since $\sigma^{-1/2} \ge \tau^{\epsilon_2/4} \ge \tau^{\epsilon_3} \ge 2^l$, for $\sigma \in [\tau^{-3/2}, \tau^{-\epsilon_2/2}]$ we have

$$\lambda^l \times \lambda^l\{(x,y) \in \mathbb{R}^2 ||x-y| \le \sigma\} \le C_0(2^l\sigma)^{\kappa_0} \le C_0\sigma^{\kappa_0/2}. \tag{3.11}$$

Summing up over $|l| \le \epsilon_3 \log_1 \tau$, we have

$$\begin{aligned}\left|\int \exp(2i\pi\varsigma x_1\cdots x_k)\mathrm{d}\lambda_1(x_1)\cdots \mathrm{d}\lambda_k(x_k)\right| &\le \sum_{l_j}\left|\int \exp(2i\pi\varsigma x_1\cdots x_k)\mathrm{d}\lambda_1^{l_1}(2^{-l_1}x_1)\cdots \mathrm{d}\lambda_k^{l_k}(2^{-l_k}x_k)\right| \\ &= \sum_{l_j}\left|\int \exp(2i\pi\varsigma 2^{l_1+\cdots+l_k} y_1\cdots y_k)\mathrm{d}\lambda_1^{l_1}(y_1)\cdots \mathrm{d}\lambda_k^{l_k}(y_k)\right|\end{aligned}$$

Let $\tau_1 = \varsigma 2^{l_1+\cdots+l_k} \in [\tau^{3/4-k\epsilon_3}, \tau^{5/4+k\epsilon_3}]$. Then we have $[\tau_1^{-1}, \tau_1^{-\epsilon_2}] \subset [\tau^{-3/2}, \tau^{-\epsilon_2/2}]$. The condition of Proposition 3.6 is verified by (3.11) with τ replaced by τ_1. Therefore

$$\begin{aligned}\sum_{l_j}\left|\int \exp(2i\pi\varsigma 2^{l_1+\cdots+l_k} y_1\cdots y_k)\mathrm{d}\lambda_1^{l_1}(y_1)\cdots \mathrm{d}\lambda_k^{l_k}(y_k)\right| &\le C_1 \sum_{l_j} |\varsigma 2^{l_1+\cdots+l_k}|^{-\epsilon_2} \\ \le C_2\varsigma^{-\epsilon_2}\Big(\sum_{|l|\le\epsilon_3\log_1\tau}(2^{-l})^{-\epsilon_2}\Big)^k &\le C_2\varsigma^{-\epsilon_2}\tau^{k\epsilon_3\epsilon_2}(1-2^{-\epsilon_2})^{-k} \le C_2\tau^{-\epsilon_2/4}.\end{aligned}$$

The proof is complete. $\square$

3.4 Application to our measure

Let (ϵ_1, c_1) be constants in non-concentration condition. Let $(\epsilon_1/4, c')$ be constants in large deviation principle. Take

$$\kappa_0 = \frac{1}{10}\min\{c, c'\}.$$

Using Proposition 3.7, we get ϵ_2, ϵ_3.

For g, h in G and x in X, let $\psi_g(h,x) = \exp(-2\sigma(gh,x) + 2\kappa(g) + 2n\sigma_\mu)$. Let $\lambda_{g_0,x}$ be a pushforward measure on $\mathbb{R}$ of μ^{*n} restricted on a subset $G_{n,g_0,x}$ of G, defined by

$$\lambda_{g_0,x}(E) = \mu^{*n}\{h \in G_{n,g_0,x} | \psi_{g_0}(h,x) \in E\},$$

for Borel subset E, and where

$$G_{n,g_0,x} = \{h \in G | d(y_{g_0}^m, x_h^M), d(y_h^m, x) \ge 2\delta,\ \|h\| \in \beta^{-1}[\delta, \delta^{-1}]\} \tag{3.12}$$

and $\beta = e^{-\sigma_\mu n}, \delta = e^{-\epsilon n}$, where $\sigma_\mu/4 \ge \epsilon > 0$ will be determined later.

The non-concentration condition is only at one scale, we need to verify all the scale needed in the sum-product estimate. The idea is to separate the random variable and try using the non-concentration condition in other scale, where we use the cocycle property to change scale.

Proposition 3.8. *With ϵ small enough depending on $\epsilon_3\epsilon_1$, there exists C_0 such that the measure $\lambda_{g_0,x}$ satisfies the conditions in Proposition 3.7 with constant $\tau = e^{\epsilon_1 n}$.*

Proof. We abbreviate $\lambda_{g_0,x}$ to λ. We first verify the condition on the support of λ. By hypothesis 3.12, the upper bound only need the control of Cartan projection

$$\sigma(g_0h,x)-\kappa(g_0)-n\sigma_\mu\le\kappa(h)-n\sigma_\mu\le\epsilon n.$$

For the lower bound, we need to control the position of y_h^m, hx. By hypothesis 3.12 and Lemma 2.3, we have $hx\in b_h^M(\beta/\delta^2)\subset b_h^M(\delta)\subset B_{g_0}^m(\delta)$. Hence by (2.1)

$$\sigma(g_0h,x)-\kappa(g_0)-n\sigma_\mu\ge\log\delta(hx,y_{g_0}^m)+\kappa(h)+\log\delta(y_h^m,x)-n\sigma_\mu\ge-3\epsilon n$$

Hence taking ϵ small depend on $\epsilon_3\epsilon_1$, we have $|\sigma(g_0h,x)-\kappa(g_0)-n\sigma_\mu|\le\epsilon_3\log\tau$. Therefore the support of λ is contained in the interval $[|\tau|^{-\epsilon_3},|\tau|^{\epsilon_3}]$.

Let $\sigma\in[|\tau|^{-2},|\tau|^{-\epsilon_2}]$. Since the support of λ is restricted in an interval, for the condition 3.9, we could suppose $a\pm\sigma\in[|\tau|^{-\epsilon_3},|\tau|^{\epsilon_3}]$. Let $m=\frac{|\log\sigma|}{2\epsilon_1}$. Then m lies in $[\epsilon_2 n/2,n]$. We separate $h=h_1h_2$ with h_1 the law μ^{*m} and h_2 the law μ^{*n-m}. We have

$$\psi_{g_0}(h,x)=\psi_{g_0}(h_1,h_2x)e^{-2\sigma_1(h_2,x)},$$

where $\sigma_1(h_2,x)=\sigma(h_2,x)-(n-m)\sigma_\mu$. Then

$$\lambda(B(a,\sigma))\le\sup_{h_2}\mu^{*m}\{h_1|\psi_{g_0}(h_1,h_2x)\in e^{2\sigma_1(h_2,x)}B(a,\sigma)\}\tag{3.13}$$

- If $\sigma_1(h_2,x)\le\epsilon_1m/2$, then $\sigma e^{2\sigma_1(h_2,x)}\le\sigma^{1/2}=e^{-\epsilon_1m}$. It follows by the non-concentration condition at scale m that

$$\mu^{*m}\{h_1|\psi_{g_0}(h_1,h_2x)\in B(a,e^{-\epsilon_1m})\}\ll_{\epsilon_1}e^{-c_1\epsilon_1m}\le\sigma^{\kappa_0}$$

- If $\sigma_1(h_2,x)\ge\epsilon_1m/2$, then $(a-\sigma)e^{2\sigma_1(h_2,x)}\ge|\tau|^{-\epsilon_3}e^{\epsilon_1m}\ge e^{\epsilon_1m/2}=\sigma^{-1/4}$, where we use the fact that $\epsilon_2\ge2\epsilon_3$. It follows by large deviation that

$$\begin{aligned}\mu^{*m}\{h_1|\psi_{g_0}(h_1,h_2x)\in e^{2\sigma_1(h_2,x)}B(a,\sigma)\}&\le\mu^{*m}\{h_1|\psi_{g_0}(h_1,h_2x)\ge e^{\epsilon_1m/2}\}\\&\ll_{\epsilon_1}e^{-c'\epsilon_1m}\le\sigma^{\kappa_0},\end{aligned}$$

  where the large deviation principle applies to random variable h_1 with quantities $\sigma(h_1,h_2x)\le m(\sigma_\mu-\epsilon_1/4)$ and $\delta(h_1h_2x,y_{g_0}^m)\le e^{-\epsilon_1m/4}$.

The proof is complete. □

4 Proof of the mains theorems

4.1 From sum-product estimates to Fourier decay

In this subsection we prove Theorem 1.3, an estimate of Fourier decay, by using the result in section 3. We will say that a property $P_n(b)$ is true except on an exponentially small set if there exists $C,c>0$ such that for $n\in\mathbb{N}$ we have

$$\mu^{*n}\{g\in G|P_n(g)\text{ is true }\}\ge1-Ce^{-cn}$$

Recall the definition that

$$\kappa_0=\frac{1}{10}\min\{c,c'\},$$

where (ϵ_1,c_1) is constants in non-concentration condition and $(\epsilon_1/4,c')$ is constants in large deviation principle. Take k,ϵ_2,ϵ_3 from Proposition 3.7 with this κ_0. Let

$$\tau=|\xi|^{\frac{\epsilon_1}{(4k+2)\sigma_\mu+\epsilon_1}},n=\log\tau/\epsilon_1\tag{4.1}$$

Let ϵ be a positive number to be determined later (the only constant which is not fixed yet), and let

$$\beta = e^{-\sigma_\mu n}, \delta = e^{-\epsilon n}. \tag{4.2}$$

With these choices of constants, we have

$$\frac{1}{|\xi|} = \beta^{4k+2+\frac{\epsilon_1}{\sigma_\mu}} = \beta^{4k+2}\tau^{-1}. \tag{4.3}$$

The constant ϵ_0 in the hypothesis of Theorem 1.3 is defined as $\epsilon/((4k+2)\sigma_\mu + \epsilon_1)$, which will be fixed once ϵ is fixed. Hence regularity scale equals

$$|\xi|^{\epsilon_0} = e^{-\epsilon n} = \delta^{-1}. \tag{4.4}$$

Notation: We state some notation which will be used throughout this section.

- Let $\mathbf{g} = (g_0, \ldots, g_k)$ be an element in $G^{\times(k+1)}$.
- Let $\mathbf{h} = (h_1, \ldots, h_k)$ be an element in $G^{\times k}$.
- We write $\mathbf{g} * \mathbf{h} = g_0 h_1 \cdots h_k g_k \in G$ be the product of $\mathbf{g}, \mathbf{h}$.
- We write $T\mathbf{g} * \mathbf{h} = g_0 h_1 \cdots g_{k-1} h_k \in G$.
- For $l \in \mathbb{N}$, let $\mu_{l,n}$ be the product measure on $G^{\times l}$ given by $\mu_{l,n} = \underbrace{\mu^{*n} \otimes \cdots \otimes \mu^{*n}}_{l \text{ times}}$.
- Recall that for g, h in G and x in X, we define $\psi_g(h,x) = \exp(-2\sigma(gh,x) + 2\kappa(g) + 2n\sigma_\mu)$.
- Recall the definition of $\lambda_{g,x}$. For g in G and x in $\mathbb{P}(\mathbb{R}^2)$, let $\lambda_{g,x}$ be a pushforward measure on $\mathbb{R}$ of μ^{*n} restricted on a subset $G_{n,g,x}$ defined by

  $$\lambda_{g,x}(E) = \mu^{*n}\{h \in G_{n,g,x} | \psi_g(h,x) \in E\},$$

  for Borel subset E, and where $G_{n,g,x} = \{h \in G | d(y_g^m, x_h^M), d(y_h^m, x) \geq 2\delta,\ \|h\| \in \beta^{-1}[\delta, \delta^{-1}]\}$.
- For $j = 1, \ldots, k$, let $\lambda_j = \lambda_{g_{j-1}, x_{g_j}^M}$ and let

4.1.1 Regularization procedure

First step: Let

$$f(x,x') = \int e^{i\xi(\varphi(\mathbf{g}*\mathbf{h}x) - \varphi(\mathbf{g}*\mathbf{h}x'))} r(\mathbf{g}*\mathbf{h}x) r(\mathbf{g}*\mathbf{h}x') \mathrm{d}\mu_{k+1,n}(\mathbf{g}) \mathrm{d}\mu_{k,n}(\mathbf{h}). \tag{4.5}$$

Lemma 4.1. *We have*

$$|\int e^{i\xi\varphi(x)} r(x) \mathrm{d}\nu(x)|^2 \leq \int f(x,x') \mathrm{d}\nu(x) \mathrm{d}\nu(x') \tag{4.6}$$

Proof. By the Cauchy-Schwarz equuality,

$$\begin{aligned} &|\int e^{i\xi\varphi(x)} r(x) \mathrm{d}\nu(x)|^2 \\ &= |\int e^{i\xi\varphi(gx)} r(gx) \mathrm{d}\mu^{*(2k+1)n}(g) \mathrm{d}\nu(x)|^2 \leq \int |\int e^{i\xi\varphi(gx)} r(gx) \mathrm{d}\nu(x)|^2 \mathrm{d}\mu^{*(2k+1)n}(g) \\ &= \int\int e^{i\xi(\varphi(gx) - \varphi(gx')} r(gx) r(gx') \mathrm{d}\mu^{*(2k+1)n}(g) \mathrm{d}\nu(x) \mathrm{d}\nu(x'). \end{aligned}$$

With x, x' fixed considering the following formula, we rewrite the formula

$$\int e^{i\xi(\varphi(gx) - \varphi(gx')} r(gx) r(gx') \mathrm{d}\mu^{*(2k+1)n}(g) = f(x,x')$$

The proof is complete. ∎

Definition 4.2 (Good Position)**.** *Let x, x' be in X, we say they are in good position if*

$$d(x, x') \geq \delta \tag{4.7}$$

We fix x, x' in good position, which means x, x' are separated, and rewrite the formula

Lemma 4.3. *We have*

$$|\int e^{i\xi\varphi(x)} r(x) \mathrm{d}\nu(x)|^2 \leq \int_{d(x,x')\geq\delta} f(x, x') \mathrm{d}\nu(x) \mathrm{d}\nu(x') + O(\delta^c) \tag{4.8}$$

Proof. If x, x' are in bad position, that is

$$d(x, x') \leq \delta,$$

then by regularity of stationary measure (2.11), this part has exponentially small $\nu \otimes \nu$ measure, that is

$$\nu \otimes \nu \ll \delta^c. \tag{4.9}$$

The proof is complete. □

Second step: Here we mimic the proof of [BD17], where they heavily use the property of Schottky group and symbolic dynamics. But in our case, the group is much more complicate in the point of view of dynamics. We use the large deviation principle to get a same formula.

By very careful control of g_l, with a loss exponentially small measure, we are able to rewrite the formula in a form to use the sum-product estimates. The key point is that by control the Cartan projection and the position of x_g^M and y_g^m of each g_l, we are able to get a good control of their product $\mathbf{g} * \mathbf{h}$.

We should be careful that the element with even index will be fixed, and we will integrate first the elements with odd index. This gives the independence of the cocycle $\sigma(g_{j-1}h_j, x_{g_j}^M)$, that is for different j they are independent, which is an important point to apply sum-product estimates.

By large deviation principle, the property

$$\|g_l\| \in \beta^{-1}[\delta, \delta^{-1}]. \tag{4.10}$$

is true except on an exponentially small set. Then we fix g_l for $l = 0, \ldots k-1$.The following property of h_{l+1} with respect to g_l

$$\delta(y_{h_{l+1}}^m, x_{g_{l+1}}^M), \delta(y_{g_l}^m, x_{h_{l+1}}^M) \geq 2\delta, \|h_l\| \in \beta^{-1}[\delta, \delta^{-1}] \tag{4.11}$$

is true except on an exponentially small set, due to large deviation principle.

We write the main result of this part

Lemma 4.4. *With $g_0, \ldots, h_k$ satisfying the above condition (4.10) (4.11), for $x \in b_{g_k}^M(\beta/\delta)$, let $x_l = g_l h_{l+1} \cdots h_k x$ for $l = 0, \ldots, k$, where we let $x_k = x$. We have*

$$x_l \in b_{g_l}^M(\beta/\delta^2) \tag{4.12}$$

$$|\sigma(g_l h_{l+1}, x_{l+1}) - \sigma(g_l h_{l+1}, x_{g_{l+1}}^M)| \ll \beta/\delta^3. \tag{4.13}$$

$$\exp(-\sigma(g_l h_{l+1}, x_{g_{l+1}}^M)) \leq \beta^2/\delta^{O(1)} \tag{4.14}$$

Proof. We use induction to prove the inclusion. For $l = k$, it is trivial.

Suppose the property holds for $l+1$, then $x_l = g_l h_{l+1} x_{l+1}$. We abbreviate $h_l, g_{l+1}, x_{l+1}, x_{g_l}^M$ to h, g, x, x'. The condition becomes

$$d(x, x') \leq \beta/\delta^2, d(y_g^m, x_h^M), d(y_h^m, x') \geq 4\delta \text{ and } \|g\|, \|h\| \geq \beta/\delta.$$

By (4.11) and Lemma 2.3, due to $x \in B(x',\beta/\delta) \subset B(x',\delta) \subset B_h^m(\delta)$, we have $hx \in b_h^M(\beta/\delta^2) \subset B_g^m(\delta)$. Therefore $ghx \in b_g^M(\beta/\delta^2)$.

By (2.1), we have

$$|\sigma(gh,x) - \sigma(gh,x')| \ll \frac{d(x,x')}{\min(d(x,y_h^m),d(x',y_h^m))} + \frac{d(hx,hx')}{\min(d(hx,y_g^m),d(hx',y_g^m))}.$$

By Lemma 2.3, we have $x,x' \in B(x',\beta/\delta^2) \subset B_h^m(4\delta)$ and $hx,hx' \in b_h^M(\beta/\delta^2) \subset B_g^m(\delta)$. Therefore

$$|\sigma(gh,x) - \sigma(gh,x')| \ll \beta/\delta^3$$

.

The third equuality, by (2.2) (2.5)

$$\exp(\sigma(gh,x')) \geq \|g\|\|h\|d(y_h^m,x')d(y_g^m,hx') \geq \beta^{-2}\delta^{O(1)}.$$

The proof is complete. □

Remark 4.5. *The intuition here is that by controlling $\kappa(g), x_g^M, y_g^m$, all the other position or length will also be controlled, which is similar with hyperbolic dynamics.*

4.1.2 End of the proof

Third step: We collect the results in the above two steps to give a new formula of the main term, to which we can apply the sum-product estimates.

We return to (4.8). We call g "good " if

$$\mathbf{g} \text{ satisfies (4.10) and } \delta(y_{g_k}^m,x),\delta(y_{g_k}^m,x') \geq 4\delta, |\varphi'(x_{g_0}^M)| \geq \delta \tag{4.15}$$

We call $\mathbf{h}$ is $\mathbf{g}$-regular if $\mathbf{h}$ satisfies (4.11). Let

$$f_{\mathbf{g}}(x,x') = \int_{\mathbf{g}-regular} e^{i\xi(\varphi(\mathbf{g}*\mathbf{h}x)-\varphi(\mathbf{g}*\mathbf{h}x'))}\mathrm{d}\mu_{k,n}(\mathbf{h}) \tag{4.16}$$

Lemma 4.6. *For x,x' in X with $d(x,x') \geq \delta$*

$$|f(x,x')| \leq \int_{\mathbf{g}\text{"}good\text{"}} |f_{\mathbf{g}}(x,x')|\mathrm{d}\mu_{k+1,n}(\mathbf{g}) + O_\epsilon(\delta^c), \tag{4.17}$$

if ϵ is small enough with respect to γ.

Proof. Let

$$\tilde{f}_{\mathbf{g}}(x,x') = \int_{\mathbf{g}-regular} e^{i\xi(\varphi(\mathbf{g}*\mathbf{h}x)-\varphi(\mathbf{g}*\mathbf{h}x'))}r(\mathbf{g}*\mathbf{h}x)r(\mathbf{g}*\mathbf{h}x')\mathrm{d}\mu_{k,n}(\mathbf{h}) \tag{4.18}$$

We call $\mathbf{g}$ "semi-good" if $\mathbf{g}$ satisfies conditions of "good" without the last inequality on φ'. Then by Large deviation principle,

$$|f(x,x')| \leq |\int_{\mathbf{g}} \left(|\tilde{f}_{\mathbf{g}}(x,x')| + O_\epsilon(\delta^c)\right)\mathrm{d}\mu_{k+1,n}(\mathbf{g})| \leq \int_{\mathbf{g}\text{"}semi-good\text{"}} |\tilde{f}_{\mathbf{g}}(x,x')|\mathrm{d}\mu_{k+1,n}(\mathbf{g}) + O_\epsilon(\delta^c) \tag{4.19}$$

Lemma 4.4 and the Hölder norm of r imply

$$|r(x_{g_0}^M)^2 - r(\mathbf{g}*\mathbf{h}x)r(\mathbf{g}*\mathbf{h}x')| \leq 2|r|_\infty c_\gamma(r)(\beta/\delta)^\gamma \leq 2\beta^\gamma\delta^{-1-\gamma} \leq 2\delta,$$

if ϵ is small enough with respect to γ. Hence

$$\begin{aligned}|\tilde{f}_{\mathbf{g}}(x,x')| &\leq |\int_{\mathbf{g}-regular} e^{i\xi(\varphi(\mathbf{g}*\mathbf{h}x)-\varphi(\mathbf{g}*\mathbf{h}x'))} r(x_{g_0}^M)^2 \mathrm{d}\mu_{k,n}(\mathbf{h})| + O_\epsilon(\delta^c) \\ &\leq r(x_{g_0}^M)^2 |f_{\mathbf{g}}(x,x')| + O_\epsilon(\delta^c)\end{aligned} \tag{4.20}$$

By the hypothesis on φ' and (4.4), we have that $|\varphi'| \geq \delta^{-1}$ on the support of r. So if $r(x_{g_0}^M) \neq 0$, then that $\mathbf{g}$ is "semi-good" implies $\mathbf{g}$ is "good". Combined with (4.19) (4.20), by $|r| \leq 1$, we have

$$\begin{aligned}|f(x,x')| &\leq \int_{\mathbf{g}\text{"}semi-good\text{"}} \left(r(x_{g_0}^M)^2 |f_{\mathbf{g}}(x,x')| + O_\epsilon(\delta^c)\right) \mathrm{d}\mu_{k+1,n}(\mathbf{g}) + O_\epsilon(\delta^c) \\ &\leq \int_{\mathbf{g}\text{"}good\text{"}} |f_{\mathbf{g}}(x,x')| \mathrm{d}\mu_{k+1,n}(\mathbf{g}) + O_\epsilon(\delta^c).\end{aligned}$$

The proof is complete. □

Proposition 4.7. *Let $I_\tau = [\tau^{3/4}, \tau^{5/4}]$. The following formula is true for $\mathbf{g}$ good,*

$$|f_{\mathbf{g}}(x,x')| \leq \sup_{\varsigma\in I_\tau} |\int e^{i\varsigma x_1\cdots x_k} \mathrm{d}\lambda_1(x_1)\cdots\lambda_k(x_k)| + O(\beta\delta^{-O(1)}\tau), \tag{4.21}$$

when ϵ is small enough with respect to ϵ_1.

Proof. The element x, x' and g_k are already fixed. Let $\tilde{x} = g_k x$ and $\tilde{x'} = g_k x'$. By (2.4), we have

$$d(\tilde{x},\tilde{x'}) \in \|g_k\|^{-2}[\delta,\delta^{-2}] \subset \beta^2[\delta^{O(1)},\delta^{-O(1)}].$$

Therefore the arc length distance, defined by $d_a(x,x') = \arcsin d(x,x')$, satisfies

$$d_a(\tilde{x},\tilde{x'}) \in \beta^2[\delta^{O(1)},\delta^{-O(1)}]. \tag{4.22}$$

By Newton-Leibniz's formula on the circle, we have

$$\varphi(\mathbf{g}*\mathbf{h}x) - \varphi(\mathbf{g}*\mathbf{h}x') = \int \varphi(T\mathbf{g}*\mathbf{h}\gamma(s))'\mathrm{d}s, \tag{4.23}$$

where γ is an arc connecting $\tilde{x}, \tilde{x}'$ with unit speed with length less than $\pi/2$. Let $s_j = g_j h_{j+1}\cdots h_k\gamma(s)$. By the chain rule, we have

$$\varphi(T\mathbf{g}*\mathbf{h}\gamma(s))' = \varphi'(s_0)(g_0h_1)'(s_1)\cdots(g_{k-1}h_k)'(\gamma(s))\gamma'(s). \tag{4.24}$$

Using the trivialization in Lemma 2.8, we know that the unit tangent vector $\gamma'(s)$ equals ± 1, depending on the position of $\tilde{x}, \tilde{x}'$. Without loss of generality, we suppose that $\gamma'(s) = 1$.

By $|\varphi'(x_{g_0}^M)| \geq \delta$, $|\varphi''| \leq 1/\delta$ and (4.12), we have

$$|\varphi'(s_0)/\varphi'(x_{g_0}^M)| \in [1 \pm \beta/\delta^3]$$

By Lemma 4.4, we have

$$(1-\beta/\delta^3)e^{-O(\beta/\delta)} \leq \frac{\varphi'(s_0)(g_0h_1)'(s_1)\cdots(g_{k-1}h_k)'(s_k)}{\varphi'(x_{g_0}^M)(g_0h_1)'(x_{g_1}^M)\cdots(g_{k-1}h_k)'(x_{g_k}^M)} \leq (1+\beta/\delta^3)e^{O(\beta/\delta)} \tag{4.25}$$

By (4.14), we also have

$$\varphi'(x_{g_0}^M)(g_0h_1)'(x_{g_1}^M)\cdots(g_{k-1}h_k)'(x_{g_k}^M) \leq \beta^{4k}\delta^{-O(1)}. \tag{4.26}$$

Together with (4.22), (4.23), (4.24), (4.25) and (4.26)

$$|\varphi(\mathbf{g}*\mathbf{h}x)-\varphi(\mathbf{g}*\mathbf{h}x')-d_a(\tilde{x},\tilde{x'})\varphi'(x^M_{g_0})(g_0h_1)'(x^M_{g_1})\cdots(g_{k-1}h_k)'(x^M_{g_k})|\leq\beta^{4k+3}\delta^{-O(1)} \quad (4.27)$$

Let

$$\varsigma=\frac{\xi d_a(\tilde{x},\tilde{x'})\varphi'(x^M_{g_0})(g_0h_1)'(x^M_{g_1})\cdots(g_{k-1}h_k)'(x^M_{g_k})}{\Pi_{l=1}^k\psi_{g_{l-1}}(h_l,x^M_{g_l})}=\xi d_a(\tilde{x},\tilde{x'})\varphi'(x^M_{g_0})\beta^{2k}\|g_0\|^{-2}\cdots\|g_{k-1}\|^{-2}$$

Due to $\xi=\tau\beta^{-(4k+2)}$, by (4.10) (4.22) we have $\varsigma\in|\tau|[\delta^{O(1)},\delta^{-O(1)}]\in[|\tau|^{3/4},|\tau|^{5/4}]$, when δ is large enough with respect to τ.

Hence by (4.27)

$$|\xi(\varphi(\mathbf{g}*\mathbf{h}x)-\varphi(\mathbf{g}*\mathbf{h}x'))-\varsigma\Pi_{l=1}^k\psi_{g_{l-1}}(h_l,x^M_{g_l})|\leq\beta\delta^{-O(1)}\tau \quad (4.28)$$

By definition, the distribution of $\psi_{g_{l-1}}(h_l,x^M_{g_l})$ with h_l of law μ^{*n} is the measure λ_l. At last, due to $|e^{ix}-e^{iy}|\leq|x-y|$ for $x,y\in\mathbb{R}$, the inequality (4.28) implies (4.21). □

Fourth step: Another difference with [BD17] is that we avoid using the renewal idea, which simplifies the proof of this part. The renewal idea is that instead of using μ^{*n}, we use a renewal measure μ_t, which is defined to be the law of $g_1\cdots g_n$ for the first time that its Cartan projection exceeds t. Because we generalize the sum-product estimate to a form that the measure can have a support depend on the frequency, and we use the large deviation principle to prove that our measure has a support not too large with respect to the frequency.

We are able to apply sum-product estimates.

For $j=1,2,\dots k$, Proposition 3.8 tells us that with ϵ small enough depending on $\epsilon_3\epsilon$, there exists C_0 such that the measures λ_j satisfy the conditions in Proposition 3.7 with τ.

Proof of Theorem 1.3. Proposition 3.7 implies

$$|\int\exp(i\varsigma x_1\cdots x_k)\mathrm{d}\lambda_1(x_1)\dots\mathrm{d}\lambda_k(x_k)|\leq C_0|\tau|^{-\epsilon_2}.$$

By (4.8) (4.17) (4.21), we have

$$|\int e^{i\xi\varphi(x)}r(x)\mathrm{d}\nu(x)|^2\ll_\epsilon\delta^c+\beta\delta^{-O(1)}\tau+|\tau|^{-\epsilon_2}.$$

Due to $\beta\delta^{-O(1)}\tau=e^{(-\sigma_\mu+O(1)\epsilon+\epsilon_1)n}$, take ϵ small enough. The proof is finished. □

4.2 From Fourier decay to spectral gap

In this section, we will prove the theorem of uniform spectral gap. The first part is classic, where we use some ideas of Dolgopyat [Dol98] to transform the problem to an effective estimate Proposition 4.12, see also [Nau05] [Sto11]. The key observation is that this effective estimate (Proposition 4.12) can be obtained by the Fourier decay, regarding the difference of cocycle as an function on X.

The intuition here is from Lemma 2.9. When g,h are in general position and x not too close to y^m_g,y^m_h, the difference $\sigma(g,x)-\sigma(h,x)$ will satisfy the conditions in Theorem 1.3.

We state our main result of this section

Proposition 4.8. *With the same assumption as in , there exists $\rho<1,C>0$ such that for all b large enough, a small enough and f in $C^\gamma(X)$, we have*

$$|P^n_{a+ib}f|_\gamma\leq C|b|^{2\gamma}\rho^n|f|_\gamma \quad (4.29)$$

Theorem 1.2 follows directly from this proposition.

Definition 4.9 (μ, γ contraction)**.** *[BQ16] If there exist $C > 0, \rho < 1$ such that for all $x \neq x'$ in X*

$$\int \left(\frac{d(gx, gx')}{d(x, x')} \right)^{\gamma} \mathrm{d}\mu^{*n}(g) \leq C\rho^n \tag{4.30}$$

Proposition 4.10. *[BL85, V, Thm.2.5][BQ16, Prop 11.10, Lem.13.5] For every γ small enough, there exist $C > 0$ and $0 < \rho < 1$ for all f in $C^{\gamma}(X)$*

$$|P^n f|_{\infty} \leq |\int_X f \mathrm{d}\nu| + C\rho^n |f|_{C^{\gamma}} \tag{4.31}$$

We start to consider complex perturbation of the transfer operator. For $z \in \mathbb{C}$, write $z = a + ib$. This is classic priori estimate, we recall a proof for completeness.

Proposition 4.11. *[GLP16, Cor.3.21] For every γ small enough, there exist $\rho < 1$ and $C > 0$ such that for f in $C^{\gamma}(X)$ and $|a|$ small enough*

$$|P_z^n f|_{\infty} \leq C^{|a|n} |f|_{\infty} \tag{4.32}$$

$$c_{\gamma}(P_z^n f) \leq C(C^{|a|n} |b|^{\gamma} |f|_{\infty} + \rho^n c_{\gamma}(f)) \tag{4.33}$$

Proof. The first inequality is due to the finiteness of the exponential moment and the Jensen inequality.

For the γ norm

$$e^{z\sigma(g,x)} f(gx) - e^{z\sigma(g,y)} f(gy) = (e^{z\sigma(g,x)} - e^{z\sigma(g,y)}) f(gx) + e^{z\sigma(g,y)}(f(gx) - f(gy))$$

Let $A_n = |\int_G \frac{e^{z\sigma(g,y)}(f(gx)-f(gy))}{d(x,y)^{\gamma}} \mathrm{d}\mu^{*n}(g)|$ and $B_n = |\int_G \frac{(e^{z\sigma(g,x)}-e^{z\sigma(g,y)})f(gx)}{d(x,y)^{\gamma}} \mathrm{d}\mu^{*n}(g)|$. By Cauchy-Schwarz's inequality

$$\begin{aligned} A_n &\leq c_{\gamma}(f) \int_G e^{a\sigma(g,y)} \frac{d(gx, gy)^{\gamma}}{d(x,y)^{\gamma}} \mathrm{d}\mu^{*n}(g) \\ &\leq c_{\gamma}(f) \left(\int_G e^{2a\sigma(g,y)} \mathrm{d}\mu^{*n}(g) \right)^{1/2} \left(\int_G \left(\frac{d(gx, gy)}{d(x,y)} \right)^{2\gamma} \mathrm{d}\mu^{*n}(g) \right)^{1/2} \end{aligned}$$

One term is controlled by (4.32), the other term is due to (μ, γ) contraction. Therefore when a small enough, there exists $\rho_1 < 1$ such that $A_n \leq C_1 \rho_1^n c_{\gamma}(f)$, where $C_1 > 0$.

Since

$$|e^c - e^d| \leq (2\max(e^{\Re c}, e^{\Re d}))^{1-\gamma} (\max(e^{\Re c}, e^{\Re d})|c-d|)^{\gamma}$$

for c, d in $\mathbb{C}$, we have

$$\frac{|e^{z\sigma(g,\eta)} - e^{z\sigma(g,y)}|}{d(\eta, y)^{\gamma}} \leq (2e^{|a|\kappa(g)})^{1-\gamma} (e^{|a|\kappa(g)} |z| Lip(\sigma(g, \cdot)))^{\gamma} \leq 2e^{|a|\kappa(g)+\gamma\kappa_0(g)} |b|^{\gamma},$$

where $\kappa_0(g)$ is the Lipschitz norm of $\sigma(g, \cdot)$ and $\kappa_0(g) \leq C\|\kappa(g)\|$ by [BQ16, Lemma 13.1]. Then by the hypothesis of finite exponential moment and Hölder's inequality, we have

$$B_n \leq |b|^{\gamma} |f|_{\infty} C_1^{(|a|+\gamma)n}$$

(we take the same constant C_1). Therefore

$$c_{\gamma}(P_z^n f) \leq C_1 \rho_1^n c_{\gamma}(f) + |b|^{\gamma} C_1^{1+(|a|+\gamma)n} |f|_{\infty} \tag{4.34}$$

We want the term $C_1^{(|a|+\gamma)n}$ does not depend on γ. Fix n large enough such that $C_1\rho_1^n = \rho_2 < 1$. For natural number N, iterate (4.34) N times and use (4.32). We have

$$\begin{aligned} c_\gamma(P_z^{nN} f) &\le \rho_2 c_\gamma(P_z^{n(N-1)} f) + |b|^\gamma C_1^{1+(|a|+\gamma)n} |P_z^{n(N-1)} f|_\infty \\ &\le \rho_2 c_\gamma(P_z^{n(N-1)} f) + |b|^\gamma C_1^{1+(|a|+\gamma)n} |f|_\infty C_1^{|a|n(N-1)} \\ &\le c_\gamma(f)\rho_2^N + |b|^\gamma C_1^{1+(|a|+\gamma)n} |f|_\infty \frac{C_1^{|a|nN}}{1-\rho_2 C_1^{-|a|n}} \le c_\gamma(f)\rho_2^N + O_n(|b|^\gamma C_1^{|a|nN})|f|_\infty \end{aligned} \tag{4.35}$$

Given $m \in \mathbb{N}$, we can write $m = nN + r$ with $r \in [0, n-1]$. Therefore by (4.35) (4.34)

$$\begin{aligned} c_\gamma(P_z^m f) = c_\gamma(P^{nN+r} f) &\le \rho_2^N c_\gamma(P_z^r f) + O_n(|b|^\gamma C^{|a|nN})|P_z^r f|_\infty \\ &\le \rho_2^N (C_1\rho_1^r c_\gamma(f) + |b|^\gamma C_1^{(1+(a+\gamma)r)}|f|_\infty) + O_n(|b|^\gamma C_1^{|a|m})|f|_\infty. \end{aligned}$$

By setting $\rho = \rho_2^{1/n}$ and choosing C large enough, we have (4.33). □

Recall the norm $|f|_{\gamma,b} = |f|_\infty + |f|_\gamma / b^\gamma$. We reduce Proposition 4.8 to the following proposition.

Proposition 4.12. *For γ small enough, for $|b|$ large enough and $|a|$ small enough, there exists ϵ_1, $C_1 > 0$ such that for f holder, satisfies $|f|_{\gamma,b} \le 1$, we have*

$$\int \left| P_{a+ib}^{C_1 \ln |b|} f \right|^2 \mathrm{d}\nu \le e^{-\epsilon_1 \ln |b|} \tag{4.36}$$

From Proposition 4.12 to Proposition 4.8. We set $N = C_1 \ln |b|$, using (4.31) for P^{mN}, (4.36) for $P_z^N f$ and (4.33) for P_z^N

$$\begin{aligned} |P_z^{(m+1)N} f|_\infty^2 &\le C^{|a|mN} |P^{mN}|P_z^N f|^2|_\infty \le C^{|a|mN}\Big(\int |P_z^N f|^2 \mathrm{d}\nu + \rho^{mN} |P_z^N f|_{C^\gamma}^2\Big) \\ &\le C^{|a|mN}\left(e^{-\epsilon_1 N/C_1} + \rho^{mN}(C^{1+|a|N}(1+b^\gamma) + C\rho^N b^\gamma)^2\right) \end{aligned} \tag{4.37}$$

So we can choose m large such that $\rho^{mN}|b|^{2\gamma} = \rho^{mC\ln b}|b|^{2\gamma} < 1$. This m is only depend on γ, C and ρ. By continuity of a we obtain the equality for infinity norm. That is when m large enough and a small enough depend on m we have $|P_z^{(m+1)N} f|_\infty^2 \ll |b|^{-\epsilon_2}$, where $\epsilon_2 > 0$

For γ norm, we use (4.33) for $(P_z^N,\ P_z^{(m+1)N} f)$ and $(P_z^{(m+1)N},\ f)$

$$\begin{aligned} c_\gamma(P_z^{(m+2)N} f)/|b|^\gamma &\le C^{|a|N} |P_z^{(m+1)N} f|_\infty + \rho^N c_\gamma(P_z^{(m+1)N} f)/|b|^\gamma \\ &\le C^{|a|N} |P_z^{(m+1)N} f|_\infty + \rho^N (C^{1+|a|mN}|b|^\gamma + \rho^{mN}|b|^\gamma)/|b|^\gamma \end{aligned}$$

Then, when $|b|$ is large enough and a is small enough, we have

$$|P_z^{(m+2)N} f|_{\gamma,b} \le |b|^{-\epsilon_3} \tag{4.38}$$

(where we should use (4.37) with m replaced by $m+1$).

Let $N_1 = (m+2)N = (m+2)C_1 \ln |b|$. Given n, we can write $n = dN_1 + r$ with $0 \le r < N_1$. By (4.38), (4.32), (4.33)

$$|P_z^n f|_{\gamma,b} \le |b|^{-\epsilon_3 d} |P^r f|_{\gamma,b} \le |b|^{-\epsilon_3 d} C^{1+|a|r} \le C|b|^{\epsilon_3}\rho^n,$$

where $\rho = |b|^{-\epsilon_3/N_1} C^{|a|} = e^{-\frac{\epsilon_3}{(m+2)C_1}} C^{|a|}$. The result follows by taking $|a|$ small enough. □

From Fourier decay to Proposition 4.12. We need to reduce (4.36) to Fourier decay. Let $n = C_1 \log |b|$ (with $C_1 \geq \max\{1/\sigma_\mu, 1\}$), let $\delta = e^{-\epsilon n}$ (with $\epsilon > 0$ to be determined later), and let $G_{n,\epsilon}$ be defined by

$$G_{n,\epsilon} = \{(g,h) \in G^2 | d(y_g^m, y_h^m) > \delta, \|g\|, \|h\| \in e^{\sigma_\mu n}[\delta, \delta^{-1}]\}. \tag{4.39}$$

For $|f|_{\gamma,b} \leq 1$

$$\begin{aligned} &\int |P_z^n f|^2 \mathrm{d}\nu = \int e^{iz(\sigma(g,x)-\sigma(h,x))} f(gx) f(hx) \mathrm{d}\nu(x) \mathrm{d}\mu^{*n}(g) \mathrm{d}\mu^{*n}(h) \\ &= \int_{G_{n,\epsilon}} \int_X e^{iz(\sigma(g,x)-\sigma(h,x))} f(gx) f(hx) \mathrm{d}\nu(x) \mathrm{d}\mu^{*n}(g) \mathrm{d}\mu^{*n}(h) + \mu^{*2n}(G_{n,\epsilon}^c) \end{aligned} \tag{4.40}$$

Since the support of the μ-stationary measure is contained in the image of the flag variety X. Hence a function on the support of ν can be lifted to a function on X. We will use the same character, and from now on all the functions are on X. Let

$$X_{g,h} = X - \{x \in X | \delta(x, y_g^m) \leq \delta\} - \{x \in X | \delta(x, y_h^m) \leq \delta\}$$

and let ρ be a smooth function on $\mathbb{R}$ such that $\rho|_{[0,\infty)} = 1$, ρ takes values in $[0,1]$, $\mathrm{supp}\rho \subset [-1,\infty)$ and $|\rho'| \leq 2$. Let

$$\varphi(x) = (\sigma(g,x) - \sigma(h,x))\rho(\frac{d(x,y_g^m)}{\delta} - 2)\rho(\frac{d(x,y_h^m)}{\delta} - 2)$$

and

$$r(x) = f(gx)f(hx)e^{a(\sigma(g,x)-\sigma(h,x)}\rho(\frac{d(x,y_g^m)}{\delta} - 3)\rho(\frac{d(x,y_h^m)}{\delta} - 3).$$

Then $e^{ib\varphi}r(x)$ equals $e^{iz(\sigma(g,x)-\sigma(h,x))}f(gx)f(hx)$ on $X - B(y_g^m, 3\delta) - B(y_h^m, 3\delta)$. By Lemma 2.9, we have $|\varphi'| \geq \delta$ on $X_{g,h}$, which contains the support of r.

Lemma 4.13. *With $C_1 > 1/\sigma_\mu$ and $|a|$ small enough depending on ϵ, we have*

$$|\varphi|_{C^2} + c_\gamma(r) \leq e^{\epsilon O(1) n}.$$

The functions φ and r satisfy the condition in Theorem 1.3 with $\epsilon_0 = \epsilon O(1) C_1$. Hence taking ϵ small enough according to ϵ_0, by Theorem 1.3, we have

$$|\int e^{ib\varphi(x)} r(x) \mathrm{d}\nu(x)| \leq |b|^{-\delta_1}. \tag{4.41}$$

Let $A_{g,h} = \int_X e^{iz(\sigma(g,x)-\sigma(h,x))} f(gx) f(hx) \mathrm{d}\nu(x)$. The difference of $A_{g,h}$ and $\int e^{ib\varphi(x)} r(x) \mathrm{d}\nu(x)$ is bounded by $\nu(B(y_g^m, 3\delta) \cup B(y_h^m, 3\delta)) \ll e^{-c\epsilon n} = |b|^{-c/C_1}$. Therefore

$$A_{g,h} \ll |b|^{-\delta_1} + |b|^{-c/C_1}.$$

Combined with (4.40), the proof is complete. □

It remains to prove Lemma 4.13.

Proof of Lemma 4.13. We need a lemma to control the derivative of the function multiplied with a cutoff.

Lemma 4.14. *Let f be a γ-Hölder function on X, and let ρ be a smooth truncate function (that is $0 \leq \rho \leq 1$), then*

$$c_\gamma(\rho f) \leq c_\gamma(\rho)|f|_\infty + c_\gamma(f|_{\mathrm{supp}\rho}) \tag{4.42}$$

Proof. For $x \neq x'$ in X, we have

$$A(x,x') = |\frac{\rho f(x) - \rho f(x')}{d(x,x')^\gamma}| \leq |\frac{f(x)(\rho(x) - \rho(x'))}{d(x,x')^\gamma}| + |\frac{\rho(x')(f(x) - f(x'))}{d(x,x')^\gamma}| \tag{4.43}$$

If x or x' is not in suppρ, by symmetry we can suppose that x' is not in suppρ, then (4.43) implies that $A(x,x') \leq |f|_\infty c_\gamma(\rho)$

Else, we have $A(x,x') \leq c_\gamma(\rho)|f|_\infty + c_\gamma(f|_{\mathrm{supp}\rho})$. □

By Lemma 4.14, we only need to control the C^γ norm on the support of the cutoff function and the maximal norm.

The term $|\varphi|_{C^2}$ is controlled by Lemma 2.9, and we have $|\varphi|_{C^2} \leq e^{O(1)\epsilon n}$.

The term $c_\gamma(r)$ is more complicated. We have

$$c_\gamma(f(g\cdot)|_{X_{g,h}}) \leq c_\gamma(f) \sup_{x_1 \neq x_2, x_1, x_2 \in X_{g,h}} (\frac{d(gx_1, gx_2)}{d(x_1,x_2)})^\gamma \leq (|b|e^{-2\kappa(g)+2\epsilon n})^\gamma \leq (|b|e^{(-2\sigma_\mu+4\epsilon)n})^\gamma$$

Due to $n = C_1 \log|b|$, we have $|b|e^{(-2\sigma_\mu+4\epsilon)n} = |b|^{1+C_1(-2\sigma_\mu+4\epsilon)}$. Hence, take ϵ small and take $C_1 \geq 1/\sigma_\mu$. This implies $c_\gamma(f(g\cdot)) \leq 1$.

Using Lemma 2.9 with c equal to δ, one has

$$\begin{aligned} c_\gamma(e^{a\sigma(g,x)}|_{X_{g,h}}) &\leq e^{|a|\kappa(g)(1-\gamma)}(e^{|a|\kappa(g)}|a|Lip(\sigma(g,\cdot)|_{X_{g,h}}))^\gamma \leq e^{|a|\kappa(g)}(|a|\delta^{-1})^\gamma \\ &\leq e^{|a|n\sigma_\mu}\delta^{-(a+\gamma)}|a|^\gamma = e^{(|a|\sigma_\mu+(|a|+\gamma)\epsilon)n}|a|^\gamma \end{aligned}$$

Hence, when $|a|$ is small enough depending on ϵ, we have $c_\gamma(e^{a\sigma(g,x)}|_{X_{g,h}}) \leq e^{\epsilon O(1)n}$. The maximal value is bounded by $e^{|a|\kappa(g)} \leq e^{|a|(\sigma_\mu+\epsilon)n}$. Therefore $|e^{a\psi(g,x)}|_\infty \leq e^{\epsilon O(1)n}$. □

5 Applications

5.1 Exponential decay in the Renewal theorem

We define a renewal operator R as follows. For a positive bounded Borel function f on $X \times \mathbb{R}$, a point x in X and a real number t, we set

$$Rf(x,t) = \sum_{n=0}^{+\infty} \int_G f(gx, \sigma(g,x) - t) \mathrm{d}\mu^{*n}(g).$$

Because of the positivity of f, this sum is well defined. In [Kes74], Kesten proved a renewal theorem for Markov chains, which is valid in our case [GLP16]. In this section, we will prove that the speed of convergence is exponential using our result on the spectral gap.

The main properties of $P(z)$ are summarized as follows

Proposition 5.1. *For any $\gamma > 0$ small enough, there exists $\eta > 0$ such that when $|\Re z| < \eta$, the transfer operator $P(z)$ is a bounded operator on $\mathcal{H}^\gamma(X)$ and depends analytically on z. Moreover there exists an analytic operator $U(z)$ on a neighborhood of $|\Re z| < \eta$ such that the following holds for $|\Re z| < \eta$*

$$(I - P(z))^{-1} = \frac{1}{\sigma_\mu z} N_0 + U(z), \tag{5.1}$$

where N_0 is the operator defined by $N_0 f = \int f \mathrm{d}\nu$. For z. There exists $C > 0$ such that for $|\Re z| \leq \eta$

$$\|U_z\|_\gamma \leq C(1 + |\Im z|)^{2\gamma}. \tag{5.2}$$

This is generalization of [LI17, Prop. 4.1] [Boy16, Theorem 4.1], the proof is exactly the same. The only difference is that the spectral radius of P_z is bounded below 1 in a strip of imaginary line (except at 0), from which we have analytic continuation of U_z to the strip. (?? Proposition 4.8) The idea is due to Guivarch and Le Page.

Proposition 5.2. *There exists $\epsilon > 0$ such that for $\varphi \in C_c^\infty(\mathbb{R})$, we have*

$$\int \varphi(\sigma(g,x)-t)\mathrm{d}\bar{\mu}(g) = \frac{1}{\sigma_\mu}\int_{-t}^{\infty}\varphi(u)\mathrm{d}u + e^{-\epsilon|t|}O(e^{\epsilon|\mathrm{supp}\varphi|}(|\varphi''|_{L^1}+|\varphi|_{L^1})). \tag{5.3}$$

Proof. By [LI17, Lemma 4.5], we have

$$Rf(x,t) = \frac{1}{\sigma_\mu}\int_t^\infty f(y,u)\mathrm{d}u\mathrm{d}\nu(y) + \frac{1}{2\pi}\int e^{it\xi}U(i\xi)\hat{f}(x,\xi)\mathrm{d}\xi.$$

Hence, we only need to control the error term.

By Proposition 5.1, we have that $U(z)$ is analytical on $\{z \in \mathbb{C} | |\Re z| \leq \eta\}$ and uniformly bounded by $(1+|\Im z|)^{2\gamma}$.

Lemma 5.3. *[RS75, Thm.IX14] If T is in $\mathcal{S}'(\mathbb{R})$, the distribution T has analytic continuation to $|\Im\xi| < a$ and $\sup_{|x|<a}\int |T(ix+y)|\mathrm{d}y < \infty$, then $\check{T}$ is a continuous function. For all $b < a$, let $C_b = \max\int |T(\pm ib+y)|\mathrm{d}y$. We have*

$$|\check{T}(x)| \leq C_b e^{-|b|x}. \tag{5.4}$$

By $|\hat{\varphi}(i\epsilon+\xi)| \leq e^{\epsilon|\mathrm{supp}\varphi|}\frac{1}{|\xi|^2}|\varphi''|_{L^1}$, and $|\hat{\varphi}(\xi)| \leq e^{\epsilon|\mathrm{supp}\varphi|}|\varphi|_{L^1}$, we have

$$|\hat{\varphi}(\xi)| \leq e^{\epsilon|\mathrm{supp}\varphi|}\frac{2}{1+|\xi|^2}(|\varphi''|_{L^1}+|\varphi|_{L^1}).$$

Then by Lemma 5.3, we have

$$\begin{aligned}|\int \hat{\varphi}(\xi)U(i\xi)\mathbb{1}(x)e^{-it\xi}\mathrm{d}\xi| = |\mathcal{F}^{-1}(\hat{\varphi}(\xi)U(i\xi)\mathbb{1}(x))(t)| &\leq e^{-\epsilon|t|}\max|\hat{\varphi}(\pm i\epsilon+\xi)U(\mp\epsilon+i\xi)\mathbb{1}(x)|_{L^1(\xi)} \\ &\leq e^{\epsilon|\mathrm{supp}\varphi|}e^{-\epsilon|t|}\int\frac{2}{1+|\xi|^2}(|\varphi''|_{L^1}+|\varphi|_{L^1})|U(\mp\epsilon+i\xi)\mathbb{1}(x)|\mathrm{d}\xi \\ &\ll_\gamma e^{\epsilon|\mathrm{supp}\varphi|}e^{-\epsilon|t|}(|\varphi''|_{L^1}+|\varphi|_{L^1}).\end{aligned}$$

The proof is complete. □

Corollary 5.4. *There exists $\epsilon > 0$ such that for $I = [0,a]$ an interval in $\mathbb{R}$ we have*

$$R\mathbb{1}_I(x,t) = \frac{a}{\sigma_\mu} + O_a(e^{-\epsilon t}), \tag{5.5}$$

where $\mathbb{1}_I$ is the characteristic function of I.

Proof. The idea is rather simple, using to smooth function to bound $\mathbb{1}_I$ from above and from below. For $\delta > 0$, let $I^{(\delta)} = [-\delta, a+\delta]$ and $I_{(\delta)} = [\delta, a-\delta]$. Let $\rho_{\delta,1}$, ρ_{δ_2}, be two smooth function such that $\rho_{\delta,1}|_I = 1$, $\mathrm{supp}\rho_{\delta,1} \subset I^{(\delta)}$ and $\rho_{\delta,2}|_{I_{(\delta)}} = 1$, $\mathrm{supp}\rho_{\delta,2} \subset I$. Then

$$R\mathbb{1}_I(x,t) \leq R\rho_{\delta,1}(x,t) \leq \frac{a+2\delta}{\sigma_\mu} + O(e^{-\epsilon(t-a)}(|\rho''_{\delta,1}|_{L^1}+|\rho_{\delta,1}|_{L^1}))$$

Due to $|\rho''_{\delta,1}|_{L^1} \ll 1/\delta$ and $|\rho_{\delta,1}|_{L^1} \leq a+2\delta$. Taking $\delta = e^{-\epsilon t/2}$, we have $R\mathbb{1}_I(x,t) \leq a/\sigma_\mu + O_a(e^{-\epsilon t/2})$. The other direction of $R\mathbb{1}_I$ is due to the same estimate by replacing $\rho_{\delta,1}$ by $\rho_{\delta,2}$. □

References

[Aou13] Richard Aoun. Transience of algebraic varieties in linear groups-applications to generic zariski density. In *Annales de l'institut Fourier*, volume 63, pages 2049–2080, 2013.

[BD17] Jean Bourgain and Semyon Dyatlov. Fourier dimension and spectral gaps for hyperbolic surfaces. *Geometric and Functional Analysis*, 27(4):744–771, 2017.

[BL85] Philippe Bougerol and Jean Lacroix. *Products of random matrices with applications to Schrödinger operators*, volume 8. Springer Science & Business Media, 1985.

[Bou10] Jean Bourgain. The discretized sum-product and projection theorems. *Journal d'Analyse Mathématique*, 112(1):193–236, 2010.

[Boy16] Jean-Baptiste Boyer. The rate of convergence for the renewal theorem in $\mathbb{R}^d$. *arXiv preprint arXiv:1603.07214*, 2016.

[BQ16] Yves Benoist and Jean-François Quint. *Random Walks on Reductive Groups*, volume 62. Springer, 2016.

[Dol98] D. Dolgopyat. On decay of correlations in Anosov flows. *Annals of mathematics*, pages 357–390, 1998.

[GLP16] Y. Guivarc'h and É. Le Page. Spectral gap properties for linear random walks and Pareto's asymptotics for affine stochastic recursions. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(2):503–574, 2016.

[Gui90] Yves Guivarc'h. Produits de matrices aléatoires et applications aux propriétés géométriques des sous-groupes du groupe linéaire. *Ergodic theory and dynamical systems*, 10(03):483–512, 1990.

[Kes74] Harry Kesten. Renewal theory for functionals of a Markov chain with general state space. *The Annals of Probability*, pages 355–386, 1974.

[LI17] Jialun LI. Decrease of Fourier coefficients of stationary measures. *arXiv preprint arXiv:1706.071849*, 2017.

[LP82] P. D. Lax and R. S. Phillips. The Asymptotic Distribution of Lattice Points in Euclidean and Non-Euclidean Spaces. *J. Funct. Anal.*, 46(280-350), 1982.

[Nau05] F. Naud. Expanding maps on Cantor sets and analytic continuation of zeta functions. In *Annales scientifiques de l'Ecole normale supérieure*, volume 38, pages 116–153, 2005.

[RS75] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics: Vol.: 2.: Fourier Analysis, Self-Adjointness.* Academic Press, 1975.

[Sto11] Luchezar Stoyanov. Spectra of Ruelle transfer operators for Axiom A flows. *Nonlinearity*, 24(4):1089, 2011.