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# Analyse harmonique des mesures stationnaires

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# Harmonic analysis of stationary measures

Thesis prepared under the supervision of
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# Contents

# Introduction en français

*Cette thèse est constituée de quatre articles de recherche. La première partie va être publiée dans Mathematische Annalen et la dernière a été soumise.*

Soit $V$ une représentation irréductible de dimension finie du groupe de Lie $G = \mathrm{SL}_{m+1}(\mathbb{R})$. Soit $X = \mathbb{P}V$ l'espace projectif réel de $V$, qui est l'ensemble des droites vectorielles de $V$. Alors, nous avons une action du groupe $G$ sur $X$. Soit $\mu$ une mesure de probabilité borélienne sur $G$ et soit $\Gamma_\mu$ le sous-groupe engendré par le support de $\mu$. Nous dirons que $\mu$ est Zariski dense si $\Gamma_\mu$ est un sous-groupe Zariski dense de $G$. Cela signifie que la mesure $\mu$ n'est pas concentrée sur un sous-groupe algébrique de $G$. Nous pouvons définir sur $X$ une marche aléatoire associée à $\mu$. Fixons un point $x$ de $X$. Lors de chaque étape, nous nous déplaçons vers un point aléatoire $gx$, où $g$ est un élément aléatoire de $G$, de loi $\mu$. D'après un théorème de Furstenberg, cette marche aléatoire admet une unique mesure stationnaire $\nu$ sur $X$, appelée la mesure de Furstenberg ou la mesure $\mu$-stationnaire. Cela signifie que la mesure $\nu$ vérifie $\nu = \mu * \nu := \int_G g_*\nu \mathrm{d}\mu(g)$, où $g_*\nu$ est l'image de $\nu$ par l'action de $g$ sur $X$. Cette mesure a été introduite par Furstenberg pour établir la loi des grands nombres pour les produits de matrices aléatoires. Les propriétés de la mesure $\mu$-stationnaire sont importantes dans d'autres théorèmes limites pour les produits de matrices aléatoires.

Il y a beaucoup d'exemples intéressants de mesures $\mu$-stationnaires. Nous nous restreignons à la basse dimension, c'est-à-dire au cas où $G = \mathrm{SL}_2(\mathbb{R})$ et $X = \mathbb{P}(\mathbb{R}^2)$, la droite projective réelle. Soit $\Gamma$ un réseau de $\mathrm{SL}_2(\mathbb{R})$ (par exemple $\mathrm{SL}_2(\mathbb{Z})$). Furstenberg a construit des exemples où $\mu$ est une mesure portée par $\Gamma$ et la mesure stationnaire $\nu$ est exactement la mesure de Lebesgue de $X$. Cette construction a été utilisée dans le travail de Furstenberg sur le bord de Poisson, qui a donné des propriétés de rigidité des réseaux. Récemment, les gens se sont intéressés aux propriétés de dimension et d'absolue continuité de la mesure $\mu$-stationnaire $\nu$ quand $\mu$ a un support fini. Voir [BPS12] et [Bou12] pour des exemples de mesures stationnaires absolument continues et [HS17] pour des exemples de mesure stationnaires de dimension totale.

Mentionnons aussi une autre classe de mesures, les convolutions de Bernoulli. Soient $X_0, X_1, \ldots$ des variables aléatoires i.i.d telles que $\mathbb{P}(X_0 = 1) = \mathbb{P}(X_0 = -1) = 1/2$. Soit $\nu_\lambda$ la convolution de Bernoulli de paramètre $\lambda \in (0,1)$, qui est définie comme la distribution de la variable aléatoire $\sum_{j\geq 0} X_j \lambda^j$. Elle peut être vue comme une mesure stationnaire sur $\mathbb{R}$ pour l'action d'un groupe résoluble. Voir l'exemple 1.1.6. Des auteurs se sont intéressés à la dimension et la régularité des convolutions de Bernoulli. Il y a beaucoup de travaux récents dans ce cadre. Voir par exemple [SS16], [Hoc14] et [Var16].

Avant d'énoncer notre question principale, nous introduisons une autre propriété de la mesure stationnaire. Nous aurons besoin d'une hypothèse de moment exponentiel fini, c'est-à-dire qu'il existe $\epsilon$ strictement positif tel que

$$\int_G \|g\|^\epsilon \mathrm{d}\mu(g) < +\infty.$$

Dorénavant, nous supposons toujours que notre mesure $\mu$ est Zariski dense avec un moment exponentiel fini. Rappelons que $X = \mathbb{P}V$. Guivarc'h a établi la régularité höldérienne des mesures stationnaires. Cela signifie qu'il existe des nombres $C, c > 0$ tels que, pour tout $r > 0$, le $r$-voisinage de tout hyperplan de $X$ a une $\nu$-mesure inférieure à $Cr^c$. Cela implique que la mesure stationnaire $\nu$ a une dimension positive. Cela implique aussi que $\nu$ n'est pas concentrée sur un hyperplan, ce qui est raisonnable vue l'hypothèse de Zariski densité de $\mu$.

## Décroissance de Fourier

Notre problème principal ici est la décroissance de Fourier de la mesure stationnaire. Considérons d'abord l'exemple $G = \mathrm{SL}_2(\mathbb{R})$ et $X = \mathbb{P}(\mathbb{R}^2)$. Fixons l'identification de $\mathbb{P}(\mathbb{R}^2)$ avec le cercle $\mathbb{T} \simeq \mathbb{R}/\pi\mathbb{Z}$, donnée par l'action transitive du groupe $\mathrm{PSO}_2$. Nous pouvons alors définir les coefficients de Fourier de la mesure stationnaire $\nu$ par

$$\hat{\nu}(k) = \int_{\mathbb{T}} e^{2ikx} \mathrm{d}\nu(x), \quad \text{pour } k \in \mathbb{Z}.$$

**Théorème** (Theorem 1.1.1, Theorem 2.1.2)**.** *Soit $\mu$ une mesure de probabilité borélienne Zariski dense sur $\mathrm{SL}_2(\mathbb{R})$ avec un moment exponentiel fini. Soit $\nu$ la mesure $\mu$-stationnaire sur $\mathbb{T}$. Alors il existe $\epsilon > 0$ tel que*

$$|\hat{\nu}(k)| = O(|k|^{-\epsilon}). \tag{0.0.1}$$

En d'autres termes, les coefficients de Fourier de la mesure stationnaire ont une décroissance polynomiale. Par un argument général, la décroissance polynomiale des coefficients de Fourier implique la régularité de Guivarc'h. En réalité, la régularité est un ingrédient essentiel de la démonstration. La décroissance de Fourier pour des mesures reliées à l'algorithme des fractions continues a été études par Kaufman [Kau80], Queffélec-Ramaré [QR03] et Jordan-Sahlsten [JS16]. Récemment, la décroissance de Fourier des mesures de Patterson-Sullivan a été démontrée par Bourgain-Dyatlov [BD17]. Notre deuxième approche est inspirée par leurs méthodes.

Nous avons deux approches pour ce problème. La première (Chapitre 1) est plus élémentaire, nous utilisons le théorème de renouvellement pour les processus stochastiques. Mais le résultat est plus faible, nous pouvons seulement établir une version qualitative, à savoir $|\hat{\nu}(k)| \to 0$ quand $|k| \to +\infty$. Car la décroissance exponentielle dans le théorème de renouvellement n'est pas encore connue. Par la suite, nous établissons l'existence de ce terme d'erreur grâce à notre seconde approche.

La seconde approche (Chapitre 2) est inspirée par la méthode de Bourgain et Dyatlov. L'ingrédient principal, l'estimées sommes-produits, vient de la combinatoire additive. Nous expliquerons cette approche plus loin.

Un exemple intéressant est la mesure de Patterson-Sullivan sur l'ensemble limite d'un groupe fuchsien convexe co-compact. En combinant les méthodes de Connell-Muchnik [CM07] et de Lalley [Lal86], pour un groupe fuchsien convexe co-compact, nous pouvons trouver une mesure $\mu$ telle que la mesure de Patterson-Sullivan soit $\mu$-stationnaire. Grâce à cette observation, nous pouvons retrouver le résultat de Bourgain-Dyatlov sur la décroissance de Fourier des mesures de Patterson-Sullivan. Mais notre vitesse de décroissance est plus faible.

En dimension plus grande, nous considérons la décroissance de la transformée de Fourier dans une carte affine. Soit $v_0$ un vecteur unitaire de $V$. Soit $v_0^\perp$ le sous-espace vectoriel orthogonal de $v_0$ dans $V$. Soit $U$ le sous-ensemble ouvert de $\mathbb{P}V$ qui est le complémentaire de l'hyperplan $\mathbb{P}v_0^\perp$. Nous prenons la carte locale affine $(\psi, U)$ of $\mathbb{P}V$, donnée par

$$\psi : \mathbb{P}V \supset U \to v_0^\perp,\ \mathbb{R}v \mapsto \frac{v - \langle v_0, v\rangle v_0}{\langle v_0, v\rangle},$$

qui est bien définie sur $U$. L'inverse de $\psi$ est donnée simplement par $\psi^{-1} : v_0^\perp \to U \subset \mathbb{P}V,\ u \mapsto \mathbb{R}(u + v_0)$.

**Théorème** (Theorem 2.1.1)**.** *Soit $\mu$ une mesure de probabilité borélienne Zariski dense sur $\mathrm{SL}_{m+1}(\mathbb{R})$ avec un moment exponentiel fini. Soit $V$ une representation irréductible de dimension fini de $\mathrm{SL}_{m+1}(\mathbb{R})$. Soit $\nu$ la mesure $\mu$-stationnaire sur $\mathbb{P}V$. Soit $r$ une fonction $C^1$ dont le support est contenu dans $U$ et qui vérifie $\|r\|_\infty \leq 1$. Alors, il existe $\epsilon > 0$ tel que, pour tout $\varsigma \in v_0^\perp$ de norme $\|\varsigma\|$ suffisamment grande, on ait*

$$\left| \int_{v_0^\perp} e^{i\langle \varsigma, u\rangle} r(u) \mathrm{d}\nu(u) \right| \leq \|\varsigma\|^{-\epsilon}.$$

Notre méthode ne permet pas de traiter le cas des groupes de Lie non déployés comme par exemple $\mathrm{SL}_2(\mathbb{C})$. Il serait intéressant d'établir une décroissance de Fourier analogue pour $\mathrm{SL}_2(\mathbb{C})$. Nous pouvons considérer le groupe $\mathrm{SL}_2(\mathbb{Q}_p)$ et les mesures stationnaires sur $\mathbb{Q}_p$. Il serait aussi intéressant d'établir une décroissance de Fourier semblable pour ce groupe.

## Théorème de renouvellement

Rappelons que $X = \mathbb{P}V$. Nous définissons la fonction cocycle $\sigma : G \times X \to \mathbb{R}$ par, pour $x = \mathbb{R}v$ dans $X$ et $g$ dans $G$, $\sigma(g, x) = \log \frac{\|gv\|}{\|v\|}$. Soit $f$ une fonction lisse à support compact sur $\mathbb{R}$. L'opérateur de renouvellement $R$ est défini par

$$Rf(x,t) = \sum_{n=0}^{+\infty} \int_G f(\sigma(g,x) - t) \mathrm{d}\mu^{*n}(g),\ \text{pour } x \in X \text{ et } t \in \mathbb{R}.$$

Le théorème de renouvellement a été introduit pour la première fois par Blackwell et dans notre situation par Kesten [Kes74]. Le résultat principal (dû à Guivarc'h et Le Page [GLP16]) est que, quand le temps $t$ tend vers l'infini, la somme de renouvellement $Rf(x,t)$ tend vers $\frac{1}{\sigma_\mu}\int f$, où $\sigma_\mu$ est la constante de Lyapunov définie par $\sigma_\mu := \int_{G\times X}\sigma(g,x)\mathrm{d}\mu(g)\mathrm{d}\nu(x)$. Par définition, la constante de Lyapunov $\sigma_\mu$ est une moyenne de la fonction cocycle $\sigma(g,x)$ pour la mesure $\mu\otimes\nu$. Le théorème de renouvellement nous donne un phénomène d'équidistribution quand le temps $t$ est assez grand. Dans notre première approche, le théorème de renouvellement est utilisé pour borner la somme de renouvellement $Rf$ pour une fonction $f$ fortement oscillante. Dans notre seconde approche, nous sommes en mesure de donner un terme d'erreur exponentiel.

**Théorème** (Theorem 2.1.4)**.** *Soit $\mu$ une mesure de probabilité borélienne Zariski dense sur $\mathrm{SL}_{m+1}(\mathbb{R})$ avec un moment exponentiel fini. Soit $V$ une représentation irréductible de $\mathrm{SL}_{m+1}(\mathbb{R})$. Il existe $\epsilon>0$ tel que, pour $f\in C_c^\infty(\mathbb{R})$ et $t\in\mathbb{R}$, on ait*

$$Rf(x,t)=\frac{1}{\sigma_\mu}\int_{-t}^{\infty} f(u)\mathrm{d}Leb(u)+O_f(e^{-\epsilon|t|}),$$

*où $O_f$ dépend du support de $f$ et de sa norme de Sobolev.*

Ce théorème est à comparer avec le théorème de renouvellement dans $\mathbb{R}$ (le cas commutatif). Si $\mu$ est une mesure sur $\mathbb{R}$ dont le support est fini, le terme d'erreur dans le théorème de renouvellement n'est jamais exponentiel.

Nous espérons que ce type de résultat peut permettre d'obtenir un terme d'erreur exponentiel dans le comptage orbital en rang supérieur. Étant donné un sous-groupe discret $\Gamma$ de $G$, nous cherchons une asymptotique pour la croissance de $\#\{\gamma\in\Gamma|\ d(\gamma o,o)\leq R\}$, où $o$ est le point base de $\mathrm{SL}_{m+1}(\mathbb{R})/\mathrm{SO}(m+1)$. Voir par exemple Lalley [Lal89], Quint [Qui05] et Sambarino [Sam15]. Ce type de terme d'erreur est toujours relié à une propriété de trou spectral.

## Trou spectral

Munissons $\mathbb{P}V$ d'une distance riemannienne. Pour $\gamma>0$, soit $C^\gamma(\mathbb{P}V)$ l'espace des fonctions $\gamma$-höldériennes. Nous introduisons l'opérateur de transfert qui est un analogue de la fonction caractéristique dans notre cas.

**Définition.** *Pour $z$ dans $\mathbb{C}$ avec une partie réelle $|\Re z|$ suffisamment petite, soit $P_z$ l'opérateur sur l'espace des fonctions continues donné par*

$$P_z f(x)=\int_G e^{z\sigma(g,x)}f(gx)\mathrm{d}\mu(g),\ \textit{pour } x\in\mathbb{P}V.$$

Nous conservons l'hypothèse que $\mu$ est une mesure de probabilité borélienne Zariski dense sur $\mathrm{SL}_{m+1}(\mathbb{R})$ avec un moment exponentiel fini. L'utilisation de cet opérateur de transfert pour l'étude des produits de matrices aléatoires a été introduite par Guivarc'h et Le Page. En raison de la propriété de moment exponentiel, l'opérateur $P_z$ préserve l'espace de Banach $C^\gamma(\mathbb{P}V)$ pour $\gamma > 0$ suffisamment petit. En raison des propriétés de contraction de l'action de $G$ dans $X$, pour $z$ dans une petite boule centrée en 0, le rayon spectral de $P_z$ dans $C^\gamma(\mathbb{P}V)$ est $< 1$ sauf pour $z = 0$. En raison de la non-arithméticité de $\Gamma_\mu$, sur l'axe imaginaire, l'opérateur $P_z$ a aussi un rayon spectral $< 1$, sauf en 0. Ces faits ont été utilisé par Le Page et Guivarc'h pour donner des théorèmes limites pour des produits de matrices aléatoires (voir [LP82b] et [BQ16]).

**Théorème** (Theorem 2.1.5). *Soit $\mu$ une mesure de probabilité borélienne Zariski dense sur $\mathrm{SL}_{m+1}(\mathbb{R})$ avec un moment exponentiel fini. Soit $V$ une représentation irréductible de $\mathrm{SL}_{m+1}(\mathbb{R})$. Pour tout $\gamma > 0$ suffisamment petit, il existe $\delta > 0$ tel que, pour tous $|b| > 1$ et $|a|$ suffisamment petit, le rayon spectral de $P_{a+ib}$ agissant dans $C^\gamma(\mathbb{P}V)$ vérifie*

$$\rho(P_{a+ib}) < 1 - \delta.$$

Même dans le cas de $\mathrm{SL}_2(\mathbb{R})$, ce résultat est nouveau et connu seulement dans des cas particuliers. Quand $\mu$ est portée par un nombre fini d'éléments de $\mathrm{SL}_2(\mathbb{R})$ et que ces éléments engendrent un sous-semi-groupe de Schottky, ce résultat est dû à Naud [Nau05]. Quand $\mu$ est absolument continue par rapport à la mesure de Haar de $\mathrm{SL}_2(\mathbb{R})$, ce résultat peut être obtenu directement en utilisant des propriétés d'oscillation forte.

Il est intéressant de constater que ces trois propriétés, la décroissance de Fourier, le théorème de renouvellement et le trou spectral sont essentiellement équivalentes. Dans la première approche, nous utilisons le théorème de renouvellement pour prouver la décroissance de Fourier. Dans la seconde approche, nous utilisons la décroissance de Fourier pour démontrer le trou spectral, puis utilisons le trou spectral pour établir le théorème de renouvellement. Ces propriétés sont analogues à des phénomènes de la théorie des surfaces convexes co-compactes. Dans ce contexte plus géométrique, la décroissance de Fourier a été étudiée récemment par Bourgain-Dyatlov ; le trou spectral peut être interprété comme l'existence d'une zone sans zéro pour la fonction zêta de Selberg ou le trou dans les valeurs propres de l'opérateur de Laplace de la surface ; le théorème de renouvellement est remplacé par le problème de comptage des orbites du groupe fondamental ou celui des géodésiques fermées primitives. Voir [Bor07] et les références qui y sont données.

## Estimées sommes-produits

Nous expliquons à présent notre deuxième approche, qui est inspirée par le travail de Bourgain et Dyatlov. L'ingrédient essentiel est une propriété de décroissance de Fourier des convolutions multiplicatives de mesures sur $\mathbb{R}$, qui découle de l'estimées sommes-produits discrétisée de Bourgain.

Grossièrement, l'estimées sommes-produits nous dit que la taille d'un sous-ensemble de $\mathbb{Z}$ doit grandir rapidement par des sommes ou des produits. Cette propriété vient de la combinatoire additive. La propriété discrétisée a été introduite par Katz et Tao. Bourgain a démontré l'estimées sommes-produits discrétisée dans [Bou03]. Une de ses nombreuses applications est un résultat de trou spectral dans $\mathrm{SU}(2)$ [BG08], qui a été ensuite généralisé dans [BdS16] et [BISG17]. Notre seconde approche peut être vue comme un analogue dans $\mathrm{SL}_{m+1}(\mathbb{R})$.

Pour des groupes de Lie de rang supérieur, nous avons besoin de généraliser la décroissance de Fourier des convolutions multiplicatives à $\mathbb{R}^n$. He-de Saxcé ont démontré une version de l'estimées sommes-produits discrétisée dans $\mathbb{R}^n$. En utilisant leur résultat, nous pouvons démontrer le

**Théorème** (Theorem 3.1.1)**.** *Étant donné $\kappa_0 > 0$, il existe $\epsilon, \epsilon_1 > 0$ et $k \in \mathbb{N}$ tels que la propriété suivante est vérifiée pour $\delta > 0$ suffisamment petit. Soit $\mu$ une mesure de probabilité sur $[1/2,1]^n \subset \mathbb{R}^n$ qui vérifie la condition de $(\delta, \kappa_0, \epsilon)$ non concentration projective, à savoir*

$$\forall \rho \geq \delta, \qquad \sup_{a\in\mathbb{R}, v\in\mathbb{S}^{n-1}} (\pi_v)_*\mu(B_{\mathbb{R}}(a,\rho)) = \sup_{a,v} \mu\{x | \langle v, x\rangle \in B_{\mathbb{R}}(a,\rho)\} \leq \delta^{-\epsilon}\rho^{\kappa_0}. \tag{0.0.2}$$

*Alors, pour tout $\xi \in \mathbb{R}^n$ avec $\|\xi\| \in [\delta^{-1}/2, \delta^{-1}]$,*

$$\left| \int \exp(2i\pi\langle \xi, x^1 \cdots x^k\rangle) \mathrm{d}\mu(x^1)\cdots \mathrm{d}\mu(x^k)\right| \leq \delta^{\epsilon_1}. \tag{0.0.3}$$

Ce théorème est utilisé dans la démonstration de la décroissance de Fourier pour $\mathrm{SL}_{m+1}(\mathbb{R})$. Comme nous l'avons déjà dit, le cas de $\mathbb{R}$ est dû à Bourgain [Bou10].

## Finitude des petites valeurs propres

Notre dernier résultat concerne une classe de variétés qui contient les surfaces hyperboliques convexes cocompactes. Soit $\mathbb{H}^3$ la variété hyperbolique de dimension 3 simplement connexe. La notion de finitude géométrique a été introduite par Ahlfors lors de l'étude du problème suivant: soit $\Gamma$ un sous-groupe discret de type fini du groupe des isométries directes de $\mathbb{H}^3$ (isomorphe à $\mathrm{PSL}_2(\mathbb{C})$). Le problème consiste à savoir si l'ensemble limite de $\Gamma$ est constitué de la sphère entière ou s'il a mesure de de Lebesgue nulle. On l'appelle la conjecture mesurée d'Ahlfors et c'est maintenant un théorème grâce à des progrès en géométrie hyperbolique. La définition originale de la finitude géométrique est qu'il existe un domaine fondamental pour l'action de $\Gamma$ sur $\mathbb{H}^3$ qui est un polyèdre possédant un nombre fini de côtés. Mais cette définition en termes de polyèdres est difficile à généraliser. Nous utiliserons la définition de Bowditch [Bow95], qui demande que la partie épaisse de l'enveloppe convexe de l'ensemble limite soit compacte modulo $\Gamma$.

Nous nous donnons une variété géométriquement finie localement symétrique de rang un, ce qui signifie que son revêtement universel est $X = \mathbb{H}^n_{\mathbb{F}}$ pour $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ ou $\mathbb{H}^n_{\mathbb{F}} = \mathbb{H}^2_{\mathbb{O}}$. Soit $\delta(X)$ l'exposant de croissance, qui vaut $(n+1)\dim_{\mathbb{R}} \mathbb{F} - 2$. Le spectre de l'opérateur de Laplace est relié à de nombreuses quantités, comme le taux de croissance des orbites du groupe fondamental et la dimension de Hausdorff de l'ensemble limite. Nous généralisons un résultat de Lax et Phillips pour $X = \mathbb{H}^n_{\mathbb{R}}$ [LP82a] et un résultat de Hamenstädt dans le cas général [Ham04].

**Théorème** (Theorem 4.1.1)**.** *Soit* $M = \Gamma\backslash X$ *une variété géométriquement finie localement symétrique de rang* 1*. Alors, l'intersection du spectre de l'opérateur de Laplace et de l'intervalle critique* $(-\delta(X)^2/4, 0]$ *est constituée d'un nombre fini de valeurs propres de multiplicité finie.*

Ce résultat peut être utilisé pour donner un trou spectral pour l'opérateur de Laplace. Alors le trou spectral donne un terme d'erreur pour le problème de comptage orbital.

# Introduction

*This thesis consists of four research papers. The first part will appear in Mathematische Annalen and the last part has been submitted.*

Let $V$ be a finite dimensional irreducible representation of the Lie group $G = \mathrm{SL}_{m+1}(\mathbb{R})$. Let $X = \mathbb{P}V$ be the real projective space of $V$, which is the set of lines of $V$. Then we have a group action of $G$ on $X$. Let $\mu$ be a Borel probability measure on $G$ and let $\Gamma_\mu$ be the subgroup generated by the support of $\mu$. We call $\mu$ Zariski dense if $\Gamma_\mu$ is a Zariski dense subgroup of $G$. This means that the measure $\mu$ does not concentrate on any algebraic subgroup of $G$. We can give a random walk on $X$ induced by $\mu$. Fix a point $x$ in $X$. At each step, we go to a random point $gx$, where $g$ is a random element in $G$ with the law of $\mu$. By a theorem of Furstenberg, this random walk has a unique stationary measure $\nu$ on $X$, called the Furstenberg measure or the $\mu$-stationary measure. That is to say, the measure $\nu$ satisfies $\nu = \mu * \nu := \int_G g_*\nu \mathrm{d}\mu(g)$, where $g_*\nu$ is the push-forward of $\nu$ by the action of $g$ on $X$. This measure was introduced by Furstenberg when he established the law of large numbers for products of random matrices. The properties of the $\mu$-stationary measure is important in other limit theorems for products of random matrices.

There are many interesting examples of $\mu$-stationary measures. We restrict our attention to low dimension, that is $G = \mathrm{SL}_2(\mathbb{R})$ and $X = \mathbb{P}(\mathbb{R}^2)$, the real projective line. Let $\Gamma$ be a lattice in $\mathrm{SL}_2(\mathbb{R})$ (for example $\mathrm{SL}_2(\mathbb{Z})$). Furstenberg constructed examples where $\mu$ is a measure supported on $\Gamma$ and the $\mu$-stationary measure $\nu$ is exactly the Lebesgue measure on $X$. This construction was used in Furstenberg's work on the Poisson boundary, which gave some rigidity properties of lattices. Recently, people have been interested in the dimension and the absolute continuity of the $\mu$-stationary measure $\nu$ when $\mu$ has finite support. Please see [BPS12] and [Bou12] for examples of absolutely continuous stationary measures, and [HS17] for examples of stationary measures with full dimension.

We also mention another class of measures, the Bernoulli convolutions. Let $X_0, X_1, \ldots$ be i.i.d. random variables such that $\mathbb{P}(X_0 = 1) = \mathbb{P}(X_0 = -1) = 1/2$. Let $\nu_\lambda$ be the Bernoulli convolution with parameter $\lambda \in (0, 1)$, defined to be the distribution of the random variable $\sum_{j \geq 0} X_j \lambda^j$. This can be seen as a stationary measure on $\mathbb{R}$ for the action of a solvable group. Please see Example 1.1.6. People are interested in the dimension and the regularity of the Bernoulli convolutions. There are many recent works on this setting. See for instance [SS16], [Hoc14] and [Var16].

Before stating our main question, we introduce another property of the stationary measure. We need the hypothesis of finite exponential moment, that is there exists $\epsilon$ positive such that

$$\int_G \|g\|^\epsilon \mathrm{d}\mu(g) < +\infty.$$

From now on, we always suppose that our measure $\mu$ is Zariski dense with a finite exponential moment. Recall that $X = \mathbb{P}V$. Guivarc'h proved Hölder regularity of stationary measures. This means that there exist $C, c$ positive such that for every $r$ positive, the $r$ neighbourhood of any hyperplane in $X$ has $\nu$ measure less than $Cr^c$. This implies that the stationary measure $\nu$ has positive dimension. This also says that $\nu$ does not concentrate on some hyperplane, which is reasonable due to the hypothesis of Zariski density of $\mu$.

## Fourier decay

Our main problem here is the Fourier decay of the stationary measure. Let us first see the example $G = \mathrm{SL}_2(\mathbb{R})$ and $X = \mathbb{P}(\mathbb{R}^2)$. Fix the identification of $\mathbb{P}(\mathbb{R}^2)$ with the circle $\mathbb{T} \simeq \mathbb{R}/\pi\mathbb{Z}$, given by the transitive action of the group $\mathrm{PSO}_2$. We can define the Fourier coefficients of the stationary measure $\nu$ by

$$\hat{\nu}(k) = \int_{\mathbb{T}} e^{2ikx} \mathrm{d}\nu(x), \quad \text{for } k \in \mathbb{Z}.$$

**Theorem** (Theorem 1.1.1, Theorem 2.1.2)**.** *Let $\mu$ be a Zariski dense Borel probability measure on $\mathrm{SL}_2(\mathbb{R})$ with a finite exponential moment. Let $\nu$ be the $\mu$-stationary measure on $\mathbb{T}$. Then there exists $\epsilon$ positive such that*

$$|\hat{\nu}(k)| = O(|k|^{-\epsilon}). \tag{0.0.4}$$

In other words, the Fourier coefficients of the stationary measure have polynomial decay. By a general argument, the polynomial decay of Fourier coefficients implies Guivarc'h's regularity. In fact, the regularity is a crucial ingredient in the proof. The Fourier decay of similar measures related to continued fractions have been studied by Kaufman [Kau80], Queffélec-Ramaré [QR03] and Jordan-Sahlsten [JS16]. Recently, the Fourier decay of Patterson-Sullivan measures was proved by Bourgain-Dyatlov [BD17]. Our second approach is inspired by their methods.

We have two different approaches for this problem. The first approach (Chapter 1) is more elementary, we use the renewal theorem from the theory of stochastic processes. But the result is weaker, we can only prove a qualitative version, that is $|\hat{\nu}(k)| \to 0$ as $|k| \to +\infty$. Because the exponential error term in the renewal theorem was not yet known. Later, the exponential error term is proved by our second approach through the Fourier decay.

The second approach (Chapter 2) is inspired by the method of Bourgain and Dyatlov. The main ingredient comes from additive combinatorics, the sum-product estimate. We will explain this approach later.

An interesting example is the Patterson-Sullivan measure on the limit set of convex cocompact Fuchsian groups. Combining the method of Connell-Muchnik [CM07] and Lalley [Lal86], we can find a measure $\mu$ for convex cocompact Fuchsian groups, such that the Patterson-Sullivan measure is $\mu$-stationary. With this observation, we can recover the result of Bourgain-Dyatlov on the Fourier decay of Patterson-Sullivan measures. But the decay rate is weaker.

In higher dimension, we consider the decay of the Fourier transform on an affine chart. Let $v_0$ be a unit vector in $V$. Let $v_0^\perp$ be the linear subspace of $V$, which is orthogonal to $v_0$. Let $U$ be the open subset of $\mathbb{P}V$, which is the complement of the hyperplane $\mathbb{P}v_0^\perp$. We take the affine local chart $(\psi, U)$ of $\mathbb{P}V$, given by

$$\psi : \mathbb{P}V \supset U \to v_0^\perp,\ \mathbb{R}v \mapsto \frac{v - \langle v_0, v\rangle v_0}{\langle v_0, v\rangle},$$

which is well defined on $U$. The inverse of $\psi$ is simply given by $\psi^{-1} : v_0^\perp \to U \subset \mathbb{P}V,\ u \mapsto \mathbb{R}(u + v_0)$.

**Theorem** (Theorem 2.1.1)**.** *Let $\mu$ be a Zariski dense Borel probability measure on $\mathrm{SL}_{m+1}(\mathbb{R})$ with a finite exponential moment. Let $V$ be an irreducible representation of $\mathrm{SL}_{m+1}(\mathbb{R})$ and let $\nu$ be the $\mu$-stationary measure on $\mathbb{P}V$. Let $r$ be a $C^1$ function whose support is in $U$ and $\|r\|_\infty \leq 1$. Then there exists $\epsilon > 0$ such that for every $\varsigma \in v_0^\perp$ with the norm $\|\varsigma\|$ sufficiently large, we have*

$$\left| \int_{v_0^\perp} e^{i\langle \varsigma, u\rangle} r(u) \mathrm{d}\nu(u) \right| \leq \|\varsigma\|^{-\epsilon}.$$

Our method cannot treat the Lie groups which are not split, for example $\mathrm{SL}_2(\mathbb{C})$. It would be interesting to establish a similar Fourier decay for $\mathrm{SL}_2(\mathbb{C})$. We can consider the group $\mathrm{SL}_2(\mathbb{Q}_p)$ and the stationary measure on $\mathbb{Q}_p$. It would also be interesting to establish a similar Fourier decay for this group.

## Renewal theorem

Recall that $X = \mathbb{P}V$. We define the cocycle function $\sigma : G \times X \to \mathbb{R}$ by, for $x = \mathbb{R}v$ in $X$ and $g$ in $G$, $\sigma(g, x) = \log \frac{\|gv\|}{\|v\|}$. Let $f$ be a smooth compactly supported function on $\mathbb{R}$. The renewal operator $R$ is defined by

$$Rf(x, t) = \sum_{n=0}^{+\infty} \int_G f(\sigma(g, x) - t) \mathrm{d}\mu^{*n}(g), \text{ for } x \in X \text{ and } t \in \mathbb{R}.$$

The renewal theorem was first introduced by Blackwell and in our situation by Kesten [Kes74]. The main result (due to Guivarc'h and Le Page [GLP16]) is that when time $t$ tends to infinite, the renewal sum $Rf(x, t)$ tends to $\frac{1}{\sigma_\mu} \int f$, where $\sigma_\mu$ is the Lyapunov constant defined by $\sigma_\mu := \int_{G\times X} \sigma(g, x) \mathrm{d}\mu(g) \mathrm{d}\nu(x)$. From the definition, we see that the

Lyapunov constant $\sigma_\mu$ is an average of the cocycle function $\sigma(g,x)$ with respect to the measure $\mu\otimes\nu$. The renewal theorem gives us a phenomenon of equidistribution when the time $t$ is large enough. In our first approach, the renewal theorem is used to bound the renewal sum $Rf$ when $f$ is a function with high oscillation. With our second approach, we are able to give an exponential error term.

**Theorem** (Theorem 2.1.4)**.** *Let $\mu$ be a Zariski dense Borel probability measure on $\mathrm{SL}_{m+1}(\mathbb{R})$ with finite exponential moment. Let $V$ be an irreducible representation of $\mathrm{SL}_{m+1}(\mathbb{R})$. There exists $\epsilon>0$ such that for $f\in C_c^\infty(\mathbb{R})$ and $t\in\mathbb{R}$, we have*

$$Rf(x,t)=\frac{1}{\sigma_\mu}\int_{-t}^{\infty}f(u)\mathrm{d}Leb(u)+O_f(e^{-\epsilon|t|}),$$

*where $O_f$ depends on the support and some Sobolev norm of $f$.*

We should compare this result with the renewal theorem on $\mathbb{R}$ (the commutative case). If $\mu$ is a measure on $\mathbb{R}$ whose support is finite, then the error term in the renewal theorem is never exponential.

Our result improves a result of Boyer, where the error term is polynomial on $t$. We hope this type of result can give some exponential error terms in the orbital counting problem of higher rank. Given a discrete subgroup $\Gamma$ of $\mathrm{SL}_{m+1}(\mathbb{R})$, we are interested in the asymptotic for the growth of $\#\{\gamma\in\Gamma|\ d(\gamma o,o)\le R\}$, where $o$ is the base point in $\mathrm{SL}_{m+1}(\mathbb{R})/\mathrm{SO}(m+1)$. See for instance Lalley [Lal89], Quint [Qui05] and Sambarino [Sam15]. This type of error term is always connected with some spectral gap property.

## Spectral gap

Equip $\mathbb{P}V$ with a Riemannian distance. For $\gamma$ positive, let $C^\gamma(\mathbb{P}V)$ be the space of $\gamma$-Hölder functions. We introduce the transfer operator, which is an analogue of the characteristic function in our case.

**Definition.** *For $z$ in $\mathbb{C}$ with the real part $|\Re z|$ small enough, let $P_z$ be the operator on the space of continuous functions, which is given by*

$$P_zf(x)=\int_G e^{z\sigma(g,x)}f(gx)\mathrm{d}\mu(g),\ \text{for } x\in\mathbb{P}V.$$

We keep the assumption that $\mu$ is a Zariski dense Borel probability measure on $\mathrm{SL}_{m+1}(\mathbb{R})$ with a finite exponential moment. The use of this transfer operator on the products of random matrices has been introduced by Guivarc'h and Le Page. Due to the property of exponential moment, when $|\Re z|$ is small enough, the operator $P_z$ preserves the Banach space $C^\gamma(\mathbb{P}V)$ for $\gamma>0$ small enough. Due to the contracting action of $G$ on $X$, for $z$ in a small ball centred at 0, the spectral radius of $P_z$ on $C^\gamma(\mathbb{P}V)$ is less than 1 except at 0. Due to the non-arithmeticity of $\Gamma_\mu$, on the imaginary line, the operator $P_z$ also has spectral radius less than 1 except at 0. These were used to give limit theorems for products of random matrices by Le Page and Guivarc'h (Please see [LP82b] and [BQ16]).

**Theorem** (Theorem 2.1.5). *Let $\mu$ be a Zariski dense Borel probability measure on $\mathrm{SL}_{m+1}(\mathbb{R})$ with finite exponential moment. Let $V$ be an irreducible representation of $\mathrm{SL}_{m+1}(\mathbb{R})$. For every $\gamma > 0$ small enough, there exists $\delta > 0$ such that for all $|b| > 1$ and $|a|$ small enough the spectral radius of $P_{a+ib}$ acting on $C^\gamma(\mathbb{P}V)$ satisfies*

$$\rho(P_{a+ib}) < 1 - \delta.$$

Even in the case of $\mathrm{SL}_2(\mathbb{R})$, the result is new and only known in some special case. When $\mu$ is supported on a finite number of elements of $\mathrm{SL}_2(\mathbb{R})$ and these elements generate a Schottky semi group, this result is due to Naud [Nau05]. When $\mu$ is absolutely continuous with respect to the Haar measure on $\mathrm{SL}_2(\mathbb{R})$, this result can be obtained directly using high oscillations.

It is interesting that the three objects, the Fourier decay, the Renewal theorem and the spectral gap are roughly equivalent. In the first approach, we use the Renewal theorem to prove the Fourier decay. In the second approach, we use the Fourier decay to prove the spectral gap, and then use the spectral gap to prove the Renewal theorem. They are analogue with similar objects for convex cocompact surfaces. In this more geometric setting, the Fourier decay was recently studied by Bourgain-Dyatlov; the spectral gap can be interpreted as the zero free region of the Selberg zeta function or the gap of the eigenvalues of the Laplace operator on the surface; the renewal theorem is replaced by the counting problem of the lattice points or the primitive closed geodesics. Please see Borthwick [Bor07] and the references there.

## Sum-product estimates

Now we explain our second approach, which is inspired by the work of Bourgain and Dyatlov. The key ingredient is a Fourier decay of multiplicative convolution of measures on $\mathbb{R}$, which is a consequence of the discretized sum-product estimate of Bourgain.

The sum-product estimate roughly says that a subset of $\mathbb{Z}$ must grow rapidly under sum or product. This comes from additive combinatorics. The discretized setting was introduced by Katz and Tao. Bourgain proved the discretized sum-product estimate in [Bou03]. One of its many applications is a spectral gap result in $\mathrm{SU}(2)$ [BG08], which was further generalized in [BdS16] and [BISG17]. Our second approach can be seen as an analogue in $\mathrm{SL}_{m+1}(\mathbb{R})$.

For Lie groups of higher rank, we need to generalize the Fourier decay of multiplicative convolutions to $\mathbb{R}^n$. He-de Saxcé have proved a version of discretized sum-product estimate in $\mathbb{R}^n$ [HdS18]. Using their result, we are able to prove the following:

**Theorem** (Theorem 3.1.1). *Given $\kappa_0 > 0$, there exist $\epsilon, \epsilon_1 > 0$ and $k \in \mathbb{N}$ such that the following holds for $\delta > 0$ small enough. Let $\mu$ be a probability measure on $[1/2, 1]^n \subset \mathbb{R}^n$ which satisfies $(\delta, \kappa_0, \epsilon)$ projective non concentration assumption, that is*

$$\forall \rho \geq \delta, \qquad \sup_{a\in\mathbb{R}, v\in\mathbb{S}^{n-1}} (\pi_v)_*\mu(B_{\mathbb{R}}(a,\rho)) = \sup_{a,v} \mu\{x | \langle v, x\rangle \in B_{\mathbb{R}}(a,\rho)\} \leq \delta^{-\epsilon}\rho^{\kappa_0}. \qquad (0.0.5)$$

*Then for all* $\xi \in \mathbb{R}^n$ *with* $\|\xi\| \in [\delta^{-1}/2, \delta^{-1}]$,

$$\left| \int \exp(2i\pi\langle \xi, x^1 \cdots x^k \rangle) \mathrm{d}\mu(x^1) \cdots \mathrm{d}\mu(x^k) \right| \leq \delta^{\epsilon_1}. \tag{0.0.6}$$

This theorem is used in the proof of Fourier decay for $\mathrm{SL}_{m+1}(\mathbb{R})$. As we have already mentioned, the case of $\mathbb{R}$ is due to Bourgain [Bou10].

## Finiteness of small eigenvalues

Our last result concerns a class of manifolds which contains convex cocompact hyperbolic surfaces. Let $\mathbb{H}^3$ be the simply connected hyperbolic three manifold. The definition of geometric finiteness was first introduced by Ahlfors in studying the following problem: Let $\Gamma$ be a discrete finitely generated subgroup of the oriented isometry group of $\mathbb{H}^3$ (isomorphic to $\mathrm{PSL}_2(\mathbb{C})$). The problem is whether the limit set of $\Gamma$ is a full sphere or has Lebesgue measure zero. This is called Ahlfors' measure conjecture, which is a theorem now due to progresses of hyperbolic geometry. The original definition of geometric finiteness is that there exists a fundamental domain of $\Gamma$ acting on $\mathbb{H}^3$, which is a finitely sided polyhedra. But this definition of fundamental polyhedra is hard to generalize. We will use the definition of Bowditch [Bow95], that the thick part of the convex hull is cocompact.

We are given a geometrically finite rank one locally symmetric manifold, which means the universal cover is $X = \mathbb{H}^n_{\mathbb{F}}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{H}^n_{\mathbb{F}} = \mathbb{H}^2_{\mathbb{O}}$. Let $\delta(X)$ be the exponent of growth, which equals $(n+1)\dim_{\mathbb{R}} \mathbb{F} - 2$. The spectrum of the Laplace operator is related to many quantities, such as the growth rate of the number of lattice points and the Hausdorff dimension of the limit set. We generalize a result of Lax and Phillips on $X = \mathbb{H}^n_{\mathbb{R}}$ [LP82a] and a result of Hamenstädt in the general case [Ham04].

**Theorem** (Theorem 4.1.1)**.** *Let* $M = \Gamma\backslash X$ *be a geometrically finite rank one locally symmetric manifold. Then the intersection of the spectrum of the Laplace operator and the critical interval* $(-\delta(X)^2/4, 0]$ *consists of finitely many eigenvalues of finite multiplicities.*

This result can be used to give a spectral gap for the spectrum of the Laplace operator. Then the spectral gap gives an exponential error in the orbital counting problem.

# Chapter 1

# Decrease of Fourier coefficients of stationary measures

Let $\mu$ be a Borel probability measure on $\mathrm{SL}_2(\mathbb{R})$ with a finite exponential moment, and assume that the subgroup $\Gamma_\mu$ generated by the support of $\mu$ is Zariski dense. Let $\nu$ be the unique $\mu-$stationary measure on $\mathbb{P}^1$. We prove that the Fourier coefficients $\hat{\nu}(k)$ of $\nu$ converge to 0 as $|k|$ tends to infinity. Our proof relies on a generalized renewal theorem for the Cartan projection.

## 1.1 Introduction

Let $\mu$ be a Borel probability measure on $\mathrm{SL}_2(\mathbb{R})$. The linear action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$ induces an action on $\mathbb{P}^1 = \mathbb{P}(\mathbb{R}^2)$. For a Borel probability measure $\nu$ on $\mathbb{P}^1$, we define its convolution with $\mu$ by

$$\mu * \nu = \int_{\mathrm{SL}_2(\mathbb{R})} g_* \nu \mathrm{d}\mu(g),$$

where $g_*\nu$ is the pushforward of $\nu$ by $g$. The measure $\nu$ is called $\mu-$stationary if $\mu * \nu = \nu$. We add the condition that the subgroup $\Gamma_\mu$ generated by the support of $\mu$ is Zariski dense in $\mathrm{SL}_2(\mathbb{R})$. In the case of $\mathrm{SL}_2(\mathbb{R})$, Zariski density is equivalent to unsolvability. When $\Gamma_\mu$ is Zariski dense in $\mathrm{SL}_2(\mathbb{R})$, there is a unique $\mu-$stationary measure (see [Fur63],[GR85]).

This stationary measure is also called the Furstenberg measure. It was first considered by Furstenberg in the study of the noncommutative law of large numbers. The stationary measure takes part in the subtle properties of random products of matrices. Please see [Fur63],[GR85] and [BL85].

In this paper, we are interested in the decay of the Fourier coefficients of stationary measures. The action of $\mathrm{PSO}_2 = \mathrm{SO}_2/\{\pm Id\}$ on $\mathbb{P}^1$ is transitive and free. We fix the point $x_o = [1 : 0]$ in $\mathbb{P}^1$, then identify $\mathbb{P}^1$ as the orbit space $\mathrm{PSO}_2 x_o$. As a group, $\mathrm{PSO}_2$ is isomorphic to the circle $\mathbb{T} \simeq \mathbb{R}/\pi\mathbb{Z}$. This is given by the map from $\mathbb{T}$ to $\mathrm{PSO}_2$,

$$\theta \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} /\{\pm Id\}.$$

So we have a homeomorphism from $\mathbb{T}$ to $\mathbb{P}^1$, that is $\theta \mapsto [\cos\theta : \sin\theta]$. We can define the Fourier coefficients of the stationary measure $\nu$ by the following formula

$$\widehat{\nu}(k) = \int_{\mathbb{T}} e^{2ik\theta} \mathrm{d}\nu(\theta).$$

We also demand that $\mu$ has a finite exponential moment, which means that there exists a constant $\epsilon_1 > 0$ such that $\int \|g\|^{\epsilon_1} \mathrm{d}\mu(g) < \infty$. We will prove

**Theorem 1.1.1.** *Let $\mu$ be a Borel probability measure on $\mathrm{SL}_2(\mathbb{R})$ with a finite exponential moment, and assume that the subgroup $\Gamma_\mu$ is Zariski dense. Then the $\mu-$stationary measure $\nu$ is a Rajchman measure, in other words*

$$\widehat{\nu}(k) \to 0 \quad \text{as } |k| \to +\infty. \tag{1.1.1}$$

**Remark 1.1.2.** *Fourier decay of measures on fractal sets and its applications have been studied in [Kau80],[QR03],[JS16] and [BD17]. Our situation is much general and we introduce a quite different method.*

Being a Rajchman measure is a local property (see [KL87]): Indeed, let $\nu_1$ be a Rajchman measure. If $\nu_2$ is absolutely continuous with respect to $\nu_1$, then $\nu_2$ is also a Rajchman measure. Conversely, the sum of two Rajchman measures is a Rajchman measure.

In this spirit, we have the following theorem:

**Theorem 1.1.3.** *Let $\mu$ be a Borel probability measure on $\mathrm{SL}_2(\mathbb{R})$ with a finite exponential moment, and assume that the subgroup $\Gamma_\mu$ is Zariski dense. Let $\nu$ be the unique $\mu-$stationary measure. Assume that $r$ is a $C^1$ function on $\mathbb{P}^1$ and $\phi$ is a $C^2$ function on $\mathbb{P}^1$ such that $|\phi'| \geq 1/C_1 > 0$ on the support of $r$ and*

$$\|r\|_{C^1}, \|\phi\|_{C^2} \leq C_1 \text{ for some constant } C_1 > 0.$$

*Then we have*

$$\int e^{i\xi\phi(x)} r(x) \mathrm{d}\nu(x) \to 0 \text{ as } |\xi| \to \infty, \tag{1.1.2}$$

*uniformly with respect to $C_1$.*

This is the main theorem of this paper. It will be proved in Section 1.3.

**Corollary 1.1.4.** *Let $\mu$ be a Borel probability measure on $\mathrm{SL}_2(\mathbb{R})$ with a finite exponential moment, and assume that the subgroup $\Gamma_\mu$ is Zariski dense. Let $\nu$ be the unique $\mu-$stationary measure. Then for a $C^2-$diffeomorphism $\phi$ on $\mathbb{P}^1$, the pushforward of the stationary measure $\phi_*\nu$ is a Rajchman measure. In other words*

$$\widehat{\phi_*\nu}(k) \to 0 \text{ as } |k| \to +\infty. \tag{1.1.3}$$

Theorem 1.1.1 is a special case of this corollary, where $\phi$ is the identity function.

*Proof of Corollary 1.1.4 from Theorem 1.1.3.* By the identification $\mathbb{P}^1 \simeq \mathbb{T}$, we may consider all the objects as living on $\mathbb{T}$. Take a partition of unity of $\mathbb{T}$: let $r_1, r_2$ be non negative Lipschitz functions on $\mathbb{T}$ such that $r_1 + r_2 = 1$, and the supports of $r_1, r_2$ are connected subintervals of $\mathbb{T}$. For $j = 1, 2$, we can lift the function $\phi|_{\mathrm{supp} r_j}$ to a function $\phi_j$ from $\mathrm{supp} r_j$ to $\mathbb{R}$. Then

$$\int_{\mathbb{T}} e^{2ik\phi(\theta)} \mathrm{d}\nu(\theta) = \int_{\mathbb{T}} (r_1(\theta) + r_2(\theta)) e^{2ik\phi(\theta)} \mathrm{d}\nu(\theta) = \int_{\mathbb{T}} \left( e^{2ik\phi_1(\theta)} r_1(\theta) + e^{2ik\phi_2(\theta)} r_2(\theta) \right) \mathrm{d}\nu(\theta).$$

Since $\phi$ is a diffeomorphism, the functions $\phi_j, r_j$ satisfy the conditions in Theorem 1.1.3. We use this theorem twice to conclude. □

Let us use another coordinate system on $\mathbb{P}^1$. We identify $\mathbb{P}^1$ with $\mathbb{R} \cup \{\infty\}$ through the map $\varphi(x) = v_1/v_2$, where $x = \mathbb{R}v$ is a point in $\mathbb{P}^1$. Then the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{P}^1$ reads as the Möbius action, that is for $r \in \mathbb{R} \cup \{\infty\}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{R})$, we have $gr = \frac{ar+b}{cr+d}$.

If the support of a $\mu-$stationary measure $\nu$ does not contain $[1:0]$, then $\varphi_* \nu$ is a stationary measure on $\mathbb{R}$. From Theorem 1.1.3, we get

**Corollary 1.1.5.** *Let $\mu$ be a Borel probability measure on $\mathrm{SL}_2(\mathbb{R})$ with a finite exponential moment, and assume that the subgroup $\Gamma_\mu$ is Zariski dense. Let $\nu$ be the unique $\mu-$stationary measure. If the support of $\nu$ does not contain $[1:0]$, then the $\mu-$stationary measure $\varphi_*\nu$ is a Rajchman measure on $\mathbb{R}$. In other words*

$$\widehat{\varphi_*\nu}(\xi) = \int_{\mathbb{P}^1} e^{i\varphi(x)\xi} \mathrm{d}\nu(x) \to 0 \text{ as } |\xi| \to +\infty. \tag{1.1.4}$$

**Example 1.1.6** (Solvable case)**.** *For stationary measures on $\mathbb{R}$, consider the following*

$$\mu = \frac{1}{2}(\delta_{g_1} + \delta_{g_2}) = \frac{1}{2}\delta_{\begin{pmatrix} \sqrt{\lambda} & -1/\sqrt{\lambda} \\ 0 & 1/\sqrt{\lambda} \end{pmatrix}} + \frac{1}{2}\delta_{\begin{pmatrix} \sqrt{\lambda} & 1/\sqrt{\lambda} \\ 0 & 1/\sqrt{\lambda} \end{pmatrix}},$$

*where $\lambda \in (0,1)$. Then the actions of $g_1, g_2$ are given by $g_1 r = \lambda r - 1$, $g_2 r = \lambda r + 1$ for $r \in \mathbb{R}$. By definition, a $\mu-$stationary measure $\nu$ on $\mathbb{R}$ must satisfy the equation*

$$\nu = \mu * \nu = \frac{1}{2}((g_1)_*\nu + (g_2)_*\nu). \tag{1.1.5}$$

*Let $X_0, X_1, \ldots$ be i.i.d. random variables such that $\mathbb{P}(X_0 = 1) = \mathbb{P}(X_0 = -1) = 1/2$. Let $\nu_\lambda$ be the Bernoulli convolution with parameter $\lambda$, defined to be the distribution of $\sum_{j \geq 0} X_j \lambda^j$. The measure $\nu_\lambda$ satisfies (1.1.5), thus it is a $\mu$-stationary measure on $\mathbb{R}$. In [Erd39], Erdös proved that when $\lambda^{-1}$ is a Pisot number, the Fourier transform of $\nu_\lambda$ does not converge to zero. In this example $\Gamma_\mu$ is solvable, so the Zariski density condition is necessary in the theorem.*

**Remark 1.1.7.** *1. A similar result for Bernoulli convolutions was obtained in [Kau76]. Kaufman proved that for Bernoulli convolutions* $\nu_\lambda$*, if* $\lambda^{-1}$ *is not a Pisot number, then it satisfies the same conclusion as in Corollary 1.1.3. That is, the pushforward measure* $\phi_*\nu_\lambda$ *is a Rajchman measure, where* $\phi$ *is a* $C^1$ *function on* $\mathbb{R}$ *with* $\phi' > 0$ *everywhere.*
*2.Our result for the measure* $\nu$ *is stronger than being a Rajchman measure. Indeed, for a probability measure on* $\mathbb{T}$*, being a Rajchman measure is not invariant by diffeomorphisms. We can find examples in [Kau84]. A typical example is the standard* $\frac{1}{3}-$*Cantor measure* $\nu$*, which is not a Rajchman measure. Let* $\phi$ *be the quadratic map* $r \mapsto r^2$*. Then the pushforward measure* $\phi_*\nu$ *becomes a Rajchman measure with polynomial decay.*

One of our motivations for establishing Theorem 1.1.1 comes from the theory of Bernoulli convolutions. One of the main questions of this theory is to determine for which parameter $\lambda$, the measure $\nu_\lambda$ is absolutely continuous with respect to the Lebesgue measure. We have already mentioned that when $\lambda^{-1}$ is a Pisot number, Erdös proved that $\nu_\lambda$ is not a Rajchman measure. Thus, in particular, $\nu_\lambda$ is not absolutely continuous with respect to the Lebesgue measure. Recently, people have been interested in the same problem for stationary measures for random walks on $\mathrm{SL}_2(\mathbb{R})$, see [Bou12],[KLP11]. Our result shows that we cannot generalize the method of Erdös to the Zariski dense case.

Our other motivation is the same question for the Patterson-Sullivan measure on the limit set of Fuchsian groups. With Theorem 1.1.1, it suffices to prove that there exists a probability measure $\mu$ on $\mathrm{SL}_2(\mathbb{R})$ such that the Patterson-Sullivan measure is $\mu$-stationary, and $\mu$ has a finite exponential moment.

In [Lal89] and [Lal86], Lalley announced the existence of such a $\mu$ for Schottky groups. But Lalley's proof only works for Schottky semigroups. In [CM07], the authors proved the existence of such a $\mu$ without the moment condition in geometrically finite cases. Combining the methods of Connell, Muchnik and Lalley, we can prove the existence of such a measure $\mu$ for convex cocompact Fuchsian groups, see [Li]. Therefore, we have

**Corollary 1.1.8.** *Let* $\Gamma$ *be a convex cocompact Fuchsian group. Then the Patterson-Sullivan measure associated to* $\Gamma$ *is a Rajchman measure.*

**Remark 1.1.9.** *Corollary 1.1.8 also holds if we replace the Patterson-Sullivan measure by any Gibbs measure. In [Li], we have a similar realization for any Gibbs measure associated to a convex cocompact Fuchsian group, as it is done by Lalley for any Gibbs measure on the limit set of a Schottky semigroup in [Lal86].*

**Remark 1.1.10.** *Using the uniform spectral gap proved in [Nau05], we can prove a polynomial decay in the convergence to zero of the Fourier coefficients of the* $\mu$*-stationary measure, when the support of* $\mu$ *is the set of generators of a Schottky semigroup. In this case, the uniform spectral gap implies an exponential error term in the renewal theorem, which is the only obstacle for polynomial decay. Please see Remark 1.3.10 for more details. We believe it is true for the general case, but the question is still open.*

**Remark 1.1.11.** *Very recently, Bourgain and Dyatlov [BD17] have proved a polynomial decay of the Fourier coefficients of the Patterson-Sullivan measure associated to a convex* *cocompact Fuchsian group. Their method, which comes from additive combinatorics, is totally different from ours. They use the Fourier decay bound and the fractal uncertainty principle to obtain an essential spectral gap for a convex cocompact hyperbolic surface. We can not recover their result directly as in Remark 1.1.10. It is possible if we modifier some steps and use the uniform spectral gap in [Nau05], but we do not pursue in this direction in this work.*

On the other hand, in the geometrically finite case, this approach can not work. The finite exponential moment condition is impossible for noncompact lattice $\Gamma$ in $\mathrm{SL}_2(\mathbb{R})$ (see [GLJ93], [DKN09], [BHM11]). That is, if $\mu$ is a measure on $\Gamma$ with a finite first moment, then the $\mu$-stationary measure $\nu$ is singular with respect to the Lebesgue measure. Maybe the generalization of the method of [JS16] works in this case, where they proved the Gibbs measures for the Gauss map which has dimension greater than $1/2$ are Rajchman measures.

In this paper, our main idea is to obtain the convergence to zero of Fourier coefficients from a renewal type result.

**The strategy of proof**: To simplify, identify $\mathbb{P}^1$ with $\mathbb{T} = \mathbb{R}/\pi\mathbb{Z}$ as before. The starting point is the relation $\nu = \mu * \nu$. Consider a random walk on $\mathrm{SL}_2(\mathbb{R})$, $X_n = b_1 b_2 \cdots b_n$, where $b_j$ are independent random variables with the same law $\mu$. Let $\mathscr{B}_n$ be the Borel $\sigma-$algebra generated by $X_1, \dots, X_n$. Let $Y_n = (X_n)_*\nu$. They are random variables which take values in the space of Borel measures on $\mathbb{T}$. By definition, we have

$$\mathbb{E}(Y_{n+1}|\mathscr{B}_n) = \mathbb{E}((X_n)_*(b_{n+1})_*\nu|\mathscr{B}_n) = (X_n)_*\mathbb{E}((b_{n+1})_*\nu) = Y_n.$$

Therefore $\{Y_n\}$ is a martingale. For $t > 0$, we define the stopping time by $\tau = \inf\{n \in \mathbb{N} | \log \|X_n\| \geq t\}$. Then the martingale property implies that

$$\mathbb{E}((X_\tau)_*\nu) = \mathbb{E}(Y_\tau) = \mathbb{E}(Y_0) = \nu.$$

(See Proposition 1.3.5). Thus for the Fourier coefficients, we have for $k \in 2\mathbb{Z}$ (since $\overline{\hat{\nu}(k)} = \hat{\nu}(-k)$, we only consider $k \geq 0$.)

$$\hat{\nu}(k) = \int e^{ikx} \mathrm{d}\nu(x) = \int e^{ikx} \mathrm{d}\mathbb{E}((X_\tau)_*\nu)(x) = \int \mathbb{E}(e^{ikX_\tau x}) \mathrm{d}\nu(x).$$

Recall our circle $\mathbb{T}$ is $\mathbb{R}/\pi\mathbb{Z}$. The idea is to find some cancellations in the "trigonometric series" $\mathbb{E}(e^{ikX_\tau x})$. By the Cauchy-Schwarz inequality, it suffices to prove $\mathbb{E}(e^{ik(X_\tau x - X_\tau y)}) \to 0$ as $k \to \infty$.

By analogy with the case of classical random walks on $\mathbb{R}$, we expect that there exists a measurable density function $p$ on $\mathbb{R}^+$ such that for a continuous compactly supported function $f$ on $\mathbb{R}$ and $t \in \mathbb{R}$,

$$\mathbb{E}(f(\log \|X_\tau\| - t)) \longrightarrow \int_{\mathbb{R}^+} f(u)p(u)\mathrm{d}u \text{ as } t \to +\infty.$$

Then absolute continuity of the limit distribution would imply the convergence to zero of $\hat{\nu}(k)$.

In the actual proof, we do not use this stopping time, but a residue process. Indeed, the latter is easier to treat with transfer operators and Fourier analysis. We will establish a limit theorem for the residue process, a generalization of the renewal theorem, in Section 1.4.

**Notation**: When $f$ and $g$ are functions on a set $X$, we write $f(x) \lesssim g(x)$, if there exists $C > 0$ independent of $x \in X$ such that $f(x) \le Cg(x)$, and $f(x) = O(g(x))$ means $|f(x)| \lesssim g(x)$. We also write $f(x,y) = O_y(g(x,y))$, which means $|f(x,y)| \le C_y g(x,y)$, where $C_y$ is a constant only depending on $y$.

We introduce a notation $O_{\exp,\epsilon}(s)$. We write $f(\epsilon, s) = O_{\exp,\epsilon}(s)$ if for $\epsilon > 0$ and $s \in \mathbb{R}$, there exists a constant $\epsilon' > 0$ such that $f(\epsilon, s) = O(e^{-\epsilon' s})$, where all the constants only depend on $\epsilon$. We write $f(s) = O_{\exp}(s)$, if there exists a uniform constant $\epsilon'$ such that $f(s) = O(e^{-\epsilon' s})$.

## 1.2 Preliminaries on random walks on $\mathbb{P}^1$

Fix the norm induced by the standard inner product on $\mathbb{R}^2$, $\|v\| = \sqrt{v_1^2 + v_2^2}$, which is $\mathrm{SO}_2(\mathbb{R})$ invariant. Then define a metric on $\mathbb{P}^1$. For two points $x = \mathbb{R}v,\ x' = \mathbb{R}w$, we set

$$d(x,x') = \frac{|\det(v,w)|}{\|v\|\|w\|}.$$

This is a sine distance. If we write $x = \mathbb{R}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$ and $x' = \mathbb{R}\begin{pmatrix}\cos\theta' \\ \sin\theta'\end{pmatrix}$, then $d(x,x') = |\sin(\theta - \theta')|$. From now on, we write $G = \mathrm{SL}_2(\mathbb{R})$ and $X = \mathbb{P}^1$.

**Definition 1.2.1.** *For $g$ in $G$ and $x = \mathbb{R}v$ in $X$, define the function $\sigma : G \times X \to \mathbb{R}$ by $\sigma(g,x) = \log \frac{\|gv\|}{\|v\|}$.*

This function $\sigma$ is a cocycle, because for $g, h$ in $G$ we have

$$\sigma(gh,x) = \log\frac{\|ghv\|}{\|v\|} = \log\frac{\|ghv\|}{\|hv\|} + \log\frac{\|hv\|}{\|v\|} = \sigma(g,hx) + \sigma(h,x),$$

where we use the fact that the action is linear, $hx = \mathbb{R}hv$.

**Lemma 1.2.2.** *For $g$ in $G$ and $x, x'$ in $X$ with $x \neq x'$, we have*

$$\frac{d(gx,gx')}{d(x,x')} = \exp(-\sigma(g,x) - \sigma(g,x')). \tag{1.2.1}$$

*Proof.* As in the definition of the distance $d(\cdot,\cdot)$, we take two non zero vectors $v$ and $w$ in $x$ and $x'$ respectively. By definition,

$$\frac{d(gx,gx')}{d(x,x')} = \left|\frac{\det(gv,gw)}{\det(v,w)}\right| \frac{\|v\|\|w\|}{\|gv\|\|gw\|} = |\det g| \frac{\|v\|\|w\|}{\|gv\|\|gw\|} = \exp\left(-\sigma(g,x) - \sigma(g,x')\right).$$

The proof is complete. $\square$

If the point $x$ is near $x'$, we know from the above equation that the cocycle $\sigma$ is essentially the logarithm of the contracting or expanding ratio. Let $\mu$ be a Borel probability measure on $G$, and let $b_1, b_2, \cdots$ be independent random variables with the same law $\mu$. Then the behavior of the mean value of the cocycle,

$$\frac{1}{n}\sigma(b_n b_{n-1}\cdots b_1, x) = \frac{1}{n}\left(\sum_{j=1}^{n}\sigma(b_j, b_{j-1}\cdots b_1 x)\right),$$

follows an asymptotic law similar to the law of large numbers. In particular,

**Theorem 1.2.3.** *[Fur63][GR85] Let $\mu$ be a Borel probability measure on $G$ having an exponential moment. Assume that the subgroup $\Gamma_\mu$ is Zariski dense. Then for all $x$ in $X$, random variables $b_j$ defined as above, we have*

$$\lim_{n\to\infty}\frac{\sigma(b_n b_{n-1}\cdots b_1, x)}{n} = \int_G\int_X \sigma(g,y)\mathrm{d}\mu(g)\mathrm{d}\nu(y) = \sigma_\mu > 0 \quad a.s\ \mu^{\otimes\mathbb{N}^*}. \tag{1.2.2}$$

The constant $\sigma_\mu$ is called the Lyapunov exponent of $\mu$.

**Theorem 1.2.4** (Hölder regularity)**.** *[Gui90][BL85, Chapter 6,Proposition 4.1] Under the assumptions of Theorem 1.1.1, there exist constants $C > 0$, $\alpha > 0$ such that for every $x$ in $X$ and $r > 0$ we have*

$$\nu(B(x,r)) \leq Cr^\alpha. \tag{1.2.3}$$

We need the Cartan decomposition of the Lie group $G$, i.e. $G = \mathrm{SO}_2 A^+ \mathrm{SO}_2$, where $A^+ = \{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}, t \geq 0\}$. For $g$ in $G$, we can write $g = k_g a_g l_g$, where $k_g$, $l_g$ are in $\mathrm{SO}_2$, and $a_g = diag\{e^{\kappa(g)}, e^{-\kappa(g)}\}$ is the diagonal matrix whose diagonal elements are $e^{\kappa(g)}$ and $e^{-\kappa(g)}$ with $\kappa(g) \geq 0$. The positive number $\kappa(g)$ is called the Cartan projection. Identify the two spaces $X$ and $\mathbb{T} \simeq \mathbb{R}/\pi\mathbb{Z}$. For an element $x$ in $X$, associate it to the unique element $\theta(x)$ in $\mathbb{R}/\pi\mathbb{Z}$ satisfying $x = \mathbb{R}\begin{pmatrix}\cos\theta(x) \\ \sin\theta(x)\end{pmatrix}$. When there is no ambiguity, we will abbreviate $\theta(x)$ to $x$.

Let $e_1 = \mathbb{R}\begin{pmatrix}1\\0\end{pmatrix}$, $e_2 = \mathbb{R}\begin{pmatrix}0\\1\end{pmatrix}$, which mean elements in $X$. Let $r_\theta = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$ be a rotation matrix in $G$. For $g$ in $G$, choosing a decomposition $g = k_g a_g l_g$, we define $x_g^m = l_g^{-1}e_2, x_g^M = k_g e_1$. If $\kappa(g) > 0$, then $x_g^M, x_g^m$ are uniquely defined.

**Proposition 1.2.5.** *For $g$ in $G$ with $\kappa(g) > 0$, we have*

$$x_g^m = x_{g^{-1}}^M. \tag{1.2.4}$$

*Proof.* For a real number $a \neq 0$, we have

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} = r_{\pi/2}\begin{pmatrix} a^{-1} & 0 \\ 0 & a\end{pmatrix} r_{3\pi/2}.$$

This implies that

$$g^{-1} = (k_g a_g l_g)^{-1} = l_g^{-1} a_g^{-1} k_g^{-1} = l_g^{-1} r_{\pi/2} a_g r_{3\pi/2} k_g^{-1}.$$

Therefore $x^M_{g^{-1}} = l_g^{-1} r_{\pi/2} e_1 = l_g^{-1} e_2 = x_g^m$. □

**Lemma 1.2.6.** *For $g$ in $G$ and $x = \mathbb{R}v$ in $X$, we have*

$$d(x_g^m, x) \le \frac{\|gv\|}{\|g\|\|v\|} \le d(x_g^m, x) + e^{-2\kappa(g)}. \tag{1.2.5}$$

*Another form that will be used frequently is*

$$\sigma(g, x) \ge \kappa(g) + \log d(x_g^m, x).$$

*Proof.* Suppose that the vector $v$ has norm 1, then

$$\frac{\|gv\|}{\|v\|} = \frac{\|k_g a_g l_g v\|}{\|v\|} = \frac{\|a_g l_g v\|}{\|l_g v\|}.$$

Since $d(x_g^m, x) = d(l_g^{-1} e_2, x) = d(e_2, l_g x)$, it suffices to prove this inequality for diagonal elements, in other words $g = diag\{e^{\kappa(g)}, e^{-\kappa(g)}\}$. Hence

$$\frac{\|gv\|}{\|v\|} = \left| \begin{pmatrix} e^{\kappa(g)} & 0 \\ 0 & e^{-\kappa(g)} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right| = |e^{2\kappa(g)} v_1^2 + e^{-2\kappa(g)} v_2^2|^{1/2}. \tag{1.2.6}$$

The equality $d(x_g^m, x) = d(e_2, x) = |v_1|$ implies that

$$\begin{aligned} \frac{\|gv\|}{\|g\|\|v\|} &\ge |v_1| = d(x_g^m, x), \\ \frac{\|gv\|}{\|g\|\|v\|} &\le |v_1| + e^{-2\kappa(g)} = d(x_g^m, x) + e^{-2\kappa(g)}. \end{aligned}$$

The proof is complete. □

The following lemma is an important tool, which gives a precise approximation of the cocycle by the Cartan projection and distance.

**Lemma 1.2.7.** *Let $x, x'$ be two points in $X$ and let $g$ be in $G$. Assume that*

$$e^{-2\kappa(g)} + d(x_g^m, g^{-1}x) \le \frac{1}{2} d(g^{-1}x', x),$$

*then*

$$|\sigma(g, x) - \kappa(g) - \log d(g^{-1}x', x)| \le 2 \frac{e^{-2\kappa(g)} + d(x_g^m, g^{-1}x')}{d(g^{-1}x', x)}. \tag{1.2.7}$$

*Proof.* Inequality (1.2.5) implies that

$$\begin{aligned} |e^{\sigma(g,x)-\kappa(g)} - d(g^{-1}x', x)| &\le \max\{|d(x_g^m, x) - d(g^{-1}x', x)|, |e^{-2\kappa(g)} + d(x_g^m, x) - d(g^{-1}x', x)|\} \\ &\le e^{-2\kappa(g)} + d(x_g^m, g^{-1}x'). \end{aligned}$$

Thus by hypothesis, we have

$$|\exp(\sigma(g,x) - \kappa(g)) - d(g^{-1}x', x)| \le 1/2 d(g^{-1}x', x).$$

Since $|\log(1+t)| \le 2|t|$ for $t > -1/2$, we obtain

$$\begin{aligned} &|\sigma(g,x) - \kappa(g) - \log d(g^{-1}x', x)| = \log|1 + \frac{\exp(\sigma(g,x) - \kappa(g)) - d(g^{-1}x', x)}{d(g^{-1}x', x)}| \\ &\le 2\frac{|\exp(\sigma(g,x) - \kappa(g)) - d(g^{-1}x', x)|}{d(g^{-1}x', x)} \le 2\frac{e^{-2\kappa(g)} + d(x_g^m, g^{-1}x')}{d(g^{-1}x', x)}. \end{aligned}$$

The proof is complete. □

In the next proposition we summarize the large deviations principle for the cocycle and for the Cartan projection,

**Proposition 1.2.8.** *[BQ16, Thm13.11, Thm 13.17] Under the assumptions of Theorem 1.1.1, for every $\epsilon > 0$ we have*

$$\mu^{*n}\{g \in G|\ |\sigma(g,x) - n\sigma_\mu| \ge n\epsilon\} = O_{\exp,\epsilon}(n), \tag{1.2.8}$$

$$\mu^{*n}\{g \in G|\ |\kappa(g) - n\sigma_\mu| \ge n\epsilon\} = O_{\exp,\epsilon}(n), \tag{1.2.9}$$

*uniformly for all $x$ in $X$ and $n \ge 1$.*

Let $t$ be a real number. Write $[t]$ for the integer part of $t$.

**Corollary 1.2.9.** *Under the assumptions of Theorem 1.1.1, for every $\epsilon > 0$ we have*

$$\sum_{m \ge n} \mu^{*m}\{g \in G|\sigma(g,x) \le t\} = O_{\exp,\epsilon}(n),$$

$$\sum_{m \ge n} \mu^{*m}\{g \in G|\kappa(g) \le t\} = O_{\exp,\epsilon}(n),$$

*uniformly for all $x$ in $X$, $t > 0$ and $n \ge [\frac{t}{\sigma_\mu - \epsilon}]$.*

By the hypothesis of finite exponential moment and the Chebyshev inequality, we have

**Lemma 1.2.10.** *Under the assumptions of Theorem 1.1.1, let $M_\mu$ be the finite exponential moment of $\mu$ defined by $M_\mu = \int \|g\|^{\epsilon_1} \mathrm{d}\mu(g)$. For $s > 0$, we have*

$$\mu\{g \in G|\kappa(g) \ge s\} \le M_\mu e^{-\epsilon_1 s}. \tag{1.2.10}$$

**Corollary 1.2.11.** *Under the assumptions of Theorem 1.1.1, for every* $\epsilon > 0$ *we have*

$$\sum_{m\leq n} \mu^{*m}\{g \in G|\sigma(g,x) \geq t\} = O_{\exp,\epsilon}(t),$$
$$\sum_{m\leq n} \mu^{*m}\{g \in G|\kappa(g) \geq t\} = O_{\exp,\epsilon}(t),$$

*uniformly for all* $x$ *in* $X$, $t > 0$ *and* $n = [\frac{t}{\sigma_\mu+\epsilon}]$.

*Proof.* The inequality about the cocycle follows from the one about the Cartan projection, because $\kappa(g) \geq \sigma(g,x)$. It suffices to prove the second inequality:

- When $m \leq \epsilon_2 t$, where $\epsilon_2 > 0$ is a small constant such that $\epsilon_2 \leq \epsilon_1/(2\log M_\mu)$, from Chebyshev's inequality and the subadditivity of the Cartan projection, we have
$$\sum_{m=1}^{[\epsilon_2 t]} \mu^{\otimes m}\{\kappa(g) \geq t\} \leq \sum_{m=1}^{[\epsilon_2 t]} e^{-\epsilon_1 t}\int e^{\epsilon_1\kappa(g)}\mathrm{d}\mu^{\otimes m}(g) \leq \sum_{m=1}^{[\epsilon_2 t]} e^{-\epsilon_1 t}\left|\int \|g\|^{\epsilon_1}\mathrm{d}\mu(g)\right|^m$$
$$\leq e^{-\epsilon_1 t}M_\mu^{[\epsilon_2 t]}/(M_\mu - 1).$$
This implies that $\sum_{m=1}^{[\epsilon_2 t]} \mu^{\otimes m}\{\kappa(g) \geq t\} \lesssim e^{-t\epsilon_1/2}$.
- When $m \in [\epsilon_2 t, t/(\sigma_\mu + \epsilon)]$, we have $\kappa(g) > t \geq m(\sigma_\mu + \epsilon)$. Then use (1.2.9) to deduce that the measure of this part is less than $\sum_{m\in[\epsilon_2 t,t/(\sigma_\mu+\epsilon)]} O_{\exp,\epsilon}(m) = O_{\exp,\epsilon}(t)$.

The proof is complete. □

The following proposition describes regularity properties of $\mu^{*n}$, which is a corollary of the large deviations principle.

**Proposition 1.2.12.** *[BQ16, Prop14.3] Under the assumptions of Theorem 1.1.1, for every* $\epsilon > 0$ *we have*

$$\mu^{*n}\{g \in G|\ d(gx, x') \leq e^{-n\epsilon}\} = O_{\exp,\epsilon}(n), \tag{1.2.11}$$
$$\mu^{*n}\{g \in G|\ d(x_g^M, x) \leq e^{-n\epsilon}\} = O_{\exp,\epsilon}(n), \tag{1.2.12}$$
$$\mu^{*n}\{g \in G|\ d(x_g^m, g^{-1}x) \geq e^{-(2\sigma_\mu-\epsilon)n}\} = O_{\exp,\epsilon}(n), \tag{1.2.13}$$
$$\mu^{*n}\{g \in G|\ d(x_g^M, gx) \geq e^{-(2\sigma_\mu-\epsilon)n}\} = O_{\exp,\epsilon}(n), \tag{1.2.14}$$

*uniformly for all* $x$, $x'$ *in* $X$ *and* $n \geq 1$.

**Corollary 1.2.13.** *Under the assumptions of Theorem 1.1.1, for every* $\epsilon > 0$ *we have*

$$\mu^{*n}\{g \in G|\ d(gx, x') \leq e^{-t}\} = O_{\exp,\epsilon}(t), \tag{1.2.15}$$

*uniformly for all* $x, x'$ *in* $X$, $t > 0$ *and* $n \geq t/\epsilon$.

*For every* $\epsilon > 0$ *we have*

$$\mu^{*n}\{g \in G|\ d(x_g^M, gx) \geq e^{-t}\} = O_{\exp,\epsilon}(t), \tag{1.2.16}$$

*uniformly for all* $x$ *in* $X$, $t > 0$ *and* $n \geq t/(2\sigma_\mu - \epsilon)$.

*Proof.* There exists an integer $n_t \leq n$ such that $\epsilon n_t < t \leq \epsilon(n_t+1)$. By inequality (1.2.11), we have $\mu^{*n_t}\{d(gx,x') \leq e^{-\epsilon n_t}\} \lesssim e^{-\epsilon' n_t}$. This implies that

$$\begin{aligned}\mu^{*n}\{g\in G|d(gx,x')\leq e^{-t}\} &= \int_G \mu^{*n_t}\{l\in G|d(l(hx),x')\leq e^{-t}\}\mathrm{d}\mu^{*(n-n_t)}(h)\\ &\leq \int_G \mu^{*n_t}\{l\in G|d(l(hx),x')\leq e^{-\epsilon n_t}\}\mathrm{d}\mu^{*(n-n_t)}(h)\\ &\lesssim e^{-\epsilon' n_t} \lesssim e^{-\epsilon'(t/\epsilon-1)} \lesssim e^{-\epsilon' t/\epsilon}.\end{aligned}$$

The second inequality follows from the same argument. $\square$

The following lemma describes the difference between the cocycle and the Cartan projection.

**Lemma 1.2.14.** *[BQ16, Lemma 17.8] Under the assumptions of Theorem 1.1.1, for every $\epsilon > 0$, there exist $C > 0, \epsilon' > 0$ such that for all $n \geq l > 0$ and $x$ in $X$, there exists a subset $S_{n,l,x} \subset G \times G$, which satisfies*

$$\mu^{*(n-l)} \otimes \mu^{*l}(S^c_{n,l,x}) \leq Ce^{-\epsilon' l} = O_{\exp,\epsilon}(l),$$

*and for all $(g_1, g_2) \in S_{n,l,x}$, we have*

$$|\kappa(g_1g_2) - \sigma(g_1, g_2x) - \kappa(g_2)| \leq e^{-\epsilon l}.$$

By the identification $X \simeq \mathbb{T}$, we can work on $\mathbb{T}$. Since the circle $\mathbb{T}$ is a quotient space of $\mathbb{R}$, it has the induced orientation. For two different points $x, y$ in $\mathbb{T}$, which are not the two endpoints of a diameter, they divide the circle into two arcs. Call the arc with longer length the large arc, and the other arc the small arc $x \frown y$. For a function $\phi$ on $\mathbb{T}$, it can be seen as a function $\Phi$ on $\mathbb{R}$ with period $\pi$. Define $\phi'(\theta)$ as the derivative of $\Phi$.

We introduce a sign for two different points $x, y$ in $X$, where $x, y$ are not the two endpoints of a diameter. If in the small arc $x \frown y$, the point $x$ is the start point in the orientation sense, then we define $\mathrm{sign}(x, y) = 1$; otherwise, we define $\mathrm{sign}(x, y) = -1$. We have a Newton-Leibniz formula on the circle

$$\phi(y) - \phi(x) = \mathrm{sign}(x,y)\int_{x\frown y}\phi'(\theta)\mathrm{d}\theta, \tag{1.2.17}$$

where $\mathrm{d}\theta$ is the Lebesgue measure on $\mathbb{T}$ induced by the Lebesgue measure on $\mathbb{R}$ with total mass $\pi$.

**Definition 1.2.15** (Orientation)**.** *Let $x, y, z$ be three points in $X$. Define*

$$\mathrm{sign}(x,y,z) = \begin{cases} 0 & \textit{if any two points coincide,}\\ 1 & \textit{if } \{x,y,z\} \textit{ is counterclockwise,}\\ -1 & \textit{otherwise.}\end{cases}$$

**Proposition 1.2.16.** *Let $x, y$ be two different points in $X$, and let $g$ be in $G$ such that $\kappa(g) > 2$ and $d(x_g^m, x), d(x_g^m, y) > e^{-\kappa(g)}$. Then*

$$\mathrm{sign}(gx, gy) = \mathrm{sign}(x, y, x_g^m). \tag{1.2.18}$$

*Proof.* With the same argument as in the proof of Lemma 1.2.6, it suffices to prove the statement in case $g = a_g$, that is $\mathrm{sign}(a_g x, a_g y) = \mathrm{sign}(x, y, e_2)$.

If $x'$ is a point in $X$ such that $d(e_2, x') > e^{-\kappa(g)}$, then

$$d(a_g x', e_1) = d(a_g x', a_g e_1) = d(x', e_1)\exp(-\sigma(a_g, x') - \sigma(a_g, e_1)).$$

By (1.2.5), we obtain $\sigma(a_g, x') \geq \kappa(g) + \log d(e_2, x') > 0$, so

$$d(a_g x', e_1) \leq \exp(-\kappa(g)) \leq e^{-2}.$$

Thus the action of $a_g$ on the interval $B(e_2, e^{-\kappa(g)})^c$ is contracting with fixed point $e_1$, and the image is in the interval $B(e_1, e^{-\kappa(g)})$. Especially, $e_2$ is not in $B(e_1, e^{-\kappa(g)})$ and the small arc $a_g x \frown a_g y$ is contained in $B(e_1, e^{-\kappa(g)})$. By definition we have

$$\mathrm{sign}(a_g x, a_g y) = \mathrm{sign}(a_g x, a_g y, e_2).$$

Since the action of $a_g$ on $\mathbb{T}$ preserves the orientation, we have $\mathrm{sign}(a_g x, a_g y, e_2) = \mathrm{sign}(x, y, e_2)$. The proof is complete. $\square$

## 1.3 Decrease of the Fourier transform

Here we give a proof of Theorem 1.1.3, by admitting the technical results that will be proved in the following two sections. Recall the notations $G = \mathrm{SL}_2(\mathbb{R})$ and $X = \mathbb{P}^1$.

**Definition 1.3.1.** *Let $\Sigma = \bigcup_{n\in\mathbb{N}} G^{\times n}$ be the symbol space of all finite sequences with elements in $G$. Let $\mu$ be a Borel probability measure on $G$, and let $\mu^{\otimes n}$ be the product measure on $G^{\times n}$. Then $\mu^{\otimes n}$ can be seen as a measure on $\Sigma$ which is nonzero only on $G^{\times n}$. Let $\bar{\mu}$ be the measure on $\Sigma$ defined by $\bar{\mu} = \delta_\emptyset + \mu + \mu^{\otimes 2} + \cdots$.*

*Let the integer $\omega(g)$ be the length of an element $g$ in $\Sigma$. Then an element $g$ can be written as $(g_1, g_2, \ldots, g_\omega)$, where $\omega$ is the abbreviation of $\omega(g)$.*

*Let $T$ be the shift map on $\Sigma$, defined by $Tg = T(g_1, g_2, \ldots, g_\omega) = (g_1, g_2, \ldots, g_{\omega-1})$, when $\omega(g) \geq 2$, and $Tg = \emptyset$, when $\omega(g) = 1, 0$.*

*Let $L$ be the left shift map on $\Sigma$, defined by $Lg = L(g_1, g_2, \ldots, g_\omega) = (g_2, \ldots, g_{\omega-1}, g_\omega)$, when $\omega(g) \geq 2$, and $Lg = \emptyset$, when $\omega(g) = 1, 0$.*

*When considering the action of $g$ on $X$, we write $gx = g_1 \cdots g_\omega x$, $\sigma(g, x) = \sigma(g_1 \cdots g_\omega, x)$, $x_g^m = l_{g_1\cdots g_\omega}^{-1} e_2$, as well as the Cartan projection $\kappa(g) = \kappa(g_1 \cdots g_\omega)$.*

**Remark 1.3.2.** *When using this definition, we may meet the convolution measure $\mu^{*n}$ on $G$ or the product measure $\mu^{\otimes n}$ on $G^{\times n}$. Denote $F : G^{\times n} \to G$ by $F(g_1, g_2, \ldots, g_n) = g_1 \cdots g_n$, then $F_*(\mu^{\otimes n}) = \mu^{*n}$.*

**Definition 1.3.3.** *For $t>0$, define two sets that contain all the sequences which make the value of the Cartan projection pass $t$,*

$$M_t^+=\{g\in\Sigma|\ \kappa(Tg)<t\le\kappa(g)\},\ M_t^-=\{g\in\Sigma|\ \kappa(Tg)\ge t>\kappa(g)\}.$$

**Remark 1.3.4.** *In some special cases, for $b_j$ in $\operatorname{supp}\mu$, the Cartan projection $\kappa(b_1b_2\cdots b_n)$ is increasing with respect to $n$. Then $M_t^-$ has $\bar\mu$ measure zero. Let $X_n=b_1b_2\cdots b_n$ be a random walk on $G$, where $b_j$ are i.i.d. random variables taking values in $G$ with the same law $\mu$. Let $\tau$ be the stopping time defined by $\tau=\inf\{n\in\mathbb{N}|\kappa(X_n)\ge t\}$. In such special case*

$$\bar\mu(M_t^+\cap G^{\times n})=\mathbb{P}(\tau=n).$$

*So in the measure sense, $M_t^+$ is a set of the steps. That is for $\bar\mu-$almost every $g$ in $M_t^+$, it is of the form $g=(b_1,b_2,\dots,b_\tau)=(X_1,X_1^{-1}X_2,\dots,X_{\tau-1}^{-1}X_\tau)$ which corresponds to the set of steps of the trajectory $(X_1,X_2,\dots,X_\tau)$. But this is not always true for general cases.*

By Corollary 1.2.9, these two sets $M_t^+$, $M_t^-$ have finite $\bar\mu$ measure. We have a property of $M_t^+$, $M_t^-$ due to the definition of stationary measures. Our proof is a generalization of the property of the stopping time for martingales.

**Proposition 1.3.5.** *Under the assumptions of Theorem 1.1.1, for a real number $t>0$ and a continuous function $f$ on $X$, we have*

$$\int_X f(x)\mathrm{d}\nu(x)=\int_X\left(\int_{g\in M_t^+}f(gx)\mathrm{d}\bar\mu(g)-\int_{g\in M_t^-}f(gx)\mathrm{d}\bar\mu(g)\right)\mathrm{d}\nu(x).$$

*Proof.* For a natural number $N$, let

$$\begin{aligned}F_N=\int_X\Bigg(&\int_{g\in M_t^+,\omega(g)\le N}f(gx)\mathrm{d}\bar\mu(g)-\int_{g\in M_t^-,\omega(g)\le N}f(gx)\mathrm{d}\bar\mu(g)\\&+\int_{\omega(g)=N,\kappa(g)<t}f(gx)\mathrm{d}\bar\mu(g)\Bigg)\mathrm{d}\nu(x).\end{aligned}$$

Then $F_o=\int_X f(x)\mathrm{d}\nu(x)$. Since all the terms are finite, we have

$$\begin{aligned}F_{N+1}-F_N=\int_X\Bigg(&\int_{g\in M_t^+,\omega(g)=N+1}f(gx)\mathrm{d}\bar\mu(g)-\int_{g\in M_t^-,\omega(g)=N+1}f(gx)\mathrm{d}\bar\mu(g)\\&+\int_{\omega(g)=N+1,\kappa(g)<t}f(gx)\mathrm{d}\bar\mu(g)-\int_{\omega(g)=N,\kappa(g)<t}f(gx)\mathrm{d}\bar\mu(g)\Bigg)\mathrm{d}\nu(x).\end{aligned}$$

By the relation $\nu = \mu * \nu$, the set of integration of the last term becomes $\{\omega(g) = N+1, \kappa(Tg) < t\}$. Compare these sets of integration

$$\begin{aligned}\{g \in &M_t^+, \omega(g) = N+1\} \cup \{\omega(g) = N+1, \kappa(g) < t\}\\ =&\{\omega(g) = N+1, \kappa(Tg) < t, \kappa(g) \geq t\} \cup \{\omega(g) = N+1, \kappa(g) < t\}\\ =&\{\omega(g) = N+1, \kappa(Tg) \geq t, \kappa(g) < t\} \cup \{\omega(g) = N+1, \kappa(Tg) < t\}\\ =&\{g \in M_t^-, \omega(g) = N+1\} \cup \{\omega(g) = N+1, \kappa(Tg) < t\}.\end{aligned}$$

Therefore, $F_{N+1} = F_N = \cdots = F_o$. Corollary 1.2.9 and Inequality (1.2.9) imply that $\bar{\mu}\{g \in M_t^{\pm}, \omega(g) > N\}$, $\bar{\mu}\{\omega(g) = N, \kappa(g) < t\} \to 0$, as $N \to \infty$. Thus

$$F_N \to \int_X \left( \int_{g \in M_t^+} f(gx) \mathrm{d}\bar{\mu}(g) - \int_{g \in M_t^-} f(gx) \mathrm{d}\bar{\mu}(g) \right) \mathrm{d}\nu(x) \text{ as } N \to \infty,$$

which completes the proof. □

With these preparations, we start to prove Theorem 1.1.3, by admitting Lemma 1.5.2, Corollary 1.5.5 and Proposition 1.4.28.

*Proof of Theorem 1.1.3.* We will prove that there exist constants $\epsilon_0 > 0, C_0 > 0$ such that for every $s > 0$, the Fourier transform $\int e^{i\xi\phi(\theta)} r(\theta) \mathrm{d}\nu(\theta)$ is less than $C_0 e^{-\epsilon_0 s}$ for all $|\xi|$ large enough depending on $s$.

Fix a constant $\epsilon_3 \leq 1/10$. Write $t = (\log|\xi| - s)/2$, and take $|\xi|$ large enough such that $t > 10s$.

**Step 1**: Let $e_\xi(x)$ be the function $e^{i\xi\phi(x)} r(x)$. Using Proposition 1.3.5 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left| \int_X e_\xi(x) \mathrm{d}\nu(x) \right| &= \left| \int_{g \in M_t^+} \int_X e_\xi(gx) \mathrm{d}\nu(x) \mathrm{d}\bar{\mu}(g) - \int_{g \in M_t^-} \int_X e_\xi(gx) \mathrm{d}\nu(x) \mathrm{d}\bar{\mu}(g) \right|\\ &\leq \bar{\mu}(M_t^+)^{1/2} \left( \int_{M_t^+} \left| \int_X e_\xi(gx) \mathrm{d}\nu(x) \right|^2 \mathrm{d}\bar{\mu}(g) \right)^{1/2}\\ &\quad + \bar{\mu}(M_t^-)^{1/2} \left( \int_{M_t^-} \left| \int_X e_\xi(gx) \mathrm{d}\nu(x) \right|^2 \mathrm{d}\bar{\mu}(g) \right)^{1/2}.\end{aligned}$$

By Lemma 1.5.2 and Proposition 1.3.5, $\bar{\mu}(M_t^+)$, $\bar{\mu}(M_t^-)$ are uniformly bounded with $t$. Change the order of integration, then

$$\begin{aligned}\left| \widehat{\phi_*(r\mathrm{d}\nu)}(\xi) \right| \lesssim& \left( \int_{X^2} \int_{M_t^+} e^{i\xi(\phi(gx) - \phi(gy))} r(gx) r(gy) \mathrm{d}\bar{\mu}(g) \mathrm{d}\nu(x) \mathrm{d}\nu(y) \right)^{1/2}\\ &+ \left( \int_{X^2} \int_{M_t^-} e^{i\xi(\phi(gx) - \phi(gy))} r(gx) r(gy) \mathrm{d}\bar{\mu}(g) \mathrm{d}\nu(x) \mathrm{d}\nu(y) \right)^{1/2}. \quad (1.3.1)\end{aligned}$$

From now on, we only consider $M_t^+$. The set $M_t^-$ has similar properties, and the needed changes will be discussed in remarks, which appear at the end of each section.

**Step 2**: The main approximation, which will be proved in Section 1.5, replaces the distance $\phi(gx) - \phi(gy)$ with $\phi' e^{-2\kappa(g)} d(x,y)$. The intuition here is that in a large set, whose complement has exponentially small measure, the behavior is nice.

To apply replacement, some regularity conditions on $x, y$ and $g$ are needed. Define a subset of $M_t^+$ for $x, y$ in $X$ by

$$M_t^+(x,y) = \{g \in M_t^+ | |\kappa(g) - \kappa(Tg)| < \epsilon_3 s, d(x_g^m, g^{-1}x) < e^{-t}, d(g^{-1}x, x), d(g^{-1}x, y) > 2e^{-\epsilon_3 s}\}. \tag{1.3.2}$$

For fixed $x, y$, set

$$\Lambda_0(g) = e^{i\xi(\phi(gx) - \phi(gy))} r(gx) r(gy),$$
$$\Lambda_1(g) = e^{i\xi \mathrm{sign}(g^{-1}x,x,y)\phi'(gx)d(x,y)\exp(-2\kappa(g))/(d(g^{-1}x,x)d(g^{-1}x,y))} r(gx)^2.$$

We give a control of the error, which appears in the replacement.

**Proposition 1.3.6.** *Assume that $t > 2s$. We have an exponential decay for all $g$ in $M_t^+(x,y)$. That is*

$$|\Lambda_0(g) - \Lambda_1(g)| = O_{\exp}(s). \tag{1.3.3}$$

This property will be proved in Section 1.5. We want to use some smooth cutoffs to regularize the function $\Lambda_1(g,x,y)$. Let $\rho$ be a smooth function on $\mathbb{R}$ such that $\rho|_{[-1,1]} = 1$, $\rho$ takes values in $[0,1]$, $\mathrm{supp}\rho \subset [-2,2]$ and $|\rho'| \leq 2$. Let

$$\Lambda_2(g) = \Lambda_1(g)(1 - \rho(d(g^{-1}x,x)e^{\epsilon_3 s}))(1 - \rho(d(g^{-1}x,y)e^{\epsilon_3 s}))\rho(\frac{\kappa(g) - \kappa(Tg)}{\epsilon_3 s})\rho(\frac{\kappa(Tg) - t}{2\epsilon_3 s}). \tag{1.3.4}$$

When $d(g^{-1}x,x) < e^{-\epsilon_3 s}$ or $d(g^{-1}x,y) < e^{-\epsilon_3 s}$, the function $\Lambda_2$ will be 0. With fixed $x, y$, $\mathrm{sign}(g^{-1}x,x,y)$ is a function of $g^{-1}x$, and the discontinuity is at $x$ and $y$. Hence the discontinuity of $\mathrm{sign}(g^{-1}x,x,y)$ is removed in $\Lambda_2$.

If $g \in M_t^+(x,y)$, it follows from definition that $|\kappa(Tg) - t| \leq |\kappa(g) - \kappa(Tg)| \leq \epsilon_3 s$. Then $\Lambda_2 = \Lambda_1$. Since $t > 10s$, using Corollary 1.5.5, Lemma 1.5.2 and (1.3.3), we get

$$\begin{aligned}\left|\int_{M_t^+} (\Lambda_0 - \Lambda_2) \mathrm{d}\bar{\mu}(g)\right| &\leq \bar{\mu}(M_t^+ - M_t^+(x,y)) + \left|\int_{M_t^+(x,y)} (\Lambda_0 - \Lambda_2) \mathrm{d}\bar{\mu}(g)\right| \\ &= \bar{\mu}(M_t^+ - M_t^+(x,y)) + \left|\int_{M_t^+(x,y)} (\Lambda_0 - \Lambda_1) \mathrm{d}\bar{\mu}(g)\right| = O_{\exp}(s).\end{aligned} \tag{1.3.5}$$

**Step 3**: Introduce the residue process for the Cartan projection. This is inspired by the stopping time. For the stopping time, the existence of the limit distribution of the residual waiting time was proved in [Kes74], but in that paper we do not have a rate of convergence, which is necessary in our method. Here we use the transfer operator to get a uniform rate of convergence. It is difficult to treat the stopping time with transfer operators, because the operator will no longer be continuous. However, the residue process, which will be introduced here, can be routinely analyzed by the transfer operator. What's more, we will get the limit distribution of $gx$ and $g^{-1}y$ simultaneously, which is important to us.

We generalize the inverse action on $\Sigma$, letting $g^{-1}=(g_1,\dots,g_\omega)^{-1}=(g_\omega^{-1},\dots,g_1^{-1})$ for $g$ in $\Sigma$. For a subset $M$ of $\Sigma$, set $\iota(M)=\{g^{-1}|g\in M\}$. Let $\check{\mu}$ be the pushforward of $\mu$ by the inverse action. Let $t$ be a positive number. Consider the limit of the following quantity as $t\to\infty$

$$\sum_{n\geq 0}\int_{\kappa(g)<t\leq\kappa(hg)} f((hg)^{-1}x', hgx, \kappa(hg)-\kappa(g), \kappa(g)-t)\mathrm{d}\mu(h)\mathrm{d}\mu^{*n}(g),$$

where $x,x'$ are points in $X$ and $f$ is a smooth, compactly supported function on $X^2\times\mathbb{R}^2$. Our result is similar to renewal theory. By Proposition 1.4.28, when $t$ tends to infinity, the limit is

$$\int_{X^2}\int_G\int_{-\sigma(h,y)}^0 f(y', hy, \sigma(h,y), u)\mathrm{d}u\mathrm{d}\mu(g)\mathrm{d}\nu(y)\mathrm{d}\check{\nu}(y'),$$

where $\check{\nu}$ is the stationary measure of $\check{\mu}$ and the integral $\int_{-\sigma(h,y_1)}^0=0$ if $\sigma(h,y_1)<0$.

Since $(Tg)^{-1}=L(g^{-1})$ and $\kappa(g^{-1})=\kappa(g)$, we can define

$$N_t^+=\iota(M_t^+)=\{g\in\Sigma|\kappa(Lg)<t\leq\kappa(g)\}. \tag{1.3.6}$$

Therefore

$$\int_{M_t^+}\Lambda_2(g)\mathrm{d}\bar{\mu}(g)=\int_{N_t^+}\Lambda_2(g^{-1})\mathrm{d}\check{\bar{\mu}}(g).$$

Recall that $x,y,\rho$ are fixed. For $x_1,x_2$ in $X$ and $v,u$ in $\mathbb{R}$, define

$$\begin{aligned}&\lambda(x_1,x_2)=d(x,y)e^s\mathrm{sign}(\xi)\mathrm{sign}(x,y,x_2)\phi'(x_1)/(d(x_2,x)d(x_2,y)),\\&\varphi(x_1,x_2,v,u)=r(x_1)^2\times(1-\rho(d(x_2,x)e^{\epsilon_3 s}))(1-\rho(d(x_2,y)e^{\epsilon_3 s}))\rho(\frac{v}{\epsilon_3 s})\rho(\frac{u}{2\epsilon_3 s}).\end{aligned}$$

By the relation $\xi=\mathrm{sign}(\xi)e^{2t+s}$, regroup the terms and rewrite the function

$$\Lambda_2(g^{-1})=e^{i\lambda(g^{-1}x,gx)\exp(-2(\kappa(g)-t))}\varphi(g^{-1}x,gx,\kappa(g)-\kappa(Lg),\kappa(Lg)-t). \tag{1.3.7}$$

Note that the function $\lambda$ is not continuous, but the function $\varphi$ will remove the discontinuity as we have discussed in Step 2. In the language of the residue process, let $f$ be the function on $X^2\times\mathbb{R}^2$ defined by

$$f(x_1,x_2,v,u)=e^{i\lambda(x_1,x_2)\exp(-2(u+v))}\varphi(x_1,x_2,v,u). \tag{1.3.8}$$

Thus the function $\Lambda_2(g^{-1})$ can be written as

$$\Lambda_2(g^{-1}) = f(g^{-1}x, gx, \kappa(g) - \kappa(Lg), \kappa(Lg) - t).$$

By Proposition 1.4.28, for $\delta > 0$, $t > 2(|K| + \delta)$ (where $K$ is the projection of supp$f$ onto $\mathbb{R}_v$), we have

$$\begin{aligned}
&\int_{M_t^+} \Lambda_2 \mathrm{d}\bar{\mu}(g) = \int_{N_t^+} f \mathrm{d}\bar{\check{\mu}}(g) \\
&= \int_{X^2} \int_G \int_{-\sigma(h,x_2)}^0 f(x_1, hx_2, \sigma(h, x_2), u) \mathrm{d}u \mathrm{d}\mu(g) \mathrm{d}\nu(x_2) \mathrm{d}\check{\nu}(x_1) + O_K(\delta + O_\delta/t)|f|_{Lip}.
\end{aligned} \tag{1.3.9}$$

Here $|f|_{Lip}$ is the Lipschitz norm defined by

$$|f|_{Lip} = |f|_\infty + \sup_{(x_1,x_2,v,u) \neq (x_1',x_2',v',u')} \frac{|f(x_1, x_2, v, u) - f(x_1', x_2', v', u')|}{d(x_1, x_1') + d(x_2, x_2') + |v - v'| + |u - u'|}.$$

**Lemma 1.3.7.** *There exist constants $\delta_0(s)$ and $t(\delta, s)$ such that if $\delta < \delta_0(s)$ and $t > t(\delta, s)$, then*

$$O_K(\delta + O_\delta/t)|f|_{Lip} \leq e^{-s}. \tag{1.3.10}$$

*Proof.* By the definition of $\rho$ and $f$, the support of $f$ is in the compact set $X^2 \times [-4\epsilon_3 s, 4\epsilon_3 s]^2$. The size of $K$, the projection of supp$f$ onto $\mathbb{R}_v$, is bounded by $8\epsilon_3 s$. The definition of $\rho$ implies that $f$ is locally Lipschitz. Together with the fact that $f$ is compactly supported, we conclude that $|f|_{Lip}$ is controlled by $e^{2s}$ independently of $x, y$. Take $\delta$ small enough according to $s$, then take $t$ large enough according to $\delta$ and $s$. We get the inequality. □

**Step 4:** For the major term in (1.3.9), use the following lemma.

**Lemma 1.3.8.** *For $b_1 < b_2$ and $\lambda$ nonzero, we have*

$$\left|\int_{b_1}^{b_2} e^{i\lambda \exp(-u)} \mathrm{d}u\right| \leq \frac{2(e^{b_1} + e^{b_2})}{|\lambda|}. \tag{1.3.11}$$

*Proof.* Integration by parts gives

$$\int_{b_1}^{b_2} e^{i\lambda \exp(-u)} \mathrm{d}u = \int_{b_1}^{b_2} \frac{\partial_u (e^{i\lambda \exp(-u)})}{-i\lambda e^{-u}} \mathrm{d}u = \frac{e^{i\lambda \exp(-u)}}{-i\lambda e^{-u}}\Big|_{b_1}^{b_2} + \int_{b_1}^{b_2} e^{i\lambda \exp(-u)} \partial_u \left(\frac{1}{-i\lambda e^{-u}}\right) \mathrm{d}u.$$

This implies that

$$\left|\int_{b_1}^{b_2} e^{i\lambda \exp(-u)} \mathrm{d}u\right| \leq \left|\frac{e^{i\lambda \exp(-u)}}{\lambda e^{-u}}\Big|_{b_1}^{b_2}\right| + \int_{b_1}^{b_2} \partial_u (\frac{e^u}{|\lambda|}) \mathrm{d}u \leq \frac{2(e^{b_1} + e^{b_2})}{|\lambda|}.$$

The proof is complete. □

When $d(x,y) > e^{-\epsilon_3 s}$, due to the definition of $\rho(\frac{v}{\epsilon_3 s})$, the major term only integrates on $h, x_2$ such that $|\sigma(h,x_2)| \leq 2\epsilon_3 s$. The inequality $|u| \leq |\sigma(h,x_2)| \leq 2\epsilon_3 s$ implies that $\rho(\frac{u}{2\epsilon_3 s}) = 1$. By the hypotheses on $\phi$, when $r(x_1) \neq 0$, we have $|\phi'(x_1)| \geq 1/C_1 > 0$. Therefore

$$|\lambda(x_1, hx_2)| = |d(x,y)e^s \mathrm{sign}(\xi)\mathrm{sign}(x,y,hx_2)\phi'(x_1)/(d(hx_2,x)d(hx_2,y))| \geq e^{(1-\epsilon_3 s)}/C_1.$$

We use Lemma 1.3.8 to obtain

$$|\int_{-\sigma(h,x_2)}^0 f\mathrm{d}u| \leq |r|_\infty^2 \left|\int_0^{\sigma(h,x_2)} e^{i\lambda \exp(-2u)}\mathrm{d}u\right| \leq C_1|r|_\infty^2 \frac{1+e^{2\epsilon_3 s}}{e^{1-\epsilon_3 s}} \leq |r|_\infty^2 2e^{(3\epsilon_3-1)s}C_1.$$

Combined with (1.3.9), they imply that $\int_{M_t^+} \Lambda_2 \mathrm{d}\bar\mu(g) = O_{\exp}(s)$. When $d(x,y) \leq e^{-\epsilon_3 s}$, the Hölder regularity of stationary measure (1.2.3) implies that

$$\int_{X\times X} \mathbb{1}_{d(x,y)\leq e^{-\epsilon_3 s}} \mathrm{d}\nu(x)\mathrm{d}\nu(y) \leq \int_X \nu(B(x,e^{-\epsilon_3 s}))\mathrm{d}\nu(x) = O_{\exp}(s).$$

Finally we obtain

$$\begin{aligned}
&\int_{X^2}\int_{M_t^+} \Lambda_2(x,y)\mathrm{d}\bar\mu(g)\mathrm{d}\nu(x)\mathrm{d}\nu(y) \\
&\leq \int_{X^2} \mathbb{1}_{d(x,y)>e^{-\epsilon_3 s}} \int_{M_t^+} \Lambda_2(x,y)\mathrm{d}\bar\mu(g)\mathrm{d}\nu(x)\mathrm{d}\nu(y) \\
&+ \int_{X^2} \mathbb{1}_{d(x,y)\leq e^{-\epsilon_3 s}} \int_{M_t^+} \Lambda_2(x,y)\mathrm{d}\bar\mu(g)\mathrm{d}\nu(x)\mathrm{d}\nu(y) \leq O_{\exp}(s)(1+\bar\mu(M_t^+)).
\end{aligned}$$

By Lemma 1.5.2, the measure $\bar\mu(M_t^+)$ is uniformly bounded. By using (1.3.1) and (1.3.5), the proof is complete. □

**Remark 1.3.9** (Minus case). *For $M_t^-$, we have another version of Lemma 1.5.2, Corollary 1.5.5 and Proposition 1.4.28. The integral $|\int_{-\sigma(h,y_1)}^0 f\mathrm{d}u|$ is replaced by $|\int_0^{-\sigma(h,y_1)} f\mathrm{d}u|$.*

**Remark 1.3.10.** *When $s$ is large and $\xi$ is of size $e^{Cs}$, all the error terms have polynomial decay except the one from Proposition 1.4.28. As we have mentioned in Remark 1.1.10, a uniform spectral gap makes Proposition 1.4.28 effective. Then we will have a polynomial decay.*

*The uniformity with respect to $\|r\|_{C^1}, \|\phi\|_{C^2}$ and $1/\inf_{\mathrm{supp} r}|\phi'|$ is due to the fact that all the terms depend only on these norms and the measure $\mu$.*

## 1.4 Renewal theory

We define a renewal operator $R$ as follows. For a positive bounded Borel function $f$ on $X\times\mathbb{R}$, a point $x$ in $X$ and a real number $t$, we set

$$Rf(x,t) = \sum_{n=0}^{+\infty}\int_G f(gx, \sigma(g,x)-t)\mathrm{d}\mu^{*n}(g).$$

Because of the positivity of $f$, this sum is well defined. In [Kes74], Kesten proved a renewal theorem for Markov chains, which is valid in our case [GLP16]. But a uniform speed of convergence is needed. We will give a proof using the complex transfer operator, which fulfills our demands. The treatment of the transfer operator will be along the path in [Boy16]. The renewal theorem will give us an equidistribution phenomenon, where the key input is non-arithmeticity.

First we give a proof of renewal theorem for good functions. Then we prove some regularity properties and independence properties for the renewal process. These will imply a version of residue process. Finally, we prove a theorem for the Cartan projection from a similar theorem for the cocycle.

Fix the constant $\epsilon = \sigma_\mu/4$ in this section. Keep in mind that the assumptions of Theorem 1.1.1 are always satisfied.

### 1.4.1 Complex transfer operators

We introduce the complex transfer operator $P(z)$. Let $\mathcal{H}^\gamma(X)$ be the space of $\gamma$-Hölder functions on $X$, a Banach space with the norm $|f|_\gamma = |f|_\infty + m_\gamma(f) = |f|_\infty + \sup_{x\neq y}\frac{|f(x)-f(y)|}{d(x,y)^\gamma}$. For $f$ in $\mathcal{H}^\gamma(X)$ and a complex number $z$, define

$$P(z)f(x) = \int_G e^{-z\sigma(g,x)} f(gx)\mathrm{d}\mu(g).$$

The main properties of $P(z)$ are summarized as follows

**Proposition 1.4.1.** *[Boy16, Theorem 4.1, Lemma 4.7] For any $\gamma > 0$ small enough, there exists $\eta > 0$ such that when $|\Re z| < \eta$, the transfer operator $P(z)$ is a bounded operator on $\mathcal{H}^\gamma(X)$ and depends analytically on $z$. Moreover there exists an analytic operator $U(z)$ on a neighborhood of $0 \leq \Re z < \eta$ such that the following equality holds for $0 \leq \Re z < \eta$*

$$(I - P(z))^{-1} = \frac{1}{\sigma_\mu z} N_0 + U(z), \tag{1.4.1}$$

*where $N_0$ is the operator defined by $N_0 f = \int f \mathrm{d}\nu$*

**Remark 1.4.2.** *In Proposition 1.4.1, the non-arithmeticity is crucial to prove that $(I - P(z))^{-1}$ has only one pole in the imaginary axis, which is 0. The non-arithmeticity follows from Zariski density. See for instance [Ben00] and [Dal00].*

The assumption of Theorem 4.1 in [Boy16] are complicated. It is verified, in the proof of theorem 1.4, page 8 [Boy16], that our condition on $\mu$ is enough to apply Theorem 4.1. The idea is due to Guivarch and Le Page.

**Proposition 1.4.3.** *[Boy16, Lemma 4.4] For any $\gamma > 0$ small enough, there exist $\eta > 0$, $0 < \rho < 1$, $C > 0$ such that when $0 \leq \Re z < \eta$, for a natural number $n$ and a $\gamma$-Hölder function $f$, we have*

$$|P(z)^n f|_\infty \leq (C\rho^n)^{\Re z} |f|_\infty \tag{1.4.2}$$

**Remark 1.4.4.** *For further usage, we need a bound on $\gamma$. Let $\epsilon, \epsilon'(\epsilon)$ be the two constants in (1.2.11), that is $\mu^{*n}\{d(gx,x') \leq e^{-\epsilon s}\} \leq Ce^{-\epsilon' s}$, and $\epsilon_1$ the constant in exponential moment. Choose a small $\gamma$ such that $\gamma \leq \frac{1}{4}\max\{\frac{\sigma_\mu/4}{\epsilon'(\sigma_\mu/4)}, \epsilon_1\}$.*

### 1.4.2 Renewal theory for regular functions

We start to compute the renewal operator. A result for the renewal operator for "good" functions will be proved. Let $f$ be a function on $X \times \mathbb{R}$. Define a norm by $|f|_{L^\infty\mathcal{H}^\gamma} = \sup_{\xi\in\mathbb{R}} |f(x,\xi)|_{\mathcal{H}^\gamma}$, which is the supremum of the Hölder norm of $f(\cdot,\xi)$. Define another norm $|f|_{W^{1,\infty}\mathcal{H}^\gamma} = |f|_{L^\infty\mathcal{H}^\gamma} + |\partial_\xi f|_{L^\infty\mathcal{H}^\gamma}$. Write the Fourier transform $\hat{f}(x,\xi) = \int e^{i\xi u} f(x,u)\mathrm{d}u$.

**Proposition 1.4.5.** *Let $f$ be a positive bounded continuous function in $L^1(X\times\mathbb{R}, \nu\otimes Leb)$ such that its Fourier transform satisfies $\hat{f} \in L^\infty\mathcal{H}^\gamma$ and $\partial_\xi \hat{f} \in L^\infty\mathcal{H}^\gamma$. Assume that the projection of $\mathrm{supp}\hat{f}$ onto $\mathbb{R}$ is in a compact set $K$. Then for all $t > 0$ and $x$ in $X$, we have*

$$Rf(x,t) = \frac{1}{\sigma_\mu}\int_{-t}^{\infty} f(y,u)\mathrm{d}u\mathrm{d}\nu(y) + \frac{1}{t}O_K(|\hat{f}|_{W^{1,\infty}\mathcal{H}^\gamma}).$$

*Proof.* Combine the following two lemmas. □

**Lemma 1.4.6.** *Under the same assumption as in Proposition 1.4.5, we have*

$$Rf(x,t) = \frac{1}{\sigma_\mu}\int_{t}^{\infty} f(y,u)\mathrm{d}u\mathrm{d}\nu(y) + \frac{1}{2\pi}\int e^{it\xi}U(i\xi)\hat{f}(x,\xi)\mathrm{d}\xi.$$

*Proof.* Introduce a local notation: for $(x,t)$ in $X\times\mathbb{R}$ and $s \geq 0$, write

$$B_s f(x,t) = \int_G e^{-s\sigma(g,x)} f(gx, \sigma(g,x)+t)\mathrm{d}\mu(g).$$

When $s = 0$, we abbreviate the notation $B_0$ to $B$. We want to prove the following equality,

$$\sum_{n\geq 0} B^n(f)(x,t) = \lim_{s\to 0^+}\sum_{n\geq 0} B_s^n(f)(x,t). \tag{1.4.3}$$

By definition, one has

$$\begin{aligned} B_s^n(f)(x,t) &= \int_G e^{-s\sigma(g,x)} f(gx,\sigma(g,x)+t)\mathrm{d}\mu^{*n}(g) \\ &= \int_G e^{-s\sigma(g,x)}(\mathbb{1}_{\sigma(g,x)>0} + \mathbb{1}_{\sigma(g,x)\leq 0}) f(gx,\sigma(g,x)+t)\mathrm{d}\mu^{*n}(g). \end{aligned}$$

- The part $\mathbb{1}_{\sigma(g,x)>0}$, since $f \geq 0$, use the monotone convergence theorem. When $s \to 0^+$ then

$$\sum_{n\geq 0}\int_G e^{-s\sigma(g,x)}\mathbb{1}_{\sigma(g,x)>0} f(gx,\sigma(g,x)+t)\mathrm{d}\mu^{*n}(g) \to \sum_{n\geq 0}\int_G \mathbb{1}_{\sigma(g,x)>0} f(gx,\sigma(g,x)+t)\mathrm{d}\mu^{*n}(g).$$

- For the part $\mathbb{1}_{\sigma(g,x)\leq 0}$, take $s$ in $[0,\eta/2]$. Proposition 1.4.3 implies that

$$\int_G e^{-s\sigma(g,x)}\mathbb{1}_{\sigma(g,x)\leq 0}f(gx,\sigma(g,x)+t)\mathrm{d}\mu^{*n}(g)\leq \int_G e^{-\eta\sigma(g,x)/2}|f|_\infty \mathrm{d}\mu^{*n}(g)\leq (C\rho^n)^{\eta/2}|f|_\infty.$$

  Since $\sum_{n\geq 0}\rho^{n\eta/2}$ is finite, take $e^{-\eta\sigma(g,x)/2}|f|_\infty$ as the dominant function. Then use the dominated convergence theorem to conclude.

This proves equation (1.4.3).

Using the inverse Fourier transform, we have

$$\begin{aligned}\sum_{n\geq 0}B_s^n(f)(x,t)&=\sum_{n\geq 0}\int_G e^{-s\sigma(g,x)}f(gx,\sigma(g,x)+t)\mathrm{d}\mu^{*n}(g)\\&=\sum_{n\geq 0}\int_G e^{-s\sigma(g,x)}\frac{1}{2\pi}\int_{\mathbb{R}}e^{i\xi(\sigma(g,x)+t)}\hat{f}(gx,\xi)\mathrm{d}\xi\mathrm{d}\mu^{*n}(g).\end{aligned}\tag{1.4.4}$$

Since $\hat{f}(x,\xi)$ has compact support, $|\hat{f}(x,\xi)|\leq|\hat{f}(x,\xi)|_{L^\infty_\xi\mathcal{H}^\gamma}$ and $|P(s)^n\mathbb{1}|\leq C\rho^{sn}$ for $s$ in $[0,\eta/2]$ (Proposition 1.4.3), we have

$$\sum_{n\geq 0}\int_G e^{-s\sigma(g,x)}\int_{\mathbb{R}}|\hat{f}(gx,\xi)|\mathrm{d}\xi\mathrm{d}\mu^{*n}(g)\leq C_f\sum_{n\geq 0}\int_G e^{-s\sigma(g,x)}\mathrm{d}\mu^{*n}(g)=C_f\sum_{n\geq 0}P(s)^n(\mathbb{1})<\infty,$$

which implies that the right hand side of (1.4.4) is absolutely convergent. Consequently, we can use the Fubini theorem to change the order of the integration. By the hypothesis $\hat{f}(x,\xi)\in\mathcal{H}^\gamma(X)$, Proposition 1.4.1 implies that

$$\begin{aligned}\sum_{n\geq 0}B_s^n(f)(x,t)&=\frac{1}{2\pi}\int_{\mathbb{R}}\sum_{n\geq 0}\int_G e^{(-s+i\xi)\sigma(g,x)}\hat{f}(gx,\xi)\mathrm{d}\mu^{*n}(g)e^{it\xi}\mathrm{d}\xi\\&=\frac{1}{2\pi}\int_{\mathbb{R}}\sum_{n\geq 0}P^n(s-i\xi)\hat{f}(x,\xi)e^{it\xi}\mathrm{d}\xi\\&=\frac{1}{2\pi}\int_{\mathbb{R}}(1-P(s-i\xi))^{-1}\hat{f}(x,\xi)e^{it\xi}\mathrm{d}\xi\\&=\frac{1}{2\pi}\int_{\mathbb{R}}(\frac{N_0}{\sigma_\mu(s-i\xi)}+U(s-i\xi))\hat{f}(x,\xi)e^{it\xi}\mathrm{d}\xi.\end{aligned}$$

Since $\frac{1}{s-i\xi}=\int_0^{+\infty}e^{-(s-i\xi)u}\mathrm{d}u$ for $s>0$, together with the property $\hat{f}\in L^1(\mathbb{R})$, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}}\frac{N_0}{\sigma_\mu(s-i\xi)}\hat{f}(x,\xi)e^{it\xi}\mathrm{d}\xi&=\frac{1}{2\pi}\frac{1}{\sigma_\mu}\int_X\int_{\mathbb{R}}\frac{\hat{f}(y,\xi)}{s-i\xi}e^{it\xi}\mathrm{d}\xi\mathrm{d}\nu(y)\\&=\frac{1}{\sigma_\mu}\int_X\int_0^\infty f(y,u+t)e^{-su}\mathrm{d}u\mathrm{d}\nu(y).\end{aligned}$$

When $s \to 0^+$, since $f$ is integrable with respect to the product measure $\nu \otimes Leb$, by monotone convergence theorem, the limit is $\frac{1}{\sigma_\mu}\int_X \int_t^\infty f(y,u)\mathrm{d}u\mathrm{d}\nu(y)$. Since $\hat{f}(x,\xi)$ is compactly supported, we have

$$\lim_{s\to 0^+}\int_{\mathbb{R}} U(-s+i\xi)\hat{f}(x,\xi)e^{it\xi}\mathrm{d}\xi = \int_{\mathbb{R}} U(i\xi)\hat{f}(x,\xi)e^{it\xi}\mathrm{d}\xi.$$

The proof is complete. □

**Lemma 1.4.7.** *Under the same assumption as in Proposition 1.4.5, we have*

$$\left|\int e^{-it\xi}U(i\xi)\hat{f}(x,\xi)\mathrm{d}\xi\right| \leq \frac{1}{t}O_K\left(|\hat{f}|_{L^\infty\mathcal{H}^\gamma} + |\partial_\xi \hat{f}|_{L^\infty\mathcal{H}^\gamma}\right).$$

*Proof.* Use the fact that $\hat{f}(x,\xi)$ is compactly supported and $|\hat{f}(x,\xi)|_{\mathcal{H}^\gamma}$, $|\partial_\xi\hat{f}(x,\xi)|_{\mathcal{H}^\gamma} < \infty$. Then applying integration by parts, we have

$$\begin{aligned}\int e^{-it\xi}U(i\xi)\hat{f}(x,\xi)\mathrm{d}\xi &= \frac{1}{it}\int e^{-it\xi}\partial_\xi(U(i\xi)\hat{f}(x,\xi))\mathrm{d}\xi \\ &= \frac{1}{it}\int e^{-it\xi}\left(\partial_\xi(U(i\xi))\hat{f}(x,\xi) + U(i\xi)\partial_\xi\hat{f}(x,\xi)\right)\mathrm{d}\xi.\end{aligned}$$

Since the operator norms of $U(i\xi)$ and $\partial_\xi U(i\xi)$ are uniformly bounded on compact regions, the result follows. □

### 1.4.3 Regularity properties of renewal measures

We have two principles in this subsection. **Principle 1**: Let $f$ be a bounded Borel function supported in $X\times[0,a]$. When we take the renewal sum outside of the interval $I_t = [\frac{t}{\sigma_\mu+\epsilon}, \frac{t+a}{\sigma_\mu-\epsilon}]$,

$$\sum_{n\in\mathbb{N}-I_t}\int_G f(gx,\sigma(g,x)-t)\mathrm{d}\mu^{*n}(g) = \sum_{n\in\mathbb{N}-I_t}\int_G f(gx,\sigma(g,x)-t)\mathbb{1}_{[0,a]}(\sigma(g,x)-t)\mathrm{d}\mu^{*n}(g),$$

this sum decays exponentially with $t$. This is given by the large deviations principle (Corollary 1.2.9, 1.2.11). For $n$ in the interval $I_t$, if some property is valid for each $n$ with an exponential error of $n$, we sum up. Since the length of this interval is comparable with $t$, this property is also valid for the renewal sum with an exponential error of $t$.

**Principle 2**: The other is independence. By Proposition 1.4.5, the limit distribution of $(\sigma(g,x)-t, gx)$ is $\frac{1}{\sigma_\mu}\nu\otimes Leb$, which is a product measure. That roughly means the following: As in Remark 1.3.4, let $X_n = b_n\cdots b_1$ be a random walk on $G$. Let $F = F_1\times F_2$ where $F_1$, $F_2$ are Borel subsets of $X$, $\mathbb{R}$ respectively. Then

$$\sum_{n\geq 0}\mathbb{P}\{(X_n x, \sigma(X_n,x)-t)\in F_1\times F_2\} \to \frac{1}{\sigma_\mu}\nu(F_1)\otimes Leb(F_2) \text{ as } t\to+\infty.$$

More concretely, we could expect that $R(\mathbb{1}_{F_1\times F_2})(x,t)$ is almost $\frac{1}{\sigma_\mu}\nu(F_1)\otimes Leb(F_2)$ when $t$ is large.

We want to use convolution to smooth out the target function. There exists an even function $\psi$ such that it is a probability density, and the Fourier transform $\hat{\psi}$ is compactly supported. Let $\psi_\delta(t)=\frac{1}{\delta^2}\psi(\frac{t}{\delta^2})$. Then $\int_{-\delta}^{\delta}\psi_\delta(t)\mathrm{d}t=\int_{-1/\delta}^{1/\delta}\psi(t)\mathrm{d}t>1-C\delta$.

**Proposition 1.4.8.** *Let $\delta\leq 1/3$ and $b_2\geq b_1$. If $b_2-b_1\geq 2\delta$, then for $x$ in $X$ and $t>0$, we have*

$$R(\mathbb{1}_{[b_1,b_2]})(x,t)\lesssim(b_2-b_1)(1/\sigma_\mu+C_\delta(1+|b_2|+|b_1|)/t). \tag{1.4.5}$$

*If $0\leq b_2-b_1<2\delta$, then for $x$ in $X$ and $t>0$, we have*

$$R(\mathbb{1}_{[b_1,b_2]})(x,t)\lesssim\delta(1/\sigma_\mu+C_\delta(1+|b_1|)/t). \tag{1.4.6}$$

*Proof.* When $b_2-b_1\geq 2\delta$, if $u$ is in $[b_1,b_2]$, then $[u-b_2,u-b_1]$ contains at least one of $[0,\delta]$ or $[-\delta,0]$. Therefore

$$\psi_\delta*\mathbb{1}_{[b_1,b_2]}(u)=\int_{b_1}^{b_2}\psi_\delta(u-v)\mathrm{d}v\geq\int_0^\delta\psi(v)\mathrm{d}v\geq(1-\delta)/2.$$

Then

$$\mathbb{1}_{[b_1,b_2]}\leq 3\psi_\delta*\mathbb{1}_{[b_1,b_2]}. \tag{1.4.7}$$

It is sufficient to bound $R(\psi_\delta*\mathbb{1}_{[b_1,b_2]})$. Proposition 1.4.5 implies that

$$R(\psi_\delta*\mathbb{1}_{[b_1,b_2]})=\frac{1}{\sigma_\mu}\int_{-t}^{\infty}\psi_\delta*\mathbb{1}_{[b_1,b_2]}+\frac{O_\delta}{t}|\hat{\psi}_\delta\hat{\mathbb{1}}_{[b_1,b_2]}|_{W^{1,\infty}\mathcal{H}^\gamma}.$$

The first term is less than $\int\psi_\delta*\mathbb{1}_{[b_1,b_2]}=(b_2-b_1)$. For the second term, we have

$$\begin{aligned}|\hat{\psi}_\delta\hat{\mathbb{1}}_{[b_1,b_2]}|_{W^{1,\infty}\mathcal{H}^\gamma}&=|\hat{\psi}_\delta\hat{\mathbb{1}}_{[b_1,b_2]}|_{L^\infty\mathcal{H}^\gamma}+|\partial_\xi\hat{\psi}_\delta\hat{\mathbb{1}}_{[b_1,b_2]}|_{L^\infty\mathcal{H}^\gamma}\\&=|\hat{\psi}_\delta\hat{\mathbb{1}}_{[b_1,b_2]}|_{L^\infty}+|\partial_\xi\hat{\psi}_\delta\hat{\mathbb{1}}_{[b_1,b_2]}|_{L^\infty}\\&\leq C'_\delta(|\mathbb{1}_{[b_1,b_2]}(u)|_{L^1}+|u\mathbb{1}_{[b_1,b_2]}(u)|_{L^1})\leq C'_\delta(b_2-b_1)(1+|b_1|+|b_2|).\end{aligned}$$

When $b_2-b_1\in[0,2\delta]$, the renewal sum $R(\mathbb{1}_{[b_1,b_2]})$ is bounded by $R(\mathbb{1}_{[b_1,b_1+2\delta]})$. Then use the previous case. □

In Proposition 1.4.5, since we do not have a good control of the spectral radius of the operator $U(i\xi)$ for large $|\xi|$, the estimates are effective only for large $t$, which means that when $t$ is small the error term will be out of control. The following lemma combines the transfer operator and the large deviations principle to give a uniform estimate.

**Lemma 1.4.9.** *For real numbers $s,t$ and a point $x$ in $X$, we have*

$$R(\mathbb{1}_{[0,s]})(x,t)\lesssim\max\{1,s\}. \tag{1.4.8}$$

*Proof.* We can suppose that $s > 1$. If not, then $R(\mathbb{1}_{[0,s]})(x,t) \leq R(\mathbb{1}_{[0,1]})(x,t)$. When $t \geq s$, this is a direct corollary of Proposition 1.4.8. Fixing $\delta = 1/3$, we get

$$\frac{1+|b_1|+|b_2|}{t} \leq \frac{1+s}{t} \leq 2.$$

Then $R(\mathbb{1}_{[0,s]})(x,t) \lesssim s(1/\sigma_\mu + 2C_\delta)$.

When $t < s$, let $m = [\max\{0, (t+s)/(\sigma_\mu - \epsilon)\}] + 1$. By Corollary 1.2.9, we have

$$\begin{aligned} R(\mathbb{1}_{[0,s]})(x,t) \leq R(\mathbb{1}_{[0,2s]})(x,0) &\leq \sum_{n\leq m} \mu^{\otimes n}\{\sigma(g,x) \leq 2s\} + \sum_{n>m} \mu^{\otimes n}\{\sigma(g,x) \leq 2s\} \\ &\lesssim m + e^{-\epsilon' m} \lesssim s. \end{aligned}$$

The proof is complete. □

In the renewal theorem, the limits of the scalar part $\sigma(g,x)$ and the angle part $gx$ are independent. Using this spirit, we give the following lemma, which quantifies this independence. In the proof, when $t$ is large enough, using Proposition 1.4.5, the remainder term will be small. When $t$ is small, we have another estimate from the regularity of the convolution measure $\mu^{\otimes n}$.

**Proposition 1.4.10.** *For $s > 0$, $a > 0$, $t > 5s$, and $x, x'$ in $X$, we have*

$$R(\mathbb{1}_{B(x',e^{-s})\times[0,a]})(x,t) = (1+a)^2 O_{\exp}(s). \tag{1.4.9}$$

*Proof.* Decompose the region of $t$ into two parts:

- When $5s < t \leq e^{2\gamma s}$, by Corollaries 1.2.9, 1.2.11, it suffices to consider $n \in [t/(\sigma_\mu + \epsilon), (t+a)/(\sigma_\mu - \epsilon)]$. Due to the hypothesis in this situation $s \leq t/5 = \epsilon t/(\sigma_\mu + \epsilon) \leq \epsilon n$, we can use Corollary 1.2.13 to obtain

  $$\mu^{*n}\{d(gx,x') \leq e^{-s}\} \lesssim e^{-\epsilon' s/\epsilon}.$$

  Then the measure of this part, summing up the above inequality over all $n \in [t/(\sigma_\mu + \epsilon), (t+a)/(\sigma_\mu - \epsilon)]$, is less than $C(t+a)e^{-\epsilon' s/\epsilon} \lesssim (1+a)e^{-\gamma s}$ (here we use the Remark 1.4.4, $4\gamma \leq \epsilon/\epsilon'$).

- When $t \geq e^{2\gamma s}$, we take $f = \mathbb{1}_{[0,a]}\varpi(x)$ where $\varpi(x)$ is a function on $X$ such that $\varpi_{B(x',e^{-s})} = 1$, $\mathrm{supp}\varpi \subset B(x', 2e^{-s})$ and $|\varpi|_\gamma \leq e^{\gamma s}$. As in the proof of Proposition 1.4.8, we use $\psi_\delta$ to regularize this function. By (1.4.7), we have

  $$3R(\psi_\delta * f)(x,t) \geq R(\mathbb{1}_{B(x',e^{-s})\times[0,a]})(x,t).$$

  Proposition 1.4.5 implies

  $$R(\psi_\delta * f) = \frac{1}{\sigma_\mu}\int_X \int_{-t}^{\infty} \psi_\delta * f(x,u)\mathrm{d}u\mathrm{d}\nu(x) + \frac{C_\delta}{t}(|\widehat{\psi_\delta * f}|_{W^{1,\infty}\mathcal{H}^\gamma}).$$

Since $\widehat{\psi_\delta * f}(x,\xi) = \hat{\psi}_\delta(\xi)\hat{\mathbb{1}}_{[0,a]}(\xi)\varpi(x)$, the two functions are independent. We can use the same estimate as in the proof of Proposition 1.4.8. So the rest term is less than $C'_\delta(1+a)^2 e^{\gamma s}/t$. The major term, due to the regularity of the stationary measure (1.2.3), is controlled by $ae^{-\alpha s}/\sigma_\mu$. The result follows from the hypothesis $t > e^{2\gamma s}$.

The proof is complete. □

We also need the independence of $\sigma(g,x)$ and $g^{-1}x_o$, where $x, x_o$ are two points in $X$. For proving this property, we pass through the Cartan projection, because the order of products in the Cartan projection can be reversed. The following proof uses Lemma 1.2.14, which is a central tool to prove a renewal type theorem for the Cartan projection from a renewal type theorem for the cocycle.

Let $f$ be a positive bounded Borel function on $X \times \mathbb{R}$. For $(x,t) \in X \times \mathbb{R}$, we define

$$R_P(f)(x,t) = \sum_{n\geq 0} \int_G f(gx, \kappa(g) - t)\mathrm{d}\mu^{*n}(g).$$

**Lemma 1.4.11.** *For $s > 0$, $a > 0$, $t > 10s$, and $x$, $x'$ in $X$, we have*

$$R_P(\mathbb{1}_{B(x',e^{-s})\times[0,a]})(x,t) = (1+a)^2 O_{\exp}(s). \tag{1.4.10}$$

*Proof.* Due to Corollary 1.2.9 and Corollary 1.2.11, the sum of the integral of $n \leq t/(\sigma_\mu + \epsilon)$ and $n \geq (t+a)/(\sigma_\mu - \epsilon)$ is exponentially small.

If suffices to consider $n$ in the interval $I_t = [t/(\sigma_\mu + \epsilon), (t+a)/(\sigma_\mu - \epsilon)]$. Fix $l = [\epsilon_4 t/\sigma_\mu]$ with $\epsilon_4 = 1/10$. By Lemma 1.2.14, there exists $S_{n,l,x} \subset G^{\times n}$ such that $\mu^{\otimes n} S^c_{n,l,x} = O_{\exp}(l)$, and for $(g_n, \dots, g_1)$ in $S_{n,l,x}$, letting $g = (g_n, \dots, g_{l+1})$ and $j = (g_l, \dots, g_1)$, we have

$$|\kappa(gj) - \sigma(g, jx) - \kappa(j)| \leq e^{-\epsilon l} \leq 1.$$

Thus

$$\begin{aligned}
&\mu^{\otimes n}\{\kappa(gj) \in [t, t+a], d(gjx, x') \leq e^{-s}\} \\
\leq &\mu^{\otimes n}\{S^c_{n,l,x}\} + \mu^{\otimes n}\{gj \in S_{n,l,x} | \kappa(gj) \in [t, t+a], d(gjx, x') \leq e^{-s}\} \\
\leq &O_{\exp}(l) + \mu^{\otimes n}\{\sigma(g, jx) + \kappa(j) \in [t-1, t+a+1], d(gjx, x') \leq e^{-s}\}.
\end{aligned}$$

Therefore summing over $n$ and integrating first with respect to $g$, we get

$$\begin{aligned}
&\sum_{n \in I_t} \mu^{\otimes n}\{\kappa(gj) \in [t, t+a], d(gjx, x') \leq e^{-s}\} \\
\leq &\sum_{n \in I_t} \mu^{\otimes n}\{\sigma(g, jx) + \kappa(j) \in [t-1, t+a+1], d(gjx, x') \leq e^{-s}\} + tO_{\exp}(l) \\
\leq &tO_{\exp}(l) + \int R(\mathbb{1}_{B(x',e^{-s})\times[-1,a+1]})(jx, t - \kappa(j))\mathrm{d}\mu^{*l}(j).
\end{aligned} \tag{1.4.11}$$

Hence, it is sufficient to bound $\int R(\mathbb{1}_{B(x',e^{-s})\times[-1,a+1]})(jx, t-\kappa(j))\mathrm{d}\mu^{*l}(j)$. Let $G_{l,\epsilon} = \{j \in G^{\times l} | \kappa(j) \leq l(\sigma_\mu + \epsilon)\}$. By the large deviations principle (Corollary 1.2.11), we have $\mu^{*l} G_{l,\epsilon}^c = O_{\exp}(l)$.

- For $j \in G_{l,\epsilon}$, we have $t - \kappa(j) \geq t - l(\sigma_\mu + \epsilon) = t - \epsilon_4(\sigma_\mu + \epsilon)t/\sigma_\mu > t/2 \geq 5s$. Hence, Proposition 1.4.10 implies that

$$R(\mathbb{1}_{B(x',e^{-s})\times[-1,a+1]})(jx, t-\kappa(j)) \lesssim (1+a)^2 O_{\exp}(s).$$

- For $j \in G_{l,\epsilon}^c$, Lemma 1.4.9 implies that

$$R(\mathbb{1}_{B(x',e^{-s})\times[-1,a+1]})(jx, t-\kappa(j)) \lesssim (1+a).$$

Combining the above two inequalities, we have

$$\int R(\mathbb{1}_{B(x',e^{-s})\times[-1,a+1]})(jx, t-\kappa(j))\mathrm{d}\mu^{*l}(j) \lesssim (1+a)^2 O_{\exp}(s) + O_{\exp}(l)(1+a). \tag{1.4.12}$$

The proof is complete. □

There is a byproduct of the above lemma. When the function $f$ does not depend on $X$, abbreviate $R_P(f)(x,t)$ by $R_P(f)(t)$.

**Lemma 1.4.12.** *For real numbers $s, t$, we have*

$$R_P(\mathbb{1}_{[0,s]})(t) \lesssim \max\{1, s^2\}. \tag{1.4.13}$$

**Remark 1.4.13.** *Here the term $s^2$ is not optimal. With some extra work, it can be improved to $s$.*

*Proof.* Suppose that $s \geq 1$. If not, then $R_p(\mathbb{1}_{[0,s]})(x,t) \leq R_p(\mathbb{1}_{[0,1]})(x,t)$. When $t \geq 10$, apply Lemma 1.4.11 with $a = s, e^{-s} = e^{-1}, x' = x_j, j \in J$, where $J$ is a finite set such that $\cup_{j\in J} B(x_j, e^{-1})$ covers $X$. So we get $R_P(\mathbb{1}_{[0,s]})(t) \lesssim s^2$.

When $t < 10 \leq 10s$, let $m = [\max\{0, (t+s)/(\sigma_\mu - \epsilon)\}] + 1$. By Corollary 1.2.11, we have

$$\begin{aligned} R_P(\mathbb{1}_{[0,s]})(t) \leq R_P(\mathbb{1}_{[0,2s]})(0) &\leq \sum_{n\leq m} \mu^{*n}\{\kappa(g) \leq t+s\} + \sum_{n>m} \mu^{*n}\{\kappa(g) \leq t+s\} \\ &\lesssim m + e^{-\epsilon' m} \lesssim s. \end{aligned}$$

The proof is complete. □

Now we are going to prove the independence of $\sigma(g,x)$ and $g^{-1}x$. Recall that $\check{\mu}$ is the pushforward of $\mu$ by the inverse action. Let $f$ be a positive bounded Borel function on $X\times\mathbb{R}$. For $(x_o,x,t)\in X^2\times\mathbb{R}$, we define

$$R_I(f)(x_o,x,t)=\sum_{n\geq 0}\int_G f(g^{-1}x_o,\sigma(g,x)-t)\mathrm{d}\mu^{*n}(g).$$

**Proposition 1.4.14.** *For $s>0$, $a>0$, $t>\max\{10s,10\}$, and $x$, $x'$, $x_o$ in $X$, we have*

$$R_I(\mathbb{1}_{B(x',e^{-s})\times[0,a]})(x_o,x,t)=(1+a)^2O_{\exp}(s). \tag{1.4.14}$$

*Proof.* Due to Corollary 1.2.9 and Corollary 1.2.11, the sums of the integral of $n\leq t/(\sigma_\mu+\epsilon)$ and $n\geq(t+a)/(\sigma_\mu-\epsilon)$ is exponentially small.

It suffices to consider $n$ in the interval $I_t=[t/(\sigma_\mu+\epsilon),(t+a)/(\sigma_\mu-\epsilon)]$. Let

$$G_{\epsilon,n}=\{g\in G^{\times n}|\kappa(g)\geq n(\sigma_\mu-\epsilon/2),d(g^{-1}x_o,x)>e^{-\epsilon n},d(g^{-1}x_o,x_g^m)\leq e^{-(2\sigma_\mu-\epsilon)n}\}.$$

By inequalities (1.2.9), (1.2.11) and (1.2.13), we have $\mu^{\otimes n}G_{\epsilon,n}\geq 1-O_{\exp}(n)$. Since $t>10$, for $n$ in $I_t$, we have $n\geq t/(\sigma_\mu+\epsilon)\geq 10/(\sigma_\mu+\epsilon)$. For $g\in G_{\epsilon,n}$, we have

$$\frac{e^{-2\kappa(g)}+d(x_g^m,g^{-1}x_o)}{d(g^{-1}x_o,x)}\leq\frac{2e^{-(2\sigma_\mu-\epsilon)n}}{e^{-\epsilon n}}=2e^{-(2\sigma_\mu-2\epsilon)n}\leq 2e^{-20(\sigma_\mu-\epsilon)/(\sigma_\mu+\epsilon)}\leq 1/2.$$

Using Lemma 1.2.7 with $g\in G_{\epsilon,n}$, we have

$$|\sigma(g,x)-\kappa(g)-\log d(g^{-1}x_o,x)|\leq 2\frac{e^{-2\kappa(g)}+d(x_g^m,g^{-1}x_o)}{d(g^{-1}x_o,x)}\leq 4e^{-(2\sigma_\mu-2\epsilon)n}\leq 1.$$

Therefore,

$$\begin{aligned}&\mu^{\otimes n}\{\sigma(g,x)\in[t,t+a],\ d(g^{-1}x_o,x')\leq e^{-s}\}\leq O_{\exp}(n)+\\&\mu^{\otimes n}\{\kappa(g)\in[t-1,t+a+1]-\log d(g^{-1}x_o,x),d(g^{-1}x_o,x')\leq e^{-s}\}.\end{aligned}$$

Summing up over $I_t$ and using the definition of $\check{\mu}$, we have

$$\begin{aligned}&\sum_{n\in I_t}\mu^{\otimes n}\{\sigma(g,x)\in[t,t+a],\ d(g^{-1}x_o,x')\leq e^{-s}\}\\\leq&O_{\exp}(t)+\sum_{n\geq 0}\mu^{\otimes n}\{\kappa(g)\in[t-1,t+a+1]-\log d(g^{-1}x_o,x),d(g^{-1}x_o,x')\leq e^{-s}\}\\=&O_{\exp}(t)+\sum_{n\geq 0}\check{\mu}^{\otimes n}\{\kappa(g)\in[t-1,t+a+1]-\log d(gx_o,x),d(gx_o,x')\leq e^{-s}\}.\end{aligned} \tag{1.4.15}$$

Hence, it is sufficient to bound $R_P(\mathbb{1}_{u+\log d(y,x)\in[-1,a+1],d(y,x')\leq e^{-s}})(x_o,t)$, where we use $(y,u)$ to denote the variables, and the measure $\mu$ is replaced by $\check{\mu}$. For simplicity, we use the same notation $R_P$. Cutting the region along $\{y\in X|\log d(y,x)\leq -t_1\}$ and the subsets $\{y\in X|\log d(y,x)\in[-(k+1)s,-ks]\}$ for $0\leq k<t_1/s$, where $t_1=(t-1)/9$.

- When $k=0$, since $t-1>10s$, we can use Lemma 1.4.11 to obtain

$$\begin{aligned}&R_P(\mathbb{1}_{u+\log d(y,x)\in[-1,a+1],d(y,x')\le e^{-s},d(y,x)\ge e^{-s}})(x_o,t)\\ \le &R_P(\mathbb{1}_{d(y,x')\le e^{-s},u\in[-1,s+a+1]})(x_o,t)\lesssim (1+s+a)^2e^{-\epsilon' s}.\end{aligned}$$

- When $0<k<t_1/s$, since $t+ks-1>10ks$, again we use Lemma 1.4.11

$$\begin{aligned}&R_P(\mathbb{1}_{u+\log d(y,x)\in[-1,a+1],d(y,x')\le e^{-s},d(y,x)\in[e^{-(k+1)s},e^{-ks}]})(x_o,t)\\ \le &R_P(\mathbb{1}_{d(y,x)\le e^{-ks},u\in[-1+ks,a+1+(k+1)s]})(x_o,t)\lesssim (1+s+a)^2e^{-\epsilon' ks}.\end{aligned}$$

- In the last case, $\log d(y,x)\le -t_1$, we have

$$\begin{aligned}&R_P(\mathbb{1}_{u+\log d(y,x)\in[-1,a+1],d(y,x')\le e^{-s},d(y,x)\le e^{-t_1}})(x_o,t)\\ \le &R_P(\mathbb{1}_{u+\log d(y,x)\in[-1,a+1],d(y,x)\le e^{-t_1}})(x_o,t).\end{aligned}$$

  This is similar to the original quantity $R_P(\mathbb{1}_{u+\log d(y,x)\in[-1,a+1],d(y,x')\le e^{-s}})(x_o,t)$. The difference is that here $t_1$ is comparable with $t$, which is crucial in the following argument. Return to the definition of $R_P$, and discuss on the length $n=\omega(g)$.

  - When $n>(t+a+1)/(\sigma_\mu-2\epsilon)$, by inequality (1.2.9) and (1.2.11), we have $\check{\mu}^{\otimes n}\{g\in G^{\times n}|\kappa(g)-n\sigma_\mu\le n\epsilon,\ d(gx_o,x)\ge e^{-\epsilon n}\}>1-Ce^{-\epsilon' n}$. By hypothesis $n>(t+a+1)/(\sigma_\mu-2\epsilon)$ , the element in this set satisfies

$$\kappa(g)\ge(\sigma_\mu-\epsilon)n>t+a+1+n\epsilon\ge t+a+1-\log d(gx_o,x).$$

    Thus $\check{\mu}^{\otimes n}\{g\in G^{\times n}|\kappa(g)\in[t-1,t+a+1]-\log d(gx_o,x)\}=O_{\exp}(n)$. Summing over $n$, we see that the measure of this part is $O_{\exp}(t)$.
  - When $n\in[(t-1)/(\sigma_\mu+\epsilon),(t+a+1)/(\sigma_\mu-2\epsilon)]$, since $\epsilon n\ge\epsilon(t-1)/(\sigma_\mu+\epsilon)>(t-1)/9=t_1$, Corollary 1.2.13 implies that

$$\check{\mu}^{\otimes n}\{g\in G^{\times n}|d(gx_o,x)\le e^{-t_1}\}=O_{\exp}(t_1)=O_{\exp}(t).$$

  - When $n\le(t-1)/(\sigma_\mu+\epsilon)$, Corollary 1.2.11 implies the measure of this part is $O_{\exp}(t)$.

  Therefore we have

$$R_P(\mathbb{1}_{u+\log d(y,x)\in[-1,a+1],d(y,x)\le e^{-t_1}})(x_o,t)=O_{\exp}(t_1).$$

Combining the three cases, we have finished the proof. □

### 1.4.4 Residue process

We introduce the residue process, which not only deals with $\sigma(g_n g_{n-1}\cdots g_1, x)$ but also takes into account the next step $\sigma(g_{n+1}, g_n g_{n-1}\cdots g_1 x)$. Let $f$ be a positive bounded Borel function on $X\times\mathbb{R}^2$. For $(x,t)\in X\times\mathbb{R}$, we define the residue operator by

$$Ef(x,t)=\sum_{n\geq 0}\int f(hgx,\sigma(h,gx),\sigma(g,x)-t)\mathrm{d}\mu^{*n}(g)\mathrm{d}\mu(h). \tag{1.4.16}$$

Let $\mathcal{F}_u f(x,v,\xi)=\int f(x,v,u)e^{iu\xi}\mathrm{d}u$ be the Fourier transform on $\mathbb{R}_u$. Let $F$ be a function on $X\times\mathbb{R}_v\times\mathbb{R}_{\xi},$. Define a partial Lipschitz norm by

$$|F|_{L^\infty Lip}=\sup_{\xi\in\mathbb{R}}|F(\xi)|_{Lip}=\sup_{\xi\in\mathbb{R}}\left(|F(\xi)|_\infty+\sup_{(x,v),(x',v')\in X\times\mathbb{R}}\frac{|F(x,v,\xi)-F(x',v',\xi)|}{d(x,x')+|v-v'|}\right).$$

**Proposition 1.4.15** (Residue process)**.** *If $f$ is a positive bounded continuous function on $X\times\mathbb{R}^2$. Assume that the projection of* $\mathrm{supp}\mathcal{F}_u(f)$ *onto $\mathbb{R}_\xi$ is contained in a compact set $K$, and $|\mathcal{F}_u(f)|_{L^\infty Lip}, |\partial_\xi\mathcal{F}_u(f)|_{L^\infty Lip}$ are finite. Then for $t>0$ and $x\in X$, we have*

$$\begin{aligned}Ef(x,t)=&\frac{1}{\sigma_\mu}\int_{-t}^{\infty}\int_G\int_X f(hy,\sigma(h,y),u)\mathrm{d}\nu(y)\mathrm{d}\mu(h)\mathrm{d}u\\&+\frac{1}{t}O_K\left(|\mathcal{F}_u(f)|_{L^\infty Lip}+|\partial_\xi\mathcal{F}_u(f)|_{L^\infty Lip}\right).\end{aligned}\tag{1.4.17}$$

*Proof.* For a bounded continuous function $f$ on $X\times\mathbb{R}^2$ and $(x,u)\in X\times\mathbb{R}$, we define an operator $Q$ by

$$Qf(x,u)=\int_G f(hx,\sigma(h,x),u)\mathrm{d}\mu(h).$$

Then

$$Ef(x,t)=\sum_{n\geq 0}\int Qf(gx,\sigma(g,x)-t)\mathrm{d}\mu^{*n}(g)=R(Qf)(x,t).$$

We want to use Proposition 1.4.5, so we need to verify the hypotheses. The function $Qf$ is bounded and integrable by the hypotheses on $f$. Then

$$\begin{aligned}\widehat{Qf}(x,\xi)&=\int Qf(x,u)e^{iu\xi}\mathrm{d}u=\int f(hx,\sigma(h,x),u)e^{iu\xi}\mathrm{d}u\mathrm{d}\mu(h)\\&=\int_G\mathcal{F}_u f(hx,\sigma(h,x),\xi)\mathrm{d}\mu(h).\end{aligned}$$

Thus $\widehat{Qf}$ is also compactly supported on $\xi$. It remains to estimate the Hölder norm of $\widehat{Qf}$. Since $\mathcal{F}_u f(x,v,\xi)$ is Lipschitz on $(x,v)\in X\times\mathbb{R}$, this implies that

$$\begin{aligned}|\widehat{Qf}(x,\xi)-\widehat{Qf}(y,\xi)|&\leq\int_G|\mathcal{F}_u f(hx,\sigma(h,x),\xi)-\mathcal{F}_u f(hy,\sigma(h,y),\xi)|\mathrm{d}\mu(h)\\&\leq\int_G|\mathcal{F}_u f|_{L^\infty Lip}(d(hx,hy)+|\sigma(h,x)-\sigma(h,y)|)\mathrm{d}\mu(h).\end{aligned}$$

Using Lipschitz property of the distance and the cocycle, and finite exponential moment, we have

$$|\widehat{Qf}(x,\xi)-\widehat{Qf}(y,\xi)| \leq |\mathcal{F}_u f|_{L^\infty Lip} d(x,y)^\gamma \int_G (1+\kappa(h))\|h\|^{2\gamma}\mathrm{d}\mu(h) \lesssim |\mathcal{F}_u f|_{L^\infty Lip} d(x,y)^\gamma,$$

where we use the Remark 1.4.4 that $4\gamma \leq \epsilon_1$. Therefore

**Lemma 1.4.16** (Change of norm)**.** *Under the assumptions of Proposition 1.4.15, we have*

$$|\widehat{Qf}|_{L^\infty_\xi \mathcal{H}^\gamma} \lesssim |\mathcal{F}_u(f)|_{L^\infty Lip}, \ |\partial_\xi \widehat{Qf}|_{L^\infty_\xi \mathcal{H}^\gamma} \lesssim |\partial_\xi \mathcal{F}_u f|_{L^\infty Lip}.$$

*Proof.* The second inequality follows by the same computation. □

By Proposition 1.4.5, we have

$$\begin{aligned} R(Qf)(x,t) &= \frac{1}{\sigma_\mu}\int_X\int_{-t}^{\infty} Qf(y,u)\mathrm{d}u\mathrm{d}\nu(y) + \frac{1}{t}O_K\left(|\widehat{Qf}|_{L^\infty_\xi \mathcal{H}^\gamma} + |\partial_\xi \widehat{Qf}|_{L^\infty_\xi \mathcal{H}^\gamma}\right) \\ &= \frac{1}{\sigma_\mu}\int_X\int_{-t}^{\infty} Qf(y,u)\mathrm{d}u\mathrm{d}\nu(y) + \frac{1}{t}O_K\left(|\mathcal{F}_u(f)|_{L^\infty Lip} + |\partial_\xi \mathcal{F}_u(f)|_{L^\infty Lip}\right). \end{aligned}$$

The proof is complete. □

### 1.4.5 Residue process with cutoff

In this section, we restrict the residue process to the sequences $(g_{n+1}, g_n, \dots, g_1)$ such that $\sigma(g_n\cdots g_1, x) < t \leq \sigma(g_{n+1}\cdots g_1, x)$. Let $f$ be a function on $X\times\mathbb{R}^2$. Define a Lipschitz norm by

$$|f|_{Lip} = |f|_\infty + \sup_{(x,v,u)\neq(x',v',u')} \frac{|f(x,v,u)-f(x',v',u')|}{d(x,x')+|v-v'|+|u-u'|}. \tag{1.4.18}$$

Define an operator from bounded Borel functions on $X\times\mathbb{R}^2$ to functions on $X\times\mathbb{R}$ by

$$E_C f(x,t) = \sum_{n\geq 0}\int_{\sigma(g,x)<t\leq\sigma(hg,x)} f(hgx, \sigma(h,gx), \sigma(g,x)-t)\mathrm{d}\mu(h)\mathrm{d}\mu^{*n}(g).$$

By Lemma 1.4.21, which will be proved later, this operator is well defined. Let $K$ be a compact set in $\mathbb{R}$. We denote $|K|$ by the supremum of the distance between a point in $K$ and 0.

**Proposition 1.4.17.** *Let $f$ be a continuous function on $X\times\mathbb{R}^2$ with $|f|_{Lip}$ finite. Assume that the projection of* $\mathrm{supp} f$ *on $\mathbb{R}_v$ is contained in a compact set $K$. For all $\delta>0$, $t>|K|+\delta$ and $x\in X$, we have*

$$E_C f(x,t) = \int_X\int_G\int_{-\sigma(h,y)}^{0} f(hy, \sigma(h,y), u)\mathrm{d}u\mathrm{d}\mu(h)\mathrm{d}\nu(y) + O_K(\delta + O_\delta/t)|f|_{Lip}, \tag{1.4.19}$$

*where $O_K$ does not depend on $\delta, f, t, x$, and the integral $\int_{-\sigma(h,y)}^0 = 0$ if $\sigma(h,y)<0$.*

**Remark 1.4.18.** *We decompose $f$ into real and imaginary parts, then decompose these two parts into positive and negative parts. Each part satisfies the hypotheses of Proposition 1.4.17, with the support and the Lipschitz norm bounded by the original one. Thus, it is sufficient to prove this proposition for $f$ positive.*

The following lemma connects the operator $E_c$ with $E$.

**Lemma 1.4.19.** *Under the assumptions of Proposition 1.4.17, let $f_o(x,v,u) = \mathbb{1}_{-v\leq u<0} f(x,v,u)$. Then*

$$E_C f(x,t) = E f_o(x,t).$$

Before proving this proposition, we describe some regularity and independence properties. They are corollaries of analogous properties for the renewal process. The idea is to decompose the integral according to the last letter. The following lemma means that the residue process with cutoff has exponential decay with respect to the last jump.

**Lemma 1.4.20.** *For $t,s$ in $\mathbb{R}$ and $x$ in $X$, we have*

$$E_C(\mathbb{1}_{v\geq s})(x,t) = E(\mathbb{1}_{-v\leq u<0, v\geq s})(x,t) = O_{\exp}(s). \tag{1.4.20}$$

*Proof.* By Lemma 1.4.9 and finiteness of the exponential moment, we have

$$\begin{aligned}
&\sum_{n\geq 0} \mu\otimes\mu^{*n}\{(h,g)\in G^2 | \sigma(g,x)-t\in[-\sigma(h,gx),0], \sigma(h,gx)\geq s\} \\
&\leq \sum_{n\geq 0} \mu\otimes\mu^{*n}\{(h,g)\in G^2 | \sigma(g,x)-t\in[-\kappa(h),0], \kappa(h)\geq s\} \\
&= \int_{\kappa(h)>s} R(\mathbb{1}_{[-\kappa(h),0]})(x,t)\mathrm{d}\mu(h) \lesssim \int_{\kappa(h)>s} \max\{1,\kappa(h)\}\mathrm{d}\mu(h) = O_{\exp}(s).
\end{aligned}$$

The proof is complete. □

**Lemma 1.4.21.** *There exists $C>0$ such that for all $t\in\mathbb{R}$ and $x\in X$, we have*

$$E_C(\mathbb{1})(x,t) = E(\mathbb{1}_{-v\leq u<0})(x,t) \leq C. \tag{1.4.21}$$

This is a special case of Lemma 1.4.20. The following lemma quantifies the independence of the scalar part and the angle part. Abbreviate $1_{d(y,x')\leq e^{-s}, -v\leq u<0}(y,v,u)$ to $1_{d(y,x')\leq e^{-s}, -v\leq u<0}$, and others are similar.

**Lemma 1.4.22.** *For $t>5s>0$ and $x,x'$ in $X$, we have*

$$E_C(\mathbb{1}_{d(y,x')\leq e^{-s}})(x,t) = E(\mathbb{1}_{d(y,x')\leq e^{-s}, -v\leq u<0})(x,t) = O_{\exp}(s). \tag{1.4.22}$$

*Proof.* Since

$$\mathbb{1}_{-v\leq u<0} \leq \mathbb{1}_{d(y,x')\leq e^{-s}, -v\leq u<0, v\geq s} + \mathbb{1}_{d(y,x')\leq e^{-s}, 0\leq u+v<s},$$

we have

$$E(\mathbb{1}_{d(y,x')\leq e^{-s},-v\leq u<0})(x,t)\leq E(\mathbb{1}_{-v\leq u<0,v\geq s})(x,t)+E(\mathbb{1}_{d(y,x')\leq e^{-s},0\leq u+v<s})(x,t).$$

By definition, we have

$$\begin{aligned}E(\mathbb{1}_{d(y,x')\leq e^{-s},0\leq u+v<s})(x,t)&=\sum_{n\geq 0}\int \mathbb{1}_{d(hgx,x')\leq e^{-s},\sigma(h,gx)+\sigma(g,x)-t\in[0,s]}\mathrm{d}\mu^{*n}(g)\mathrm{d}\mu(h)\\&=\sum_{n\geq 1}\int \mathbb{1}_{d(gx,x')\leq e^{-s},\sigma(g,x)-t\in[0,s]}\mathrm{d}\mu^{*n}(g)=R(\mathbb{1}_{B(x',e^{-s}),[0,s]})(x,t).\end{aligned}$$

By Lemma 1.4.20 and Proposition 1.4.10, the result follows. □

**Lemma 1.4.23.** *For $s>0$, $t>\max\{10s,10\}$ and $x,x_o,x'\in X$, we have*

$$\sum_{n\geq 0}\mu\otimes\mu^{*n}\{(h,g)\in G\times G|\sigma(hg,x)\geq t,\sigma(g,x)<t,d((hg)^{-1}x_o,x')\leq e^{-s}\}=O_{\exp}(s).$$

By the same argument as in the proof of Lemma 1.4.22, we only need to replace Proposition 1.4.10 by Proposition 1.4.14. The difference between this lemma and Lemma 1.4.22 is the angle part $(hg)^{-1}x$.

Using $\psi_\delta$ to regularize these functions, we write $f_\delta(x,v,u)=\int f_o(x,v,u-u_1)\psi_\delta(u_1)\mathrm{d}u_1=\psi_\delta * f_o(x,v,u)$.

**Lemma 1.4.24.** *Under the same hypotheses as in Proposition 1.4.17, we have*

$$E(f_\delta)(x,t)=\int_X\int_G\int_{-\sigma(h,y)}^0 f(hy,\sigma(h,y),u)\mathrm{d}u\mathrm{d}\mu(g)\mathrm{d}\nu(y)+O_K(\delta+\frac{O_\delta}{t})|f|_{Lip}.$$

*Proof.* We want to verify the conditions in Proposition 1.4.15 and then use this proposition. The integrable condition is valid because $|\int_{\mathbb{R}_u}f_\delta|=|\int_{\mathbb{R}_u}f_o(x,v,u)\mathrm{d}u|=|\int_{-v}^0 f(x,v,u)\mathrm{d}u|\leq |K||f|_\infty$. For the Fourier transform, we have

$$\mathcal{F}_u f_\delta=\mathcal{F}_u(\psi_\delta * f_o)=\hat{\psi}_\delta\mathcal{F}_u f_o.$$

We need to estimate the Lipschitz norm of $\mathcal{F}_u f_o$. This function equals

$$\int f_o(x,v,u)e^{i\xi u}\mathrm{d}u=\int_{-v}^0 f(x,v,u)e^{i\xi u}\mathrm{d}u.$$

Taking $(x,v)\neq(x',v')$, we have

$$\begin{aligned}&|\int_{-v}^0 f(x,v,u)e^{i\xi u}\mathrm{d}u-\int_{-v'}^0 f(x',v',u)e^{i\xi u}\mathrm{d}u|\\ \leq&|\int_{-v}^0(f(x,v,u)-f(x',v',u))e^{i\xi u}\mathrm{d}u|+|v'-v||f|_\infty\lesssim |K||f|_{Lip}(d(x,x')+|v-v'|).\end{aligned}$$

Then we have

**Lemma 1.4.25** (Change of norm)**.** *Under the same hypotheses as in Proposition 1.4.17, we have*

$$|\mathcal{F}_u f_\delta|_{L^\infty Lip} \leq |K||f|_{Lip}, \ |\partial_\xi \mathcal{F} f_\delta|_{L^\infty Lip} \leq |K|^2 |f|_{Lip}.$$

*Proof.* Noting that in the integration $|u| \leq |v|$, we get the second inequality by the same computation. □

Therefore by Proposition 1.4.15, we have

$$E(f_\delta)(x,t) = \frac{1}{\sigma_\mu}\int_{-t}^{\infty}\int_G\int_X f_\delta(hy,\sigma(h,y),u)\mathrm{d}\nu(y)\mathrm{d}\mu(h)\mathrm{d}u + \frac{O_\delta}{t}\left(|f|_{Lip}(|K|+|K|^2)\right).$$

Then

$$\begin{aligned}\int_{-t}^{\infty} f_\delta(x,v,u)\mathrm{d}u &= \int_{-t}^{\infty}\int_{-v}^{0} f(x,v,u_1)\psi_\delta(u-u_1)\mathrm{d}u_1\mathrm{d}u = \int_{-v}^{0} f(x,v,u_1)\int_{-t}^{\infty}\psi_\delta(u-u_1)\mathrm{d}u\mathrm{d}u_1\\ &= \int_{-v}^{0} f(x,v,u_1)\mathrm{d}u_1 - \int_{-v}^{0} f(x,v,u_1)\int_{-\infty}^{-t-u_1}\psi_\delta(u)\mathrm{d}u\mathrm{d}u_1.\end{aligned}$$

Since $t-\delta \geq |K|$, we have $-t-u_1 \leq -t+v \leq -\delta$. By $\int_{-\infty}^{-\delta}\psi_\delta \leq C_\psi \delta$, this implies that $\int_{-t}^{\infty} f_\delta(x,v,u)\mathrm{d}u = \int_{-v}^{0} f_\delta(x,v,u)\mathrm{d}u(1+O(\delta))$. Using Lemma 1.4.21, we have

$$|\int_X\int_G\int_{-\sigma(h,y)}^{0} f(hy,\sigma(h,y),u)\mathrm{d}u\mathrm{d}\mu(g)\mathrm{d}\nu(y)| \leq |f|_\infty E_C(\mathbb{1}) = O(|f|_\infty).$$

Therefore

$$\begin{aligned}&\int_{-t}^{\infty}\int_G\int_X f_\delta(hy,\sigma(h,y),u)\mathrm{d}\nu(y)\mathrm{d}\mu(h)\mathrm{d}u\\ &\qquad = \int_X\int_G\int_{-\sigma(h,y)}^{0} f(hy,\sigma(h,y),u)\mathrm{d}u\mathrm{d}\mu(g)\mathrm{d}\nu(y) + O(\delta|f|_\infty).\end{aligned}$$

The proof is complete. □

Next lemma gives the difference between a function and its regularization.

**Lemma 1.4.26.** *Let $\varphi_0(u) = \mathbb{1}_{[b_1,b_2]}(u)\varphi(u)$, where $b_2 > b_1$ and $|\varphi'|_{L^\infty} < \infty$, $|\varphi|_{L^\infty} \leq 1$. Then we have*

$$|\psi_\delta * \varphi_0(u) - \varphi_0(u)| \leq \begin{cases}(|\varphi'|_\infty + 2)\delta & u \in [b_1+\delta, b_2-\delta],\\ 2 & u \in [b_1-\delta, b_1+\delta]\cup[b_2-\delta,b_2+\delta],\\ \psi_\delta * \mathbb{1}_{[b_1,b_2]}(u) & u \in [b_1-\delta, b_2+\delta]^c.\end{cases} \tag{1.4.23}$$

*Proof.* We will prove this inequality in each interval.

- When $u$ is in $[b_1+\delta, b_2-\delta]$, we have

$$|(\psi_\delta * \varphi_0 - \varphi_0)(u)| = \left|\int \psi_\delta(t)(\varphi_0(u-t)-\varphi_0(u))\mathrm{d}t\right| \leq \int_{-\delta}^{\delta} \psi_\delta(t)|\varphi_0(u-t)-\varphi_0(u)|\mathrm{d}t + 2\delta.$$

  When $|t|\leq\delta$, we have $u-t\in[b_1,b_2]$. Since $|\varphi_0'(u)|\leq|\varphi'|_\infty$ for $u\in[b_1,b_2]$, this implies that

$$\int_{-\delta}^{\delta} \psi_\delta(t)|\varphi_0(u-t)-\varphi_0(u)|\mathrm{d}t \leq \int_{-\delta}^{\delta}\psi_\delta(t)|t||\varphi'|_\infty\mathrm{d}t \leq \delta|\varphi|_\infty.$$

- When $u\in[b_1-\delta,b_1+\delta]\cup[b_2-\delta,b_2+\delta]$, we use the trivial bound $|\psi_\delta*\varphi_0(u)-\varphi_0(u)|\leq 2$.

- When $u\in(-\infty,b_1-\delta]\cup[b_2+\delta,\infty]$, we have $\varphi_0(u)=0$, then $|\psi_\delta*\varphi_0|\leq|\psi_\delta*\mathbb{1}_{[b_1,b_2]}|$.

Thus collecting all together, we get the inequality. □

*Proof of Proposition 1.4.17.* To simplifier the notation, we normalize $f$ in such a way that $|f|_\infty=1$. By Lemma 1.4.24, we only need to give an estimate of $E(|f_\delta-f_o|)(x,t)$.

Since $f_o(x,v,u)=\mathbb{1}_{-v\leq u<0}(u)f(x,v,u)$ with $(x,v)$ fixed, Lemma 1.4.26 implies that

$$|f_\delta-f_o|(u)\leq\begin{cases}(|\partial_u f|_\infty+2)\delta & u\in[-v+\delta,-\delta],\\ 2 & u\in[-v-\delta,-v+\delta]\cup[-\delta,\delta],\\ \psi_\delta*\mathbb{1}_{[-v,0]}(u) & u\in[-v-\delta,\delta]^c.\end{cases}$$

By definition of $|K|$, the first term is less than $(|\partial_u f|_\infty+2)\delta\mathbb{1}_{[-|K|+\delta,-\delta]}$. The third term equals

$$\begin{aligned}\mathbb{1}_{[-\infty,-v-\delta]\cup[\delta,\infty]}\psi_\delta*\mathbb{1}_{[-v,0]}(u) &= \mathbb{1}_{[-\infty,-v-\delta]\cup[\delta,\infty]}(u)\int_{-v}^{0}\psi_\delta(u-u_1)\mathrm{d}u_1\\ &= \mathbb{1}_{[-\infty,-v-\delta]\cup[\delta,\infty]}(u)\int_{u}^{u+v}\psi_\delta(u_1)\mathrm{d}u_1.\end{aligned}$$

By definition and the above arguments, we have

$$\begin{aligned}E(|f_\delta-f_o|)(x,t) &= \sum_{n\geq0}\int|f_\delta-f_o|(hgx,\sigma(h,gx),\sigma(g,x)-t)\mathrm{d}\mu^{*n}(g)\mathrm{d}\mu(h)\\ &\leq\sum_{n\geq0}\int\Big((|\partial_u f|_\infty+2)\delta\mathbb{1}_{[-|K|,-\delta]}(\sigma(g,x)-t)+\\ &+2\mathbb{1}_{[-\sigma(h,gx)-\delta,-\sigma(h,gx)+\delta]\cup[-\delta,\delta]}(\sigma(g,x)-t)\\ &+\mathbb{1}_{[-\infty,-\sigma(h,gx)-\delta]\cup[\delta,\infty]}(\sigma(g,x)-t)\int_{\sigma(g,x)-t}^{\sigma(hg,x)-t}\psi_\delta(u_1)\mathrm{d}u_1\Big)\mathrm{d}\mu^{*n}(g)\mathrm{d}\mu(h).\end{aligned}$$

By Lemma 1.4.9, the first term is controlled by $(|\partial_u f|_\infty+2)\delta|K|$. The second term is less than $R(\mathbb{1}_{[-\delta,\delta]})(x,t)$. Due to Proposition 1.4.8, it is controlled by $6\delta(1/\sigma_\mu+C_\delta(1+2\delta)/t)$.

For the third term, we need to change the order of integration. Since $\sigma(g,x)-t>\delta$ or $\sigma(g,x)-t<-\sigma(h,gx)-\delta$, we have $u_1\geq\sigma(g,x)-t>\delta$ or $u_1\leq\sigma(hg,x)-t=\sigma(h,gx)+\sigma(g,x)-t\leq-\delta$. We integrate first with respect to $u_1$, then the third term is less than

$$\int_{[-\infty,-\delta]\cup[\delta,\infty]}\psi_\delta(u_1)\sum_{n\geq 0}\mu\otimes\mu^{*n}\{(h,g)|\sigma(hg,x)\geq u_1+t,\sigma(g,x)\leq u_1+t\}\mathrm{d}u_1.$$

By Lemma 1.4.21, the above quantity is less than $C\int_{[-\infty,-\delta]\cup[\delta,\infty]}\psi_\delta(u_1)\mathrm{d}u_1\lesssim\delta$.

Therefore, we have

$$E(|f_\delta-f|)(x,t)=O_K(\delta+C_\delta/t)|f|_{Lip}.$$

The proof is complete. □

**Remark 1.4.27** (Minus case)**.** *The lemmas in this part concern plus and minus. The another version we need is for $E_C^-(f)(x,t)=E(\mathbb{1}_{0<u\leq -v}f)(x,t)$, the proofs are exactly the same.*

**Proposition* 1.4.17.** *Under the assumptions of Proposition 1.4.17, we have*

$$E_C^-(f)(x,t)=\int_X\int_G\int_0^{-\sigma(h,y)}f(hy,\sigma(h,y),u)\mathrm{d}u\mathrm{d}\mu(h)\mathrm{d}\nu(y)+O_K(\delta+O_\delta/t)|f|_{Lip}.$$

### 1.4.6 Residue process for the Cartan Projection

We consider the residue process for the cutoff of a function $f$ on $X^2\times\mathbb{R}^2$, where the cocycle is replaced by the Cartan projection. We will give a limit not only with $gx$, but also with $g^{-1}x'$.

As in the previous subsection, we can define a similar Lipschitz norm on the space of Lipschitz functions on $X^2\times\mathbb{R}^2$, using the same name $|f|_{Lip}$. Define the operator from bounded Borel functions on $X^2\times\mathbb{R}^2$ to functions on $X^2\times\mathbb{R}$ by

$$E_Pf(x',x,t)=\sum_{n\geq 0}\int_{\kappa(g)<t\leq\kappa(hg)}f((hg)^{-1}x',hgx,\kappa(hg)-\kappa(g),\kappa(g)-t)\mathrm{d}\mu(h)\mathrm{d}\mu^{*n}(g).$$

**Proposition 1.4.28.** *Let $f$ be a continuous function on $X^2\times\mathbb{R}^2$ with $|f|_{Lip}$ finite. Assume that the projection of* $\mathrm{supp}f$ *on $\mathbb{R}_v$ is contained in a compact set $K$. For all $\delta>0$, $t>\max\{2(|K|+\delta),20\}$ and $x',x$ in $X$, we have*

$$E_Pf(x',x,t)=\int_{X^2}\int_G\int_{-\sigma(h,y)}^0 f(y',hy,\sigma(h,y),u)\mathrm{d}u\mathrm{d}\mu(h)\mathrm{d}\nu(y)\mathrm{d}\check\nu(y')+O_K(\delta+O_\delta/t)|f|_{Lip},\tag{1.4.24}$$

*where $O_K$ does not depend on $\delta,f,t,x,x'$, the integral $\int_{-\sigma(h,y)}^0=0$ if $\sigma(h,y)<0$.*

*Proof.* We introduce local notations here: for an element $g$ in $G$ and a continuous function $f$ on $X^2 \times \mathbb{R}^2$, define $gf(x', x, v, u) = f(g^{-1}x', x, v, u)$. Let $f_{x'}(x, v, u) = f(x', x, v, u)$, which emphasizes that the first coordinate is fixed. Let $l = [\epsilon_5 t/\sigma_\mu]$, where $\epsilon_5 < 1/10$. We use the decomposition

$$h = g_{n+1}, g = (g_n, \dots, g_{l+1}), j = (g_l, \dots, g_1).$$

Recall that $N_t^+ = \bigcup_{n\geq 0}\{(g_{n+1}, g_n, \dots, g_1)|\kappa(g_{n+1}\cdots g_1) \geq t > \kappa(g_n\cdots g_1)\}$. Let $N_t^+(n) = N_t^+ \cap G^{\times(n+1)} = \{(g_{n+1}, \dots, g_1)|\kappa(gj) \leq t, \kappa(hgj) > t\}$. Let

$$T_n(x, t) = \{(g_{n+1}, \dots, g_1) \in G^{\times(n+1)}|\sigma(hg, jx) > t - \kappa(j), \sigma(g, jx) \leq t - \kappa(j)\},$$

and let $G_{\epsilon,l} = \{(g_l, \dots, g_1)||\kappa(j) - l\sigma_\mu| \leq l\epsilon, d(x^M_{g_l\cdots g_1}, x') \geq e^{-\epsilon l}\}$, as well as

$$T_{n,\epsilon} = \{(g_{n+1}, \dots, g_1) \in T_n|(g_l, \dots, g_1) \in G_{\epsilon,l}\}.$$

**Step 1:** Due to Corollary 1.2.9 and Corollary 1.2.11, the sum of the integrals $\int_{N_t^+(n)}$ for $n$ ranging from $t/(\sigma_\mu + \epsilon) - 1$ to $t/(\sigma_\mu - \epsilon)$ is exponentially small in $t$. In other words, we have

$$\begin{aligned}|\sum_{n=[t/(\sigma_\mu+\epsilon)]}^{[t/(\sigma_\mu-\epsilon)]} \int_{N_t^+(n)} f((hgj)^{-1}x', hgjx, \kappa(hgj) - \kappa(gj), \kappa(gj) - t)\mathrm{d}\mu^{\otimes(n+1)}&\\ -E_P f(x', x, t)| = O_{\exp}(t)|f|_\infty&\end{aligned} \tag{1.4.25}$$

The following lemma replaces the Cartan projection with the cocycle.

**Lemma 1.4.29.** *Under the same assumption as in Proposition 1.4.28, we have*

$$\begin{aligned}&|\sum_{n=[t/(\sigma_\mu+\epsilon)]}^{[t/(\sigma_\mu-\epsilon)]} \int_{N_t^+(n)} f((hgj)^{-1}x', hgjx, \kappa(hgj) - \kappa(gj), \kappa(gj) - t)\mathrm{d}\mu^{\otimes(n+1)}(hgj)\\ &- \sum_{n=[t/(\sigma_\mu+\epsilon)]}^{[t/(\sigma_\mu-\epsilon)]} \int_{T_{n,\epsilon}} jf((hg)^{-1}x', hgjx, \sigma(h, gjx), \sigma(g, jx) - (t - \kappa(j)))\mathrm{d}\mu^{\otimes(n+1)}(hgj)|\\ &= O(\delta + O_\delta/t)|f|_{Lip}.\end{aligned} \tag{1.4.26}$$

This lemma will be proved later. We will decompose $T_{n,\epsilon}(x, t)$ to apply the residue process for the cocycle. The space $T_{n,\epsilon}(x, t)$ can be seen as a fibered space over $G_{\epsilon,l}$. When the first $l$ elements are fixed, the elements $(h, g)$ such that $hgj = (g_{n+1}, \dots, g_1) \in T_{n,\epsilon}(x, t)$, are the admitted elements in the residue process with cutoff, whose start point is $jx$ and time is $t - \kappa(j)$. Since $(n-l)(\sigma_\mu + \epsilon) \leq t - \kappa(j)$ and $(n-l)(\sigma_\mu - \epsilon) \geq t - \kappa(j)$, we can apply Principle 1 to this residue process. Integrating over $G_{\epsilon,l}$ implies that

$$|\sum_{n=[t/(\sigma_\mu+\epsilon)]}^{[t/(\sigma_\mu-\epsilon)]}\int_{T_{n,\epsilon}} jf((hg)^{-1}x', hgjx, \sigma(h,gjx), \sigma(g,jx)-(t-\kappa(j)))\mathrm{d}\mu^{\otimes(n+1)}(hgj)$$
$$-\int_{G_{\epsilon,l}} E_I jf(x', jx, t-\kappa(j))\mathrm{d}\mu^{\otimes l}(j)| = O_{\exp}(t)|f|_\infty. \tag{1.4.27}$$

where

$$E_I f(x',x,t) = \sum_{n\geq 0}\int_{\sigma(g,x)<t\leq\sigma(hg,x)} f((hg)^{-1}x', hgx, \sigma(h,gx), \sigma(g,x)-t)\mathrm{d}\mu(h)\mathrm{d}\mu^{*n}(g).$$

The following inequality, whose proof relies on Lemma 1.4.23, will give a major term.

**Lemma 1.4.30.** *Under the same assumption as in Proposition 1.4.28, for all $j\in G_{\epsilon,l}$, we have*

$$|E_C f_{j^{-1}x'}(jx, t-\kappa(j)) - E_I jf(x', jx, t-\kappa(j))| \leq |f|_{Lip} O_{\exp}(l), \tag{1.4.28}$$

This lemma will be proved later. Integrating (1.4.28) over $G_{\epsilon,l}$, we obtain

$$|\int_{G_{\epsilon,l}} E_C f_{j^{-1}x'}(jx, t-\kappa(j)) - E_I jf(x', jx, t-\kappa(j))\mathrm{d}\mu^{\otimes l}(j)| \leq |f|_{Lip} O_{\exp}(t). \tag{1.4.29}$$

By (1.4.25)(1.4.26)(1.4.27), it suffices to compute the major term

$$\int_{G_{\epsilon,l}} E_C f_{j^{-1}x'}(jx, t-\kappa(j))\mathrm{d}\mu^{\otimes}(j).$$

**Step 2:** Recall that $N_0, P(0)$ are the two operators defined by $N_0\varphi = \int\varphi\mathrm{d}\nu$, $P(0)\varphi(x) = \int\varphi(gx)\mathrm{d}\nu(g)$, where $\varphi$ is a function in $\mathcal{H}^\gamma(X)$. We have another property of transfer operators [BQ16, Lemma 11.18]: The spectral radius of $P=P(0)$ restricted to $\ker N_0$ is less than 1, which means that there exist $\rho<1, C>0$ such that for every function $\varphi$ in $\mathcal{H}^\gamma(X)$, we have

$$|P^n\varphi - \int\varphi\mathrm{d}\nu|_\infty \leq C\rho^n|\varphi|_\gamma.$$

Thus by $\mu^{\otimes l}G_{\epsilon,l} = O_{\exp}(l)$, we have

$$|\int_{G_{\epsilon,l}}\varphi(j^{-1}x)\mathrm{d}\mu^{\otimes l} - \int\varphi\mathrm{d}\check{\nu}| = |\int_{G^{\times l}}\varphi(j^{-1}x)\mathrm{d}\mu^{\otimes l}(j) - \int\varphi\mathrm{d}\check{\nu}| + O_{\exp}(l)|\varphi|_\infty = O_{\exp}(l)|\varphi|_{Lip}. \tag{1.4.30}$$

By the definition of $|\cdot|_{Lip}$ on $X\times\mathbb{R}^2$, the function $f_{j^{-1}x'}(x,v,u)$ has a finite $|\cdot|_{Lip}$ value. Together with $t-\kappa(j)\geq t/2\geq |K|+\delta$, Proposition 1.4.17 implies that

$$\begin{aligned}
&\int_{G_{\epsilon,l}} E_C f_{j^{-1}x'}(jx,t-\kappa(j))\mathrm{d}\mu^{\otimes}(j)\\
&=\int_{G_{\epsilon,l}}\left(\int_X\int_G\int_{-\sigma(h,y)}^0 f_{j^{-1}x'}(y,\sigma(h,y),u)\mathrm{d}u\mathrm{d}\mu(h)\mathrm{d}\nu(y)\mathrm{d}\mu^{\otimes l}(j)+O_K(\delta+O_\delta/t)|f_{j^{-1}x'}|_{Lip}\right)\\
&=\int_X\int_G\int_{-\sigma(h,y)}^0\int_{G_{\epsilon,l}} f(j^{-1}x',y,\sigma(h,y),u)\mathrm{d}\mu^{\otimes l}(j)\mathrm{d}u\mathrm{d}\mu(h)\mathrm{d}\nu(y)+O_K(\delta+O_\delta/t)|f|_{Lip}).
\end{aligned}\tag{1.4.31}$$

With $(x,v,u)$ fixed, $f(x',x,v,u)$ is a Lipschitz function on $x'$, so it is a Hölder function. Together with Lemma 1.4.21 and inequality (1.4.30), we have

$$\begin{aligned}
&\int_{G_{\epsilon,l}} E_C f_{j^{-1}x'}(jx,t-\kappa(j))\mathrm{d}\mu^{\otimes}(j)\\
&=\int_X\int_G\int_{-\sigma(h,y)}^0\int_X f(u,\sigma(h,y),y,y')\mathrm{d}\check{\nu}(y')\mathrm{d}u\mathrm{d}\mu(h)\mathrm{d}\nu(y)+(O_{\exp}(l)+O_K(\delta+O_\delta/t))|f|_{Lip}.
\end{aligned}\tag{1.4.32}$$

The result follows. □

It remains to prove Lemma 1.4.29 and Lemma 1.4.30.

*Proof of Lemma 1.4.29.* There exist $S_{n+1,l,x}\subset G^{\times(n+1)}$ and $S_{n,l,x}\subset G^{\times n}$ which satisfy the conditions in Lemma 1.2.14. Let $S_n(x)=S_{n+1,l,x}\cap(G\times S_{n,l,x})$. Then

$$\mu^{\otimes(n+1)}S_n(x)^c=O_{\exp}(l),\tag{1.4.33}$$

and for $(g_{n+1},\dots,g_1)$ in $S_n(x)$, we have

$$\begin{aligned}
&|\kappa(hgj)-\sigma(hg,jx)-\kappa(j)|\leq e^{-\epsilon l}\\
&|\kappa(gj)-\sigma(g,jx)-\kappa(j)|\leq e^{-\epsilon l}.
\end{aligned}$$

In $N_t^+(n)\cap S_n(x)\cap T_n(x,t)$, we can replace the Cartan projection by the cocycle with exponentially small error. Fortunately, the difference of this set with $N_t^+(n)$ and $T_n(x,t)$ has exponentially small measure. By definition, we have

$$N_t^+(n)\cap S_n(x)\subset\{\sigma(hg,jx)>t-e^{-l\epsilon}-\kappa(j),\sigma(g,jx)\leq t+e^{-l\epsilon}-\kappa(j)\},$$

and

$$N_t^+(n)\supset\{\sigma(hg,jx)>t+e^{-l\epsilon}-\kappa(j),\sigma(g,jx)\leq t-e^{-l\epsilon}-\kappa(j)\}\cap S_n(x).$$

Therefore

$$\begin{aligned}(N_t^+(n)\cap S_n(x)-T_n(x,t))\subset&\{\sigma(hg,jx)\in[-e^{-\epsilon l},0]+t-\kappa(j)\}\\&\cup\{\sigma(g,jx)\in[0,e^{-\epsilon l}]+t-\kappa(j)\},\end{aligned}$$

and

$$\begin{aligned}(T_n(x,t)\cap S_n(x)-N_t^+(n))\subset&\{\sigma(hg,jx)\in[0,e^{-\epsilon l}]+t-\kappa(j)\}\\&\cup\{\sigma(g,jx)\in[-e^{-\epsilon l},0]+t-\kappa(j)\}.\end{aligned}$$

Hence, these imply that

$$\begin{aligned}&\mu^{\otimes(n+1)}(N_t^+(n)-N_t^+(n)\cap S_n(x)\cap T_n(x,t))\le\mu^{\otimes(n+1)}S_n(x)^c\\&\qquad+\mu^{\otimes(n+1)}(N_t^+(n)\cap S_n(x)-T_n(x,t))\\&\le O_{\exp}(l)+\mu^{\otimes(n+1)}\{\sigma(hg,jx)\in[-e^{-\epsilon l},0]+t-\kappa(j)\}\cup\{\sigma(g,jx)\in[0,e^{-\epsilon l}]+t-\kappa(j)\}.\end{aligned}$$

and

$$\begin{aligned}&\mu^{\otimes(n+1)}(T_n(x,t)-N_t^+(n)\cap S_n(x)\cap T_n(x,t))\le\mu^{\otimes(n+1)}S_n(x)^c\\&\qquad+\mu^{\otimes(n+1)}(T_n(x,t)\cap S_n(x)-N_t^+(n))\\&\le O_{\exp}(l)+\mu^{\otimes(n+1)}\{\sigma(hg,jx)\in[0,e^{-\epsilon l}]+t-\kappa(j)\}\cup\{\sigma(g,jx)\in[-e^{-\epsilon l},0]+t-\kappa(j)\}.\end{aligned}$$

Moreover, for $(g_{n+1},\dots,g_1)$ in the set $N_t^+(n)\cap S_n(x)\cap T_n(x,t)$, the definition of $S_n(x)$ implies that

$$\begin{aligned}&|f((hgj)^{-1}x',hgjx,\kappa(hgj)-\kappa(gj),\kappa(gj)-t)\\&\qquad-jf((hg)^{-1}x',hgjx,\sigma(h,gjx),\sigma(g,jx)-(t-\kappa(j)))|\le e^{-\gamma l\epsilon}|f|_{Lip}.\end{aligned}\tag{1.4.34}$$

Thus, for $n\in[t/(\sigma_\mu+\epsilon)-1,t/(\sigma_\mu-\epsilon)]$, we have

$$\begin{aligned}&|\int_{N_t^+(n)}f((hgj)^{-1}x',hgjx,\kappa(hgj)-\kappa(gj),\kappa(gj)-t)\mathrm{d}\mu^{\otimes(n+1)}\\&-\int_{T_n(x,t)}jf((hg)^{-1}x',hgjx,\sigma(h,gjx),\sigma(g,jx)-(t-\kappa(j)))\mathrm{d}\mu^{\otimes(n+1)}|\\&\le\mu^{\otimes}(N_t^+(n)-N_t^+(n)\cap S_n(x)\cap T_n(x,t))\cup(T_n(x,t)-N_t^+(n)\cap S_n(x)\cap T_n(x,t))\\&\qquad+\mu^{\times(n+1)}N_t^+(n)\cap S_n(x)\cap T_n(x,t)O_{\exp}(l)|f|_{Lip}\\&\le(O_{\exp}(l)+\mu^{\otimes(n+1)}\{|\sigma(hg,jx)-t+\kappa(j)|,|\sigma(g,jx)-t+\kappa(j)|\le e^{-l\epsilon}\})|f|_\infty+O_{\exp}(l)|f|_{Lip}.\end{aligned}$$

Sum up over all $n \in [t/(\sigma_\mu+\epsilon)-1, t/(\sigma_\mu-\epsilon)]$. Then the above inequality becomes

$$\begin{aligned}&|\sum_{n=[t/(\sigma_\mu+\epsilon)]}^{[t/(\sigma_\mu-\epsilon)]}\int_{N_t^+(n)} f((hgj)^{-1}x', hgjx, \kappa(hgj)-\kappa(gj), \kappa(gj)-t)\mathrm{d}\mu^{\otimes(n+1)}(hgj)| \\ &-\sum_{n=[t/(\sigma_\mu+\epsilon)]}^{[t/(\sigma_\mu-\epsilon)]}\int_{T_n} jf((hg)^{-1}x', hgjx, \sigma(h,gjx), \sigma(g,jx)-(t-\kappa(j)))\mathrm{d}\mu^{\otimes(n+1)}(hgj) \\ &\leq tO_{\exp}(l)|f|_{Lip} + |f|_\infty \int_{G^{\times l}} 2R(\mathbb{1}_{[-e^{-\epsilon l}, e^{-\epsilon l}]})(jx, t-\kappa(j))\mathrm{d}\mu^{\otimes l}(j).\end{aligned} \tag{1.4.35}$$

By (1.2.9), (1.2.12), we have $\mu^{\otimes l}G_{\epsilon,l} \geq 1 - O_{\exp}(l)$. Thus combined with Lemma 1.4.21, we get

$$\sum_{n\geq l}\mu^{\otimes(n+1)}(T_n(x,t)-T_{n,\epsilon}(x,t)) = \int_{G^c_{\epsilon,l}} E\mathbb{1}(jx, t-\kappa(j))\mathrm{d}\mu^{\otimes l}(j) = O_{\exp}(l).$$

This enables us to replace the integration domain $T_n$ by $T_{n,\epsilon}$ with exponentially small error. It is sufficient to control the right hand side of (1.4.35).

The last term can be bounded by the similar argument as in (1.4.12), with Proposition 1.4.10 replaced by inequality (1.4.6). It follows that

$$\int_{G^{\times l}} 2R(\mathbb{1}_{[-e^{-\epsilon l}, e^{-\epsilon l}]})(jx, t-\kappa(j))\mathrm{d}\mu^{\otimes l}(j) = O_{\exp}(l) + \delta O(1 + O_\delta/t). \tag{1.4.36}$$

The proof is complete. □

*Proof of Lemma 1.4.30.* We want to replace $(hgj)^{-1}x'$ with $(j)^{-1}x'$ in the first coordinate in order to find the residue process with cutoff. The idea is always similar. We have a good approximation in a large set, whose complement has exponentially small measure. Let

$$\Sigma_l = \bigcup_{n\geq 0}\{(h,g)\in G\times G^{\times n} | \sigma(g,jx) < t-\kappa(j) \leq \sigma(hg,jx), d((hg)^{-1}x', x_j^M) \leq e^{-\epsilon l}\}.$$

Since $t-\kappa(j) \geq t-(\sigma_\mu+\epsilon)l \geq 10\epsilon l$ and $t-\kappa(j) \geq t/2 > 10$, we can use Lemma 1.4.23 with $s=\epsilon l$ and $jx, x', x_j^M$ to obtain

$$\mu\otimes\bar{\mu}\Sigma_l = O_{\exp}(l). \tag{1.4.37}$$

The definition of $G_{\epsilon,l}$ implies that $d(x_j^M, x') \geq e^{-\epsilon l}$ and $\kappa(j) \geq (\sigma_\mu-\epsilon)l$. It follows from (1.2.4) that $x_j^M = x_{j^{-1}}^m$. Together with (1.2.1),(1.2.5), for $(h,g)$ outside of the set $\Sigma_l$, we have

$$\begin{aligned}&d((hgj)^{-1}x', j^{-1}x') = d(j^{-1}(hg)^{-1}x', j^{-1}x') \\ &\leq \exp(-2\kappa(j^{-1}) - \log d(x_{j^{-1}}^m, x') - \log d(x_{j^{-1}}^m, (hg)^{-1}x')) \leq \exp(-2(\sigma_\mu-\epsilon)l + 2\epsilon l).\end{aligned}$$

Therefore

$$|f(j^{-1}x', x, v, u) - f((hgj)^{-1}x', x, v, u)| \le |f|_{Lip} d(j^{-1}x', (hgj)^{-1}x') = |f|_{Lip} O_{\exp}(l). \tag{1.4.38}$$

In the bad part $\Sigma_l$, we use inequality (1.4.37) to control. Outside of $\Sigma_l$, we apply inequality (1.4.38). Thus we have

$$\begin{aligned} &|\sum_{n\ge 0} \int_{\sigma(hg,jx)>t-\kappa(j)\ge\sigma(g,jx)} f(j^{-1}x', hgjx, \sigma(h, gjx), \sigma(g, jx) + \kappa(j) - t) \\ &\quad - f((hgj)^{-1}x', hgjx, \sigma(h, gjx), \sigma(g, jx) + \kappa(j) - t) \mathrm{d}\mu(h) \mathrm{d}\mu^{*n}(g)| \\ &\quad \le |f|_{Lip}(O_{\exp}(l) + O_{\exp}(l) E_C \mathbb{1}(jx, t - \kappa(j))). \end{aligned}$$

Then by Lemma 1.4.21, the proof is complete. □

**Remark 1.4.31** (Minus case). *Let*

$$E_P^- f(x', x, t) = \sum_{n\ge 0} \int_{\kappa(g)\ge t>\kappa(hg)} f((hg)^{-1}x', hgx, \kappa(hg) - \kappa(g), \kappa(g) - t) \mathrm{d}\mu(h) \mathrm{d}\mu^{*n}(g).$$

*Then by the same proof, we have*

**Proposition* 1.4.28.** *Under the assumptions of Proposition 1.4.28, we have*

$$E_P^- f(x', x, t) = \int_{X^2} \int_G \int_0^{-\sigma(h,y)} f(y', hy, \sigma(h, y), u) \mathrm{d}u \mathrm{d}\mu(h) \mathrm{d}\nu(y) \mathrm{d}\check{\nu}(y') + O_K(\delta + O_\delta/t)|f|_{Lip}.$$

## 1.5 Main Approximation

In this section, we want to complete the proof in Section 1.3. It remains to prove Proposition 1.3.6 and the following Lemma 1.5.2 and Corollary 1.5.5.

Recall the definitions in Section 1.3: Let $\mu$ be a Borel probability measure on $\mathrm{SL}_2(\mathbb{R})$ with a finite exponential moment, and assume that the subgroup $\Gamma_\mu$ is Zariski dense. Let $\Sigma = \bigcup_{n\in\mathbb{N}} G^{\times n}$ be the symbol space of all finite sequences with elements in $G$. Let $\bar{\mu}$ be the measure on $\Sigma$ defined by

$$\bar{\mu} = \sum_{n=0}^{+\infty} \mu^{\otimes n}, \text{ where } \mu^{\otimes 0} = \delta_{\emptyset}.$$

Let the integer $\omega(g)$ be the length of an element $g$ in $\Sigma$. Let $T$ be the shift map on $\Sigma$, defined by $Tg = T(g_1, g_2, \dots, g_\omega) = (g_1, g_2, \dots, g_{\omega-1})$, when $\omega(g) \ge 2$, and $Tg = \emptyset$, when $\omega(g) = 1, 0$. Let $L$ be the left shift map on $\Sigma$, defined by $Lg = L(g_1, g_2, \dots, g_\omega) = (g_2, \dots, g_{\omega-1}, g_\omega)$, when $\omega(g) \ge 2$, and $Lg = \emptyset$, when $\omega(g) = 1, 0$.

The sets $M_t^+, N_t^+$ are defined by

$$\begin{aligned} M_t^+ &= \{g \in \Sigma |\ \kappa(Tg) < t \le \kappa(g)\}, \\ N_t^+ &= \iota(M_t^+) = \{g \in \Sigma | \kappa(Lg) < t \le \kappa(g)\}, \end{aligned}$$

where $\iota(M)$ equals $\{g^{-1}|g \in M\}$ for any subset $M$ of $\Sigma$.

Let $\check{\mu}$ be the pushforward of $\mu$ by the inverse action. It also satisfies the assumptions of Theorem 1.1.1. By definition $\bar{\mu}(M_t^+) = \bar{\check{\mu}}(N_t^+)$.

For $x, y$ in $X$, write $s_1 = \epsilon_3 s$ and

$$M_t^+(x,y) = \{g \in M_t^+ || \kappa(g) - \kappa(Tg)| < s_1, d(x_g^m, g^{-1}x) < e^{-t}, d(g^{-1}x, x), d(g^{-1}x, y) > 2e^{-s_1}\}.$$

We need some regularity properties of $N_t^+$. These lemmas are of the same type as the ones with the cocycle, using the Cartan projection instead. The correspondences are: Lemma 1.5.1 with Lemma 1.4.20, Lemma 1.5.2 with Lemma 1.4.21, Lemma 1.5.3 with Lemma 1.4.22. In fact, for all the regularity properties, there are similar versions for the Cartan projection. The subadditivity is sufficient. We follow the same procedure as in the proof for the cocycle.

**Lemma 1.5.1.** *For $s$ in $\mathbb{R}$, we have*

$$\bar{\mu}\{g \in N_t^+ || \kappa(g) - \kappa(Lg)| > s\} = O_{\exp}(s). \tag{1.5.1}$$

*Proof.* Subadditivity of Cartan projection implies $\kappa(g_\omega) \geq |\kappa(g_\omega \cdots g_1) - \kappa(g_{\omega-1} \cdots g_1)| = |\kappa(g) - \kappa(Lg)| > s$ and $\kappa(Lg) \geq \kappa(g) - \kappa(g_\omega)$. Then

$$\begin{aligned} &\bar{\mu}\{g \in N_t^+ || \kappa(g) - \kappa(Lg)| > s\} \\ &= \sum_{n \geq 0} \mu \otimes \mu^{*n}\{(h,g) \in G \times G | \kappa(g) < t \leq \kappa(hg), |\kappa(hg) - \kappa(g)| > s\} \\ &\leq \sum_{n \geq 0} \mu \otimes \mu^{*n}\{(h,g) \in G \times G | t - \kappa(h) \leq \kappa(g) < t, \kappa(h) > s\} \\ &= \int_{\kappa(h) > s} R_p(\mathbb{1}_{[-\kappa(h),0]})(t) \mathrm{d}\mu(h). \end{aligned}$$

By Lemma 1.4.12 and finite exponential moment, we have

$$\bar{\mu}\{g \in N_t^+ || \kappa(g) - \kappa(Lg)| > s\} \lesssim \int_{\kappa(h) > s} \max\{1, \kappa(h)^2\} \mathrm{d}\mu(h) = O_{\exp}(s).$$

The proof is complete. □

A special case is when $s = 0$. Applying the above lemma with $\bar{\check{\mu}}$, we have

**Lemma 1.5.2.** *The measure $\bar{\mu}(M_t^+) = \bar{\check{\mu}}(N_t^+)$ is uniformly bounded with $t$.*

The following lemma quantifies the independence of the scalar part and the angle part of residue process for the Cartan projection.

**Lemma 1.5.3.** *For $s > 0$, $t > 10s$ and $x, x_o \in X$, we have*

$$\bar{\mu}\{g \in N_t^+ | d(x_g^M, gx) \geq e^{-t}\} = O_{\exp}(t), \tag{1.5.2}$$

$$\bar{\mu}\{g \in N_t^+ | d(gx_o, x) \leq e^{-s}\} = O_{\exp}(s). \tag{1.5.3}$$

The proof of the second inequality follows the same procedure as in the proof of Lemma 1.4.22, replacing Lemma 1.4.20 and Proposition 1.4.10 with Lemma 1.5.1 and Lemma 1.4.11. The first inequality is standard, using Principle 1 and Principle 2. When $n \in [\frac{t}{\sigma_\mu+\epsilon} - 1, \frac{t}{\sigma_\mu-\epsilon}]$, use Corollary 1.2.13, and when $n$ is outside of this interval, use Corollary 1.2.9 and Corollary 1.2.11.

Joining Lemma 1.5.1 and Lemma 1.5.3, we have the following corollary

**Corollary 1.5.4.** *Let $s > 0$, $t > 10s$ and let $x, y$ be in $X$. Let*

$$N_t^+(x,y) = \{g \in N_t^+ || \kappa(g) - \kappa(Lg)| < s, d(x_g^M, gx) < e^{-t}, d(gx,x), d(gx,y) > 2e^{-s}\}. \tag{1.5.4}$$

*Then we have*

$$\bar{\mu}(N_t^+) - \bar{\mu}(N_t^+(x,y)) = O_{\exp}(s). \tag{1.5.5}$$

**Corollary 1.5.5.** *For $s > 0$, $t > 10s$ and $x, y$ in $X$, we have*

$$\bar{\mu}(M_t^+) - \bar{\mu}(M_t^+(x,y)) = O_{\exp}(s).$$

*Proof.* By definition, we have

$$\bar{\mu}(M_t^+) - \bar{\mu}(M_t^+(x,y)) = \bar{\check{\mu}}(N_t^+) - \bar{\check{\mu}}(N_t^+(x,y)).$$

Applying the above corollary with $\bar{\check{\mu}}$, we have completed the proof. □

We start to proof Proposition 1.3.6. The central tool here is Lemma 1.2.7, which enables us to replace the cocycle with the sum of the scalar part and the angle part.

*Proof of Proposition 1.3.6.* We first replace the distance with the cocycle. By hypothesis, we have

$$d(x_g^m, x) \geq d(g^{-1}x, x) - d(x_g^m, g^{-1}x) \geq 2e^{-s_1} - e^{-t} \geq e^{-s_1}.$$

Using the same argument, we have $d(x_g^m, y), d(x_g^m, x) \geq e^{-s_1}$. Then (1.2.1) and (1.2.5) imply

$$d(gx, gy) = d(x,y)\exp(-\sigma(g,x) - \sigma(g,y)) \leq \frac{\exp(-2\kappa(g))}{d(x_g^m, x)d(x_g^m, y)} \leq e^{-2(t-s_1)}.$$

Applying the Newton-Leibniz formula (1.2.17) to $\phi$ at $gx, gy$, we have

$$\phi(gx) - \phi(gy) = \mathrm{sign}(gx, gy)\int_{gx \frown gy} \phi'(\theta)\mathrm{d}\theta.$$

Since $\kappa(g) > t > s_1$, we have $d(x_g^m, x) \geq e^{-s_1} \geq e^{-\kappa(g)}$. Then (1.2.18) implies that

$$\phi(gx) - \phi(gy)) = \mathrm{sign}(x, y, x_g^m)\int_{gx \frown gy} \phi'(\theta)\mathrm{d}\theta.$$

We need the arc length distance $d_a(\cdot,\cdot)$ on $\mathbb{R}/\pi\mathbb{Z}$. Since $d(gx,gy)\leq e^{-2(t-s_1)}$, for $\theta$ in the small arc $gx\frown gy$, we have $d_a(\theta,gx)\leq e^{-2(t-s_1)}$. Therefore

$$|\phi(gx)-\phi(gy)-\mathrm{sign}(x,y,x_g^m)\phi'(gx)d_a(gx,gy)|\leq|\phi''|_\infty e^{-4(t-s_1)}. \tag{1.5.6}$$

By equality $\sin d_a(gx,gy)=d(gx,gy)$, we have

$$|d_a(gx,gy)-d(gx,gy)|=O(d(gx,gy)^3).$$

So we can replace the arc length distance with the sine distance. Again by hypothesis, we have $d(x_g^m,g^{-1}x)\leq e^{-t}<d(g^{-1}x,x),d(g^{-1}x,y)$. When changing $x_g^m$ to $g^{-1}x$, the relative place with respect to $x,y$ does not change, therefore we get

$$\mathrm{sign}(x,y,x_g^m)=\mathrm{sign}(x,y,g^{-1}x).$$

Inequality (1.2.1), together with the above two inequalities, implies

$$|\phi(gx)-\phi(gy)-\mathrm{sign}(x,y,g^{-1}x)\phi'(gx)d(x,y)\exp(-\sigma(g,x)-\sigma(g,y))|\leq|\phi''|_\infty 2e^{-4(t-s_1)}. \tag{1.5.7}$$

We may now replace the cocycle with the Cartan projection and the angle part. Since

$$\frac{e^{-2\kappa(g)}+d(x_g^m,g^{-1}x)}{d(g^{-1}x,x)}\leq 2e^{-t+s_1}<1/2,$$

Lemma 1.2.7 implies that

$$|\sigma(g,x)-\kappa(g)-\log d(g^{-1}x,x)|\leq 2\frac{e^{-2\kappa(g)}+d(x_g^m,g^{-1}x)}{d(g^{-1}x,x)}\leq 4e^{-t+s_1},$$
$$|\sigma(g,y)-\kappa(g)-\log d(g^{-1}x,y)|\leq 2\frac{e^{-2\kappa(g)}+d(x_g^m,g^{-1}x)}{d(g^{-1}x,y)}\leq 4e^{-t+s_1}.$$

We have an inequality for $z_1,z_2$ in $\mathbb{C}$,

$$|e^{z_1}-e^{z_2}|\leq\max\{e^{\Re z_1},e^{\Re z_2}\}|z_1-z_2|.$$

Since $\sigma(g,x)\geq\kappa(g)+\log d(x_g^m,x)\geq t-s_1$ and $\kappa(g)+\log d(g^{-1}x,x)\geq t-s_1$, we have

$$|\exp(-\sigma(g,x))-\exp(-\kappa(g))/d(g^{-1}x,x)|\leq e^{-t+s_1}4e^{-t+s_1}.$$

Therefore by inequality $|a_1a_2-b_1b_2|\leq|(a_1-b_1)a_2|+|(a_2-b_2)b_1|$, we have

$$|e^{-\sigma(g,x)-\sigma(g,y)}-e^{-2\kappa(g)}/(d(g^{-1}x,x)d(g^{-1}x,y))|\leq 8e^{-3(t-s_1)}.$$

Then by the hypothesis $|\xi|=e^{2t+s}$ and (1.5.7), we have

$$\begin{aligned}
&\left|e^{i\xi(\phi(gx)-\phi(gy))}-e^{i\xi\phi'(gx)\mathrm{sign}(x,y,g^{-1}x)d(x,y)\exp(-2\kappa(g))/(d(g^{-1}x,x)d(g^{-1}x,y))}\right|\\
\leq&|\xi||\phi(gx)-\phi(gy)-\phi'(gx)\mathrm{sign}(x,y,g^{-1}x)d(x,y)\exp(-2\kappa(g))/(d(g^{-1}x,x)d(g^{-1}x,y))|\\
\leq&|\xi||\phi''|_\infty 2e^{-4(t-s_1)}+|\xi||\phi'd(x,y)||e^{-\sigma(g,x)-\sigma(g,y)}-e^{-2\kappa(g)}/(d(g^{-1}x,x)d(g^{-1}x,y))|\\
\leq&|\xi|(|\phi''|_\infty 2e^{-4(t-s_1)}+8|\phi'|_\infty e^{-3(t-s_1)})\leq 8(|\phi''|_\infty+|\phi'|_\infty)e^{-t+s+3s_1}.
\end{aligned}$$

Finally, for $|\varLambda_0 - \varLambda|$, it suffices to add the difference

$$|r(gx)r(gy) - r(gx)^2| \leq |r|_\infty |r'|_\infty e^{-2(t-s_1)}.$$

Then

$$|\varLambda_0 - \varLambda| \leq |r|_\infty |r'|_\infty e^{-2(t-s_1)} + |r|_\infty^2 (|\phi''|_\infty + |\phi'|_\infty) e^{-t+s+3s_1} = O_{\exp}(s),$$

where $O_{\exp}(s)$ does not depend on $t$, but depends on $r, \phi$. The proof is complete. □

**Remark 1.5.6** (Minus case)**.** *The proof works the same for* $M_t^-$.

# Chapter 2

# Fourier dimension and spectral gaps for random walks on $\mathrm{SL}_{m+1}(\mathbb{R})$

# List of Abbreviations and Symbols in Chapter 2

## 2.1 Introduction

The purpose of this manuscript is to study the Fourier decay of stationary measures on projective spaces and some spectral properties of random walks on $G = \mathrm{SL}_{m+1}(\mathbb{R})$, the special linear group of degree $m+1$. Let $\mu$ be a Borel probability measure on $G$. Let $\Gamma_\mu$ be the subsemigroup of $G$ generated by the support of $\mu$. If $\Gamma_\mu$ is Zariski dense in $G$, then we call $\mu$ a Zariski dense measure. We say that $\mu$ has a finite exponential moment if there exists $\epsilon > 0$ such that

$$\int_G \|g\|^\epsilon \mathrm{d}\mu(g) < \infty.$$

If we have a group action of $G$ on a compact manifold $X$, then a Borel probability measure $\nu$ on $X$ is called $\mu$-stationary if $\nu = \mu * \nu$, which means

$$\nu = \int_G g_*\nu \mathrm{d}\mu(g)$$

and where $g_*\nu$ is the pushforward measure, that is $g_*\nu(E) = \nu(g^{-1}E)$ for any Borel subset $E$ of $X$. For a metric space $X$, let $C^\gamma(X)$ be the space of $\gamma-$Hölder functions on $X$. For $f$ in $C^\gamma(X)$ let $c_\gamma(f) = \sup_{x\neq x'} \frac{|f(x)-f(x')|}{d(x,x')^\gamma}$ and $|f|_\gamma = |f|_\infty + c_\gamma(f)$.

Let $(\rho, V)$ be a finite dimensional irreducible linear representation of $G$ with a norm. (For example $V = \mathbb{R}^{m+1}$) Let $\mathbb{P}V$ be the real projective space defined by $(V\backslash\{0\})/\mathbb{R}^*$, the set of all the directions in $V$. Then we have the group action of $G$ on $\mathbb{P}V$, given by $g\mathbb{R}v = \mathbb{R}\rho(g)v$ for $g$ in $G$ and $\mathbb{R}v$ in $\mathbb{P}V$. A result of Furstenberg says that when $\mu$ is Zariski dense, there exists a unique $\mu$-stationary measure $\nu_V$ on $\mathbb{P}V$.

Let $v_0$ be a unit vector in $V$. Let $v_0^\perp$ be the linear subspace of $V$, which is orthogonal to $v_0$. Let $U$ be the open subset of $\mathbb{P}V$, which is the complement of the hyperplane $\mathbb{P}v_0^\perp$. We take an affine local chart $(\psi, U)$ of $\mathbb{P}V$, given by

$$\psi\colon \mathbb{P}V \supset U \to v_0^\perp,\ \mathbb{R}v \mapsto \frac{v - \langle v_0, v\rangle v_0}{\langle v_0, v\rangle},$$

which is well defined on $U$. The inverse of $\psi$ is simply given by $\psi^{-1}\colon v_0^\perp \to U \subset \mathbb{P}V,\ u \mapsto \mathbb{R}(u + v_0)$.

**Theorem 2.1.1.** *Let $\mu$ be a Zariski dense Borel probability measure on $\mathrm{SL}_{m+1}(\mathbb{R})$ with a finite exponential moment. Let $V$ be an irreducible representation of $\mathrm{SL}_{m+1}(\mathbb{R})$. Let $\nu_V$ be the $\mu$-stationary measure on $\mathbb{P}V$. Let $r$ be a $C^1$ continuous function whose support is in $U$ and $\|r\|_\infty \leq 1$. Then there exists $\epsilon > 0$ such that for every $\varsigma \in v_0^\perp$ with the norm $\|\varsigma\|$ sufficiently large, we have*

$$\left|\int_{v_0^\perp} e^{i\langle\varsigma,u\rangle} r(u)\mathrm{d}\nu_V(u)\right| \leq \|\varsigma\|^{-\epsilon}.$$

**Remark.** *For simplicity, we use the same notation $\nu_V$ for the measure on $\mathbb{P}V$ and the measure on $v_0^\perp$. More precisely, the integral actually means $\int_{\mathbb{P}V} e^{i\langle\varsigma,\psi(x)\rangle} r(\psi(x))\mathrm{d}\nu_V(x)$.*

*The constant $\epsilon$ only depends on $\mu$ and $V$, and inequality holds for $\|\varsigma\|$ sufficiently large only depending on $\mu$, $V$, the support of $r$ and $c_\gamma(r)$.*

We state a stronger version for $m = 1$, $\mathrm{SL}_2(\mathbb{R})$, which is a quantitative version of the main result in Chapter 1

**Theorem 2.1.2.** *Let $\mu$ be a Zariski dense Borel probability measure on $\mathrm{SL}_2(\mathbb{R})$ with a finite exponential moment. Let $X = \mathbb{P}(\mathbb{R}^2)$ and let $\nu$ be the $\mu$-stationary measure on $X$.*

*For every $\gamma > 0$, there exist $\epsilon_0 >, \epsilon_1 > 0$ depending on $\mu$ such that the following holds. For any $f \in C^2(X)$, $r \in C^\gamma(X)$ such that $|\varphi'| \geq |\xi|^{-\epsilon_0}$ on the support of $r$, $\|r\|_\infty \leq 1$ and*

$$\|\varphi\|_{C^2} + c_\gamma(r) \leq |\xi|^{\epsilon_0},$$

*then*

$$\left| \int e^{i\xi\varphi(x)} r(x) \mathrm{d}\nu(x) \right| \leq |\xi|^{-\epsilon_1} \quad \text{for all } |\xi| \text{ large enough.}$$

**Remark 2.1.3.** *As a consequence of the case of $\mathrm{SL}_2(\mathbb{R})$, the Fourier coefficients of the stationary measure $\nu$ on the circle converge to zero with a power decay. This is also a generalization of the same theorem for the Patterson-Sullivan measures as in [BD17].*

*This stronger version is not valid if we replace $\mathbb{R}^2$ by higher dimensional representation $V$ of $\mathrm{SL}_2(\mathbb{R})$. Because the support of the stationary measure $\nu_V$ is in a one dimensional subvariety of $\mathbb{P}V$. We can always find a $\varphi$ which is constant on the subvariety and satisfies similar assumptions in Theorem 2.1.2. Then we have no Fourier decay for this function $\varphi$.*

Another result is an exponential remainder term in the renewal theorem. Define the renewal operator $R$ as follows. For a positive bounded Borel function $f$ on $\mathbb{P}V \times \mathbb{R}$, a point $x = \mathbb{R}v$ in $\mathbb{P}V$ and a real number $t$, we set

$$Rf(x,t) = \sum_{n=0}^{+\infty} \int_G f(gx, \log \frac{\|gv\|}{\|v\|} - t) \mathrm{d}\mu^{*n}(g).$$

Here $\log \frac{\|gv\|}{\|v\|}$ is an analogue of the sum of i.i.d. real random variables. Because of the positivity of $f$, this sum is well defined. In [Kes74], Kesten proved a renewal theorem for Markov chains, which is valid in our case. The assumptions of [Kes74] were verified in [GLP16]. Using spectral gap, more precisely by Proposition 2.4.22, we can give a version with exponential remainder term. Let $\sigma_{V,\mu}$ be the Lyapunov constant defined by

$$\sigma_{V,\mu} = \int_G \int_{\mathbb{P}V} \log \frac{\|gv\|}{\|v\|} \mathrm{d}\nu(x) \mathrm{d}\mu(g), \text{ where } x = \mathbb{P}v.$$

**Theorem 2.1.4** (Renewal theorem)**.** *Let $\mu$ be a Zariski dense Borel probability measure on $\mathrm{SL}_{m+1}(\mathbb{R})$ with finite exponential moment. Let $V$ be an irreducible representation of $\mathrm{SL}_{m+1}(\mathbb{R})$. There exists $\epsilon > 0$ such that for $f \in C_c^\infty(\mathbb{R})$ and $t \in \mathbb{R}$, we have*

$$Rf(x,t) = \frac{1}{\sigma_{V,\mu}} \int_{-t}^{\infty} f(u) \mathrm{d}Leb(u) + O_f(e^{-\epsilon|t|}),$$

*where $O_f$ depends on the support and some Sobolev norm of $f$.*

It is a standard Fourier analysis argument which follows from the spectral gap. We give a proof in Section 2.4.5 for completeness.

Now, we will introduce our results on spectral gaps. On $\mathbb{P}V$, we fix a Riemannian distance and we define the transfer operator.

**Definition.** *For $z \in \mathbb{C}$ with $|\Re z|$ small enough, let $P_z$ be the operator on the space of continuous functions, which is given by*

$$P_z f(x) = \int_G e^{z \log \frac{\|gv\|}{\|v\|}} f(gx)\mathrm{d}\mu(g), \text{ where } x = \mathbb{R}v \in \mathbb{P}V.$$

We keep the assumption that $\mu$ is a Zariski dense Borel probability measure on $\mathrm{SL}_{m+1}(\mathbb{R})$ with a finite exponential moment. The use of this transfer operator on the products of random matrices has been introduced by Guivarc'h. Due to the contracting action of $G$ on $X$, when $|\Re z|$ is small enough, the operator $P_z$ preserves the Banach space $C^\gamma(\mathbb{P}V)$ for $\gamma > 0$ small enough. For $z$ in a small ball centred at 0, the spectral radius of $P_z$ on $C^\gamma(\mathbb{P}V)$ is less than 1 except at 0. Due the non-arithmeticity of $\Gamma_\mu$, on the imaginary line, the operator $P_z$ also has spectral radius less than 1 except at 0. These were used to give limit theorems for products of random matrices by Le Page and Guivarc'h (Please see [LP82b] and [BQ16]).

**Theorem 2.1.5** (Spectral gap). *Let $\mu$ be a Zariski dense Borel probability measure on $\mathrm{SL}_{m+1}(\mathbb{R})$ with finite exponential moment. Let $V$ be an irreducible representation of $\mathrm{SL}_{m+1}(\mathbb{R})$. For every $\gamma > 0$ small enough, there exists $\delta > 0$ such that for all $|b| > 1$ and $|a|$ small enough the spectral radius of $P_{a+ib}$ acting on $C^\gamma(\mathbb{P}V)$ satisfies*

$$\rho(P_{a+ib}) < 1 - \delta.$$

Even in the case of $\mathrm{SL}_2(\mathbb{R})$, the result is new and only known in some special case. When $\mu$ is supported on a finite number of elements in $\mathrm{SL}_2(\mathbb{R})$ and these elements generate a Schottky semigroup, this result is due to Naud [Nau05]. When $\mu$ is absolutely continuous with respect to the Haar measure on $\mathrm{SL}_2(\mathbb{R})$, this result can be obtained directly using oscillation integral.

This result should be compared with similar results for random walks on $\mathbb{R}$. Let $\mu$ be a Borel probability measure on $\mathbb{R}$ with finite support. Then

$$\liminf_{|b|\to\infty} |1 - \hat{\mu}(ib)| = 0,$$

which is totally different from our case and where $\hat{\mu}(z)$ is the Laplace transform of the measure $\mu$, given by

$$\hat{\mu}(z) = \int_{\mathbb{R}} e^{zx}\mathrm{d}\mu(x).$$

The proof is direct. Let $\{x_1, \dots, x_l\}$ be the support of $\mu$. Then $\hat{\mu}(ib) = \sum_{1\leq j\leq l} \mu(x_j)e^{ibx_j}$, and we only need to find $b$ such that all the terms are uniformly near 1. Using the fact that $\liminf_{b\to\infty} d_{\mathbb{R}^l}(b(x_1, \dots, x_l), 2\pi\mathbb{Z}^l) = 0$, we have the claim.

We can also compare with the counting problem in the setting of hyperbolic surfaces. The spectral gap is used to obtain an exponential remainder term in the counting problem as in [LP82a], [Nau05].

An analogous result is valid if we replace the projective space $\mathbb{P}V$ by the flag variety $\mathscr{P}$. Let $\mathscr{P}$ be the full flag variety of $\mathrm{SL}_{m+1}(\mathbb{R})$ and let $\mathfrak{a}$ be a Cartan subspace of Lie algebra $\mathfrak{sl}_{m+1}\mathbb{R}$. For $g \in G$ and $\eta \in \mathscr{P}$, let $\sigma(g,\eta)$ be the Iwasawa cocycle, which takes values in $\mathfrak{a}$. We fix a Riemannian distance on $\mathscr{P}$. We can similarly define the space of $\gamma$-Hölder functions $C^\gamma(\mathscr{P})$. Let $\varpi, \theta$ be in $\mathfrak{a}^*$. For a continuous function $f$ on $\mathscr{P}$ and $|\varpi|$ small enough, the transfer operator $P_{\varpi+i\theta}$ on the flag variety is defined by

$$P_{\varpi+i\theta}f(\eta) = \int_G e^{(\varpi+i\theta)\sigma(g,\eta)} f(g\eta)\mathrm{d}\mu(g).$$

**Theorem 2.1.6** (Spectral gap)**.** *Let $\mu$ be a Zariski dense Borel probability measure on $\mathrm{SL}_{m+1}(\mathbb{R})$ with finite exponential moment. For every $\gamma > 0$ small enough, there exists $\delta > 0$ such that for all $\theta, \varpi$ in $\mathfrak{a}^*$ with $|\theta| > 1$ and $|\varpi|$ small enough the spectral radius of $P_{\varpi+i\theta}$ acting on $C^\gamma(\mathscr{P})$ satisfies*

$$\rho(P_{\varpi+i\theta}) < 1 - \delta.$$

### Fourier decay

The key ingredient of the proof of the above results is the following Fourier decay property of the $\mu$-stationary measure on the flag variety $\mathscr{P}$. In order to state the Fourier decay on the flag variety, we need to introduce a special condition. Let $r$ be a continuous function on $\mathscr{P}$ and let $C > 0$. For a $C^2$ function $\varphi$ on $\mathscr{P}$, we say $\varphi$ is $(C, r)$ good if it satisfies some assumptions on the Lipschitz norm and derivative, which will be defined later (Definition 2.4.1). When $G = \mathrm{SL}_2(\mathbb{R})$, the $(C, r)$ goodness is exactly the assumption of $\varphi$ in Theorem 2.1.2, which is natural for having a Fourier decay. Recall that for a $\gamma$-Hölder function $f$, we have defined $c_\gamma(f) = \sup_{x \neq x'} \frac{|f(x)-f(x')|}{d(x,x')^\gamma}$.

**Theorem 2.1.7** (Fourier decay)**.** *Let $\mu$ be Zariski dense Borel probability measure on $\mathrm{SL}_{m+1}(\mathbb{R})$ with finite exponential moment. Let $\nu$ be the $\mu$-stationary measure on the flag variety $\mathscr{P}$.*

*For every $\gamma > 0$, there exist $\epsilon_0 > 0, \epsilon_1 > 0$ depending on $\mu$ such that the following holds. For any real function $\varphi \in C^2(\mathscr{P})$, $r \in C^\gamma(\mathscr{P})$ and $\xi > 0$ such that $\varphi$ is $(\xi^{\epsilon_0}, r)$ good, $\|r\|_\infty \leq 1$ and $c_\gamma(r) \leq \xi^{\epsilon_0}$, then*

$$\left| \int e^{i\xi\varphi(\eta)} r(\eta)\mathrm{d}\nu(\eta) \right| \leq \xi^{-\epsilon_1} \quad \text{for all } \xi \text{ large enough.} \tag{2.1.1}$$

**Remark 2.1.8.** *The decay rate only depends on the constants in the large deviation principles and the regularity of stationary measures. This should be compared with [BD17], where the spectral gap and the decay rate only depend on the dimension of the Patterson-Sullivan measure.*

Now we explain the $(C, r)$ good condition. In higher dimension, we observe that under the action of $G$ there are some directions contracting slower than other directions. Roughly speaking, we will only consider these principal directions in the flag variety $\mathscr{P}$ and generalize the condition of $\mathrm{SL}_2(\mathbb{R})$ to higher dimension. The exact definition is a little technique and all the notation will be explained in Section 2.4.1.

A key ingredient in the proof of Theorem 2.1.7 comes from the discretized sum-product estimate, Proposition 2.3.17, which is a generalized version of a result of Bourgain in [Bou10]. The key input to use the machine of the discretized sum-product estimate is a non concentration hypothesis. A analogue hypothesis for measures on $\mathbb{R}$ is as follows.

**Definition** (non concentration). *Let $\mu$ be a Borel probability measure on $\mathbb{R}$. We say that $\mu$ satisfies non concentration hypothesis if there exist $\epsilon, \kappa, C > 0$ such that for every $n \in \mathbb{N}$ and $\rho = e^{-\epsilon n}$,*

$$\sup_{a\in\mathbb{R}} \mu^{*n}\{x \in \mathbb{R} |\ |x - a| \leq \rho\} \leq C\rho^{\kappa}.$$

But this hypothesis is never satisfied when the measure $\mu$ supports on a finite set, for example $\{x_1, \cdots, x_l\} \subset \mathbb{R}$. Because the convolution $\mu^{*n}$ is supported on at most $n^l$ points, there exists a point $y$ such that $\mu^{*n}\{y\} \gg n^{-l}$. The decay rate of $\sup_{a\in R} \mu^{*n}(B(a, \rho))$ is at most polynomial in $n$, which does not satisfy the definition of non concentration.

In Section 2.3, we will introduce a similar non concentration hypothesis and we will verify the hypothesis for our measure $\mu$. The main ingredients are the large deviation principle, a Hölder regularity for stationary measures and highest weight representations. The strategy can be roughly summarized by saying that once the non concentration of the Iwasawa cocycle is verified, by the discretized sum-product estimate, we will have a Fourier decay and a speed in the equidistribution of the Iwasawa cocycle (the renewal theorem). This is in the similar spirit of the work of Bourgain and Gamburd on the spectral gap for compact Lie groups [BG08].

The action of the group $G$ on the flag variety $\mathscr{P}$ is not conformal if the rank $m$ is greater than 1, which is quite different from the theory of Kleinian groups. In Section 2.2, we will study the action of $G$ on the tangent bundle of $\mathscr{P}$ and we will find directions of slowest contraction speed.

We will make use of some classical notation: for two real functions $A$ and $B$, we write $A = O(B), A \ll B$ or $B \gg A$ if there exists a constant $C > 0$ such that $|A| \leq CB$, where $C$ only depends on the ambient group $G$ and the measure $\mu$. We write $A \asymp B$ if $A \ll B$ and $B \ll A$. We write $A = O_\epsilon(B), A \ll_\epsilon B$ or $B \gg_\epsilon A$ if the constant $C$ depends on an extra parameter $\epsilon > 0$.

We always use $0 < \delta < 1$ to denote an error term and $0 < \beta < 1$ to denote the magnitude. The quantity $\beta^{-1}$ is supposed to be greater than $\delta^{-O(1)}$. If $\delta^{O(1)}A \leq B \leq \delta^{-O(1)}A$, then we say that $A$ and $B$ are of the same size.

## 2.2 Random walks on Lie groups

In this manuscript, we only consider $G = \mathrm{SL}_{m+1}(\mathbb{R})$.

### 2.2.1 Semisimple Lie groups and representations

We will introduce the vocabulary of semisimple algebraic connected real Lie groups. Please see [Hel79], [Bor90] and [BQ16] for more details.

#### Semisimple Lie groups

Let $G$ be a semisimple algebraic connected real Lie group. Let $\mathfrak{g}$ be its Lie algebra. Since all the maximal compact subgroups are conjugate, we fix a maximal compact subgroup $K$ of $G$. Let $\mathfrak{k}$ be its Lie algebra. For $X, Y$ in $\mathfrak{g}$, the Killing form is defined as

$$K(X,Y) = tr(\mathrm{ad}X\mathrm{ad}Y).$$

The Killing form is non degenerate on $\mathfrak{g}$ and negative definite on $\mathfrak{k}$. Let $\mathfrak{s}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then the Killing form is positive definite on $\mathfrak{s}$. Let $\tau$ be the Cartan involution which fixes $\mathfrak{k}$ and equals $-id$ on $\mathfrak{s}$.

We say an element $X$ in $\mathfrak{g}$ is hyperbolic, if $\mathrm{ad}X$ is diagonalizable over $\mathbb{R}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{s}$ whose elements are hyperbolic. Such subalgebras are called Cartan subspaces, and they are conjugate under $K$. The dimension of $\mathfrak{a}$ is called the **real rank** of $G$. The real rank of the group $G$ will always be denoted by $m$. Endowed with the Killing form, the Cartan subalgebra $\mathfrak{a}$ and its dual $\mathfrak{a}^*$ become Euclidean spaces. Let $A$ be the algebraic subgroup of $G$ with the Lie algebra $\mathfrak{a}$. We write exp for the exponential map from $\mathfrak{a}$ to $A$.

#### Root systems and the Weyl group

Let $R$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$, which is a finite subset of $\mathfrak{a}^*$. Fix a choice of positive roots $R^+$. Let $\Pi$ be the collection of primitive simple roots of $R^+$. Let $\mathfrak{a}^+$ be the Weyl chamber defined by $\{X \in \mathfrak{a} | \alpha(X) \geq 0,\ \forall \alpha \in \Pi\}$. Let $\mathfrak{a}^{++}$ be the interior of Weyl chamber defined by $\{X \in \mathfrak{a} | \alpha(X) > 0,\ \forall \alpha \in \Pi\}$. Using the root system, we have a decomposition of $\mathfrak{g}$ into eigenspaces of $\mathfrak{a}$,

$$\mathfrak{g} = \mathfrak{z} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha,$$

where $\mathfrak{z}$ is the centralizer of $\mathfrak{a}$ and $\mathfrak{g}^\alpha$ is the eigenspace given by

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} |\ [Y, X] = \alpha(Y)X \text{ for all } Y \in \mathfrak{a}\}.$$

The real Lie group is called **split** if $\mathfrak{a} = \mathfrak{z}$, which is equivalent to saying that $\mathfrak{g}^\alpha$ are of dimension 1. The groups $\mathrm{SL}_{m+1}(\mathbb{R})$ are split groups.

Recall that for every root $\alpha$ in $R$, there is an orthogonal symmetry $s_\alpha$ which preserves $R$ and $s_\alpha(\alpha) = -\alpha$. For $\alpha \in R$, let $H_\alpha$ be the unique element in $\mathfrak{a}$ such that $s_\alpha(\alpha') =$

$\alpha' - \alpha'(H_\alpha)\alpha$ for $\alpha' \in \mathfrak{a}^*$. The set $\{H_\alpha |\ \alpha \in R\}$ is called the set of duals roots in $\mathfrak{a}$. Since the Cartan involution $\tau$ equals $-id$ on $\mathfrak{a}$, this implies $\tau\mathfrak{g}^\alpha = \mathfrak{g}^{-\alpha}$ for $\alpha \in R$. Using the Killing form, we can prove that $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] = \mathbb{R}H_\alpha$. (See [Ser66, Cha. 4, Theorem 2] for more details) Hence, there is a unique choose (up to sign) $X_\alpha \in \mathfrak{g}^\alpha,\ Y_\alpha \in \mathfrak{g}^{-\alpha}$ such that

$$[X_\alpha, Y_\alpha] = H_\alpha \text{ and } \tau(X_\alpha) = -Y_\alpha.$$

Let $K_\alpha = X_\alpha - Y_\alpha$. Due to $\tau K_\alpha = K_\alpha$, the element $K_\alpha$ is in $\mathfrak{k}$.

Let $W$ be the Weyl group of $R$. Then the group $W$ acts simply transitively on the set of Weyl chambers. Let $w_0$ be the unique element in $W$ which sends the Weyl chamber $\mathfrak{a}^+$ to the Weyl chamber $-\mathfrak{a}^+$. Let $\iota = -w_0$ be the opposition involution. The Weyl group also acts on $\mathfrak{a}^*$ by the dual action. Let $N_G(A)$ be the normalizer of $A$ in $G$. An element in $N_G(A)/A$ induces an automorphism on the tangent space $\mathfrak{a}$. This gives an isomorphism from $N_G(A)/A$ to the Weyl group $W$. Hence $w_0$ can be realized as an element in $G/A$ and its action on $\mathfrak{a}$ is given by conjugation.

### The Iwasawa cocycle

Let $\mathfrak{n} = \oplus_{\alpha\in R^+}\mathfrak{g}^\alpha$ and $\mathfrak{n}^- = \oplus_{\alpha\in R^+}\mathfrak{g}^{-\alpha}$. They are nilpotent Lie algebras. Let $N$ be the connected algebraic subgroup of $G$ with Lie algebra $\mathfrak{n}$. The group $N$ is normalized by $A$. Let $P = AN$ be a minimal parabolic subgroup. The flag variety $\mathscr{P}$ is defined to be the set of conjugations of $P$ under the action of $G$. Since the normalizer of $P$ in $G$ is itself, we have an isomorphism

$$G/P \to \mathscr{P}.$$

We write $\eta_o$ for the subgroup $P$ seen as a point in $\mathscr{P}$. The action of $K$ on $\mathscr{P}$ is transitive. Hence $\mathscr{P}$ is a compact manifold. Let $M = P \cap K$. The fact that the group $G$ is split implies that $M$ is discrete.

Recall that $N$ is the nilpotent subgroup with Lie algebra $\mathfrak{n}$. We have an Iwasawa decomposition of $G$ given by

$$G = KA_eN,$$

where $A_e = \exp(\mathfrak{a})$ is the analytical connected component of $A$. This is a bijection between $G$ and $K \times A_e \times N$. Then we can define the Iwasawa cocycle $\sigma$ from $G \times \mathscr{P}$ to $\mathfrak{a}$. Let $\eta$ be in $\mathscr{P}$ and $g$ be in $G$. By the transitivity of $K$, there exists $k \in K$ such that $\eta = k\eta_o$. By the Iwasawa decomposition, there exists a unique element $\sigma(g, \eta)$ in $\mathfrak{a}$ such that

$$gk \in K \exp(\sigma(g, \eta))N.$$

We can verify that this is well defined and $\sigma$ is an additive cocycle, that is for $g, h$ in $G$ and $\eta$ in $\mathscr{P}$

$$\sigma(gh, \eta) = \sigma(g, h\eta) + \sigma(h, \eta).$$

### The Cartan decomposition

The Cartan decomposition says that $G = KA^+K$, where $A^+$ is the image of the Weyl chamber $\mathfrak{a}^+$ under the exponential map. For $g$ in $G$, by Cartan decomposition, we can write $g = k_g a_g \ell_g$ with $k_g, \ell_g$ in $K$ and $a_g$ in $A^+$. The element $a_g$ is unique and there is a unique element $\kappa(g)$ in $\mathfrak{a}^+$ such that $a_g = \exp(\kappa(g))$. We call $\kappa(g)$ the Cartan projection of $g$. Then $\kappa(g^{-1}) = \iota\kappa(g)$, where $\iota$ is the opposition involution. Since $A$ is contained in $P$, we can define $\zeta_o = w_0\eta_o$, where the element $w_0$ in the Weyl group is seen as an element in $G/A$. (As an element in $\mathscr{P}$, $\zeta_o$ is the opposite parabolic group with respect to $P$ and $A$) Let $\eta_g^M = k_g\eta_o$ and $\zeta_g^m = \ell_g^{-1}\zeta_o$. When $\kappa(g)$ is in $\mathfrak{a}^{++}$, they are uniquely defined, independently of the choice of $k_g$ and $\ell_g$.

### Representations and highest weight

Let $(\rho, V)$ be a linear finite dimensional algebraic representation of $G$. In this manuscript, we only consider finite dimensional representations. The set of restricted weights $\Sigma(\rho)$ of the representation is the set of elements $\omega$ in $\mathfrak{a}^*$ such that the eigenspace

$$V^\omega = \{v \in V | \forall X \in \mathfrak{a},\ \mathrm{d}\rho(X)v = \omega(X)v\}$$

is nonzero, where $\mathrm{d}\rho$ is the tangent map of $\rho$ from $\mathfrak{g}$ to $End(V)$. We define a partial order on the restricted weights: For $\omega_1, \omega_2$ in $\Sigma(\rho)$,

$$\omega_1 \geq \omega_2 \Leftrightarrow \omega_1 - \omega_2 \text{ is a sum of positive roots.}$$

If $\omega$ is in $\Sigma(\rho)$, then we say that $\omega$ is a weight of $V$ and a vector $v$ in $V^\omega$ is said to have weight $\omega$. We call $\rho$ proximal if there exists $\chi$ in $\Sigma(\rho)$ which is greater than the other restricted weights and such that $V^\chi$ is of dimension 1. We should pay attention that a proximal representation is not supposed to be irreducible. The advantage of the splitness of $G$ is that all the irreducible representations are proximal, which will be extensively used later on.

Let $\{\chi_\alpha\}_{\alpha\in\Pi}$ be the set of fundamental weights, which is the dual basis of the dual roots $(H_\alpha)_{\alpha\in\Pi}$, a basis of $\mathfrak{a}^*$. For an element $\chi$ in $\mathfrak{a}$, there exists a finite dimensional representation with highest weight $\chi$ if and only if it is a dominant weight, which means that $\chi$ is a sum of multiple of fundamental weights.

Suppose that $(\rho, V)$ is an irreducible representation. Let $\chi \in \mathfrak{a}^*$ be the highest weight of $(\rho, V)$. We write $V_{\chi,\eta} = \rho(g)V^\chi$ for $\eta = g\eta_o$, which is well defined because the parabolic subgroup $P$ fixes the subspace $V^\chi$. This gives a map from $\mathscr{P}$ to $\mathbb{P}V$ by

$$\mathscr{P} \to \mathbb{P}V,\ \eta \mapsto V_{\chi,\eta}. \tag{2.2.1}$$

**Lemma 2.2.1.** *Let $G$ be $\mathrm{SL}_{m+1}(\mathbb{R})$. There exists a family of representations $(\rho_\alpha, V_\alpha)_{\alpha\in\Pi}$ such that the highest weight of $\rho_\alpha$ is $\chi_\alpha$. Furthermore, the other weights of $\rho_\alpha$ are of the form*

$$\chi_\alpha - \alpha - \sum_{\beta\in\Pi} n_\beta\beta,\ \text{where } n_\beta \in \mathbb{N}_{\geq 0}.$$

*The product of the maps given by (2.2.1)*

$$\mathscr{P} \longrightarrow \prod_{\alpha\in\Pi} \mathbb{P}V_\alpha, \quad \eta \mapsto (V_{\chi_\alpha,\eta})_{\alpha\in\Pi},$$

*is an embedding of $\mathscr{P}$ to the product of projective spaces.*

Please see [Tit71]. The set of restrict weight is invariant under the Weyl group. Due to $\chi_\alpha(H_\beta) = 0$ when $\beta \neq \alpha$, we have $s_\beta(\chi_\alpha - \beta) = \chi_\alpha + \beta$. Hence $\chi_\alpha - \beta$ is not a weight of $V_\alpha$ except $\beta = \alpha$. This explains the structure of weights of $\rho_\alpha$.

**Definition 2.2.2** (Super proximal representation)**.** *Let $(V,\chi)$ be an irreducible representation of $G$. We call $V$ super proximal if the exterior square $\wedge^2 V$ is also proximal. This is equivalent to say that there is only one simple root $\alpha$ such that $\chi - \alpha$ is a weight of $V$, and $V^{\chi-\alpha}$ is of dimension 1.*

**Lemma 2.2.3.** *Fundamental representations are super proximal.*

*Proof.* Let $\alpha$ be a simple root. By Lemma 2.2.1, we only need to prove that $V^{\chi_\alpha-\alpha}$ is of dimension 1. Let $v$ be a nonzero vector with highest weight $\chi_\alpha$. By [Ser66, Chapter 7, Proposition 2], the representation $V$ is generated by vectors $Y_{\beta_1}\cdots Y_{\beta_k}v$, where $\beta_1,\dots,\beta_k$ are positive roots. Hence a vector of weight $\chi_\alpha - \alpha$ can only be obtained by $Y_\alpha v$. The dimension of $V^{\chi_\alpha-\alpha}$ is no greater than 1. Due to $\chi_\alpha - \alpha = s_\alpha(\chi_\alpha)$, the element $\chi_\alpha - \alpha$ is a weight of $V_\alpha$. The proof is complete. □

**Remark 2.2.4.** *We can prove that an irreducible representation is super proximal if and only if its highest weight is a multiple of fundamental weight, by using Freudenthal's multiplicity formula.*

### Representations and good norms

Let $\|\cdot\|$ be an euclidean norm on $V$. For $g$ in $GL(V)$, let $\|g\|$ be its application norm. We call $\|\cdot\|$ a good norm if $\rho(A)$ is symmetric and $\rho(K)$ preserves the norm. By [Hel79], [BQ16, Lemma 6.33], good norms exist on every representation of $G$. The application (2.2.1) enables us to get information on $\mathscr{P}$ by the representations.

**Lemma 2.2.5.** *Let $G$ be a connected algebraic semisimple real Lie group. Let $(\rho,V)$ be an irreducible linear representation of $G$ with good norm. Let $\chi$ be the highest weight of $V$. For $\eta$ in $\mathscr{P}$ and a non zero vector $v\in V_{\chi,\eta}$, we have*

$$\frac{\|\rho(g)v\|}{\|v\|} = \exp(\chi\sigma(g,\eta)), \tag{2.2.2}$$

$$\|\rho(g)\| = \exp(\chi\kappa(g)). \tag{2.2.3}$$

Please see [BQ16, Lemma 6.33] for the proof.

### Algebraic characters

Let $X(A)$ be the set of algebraic characters of $A$. For any character $\chi$ of $A$, there exists a unique weight $\chi^\omega$ in $\mathfrak{a}^*$ such that for any $X$ in $\mathfrak{a}$,

$$\chi(\exp(X)) = e^{\chi^\omega(X)}.$$

When $G = \mathrm{SL}_{m+1}(\mathbb{R})$, for every weight $\varpi$, which is in $\Gamma(G) := \oplus_{\alpha\in\Pi}\mathbb{Z}\chi_\alpha$ (the lattice generated by $\{\chi_\alpha\}_{\alpha\in\Pi}$), we can find an algebraic character $\chi$ in $X(A)$ such that $\chi^\omega = \varpi$. In fact, this is an isomorphism between $X(A)$ and $\Gamma(G)$. By the definition of eigenspace $V^\omega$, we have

**Lemma 2.2.6.** *Let $(\rho, V)$ be an irreducible representation of $G$. Let $\chi$ be an algebraic character of $A$. For $a$ in $A$ and $v \in V^{\chi^\omega}$, we have*

$$\rho(a)v = \chi(a)v.$$

Algebraic characters will be used to determiner the sign in Section 2.2.5. We will use the same symbol $\chi$ to denote a weight in $\Gamma(G)$ and $\chi^\sharp$ to denote its corresponding algebraic character in $X(A)$, that is $(\chi^\sharp)^\omega = \chi$.

### Examples

For the group $\mathrm{SL}_{m+1}(\mathbb{R})$, the maximal torus $A$ can be taken as the diagonal subgroup and the Lie algebra $\mathfrak{a}$ is the set of diagonal matrices with trace 0. For $X$ in $\mathfrak{a}$, we write $X = \mathrm{diag}(x_1, \dots, x_{m+1})$ with $x_i \in \mathbb{R}$ and $\sum_{1\le i\le m+1} x_i = 0$. The restriction of Killing form on $\mathfrak{a}$ is a multiple of the standard euclidean norm on $\mathbb{R}^{m+1}$. Let $\lambda_i$ in $\mathfrak{a}^*$ be the linear map given by $\lambda_i(X) = x_i$. The root system $R$ is given by

$$R = \{\lambda_i - \lambda_j | i \neq j, \text{ and } i, j \in \{1, \dots, m+1\}\}.$$

A choice of positive roots is $\lambda_i - \lambda_j$ with $i < j$. The set of simple roots is $\Pi = \{\lambda_i - \lambda_{i+1} | i = 1, \dots, m\}$. Let $\alpha_i = \lambda_i - \lambda_{i+1}$. The Weyl chamber is

$$\mathfrak{a}^+ = \{X \in \mathfrak{a} | x_1 \geq x_2 \geq \cdots \geq x_{m+1}\}.$$

The fundamental weights are $\chi_{\alpha_i} = \lambda_1 + \cdots + \lambda_i$ for $i = 1, \dots, m$. The fundamental representations are $V_{\alpha_i} = \wedge^i \mathbb{R}^{m+1}$ for $i = 1, \dots, m$. The maximal compact subgroup $K$ is $\mathrm{SO}(m+1)$ and the parabolic group $P$ is the upper triangular subgroup and $N$ is the subgroup of $P$ with all the diagonal entries equal to 1. The flag variety $\mathscr{P}$ is the set of all flags

$$W_1 \subset W_2 \subset \cdots \subset W_m,$$

where $W_i$ is a subspace of $\mathbb{R}^{m+1}$ of dimension $i$.

Let $\epsilon_{i,j}$ be the square matrix of dimension $m+1$ with the only nonzero entry at the $i$-th row and $j$-th column, which equals 1. The element $H_{\alpha_i}$ is $\epsilon_{i,i} - \epsilon_{i+1,i+1}$. The element $X_{\alpha_i}, Y_{\alpha_i}$ are given by $\epsilon_{i,i+1}, \epsilon_{i+1,i}$. The Cartan involution $\tau$ is the additive inverse of the transpose, that is $\tau(X) = -{}^tX$ for $X$ in $\mathfrak{a}$.

The Weyl group $W$ is isomorphic to the symmetric group $\mathcal{S}_{m+1}$. The action on $\mathfrak{a}$ is simply given by the permutation of coordinates and the element $w_0$ sends $X = \mathrm{diag}(x_1, \dots, x_{m+1})$ to $w_0 X = \mathrm{diag}(x_{m+1}, \dots, x_1)$.

### 2.2.2 Linear actions on vector spaces

Let $V$ be a vector space with euclidean norm. Then we have an induced norm on its dual space $V^*$, exterior powers $\wedge^j V$ and tensor products $\otimes^j V$.

For $x = \mathbb{R}v, x' = \mathbb{R}v'$ in $\mathbb{P}V$, we define the distance between $x, x'$ by

$$d(x, x') = \frac{\|v \wedge v'\|}{\|v\|\|v'\|}. \tag{2.2.4}$$

This distance has the advantage that it behaves well under the action of $GL(V)$. See for example Lemma 2.2.8. For $y = \mathbb{R}f$ in $\mathbb{P}V^*$, let $y^\perp = \mathbb{P}(\ker f) \subset \mathbb{P}V$ be a hyperplane in $\mathbb{P}V$. For $x = \mathbb{R}v$ in $\mathbb{P}V$, we define the distance of $x$ to $y^\perp$ by

$$\delta(x, y) = \frac{|f(v)|}{\|f\|\|v\|},$$

which is explained by $\delta(x, y) = d(x, y^\perp) = \min_{x' \in y^\perp} d(x, x')$. Let $K_V$ be the compact group which preserves the norm. Let $A_V^+$ be the set of diagonal elements such that $\{a = \mathrm{diag}(a_1, \cdots, a_d) | a_1 \geq a_2 \geq \cdots \geq a_d\}$, under the basis $\{e_1, \cdots, e_d\}$. Let $A_V^{++}$ be the interior of $A_V^+$. For $g$ in $GL(V)$, by the Cartan decomposition we can choose

$$g = k_g a_g \ell_g, \text{ where } a_g \in A_V^+ \text{ and } k_g, \ell_g \in K_V. \tag{2.2.5}$$

Let $x_g^M = \mathbb{R}k_g e_1$ and $y_g^m = \mathbb{R}\,{}^t g e_1^*$ be the density points of $g$ on $\mathbb{P}V$ and ${}^t g$ on $\mathbb{P}V^*$, which is unique and independent of the choice of basis when $a_g$ is in $A_V^{++}$. For $r > 0$ and $g$ in $GL(V)$, let

$$b_{V,g}^M(r) = \{x \in \mathbb{P}V | d(x, x_g^M) \leq r\},$$
$$B_{V,g}^m(r) = \{x \in \mathbb{P}V | \delta(x, y_g^m) \geq r\}.$$

These two sets play important role when we want to get some ping-pong property. The elements in set $B_{V,g}^m$ have distance at least $r$ to the hyperplane determined by $y_g^m$. For $g$ in $GL(V)$, let $\gamma_{1,2}(g) := \frac{\|\wedge^2 g\|}{\|g\|^2}$ be the gap of $g$.

#### Distance and norm

We start with general $g$ in $GL(V)$, where $V$ is a finite dimensional vector space with euclidean norm. We need some technical control of distance. These are quantitative versions of the same controls in [Qui02, Lemma 2.5, 4.3, 6.5].

For $g$ in $GL(V)$ and $x = \mathbb{R}v \in \mathbb{P}V$, we define an additive cocycle $\sigma_V : GL(V) \times \mathbb{P}V \to \mathbb{R}$ by

$$\sigma_V(g, x) = \log \frac{\|gv\|}{\|v\|}. \tag{2.2.6}$$

This is called cocycle, because for $g, h$ in $G$, we have

$$\sigma_V(gh, x) = \sigma_V(g, hx) + \sigma_V(h, x).$$

We fix the operator norm $\|\cdot\|$ on $GL(V)$.

**Lemma 2.2.7.** *For any $g$ in $GL(V)$ and $x$ in $\mathbb{P}V$, we have*

$$\delta(x, y_g^m) \leq \frac{\|gv\|}{\|g\|\|v\|} \leq 1. \tag{2.2.7}$$

Please see [BQ16, Lem 14.2] for the proof.

**Lemma 2.2.8.** *Let $\delta > 0$. For $g$ in $GL(V)$, if $\beta = \gamma_{1,2}(g) \leq \delta^2$, then*

- *the action of $g$ on $B_{V,g}^m(\delta)$ is $\beta\delta^{-2}$-Lipschitz and*

$$gB_{V,g}^m(\delta) \subset b_{V,g}^M(\beta\delta^{-1}) \subset b_{V,g}^M(\delta),$$

- *the restriction of the real valued function $\sigma_V(g, \cdot)$ on $B_{V,g}^m(\delta)$ is $2\delta^{-1}$-Lipschitz.*

*Proof.* Due to [BQ16, Lem 14.2],

$$d(gx, x_g^M)\delta(x, y_g^m) \leq \gamma_{1,2}(g) = \beta.$$

Hence

$$d(gx, x_g^M) \leq \beta\delta(x, y_g^m)^{-1} \leq \beta\delta^{-1},$$

which implies the inclusion.

For $x = \mathbb{R}v$ and $x' = \mathbb{R}v'$ in $B_{V,g}^m(\delta)$, by (2.2.7), we have

$$d(gx, gx') = \frac{\|gv \wedge gv'\|}{\|v \wedge v'\|} \frac{\|v \wedge v'\|}{\|v\|\|v'\|} \frac{\|v\|\|v'\|}{\|gv\|\|gv'\|} \leq \gamma_{1,2}(g) d(x, x')\delta^{-2},$$

which implies the Lipschitz property of $g$.

For the Lipschitz property of $\sigma_V(g, \cdot)$, please see [BQ16, Lemma 17.11]. □

For two different points $x = \mathbb{R}v$ and $x' = \mathbb{R}v'$ in $\mathbb{P}V$, we write $x \wedge x' = \mathbb{R}(v \wedge v') \in \mathbb{P}(\wedge^2 V)$.

**Lemma 2.2.9.** *For any $g$ in $GL(V)$ and two different points $x = \mathbb{R}v, x' = \mathbb{R}v'$ in $\mathbb{P}V$, we have*

$$\gamma_{1,2}(g)\delta(x \wedge x', y_{\wedge^2 g}^m) \leq \frac{d(gx, gx')}{d(x, x')}. \tag{2.2.8}$$

*Proof.* By definition and (2.2.7), we have

$$d(gx, gx') = \frac{\|gv \wedge gv'\|}{\|v \wedge v'\|} \frac{\|v \wedge v'\|}{\|v\|\|v'\|} \frac{\|v\|\|v'\|}{\|gv\|\|gv'\|} \geq \gamma_{1,2}(g)\delta(x \wedge x', y_{\wedge^2 g}^m) d(x, x').$$

The proof is complete. □

### 2.2.3 Actions on Flag varieties

#### Representations and Density points

Now, suppose that $V$ is a representation of $G$ with a good norm. Recall that $V^\chi$ is the eigenspace of the highest weight. Let $V^*$ be the dual space of $V$. The representation of $G$ on $V^*$ is the dual representation given by: for $g \in G$ and $f \in V^*$, let $\rho^*(g)f = {}^t\rho(g^{-1})f$. This definition gives

$$\langle \rho^*(g)f, \rho(g)v\rangle = \langle {}^t\rho(g^{-1})f, \rho(g)v\rangle = \langle f, v\rangle,$$

for $f$ in $V^*$ and $v$ in $V$. Then the highest weight of $V^*$ is $\iota\chi$. The following results explain the relation between different definitions by using combinatoric information on root systems and representations.

**Lemma 2.2.10.** *We claim that for every irreducible representation $V$ and weight $\chi$,*

$$V_{\chi,\zeta_o} = V^{w_0\chi}. \tag{2.2.9}$$

*Proof.* This can be verified as follows: For $X$ in $\mathfrak{a}$ and $v$ in $V^\chi$,

$$\mathrm{d}\rho(X)\rho(w_0)v = w_0\mathrm{d}\rho(w_0X)v = \chi(w_0X)w_0v = (w_0\chi)(X)w_0v.$$

The proof is complete. □

**Lemma 2.2.11.** *Let $V$ be a proximal representation of $G$. Then we have*

$$x^M_{\rho(g)} = \rho(k_g)V^\chi \text{ and } y^m_{\rho(g)} = {}^t\rho(\ell_g)(V^*)^{-\chi}. \tag{2.2.10}$$

*If $V$ is irreducible, then we have*

$$x^M_{\rho(g)} = V_{\chi,\eta^M_g} \text{ and } y^m_{\rho(g)} = V^*_{\iota\chi,\zeta^m_g}.$$

*Proof.* Let $\{e_1,\dots,e_d\}$ be an orthonormal basis of $V$ composed of eigenvectors of $\rho(A)$ such that $e_1 \in V^\chi$. Then $\rho(A)$ is diagonal. For $g = \exp(X) \in A^+$, since $\chi$ is the highest weight, we have

$$a_1 = \exp(\chi(X)) \geq a_2,\dots,a_d.$$

By the definition of a good norm, $\rho(K)$ preserves the norm. Hence for $g$ in $G$, the formula $\rho(g) = \rho(k_g)\rho(a_g)\rho(\ell_g)$ is a decomposition which satisfies (2.2.5) in the previous paragraph with some permutation of $\{e_2,\dots,e_d\}$. But these permutations do not change the density points. Hence we have $x^M_{\rho(g)} = \mathbb{R}\rho(k_g)e_1 = \rho(k_g)V^\chi$. If $V$ is irreducible we have $x^M_{\rho(g)} = V_{\chi,\eta^M_g}$.

In the dual space, we can verify that $e_1^*$ has weight $-\chi$, which is the lowest weight in weights of $V^*$. By the same argument as in $\mathbb{P}V$, we have

$$y^m_{\rho(g)} = \mathbb{R}\,{}^t\rho(\ell_g)e_1^* = {}^t\rho(\ell_g)(V^*)^{-\chi}.$$

We also have a map from $\mathscr{P}$ to $\mathbb{P}V^*$. Hence by (2.2.9) with representation $V^*$ and weight $\iota\chi$, we know $V^*_{\iota\chi,\zeta_o} = (V^*)^{w_0\iota\chi} = (V^*)^{-\chi}$. For $\zeta = g\zeta_o$ in $\mathscr{P}$, by definition,

$$V^*_{\iota\chi,\zeta} = gV^*_{\iota\chi,\zeta_o} = g(V^*)^{-\chi}. \tag{2.2.11}$$

Since $V$ is irreducible, by (2.2.11) we have $y^m_{\rho(g)} = {}^t\rho(\ell_g)(V^*)^{-\chi} = \rho^*(\ell_g^{-1})(V^*)^{-\chi} = V^*_{\iota\chi,\zeta^m_g}$. □

**Distance on Flag varieties**

For $\alpha$ in $\Pi$, we abbreviate $V_{\chi_\alpha,\eta}, V^*_{\iota\chi_\alpha,\zeta}$ to $V_{\alpha,\eta}, V^*_{\alpha,\zeta}$. For $g$ in $G$, by Lemma 2.2.11, we find $x^M_{\rho_\alpha(g)} = V_{\alpha,\eta^M_g}$ and $y^m_{\rho_\alpha(g)} = V^*_{\alpha,\zeta^m_g}$. For $\eta,\eta'$ in $\mathscr{P}$, let

$$d_\alpha(\eta,\eta') = d(V_{\alpha,\eta}, V_{\alpha,\eta'})$$

be its distance between their images in $\mathbb{P}V_\alpha$. We define a distance on the flag variety. It is the maximal distance induced by projections,

$$d(\eta,\eta') = \max_{\alpha\in\Pi} d(V_{\alpha,\eta}, V_{\alpha,\eta'}). \tag{2.2.12}$$

We have another embedding of the flag variety

$$\mathscr{P} \to \prod_{\alpha\in\Pi} \mathbb{P}(V^*_\alpha).$$

For $\zeta = k\zeta_o \in \mathscr{P}$, by definition, we have $V^*_{\alpha,\zeta} = kV^*_{\alpha,\zeta_o}$. For $\eta \in \mathscr{P}$ and $\zeta \in \mathscr{P}$, we set

$$\delta(\eta,\zeta) = \min_{\alpha\in\Pi} \delta(V_{\alpha,\eta}, V^*_{\alpha,\zeta}).$$

In particular, because the images of $\eta_o,\zeta_o$ in $\mathbb{P}V_\alpha, \mathbb{P}V^*_\alpha$ are $V^{\chi_\alpha}, (V^*)^{-\chi_\alpha}$, we know $\delta(V_{\alpha,\eta_o}, V^*_{\alpha,\zeta_o}) = \delta(V^{\chi_\alpha}, (V^*)^{-\chi_\alpha}) = 1$, and then

$$\delta(\eta_o,\zeta_o) = 1. \tag{2.2.13}$$

We write

$$b^M_{V_\alpha,g}(r) = \{x \in \mathbb{P}V_\alpha | d(x, x^M_{\rho_\alpha(g)}) \le r\},$$
$$B^m_{V_\alpha,g}(r) = \{x \in \mathbb{P}V_\alpha | \delta(x, y^m_{\rho_\alpha(g)}) \ge r\}.$$

They are subsets of $\mathbb{P}V_\alpha$. Write

$$b^M_g(r) = \{\eta \in \mathscr{P} | \forall\alpha \in \Pi,\ V_{\alpha,\eta} \in b^M_{V_\alpha,g}(r)\} = \{\eta \in \mathscr{P} | d(\eta, \eta^M_g) \le r\},$$

$$B^m_g(r) = \{\eta \in \mathscr{P} | \forall\alpha \in \Pi,\ V_{\alpha,\eta} \in B^m_{V_\alpha,g}(r)\} = \{\eta \in \mathscr{P} | \delta(\eta, \zeta^m_g) \ge r\}.$$

They are subsets of $\mathscr{P}$.

### Distance and norms

We need a multidimensional version of the lemmas in Section 2.2.2. Recall that $G = \mathrm{SL}_{m+1}(\mathbb{R})$. They are about the similar quantities on flag varieties. The idea is to use all the fundamental representations $\rho_\alpha$. For an element $X$ in $\mathfrak{a}$, we have

$$\sup_{\alpha\in\Pi} |\chi_\alpha(X)| \leq \|X\| \ll \sup_{\alpha\in\Pi} |\chi_\alpha(X)|. \tag{2.2.14}$$

Using Lemma 2.2.5 and (2.2.14), we deduce the following two lemmas from Lemma 2.2.7 and Lemma 2.2.8

**Lemma 2.2.12.** *For $g$ in $G$ and $\eta$ in $\mathscr{P}$,*

$$\|\sigma(g,\eta) - \kappa(g)\| \ll |\log \delta(\eta, \zeta_g^m)|.$$

For $g$ in $G$ and $\alpha \in \Pi$, by Lemma 2.2.5,

$$\gamma_{1,2}(\rho_\alpha(g)) = \frac{\| \wedge^2 \rho_\alpha(g)\|}{\|\rho_\alpha(g)\|^2} = e^{(2\chi_\alpha - \alpha - 2\chi_\alpha)\kappa(g)} = e^{-\alpha\kappa(g)}.$$

Let

$$\gamma(g) = \sup_{\alpha\in\Pi} e^{-\alpha\kappa(g)}. \tag{2.2.15}$$

We call it the gap of $g$.

**Lemma 2.2.13.** *Let $\delta > 0$. For $g$ in $G$, if $\beta = \gamma(g) = \sup_{\alpha\in\Pi}\exp(-\alpha\kappa(g)) \leq \delta^2$, then*

- *the action of $g$ on $B_g^m(\delta)$ is $\beta\delta^{-2}$-Lipschitz and*

$$gB_g^m(\delta) \subset b_g^M(\beta\delta^{-1}) \subset b_g^M(\delta),$$

- *the restriction of the $\mathfrak{a}$-valued function $\sigma(g,\cdot)$ on $B_g^m(\delta)$ is $O(\delta^{-1})$-Lipschitz.*

These properties tell us that the action of an element $g$ on a large set of the flag variety $\mathscr{P}$ behaves like uniformly contracting map.

We also need to compare the distance on the projective space and the flag variety. Recall the map from $\mathscr{P}$ to $\mathbb{P}V$ defined in (2.2.1).

**Lemma 2.2.14.** *Let $(\rho, V)$ be an irreducible representation of $G$ with highest weight $\chi$. There exists a constant $C > 0$ depending on the chosen norm such that for $\eta, \eta'$ in $\mathscr{P}$,*

$$d(V_{\chi,\eta}, V_{\chi,\eta'}) \leq Cd(\eta, \eta'). \tag{2.2.16}$$

The intuition is that a differentiable map between two compact Riemannian manifolds is Lipschitz. For more details, please see Corollary 2.5.6 in Appendix 2.5.2.

### 2.2.4 Actions on the tangent bundle of the Flag variety

In this section, we will study the action of $G$ on the tangent bundle of $\mathscr{P}$. Recall that $\mathscr{P} \simeq G/P$ is the flag variety and $P = AN$ is a parabolic subgroup.

We first study the tangent bundle of the homogeneous space

$$\mathscr{P}_0 = G/A_eN.$$

Recall that $A_e$ is the analytical connected component of $A$, given by $\exp(\mathfrak{a})$. Note that the left action of $K$ on $\mathscr{P}_0$ is simply transitive (due to the Iwasawa decomposition). Let $z_o$ be the base point $A_eN$ in $\mathscr{P}_0$. We can identify the left $K$-invariant vector fields as

$$T_{z_o}\mathscr{P}_0 = T_{z_o}(G/A_eN) \simeq \mathfrak{g}/\mathfrak{p}.$$

Hence the tangent bundle of $\mathscr{P}_0$ has an isomorphism

$$T\mathscr{P}_0 \simeq \mathscr{P}_0 \times \mathfrak{g}/\mathfrak{p},$$

that is because we can identify the tangent space at $z_o$ and $z = kz_o$ by the left action of $k$. We denote by $(z, Y)$ a point of $T\mathscr{P}_0$ where $z$ is in $\mathscr{P}_0$ and $Y$ is in $\mathfrak{g}/\mathfrak{p}$. We use elements in $\mathfrak{n}^- = \oplus_{\alpha\in R^+}\mathfrak{g}^{-\alpha}$ as representative elements in $\mathfrak{g}/\mathfrak{p}$.

Then we describe the left action of $G$ on $T\mathscr{P}_0$. Take $Y$ in $\mathfrak{g}^{-\alpha}$ and $z = kz_o$ in $\mathscr{P}_0$. For $g$ in $G$, by the Iwasawa decomposition we have a unique $k'$ in $K$ and a unique $\sigma(g, k)$ in $\mathfrak{a}$ such that $gk = k'p \in k'\exp(\sigma(g, k))N$, where $p \in A_eN$. Here $\sigma(g, k)$ is understood as $\sigma(g, k\eta_o)$. Due to

$$gk\exp(tY)z_o = k'p\exp(tY)z_o = k'\exp(t\mathrm{Ad}_pY)z_o,$$

by taking derivative at $t = 0$, the left action of $g$ on the tangent vector $(z, Y)$ satisfies

$$L_g(z, Y) = (z', \mathrm{Ad}_pY),$$

where $z' = k'\langle_o$ and Ad is the adjoint action of $P$ on $\mathfrak{g}/\mathfrak{p}$.

Now we restrict our attention to simple roots. Let $\alpha$ be a simple root. Due to $Y \in \mathfrak{g}^{-\alpha}$, we have $\mathrm{Ad}_NY \subset Y + \mathfrak{a} + \mathfrak{n}$, which implies that the unipotent part $N$ acts trivially on $(\mathfrak{g}^{-\alpha} + \mathfrak{p})/\mathfrak{p}$. By $p \in \exp(\sigma(g, k))N$, we have

$$\mathrm{Ad}_pY = \exp(-\alpha\sigma(g, k))Y \text{ on } (\mathfrak{g}^{-\alpha} + \mathfrak{p})/\mathfrak{p}. \tag{2.2.17}$$

This means that the line bundle $\mathscr{P}_0 \times \mathfrak{g}^{-\alpha}$ is stable under the left action of $G$, and we call it the $\alpha$-bundle.

The flag variety $\mathscr{P}$ is a quotient of $\mathscr{P}_0$ by the right action of group $M$, due to $A = MA_e$. We use $\pi$ to denote the quotient map. The right action of $M$ also induces an action on the tangent bundle. For $(z, Y)$ in $T\mathscr{P}_0$ and $m$ in $M$, by $k\exp(tY)mz_o = km\exp(t\mathrm{Ad}_{m^{-1}}Y)z_o$, we have

$$R_m(kz_o, Y) = (kmz_o, \mathrm{Ad}_{m^{-1}}Y). \tag{2.2.18}$$

Descending to the quotient implies the tangent bundle of $\mathscr{P}$ satisfies

$$T\mathscr{P} \simeq \mathscr{P}_0 \times_M \mathfrak{g}/\mathfrak{p},$$

which is the quotient space of $\mathscr{P}_0 \times \mathfrak{g}/\mathfrak{p}$ by the equivalence relation generated by the action of $M$, (2.2.18). Due to $M < A$, its adjoint action preserves the line $\mathfrak{g}^{-\alpha}$ in $\mathfrak{g}/\mathfrak{p}$. Hence the $\alpha$-bundle on $\mathscr{P}_0$ descends to a line bundle on $\mathscr{P}$, and we call it $\mathscr{P}_\alpha$, a subbundle of the tangent bundle. The integral curves of $\alpha$-bundle on $\mathscr{P}_0$ are closed, and we call them $\alpha$-circles on $\mathscr{P}_0$. At a point $z = kz_o$ in $\mathscr{P}_0$, it is given by

$$\gamma_\alpha : \mathbb{R} \to \mathscr{P}_0,\ t \mapsto k\exp(tK_\alpha)z_o. \tag{2.2.19}$$

This can be verified directly, because the tangent vector of the curve at time $t$ is $(\gamma_\alpha(t), K_\alpha) = (\gamma_\alpha(t), Y_\alpha)$, due to the definition of $\mathfrak{g}/\mathfrak{p}$, which belongs to the $\alpha$-bundle. The one parameter subgroup $\{\exp(tK_\alpha) : t \in \mathbb{R}\}$ is a compact subgroup of $G$, which is isomorphic to $SO(2)$. We call it $O_\alpha$.

Under the right action of $M$, the $\alpha$-circles on $\mathscr{P}_0$ descends to the $\alpha$-circles on $\mathscr{P}$.

**Lemma 2.2.15.** *Under the map (2.2.1), the image of the $\alpha$-circle containing $\eta = k\eta_0$ in $\mathbb{P}V_\alpha$ is the projective line generated by $\rho_\alpha(k)V^{\chi_\alpha}$ and $\rho_\alpha(k)V^{\chi_\alpha - \alpha}$.*

*Let $\alpha'$ be another simple root. The image of an $\alpha$-circle in $\mathbb{P}V_{\alpha'}$ is a point.*

*Proof.* Since $\alpha$-bundle is left $K$-invariant, the set of $\alpha$-circles are also left $K$-invariant. It is sufficient to consider the $\alpha$-circle containing $\eta_0$. Let $(\rho, V)$ be an irreducible representation of highest weight $\chi$. By (2.2.19) and (2.2.1), the image of $\alpha$-circle is given by $\rho(O_\alpha)V^\chi$.

Consider the Lie algebra $\mathfrak{s}_\alpha$ generated by $H_\alpha, X_\alpha, Y_\alpha$, which is isomorphic ot $\mathfrak{sl}_2$. For $v$ in $V^\chi$, we have $\mathrm{d}\rho(H_\alpha)v = \chi(H_\alpha)v$. Due to the classification of the irreducible representation of $\mathfrak{sl}_2$, the irreducible representation $V_1$ of $\mathfrak{s}_\alpha$ generating by $V^\chi$ is of dimension $\chi(H_\alpha) + 1$.

When $\chi = \chi_{\alpha'}$, we have $\chi_{\alpha'}(H_\alpha) = 0$, which implies $V_1$ is a trivial representation and $\rho(O_\alpha)$ acts trivially on $V_1$. Hence the image of the $\alpha$-circle is a point.

When $\chi = \chi_\alpha$, the same argument implies $V_1$ is of dimension 2. Another eigenspace of $V_1$ is $V^{\chi_\alpha - \alpha}$. The group $\rho(O_\alpha)$ acts as $SO(2)$ on $V_1$, which implies the result. □

**Remark 2.2.16.** *If we introduce the partial flag variety $\mathscr{P}_{\Pi - \{\alpha\}}$, then $\alpha$-circle is simply the fiber of the quotient map $\mathscr{P} \to \mathscr{P}_{\Pi - \{\alpha\}}$. This point of view also implies Lemma 2.2.15.*

Generally, the $\alpha$-bundle on $\mathscr{P}$ is non trivial in the sense of line bundle.

**Example 2.2.17.** *Let $G$ be $\mathrm{SL}_3(\mathbb{R})$. Recall that*

$$\mathfrak{a} = \{X = \mathrm{diag}(x_1, x_2, x_3) |\ x_1 + x_2 + x_3 = 0,\ x_1, x_2, x_2 \in \mathbb{R}\},$$

*and $\alpha_1, \alpha_2$ are two simple roots given by $\alpha_1 = \lambda_1 - \lambda_2$ and $\alpha_2 = \lambda_2 - \lambda_3$. The group $M$ is $\{e, \mathrm{diag}(1,-1,-1), \mathrm{diag}(-1,1,-1), \mathrm{diag}(-1,-1,1)\} \simeq (\mathbb{Z}/2\mathbb{Z})^2$. We have*

$$\mathrm{Ad}_{\mathrm{diag}(1,-1,-1)} Y_{\alpha_1} = \mathrm{Ad}_{\mathrm{diag}(1,-1,-1)} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -Y_{\alpha_1}.$$

*In this case the action of $M$ is nontrivial and it is not a normal subgroup of $K = SO(3)$. The $\alpha$-bundle on $\mathscr{P}$ restricted to an $\alpha$-circle is roughly a Möbius band.*

*In this case, $\alpha_1$-circles are given by $\{W_1 \subset W_2\}$, where $W_2$ is a fixed two dimensional subspace of $\mathbb{R}^3$ and $W_1$ varies in one dimensional subspaces of $W_2$. On the contrary, $\alpha_2$-circles are given by $\{W_1 \subset W_2\}$ with $W_1$ fixed and $W_2$ varying in two planes which contain $W_1$. From this description, we can easily see the $G$ invariance of the set of $\alpha$ circles.*

It is better to work on $\mathscr{P}_0$, where the $\alpha$-bundle is trivial. One difficulty is that in the covering space $\mathscr{P}_0$, we need to capture the missing information of group $M$. More precisely, for $h$ in $G$ and $z, z'$ in $\mathscr{P}_0$ if $h\pi(z), h\pi(z')$ are close, we do not know whether $hz, hz'$ are close or not. This will be answered at the end of Section 2.2.5.

**Remark 2.2.18.** *In an abstract language as in [BQ14, Lemma 4.8], we have a principal bundle $M \to \mathscr{P}_0 \to \mathscr{P}$, where the action of $M$ on $\mathscr{P}_0$ is a right action. We also have a left action of a semigroup $\Gamma$ in $G$ on $\mathscr{P}_0$ and $\mathscr{P}$ ($\Gamma$ will be taken as $\Gamma_\mu$ in our case). Suppose that we have a $\Gamma$-minimal set $\Lambda_\Gamma$ in $\mathscr{P}$. The lifting of $\Lambda_\Gamma$ to $\mathscr{P}_0$ has different possibilities. Let $\eta$ be a point in $\Lambda_\Gamma$ and $z = kz_o$ be a lifting in $\mathscr{P}_0$. Let $M_z = \{m \in M | \overline{\Gamma k} m = \overline{\Gamma k}\}$. Then we have a nice equivalence*

$$\{\Gamma - \textit{minimal orbit in } \mathscr{P}_0\} \longleftrightarrow M_z \backslash M.$$

*In particular, if $\Gamma$ is a semigroup of matrices of positive entries, then $M_z = \{e\}$ and $\Gamma$ has the maximal number of minimal orbits in $\mathscr{P}_0$.*

### 2.2.5 The sign group

Recall the notation for Lie groups and Lie algebras. Let $N^-$ be the subgroup with Lie algebra $\mathfrak{n}^-$. We have a Bruhat decomposition of the Lie group $G$, where the main part is given by

$$N^- \times M \times A_e \times N \to G.$$

The image $U$ is a Zariski open subset of $G$ and the map is injective. For elements in $U$, we can define a map m to the group $M$, mapping an element $g$ to the part of $M$ in the Bruhat decomposition.

In order to study the $M$ part, we will use fundamental representations defined in Lemma 2.2.1. This is in the same spirit as the treatment of the sign group $M$ in [Ben05]. Let $v_\alpha$ be a non zero eigenvector with highest weight $\chi_\alpha$ in $V_\alpha$. Let sg be the sign function on $\mathbb{R}$.

**Lemma 2.2.19.** *For $g$ in $U$, we have*

$$\mathrm{sg}\langle v_\alpha, \rho_\alpha(g)v_\alpha\rangle = \chi_\alpha^\sharp(\mathrm{m}(g)),$$

*where $\chi_\alpha^\sharp$ is the corresponding algebraic character on $A$ of the fundamental weight $\chi_\alpha$.*

*Proof.* Since $v_\alpha$ is $N$-invariant and the transpose of $N^-$ is $N$,

$$\langle v_\alpha, \rho_\alpha(N^- M A_e N)v_\alpha\rangle = \langle \rho_\alpha({}^t(N^-))v_\alpha, \rho_\alpha(MA_eN)v_\alpha\rangle = \langle v_\alpha, \rho_\alpha(MA_e)v_\alpha\rangle.$$

The action of $A_e$ does not change the sign, hence by Lemma 2.2.6 we have

$$\mathrm{sg}\langle v_\alpha, \rho_\alpha(g)v_\alpha\rangle = \mathrm{sg}\langle v_\alpha, \rho_\alpha(\mathrm{m}(g))v_\alpha\rangle = \chi_\alpha^\sharp(\mathrm{m}(g)).$$

The proof is complete. □

In the case $G = \mathrm{SL}_{m+1}(\mathbb{R})$, the algebraic character $\chi_{\alpha_i}^\epsilon$ is given by $\chi_{\alpha_i}^\sharp(a) = a_1 \cdots a_i$ for $a = \mathrm{diag}(a_1, \cdots, a_{m+1}) \in A$. Hence we have

**Lemma 2.2.20.** *The function $\Pi_{\alpha\in\Pi}\chi_\alpha^\sharp : M \to \mathbb{R}^m$ given by*

$$\Pi_{\alpha\in\Pi}\chi_\alpha^\sharp(m) = (\chi_\alpha^\sharp(m))_{\alpha\in\Pi} \quad \text{for } m \in M,$$

*is injective.*

**Definition 2.2.21.** *We define the sign function from $G \times G$ to $M \cup \{0\}$ by*

$$\mathrm{m}(g,g') = \begin{cases} \mathrm{m}({}^tgg') & \text{if } {}^tgg' \in U, \\ 0 & \text{if not,} \end{cases}$$

*where $g, g'$ in $G$.*

This definition exploits the relation between $g$ and $g'$. More precisely, for $u, v$ in $V_\alpha$ we have $\langle v,\, \rho_\alpha({}^tgg')u\rangle = \langle \rho_\alpha g v, \rho_\alpha g' u\rangle$, which explains the definition. Due to ${}^tN = N^-$, the sign function m factors through $G/A_eN \times G/A_eN = \mathscr{P}_0 \times \mathscr{P}_0$.

We now explain the sign function for the case $m = 1$, that is $\mathrm{SL}_2(\mathbb{R})$. We only need to consider the representation of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$. Let $v_0 = \begin{pmatrix}1\\0\end{pmatrix}$ be a vector with highest weight in $\mathbb{R}^2$. Then

$$\langle v_0, {}^t gg'v_0\rangle = \langle gv_0, g'v_0\rangle,$$

which is the inner product of the first column of $g$ and $g'$. The sign function is used to determine whether these two vectors $gv_0, g'v_0$ have an acute angle or not.

By the Bruhat decomposition, we have the following two lemmas.

**Lemma 2.2.22.** *For $g, g'$ in $G$ and $m$ in $M$, we have*

$$\mathrm{m}(g, g'm) = \mathrm{m}(gm, g') = \mathrm{m}(g,g')m.$$

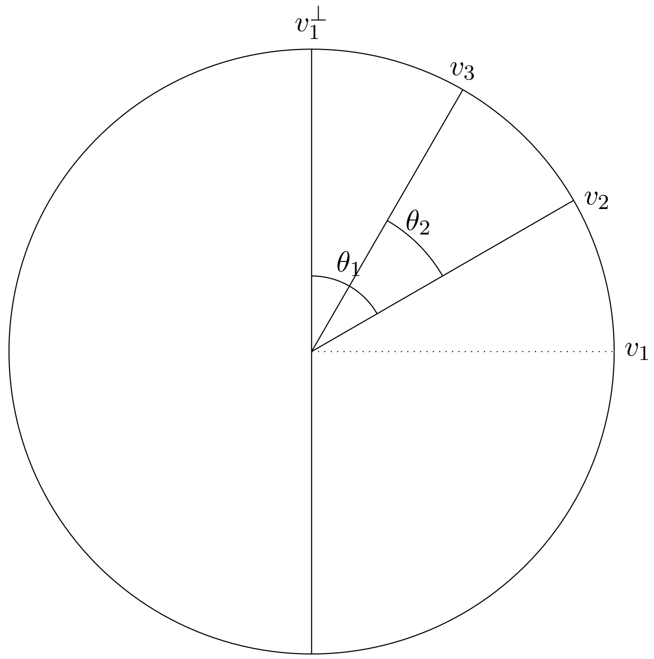


Figure 2.1: Angle

**Lemma 2.2.23.** *Take a Cartan decomposition of $g$, that is $g = k_g a_g \ell_g \in KA^+K$. Then for $h$ in $G$,*

$$\mathrm{m}(k_g, gh) = \mathrm{m}(\ell_g^{-1}, h)$$

The key observation here is that the sign function is locally constant. Recall that $\zeta_o$ is point in $\mathscr{P}$ and its image in $\mathbb{P}V_\alpha^*$ is the linear functional on $V_\alpha$ which vanishes on the hyperplane perpendicular to $V^{\chi_\alpha}$. Recall that $\delta(\eta, \zeta) = \min_{\alpha \in \Pi} \delta(V_{\alpha,\eta}, V_{\alpha,\zeta}^*)$ and $d(\eta, \eta') = \max_{\alpha \in \Pi} d(V_{\alpha,\eta}, V_{\alpha,\eta'})$.

**Lemma 2.2.24.** *For $k_1, k_2, k_3$ in $K$, if $\delta(k_2\eta_o, k_1\zeta_o) > d(k_2\eta_o, k_3\eta_o)$, then*

$$\mathrm{m}(k_1, k_2) = \mathrm{m}(k_1, k_3)\mathrm{m}(k_2, k_3).$$

*Proof.* By definition, we have $\delta(k_2\eta_o, k_1\zeta_o) = \delta({}^tk_1k_2\eta_o, \zeta_o)$ and $\mathrm{m}(k_1, k_2) = \mathrm{m}(id, {}^tk_1k_2)$. Hence, we can suppose that $k_1 = e$, the identity element in $K$. Lemma 2.2.19 and Lemma 2.2.20 imply that it is sufficient to prove that if $\delta(k_2\eta_o, \zeta_o) > d(k_2\eta_0, k_3\eta_0)$ and $\mathrm{m}(k_2, k_3) = e$, then for every simple root $\alpha$, we have

$$\mathrm{sg}\langle v_\alpha, \rho_\alpha(k_2)v_\alpha\rangle = \mathrm{sg}\langle v_\alpha, \rho_\alpha(k_3)v_\alpha\rangle.$$

Fix a simple root $\alpha$ in $\Pi$. Abbreviate $v_\alpha, \rho_\alpha(k_2)v_\alpha, \rho_\alpha(k_3)v_\alpha$ to $v_1, v_2, v_3$. Let $\theta_1$ be the angle between the vector $v_2$ and the hyperplane $v_1^\perp$ and let $\theta_2$ be the angle between

$v_2$ and $v_3$. Due to $\mathrm{m}(k_2,k_3)=e$, this implies

$$0<\langle v_1, {}^tk_2k_3v_1\rangle=\langle k_2v_1,k_3v_1\rangle=\langle v_2,v_3\rangle,$$

the angle $\theta_2$ is acute. The image of $\zeta_0$ in $\mathbb{P}V_\alpha^*$ is given by $\mathbb{R}\langle v_1,\cdot\rangle$. The hypothesis $\delta(k_2\eta_o,\zeta_o)>d(k_2\eta_0,k_3\eta_0)$ implies that

$$\sin\theta_1=\langle v_1,v_2\rangle>\|v_2\wedge v_3\|=\sin\theta_2.$$

Hence $\theta_2<\theta_1$ and $v_2,v_3$ are in the same side of the hyperplane $v_1^\perp$, which implies $\mathrm{sg}\langle v_1,v_2\rangle=\mathrm{sg}\langle v_1,v_3\rangle$. Please see figure 2.1. □

We state a consequence of Lemma 2.2.24 which will be used in Section 2.4.2 to get independence of certain measures $\lambda_j$.

**Lemma 2.2.25.** *Let $\delta<1/2$, let $g,h$ be in $G$ and $k,k'$ in $K$. If $h,k,k'$ satisfy*

$$d(k\eta_o,k'\eta_o)<\delta, k\eta_o,k'\eta_o\in B_h^m(\delta),\eta_h^M\in B_g^m(3\delta) \textit{ and } \gamma(h)<\delta^2,$$

*then*

$$\mathrm{m}(k_g,ghk)=\mathrm{m}(\ell_g^{-1},hk')\mathrm{m}(k,k').$$

*Proof.* By Lemma 2.2.22, it is sufficient to prove the case that $\mathrm{m}(k,k')=e$. By Lemma 2.2.23,

$$\mathrm{m}(k_g,ghk)=\mathrm{m}(\ell_g^{-1},hk). \tag{2.2.20}$$

Denote $k\eta_o,k'\eta_o$ by $\eta,\eta'$. Then by Lemma 2.2.13, we have $h\eta,h\eta'\in b_h^M(\delta)\subset B_g^m(2\delta)$. Hence by $d(h\eta,h\eta')<2\delta\le\delta(h\eta,\zeta_g^m)=\delta(h\eta,\ell_g^{-1}\zeta_o)$ and Lemma 2.2.24, we have

$$\mathrm{m}(\ell_g^{-1},hk)=\mathrm{m}(\ell_g^{-1},hk')\mathrm{m}(hk,hk'). \tag{2.2.21}$$

The main point here is to prove the following lemma.

**Lemma 2.2.26.** *Under the same assumption as in Lemma 2.2.25, we have*

$$\mathrm{m}(hk,hk')=\mathrm{m}(k,k').$$

Combined with (2.2.20) and (2.2.21), the proof is complete. □

*Proof of Lemma 2.2.26.* Without loss of generality, suppose that $\mathrm{m}(k,k')=e$. Due to $k\eta_0\in B_h^m(\delta)$, we can chose a $\ell_h$ in the Cartan decomposition $h=k_ha_h\ell_h$ such that $\mathrm{m}(\ell_h^{-1},k)=e$. By Lemma 2.2.24, the hypothesis that $\delta(k\eta_o,\ell_h^{-1}\zeta_o)>\delta>d(k\eta_o,k'\eta_o)$ implies $\mathrm{m}(\ell_h^{-1},k')=\mathrm{m}(\ell_h^{-1},k)=e$. By Lemma 2.2.23, we conclude that $e=\mathrm{m}(k_h,hk)=\mathrm{m}(\ell_h^{-1},k)=\mathrm{m}(\ell_h^{-1},k')=\mathrm{m}(k_h,hk')$. Here we need a distance $d_0$ on $\mathscr{P}_0$, which is defined in Appendix 2.5.2. Let $z=kz_o$ and $z'=k'z_o$. By Lemma 2.5.5,

$$d_0(hz,hz')\le d_0(hz,z_h)+d_0(z_h,hz')\le d(hk\eta_0,\eta_h^M)+d(\eta_h^M,hk'\eta_0). \tag{2.2.22}$$

Hence by (2.2.22), we have $d_0(hz,hz')\le 2\delta<1$, which implies $\mathrm{m}(hk,hk')=e$ due to Lemma 2.5.5. □

The proof of Lemma 2.2.26 also says that if $z,z'$ are close and away from the subvariety defined by $h$, the gap of $h$ is large, then $hz,hz'$ are also close.

### 2.2.6 Derivative

Let $\varphi$ be a $C^1$ function on $\mathscr{P}_0$. We will give some property of the directional derivative of $\varphi$. We write $\partial_\alpha \varphi$ for the directional derivative $\partial_{Y_\alpha}\varphi$, where $\alpha$ is a simple root. It turns out later that these directions are the major directions when we consider the action of $G$ on $\mathscr{P}_0$.

**Definition 2.2.27** (Arc length)**.** *Let $z_1, z_2$ be two points in the same $\alpha$-circle in $\mathscr{P}_0$. If $\mathrm{m}(z_1, z_2) = e$, we define the arc length distance between $z_1, z_2$ by*

$$d_A(z_1, z_2) := \arcsin d(\pi z_1, \pi z_2).$$

**Remark 2.2.28.** *This is a restriction of left $K$-invariant distance, which can be induced by the $K$-invariant Riemann metric $d_2$ in the appendix.*

**Lemma 2.2.29** (The Newton-Leibniz formula)**.** *Let $z_1, z_2$ be two points in the same $\alpha$-circle on $\mathscr{P}_0$ such that $\mathrm{m}(z_1, z_2) = e$. Let $u = d_A(z_1, z_2)$ and let $\gamma : [0, u] \to \mathscr{P}_0$ be the curve in the $\alpha$-circle connecting $z_1, z_2$ with unit speed (in the sense of arc length). Then for $g$ in $G$*

$$\varphi(gz_1) - \varphi(gz_2) = \pm \int_0^u \partial_\alpha \varphi_{g\gamma(s)} e^{-\alpha\sigma(g,\gamma(s))} \mathrm{d}s, \tag{2.2.23}$$

*where the sign depends on the direction of $\gamma$.*

**Remark 2.2.30.** *The $\alpha$-circle already has an orientation given by $Y_\alpha$. The sign is negative if the curve $\gamma$ is negatively oriented.*

*Proof.* Without loss of generality, suppose that $\gamma$ is positively oriented. Recall that $K_\alpha = Y_\alpha - X_\alpha$ for $\alpha \in \Pi$. The images of $K_\alpha$ and $Y_\alpha$ coincide in $\mathfrak{g}/\mathfrak{p}$. Then $k_2 = k_1 \exp(uK_\alpha)$ and $\gamma(s) = k_1 \exp(sK_\alpha) z_o$ for $s \in [0, u]$. By the Newton-Leibniz formula and (2.2.17) we have

$$\begin{aligned}\varphi(gz_2) - \varphi(gz_1) &= \int_0^u \mathrm{d}\varphi_{g\gamma(s)} \mathrm{d}g_{\gamma(s)} K_\alpha \mathrm{d}s = \int_0^u \mathrm{d}\varphi_{g\gamma(s)} \mathrm{d}g_{\gamma(s)} Y_\alpha \mathrm{d}s \\ &= \int_0^u \mathrm{d}\varphi_{g\gamma(s)} \exp(-\alpha\sigma(g, \gamma(s))) Y_\alpha \mathrm{d}s = \int_0^u \partial_\alpha \varphi_{g\gamma(s)} e^{-\alpha\sigma(g,\gamma(s))} \mathrm{d}s.\end{aligned}$$

The proof is complete. □

Since a root $\alpha$ lies in $\Gamma(G)$, the lattice generated by fundamental weights, there is a corresponding algebraic character $\alpha^\sharp$ of $A$. For $m$ in $M$ and $\alpha$ in $\Pi$, by Lemma 2.2.6 with the adjoint representation of $G$ on $\mathfrak{g}$, due to $Y_\alpha \in \mathfrak{g}^{-\alpha}$, we have $\mathrm{Ad}_m Y_\alpha = (-\alpha)^\sharp(m) Y_\alpha = \alpha^\sharp(m)^{-1} Y_\alpha = \alpha^\sharp(m) Y_\alpha$. The last equality is due to $\alpha^\sharp(m) \in \{\pm 1\}$. Thanks to (2.2.18), we have

**Lemma 2.2.31.** *Let $m$ be in $M$ and let $\varphi$ be a $C^1$ function on $\mathscr{P}_0$ which is right $M$-invariant. We have for $z = kz_o$ in $\mathscr{P}_0$*

$$\partial_\alpha \varphi_{kmz_o} = \alpha^\sharp(m) \partial_\alpha \varphi_z.$$

We say a function $\varphi$ on $\mathscr{P}_0$ is the lift of a function on $\mathbb{P}V_\alpha$, if there exists a function $\varphi_1$ on $\mathbb{P}V_\alpha$ such that for $z = kz_o \in \mathscr{P}_0$

$$\varphi(z) = \varphi_1(V_{\alpha,k\eta_0}).$$

By Lemma 2.2.15, we have

**Lemma 2.2.32.** *If $\varphi$ is a $C^1$ function on $\mathscr{P}_0$, which is the lift of a $C^1$ function on $\mathbb{P}V_\alpha$, then*

$$\partial_{\alpha'}\varphi = 0 \text{ for } \alpha' \neq \alpha, \alpha' \in \Pi.$$

### 2.2.7 Changing Flags

This part is trivial for $\mathrm{SL}_2(\mathbb{R})$, where the flag variety $\mathbb{P}(\mathbb{R}^2)$ is a single $\alpha$-orbit. In this section, we suppose that the rank $m$ is no less than two.

On the flag variety, we have many directions in the tangent space. Roughly speaking, the action of $g$ is contracting and the contraction speed on $Y_\alpha$ is given by $e^{-\alpha\kappa(g)}$, $\alpha \in R^+$. Due to $\kappa(g)$ being in the Weyl chamber $\mathfrak{a}^+$, the slowest directions are given by simple roots. Other directions are negligible. The main result Lemma 2.2.40 is a quantitative version of this intuition.

We have already seen that if two points $\eta, \eta'$ are in the same $\alpha$-circle, then we have a nice formula for the difference of the value of a real function $\varphi$ at $g\eta$ and $g\eta'$, where $g \in G$. We want to do this for $\eta, \eta'$ in general position. For this purpose, we need to change the point according to $g$. This is a key new observation in higher rank.

If we are on the euclidean space $\mathbf{E}^n$ and we are only allowed to move along the directions of coordinate vectors. For any two points $x, x'$, we can walk from $x$ to $x'$ with at most $n$ moves. But this is not true for the flag variety $\mathscr{P}$. Suppose that we are only allowed to move along $\alpha$ circles with $\alpha \in \Pi$. Then for two general points $\eta, \eta'$ in $\mathscr{P}$, it takes more than $m = \#\Pi$ moves to walk from one point to the other point. We try to move in each $\alpha$ circle at most one time and to make the resulting points as close as possible.

Recall that $V$ is a finite dimensional vector space with euclidean norm. Let $l = \mathbb{R}(v_1 \wedge v_2)$ be a point in $\mathbb{P}(\wedge^2 V)$, which is also a line in $\mathbb{P}V$.

**Lemma 2.2.33.** *Let $x = \mathbb{R}w_1$ be a point in $\mathbb{P}V$ and $l = \mathbb{R}(v_1 \wedge v_2)$ be a line in $\mathbb{P}V$. Then we have*

$$d(l, x) := \min_{x' \in l} d(x', x) = \frac{\|v_1 \wedge v_2 \wedge w_1\|}{\|v_1 \wedge v_2\|\|w_1\|}.$$

*Proof.* The geometric meaning of $\|v_1 \wedge v_2 \wedge w_1\|$ is the volume of the parallelepiped generated by three vectors $v_1, v_2, w_1$. This volume can also be calculated as the product of the area of the parallelogram generated by $v_1$ and $v_2$, that is $\|v_1 \wedge v_2\|$, and the distance of $w_1$ to the plane generated by $v_1$ and $v_2$, that is $d(w_1, \mathrm{Span}(v_1, v_2))$. Hence, we have the formula

$$\|v_1 \wedge v_2 \wedge w_1\| = \|v_1 \wedge v_2\| d(w_1, \mathrm{Span}(v_1, v_2)). \tag{2.2.24}$$

The distance $d(w_1, \mathrm{Span}(v_1, v_2))$ equals $\|w_1\| d(l, x)$, because the geometric sense of $d(l, x)$ is the sine of the angle between the vector $w_1$ and the plane $\mathrm{Span}(v_1, v_2)$. Together with (2.2.24), we have the result . $\square$

**Lemma 2.2.34.** *Let $x$ be a point in $\mathbb{P}V$ and $l$ be a line in $\mathbb{P}V$. If $g \in GL(V)$ satisfies that $\delta(x, y_g^m), \delta(l, y_{\wedge^2 g}^m) > \delta$, then*

$$d(gl, gx) \leq \delta^{-2} \gamma_{1,3}(g) d(l, x),$$

*where $\gamma_{1,3}(g) = \frac{\|\wedge^3 g\|}{\|\wedge^2 g\| \|g\|}$.*

Compared with Lemma 2.2.9, with more degree of freedom the contracting speed is significantly greater.

*Proof.* By definition and $l = \mathbb{R}(v_1 \wedge v_2), x = \mathbb{R}w_1$, we have

$$d(gl, gx) = \frac{\| \wedge^2 g(v_1 \wedge v_2) \wedge g w_1 \|}{\| \wedge^2 g(v_1 \wedge v_2)\| \|g w_1\|} \leq \frac{\| \wedge^3 g\| \|v_1 \wedge v_2 \wedge w_1\|}{\| \wedge^2 g(v_1 \wedge v_2)\| \|g w_1\|},$$

Then by Lemma 2.2.7, we have

$$d(gl, gx) \leq \frac{\| \wedge^3 g\| \|v_1 \wedge v_2 \wedge w_1\|}{\delta^2 \| \wedge^2 g\| \|v_1 \wedge v_2\| \|g\| \|w_1\|} = \frac{\| \wedge^3 g\|}{\delta^2 \| \wedge^2 g\| \|g\|} d(l, x).$$

The proof is complete. $\square$

Lemma 2.2.34 can also be understood that there exists a point $x' = \mathbb{R}v' \in l$ such that $v' \wedge w_1$ is orthogonal to the vector of highest weight in $\wedge^2 V$. Then the distance between $gx'$ and $gx$ will be roughly $\gamma_{1,3}(g)$.

We will start to change the flags. Recall that for $\alpha \in \Pi$ and $\eta, \eta'$ in $\mathscr{P}$, the function $d_\alpha(\eta, \eta')$ is the distance between the images of $\eta$ and $\eta'$ in $\mathbb{P}V_\alpha$. If one wants to change a flag in the $\alpha$-circle in $\mathscr{P}$, there are some constraints from the structure of flags. We introduce the following definition which explains the constraint.

**Definition 2.2.35.** *Let*

$$\eta = \{W_1 \subset W_2 \subset \cdots \subset W_{m+1} = \mathbb{R}^{m+1}\}$$

*be a flag in $\mathscr{P}$. Recall that $W_r$ are $r$-dimensional subspaces of $\mathbb{R}^{m+1}$. Let $i_r$ be the natural embedding of the Grassmannian to projective spaces, that is $\mathbb{G}_r(\mathbb{R}^{m+1}) \to \mathbb{P}(\wedge^r \mathbb{R}^{m+1})$. We write*

$$l_{r,\eta} = l_{\alpha_r,\eta} := i_r(W_{r+1} \supset W_r' \supset W_{r-1})$$

*for a line in $\mathbb{P}(\wedge^r \mathbb{R}^{m+1})$, which is the image of all the $r$ dimensional subspace $W_r'$ of $\mathbb{R}^{m+1}$ such that $W_{r-1} \subset W_r' \subset W_{r+1}$. Take $W_0 = \{0\}$. We also write $l_{r,\eta}$ when the line $l_{r,\eta}$ is seen as a point in $\mathbb{P}(\wedge^2(\wedge^r \mathbb{R}^{m+1}))$.*

Recall that $V_{\alpha_r} = \wedge^r V_{\alpha_1} = \wedge^r \mathbb{R}^{m+1}$ and $e_1, \dots, e_{m+1}$ is the standard basis of $\mathbb{R}^{m+1}$.

**Lemma 2.2.36.** *The line $l_{r,\eta}$ is the image of the $\alpha_r$-circle of $\eta$ in $\mathbb{P}V_{\alpha_r}$.*

*Proof.* Due to $l_{r,k\eta_0} = kl_{r,\eta_0}$ and the left $K$ invariance of the set of $\alpha$ circles, it is sufficient to consider $\eta_0$. The line $l_{r,\eta_0}$ is generated by two points $V^{\chi_{\alpha_r}} = \mathbb{R}e_1 \wedge \cdots \wedge e_{r-1} \wedge e_r$ and $V^{\chi_{\alpha_r}-\alpha_r} = \mathbb{R}e_1 \wedge \cdots \wedge e_{r-1} \wedge e_{r+1}$ in $\mathbb{P}V_{\alpha_r}$. By Lemma 2.2.15, this is exactly the image of $\alpha_r$ circle containing $\eta_0$ in $\mathbb{P}V_{\alpha_r}$. □

**Definition 2.2.37.** *Let $(\eta_0, \eta_1, \ldots, \eta_k)$ be a sequence of points in $\mathscr{P}$. We call it a chain if any consecutive elements $\eta_i, \eta_{i+1}$ are in the same $\alpha$-circle for some $\alpha \in \Pi$, and we write $\alpha(\eta_i, \eta_{i+1})$ for this simple root.*

By the structure of the root system of $G$, we have

**Lemma 2.2.38.** *We can separate $\Pi$ into a disjoint union $\Pi_1$ and $\Pi_2$ such that for $\alpha, \alpha'$ in the same atom $\Pi_j$,*

$$\alpha + \alpha' \text{ is not a root.}$$

*Let $l_1 = \#\Pi_1$ and $l_2 = \#\Pi_2$.*

By Lemma 2.2.15

**Lemma 2.2.39.** *Let $(\eta_0, \ldots, \eta_l)$ be a chain and let $\alpha$ be a simple root. If the set of simple roots appearing in the chain does not contain $\alpha$, then the image of the chain in $\mathbb{P}V_\alpha$ is a single point, that is*

$$V_{\alpha,\eta_j} = V_{\alpha,\eta_0},\ \forall j = 1, \ldots, l.$$

Now, we state our main result of this part, which will be used in the main approximation (Proposition 2.4.11).

**Lemma 2.2.40.** *Let $\eta, \eta'$ be two points in $\mathscr{P}$ and let $g$ be in $G$. If for $\alpha \in \Pi_1$,*

$$\delta(V_{\alpha,\eta'}, y^m_{\rho_\alpha(g)}), \delta(l_{\alpha,\eta}, y^m_{\wedge^2\rho_\alpha(g)}) > \delta,$$

*for $\alpha \in \Pi_2$,*

$$\delta(V_{\alpha,\eta}, y^m_{\rho_\alpha(g)}), \delta(l_{\alpha,\eta'}, y^m_{\wedge^2\rho_\alpha(g)}) > \delta.$$

*Then we can find two chains $(\eta = \eta_0, \eta_1, \ldots, \eta_{l_1})$ and $(\eta' = \eta'_0, \eta'_1, \ldots, \eta'_{l_2})$ such that*

$$d(g\eta_j, g\eta_{j+1}) = d_\alpha(g\eta_j, g\eta_{j+1}) = d_\alpha(g\eta, g\eta') + O(\delta^{-2}\beta e^{-\alpha\kappa(g)}), \tag{2.2.25}$$

*where $\alpha = \alpha(\eta_j, \eta_{j+1}) \in \Pi_1$ and different $j$ correspond to different roots; similarly for $\eta'$.*

*We also have that for all $\alpha \in \Pi$*

$$d_\alpha(g\eta_{l_1}, g\eta'_{l_2}) \le \beta e^{-\alpha\kappa(g)}\delta^{-2}, \tag{2.2.26}$$

*where $\beta$ is the gap of $g$, that is $\beta = \gamma(g) = \max_{\alpha\in\Pi}\{e^{-\alpha\kappa(g)}\}$.*

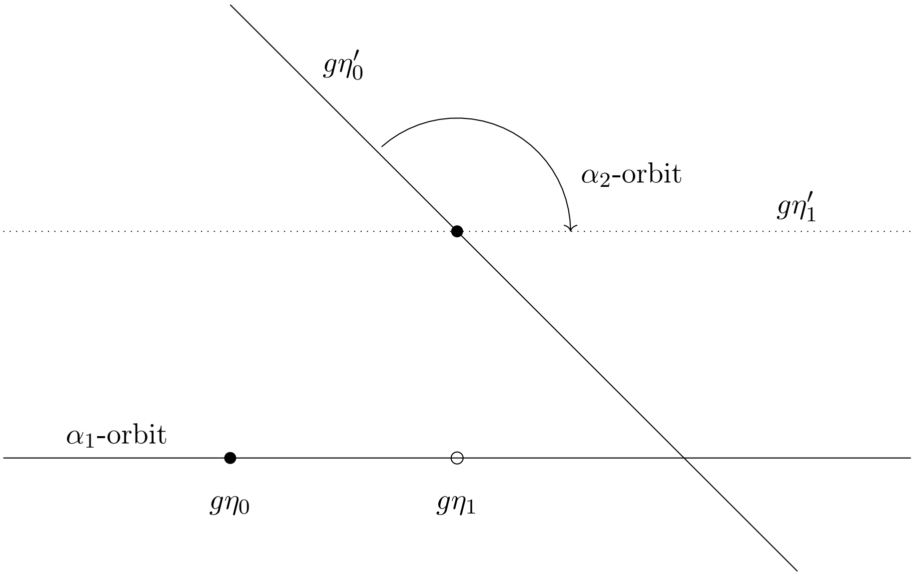


Figure 2.2: Changing Flag for $\mathrm{SL}_3(\mathbb{R})$

The point is that the contraction speed $\beta$ implies that the term $\delta^{-2}\beta e^{-\alpha\kappa(g)}$ is of smaller magnitude than $e^{-\alpha\kappa(g)}$. The objective is to walk from $g\eta$ to $g\eta'$ only through $\alpha$ circles and to preserve information of distance. Since we can neglect error term, it is simpler to walk from $g\eta$ to $g\eta_{l_1}$ through some $\alpha$ circles and to walk from $g\eta'$ to $g\eta'_{l_2}$ through the other $\alpha$ circles, which means the corresponding simple roots are different from the first walk. After this operation, the distance between $g\eta_{l_1}$ and $g\eta'_{l_2}$ is negligible, due to (2.2.26). The distance of the move in the $\alpha$ circle is approximately the distance between the images of $g\eta$ and $g\eta'$ in $\mathbb{P}V_\alpha$, due to (2.2.25).

*Proof of Lemma 2.2.40.* If we have already found $(\eta_0, \dots, \eta_j)$ and $j < l_1$, we want to find $\eta_{j+1}$. Let $\alpha = \alpha_r \in \Pi_1$ be a root that does not appear in the chain. Hence by Lemma 2.2.39,

$$V_{\alpha_r,\eta_j} = V_{\alpha_r,\eta_0} = V_{\alpha_r,\eta}. \tag{2.2.27}$$

Due to Lemma 2.2.38, the neighbour simple roots $\alpha_{r-1}$ and $\alpha_{r+1}$ are not in $\Pi_1$, since $\alpha_{r-1}+\alpha_r$ and $\alpha_{r+1}+\alpha_r$ are roots. By Lemma 2.2.39,

$$V_{\alpha_{r-1},\eta_j} = V_{\alpha_{r-1},\eta} \text{ and } V_{\alpha_{r+1},\eta_j} = V_{\alpha_{r+1},\eta}.$$

We are in the situation of Lemma 2.2.34 with $V = V_{\alpha_r} = \wedge^r\mathbb{R}^{m+1}$, $x = V_{\alpha_r,\eta'}$ and $l = l_{r,\eta}$. Due to the hypothesis, Lemma 2.2.34 and Lemma 2.2.36, we can find $\eta_{j+1}$ in the same $\alpha_r$-circle of $\eta_j$ such that

$$d_{\alpha_r}(g\eta_{j+1}, g\eta') = d(\rho_{\alpha_r} gV_{\alpha_r,\eta_{j+1}}, \rho_{\alpha_r} gV_{\alpha_r,\eta'}) \le \delta^{-2}\gamma_{1,3}(\rho_{\alpha_r} g) \le \delta^{-2}\beta e^{-\alpha_r\kappa(g)}. \tag{2.2.28}$$

Hence by (2.2.27) and (2.2.28),

$$d_{\alpha_r}(g\eta_{j+1}, g\eta_j) = d_{\alpha_r}(g\eta_{j+1}, g\eta) = d_{\alpha_r}(g\eta, g\eta') + O(\delta^{-2}\beta e^{-\alpha_r\kappa(g)}),$$

which is (2.2.25). Please see Figure 2.2, where an element in the flag variety is represented by a projective line with a point.

We need to verify the distance between $g\eta_{l_1}$ and $g\eta'_{l_2}$. Without loss of generality, suppose that $\alpha \in \Pi_1$. Then by Lemma 2.2.39, the construction and (2.2.28),

$$d_\alpha(g\eta_{l_1}, g\eta'_{l_2}) = d_\alpha(g\eta_{l_1}, g\eta') = d_\alpha(g\eta_{j+1}, g\eta') \leq \delta^{-2}\beta e^{-\alpha\kappa(g)},$$

where $j$ is the unique number such that $\alpha(\eta_j, \eta_{j+1}) = \alpha$. □

**Remark 2.2.41.** *In the case of* $\mathrm{SL}_3(\mathbb{R})$*, we know that* $\wedge^2 V_{\alpha_1}$ *and* $\wedge^2 V_{\alpha_2}$ *are isomorphic to* $V_{\alpha_2}$ *and* $V_{\alpha_1}$*, respectively. The condition in Lemma 2.2.40 is equivalent to* $\eta, \eta'$ *in* $B_g^m(\delta)$*.*

*In the case of* $\mathrm{SL}_{m+1}(\mathbb{R})$*, the representations* $V_r = \wedge^r \mathbb{R}^{m+1}$ *are fundamental representation. Since* $\mathrm{SL}_{m+1}(\mathbb{R})$ *is split,* $\wedge^2 V_r$ *is again proximal, but may not be irreducible. In Lemma 2.2.58, we will proceed to give a control on* $y^m_{\wedge^2(\wedge^r g)}$*.*

*The condition of Lemma 2.2.40 is not really important, what we need is that the condition is true with a loss of exponentially small measure when we consider the random walks on* $\mathrm{SL}_{m+1}(\mathbb{R})$*.*

**Lemma 2.2.42.** *With the same assumption and construction in Lemma 2.2.40, if we also have* $\eta, \eta' \in B_g^m(\delta)$*, then* $g\eta_j, g\eta'_l$ *are in* $b_g^M(\beta\delta^{-O(1)})$ *for* $1 \leq j \leq l_1$ *and* $1 \leq l \leq l_2$*.*

*Proof.* By hypothesis, Lemma 2.2.13 implies that $g\eta, g\eta' \in b_g^M(\beta\delta^{-1})$. By (2.2.25),

$$d(g\eta_j, g\eta_{j+1}) \leq 2\beta\delta^{-1} + O(\delta^{-2}\beta e^{-\alpha\kappa(g)}) \leq \beta\delta^{-O(1)}.$$

Hence by induction, we have $g\eta_j \in b_g^M(\beta\delta^{-O(1)})$ for all $j$. Similarly the results hold for $g\eta'_l$. □

### 2.2.8 Random walks and Large deviation principles

The study of random walks on projective spaces and flag varieties are connected by representation theory.

Let $X$ be $\mathscr{P}$ or $\mathbb{P}V$, where $V$ is an irreducible representation of $G$. There is a natural group action of $G$ on $X$. Let $\mu$ be a Borel probability measure on $G$. Then a Borel probability measure $\nu$ on $X$ is called $\mu$-stationary if

$$\nu = \mu * \nu := \int_G g_*\nu \mathrm{d}\mu(g),$$

where $g_*\nu$ is the pushforward measure of $\nu$ under the action of $g$ on $X$.

**Lemma 2.2.43** (Furstenberg)**.** *Let $\mu$ be a Zariski dense Borel probability measure on $G$. There exists a unique $\mu$-stationary probability measure $\nu$ on the flag variety and its images in the projective spaces $\mathbb{P}V$ are the unique $\mu$-stationary probability measures when $V$ is an irreducible representation of $G$.*

See [Fur73], [BQ16, Proposition 10.1] for more details. In order to distinguish stationary measures on different spaces, we use $\nu_V$ to denote a $\mu$-stationary measure on $\mathbb{P}V$.

**Definition 2.2.44.** *Let $\mu$ be a Zariski dense Borel probability measure with exponential moment on $G$. The Lyapunov constant $\sigma_\mu$ is defined as the average of the Iwasawa cocycle*

$$\sigma_\mu := \int_{G\times\mathscr{P}} \sigma(g,\eta)\mathrm{d}\mu(g)\mathrm{d}\nu(\eta).$$

**Lemma 2.2.45.** *Let $\mu$ be a Zariski dense Borel probability measure with exponential moment on $G$. Then the Lyapunov constant $\sigma_\mu$ is in $\mathfrak{a}^{++}$, the interior of the Weyl chamber. Equivalently, for any simple root $\alpha$, we have $\alpha(\sigma_\mu) > 0$.*

The maximal positivity of Lyapunov constant in Lemma 2.2.45 is due to Guivarc'h-Raugi [GR85] and Goldsheid-Margulis [GM89]. See [BQ16, Corollary 10.15] for more details. Lemma 2.2.45 will be used to show that the action of $G$ on $\mathscr{P}$ is contracting in Section 2.4.2, where the contraction speed is give by $\beta = \sup_{\alpha\in\Pi}\{e^{-\alpha\sigma_\mu}\}$.

In following proposition, we give the large deviation principle for the Cartan projection. We keep the assumption that $\mu$ **is a Zariski dense Borel probability measure on $G$ with a finite exponential moment**.

**Proposition 2.2.46.** *For every $\epsilon > 0$ there exist $C, c > 0$ such that for all $n \in \mathbb{N}$ and $\eta \in \mathscr{P}$ we have*

$$\mu^{*n}\{g \in G|\ \|\kappa(g) - n\sigma_\mu\| \geq n\epsilon\} \leq Ce^{-c\epsilon n}, \tag{2.2.29}$$

See [BQ16, Thm 13.17] for more details.

**Proposition 2.2.47.** *If $(\rho, V)$ is an irreducible representation of $G$, then for every $\epsilon > 0$ there exist $C, c$ such that for all $x$ in $\mathbb{P}V$ and $y$ in $\mathbb{P}V^*$ and $n \geq 1$ we have*

$$\begin{aligned} \mu^{*n}\{g \in G|\ \delta(x, y_g^m) \leq e^{-n\epsilon}\} \leq Ce^{-c\epsilon n}, \\ \mu^{*n}\{g \in G|\ \delta(x_g^M, y) \leq e^{-n\epsilon}\} \leq Ce^{-c\epsilon n}. \end{aligned} \tag{2.2.30}$$

See [BQ16, Prop 14.3] for more details. Attention, we need $\rho$ to be proximal in Proposition 2.2.47. Here the representation is automatically proximal due to the splitness of $G$.

**Proposition 2.2.48.** *For every $\epsilon > 0$ there exist $C, c$ such that for all $\eta$, $\eta'$ in $\mathscr{P}$ and $n \geq 1$ we have*

$$\mu^{*n}\{g \in G|\ \delta(\eta_g^M, \zeta) \leq e^{-n\epsilon}\} \leq Ce^{-c\epsilon n}, \tag{2.2.31}$$

$$\mu^{*n}\{g \in G|\ \delta(\eta, \zeta_g^m) \leq e^{-n\epsilon}\} \leq Ce^{-c\epsilon n}, \tag{2.2.32}$$

Proposition 2.2.48 is a multidimensional version of Proposition 2.2.47.

**Proposition 2.2.49** (Hölder regularity). *If $(\rho, V)$ is an irreducible representation of $G$, then there exist constants $C > 0$, $c > 0$ such that for every $y$ in $\mathbb{P}V^*$ and $r > 0$ we have*

$$\nu_V(\{x \in \mathbb{P}V|\ \delta(x, y) \leq r\}) \leq Cr^c. \tag{2.2.33}$$

The proximality of the representation is also needed in Proposition 2.2.49. This result is due to Guivarc'h [Gui90]. See [BQ16, Thm 14.1] for more details. As a corollary of Proposition 2.2.49, we have the following.

**Corollary 2.2.50.** *If $(\rho, V)$ is an irreducible representation of $G$ with highest weight $\chi$, then there exist constants $C > 0$, $c > 0$ such that for every $y$ in $\mathbb{P}V^*$ and $r > 0$ we have*

$$\nu(\{\eta \in \mathscr{P}|\ \delta(V_{\chi,\eta}, y) \leq r\}) \leq Cr^c. \tag{2.2.34}$$

*Proof.* By Lemma 2.2.43, we have

$$\nu(\{\eta \in \mathscr{P}|\ \delta(V_{\chi,\eta}, y) \leq r\}) = \nu_V(\{x \in \mathbb{P}V|\ \delta(x, y) \leq r\}).$$

Hence Corollary 2.2.50 follows from Proposition 2.2.49. □

All the results in this section mean that the quantities considered here are really flexible. We can always image that things happen as wished in a large probability, a very positive expectation. Bad things are near some algebraic subvariety and have exponential small measures. For later convenience, we introduce the following definition.

**Definition 2.2.51** (Good element). *For $n \in \mathbb{N}, \epsilon > 0$ and $\eta, \zeta \in \mathscr{P}$, we say that an element $h$ is $(n, \epsilon, \eta, \zeta)$ good if*

$$\|\kappa(h) - n\sigma_\mu\| \leq \epsilon n/C_A \text{ and } \delta(\eta, \zeta_h^m), \delta(\eta_h^M, \zeta) > 2e^{-\epsilon n/C_A}, \tag{2.2.35}$$

*where $C_A$ is a constant greater than 2, which is only depend on the whole group and will be determined in Lemma 2.2.53.*

**Lemma 2.2.52.** *We have that $h$ is $(n, \epsilon, \eta, \zeta)$ good outside an exponentially small set, that is to say there exist $C > 0, c > 0$ such that*

$$\mu^{*n}\{h \text{ is not } (n, \epsilon, \eta, \zeta) \text{ good.}\} \leq Ce^{-c\epsilon n}.$$

*Proof.* This is due to the large deviation principle (2.2.29), (2.2.31) and (2.2.32). □

**Lemma 2.2.53.** *Let $\delta = e^{-\epsilon n}$ and $\beta = \max_{\alpha \in \Pi} e^{-\alpha\sigma_\mu n}$. Suppose that $\epsilon$ is small enough such that $\beta < \delta^3$. If $h$ is $(n, \epsilon, \eta, \zeta_g^m)$ good, then*

$$\gamma(h) \leq \beta\delta^{-1} \leq \delta^2 \text{ and } \|\sigma(gh, \eta) - \kappa(g) - n\sigma_\mu\| \leq \epsilon n.$$

*Proof.* By hypothesis,

$$\gamma(h) = \max_{\alpha\in\Pi} e^{-\alpha\kappa(h)} = \sup_{\alpha\in\Pi} e^{-\alpha n\sigma_\mu} e^{\alpha(n\sigma_\mu - \kappa(h))} \leq \beta\delta^{-1},$$

if we take $C_A$ large enough such that for all simple roots $\alpha$ and $X$ in $\mathfrak{a}$, we have $|\alpha(X)| \leq C_A\|X\|$.

By Lemma 2.2.13, we have $h\eta \in b_h^M(\gamma(h)/\delta) \subset b_h^M(\delta) \subset B_g^m(\delta)$. Hence by Lemma 2.2.12

$$\begin{aligned}\|\sigma(gh,\eta) - \kappa(g) - n\sigma_\mu\| &= \|\sigma(g,h\eta) - \kappa(g) + \sigma(h,\eta) - n\sigma_\mu\| \\ &\ll |\log\delta(h\eta,\zeta_g^m)| + |\log\delta(\eta,\zeta_h^m)| + \|\kappa(h) - n\sigma_\mu\| \ll \epsilon n/C_A.\end{aligned}$$

Hence if $C_A$ is large enough depending on the whole group, the inequality holds. □

For later usage in Section 2.3, we will define another notation of goodness.

**Definition 2.2.54.** *For $n\in\mathbb{N}, \epsilon > 0$ and $\zeta\in\mathscr{P}$, we say that an element $h$ is $(n,\epsilon,\zeta)$ good if*

$$\|\kappa(h) - n\sigma_\mu\| \leq \epsilon n/C_A \text{ and } \delta(\eta_h^M,\zeta) > 2e^{-\epsilon n/C_A}. \tag{2.2.36}$$

**Lemma 2.2.55.** *Let $\delta = e^{-\epsilon n}$ and $\beta = \max_{\alpha\in\Pi} e^{-\alpha\sigma_\mu n}$. Let $\eta_d$ be a flag which is different from $\eta_o$ only in $d$-dimensional subspace, that is*

$$\eta_d = \{\mathbb{R}e_1 \subset \cdots \subset \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{d-1} \subset \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{d-1} \oplus \mathbb{R}e_{d+1} \subset \cdots\}. \tag{2.2.37}$$

*If $h$ is $(n,\epsilon,\zeta_g^m)$ good, then for $\eta = l_h^{-1}\eta_d$, we have*

$$e^{\chi_j(\sigma(gh,\eta)-\kappa(g)-n\sigma_\mu)} \in [\delta,\delta^{-1}] \text{ for } j\neq d \text{ and } e^{\chi_d(\sigma(gh,\eta)-\kappa(g)-n\sigma_\mu)} \leq \beta\delta^{-1}. \tag{2.2.38}$$

*Proof.* Without loss of generality, we can suppose that $l_h = e$. The image of $\eta_d$ in $\mathbb{P}V_j$ is the same as $\eta_o$ if $j\neq d$. Hence by (2.2.2), we have $\chi_j\sigma(gh,\eta_d) = \chi_j\sigma(gh,\eta_o)$ for $j\neq d$. By (2.2.13), that is $\delta(\eta_o,\zeta_o) = 1$, the element $h$ is $(n,\epsilon,\eta_o,\zeta_g^m)$ good. By Lemma 2.2.53, we obtain the first part of (2.2.38).

The image of $\eta_d$ in $\mathbb{P}V_d$ is $\mathbb{R}v = \mathbb{R}(e_1\wedge\cdots e_{d-1}\wedge e_{d+1})$, whose weight is $\chi_d - \alpha_d$. Hence by (2.2.2),

$$\chi_d\sigma(h,\eta_d) = \log\frac{\|hv\|}{\|v\|} = \log\frac{\|\exp(\kappa(h))v\|}{\|v\|} = (\chi_d - \alpha_d)\kappa(h). \tag{2.2.39}$$

By (2.2.2) and (2.2.3), we have $\chi_d(\sigma(g,h\eta) - \kappa(g)) \leq 0$. Together with (2.2.39),

$$e^{\chi_d(\sigma(gh,\eta)-\kappa(g)-n\sigma_\mu)} = e^{\chi_d(\sigma(g,h\eta)-\kappa(g))}e^{\chi_d(\sigma(h,\eta)-n\sigma_\mu)} \leq e^{(\chi_d-\alpha_d)\kappa(h)-n\chi_d\sigma_\mu} = e^{-\alpha_d\kappa(h)}.$$

By (2.2.36), the proof is complete. □

This Lemma tells us that by changing the image of $\eta$ in one projective space, the value of Iwasawa cocycle only changes in that space. There is some independence of the value of Iwasawa cocycle with respect to $\eta$.

Let $V$ be a representation of $G$. Let $\mathbb{G}_2(V) := \{2\text{-planes in } V\}$ be the Grassmannian variety of $V$. Let $q_\lambda : \wedge^2 V \to \wedge^2 V$ be the $G$-equivalent projection on the sum of all the irreducible subrepresentations of $\wedge^2 V$ with highest weight equal to $\lambda$.

**Lemma 2.2.56.** *Let $V$ be an irreducible representation of $G$ with highest weight $\chi$. For a simple root $\alpha$, let $q_{2\chi-\alpha}$ be the $G$-equivalent projection from $\wedge^2 V$ to $\wedge^2 V$. There exists $c > 0$ such that for all $v, v'$ in $V$*

$$\sum_{\alpha\in\Pi} \|q_{2\chi-\alpha}(v \wedge v')\| \geq c\|v \wedge v'\|.$$

*Proof.* By Lemma 2.2.57, we know that $\frac{\sum_{\alpha\in\Pi}\|q_{2\chi-\alpha}(v\wedge v')\|}{\|v\wedge v'\|} : \mathbb{G}_2(V) \to \mathbb{R}_{\geq 0}$ is a positive continuous function. Since $\mathbb{G}_2(V)$ is a compact space, on which positive continuous function has a lower bound, the result follows. □

The following lemma is similar to [BQ12, Lemma 3.3].

**Lemma 2.2.57.** *With the same assumption as in Lemma 2.2.56, then $\bigcap_{\alpha\in\Pi} q_{2\chi-\alpha}$ does not contain any pure wedge.*

*Proof.* Let $W'$ be the intersection of all the kernels, that is $W' = \bigcap_{\alpha\in\Pi} q_{2\chi-\alpha}$. The two sets $\mathbb{G}_2(V)$ and $\mathbb{P}W'$ are closed subset of $\mathbb{P}(\wedge^2 V)$ and $G$ invariant. Therefore their intersection is again a $G$ invariant closed subvariety which is complete. Let $B$ be the Borel subgroup of $G$, which is solvable. By [Bor90, Thm.10.4], the action of a solvable algebraic connected group on a complete variety has fixed points. We claim that the fixed points of $B$ on $\mathbb{G}_2(V)$ are the lines with the highest weight. Then the result follows by the fact that these lines do not belong to $W'$.

Suppose that there exit $v, u$ in $V$ such that $v \wedge u$ is $B$ invariant. We can decompose $v, u$ as a sum $v = \sum_\lambda v_\lambda$ and $u = \sum_\lambda u_\lambda$. Since we can replace $v, u$ by $bv, bu$ for $b$ in $B$, we can suppose that the component of highest weight $v_\chi$ is non zero. Since the dimension of $V^\chi$ is 1, we can suppose that $u_\chi = 0$. Let $\rho \neq \chi$ be a highest weight such that $u_\rho$ is nonzero. The $B$ invariance of $\mathbb{R}(v \wedge u)$ also implies that the action of $X_\alpha$, for $\alpha$ simple roots, fixes the line. Hence $X_\alpha(v\wedge u) = X_\alpha v \wedge u + v \wedge X_\alpha u \in \mathbb{R}v \wedge u$. The weight $\chi+\rho+\alpha$ is higher than all the weights appear in $v \wedge u$, hence $v_\chi \wedge X_\alpha u_\rho = 0$ for all simple roots $\alpha$. This implies that $\rho = \chi - \alpha$ for some simple root $\alpha$. Therefore $v \wedge u$ contains $v_\chi \wedge u_{\chi-\alpha}$. Since $v \wedge u$ is also $A$ invariant, all the components in the weight decomposition have the same weight. Hence $v \wedge u = v_\chi \wedge u_{\chi-\alpha}$ which is a vector of highest weight in $\wedge^2 V$. □

We want to prove a large deviation principle for a special reducible representation. This lemma will be used in Lemma 2.4.10 to control $y^m_{\wedge^2 g}$ in Lemma 2.2.9 and Lemma 2.2.40.

**Lemma 2.2.58.** *Let $V$ be a super proximal representation of $G$ (Definition 2.2.2). For $\epsilon > 0$ there exist $C, c > 0$ such that the following holds. For $x = \mathbb{R}v, x' = \mathbb{R}v' \in \mathbb{P}V$ with $x \neq x'$, we have*

$$\mu^{*n}\{g \in G | \delta(x \wedge x', y^m_{\wedge^2 \rho(g)}) < e^{-\epsilon n}\} \leq Ce^{-c\epsilon n}.$$

Due to Definition 2.2.2, there is only one simple root $\alpha$ such that $q_{2\chi-\alpha}(\wedge^2 V)$ is non zero. Write $\wedge^2 V = W \oplus W'$, where $W$ is the irreducible representation generated by the vector corresponding to the highest weight in $\wedge^2 V$, and $W'$ is the $G-$invariant complementary subspace. Then $q_{2\chi-\alpha}(\wedge^2 V) = W$, and we write $Pr_W = q_{2\chi-\alpha}$.

*Proof of Lemma 2.2.58.* By (2.2.10), we see that a non zero vector in $y^m_{\wedge^2 g}$ vanishes on $W'$ and $y^m_{\wedge^2 g}$ can be seen as an element in $\mathbb{P}W^*$. We only need to consider the projection of $v \wedge v'$ onto $W$ and use large deviation principle (2.2.30). By Lemma 2.2.56,

$$\delta(x \wedge x', y^m_{\wedge^2 g}) = \frac{|f(v \wedge v')|}{\|v \wedge v'\|} = \frac{|f(Pr_W(v \wedge v'))|}{\|Pr_W(v \wedge v')\|}\frac{\|Pr_W(v \wedge v')\|}{\|v \wedge v'\|} \geq c\delta(Pr_W(x \wedge x'), y^m_{\wedge^2 g}),$$

where $f$ is a unit vector in $y^m_{\wedge^2 g}$. The proof is complete. $\square$

## 2.3 Non concentration condition

We want to verify the main input for the sum-product estimate, the non concentration condition. If we want to get the non concentration directly, then this becomes an effective local limit estimate, which is difficult due to the lack of spectral gap. Hence, we transfer it to the Hölder regularity of stationary measure.

For the first time read, the reader can neglect $g$ in the left of $h$. The main idea of the proof is already there. Adding $g$ is a technical step, which is needed in its application. (We only need an additional condition on $\eta_h^M$ to control $\kappa(gh)$.)

### 2.3.1 Projective, Weak and Strong non concentration

Recall that $m$ is the real rank of $G$ and $\chi_1, \cdots, \chi_m$ are the fundamental weights, where we change the subscript from $\alpha \in \Pi$ to $i \in \{1, \cdots, m\}$. Recall that $\alpha_1, \cdots, \alpha_m$ are the simple roots of $\mathfrak{a}^*$, which are linear combinations of fundamental weights $\chi_i$ with integral coefficients.

In order to distinguish different objects, we will use capital letter $X$ to denote functions or random variables and use small letter $x$ to denote vectors or indeterminates.

Let $L$ be the $d \times d$ square matrix which changes the basis $(\chi_1, \cdots, \chi_m)$ of $\mathfrak{a}^*$ to the basis $(-\alpha_1, \cdots, -\alpha_m)$, that is $L_{ij} = -\alpha_i(H_j)$. Then $L$ is an integer matrix. Hence, we can define $E_d$, a rational map from $(\mathbb{R}^*)^m$ to $(\mathbb{R}^*)^d$, which is given by $y = E_d(x)$ for $x \in (\mathbb{R}^*)^m$ where

$$y_i = \Pi_{1 \leq j \leq m} x_j^{L_{ij}}.$$

Fix an element $g$ in $G$. Let

$$X_g(n,h,\eta) = (e^{\chi_1(\sigma(gh,\eta)-\kappa(g)-n\sigma_\mu)}, \dots, e^{\chi_m(\sigma(gh,\eta)-\kappa(g)-n\sigma_\mu)}),$$
$$Y_g^n(h,\eta) = (e^{-\alpha_1(\sigma(gh,\eta)-\kappa(g)-n\sigma_\mu)}, \dots, e^{-\alpha_m(\sigma(gh,\eta)-\kappa(g)-n\sigma_\mu)})$$

for $\eta$ in $\mathscr{P}$ and $h$ in $G$. By definition, $E_d X_g(n,h,\eta)$ is the vector which is composed of the first $d$ components of $Y_g^n(h,\eta)$, that is

$$p_d Y_g^n(h,\eta) = E_d X_g(n,h,\eta), \tag{2.3.1}$$

where $p_d : \mathbb{R}^m \to \mathbb{R}^d$ is the map which takes a vector $x$ of $\mathbb{R}^m$ to the vector of $\mathbb{R}^d$ composed of the first $d$ components of $x$. In the following argument $g$ is fixed or $g$ equals identity. Hence we will abbreviate $X_g, Y_g^n, Y_e^n$ to $X, Y^n, Y_0^n$.

We define an affine determinant $A_d$ on $(\mathbb{R}^d)^{d+1}$. For $d+1$ vectors $y^1, \cdots, y^{d+1}$ in $\mathbb{R}^d$, let $A_d$ be the determinant of the $(d+1)\times(d+1)$ matrix $\begin{pmatrix} y^1 & \cdots & y^{d+1} \\ 1 & \cdots & 1 \end{pmatrix}$, which is the volume of the $d+1$-dimensional parallelogram generated by vectors $(y^i, 1)$ for $i = 1, \dots, d+1$. Let $e_i$ be the vector in $\mathbb{R}^d$ with only i-th coordinate nonzero and equal to 1. By identifying $e_1 \wedge \cdots \wedge e_d$ with number 1, we can also define $A_d$ by

$$A_d(y^1, \cdots, y^{d+1}) = \sum_{1\le i\le d+1} (-1)^{i+d+1} y^1 \wedge \cdots \wedge \widehat{y^i} \wedge \cdots \wedge y^{d+1}.$$

For $d+1$ vectors $x^1, \cdots, x^{d+1}$ in $\mathbb{R}^m$, let $B_d$ be a rational function defined by

$$B_d(x^1, \cdots, x^{d+1}) = A_d(E_d x^1, \cdots, E_d x^{d+1}).$$

We introduce the notation

$$\mathbf{h}_{d+1} = (h_1, \dots, h_{d+1}),$$

which is an element in $G^{\times(d+1)}$. Let

$$A_d^n(\mathbf{h}_{d+1}, \eta) := B_d(X(n,h_1,\eta), \dots, X(n,h_{d+1},\eta)).$$

**Definition 2.3.1.** *We say that $\mu$ satisfies the projective non concentration (PNC) on dimension $d$, if for every $\epsilon > 0$ there exist $c, C > 0$ such that for all $n$ in $\mathbb{N}$, $\eta$ in $\mathscr{P}$ and $g$ in $G$*

$$\sup_{a\in\mathbb{R}, v\in\mathbb{S}^{d-1}} \mu^{*n}\{h \in G || \langle v, Y^n(h,\eta)\rangle - a| \le e^{-\epsilon n}\} \le C e^{-c\epsilon n},$$

*where $v$ is regarded as a vector in $\mathbb{R}^d \times \{0\}^{m-d} \subset \mathbb{R}^m$.*

More geometrically, this is equivalent to say that the measure of $Y^n(h,\eta)$ close to an affine hyperplane is exponentially small.

**Definition 2.3.2.** *We say that $\mu$ satisfies the weak non concentration (WNC) on dimension $d$, if for every $\epsilon > 0$ there exist $c, C > 0$ such that for all $n$ in $\mathbb{N}$, $\eta$ in $\mathscr{P}$ and $g$ in $G$*

$$(\mu^{*n})^{\otimes(d+2)}\{(\mathbf{h}_{d+1}, \ell) \in G^{\times(d+2)} | |A_d^n(\mathbf{h}_{d+1}, \ell\eta)| \leq e^{-\epsilon n}\} \leq Ce^{-c\epsilon n}.$$

**Definition 2.3.3.** *We say that $\mu$ satisfies the strong non concentration (SNC) on dimension $d$, if for every $\epsilon > 0$ there exist $c, C > 0$ such that for all $n$ in $\mathbb{N}$, $\eta$ in $\mathscr{P}$ and $g$ in $G$*

$$(\mu^{*n})^{\otimes(d+1)}\{\mathbf{h}_{d+1} \in G^{\times(d+1)} | |A_d^n(\mathbf{h}_{d+1}, \eta)| \leq e^{-\epsilon n}\} \leq Ce^{-c\epsilon n}.$$

We will proceed by induction. When $d = 0$, we make the convention that $A_0^d = 1$ and it is trivial that SNC holds. Then

- SNC on dimension $d \Rightarrow$ WNC on dimension $d$ (By definition)
- PNC on dimension $d \Leftrightarrow$ SNC on dimension $d$ (Lemma 2.3.7)
- WNC on dimension $d \Rightarrow$ PNC on dimension $d$ (Lemma 2.3.9)
- SNC on dimension $d - 1 \Rightarrow$ WNC on dimension $d$ (Lemma 2.3.10).

In the above implications, the constants $C, c$ will change. We can conclude

**Proposition 2.3.4.** *Let $\mu$ be a Zariski dense Borel probability measure on $G$ with exponential moment. Then $\mu$ satisfies PNC on dimension $m$.*

### 2.3.2 Away from affine hyperplanes

We need a lemma of linear algebra, which relates different non concentrations. This lemma is already known from [EMO05, Lemma 7.5]. Recall that for two subsets $A, B$ of a metric space $(X, d)$, the distance between $A$ and $B$ is defined as

$$d(A, B) = \inf_{x \in A, y \in B} d(x, y)$$

**Lemma 2.3.5.** *Let $C > 0, c > 0$. Let $u_1, \cdots, u_{d+1}$ be vectors in $\mathbb{R}^d$ with length less than $C$. Consider the following conditions:*

*i. There exists an affine hyperplane $l$ such that for $i = 1, \ldots, d+1$,*

$$d(u_i, l) \leq c.$$

*ii. We have*

$$\left\| \sum_{1 \leq i \leq d+1} (-1)^i u_1 \wedge \cdots \wedge \widehat{u_i} \wedge \cdots \wedge u_{d+1} \right\| < c,$$

*where $\widehat{u_i}$ means this term is not in the wedge product.*

iii. *There exists $i$ in $\{1,\dots,d\}$ such that*

$$d(u_i, \mathrm{Span}_{\mathrm{aff}}(u_{d+1}, u_1, \dots, u_{i-1})) < c,$$

*where* $\mathrm{Span}_{\mathrm{aff}}$ *is the affine subspace generated by the elements in the bracket.*

*Then $i(c) \Rightarrow ii(2^{d+1}C^{d-1}c)$, $ii(c) \Rightarrow iii(c^{1/d})$ and $iii(c) \Rightarrow i(c)$.*

*Proof.* We first transfer the affine problem to a linear problem. Let $v_i = u_i - u_{d+1}$ for $i = 1,\dots,d$. Then $v_i$ are vectors with length less than $2C$. The above three conditions are equivalent to (with change of constants in $i$)

i'. There exists a linear subspace $l$ of codimension 1 such that for $i = 1,\dots,d$

$$d(v_i, l) \le c.$$

ii'. We have

$$\|v_1 \wedge \cdots \wedge v_d\| < c.$$

iii'. There exists $i$ such that

$$d(v_i, \mathrm{Span}(v_1, \dots, v_{i-1})) < c,$$

where Span is the linear subspace generated by the elements in the bracket.

$iii'(c) \Rightarrow i'(c)$: Let the hyperplane $l$ be $\mathrm{Span}(v_1, \cdots, \hat{v}_i, \cdots, v_d)$. Then $i'(c)$ follows from $iii'(c)$.

$i'(c) \Rightarrow ii'(2^dC^{d-1}c)$: Due to $i'$, the volume of the parallelogram generated by $\{v_i\}_{1\le i\le d}$ is less than $(2C)^{d-1}2c$, which is $ii'$.

$ii'(c) \Rightarrow iii'(c^{1/d})$: Due to the same argument as in Lemma 2.2.33, we have a formula of volume,

$$\|v_1 \wedge \cdots \wedge v_d\| = \Pi_{1\le i\le d} d(v_i, \mathrm{Span}(v_1, \dots, v_{i-1})),$$

from which the result follows. □

As a corollary, we have the following lemma, which is general and deals with random variables.

**Corollary 2.3.6.** *Let $X_1, \dots, X_{d+1}$ be i.i.d. random vectors in $\mathbb{R}^d$ bounded by $C > 0$. Let $l$ be an affine hyperplane in $\mathbb{R}^d$. Then for any $c > 0$, we have*

$$\mathbb{P}\{d(X_1, l) < c\}^{d+1} \le \mathbb{P}\{\| \sum (-1)^i X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_{d+1}\| < 2^{d+1}C^{d-1}c\}, \tag{2.3.2}$$

*and*

$$\begin{aligned} &\mathbb{P}\{\| \sum (-1)^i X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_{d+1}\| < c\} \\ &\le \sum_{1\le i\le d} \mathbb{P}\{d(X_i, \mathrm{Span}_{\mathrm{aff}}(X_{d+1}, X_1, \cdots, X_{i-1})) < c^{1/d}\}. \end{aligned} \tag{2.3.3}$$

**Lemma 2.3.7.** *PNC on dimension $d$ is equivalent to SNC on dimension $d$.*

*Proof.* Let $X_i = E_d X(n, h_i, \eta)$ for $i = 1, \cdots, d+1$, where $h_i$ has distribution $\mu^{*n}$. Due to Lemma 2.2.53, with a loss of exponentially small measure, we can suppose that $X_i$ are bounded by $C = e^{\epsilon_2 n}$, where $\epsilon_2 = \epsilon/(2d)$.

Due to (2.3.1), we have $\langle v, Y^n(h,\eta)\rangle = \langle p_d v, E_d X(n,h,\eta)\rangle$. PNC asks exactly that the probability that $E_d X$ is close to a hyperplane is small. By (2.3.2), PNC on dimension $d$ follows from SNC on dimension $d$.

By (2.3.3), SNC on dimension $d$ follows from PNC on dimension $d$. □

**Remark 2.3.8.** *We explain that SNC implies the stronger form of SNC, which will be used later. Let $\mathrm{O}(d)$ be the orthogonal group in dimension $d$. The stronger form of SNC says that for any $(\rho_1, \cdots, \rho_{d+1}) \in \mathrm{O}(d)^{\times(d+1)}$, we have*

$$(\mu^{*n})^{\otimes(d+1)}\{\mathbf{h}_{d+1} \in G^{\times(d+1)} || A_d(\rho_1 E_d X(n,h_1,\eta), \ldots, \rho_{d+1} E_d X(n, h_{d+1}, \eta)| \le e^{-\epsilon n}\} \le Ce^{-c\epsilon n}.$$

*By Lemma 2.3.7, SNC implies PNC. We adopt the notation in the proof of Lemma 2.3.7. By (2.3.3) and the fact that $\mathrm{O}(d)$ preserves the distance,*

$$\begin{aligned}&\mathbb{P}\{\|\sum(-1)^i \rho_1 X_1 \wedge \cdots \widehat{\rho_i X_i} \cdots \wedge \rho_{d+1} X_{d+1}\| < c\}\\ &\le \sum_{1\le i\le d} \mathbb{P}\{d(\rho_i X_i, l_i) < c^{1/d}\} = \sum_{1\le i\le d} \mathbb{P}\{d(X_i, \rho_i^{-1} l_i) < c^{1/d}\},\end{aligned}$$

*where $l_i = \mathrm{Span}_{\mathrm{aff}}(\rho_{d+1}X_{d+1}, \rho_1 X_1, \cdots, \rho_{i-1}X_{i-1})$. Therefore SNC implies the stronger form of SNC.*

**Lemma 2.3.9.** *WNC on dimension $d$ implies PNC on dimension $d$.*

WNC is weaker than SNC, because WNC is not uniform on position $\eta$. Let $f(\eta)$ be $(\mu^{*n})^{\otimes(d_2)}\{\ldots\eta\}$ in SNC (Definition 2.3.3). Then WNC only asks that $\int f(\ell\eta)\mathrm{d}\mu^{*n}(\ell)$ is small, whereas SNC asks that $f(\eta)$ is small for every $\eta$. The cocycle property is the key point to obtain an estimate uniform on position from an estimate not uniform on position.

*Proof of Lemma 2.3.9.* Let $\delta = e^{-\epsilon n}$. We first prove the result for $2n$. Recall that $h$ is a random variable which takes values in $G$ with the distribution $\mu^{*2n}$. Let $h = \ell_1\ell$ such that $\ell_1$ and $\ell$ have distribution $\mu^{*n}$. Then the cocycle property implies $Y^n(h,\eta) = Y^n(\ell_1\ell,\eta) = Y^n(\ell_1,\ell\eta)Y_0^n(\ell,\eta)$. Fubini's theorem implies

$$\begin{aligned}E &:= \sup_{a,v} \mu^{*2n}\{h | \langle v, Y^{2n}(h,\eta)\rangle \in B(a,\delta)\}\\ &\le \int_G \sup_{a,v} \mu^{*n}\{\ell_1 | \langle v, Y^n(\ell_1,\ell\eta)Y_0^n(\ell,\eta)\rangle \in B(a,\delta)\}\mathrm{d}\mu^{*n}(\ell).\end{aligned}$$

The cocycle property is crucial here. Fix $\ell$ and fix $a, v$. We can write

$$\langle v, Y^n(\ell_1,\ell\eta)Y_0^n(\ell,\eta)\rangle = R\langle v', Y^n(\ell_1,\ell\eta)\rangle,$$

where $R = \|v \cdot Y_0^n(\ell,\eta)\| \geq \min_{1\leq j\leq d}|Y_0^n(\ell,\eta)_j|$. Here $v'$ is a vector of norm 1, defined by $v' = v\cdot Y_0^n(\ell,\eta)/R$, depending on $v, l$ and $\eta$. By Lemma 2.2.52 and Lemma 2.2.53, for $\ell$ outside an exponentially small set independent of $a, v$, we have $R \geq \delta^{1/2}$. Therefore

$$E \leq \int_G \sup_{a,v} \mu^{*n}\{\ell_1 | \langle v, Y^n(\ell_1,\ell\eta)\rangle \in B(a,\delta^{1/2})\} \mathrm{d}\mu^{*n}(\ell) + O_\epsilon(\delta^c), \tag{2.3.4}$$

where $c > 0$ comes from the large deviation principle (Lemma 2.2.52). By Hölder's inequality,

$$\begin{aligned} &\int_G \sup_{a,v} \mu^{*n}\{\ell_1 | \langle v, Y^n(\ell_1,\ell\eta)\rangle \in B(a,\delta^{1/2})\} \mathrm{d}\mu^{*n}(\ell) \\ &\leq \left( \int (\sup_{a,v} \mu^{*n}\{\ell_1 | \langle v, Y^n(\ell_1,\ell\eta)\rangle \in B(a,\delta^{1/2})\})^{d+1} \mathrm{d}\mu^{*n}(\ell) \right)^{1/(d+1)}. \end{aligned} \tag{2.3.5}$$

By the same argument as in Lemma 2.3.7

$$\sup_{a,v} \mu^{*n}\{\ell_1 | \langle v, Y^n(\ell_1,\ell\eta)\rangle \in B(a,\delta^{1/2})\}^{d+1} \leq \mu^{*(d+1)n}\{(\mathbf{h}_{d+1}) | |A_d^n(\mathbf{h}_{d+1},\ell\eta)| \leq 2\delta^{1/4}\} + O_\epsilon(\delta^c).$$

Therefore, by (2.3.4) and (2.3.5), we have

$$E^{d+1} \leq \mu^{*(d+2)n}\{(\mathbf{h}_{d+1},\ell) | |A_d^n(\mathbf{h}_{d+1},\ell\eta)| \leq 2\delta^{1/4}\} + O_\epsilon(\delta^c).$$

The proof for $2n$ ends by Definition 2.3.2.

It remains to prove the same result for $2n+1$. Let $h = \ell\ell$ such that $\ell$ has distribution $\mu^{*(n+1)}$ and $\ell_1$ has distribution $\mu^{*n}$. Following the same argument, we have

$$E^{d+1} \leq \mu^{*(d+1)n+(n+1)}\{(\mathbf{h}_{d+1},\ell) | |A_d^n(\mathbf{h}_{d+1},\ell\eta)| \leq 2\delta^{1/4}\} + O_\epsilon(\delta^c).$$

Since $\ell$ only changes the position $\eta$, the uniformity of WNC implies that

$$\begin{aligned} &\mu^{*(d+1)n+(n+1)}\{(\mathbf{h}_{d+1},\ell) | |A_d^n(\mathbf{h}_{d+1},\ell\eta)| \leq 2\delta^{1/4}\} \\ &= \int_{l_3\in G} \mu^{*(d+2)n}\{(\mathbf{h}_{d+1},l_2) | |A_d^n(\mathbf{h}_{d+1},l_2(l_3\eta))| \leq 2\delta^{1/4}\} \mathrm{d}\mu(l_3) \ll_\epsilon \delta^c. \end{aligned}$$

The proof is complete. □

### 2.3.3 Hölder regularity

In this section, we will prove

**Lemma 2.3.10.** *SNC on dimension $d-1$ implies WNC on dimension $d$.*

Using other representations, we can get more information on the Iwasawa cocycle. This idea has already been used in [Aou13] for problem concerning transience of algebraic subvariety of split real Lie groups. It is also used in the work of Bourgain-Gamburd on the spectral gap of dense subgroups in $\mathrm{SU}(n)$, for establishing transience of subgroups.

The key tool is the following estimate. See [BQ16, Proposition 14.3] or [Gui90] for example.

**Lemma 2.3.11.** *Let $V$ be an irreducible representation of $G$. Let $\mu$ be a Zariski dense Borel probability measure on $G$ with exponential moment. For every $\epsilon > 0$ there exist $c, C > 0$ such that for $v$ in $V$ and $f$ in $V^*$ we have*

$$\mu^{*n}\{\ell \in G | \, |f(\ell v)| \leq \|f\| \|\ell\| e^{-\epsilon n}\} \leq Ce^{-cen}.$$

The intuition is that if a function $f$ is not small at some point, then it is robustly large for almost all points.

In this part, we write $V_j = V_{\chi_j}$ for the fundamental representation and we write $V_{j,\eta}$ for the image of $\eta \in \mathscr{P}$ in $\mathbb{P}V_j$ for $j = 1, \dots, m$. Let $v^j$ be a nonzero vector in $V_{j,\eta}$. For $\ell$ in $G$, we abbreviate $\rho_j(\ell)v^j$ to $\ell v^j$. Since $v^j$ lives in $V_j$, we use the same symbol $\|\cdot\|$ for norms on different $V_j$, which makes no confusion. For a vector $x$ in $\mathbb{R}^m$, we denote by $x_i$ the $i$-th coordinate. We use upper script to denote different vectors.

Before proving Lemma 2.3.10, we introduce some linear algebras. We want to construct a linear form. Recall that $E_d$ is a rational map, $A_d$ is the affine determinant, $B_d$ is the composition of $A_d$ and $E_d$ and

$$A_d^n(\mathbf{h}_{d+1}, \eta) := B_d(X(n, h_1, \eta), \dots, X(n, h_{d+1}, \eta)),$$

where

$$X(n, h, \eta) = (e^{\chi_j(\sigma(gh,\eta) - \kappa(g) - n\sigma_\mu)})_{1 \leq j \leq m} = \left( \frac{\|ghv^j\|}{e^{\chi_j(\kappa(g) + n\sigma_\mu)} \|v^j\|} \right)_{1 \leq j \leq m}, \tag{2.3.6}$$

and the second equality is due to (2.2.3) and (2.2.2). Let

$$X^i(n, \eta) := X(n, h_i, \eta). \tag{2.3.7}$$

In order to use Lemma 2.3.11, we need to linearise some function related to $A_d^n(\mathbf{h}_{n+1}, \eta)$ with $\mathbf{h}_{n+1}$ fixed. We will multiply $B_d$ by its denominator, and all the Galois conjugate to get a polynomial on $\|X_j^i\|^2$, which can be realized as a linear functional.

The function $B_d$ can be seen as a rational function on

$$(\underline{x}) := (x^1, \cdots, x^{d+1}) = (x_j^i)_{1 \leq i \leq d+1, 1 \leq j \leq m}.$$

By definition, $B_d$ has a special form. Each term in $B_d$ can be expressed as a quotient of two monomials. Let $D_d$ be the lowest common denominator of $B_d$ such that $D_d B_d$ is a polynomial on $(\underline{x})$. In other words, suppose that

$$B_d = \sum_{\mathbf{n} \in \mathbb{Z}^{m(d+1)}} b_{\mathbf{n}} \prod_{1 \leq j \leq m, 1 \leq i \leq d+1} (x_j^i)^{n_{ij}},$$

where $\mathbf{n}$ is a multi index and $b_{\mathbf{n}}$ is the coefficient. Let $q_{ij} = \sup_{\mathbf{n} \in \mathbb{Z}^{m(d+1)}} \{-n_{ij}, 0\}$ for $1 \leq j \leq m, 1 \leq i \leq d+1$. Then $D_d = \Pi_{1 \leq j \leq m, 1 \leq i \leq d+1} (x_j^i)^{q_{ij}}$.

**Definition 2.3.12.** *Let $F$ be a polynomial on $(x^1, \cdots, x^k)$ where $x^1, \cdots x^k$ are vectors in $\mathbb{R}^n$. Then we call $F$ a multi homogeneous polynomial of degree $\mathbf{q} = (q_1, \cdots, q_n) \in \mathbb{N}^n$ if for $\xi$ in $(\mathbb{R}^*)^n$ we have*

$$F(\xi x^1, \cdots, \xi x^k) = \xi^{\mathbf{q}} F(x^1, \cdots, x^k),$$

*where $\xi^{\mathbf{q}} = \Pi_{1 \le j \le n} \xi_j^{q_j}$.*

Let $\Gamma$ be the finite group $(\mathbb{Z}/2\mathbb{Z})^{d(d+1)}$ which acts on $\mathbb{R}^{d(d+1)}$. Let $(\underline{y}) := (y^1, \cdots, y^{d+1}) = (y^i_j)_{1 \le i \le d+1, 1 \le j \le d} \in (\mathbb{R}^d)^{d+1}$. For $\rho \in \Gamma$, we write $\rho(\underline{y})$ for the action on the coefficient $y^i_j$, which is of dimension $d(d+1)$. Due to the definition of $\Gamma$, the product $\Pi_{\rho \in \Gamma} A_d \rho(y^1, \ldots, y^{d+1})$ is invariant under the action $\Gamma$, hence it is a polynomial on $(y^i_j)^2$. Let

$$F_d(x^1, \ldots, x^{d+1}) = \prod_{\rho \in \Gamma} D_d A_d \rho(E_d x^1, \ldots, E_d x^{d+1}), \tag{2.3.8}$$

then

**Lemma 2.3.13.** *$F_d$ is a multi homogeneous polynomial on $((x^1)^2, \cdots, (x^{d+1})^2)$ with degree $\mathbf{q} = (q_1, \cdots, q_m) \in \mathbb{N}^m$.*

*Proof.* We only need to verify that $F_d$ is a multi homogeneous polynomial. The fact that the determinant is a multilinear function implies that for $\lambda$ and $y^i$ in $\mathbb{R}^d$

$$A_d(\lambda y^1, \cdots, \lambda y^{d+1}) = \det(\lambda) A_d(y^1, \cdots, y^{d+1}), \tag{2.3.9}$$

where $\det(\lambda) = \lambda_1 \cdots \lambda_d$. The functions $E_d$ and $D_d$ are group morphisms due to definition. Hence we have

$$E_d(\xi x) = E_d(\xi) E_d(x) \text{ and } D_d(\xi x^1, \cdots, \xi x^{d+1}) = D_d(\xi, \cdots, \xi) D_d(x^1, \cdots, x^{d+1}). \tag{2.3.10}$$

Therefore by (2.3.8), (2.3.9) and (2.3.10), for $\xi$ and $x^i$ in $\mathbb{R}^m$,

$$\begin{aligned} F_d(\xi x^1, \cdots, \xi x^{d+1}) &= \prod_{\rho \in \Gamma} D_d A_d \rho(E_d(\xi x^1), \cdots, E_d(\xi x^{d+1})) \\ &= \prod_{\rho \in \Gamma} D_d A_d \rho(E_d(\xi) E_d(x^1), \cdots, E_d(\xi) E_d(x^{d+1})) \\ &= \prod_{\rho \in \Gamma} D_d A_d \rho(E_d(x^1), \cdots, E_d(x^{d+1})) \det(E_d(\xi)) D_d(\xi, \cdots, \xi) \\ &= \xi^{\mathbf{q}} F_d(x^1, \cdots, x^{d+1}), \end{aligned}$$

where $\mathbf{q}$ is a vector in $\mathbb{N}^m$ such that $\xi^{\mathbf{q}} = (\det(E_d(\xi)) D_d(\xi, \cdots, \xi))^{|\Gamma|}$. □

For $\mathbf{h}_{d+1} \in G^{\times(d+1)}$ and $\eta$ in $\mathscr{P}$, we write

$$F(\mathbf{h}_{d+1}, \eta) = F_d(X(n, h_1, \eta), \ldots, X(n, h_{d+1}, \eta)).$$

Fix $\mathbf{h}_{d+1}$. By (2.3.6), $F$ is a function on $v^j$ for $1 \leq j \leq m$. Recall that $v^j$ are vectors in $V_{j,\eta}$. Let

$$F_0(v^1, \cdots, v^m) = F(\mathbf{h}_{d+1}, \eta)\Pi_{1\leq j\leq m}\|v^j\|^{2q_j}.$$

Now, we want to explain how to realize $F_0$ as a linear functional.

**Lemma 2.3.14.** *Let $F$ be a multi homogeneous polynomial of degree $\mathbf{q} = (q_1, \cdots, q_{d+1}) \in (\mathbb{N})^{d+1}$. Then $F_0(v^1, \cdots, v^m) := F((X^1)^2, \cdots, (X^{d+1})^2)\|v^j\|^{2q_j}$ is a linear functional $F_1$ on the space $V_0 = \bigotimes_{1\leq j\leq m}(Sym^2 V_j)^{\otimes q_j}$, where $X^j$ is defined in (2.3.7).*

*Proof.* Since $F$ is a multi homogeneous polynomial, it is sufficient to prove that every monomial in $F$ has the same property. By Definition 2.3.12, a monomial of $F$ is of the form

$$\Pi_{1\leq j\leq m}\Pi_{1\leq i\leq d+1}(x^i_j)^{2n_{ij}},$$

with $n_{ij} \in \mathbb{N}$ and $\sum_{1\leq i\leq d+1} n_{ij} = q_j$. The term $\Pi\|v^j\|^{2q_j}$ is used to compensate $\|v^j\|$ in the denominator of $X^i_j$ in (2.3.6). Now, by multiplying $\|v^j\|$, we can view $X^i_j$ as $\|gh_i v^j\|$ with some coefficient. By (2.3.6) and $\|ghv^j\|^2 = \langle ghv^j, ghv^j\rangle$, the function $(X^i_j)^2$ is a linear functional on $Sym^2 V_j$. Hence $\Pi_{1\leq i\leq d+1}(X^i_j)^{2n_{ij}}$ is a linear functional on $(Sym^2 V_j)^{\otimes q_j}$. This is because if we have two linear functionals $f_1$ and $f_2$ on $W_1$ and $W_2$, then $f_1 f_2$ is the linear functional on $W_1 \otimes W_2$ given by $f_1 f_2(w_1 \otimes w_2) = f_1(w_1) f_2(w_2)$. Then by the same reason, the monomial $\Pi_{i,j}(X^i_j)^{2n_{ij}}$ is a linear functional on $V_0$. In order to express the linearity of $F_0$, we rewrite

$$F_1(\otimes_j((v^j)^2)^{\otimes q_j}) := F_0(v^1, \cdots, v^m),$$

where $v^j$ is in $V_{j,\eta}$ and $F_1$ is understood as a linear functional on $V_0$. $\square$

*Proof of Lemma 2.3.10.* Recall $\beta = \max_{\alpha\in\Pi} e^{-\alpha\sigma_\mu n}$. Let $\delta = e^{-\epsilon_2 n}$, where the constant $\epsilon_2$ will be determined later depending on $\epsilon$. We suppose that $n$ is large enough such that $\delta \leq 1/2$. Because for small $n$, WNC can be obtained by enlarging the constant $C$.

**Step 1:** We take into account of measures. We want to reduce the condition of WNC on $A^n_d$ to $F$, which is essentially a linear functional.

For this purpose, we will bound the measure of small $A^n_d$ by the measure of small $F$.

**Lemma 2.3.15.** *Let $f_1, f_2$ be two Borel measurable functions on a locally compact Hausdorff space $X$ and $m$ be a Borel probability measure on $X$. Then for $c > 0$*

$$m\{h \in X | |f_1(h) \leq c|\} \leq m\{h \in X | |f_1(h) f_2(h)| \leq c \sup_X |f_2|\}.$$

In order to control $F/A^n_d(\mathbf{h}_{d+1}, \eta)$, we take $\mathbf{h}_{d+1}$ which is $\eta$ good, that means for every $i$ in $\{1, \cdots, d+1\}$, the group element $h_i$ is $(n, \epsilon_2, \eta, \zeta^m_g)$ good (Definition 2.2.51). By Lemma 2.2.53 and (2.3.6), for $1 \leq i \leq d+1, 1 \leq j \leq m$

$$|X^i_j| \leq \delta^{-1}.$$

Since $F/A_d^n$ is a polynomial on $X_j^i$, for $\mathbf{h}_{d+1}$ which is $\eta$ good, we have

$$F/A_d^n = D_d \Pi_{\rho\in\Gamma,\rho\neq e} D_d A_d^n \rho \leq \delta^{-O(1)}. \tag{2.3.11}$$

Using Lemma 2.3.15 with $f_1 = A_d^n$ and $f_2 = F/A_d^n$, hence by (2.3.11) and Lemma 2.2.52, we have

$$\begin{aligned} M &:= \mu^{*(d+2)n}\{(\mathbf{h}_{d+1},\ell)||A_d^n(\mathbf{h}_{d+1},\ell\eta)| \leq e^{-\epsilon n}\} \\ &\leq \mu^{*(d+2)n}\{\mathbf{h}_{d+1} \text{ is } \ell\eta \text{ good}, \ell\in G||A_d^n(\mathbf{h}_{d+1},\ell\eta)| \leq e^{-\epsilon n}\} + O_{\epsilon_2}(\delta^c) \\ &\leq \mu^{*(d+2)n}\{\mathbf{h}_{d+1} \text{ is } \ell\eta \text{ good}, \ell\in G||F(\mathbf{h}_{d+1},\ell\eta)| \leq e^{-\epsilon n}\delta^{-O(1)}\} + O_{\epsilon_2}(\delta^c) \\ &\leq \mu^{*(d+2)n}\{(\mathbf{h}_{d+1},\ell)||F(\mathbf{h}_{d+1},\ell\eta)| \leq e^{-\epsilon n}\delta^{-O(1)}\} + O_{\epsilon_2}(\delta^c) \end{aligned} \tag{2.3.12}$$

**Step 2:** Lemma 2.3.13 implies that $F$ is a multi homogeneous polynomial on $(x_j^i)^2$ of degree $\mathbf{q} = (q_1,\dots,q_{d+1})$. Lemma 2.3.14 implies that

$$F(\mathbf{h}_{d+1},\eta) = F_1(\otimes_j((v^j)^2)^{\otimes q_j})/\Pi\|v_j\|^{2q_j},$$

where $F_1$ is a linear functional on $V_0 = \bigotimes_j (Sym^2 V_j)^{\otimes q_j}$. To be more precise, $F_1$ will be restricted to a linear form on $W$, the unique irreducible representation of $V_0$ with maximal weight. (This is specific for real split Lie groups)

**It remains to show that for most $\mathbf{h}_{d+1}$ in $G^{\times(d+1)}$, the norm of $F_1$ is robustly large.** It is sufficient to find one $\eta$ such that $|F(\mathbf{h}_{d+1},\eta)|$ is large. We will prove that $|D_d A_d \rho|$ is large for each $\rho$ in $\Gamma$, which implies that $|F(\mathbf{h}_{d+1},\eta)|$ is large.

Using the $d+1$-th column expansion of the matrix $\begin{pmatrix} y^1 & \cdots & y^{d+1} \\ 1 & \cdots & 1 \end{pmatrix}$, we have

$$\begin{aligned} A_d(y^1,\cdots,y^{d+1}) &= -A_{d-1}(r_d y^1,\cdots,r_d y^d)y_d^{d+1} + \text{ other terms}, \\ &= \sum_{1\leq j\leq d} (-1)^{j+d+1} A_{d-1}(r_j y^1,\cdots,r_j y^d) y_j^{d+1} + \det(y^1,\cdots,y^d), \end{aligned} \tag{2.3.13}$$

where $r_j : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is the map forgetting the $j$-th coordinate. Replacing $y^i$ by $E_d x^i$, due to $r_d E_d x^i = E_{d-1} x^i$, we obtain

$$A_d(E_d x^1,\cdots,E_d x^{d+1}) = -A_{d-1}(E_{d-1}x^1,\cdots,E_{d-1}x^d)(E_d x^{d+1})_d + \text{ other terms}. \tag{2.3.14}$$

Using SNC on dimension $d-1$, we are able to give a lower bound of $A_{d-1}(E_{d-1}X^1,\cdots,E_{d-1}X^d)$ with a loss of exponentially small probability of $\mathbf{h}_{d+1}$. But the problem is in other similar terms. Due to $y_j^{d+1} = \Pi_{1\leq i\leq m}(x_i^{d+1})^{-\alpha_j(H_i)}$ and the structure of root system, the degree of $x_d^{d+1}$ in $y_j^{d+1} = (E_d x^{d+1})_j$ is

$$-\alpha_d(H_d) = -2 \text{ and } -\alpha_j(H_d) \geq 0 \text{ for } j<d. \tag{2.3.15}$$

Hence, we will make $X_d^{d+1} \leq \beta$, which makes the first term in (2.3.13) greater than $\delta^{O(1)}\beta^{-2}$, and the other terms are less than $\delta^{-O(1)}$.

Now, here is the precise proof. Take $h_{d+1}$ good, that means $h_{d+1}$ is $(n, \epsilon_2, \zeta_g^m)$ good (Definition 2.2.54). We take

$$\eta = \ell_h^{-1}\eta_d \tag{2.3.16}$$

as in Lemma 2.2.55. By Lemma 2.2.55

$$X_j^{d+1} \in [\delta, \delta^{-1}] \text{ for } j \neq d \text{ and } X_d^{d+1} \leq \beta\delta^{-1}. \tag{2.3.17}$$

Let $\Gamma_{d-1} = (\mathbb{Z}/2\mathbb{Z})^{(d-1)d}$, seen as a subgroup of $\Gamma$, which acts on $\mathbb{R}^{(d-1)d}$. Then we demand that $\mathbf{h}_d$ satisfies

$$|A_{d-1}^n\rho(\mathbf{h}_d, \eta)| \geq \delta \text{ for all } \rho \in \Gamma_{d-1} \text{ and } \mathbf{h}_d \text{ is } \eta \text{ good}. \tag{2.3.18}$$

Recall that $\mathbf{h}_d$ is $\eta$ good means that $h_i$ is $(n, \epsilon_2, \eta, \zeta_g^m)$ good for $1 \leq i \leq d$. By Lemma 2.2.53 and (2.3.6),

$$X_j^i(\eta) \in [\delta, \delta^{-1}], \text{ for } 1 \leq i \leq d, 1 \leq j \leq m. \tag{2.3.19}$$

Recall that $W$ is the unique irreducible subrepresentation of $V_0$ with the highest weight.

**Lemma 2.3.16.** *We claim that if $h_{d+1}$ is good ($(n, \epsilon_2, \zeta_g^m)$ good), $\eta$ is taken as in (2.3.16) and the assumption (2.3.18) is satisfied for $\mathbf{h}_d$, then the operator norm satisfies*

$$\|F_1|_W\| \geq \delta^{O(1)}.$$

*Proof ot Lemma 2.3.16.* As we have already explained, it is sufficient to prove that for $\rho$ in $\Gamma$, we have

$$|D_d A_d^n \rho(\mathbf{h}_d, \eta)| \geq \delta^{O(1)}.$$

The proof is similar for $\rho$ in $\Gamma$, we will only prove the case $\rho = e$.

By (2.3.13) and (2.3.14)

$$\begin{aligned} D_d A_d(E_d x^1, \cdots, E_d x^{d+1}) &= -A_{d-1}(E_{d-1}x^1, \cdots, E_{d-1}x^d) D_d(E_d x^{d+1})_d \\ &+ \sum_{1 \leq j < d} (-1)^{j+d+1} A_{d-1}(r_j E_d x^1, \cdots, r_j E_d x^d) D_d(E_d x^{d+1})_j + D_d \det(E_d x^1, \cdots, E_d x^d) \end{aligned} \tag{2.3.20}$$

where $r_j : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is the map forgetting the $j$-th coordinate. Since $x_d^{d+1}$ only appears in $E_d x^{d+1}$, by (2.3.15), we know that the degree of $x_d^{d+1}$ in $D_d$ equals $\alpha_d(H_d) = 2$, which implies that

$$D_d \leq \delta^{-O(1)}\beta^2.$$

Hence by (2.3.17)-(2.3.19) and the property (2.3.15) that the degree of $X_d^{d+1}$ in $(E_d X^{d+1})_d$ is $-2$, the degree in $(E_d X^{d+1})_j$ is non negative for $j < d$, we have

$$\begin{aligned} &D_d(E_d X^{d+1})_d \geq \delta^{O(1)}, \ |A_{d-1}(E_{d-1}X^1, \cdots, E_{d-1}X^d)| \geq \delta^{O(1)}, \\ &D_d(E_d X^{d+1})_j \leq \delta^{-O(1)}\beta^2, \ |A_{d-1}(r_j E_d X^1, \cdots, r_j E_d X^d)| \leq \delta^{-O(1)} \text{ for } 1 \leq j < d \\ &\text{and } D_d \det(E_d X^1, \cdots, E_d X^d) \leq \delta^{-O(1)}\beta^2. \end{aligned} \tag{2.3.21}$$

By (2.3.20) and (2.3.21), we have

$$|D_d A_d^n| \geq \delta^{O(1)} - \delta^{-O(1)}\beta^2 \geq \delta^{O(1)}.$$

The proof is complete. □

**Step 3.** We return to the proof of Lemma 2.3.10. We write $\ell v$ for the vector $\otimes_j(\ell(v^j)^2)^{\otimes q_j}$ in $V_0$. Then $\mathbb{R}lv$ is exactly the image of $\ell\eta$ in $\mathbb{P}W$. Using the Fubini theorem and (2.3.12), we have

$$M \leq \int \mathrm{d}\mu^{*n}(h_{d+1}) \int \mathrm{d}\mu^{*(d-1)n}(\mathbf{h}_d)\mu^{*n}\left\{\ell \Big| \frac{|F_1(\ell v)|}{\|F_1|_W\|\|\ell\|} \leq e^{-\epsilon n}\delta^{-O(1)}\|F_1|_W\|^{-1}\right\} + O_{\epsilon_2}(\delta^c).$$

Using SNC on dimension $d-1$, for all $\rho \in \Gamma_{d-1}$, we have $\mu^{*(d-1)n}\{(\mathbf{h}_d)||A_{d-1}^n\rho(\mathbf{h}_d,\eta)| \leq \delta\} = O_{\epsilon_2}(\delta^c)$. (This is a stronger form of SNC on dimension $d-1$. Due to $\Gamma_{d-1} \in \mathrm{O}(d-1)^{\times d}$, it follows from Remark 2.3.8 that SNC implies this stronger form.) By Lemma 2.2.52, the set that $h_{d+1}$ is not $(n, \epsilon_2, \zeta_g^m)$ good and $\mathbf{h}_d$ is not $\eta$ good have exponentially small measure. Hence

$$M \leq \int_{good} \mathrm{d}\mu^{*n}(h_{d+1}) \int_{\mathbf{h}_d \text{ satisifes } (2.3.18)} \mathrm{d}\mu^{*(d-1)n}(\mathbf{h}_d)\mu^{*n}\left\{\ell \Big| \frac{|F_1(\ell v)|}{\|F_1|_W\|\|\ell\|} \leq e^{-\epsilon n}\delta^{-O(1)}\|F_1|_W\|^{-1}\right\} + O_{\epsilon_2}(\delta^c). \tag{2.3.22}$$

Due to Lemma 2.3.16, when $\epsilon_2$ is small enough with respect to $\epsilon$, we have ($\delta = e^{-\epsilon_2 n}$ and $\|F_1|_W\| \ll \delta^{-O(1)}$)

$$e^{-\epsilon n}\delta^{-O(1)}\|F_1|_W\|^{-1} \leq e^{-\epsilon n}\delta^{-O(1)} \leq e^{-\epsilon n/2}.$$

Using Lemma 2.3.11 with $V = W$, due to $\ell v$ in $W$ we conclude that under the condition of Lemma 2.3.16,

$$\mu^{*n}\left\{\ell \Big| \frac{F_1(\ell v)}{\|F_1|_W\|\|\ell\|} \leq e^{-\epsilon n}\delta^{-O(1)}\|F_1|_W\|^{-1}\right\} \leq_\epsilon e^{-c\epsilon n}. \tag{2.3.23}$$

By (2.3.22) and (2.3.23), the proof is complete. □

### 2.3.4 Combinatoric tool

**Proposition 2.3.17.** *Fix $\kappa_1 > 0$. Let $C_0 > 0$. Then there exist $\epsilon_3$ and $k \in \mathbb{N}, \epsilon > 0$ depending only on $\kappa_1$ such that the following holds for $\tau$ large enough depending on $C_0$. Let $\lambda_1, \dots \lambda_k$ be Borel measures on $([-\tau^{\epsilon_4}, -\tau^{-\epsilon_4}] \cup [\tau^{-\epsilon_4}, \tau^{\epsilon_4}])^m \subset \mathbb{R}^m$ where $\epsilon_4 = \min\{\epsilon_3, \epsilon_3\kappa_0\}/10k$, with total mass less than 1. Assume that for all $\rho \in [\tau^{-2}, \tau^{-\epsilon_3}]$ and $j = 1, \dots, k$*

$$\sup_{a\in\mathbb{R}, v\in\mathbb{S}^{m-1}} (\pi_v)_*\lambda_j(B_{\mathbb{R}}(a,\rho)) = \sup_{a,v} \lambda_j\{x | \langle v, x\rangle \in B_{\mathbb{R}}(a,\rho)\} \leq C_0\rho^{\kappa_1}. \tag{2.3.24}$$

*Then for all* $\varsigma \in \mathbb{R}^m, \|\varsigma\| \in [\tau^{3/4}, \tau^{5/4}]$ *we have*

$$\left| \int \exp(i\langle \varsigma, x_1 \cdots x_k \rangle) \mathrm{d}\lambda_1(x_1) \cdots \mathrm{d}\lambda_k(x_k) \right| \leq \tau^{-\epsilon_3}.$$

This is Proposition 3.4.4 in Chapter 3, based on a discretized sum-product estimate by He-de Saxcé. When $n = 1$, this is due to Bourgain in [Bou10]. The assumption (2.3.24) is called the projective non concentration in the introduction (Definition 2.1).

### 2.3.5 Application to our measure

From Proposition 2.3.4, we fix $\epsilon_2 < \frac{1}{10} \min_{\alpha \in \Pi}\{\alpha\sigma_\mu\}$ and we can find $c_1$ such that PNC holds. Let $(\epsilon_2/2, c')$ be the constants in Lemma 2.2.53. Take

$$\kappa_0 = \frac{1}{10} \min\{c_1, c'\}.$$

Using Proposition 2.3.17 with $\kappa_1 = \kappa_0$, we get $\epsilon_3, \epsilon_4$.

For $g, h$ in $G$ and $\eta$ in $\mathscr{P}$, recall that $Y^n(h, \eta) = (e^{-\alpha(\sigma(gh,\eta) - \kappa(g) - n\sigma_\mu)})_{\alpha \in \Pi} \in \mathbb{R}^m$. Let $\lambda_{g,\eta}$ be a pushforward measure on $\mathbb{R}^m$ of $\mu^{*n}$ restricted on a subset $G_{n,g,\eta}$ of $G$, which is defined by

$$\lambda_{g,\eta}(E) = \mu^{*n}\{h \in G_{n,g,\eta} | Y^n(h, \eta) \in E\},$$

for any Borel subset $E$ of $\mathbb{R}^m$, where

$$G_{n,g,\eta} = \{h \in G | h \text{ is } (n, \epsilon, \eta, \zeta_g^m) \text{ good}\} \tag{2.3.25}$$

and where $\epsilon_\mu \geq \epsilon > 0$ will be determined later.

PNC is only at one scale, we need to verify all the scales needed in the sum-product estimate. The idea is to separate the random variable and try to use PNC in other scale, where we need the cocycle property to change scale.

**Proposition 2.3.18** (Change scale)**.** *With* $\epsilon$ *small enough depending on* $\epsilon_4\epsilon_2$*, there exists* $C_0$ *independent of* $n$ *such that the measure* $\lambda_{g,\eta}$ *satisfies the conditions in Proposition 2.3.17 with constant* $\tau = e^{\epsilon_2 n}$ *for all* $n \in \mathbb{N}$.

*Proof.* We abbreviate $\lambda_{g,\eta}$ to $\lambda$. By taking $\epsilon$ small depending on $\epsilon_4\epsilon_2$, Lemma 2.2.53 implies that the support of $\lambda$ is contained in the cube $[\tau^{-\epsilon_4}, \tau^{\epsilon_4}]^m$.

Then we verify (2.3.24). Let $\rho \in [\tau^{-2}, \tau^{-\epsilon_3}]$. Let $n_1 = [\frac{|\log \rho|}{2\epsilon_2}]$. and $n_2 = n - n_1$. Then $n_1$ lies in $[\epsilon_3 n/2, n]$. We separate $h = h_1 h_2$ such that $h_1, h_2$ have distributions $\mu^{*n_1}, \mu^{*(n-n_1)}$, respectively. We have

$$Y^n(h, \eta) = Y^{n_1}(h_1, h_2\eta) Y_0^{n_2}(h_2, \eta), \tag{2.3.26}$$

We can not use the cocycle property directly to change the scale. The problem is in (2.3.26), where the term $Y_0^{n_2}$ behaves bad if $n_2 \gg n_1$, that is to say that the probability of $h_2$ such that $Y_0^{n_2}(h_2, \eta)$ is smaller than $\rho = e^{-2\epsilon_2 n_1}$ is large. In order to overcome this difficulty, we use the support of $Y^n$. We will prove that if $Y_0^{n_2}$ is too small, then the support of $Y^n$ will force $Y^{n_1}$ to become large, which can be controlled by the large deviation principle.

Now we give the details of the proof. For (2.3.24), due to the fact that the support of $\lambda$ is contained in $[\tau^{-\epsilon_4}, \tau^{\epsilon_4}]^m$, we have

$$(\pi_w)_*\lambda(B(a,\rho)) \leq \sup_{h_2,v} \mu^{*n_1}\{h_1|\langle v, Y^{n_1}(h_1, h_2\eta)\rangle \in R^{-1}B(a,\rho), Y^n(h_1h_2,\eta) \in [\tau^{-\epsilon_4}, \tau^{\epsilon_4}]^m\}, \tag{2.3.27}$$

where $R = \|wY_0^{n_2}(h_2,\eta)\|$ depends on $h_2$.

- If $R \geq \rho^{1/2}$, then $\rho R^{-1} \leq \rho^{1/2} = e^{-\epsilon_2 n_1}$. It follows by PNC at scale $n_1$ that
$$\mu^{*n_1}\{h_1|\langle v, Y^{n_1}(h_1, h_2\eta)\rangle \in B(a, e^{-\epsilon_2 n_1})\} \ll_{\epsilon_2} e^{-c_1\epsilon_2 n_1} \leq \rho^{\kappa_0}. \tag{2.3.28}$$

- If $R \leq \rho^{1/2}$. There exists one coordinate $\alpha$ such that $|Y_0^{n_2}(h_2,\eta)_\alpha| \leq \rho^{1/2}$, which implies that $Y^{n_1}(h_1, h_2\eta)_\alpha = Y^n(h,\eta)_\alpha / Y_0^{n_2}(h_2,\eta)_\alpha \geq \tau^{-\epsilon_4}\rho^{-1/2}$. Due to $\epsilon_3 \geq 4\epsilon_4$ and $n_1 \geq \epsilon_3 n/2$, we have $\epsilon_2 n_1 \geq 2\epsilon_4\epsilon_2 n$. Therefore $\tau^{-\epsilon_4}\rho^{-1/2} = \tau^{-\epsilon_4}e^{\epsilon_2 n_1} \geq e^{\epsilon_2 n_1/2}$. For such $h_2$, we have
$$\mu^{*n_1}\{h_1|Y^{n_1}(h_1h_2,\eta) \in [\tau^{-\epsilon_4}, \tau^{\epsilon_4}]^m\} \leq \sum_{\alpha\in\Pi} \mu^{*n_1}\{h_1|Y^{n_1}(h_1, h_2\eta)_\alpha \geq e^{\epsilon_2 n_1/2}\}. \tag{2.3.29}$$

It follows from Lemma 2.2.53 that

$$\begin{aligned}\mu^{*n_1}\{h_1|Y^{n_1}(h_1, h_2\eta)_\alpha \geq e^{\epsilon_2 n_1/2}\} &\leq \mu^{*n_1}\{h_1|\|\sigma(gh_1, h_2\eta) - \kappa(g) - n_1\sigma_\mu\| \geq \epsilon_2 n_1/2\} \\ &\ll_{\epsilon_2} e^{-c'\epsilon_2 n_1} \leq \rho^{\kappa_0}.\end{aligned} \tag{2.3.30}$$

By (2.3.27)-(2.3.30), for $\rho \in [\tau^{-2}, \tau^{-\epsilon_3}]$ we have

$$(\pi_w)_*\lambda(B(a,\rho)) \ll_{\epsilon_2} \rho^{\kappa_0}.$$

The proof is complete. □

## 2.4 Proof of the main theorems

In this section, we will use the results of Section 2.2 and Section 2.3 to give the proofs of the main theorems. We will add many assumptions on the elements of $G$ and $\mathscr{P}$. The assumptions seem complicate. In fact, they are not really important. They are taken to make the result work outside a set of exponentially small measure. These assumptions says that the elements are away from certain closed subvarieties of $G$ or $\mathscr{P}$, which also explains that they are true almost everywhere.

### 2.4.1 $(C, r)$ good function

For a $C^1$ function $\varphi$ on the flag variety $\mathscr{P}$. We first lift it to $\mathscr{P}_0 = G/A_eN$. Let $\partial_\alpha\varphi = \partial_{Y_\alpha}\varphi$ be the directional derivative on $\mathscr{P}_0$. By Lemma 2.2.31 the action of the group $M$ only changes the sign of the directional derivative $\partial_\alpha\varphi$, hence $|\partial_\alpha\varphi|$ is actually a function on $\mathscr{P}$. Recall that for $\eta, \eta'$ in $\mathscr{P}$ and simple root $\alpha$, we have defined $d_\alpha(\eta, \eta') = d(V_{\alpha,\eta}, V_{\alpha,\eta'})$.

**Definition 2.4.1.** *Let $r$ be a continuous function on $\mathscr{P}$. Let $J$ be the open set in $\mathscr{P}$, which is the $1/C$-neighbourhood of the support of $r$. Let $\varphi$ be a $C^2$ function on $\mathscr{P}$. For a simple root $\alpha$, let $v_\alpha = \sup_{\eta\in\mathrm{supp} r} |\partial_\alpha\varphi(\eta)|$. We say that $\varphi$ is $(C, r)$ good if:*

*(G1) For $\eta, \eta'$ in $J$ such that $d(\eta, \eta') \leq 1/C$,*

$$|\varphi(\eta) - \varphi(\eta')| \leq C \sum_{\alpha\in\Pi} d_\alpha(\eta, \eta')v_\alpha, \tag{2.4.1}$$

*(G2) For every simple root $\alpha$ and for every $\eta$ in the support of $r$, we have*

$$|\partial_\alpha\varphi(\eta)| \geq \frac{1}{C} v_\alpha, \tag{2.4.2}$$

*(G3) For $z, z'$ in $\pi^{-1}(J) \subset \mathscr{P}_0$,*

$$|\partial_\alpha\varphi(z) - \partial_\alpha\varphi(z')| \leq C d_0(z, z')v_\alpha. \tag{2.4.3}$$

*(G4)*

$$\sup_{\alpha\in\Pi} v_\alpha \in [1/C, C]. \tag{2.4.4}$$

In the above definition, only G3 assumption (2.4.3) really need $\mathscr{P}_0$, where the Lipschitz norm is defined with respect to the distance $d_0$ on $\mathscr{P}_0$. G1 assumption is new in higher dimension which means that we can bound the difference by its difference in each fundamental representation, and in a fundamental representation the directional derivative $|\partial_\alpha\varphi|$ can bound the Lipschitz norm. G2 and G3 assumptions are natural generalizations of the case $m = 1$, $\mathrm{SL}_2(\mathbb{R})$. G4 assumption is used to normalize the function.

The role of $J$ is to simplify the verification of $(C, r)$ goodness. With this definition, we only need to verify assumptions on a neighbourhood of the support of $r$.

### 2.4.2 From sum-product estimates to Fourier decay

In this subsection we will prove Theorem 2.1.7, an estimate of Fourier decay, by using the results established in Section 2.2 and Section 2.3.

Recall that we have fixed $(\epsilon_2, c_1)$ for Proposition 2.3.4 in Section 2.3.5, the constant $(\epsilon_2/2, c')$ in Lemma 2.2.52 and

$$\kappa_0 = \frac{1}{10}\min\{c_1, c'\}.$$

Take $k, \epsilon_3, \epsilon_4$ from Proposition 2.3.17 with this $\kappa_0$. Let $\epsilon$ be a positive number to be determined later (the only constant which is not fixed yet). The constant $\epsilon_0$ in the hypothesis of Theorem 2.1.7 is defined as

$$\epsilon_0 = \frac{\epsilon}{\max_{\alpha\in\Pi}\{(2k+1)\alpha\sigma_\mu + \epsilon_2\} + \epsilon} \tag{2.4.5}$$

which will be fixed once $\epsilon$ is fixed.

Here, we define and give relations of different constants. Let $v$ be the vector in $\mathbb{R}^m$ whose components are $v_\alpha = \sup_{\eta\in\mathrm{supp}r} |\partial_\alpha \varphi(\eta)|$, for $\alpha \in \Pi$. Then by G4 assumption (2.4.4), we have

$$\sup_{\alpha\in\Pi} v_\alpha \in [\xi^{-\epsilon_0}, \xi^{\epsilon_0}]. \tag{2.4.6}$$

Let $n$ be the minimal integer such that

$$e^{\epsilon_2 n} \geq \xi \max_{\alpha\in\Pi}\{v_\alpha e^{-(2k+1)\alpha\sigma_\mu n}\}. \tag{2.4.7}$$

The existence is guaranteed by the positivity of Lyapunov constant, that is $\alpha\sigma_\mu > 0$ for $\alpha \in \Pi$ (Lemma 2.2.45). Let the regularity scale $\delta$ be given by

$$\delta = e^{-\epsilon n} < 1/2,$$

where we take $\xi$ large enough such that $n$ is large enough. Let the contraction scale $\beta$ given by

$$\beta_\alpha = e^{-\alpha\sigma_\mu n}, \beta = \max_{\alpha\in\Pi}\{\beta_\alpha\}.$$

The point is that the contraction speed $\beta$ decides the magnitude of a term and $\delta$ is only an error term, much larger than $\beta$.

Let the frequency $\tau$ be defined by $\tau = e^{\epsilon_2 n}$. By (2.4.7), we have

$$\tau \geq \xi \max_{\alpha\in\Pi}\{v_\alpha \beta_\alpha^{2k+1}\} \geq C_{\epsilon_2}\tau, \tag{2.4.8}$$

where $C_{\epsilon_2} = e^{-\epsilon_2}\min_{\alpha\in\Pi}\{e^{-(2k+1)\alpha\sigma_\mu}\}$. By (2.4.6), there exists $\alpha_o$ in $\Pi$ such that $v_{\alpha_o} \geq \xi^{-\epsilon_0}$. Then (2.4.8) and (2.4.5) imply that

$$\xi \leq \tau v_{\alpha_o}^{-1}\beta_{\alpha_o}^{-2k-1} \leq \xi^{\epsilon_0}\tau\beta_{\alpha_0}^{-(2k+1)} \leq \xi^{\epsilon_0} e^{n\epsilon\frac{1-\epsilon_0}{\epsilon_0}}.$$

Hence the regularity scale satisfies

$$\xi^{\epsilon_0} \leq e^{\epsilon n} = \delta^{-1}. \tag{2.4.9}$$

**Notation**: We state some notation which will be used throughout Section 2.4.2.

- Let $\mathbf{g} = (g_0, \ldots, g_k)$ be an element in $G^{\times(k+1)}$.
- Let $\mathbf{h} = (h_1, \ldots, h_k)$ be an element in $G^{\times k}$.

- We write $\mathbf{g}\leftrightarrow\mathbf{h} = g_0h_1\cdots h_kg_k \in G$ for the product of $\mathbf{g}, \mathbf{h}$.
- We write $T\mathbf{g}\leftrightarrow\mathbf{h} = g_0h_1\cdots g_{k-1}h_k \in G$.
- For $l \in \mathbb{N}$, let $\mu_{l,n}$ be the product measure on $G^{\times l}$ given by $\mu_{l,n} = \underbrace{\mu^{*n}\otimes\cdots\otimes\mu^{*n}}_{l\text{ times}}$.
- Recall that for $g, h$ in $G$ and $\eta$ in $\mathscr{P}$, we define $Y_g^n(h,\eta)_\alpha = \exp(-\alpha(\sigma(gh,\eta) - \kappa(g) - n\sigma_\mu))$ and $Y_g^n(h,\eta) = (Y_g^n(h,\eta)_\alpha)_{\alpha\in\Pi} \in \mathbb{R}^m$.
- For $z$ in $\mathscr{P}_0$, let $\tilde{Y^n}_g(h,z)_\alpha = \alpha^\sharp(\mathrm{m}(\ell_g^{-1},hz))Y_g^n(h,\eta)_\alpha$, where $\alpha^\sharp$ is the corresponding algebraic character of the simple root $\alpha$ and we make a choice of $\ell_g$ and $\eta = \pi(z)$.
- For $g$ in $G$, $z$ in $\mathscr{P}_0$ and $\eta = \pi(z)$, let $\tilde{\lambda}_{g,z}$ be the pushforward measure on $\mathbb{R}^m$ of $\mu^{*n}$ restricted to a subset $G_{n,g,\eta}$ under the map $\tilde{Y_g^n}(\cdot,z)$. In other words, for a Borel set $E$,
$$\tilde{\lambda}_{g,z}(E) = \mu^{*n}\{h \in G_{n,g,\eta} | \tilde{Y_g^n}(h,z) \in E\}.$$
Recall that the set $G_{n,g,\eta}$ is defined by $G_{n,g,\eta} = \{h \in G | h \text{ is } (n,\epsilon,\eta,\zeta_g^m) \text{ good}\}$.
- After fixing $\mathbf{g}$, we will also fix a choice of $k_{g_j}$, $\ell_{g_j}$ for $g_j$ and let $z_{g_j} = k_{g_j}z_o$, $m_j(h) = \mathrm{m}(\ell_{g_{j-1}}^{-1}, hk_{g_j})$ and $\lambda_j = \tilde{\lambda}_{g_{j-1},z_{g_j}}$, for $j = 1,\ldots,k$.

**Lemma 2.4.2.** *The measure $\tilde{\lambda}_{g,z}$ satisfies the same property (2.3.24) as $\lambda_{g,\eta}$ with $C_0$ replaced by $2^mC_0$, where $\eta = \pi(z)$.*

*Proof.* Since the difference is only in the sign, we have
$$(\pi_v)_*\tilde{\lambda}_{g,z}(B_\mathbb{R}(a,\rho)) \le \sum_{f\in(\mathbb{Z}/2\mathbb{Z})^m} (\pi_{fv})_*\lambda_{g,\eta}(B_\mathbb{R}(a,\rho)),$$
where we identify $(\mathbb{Z}/2\mathbb{Z})^m$ with $\{-1,1\}^m \subset \mathbb{R}^m$. The result follows from this inequality. ☐

**First step:** For $\eta, \eta'$ in $\mathscr{P}$, let
$$f(\eta,\eta') = \int_G e^{i\xi(\varphi(g\eta)-\varphi(g\eta'))}r(g\eta)r(g\eta')\mathrm{d}\mu^{*(2k+1)n}(g). \tag{2.4.10}$$

**Lemma 2.4.3.** *We have*
$$\left|\int_\mathscr{P} e^{i\xi\varphi(\eta)}r(\eta)\mathrm{d}\nu(\eta)\right|^2 \le \int_{\mathscr{P}^2} f(\eta,\eta')\mathrm{d}\nu(\eta)\mathrm{d}\nu(\eta'). \tag{2.4.11}$$

*Proof.* By the definition of $\mu$-stationary measure and the Cauchy-Schwarz inequality,
$$\left|\int_\mathscr{P} e^{i\xi\varphi(\eta)}r(\eta)\mathrm{d}\nu(\eta)\right|^2$$
$$= \left|\int_{\mathscr{P}\times G} e^{i\xi\varphi(g\eta)}r(g\eta)\mathrm{d}\mu^{*(2k+1)n}(g)\mathrm{d}\nu(\eta)\right|^2 \le \int_G \left|\int_\mathscr{P} e^{i\xi\varphi(g\eta)}r(g\eta)\mathrm{d}\nu(\eta)\right|^2 \mathrm{d}\mu^{*(2k+1)n}(g)$$
$$= \int_{\mathscr{P}^2}\int_G e^{i\xi(\varphi(g\eta)-\varphi(g\eta'))}r(g\eta)r(g\eta')\mathrm{d}\mu^{*(2k+1)n}(g)\mathrm{d}\nu(\eta)\mathrm{d}\nu(\eta').$$

The proof is complete. □

Recall that for $\eta$ in $\mathscr{P}$, we write $V_{\alpha,\eta}$ for its image in $\mathbb{P}V_\alpha$ and $d_\alpha(\eta,\eta')=d(V_{\alpha,\eta},V_{\alpha,\eta'})$.

**Definition 2.4.4** (Good Position)**.** *Let $\eta,\eta'$ be in $\mathscr{P}$. We say that they are in good position if*

$$\forall\alpha\in\Pi,\ \ d_\alpha(\eta,\eta')\geq\delta.$$

We fix $\eta,\eta'$ in good position, which means that $\eta,\eta'$ are far in all $\mathbb{P}V_\alpha$. We rewrite the formula.

**Lemma 2.4.5.** *We have*

$$\left|\int_{\mathscr{P}}e^{i\xi\varphi(\eta)}r(\eta)\mathrm{d}\nu(\eta)\right|^2\leq\int_{\eta,\eta'\ good}f(\eta,\eta')\mathrm{d}\nu(\eta)\mathrm{d}\nu(\eta')+O(\delta^c). \tag{2.4.12}$$

*Proof.* By the regularity of stationary measure (2.2.34), we have

$$\nu\{\eta'\in\mathscr{P}|d_\alpha(\eta,\eta')\leq\delta\}=\nu\{\eta'\in\mathscr{P}|d(V_{\alpha,\eta},V_{\alpha,\eta'})\leq\delta\}\leq C\delta^c. \tag{2.4.13}$$

Therefore by (2.4.13) and Fubini's theorem,

$$\nu\otimes\nu\{(\eta,\eta')\in\mathscr{P}^2|\,d_\alpha(\eta',\eta)<\delta\}=\int_{\eta\in\mathscr{P}}\nu\{\eta'\in\mathscr{P}|d_\alpha(\eta,\eta')\leq\delta\}\mathrm{d}\nu(\eta)\ll\delta^c.$$

Summing over simple roots $\alpha$, we obtain the result by $\|r\|_\infty\leq 1$. □

**Second step:** The purpose of this part is to give a Ping-Pong Lemma in measure sense. We will eliminate sets with negligible measure such that the Ping-Pong condition is almost preserved by iteration on the complement.

We fix $g_j$ for $j=0,\dots,k-1$ which satisfies

$$\|\kappa(g_j)-n\sigma_\mu\|\leq\epsilon n/C_A. \tag{2.4.14}$$

Recall that $C_A$ is a constant in Definition 2.2.51. We also demand that

$$h_{j+1}\text{ is }(n,\epsilon,\eta^M_{g_{j+1}},\zeta^m_{g_j})\text{ good}. \tag{2.4.15}$$

Recall that the Cartan subspace $\mathfrak{a}$ is equipped with the norm induced by the Killing form, and with this norm $\mathfrak{a}$ is isomorphic to the euclidean space $\mathbb{R}^m$.

**Lemma 2.4.6.** *Suppose that $\mathbf{g},\mathbf{h}$ satisfy the above conditions (2.4.14) and (2.4.15). Then the action of $T\mathbf{g}\leftrightarrow\mathbf{h}$ on $b^M_{V_\alpha,g_k}(\delta)$ is $\beta_\alpha^{2k}\delta^{-O(1)}$ Lipschitz and*

$$e^{-\alpha\sigma(g_0h_1,x^M_{g_1})}\cdots e^{-\alpha\sigma(g_{k-1}h_k,x^M_{g_k})}\leq\beta_\alpha^{2k}\delta^{-O(1)}, \tag{2.4.16}$$

*for every $\alpha$ in $\Pi$. For $t\in b^M_{g_k}(\delta)$, let $t_j=g_jh_{j+1}\cdots h_kt$ for $j=0,\dots,k$, where we let $t_k=t$. Then*

$$t_j\in b^M_{g_j}(\beta\delta^{-2})\subset b^M_{g_j}(\delta), \tag{2.4.17}$$

$$\|\sigma(g_jh_{j+1},t_{j+1})-\sigma(g_jh_{j+1},\eta^M_{g_{j+1}})\|\ll\beta\delta^{-O(1)}. \tag{2.4.18}$$

**Remark 2.4.7.** *The contraction constant $\beta$ here is a little different from the gap $\gamma(g_j)$, but $\gamma(g_j)/\beta$ is in the interval $[\delta^{O(1)}, \delta^{-O(1)}]$ by Lemma 2.2.53. Hence they are of the same largeness and we will not distinguish them.*

*The intuition here is that by controlling $\kappa(g), \eta_g^M, \zeta_g^m$, all the other position or length will also be controlled, which is similar to hyperbolic dynamics.*

*Proof.* For every $\alpha$ in $\Pi$, using Lemma 2.2.8 $2k$ times, we obtain the Lipschitz property. By Lemma 2.2.53, we have (2.4.16) from (2.4.15) for all $\alpha$ in $\Pi$ at the same time.

We use induction to prove the inclusion. For $j = k$, it is due to the hypothesis of Lemma 2.4.6.

Suppose that the property holds for $j + 1$. By definition, $t_j = g_j h_{j+1} t_{j+1}$. We abbreviate $g_j, h_{j+1}, t_{j+1}, \eta_{g_{j+1}}^M$ to $g, h, \eta, \eta'$. The condition becomes

$$d(\eta, \eta') \leq \delta, \|\kappa(g) - n\sigma_\mu\| \leq \epsilon n/C_A \text{ and } h \text{ is } (n, \epsilon, \eta', \zeta_g^m) \text{ good.}$$

By Lemma 2.2.53, we have $\gamma(h) \leq \beta\delta^{-1}$. By Lemma 2.2.13, due to $\eta \in B(\eta', \delta) \subset B_h^m(\delta)$, we have $h\eta \in b_h^M(\beta/\delta^2) \subset B_g^m(\delta)$. Therefore $gh\eta \in b_g^M(\beta/\delta^2)$, which is the inclusion condition.

Then we will prove (2.4.18) and we keep the notation $g, h, \eta, \eta'$.

$$\|\sigma(gh, \eta) - \sigma(gh, \eta')\| \ll \|\sigma(g, h\eta) - \sigma(g, h\eta')\| + \|\sigma(h, \eta) - \sigma(h, \eta')\|.$$

By the same argument, due to Lemma 2.2.13 and $\eta, \eta' \in B(\eta', \beta/\delta^2) \subset B_h^m(\delta)$, we have $h\eta, h\eta' \in b_h^M(\beta/\delta^2) \subset B_g^m(\delta)$. Therefore by the Lipschitz property of Lemma 2.2.13

$$\|\sigma(gh, \eta) - \sigma(gh, \eta')\| \ll (d(\eta, \eta') + d(h\eta, h\eta'))\delta \ll \beta/\delta^3.$$

The proof is complete. □

**Lemma 2.4.8.** *Suppose that* $\mathbf{g}, \mathbf{h}$ *satisfy the conditions* (2.4.14) *and* (2.4.15)*. Let $s$ be in $\{z \in \mathscr{P}_0 | d_0(z, z_{g_k}) \leq \delta\}$. Let $s_j = g_j h_{j+1} \cdots h_k s$ for $j = 0, \dots, k$, where we let $s_k = s$. We have*

$$\mathrm{m}(s_0, k_{g_0}) = \Pi_{1 \leq j \leq k} \mathrm{m}(\ell_{g_{j-1}}^{-1}, h_j k_{g_j}) = \Pi_{1 \leq j \leq k} m_j(h_j). \tag{2.4.19}$$

*Proof.* We let $\eta = \pi(s)$, then $\eta$ is in $b_{g_k}^M(\delta)$. By (2.4.17) with $j = 1$ and (2.4.15) with $j = 0$, Lemma 2.2.25 implies

$$\mathrm{m}(s_0, k_{g_0}) = \mathrm{m}(k_{g_0}, g_0 h_1 s_1) = \mathrm{m}(\ell_{g_0}^{-1}, h_1 k_{g_1}) \mathrm{m}(s_1, k_{g_1}).$$

Iterating this formula, we obtain the result. □

**Third step:** Here we mimic the proof of [BD17], where they heavily use the properties of Schottky groups and symbolic dynamics. But in our case, the group is much more complicate from the point of view of dynamics. We use the large deviation principle to get a similar formula.

By very careful control of $g_l$, with a loss of an exponentially small measure, we are able to rewrite the formula in a form to use the sum-product estimates. The key point is that by controlling the Cartan projection and the position of $\eta_g^M$ and $\zeta_g^m$ of each $g_l$, we are able to get good control of their product $\mathbf{g}\leftrightarrow\mathbf{h}$.

We should notice that the element $g_j$ will be fixed, and we will integrate first with respect to $h_j$. This gives the independence of the cocycle $\sigma(g_{j-1}h_j, \eta_{g_j}^M)$, that is for different $j$ they are independent, which is an important point to apply sum-product estimates.

We return to (2.4.12). We call $\mathbf{g}$ "good" with respect to $\eta, \eta'$ if

$$\begin{aligned}&\mathbf{g} \text{ satisfies (2.4.14)}, g_k \text{ satisfies conditions in Lemma 2.2.40}, \ \eta_{g_0}^M \in \operatorname{supp} r\\ &\text{and } \delta(\eta, \zeta_{g_k}^m), \delta(\eta', \zeta_{g_k}^m), \delta(V_{\alpha,\eta}\wedge V_{\alpha,\eta'}, y^m_{\wedge^2\rho_\alpha g_k}) \geq 4\delta.\end{aligned} \tag{2.4.20}$$

**Lemma 2.4.9.** *If $\eta$ and $\eta'$ are in good position and $\mathbf{g}$ is "good", then $g_k\eta, g_k\eta'$ are in $b_{g_k}^M(\delta)$, and for $\alpha\in\Pi$ the $d_\alpha$ distance between $g_k\eta$ and $g_k\eta'$ is almost $\beta_\alpha$, that is*

$$d_\alpha(g_k\eta, g_k\eta') \in \beta_\alpha[\delta^{O(1)}, \delta^{-O(1)}].$$

*Proof.* The inclusion is due to Lemma 2.2.13. Since $g$ is good (2.4.20), by (2.2.8) we have the lower bound and by the Lipschitz property in Lemma 2.2.8 we have the upper bound. □

For $\eta, \eta'$ in $\mathscr{P}$, we can rewrite the formula of $f(\eta,\eta')$ as

$$f(\eta,\eta') = \int e^{i\xi(\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta)-\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta'))} r(\mathbf{g}\leftrightarrow\mathbf{h}\eta) r(\mathbf{g}\leftrightarrow\mathbf{h}\eta')\mathrm{d}\mu_{k,n}(\mathbf{h})\mathrm{d}\mu_{k+1,n}(\mathbf{g}). \tag{2.4.21}$$

We call $\mathbf{h}$ is $\mathbf{g}$-regular if $\mathbf{h}$ satisfies (2.4.15). Let

$$f_{\mathbf{g}}(\eta,\eta') = \int_{\mathbf{g}-regular} e^{i\xi(\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta)-\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta'))}\mathrm{d}\mu_{k,n}(\mathbf{h}).$$

**Lemma 2.4.10.** *For $\eta, \eta'$ in $\mathscr{P}$*

$$|f(\eta,\eta')| \leq \int_{\mathbf{g}\text{``}good''} |f_{\mathbf{g}}(\eta,\eta')|\mathrm{d}\mu_{k+1,n}(\mathbf{g}) + O_\epsilon(\delta^c), \tag{2.4.22}$$

*if $\epsilon$ is small enough with respect to $\gamma$, that is $\epsilon \leq \min_{\alpha\in\Pi}\{\alpha\sigma_\mu\gamma/(2+2\gamma)\}$.*

*Proof.* Let

$$\tilde{f}_{\mathbf{g}}(\eta,\eta') = \int_{\mathbf{g}-regular} e^{i\xi(\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta)-\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta'))} r(\mathbf{g}\leftrightarrow\mathbf{h}\eta) r(\mathbf{g}\leftrightarrow\mathbf{h}\eta')\mathrm{d}\mu_{k,n}(\mathbf{h}).$$

We call $\mathbf{g}$ "semi-good" if $\mathbf{g}$ satisfies (2.4.20) except the assumption of $\eta_{g_0}^M \in \operatorname{supp} r$ in (2.4.20). By large deviation principle (Proposition 2.2.46, Proposition 2.2.48, Lemma 2.2.58), we conclude that

$$\mu_{k+1,n}\{g \text{ not ``semi-good'' }\} \leq O_\epsilon(\delta^c). \tag{2.4.23}$$

Then by (2.4.21), Lemma 2.2.52 and (2.4.23),

$$|f(\eta,\eta')| \leq \int_{\mathbf{g}} |\tilde{f}_{\mathbf{g}}(\eta,\eta')| \mathrm{d}\mu_{k+1,n}(\mathbf{g}) + O_\epsilon(\delta^c) \leq \int_{\mathbf{g}\text{``}semi-good''} |\tilde{f}_{\mathbf{g}}(\eta,\eta')| \mathrm{d}\mu_{k+1,n}(\mathbf{g}) + O_\epsilon(\delta^c). \tag{2.4.24}$$

By Lemma 2.4.9, (2.4.17) with $j=0$ and $c_\gamma(r) \leq \xi^{\epsilon_0} \leq \delta^{-1}$,

$$|r(\eta_{g_0}^M)^2 - r(\mathbf{g}\leftrightarrow\mathbf{h}\eta) r(\mathbf{g}\leftrightarrow\mathbf{h}\eta')| \leq 2\|r\|_\infty c_\gamma(r)(\beta\delta^{-2})^\gamma \leq 2\beta^\gamma\delta^{-1-2\gamma} \leq 2\delta,$$

if $\epsilon$ is small enough with respect to $\gamma$. Hence

$$\begin{aligned} |\tilde{f}_{\mathbf{g}}(\eta,\eta')| &\leq \left| \int_{\mathbf{g}-regular} e^{i\xi(\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta)-\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta'))} r(\eta_{g_0}^M)^2 \mathrm{d}\mu_{k,n}(\mathbf{h}) \right| + O(\delta^c) \\ &\leq r(\eta_{g_0}^M)^2 |f_{\mathbf{g}}(\eta,\eta')| + O(\delta^c). \end{aligned} \tag{2.4.25}$$

If $r(\eta_{g_0}^M) \neq 0$, then that $\mathbf{g}$ is "semi-good" implies $\mathbf{g}$ is "good". Combined with (2.4.24) and (2.4.25), by $\|r\|_\infty \leq 1$, we have

$$\begin{aligned} |f(\eta,\eta')| &\leq \int_{\mathbf{g}\text{``}semi-good''} \left( r(\eta_{g_0}^M)^2 |f_{\mathbf{g}}(\eta,\eta')| + O(\delta^c) \right) \mathrm{d}\mu_{k+1,n}(\mathbf{g}) + O_\epsilon(\delta^c) \\ &\leq \int_{\mathbf{g}\text{``}good''} |f_{\mathbf{g}}(\eta,\eta')| \mathrm{d}\mu_{k+1,n}(\mathbf{g}) + O_\epsilon(\delta^c). \end{aligned}$$

The proof is complete. □

Recall that $\beta$ is the magnitude which is really small, $\delta$ is only an error term and $\tau$ is the frequency for applying the sum-product estimate, which lies between $\delta^{-1}$ and $\beta^{-1}$.

**Proposition 2.4.11.** *Let $I_\tau = [\tau^{3/4}, \tau^{5/4}]$. The following formula is true for $\eta, \eta'$ in good position and $\mathbf{g}$ "good",*

$$|f_{\mathbf{g}}(\eta,\eta')| \leq \sup_{\|\varsigma\| \in I_\tau} \left| \int e^{i\langle \varsigma, x_1 \cdots x_k \rangle} \mathrm{d}\lambda_1(x_1) \cdots \lambda_k(x_k) \right| + O(\beta\delta^{-O(1)}\tau), \tag{2.4.26}$$

*when $\epsilon$ is small enough with respect to $\epsilon_2$.*

**Remark 2.4.12.** *This is the most complicate step, where the difficulty comes from higher rank. We need to use the technique of changing flags to find the direction of slowest contraction speed, where we can use Newton-Leibniz's formula. Since the action of the sign group $M$ is non trivial on the slowest directions, we also carefully treat the sign.*

*Proof.* The element $\eta, \eta'$ and $\mathbf{g}$ are already fixed. Since $g_k$ satisfies the conditions in Lemma 2.2.40, we obtain two chains $(\eta = \eta_o, \eta_1, \dots, \eta_{l_1})$ and $(\eta' = \eta'_o, \eta'_1, \dots, \eta'_{l_2})$ as in Lemma 2.2.40. Then we write

$$\begin{aligned} \varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta) - \varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta') &= \sum_{0\leq j\leq l_1-1} (\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta_j) - \varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta_{j+1})) \\ - \sum_{0\leq j\leq l_2-1} (\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta'_j) &- \varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta'_{j+1})) + \left(\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta_{l_1}) - \varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta'_{l_2})\right), \end{aligned} \tag{2.4.27}$$

The terms for different $j$ and for $\eta,\eta'$ are similar. We fix $j$ and we simplify $\alpha(\eta_j,\eta_{j+1})$ to $\alpha$.

**We compute the term $\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta_j)-\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta_{j+1})$.** In order to treat the sign, we will work on $\mathscr{P}_0=G/A_eN$. Recall that $\pi:\mathscr{P}_0\to\mathscr{P}$ is the projection and we use $z=kz_o$ to denote the element $kA_eN$ in $\mathscr{P}_0$.

By Lemma 2.2.42 and (2.4.20), we know that $g_k\eta_j, g_k\eta_{j+1}$ are in $b^M_{g_k}(\delta)$, which satisfy the condition of Lemma 2.4.6. Let $z_0,z_1$ be preimages of $g_k\eta_j$ and $g_k\eta_{j+1}$ in $\mathscr{P}_0$ such that $\mathrm{m}(z_0,z_1)=e$. Notice that $z_0,z_1$ are in the same $\alpha$-circle. By Lemma 2.2.40 (2.2.25) and Lemma 2.4.9

$$d(g_k\eta_j,g_k\eta_{j+1})=d_\alpha(g_k\eta,g_k\eta')+O(\beta e^{-\alpha\kappa(g_k)}\delta^{O(1)})\in\beta_\alpha[\delta^{O(1)},\delta^{-O(1)}].$$

Due to $\mathrm{m}(z_0,z_1)=e$, the arc length distance also satisfies

$$d_A(z_0,z_1)=\arcsin d(g_k\eta_j,g_k\eta_{j+1})\in\beta_\alpha[\delta^{O(1)},\delta^{-O(1)}].\tag{2.4.28}$$

Now, we lift $\varphi$ to $\mathscr{P}_0$, becoming a right $M$-invariant function. By abuse of notation, we also use $\varphi$ to denote the lifted function. Let $\gamma$ be an arc connecting $z_0,z_1$ with unit speed in the $\alpha$-circle with length less than $\pi/2$. Without loss of generality, we suppose that $\gamma$ is in the positive direction (If not, we add minus in the right hand side of (2.4.29)). By Newton-Leibniz's formula (2.2.23), we have

$$\varphi(T\mathbf{g}\leftrightarrow\mathbf{h}z_0)-\varphi(T\mathbf{g}\leftrightarrow\mathbf{h}z_1)=\int_0^u\partial_\alpha\varphi(T\mathbf{g}\leftrightarrow\mathbf{h}\gamma(t))e^{-\alpha\sigma(T\mathbf{g}\leftrightarrow\mathbf{h},\gamma(t))}\mathrm{d}t,\tag{2.4.29}$$

where $u=d_A(z_0,z_1)$. Fix a time $t$ in $[0,u]$, let $s_j=g_jh_{j+1}\cdots h_k\gamma(t)$. Then $\pi(\gamma(t))$ is in $b^M_{g_k}(\delta)$, because $g_k\eta_j$ and $g_k\eta_{j+1}$ are in $b^M_{g_k}(\delta)$ and by (2.4.28). By (2.4.17), the element $\pi(s_0)$, the image of $s_0=T\mathbf{g}\leftrightarrow\mathbf{h}\gamma(t)$ in $\mathscr{P}$, is in $b^M_{g_0}(\beta\delta^{-O(1)})$.

Recall that we have made a choice of the Cartan decomposition of every $g_j$ for $0\le j\le k$. In particular, $k_{g_0}$ is given in the decomposition of $g_0=k_{g_0}a_{g_0}\ell_{g_0}\in KA^+K$. Let $m_0=\mathrm{m}(s_0,k_{g_0})$ and $\underline{s_0}=s_0m_0$, then $\mathrm{m}(\underline{s_0},k_{g_0})=e$. By Lemma 2.2.31,

$$\partial_\alpha\varphi_{s_0}=\partial_\alpha\varphi_{\underline{s_0}m_0}=\alpha^\sharp(m_0)\partial_\alpha\varphi_{\underline{s_0}}.\tag{2.4.30}$$

By Lemma 2.5.5 and $\pi s_0,\pi z_{g_0}=\eta^M_{g_0}$ in $b^M_{g_0}(\beta\delta^{-O(1)})$, we have

$$d_0(\underline{s_0},z_{g_0})\le d(\pi s_0,\pi z_{g_0})<\beta\delta^{-O(1)}.\tag{2.4.31}$$

Due to $\mathbf{g}$ good (2.4.20), we have $\eta^M_{g_0}\in\mathrm{supp}r$. By G2 assumption (2.4.2), we have $|\partial_\alpha\varphi(z_{g_0})|\ge\delta v_\alpha$. By (2.4.31), the point $\pi s_0$ is in $J$, the $\delta$ neighbourhood of $\mathrm{supp}r$. By G3 assumption (2.4.3), $|\partial_\alpha\varphi(\underline{s_0})-\partial_\alpha\varphi(z_{g_0})|\le\delta^{-1}v_\alpha d_0(\underline{s_0},z_{g_0})$, which implies

$$\partial_\alpha\varphi(\underline{s_0})/\partial_\alpha\varphi(z_{g_0})\in[1-\beta\delta^{-O(1)},1+\beta\delta^{-O(1)}].$$

By Lemma 2.4.6 (2.4.18), we have

$$(1-\beta\delta^{-O(1)})e^{-O(\beta/\delta)}\le\frac{\partial_\alpha\varphi(\underline{s_0})e^{-\alpha\sigma(g_0h_1,s_1)}\cdots e^{-\alpha\sigma(g_{k-1}h_k,s_k)}}{\partial_\alpha\varphi(z_{g_0})e^{-\alpha\sigma(g_0h_1,x^M_{g_1})}\cdots e^{-\alpha\sigma(g_{k-1}h_k,x^M_{g_k})}}\le(1+\beta\delta^{-O(1)})e^{O(\beta/\delta)}.\tag{2.4.32}$$

By (2.4.16),

$$B_\alpha := e^{-\alpha\sigma(g_0h_1,x_{g_1}^M)}\cdots e^{-\alpha\sigma(g_{k-1}h_k,x_{g_k}^M)} \leq \beta_\alpha^{2k}\delta^{-O(1)}.$$

Together with (2.4.28)-(2.4.32)

$$|\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta_j)-\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta_{j+1})-d_A(z_0,z_1)\alpha^\sharp(m_0)\partial_\alpha\varphi(z_{g_0})B_\alpha| \leq \beta\beta_\alpha^{2k+1}\delta^{-O(1)}v_\alpha. \tag{2.4.33}$$

**We deal with the error term which comes from the process of changing flags**. The Lipschitz property in Lemma 2.4.6 and Lemma 2.2.40 (2.2.26) imply that

$$d_\alpha(\mathbf{g}\leftrightarrow\mathbf{h}\eta_{l_1},\mathbf{g}\leftrightarrow\mathbf{h}\eta_{l_2}') \leq \beta_\alpha^{2k}\delta^{-O(1)}d_\alpha(g_k\eta_{l_1},g_k\eta_{l_2}') \leq \beta_\alpha^{2k+1}\beta\delta^{-O(1)},$$

Due to (2.4.17) in Lemma 2.4.6 and Lemma 2.2.42, the two points $\mathbf{g}\leftrightarrow\mathbf{h}\eta_{l_1},\mathbf{g}\leftrightarrow\mathbf{h}\eta_{l_2}'$ are in $J$, the $\delta$ neighbourhood of supp$r$. Due to G1 assumption (2.4.1)

$$|\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta_{l_1})-\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta_{l_2}')| \leq \delta^{-1}\sum_\alpha v_\alpha d_\alpha(\mathbf{g}\leftrightarrow\mathbf{h}\eta_{l_1},\mathbf{g}\leftrightarrow\mathbf{h}\eta_{l_2}').$$

Therefore

$$|\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta_{l_1})-\varphi(\mathbf{g}\leftrightarrow\mathbf{h}\eta_{l_2}')| \leq \delta^{-O(1)}\beta\sum_\alpha v_\alpha\beta_\alpha^{2k+1}. \tag{2.4.34}$$

**We collect information for different simple roots.** Recall that for a fixed $g$ in $G$ and for $h\in G$, $z\in\mathscr{P}_0$, we have defined $\tilde{Y_g^n}(h,z)_\alpha = e^{-\alpha(\sigma(gh,z)-\kappa(g)-n\sigma_\mu)}\alpha(\mathrm{m}(\ell_g,hk))$. Let

$$\varsigma_\alpha := \frac{\xi d_A(z_0,z_1)\alpha^\sharp(m_0)\partial_\alpha\varphi(z_{g_0})B_\alpha}{\Pi_{l=1}^k Y_{g_{l-1}}^{\tilde{n}}(h_l,z_{g_l})_\alpha}.$$

Let $\varsigma=(\varsigma_\alpha)_{\alpha\in\Pi}\in\mathbb{R}^m$. Hence by (2.4.27), (2.4.33), (2.4.34) and (2.4.8)

$$|\xi(\varphi(\mathbf{g}\leftrightarrow\mathbf{h}x)-\varphi(\mathbf{g}\leftrightarrow\mathbf{h}x'))-\langle\varsigma,\Pi_{l=1}^k Y_{g_{l-1}}^{\tilde{n}}(h_l,z_{g_l})\rangle| \leq \beta\delta^{-O(1)}\sum_\alpha \beta_\alpha^{2k+1}v_\alpha\xi \ll \beta\delta^{-O(1)}\tau. \tag{2.4.35}$$

We want to verify that $\|\varsigma\|\in I_\tau$. By (2.4.19), we have

$$\varsigma_\alpha = \xi d_A(z_0,z_1)\partial_\alpha\varphi(z_{g_0})\beta_\alpha^k e^{-\alpha\kappa(g_0)-\cdots-\alpha\kappa(g_{k-1})}.$$

By (2.4.14), (2.4.28), (2.4.20) and (2.4.2) we have $|\varsigma_\alpha|\in\xi v_\alpha\beta_\alpha^{2k+1}[\delta^{O(1)},\delta^{-O(1)}]$. Therefore by (2.4.8),

$$\|\varsigma\|\in\sup_\alpha \xi v_\alpha\beta_\alpha^{2k+1}[\delta^{O(1)},\delta^{-O(1)}]\in\tau[\delta^{O(1)},\delta^{-O(1)}]\subset[\tau^{3/4},\tau^{5/4}]=I_\tau.$$

By definition, the distribution of $Y_{g_{l-1}}^{\tilde{n}}(h_l,z_{g_l})$, where $h_l$ satisfies (2.4.15) with distribution $\mu^{*n}$, is the measure $\lambda_l$. Finally, due to $|e^{ix}-e^{iy}|\leq|x-y|$ for $x,y\in\mathbb{R}$, the inequality (2.4.35) implies (2.4.26). □

**Fourth step:** We are able to apply sum-product estimates.

*Proof of Theorem 2.1.7.* For $l = 1, 2, \ldots k$, Proposition 2.3.18 and Lemma 2.4.2 tell us that with $\epsilon$ small enough depending on $\epsilon_4\epsilon$, there exists $C_0$ such that the measures $\lambda_l$ satisfy the assumptions in Proposition 2.3.17 with $\tau$.

Proposition 2.3.17 implies that for $\tau$ large enough,

$$\left|\int \exp(i\langle\varsigma, x_1\cdots x_k\rangle)\mathrm{d}\lambda_1(x_1)\ldots\mathrm{d}\lambda_k(x_k)\right| \leq \tau^{-\epsilon_3}.$$

Then by (2.4.12), (2.4.22) and (2.4.26), we have

$$\left|\int e^{i\xi\varphi(\eta)}r(\eta)\mathrm{d}\nu(\eta)\right|^2 \leq O_\epsilon(\delta^c) + O(\beta\delta^{-O(1)}\tau) + \tau^{-\epsilon_3}.$$

Due to $\beta\delta^{-O(1)}\tau = \max_{\alpha\in\Pi} e^{(-\alpha\sigma_\mu + O(1)\epsilon+\epsilon_2)n}$, take $\epsilon$ small enough. The proof is complete. $\square$

**Remark 2.4.13.** *Another difference with [BD17] is that we avoid using the renewal idea, which simplifies the proof of this part. The renewal idea is that instead of using $\mu^{*n}$, we use a renewal measure $\mu_t$, which is defined to be the distribution of $g_1\cdots g_n$ for the first time that its Cartan projection exceeds $t$, where $g_1, g_2\ldots$ are i.i.d. random variables with distribution $\mu$. This is because we generalize the sum-product estimate to a form that the measure can have a support which depends on the frequency, and we use the large deviation principle to prove that our measure has a support not too large with respect to the frequency.*

### 2.4.3 Examples of Fourier decay

In this section, we give a nice application of Theorem 2.1.7, that is Theorem 2.1.1. This application also serves as a "baby case" for Section 2.4.4.

Recall that $v_0$ is a unit vector in $V$ and $\varsigma$ is a vector in $v_0^\perp$. We fix the direction, that is $u_0 := \varsigma/\|\varsigma\|$, and we let $\xi = \|\varsigma\|$. Then for $x = \mathbb{R}v$, we have $\langle\varsigma, \psi(v)\rangle = \xi\langle u_0, \psi(v)\rangle = \xi\langle u_0, v\rangle/\langle v_0, v\rangle$, and we take

$$\varphi(x) = \frac{\langle u_0, v\rangle}{\langle v_0, v\rangle}.$$

Since we are only interested in the value on the support of $\nu_V$, which is contained in the image of $\mathscr{P}$ in $\mathbb{P}V$. The functions $\varphi$, $r$ can be lifted to functions on $\mathscr{P}$. We use the same notation $\varphi$ to denote the lifted functions. We first calculate the directional derivative of $\varphi$. Recall that the inner product on the exterior square $\wedge^2 V$ is given by

$$\langle v_1\wedge v_2, w_1\wedge w_2\rangle = \langle v_1, w_1\rangle\langle v_2, w_2\rangle - \langle v_1, w_2\rangle\langle v_2, w_1\rangle, \tag{2.4.36}$$

for $v_1, v_2, w_1, w_2$ in $V$. Recall that $q_{2\chi-\alpha}$ is the projection of $\wedge^2 V$ on the subrepresentations of highest weight $2\chi - \alpha$. By the same proof as in Lemma 2.2.3, we see that the multiplicity of an irreducible representation of highest weight $2\chi-\alpha$ is at most 1 in $\wedge^2 V$. Hence the image of the projection $q_{2\chi-\alpha}$ is an irreducible subrepresentation or zero. Let $e_1$ be a unit vector of highest weight in $V$.

**Lemma 2.4.14.** *Let $v_0, u_0$ be two unit vectors in $V$. Let $\varphi$ be defined as above. Then for a simple root $\alpha$ and $z = kz_0 \in \mathscr{P}_0$,*

$$\partial_\alpha \varphi(z) = \frac{\langle v_0 \wedge u_0, v \wedge u\rangle}{\langle v_0, v\rangle^2} = \frac{\langle q_{2\chi-\alpha}(v_0 \wedge u_0), v \wedge u\rangle}{\langle v_0, v\rangle^2}, \tag{2.4.37}$$

*where $v = ke_1$ and $u = kY_\alpha e_1$.*

*Proof.* By definition,

$$\partial_\alpha \varphi(z) = \partial_t \left.\frac{\langle u_0, k\exp(tY_\alpha)e_1\rangle}{\langle v_0, k\exp(tY_\alpha)e_1\rangle}\right|_{t=0} = \frac{\langle u_0, kY_\alpha e_1\rangle\langle v_0, ke_1\rangle - \langle u_0, ke_1\rangle\langle v_0, kY_\alpha e_1\rangle}{\langle v_0, ke_1\rangle^2}.$$

By (2.4.36), we have the first equality. The vector $e_1 \wedge Y_\alpha e_1$ is a vector of weight $2\chi - \alpha$, which is in the irreducible subrepresentation of $\wedge^2 V$ with highest weight $2\chi - \alpha$. The vector $v \wedge u = k(e_1 \wedge Y_\alpha e_1)$ is also in this subrepresentation, hence

$$\langle v_0 \wedge u_0, v \wedge u\rangle = \langle q_{2\chi-\alpha}(v_0 \wedge u_0), v \wedge u\rangle.$$

The proof is complete. □

For a vector $v$ in an euclidean space $W$, let $v^*$ be the linear linear functional on $W$ given by

$$v^*(w) = \langle v, w\rangle \quad \text{for } w \in W.$$

*Proof of Theorem 2.1.1 from Theorem 2.1.7.* Let $\delta > 0$ be a constant to be fixed later. Recall that $\varphi$, $r$ have been lifted to functions on $\mathscr{P}$. In order to use Theorem 2.1.7, we need to verify the $(\xi^{\epsilon_0}, r)$ goodness assumption for $\varphi$. Let $C_0 > 0$ be a constant such that

$$c_\gamma(r) \le C_0,\ |\langle v_0, v\rangle| \ge \|v\|/C_0 \text{ for } \mathbb{R}v \in V_{\chi,\eta} \text{ and } \eta \in \operatorname{supp} r, \tag{2.4.38}$$

$$\max_{\alpha\in\Pi} \frac{\|q_{2\chi-\alpha}(v \wedge u)\|}{\|v \wedge u\|} \ge 1/C_0 \text{ for every couple } v, u \text{ in } V \text{ with } v \wedge u \ne 0. \tag{2.4.39}$$

The existence of $C_0$ for (2.4.39) is due to Lemma 2.2.56.

We want to verify that $\varphi$ is $(\xi^{\epsilon_0}, r)$ good. Let $l_\alpha = q_{2\chi-\alpha}(v_0 \wedge u_0)$ and $\theta_\alpha = \|l_\alpha\|$. The main problem is to verify G2, because $\partial_\alpha\varphi$ may vanish. We need a cutoff. Let $\tau$ be a smooth function on $\mathbb{R}$ such that $\tau|_{[0,\infty)} = 1$, $\tau$ takes values in $[0,1]$, $\operatorname{supp}\tau \subset [-1,\infty)$ and $|\tau'| \le 2$. Set $\tau_\delta(x) = \tau(x/\delta)$ for $x \in \mathbb{R}$. Let $r_1 = r \cdot \Pi_{\alpha\in\Pi}\tau_\alpha$, where

$$\tau_\alpha = \tau_\delta(\delta(V_{2\chi-\alpha,\eta}, \mathbb{R}l_\alpha^*) - 2\delta).$$

If $l_\alpha = 0$, then we let $\tau_\alpha = 1$. Let $J$ be the $\xi^{-\epsilon_0}$ neighbourhood of the support of $r_1$, an open set of $\mathscr{P}$. When $\xi$ is large enough, we can suppose that for $\eta \in J$ and $v \in V_{\chi,\eta}$ we have $|\langle v_0, v\rangle| > \|v\|/(2C_0)$.

**We claim that if $\delta = \xi^{-\epsilon_0/2}$ and $\xi$ is large enough such that $\delta^{-1} \ge CC_0^{m+4}$**, where $C$ is a constant only depending on the group $G$ and the norm on $V$, which is defined in Lemma 2.5.7. Then $\varphi, r_1$ satisfy the assumptions of Theorem 2.1.7.

For $\eta$ in the support of $r_1$, due to (2.4.37) and $\delta(V_{2\chi-\alpha,\eta}, \mathbb{R}l_\alpha^*) > \delta$, we have

$$|\partial_\alpha\varphi(\eta)| = \frac{|\langle l_\alpha, v \wedge u\rangle|}{\langle v_0, v\rangle^2} = \theta_\alpha \delta(V_{2\chi-\alpha,\eta}, \mathbb{R}l_\alpha^*)\langle v_0, v\rangle^{-2} \geq \delta\theta_\alpha. \tag{2.4.40}$$

Due to $|\langle v_0, v\rangle| \geq \|v\|/C_0$ for $\eta \in \operatorname{supp} r$ and $v \in V_{\chi,\eta}$,

$$v_\alpha = \sup_{\eta\in\operatorname{supp} r_1} |\partial_\alpha\varphi(\eta)| = \sup_{\eta\in\operatorname{supp} r_1} \frac{|\langle l_\alpha, v \wedge u\rangle|}{\langle v_0, v\rangle^2} \leq C_0^2\theta_\alpha. \tag{2.4.41}$$

Then for $\eta$ in $\operatorname{supp} r_1$, by (2.4.40) and (2.4.41) we have

$$|\partial_\alpha\varphi(\eta)| \geq C_0^{-2}\delta v_\alpha$$

which implies G2 assumption (2.4.2). The inequality (2.4.40) also implies that

$$v_\alpha \geq \delta\theta_\alpha, \tag{2.4.42}$$

that is $v_\alpha$ and $\theta_\alpha$ are of the same magnitude. Hence by (2.4.39), we have

$$\sup_{\alpha\in\Pi} v_\alpha \in [\delta, C_0^2] \sup_{\alpha\in\Pi} \theta_\alpha \subset [\delta C_0^{-1}, C_0^2],$$

which is G4 assumption (2.4.4).

Now, we verify G1 assumption (2.4.1). If $\chi$ is a fundamental weight $\chi_\alpha$, then (2.4.39) implies $\theta_\alpha = \|q_{2\chi-\alpha}(v_0 \wedge u_0)\| \geq 1/C_0$. Hence, for $\eta, \eta'$ in $J$ and unit vectors $v \in V_{\chi,\eta}$, $v' \in V_{\chi,\eta'}$, we have

$$\begin{aligned}|\varphi(\eta) - \varphi(\eta')| &= \left|\frac{\langle u_0 \wedge v_0, v \wedge v'\rangle}{\langle v_0, v\rangle\langle v_0, v'\rangle}\right| \leq 4C_0^2\|u_0 \wedge v_0\|\|v \wedge v'\| \leq 4C_0^3\theta_\alpha d(V_{\alpha,\eta}, V_{\alpha,\eta'}) \\ &\leq 4\delta^{-1}C_0^3 v_\alpha d(V_{\alpha,\eta}, V_{\alpha,\eta'}).\end{aligned} \tag{2.4.43}$$

For general case, this step is more complicate. Please see Lemma 2.5.7.

For G3 assumption (2.4.3), for $z = kz_0, z' = k'z_0$ in $\pi^{-1}(J) \subset \mathscr{P}_0$ and $v' = k'e_1, u' = k'Y_\alpha e_1$

$$\begin{aligned}|\partial_\alpha\varphi(z) - \partial_\alpha\varphi(z')| &\leq C_0^4(|\langle l_\alpha, v \wedge u - v' \wedge u'\rangle| + |\langle l_\alpha, v \wedge u\rangle\langle v_0, v - v'\rangle|) \\ &\ll C_0^4\theta_\alpha d_0(z, z')\end{aligned}$$

where the last inequality is due to Lemma 2.5.3.

We also need to calculate $c_\gamma(r_1)$. Lemma 2.2.14 implies $c_\gamma(\tau_\alpha) \ll \delta^{-\gamma}$. Hence $c_\gamma(r_1) \ll \delta^{-\gamma} + c_\gamma(r) \leq \delta^{-\gamma} + C_0$. The claim is true and Theorem 2.1.7 implies that

$$\left|\int e^{i\xi\varphi(\eta)} r_1(\eta)\mathrm{d}\nu(\eta)\right| \leq \xi^{-\epsilon_1}.$$

Finally, by regularity of stationary measure, Corollary 2.2.50, the set where $r_1 \neq r$ has measure bounded by $O(\delta^c) = O(\xi^{-\epsilon_0 c/2})$, that is there exist $C, c > 0$ such that for all $\delta > 0$

$$\nu\{\eta \in \mathscr{P} |\ \delta(V_{2\chi-\alpha,\eta}, \mathbb{R}l_\alpha^*) \leq 2\delta\} \leq C\delta^c.$$

The proof is complete. □

**Remark 2.4.15.** *In higher dimension, the differential* $\mathrm{d}\varphi$ *at a point always vanishes in some direction of the tangent space. The cutoff in the proof can be understood as removing a neighbourhood of the zero locus of* $\mathrm{d}\varphi$ *in the unit tangent bundle of* $\mathbb{P}V$*. The language of flag variety makes the proof obscure, but this language is really powerful.*

### 2.4.4 From Fourier decay to spectral gap

#### Derivative of the cocycle

This part is devoted to the derivative of the cocycle. The results of this part imply that for most $g, h$ in $G$, the difference of the Iwasawa cocycle $\sigma(g,\cdot) - \sigma(h,\cdot)$ satisfies the $(C, r)$ good condition in Definition 2.4.1 (See Lemma 2.4.25). Since the $\alpha$-bundle is trivial on $\mathscr{P}_0$, we will work on $\mathscr{P}_0$. We need to lift the Iwasawa cocycle $\sigma$ to $\mathscr{P}_0$ and we use the same notation $\sigma$.

Let $V$ be an irreducible representation of $G$ with a good norm. Recall that $\sigma_V(g, x) = \frac{\|\rho(g)v\|}{\|v\|}$ for $g$ in $G$ and $v$ in $V$. We will abbreviate $\rho g$ to $g$ in the proof, because $(\rho, V)$ is the only representation to be studied in this part. Let $\alpha$ be a simple root. Let $e_1$ be a unit vector of highest weight in $V$ and let $e_2 = Y_\alpha e_1$.

**Lemma 2.4.16.** *Let* $V$ *be an irreducible representation of* $G$ *with a good norm. For* $z = kz_o$ *in* $\mathscr{P}_0$*, we have*

$$\partial_\alpha \sigma_V(g, z) = \frac{\langle \rho g v, \rho g u\rangle}{\|\rho g v\|^2},$$

*where* $v = ke_1$ *and* $u = ke_2$*.*

*Proof.* Without loss of generality, we suppose that $z = z_o$. Since $Y_\alpha$ is a left $K$ invariant vector field on $\mathscr{P}_0$, we have

$$\begin{aligned}\partial_{Y_\alpha}\sigma_V(g, e) &= \partial_t \sigma_V(g, \exp(tY_\alpha)z_o)|_{t=0} = \partial_t\left(\log\frac{\|g\exp(tY_\alpha)e_1\|}{\|\exp(tY_\alpha)e_1\|}\right)\Big|_{t=0}\\ &= \frac{\langle ge_1, gY_\alpha e_1\rangle}{\|ge_1\|^2} - \frac{\langle e_1, Y_\alpha e_1\rangle}{\|e_1\|^2}.\end{aligned}$$

Since the norm is good, eigenvectors of different weights are orthogonal, we have $\langle e_1, Y_\alpha e_1\rangle = 0$. The result follows. □

Form this lemma, we know that the derivative of the cocycle $\sigma_V$ in the direction $Y_\alpha$ is nonzero only if $\chi - \alpha$ is a weight of $V$. Lemma 2.2.32 only implies that $\partial_\alpha \sigma_{\alpha'} = 0$ for $\alpha \neq \alpha'$, which works for fundamental representations $\sigma_{V_{\alpha'}} = \sigma_{\alpha'}$. We fix the distance $d_0$ on $\mathscr{P}_0$, which is defined in Appendix 2.5.2.

**Lemma 2.4.17.** *Let* $\delta < 1/2$*. Let* $\widetilde{B^m_{V,g}(\delta)}$ *be the preimage of* $B^m_{V,g}(\delta) \subset \mathbb{P}V$ *in* $\mathscr{P}_0$*. For* $z = kz_o \in \widetilde{B^m_{V,g}(\delta)}$*,*

$$|\partial_\alpha \sigma_V(g, z)| \le \delta^{-O(1)}. \tag{2.4.44}$$

*We also have*

$$Lip_{\mathscr{P}_0}(\partial_\alpha \sigma_V(g,\cdot)|_{\widetilde{B^m_{V,g}(\delta)}}) \le \delta^{-O(1)}. \tag{2.4.45}$$

*Proof.* By Lemma 2.4.16, the hypothesis that $\mathbb{R}ke_1 \in B^m_{V,g}(\delta)$ and (2.2.7)

$$|\partial_\alpha \sigma_V(g,z)| = \left|\frac{\langle gke_1, gke_2\rangle}{\|gke_1\|^2}\right| \leq \frac{\|Y_\alpha\|\|g\|^2\|e_1\|^2}{\|g\|^2\delta^2\|e_1\|^2}.$$

Since the operator norm of $Y_\alpha$ is bounded, we have

$$|\partial_\alpha \sigma_V(g,z)| \leq \delta^{-O(1)}.$$

The estimate of Lipschitz norm is more complicate. Let $v = ke_1, v' = k'e_1, u = ke_2, u' = k'e_2$. We have

$$|\partial_\alpha \sigma_V(g,z) - \partial_\alpha \sigma_V(g,z')| = \frac{|\langle gv, gu\rangle\|gv'\|^2 - \langle gv', gu'\rangle\|gv\|^2|}{\|gv\|^2\|gv'\|^2}.$$

By the same argument, due to $v = ke_1 \in B^m_{V,g}(\delta)$, we use (2.2.7) to give a lower bound of the denominator, that is

$$\|gv\|^2\|gv'\|^2 \geq \delta^4\|g\|^4\|v\|^2\|v'\|^2 = \delta^4\|g\|^4\|e_1\|^4.$$

Use the difference to give a upper bound of the numerator, that is

$$\begin{aligned}&|\langle gv, gu\rangle\|gv'\|^2 - \langle gv', gu'\rangle\|gv\|^2| \\ &\ll \|g\|^3\|e_1\|^3(\|gv - gv'\| + \|gu - gu'\|) \ll \|g\|^4\|v\|^3(\|v - v'\| + \|u - u'\|).\end{aligned}$$

Therefore we have

$$|\partial_\alpha \sigma_V(g,z) - \partial_\alpha \sigma_V(g,z')| \ll \delta^{-O(1)}(\|ke_1 - k'e_1\| + \|ke_2 - k'e_2\|).$$

Then by Lemma 2.5.3, the proof is complete. □

Let $V$ be a finite dimensional vector space with euclidean norm. Recall that $\wedge^2 Sym^2 V$ is the exterior square of the symmetric square of $V$. It is a linear space generated by vectors of the form $v_1v_2 \wedge v_3v_4$ where $v_i$ are in $V$, for $i = 1,2,3,4$. For $g, h$ in $GL(V)$, let $F_{g,h}$ be the linear functional on $\wedge^2 Sym^2 V$, whose action on the vector $v_1v_2 \wedge w_1w_2$ is defined by

$$F_{g,h}(v_1v_2 \wedge w_1w_2) = \langle hv_1, hv_2\rangle\langle gw_1, gw_2\rangle - \langle gv_1, gv_2\rangle\langle hw_1, hw_2\rangle.$$

This formula is well defined because $v_1, v_2$ and $w_1, w_2$ are symmetric, respectively. We also have $F_{g,h}(v_1v_2 \wedge w_1w_2) = -F_{g,h}(w_1w_2 \wedge v_1v_2)$. Since the vectors of form $v_1v_2 \wedge w_1w_2$ generate the space $\wedge^2 Sym^2 V$, the linear form $F_{g,h}$ is uniquely defined.

Suppose that $V$ is a super proximal representation of $G$ with highest weight $\chi$ (Definition 2.2.2). Let $\alpha$ be the unique simple root such that $\chi - \alpha$ is a weight of $V$. The space $\wedge^2 Sym^2 V$ may be reducible. The two highest weights of $Sym^2 V$ are $2\chi$, $2\chi - \alpha$, whose eigenspaces have dimension 1. Hence, the highest weight of $\wedge^2 Sym^2 V$ is $4\chi - \alpha$, and the eigenspace has dimension 1. Let $W$ be the irreducible subrepresentation of $\wedge^2 Sym^2 V$ with the highest weight $\chi_1 := 4\chi - \alpha$. In the following lemma, we abbreviate $\rho(g), \rho(h)$ to $g, h$.

**Lemma 2.4.18.** *Let $\delta < 1/2$. Let $V$ be a super proximal representation of $G$ and let $\alpha$ be the unique simple root such that $\chi - \alpha$ is a weight of $V$. Recall that $V_{\chi 1,\eta}$ is the image of $\eta \in \mathscr{P}$ in $\mathbb{P}W$. If $g, h$ in $G$ and $z = kz_o \in \mathscr{P}_0, \eta = \pi(z)$ satisfy*

*(1) $\ell_h^{-1}V^{\chi}, \ell_h^{-1}V^{\chi-\alpha} \in B^m_{V,g}(\delta), \gamma_{1,2}(g) \le \delta^3$,*

*(2) $\delta(V_{\chi 1,\eta}, F_{g,h}|_W) > \delta$ and $V_{\chi,\eta} \in B^m_{V,g}(\delta) \cap B^m_{V,h}(\delta)$,*

*then*

$$|\partial_\alpha(\sigma_V(g,z) - \sigma_V(h,z))| \ge \delta^{O(1)}.$$

**Remark 2.4.19.** *This is similar to the non local integrability property as defined in [Dol98] [Nau05] and [Sto11]. Although the above two conditions are complicate, we will see later that in the measure sense, most pairs $g, h$ satisfy these conditions.*

*The key idea here is to use other representation to linearise polynomial functions on $V$. As long as the function is linear, we will have a good control of it. Another point is that the image of $\mathscr{P}$ stays in the same irreducible subrepresentation.*

*Proof of Lemma 2.4.18.* By Lemma 2.4.16, let

$$L := \partial_\alpha(\sigma_V(g,z) - \sigma_V(h,z)) = \frac{F_{g,h}(v^2 \wedge vu)}{\|gv\|^2\|hv\|^2}, \tag{2.4.46}$$

where $v = ke_1$ and $u = kY_\alpha e_1$ as in Lemma 2.4.16.

**Lemma 2.4.20.** *If $g, h$ satisfy assumption* (1)*, then the operator norm satisfy*

$$\|F_{g,h}|_W\| \ge \delta^{O(1)}\|g\|^2\|h\|^2.$$

*Proof.* Using the Cartan decomposition and good norm, we can suppose that $h$ is diagonal and $h = \mathrm{diag}(a_1, a_2, \cdots, a_n)$ with $a_1 \ge a_2 \ge \cdots \ge a_n$. By Definition 2.2.2, we know that $he_1 = a_1e_1$ and $he_2 = a_2e_2$. The assumption (1) becomes

$$\delta(\mathbb{R}e_1, y_g^m), \delta(\mathbb{R}e_2, y_g^m) > \delta, \gamma_{1,2}(g) \le \delta^3. \tag{2.4.47}$$

In (2.4.46), let $z = z_o$, then $v = e_1, u = e_2$, which make

$$\langle hv, hu\rangle = \langle a_1e_1, a_2e_2\rangle = 0.$$

Therefore, due to

$$\langle v_1, v_2\rangle \ge \|v_1\|\|v_2\| - \|v_1 \wedge v_2\|,$$

for $v_1, v_2$ in $V$, we have

$$F_{g,h}(e_1^2 \wedge e_1e_2) = a_1^2\langle ge_1, ge_2\rangle \ge a_1^2(\|ge_1\|\|ge_2\| - \|ge_1 \wedge ge_2\|).$$

Then (2.2.7) and (2.4.47) imply

$$F_{g,h}(e_1^2 \wedge e_1e_2) \ge \|h\|^2\|g\|^2(\delta^2 - \gamma_{1,2}(g)).$$

The proof is complete. $\square$

By Definition 2.2.2, the representation $\wedge^2 Sym^2 V$ is a proximal representation. Due to $\mathbb{R}(v^2\wedge vu)=\mathbb{R}k(e_1^2\wedge e_1e_2)=kV^{\chi_1}$, the line $\mathbb{R}(v^2\wedge vu)$ is contained in the $K$-orbit of the subspace of highest weight $V^{\chi_1}$. Since $V^{\chi_1}$ is in $W$, we see that $v^2\wedge vu$ is also in $W$. By (2.4.46),

$$L=\frac{F_{g,h}(v^2\wedge vu)}{\|F_{g,h}|_W\|}\frac{\|g\|^2\|h\|^2}{\|gv\|^2\|hv\|^2}\frac{\|F_{g,h}|_W\|}{\|g\|^2\|h\|^2}.$$

When $\eta$ satisfies assumption (2), the result follows by applying (2.2.7) to $\|gv\|^2,\|hv\|^2$ and by Lemma 2.4.20. □

### Proof of the spectral gap

Here we will prove the theorem of uniform spectral gap. The first part is classic, where we use some ideas of Dolgopyat [Dol98] to transform the problem to an effective estimate Proposition 2.4.24, see also [Nau05] and [Sto11]. The key observation is that this effective estimate (Proposition 2.4.24) can be obtained by the Fourier decay, regarding the difference of cocycle as a function on $\mathscr{P}$. The intuition here is from Lemma 2.4.18. When $g,h$ are in general position and $\eta$ not too close to $\zeta_g^m,\zeta_h^m$, the difference $\varphi(\eta)=\sigma(g,\eta)-\sigma(h,\eta)$ will be $(C,r)$ good (Definition 2.4.1). But in order to accomplish this, we need some sophisticate cutoff, which makes the proof complicate.

Recall that the Iwasawa cocycle takes values in the Cartan subspace $\mathfrak{a}$. We can write $\theta$ in $\mathfrak{a}^*$ as a linear combination of fundamental weights, $\{\chi_\alpha|\alpha\in\Pi\}$, that is

$$\theta=\sum_{\alpha\in\Pi}\theta_\alpha\chi_\alpha.$$

Set $|\theta|=\max_{\alpha\in\Pi}|\theta_\alpha|$.

We want to treat the spectral gap on the flag variety $\mathscr{P}$ and the projective space $\mathbb{P}V$ at the same time, where $V$ is an irreducible representation of $G$ with good norm. Let $X$ be $\mathscr{P}$ or $\mathbb{P}V$. Let $\sigma:G\times X\to E$ be the cocycle, which is given by the Iwasawa cocycle and $E=\mathfrak{a}$ when $X=\mathscr{P}$, and given by $\sigma_V$ (defined in (2.2.6)) and $E=\mathbb{R}$ when $X=\mathbb{P}V$. Let $E_{\mathbb{C}}=E\otimes_{\mathbb{R}}\mathbb{C}$ and $E_{\mathbb{C}}^*$ be the dual space of $E_{\mathbb{C}}$. For $z\in E_{\mathbb{C}}^*$, write $z=\varpi+i\theta$, where $\theta,\varpi$ are elements in $E^*$. Recall that the transfer operator $P_z$ is defined as: For $|\varpi|$ small enough and for $f$ in $C^0(X)$, $x$ in $X$

$$P_zf(x)=\int_G e^{z\sigma(g,x)}f(gx)\mathrm{d}\mu(g).$$

Recall that for $f$ in $C^\gamma(X)$ let $c_\gamma(f)=\sup_{x\neq x'}\frac{|f(x)-f(x')|}{d(x,x')^\gamma}$ and $|f|_\gamma=|f|_\infty+c_\gamma(f)$.

**Remark 2.4.21.** *Here we should be careful that the distances on $\mathbb{P}V$ and $\mathscr{P}$ are defined in (2.2.4) and (2.2.12). They are not the Riemannian distances defined in the introduction. But on a compact Riemannian manifold, different Riemannian distances are equivalent. In particular, every Riemannian distance on $\mathscr{P}$ is equivalent to the $K$-invariant Riemannian distance on $\mathscr{P}$. By Corollary 2.5.6, we know it is equivalent to the distances defined (2.2.12). The case of the projective space $\mathbb{P}V$ is similar. Hence, the norm $|\cdot|_\gamma$ induced by different distances are equivalent.*

We state our main result of this section

**Proposition 2.4.22.** *Let $\mu$ be a Zariski dense Borel probability measure on $G$ with a finite exponential moment. For $\gamma > 0$ small enough, there exist $\rho < 1, C > 0$ such that for all $\theta$ and $\varpi$ in $E^*$ with $|\theta|$ large enough, $|\varpi|$ small enough and $f$ in $C^\gamma(X)$, $n$ in $\mathbb{N}$ we have*

$$|P^n_{\varpi+i\theta} f|_\gamma \leq C|\theta|^{2\gamma}\rho^n|f|_\gamma.$$

Theorem 2.1.5 and Theorem 2.1.6 follow directly from Proposition 2.4.22. The assumption on $\mu$ will be needed throughout this section.

We start with standard *a priori* estimates. When $z = 0$, we will write $P$ for $P_0$.

**Proposition 2.4.23.** *For every $\gamma > 0$ small enough, there exist $C > 0$ and $0 < \rho < 1$ such that for all $f$ in $C^\gamma(X)$, $|\varpi|$ small enough and $n \in \mathbb{N}$*

$$|P^n_z f|_\infty \leq C^{|\varpi|n}|f|_\infty, \tag{2.4.48}$$

$$|P^n f|_\infty \leq \left|\int_X f \mathrm{d}\nu\right| + C\rho^n|f|_\gamma, \tag{2.4.49}$$

$$c_\gamma(P^n_z f) \leq C(C^{|\varpi|n}|\theta|^\gamma|f|_\infty + \rho^n c_\gamma(f)). \tag{2.4.50}$$

The inequality (2.4.48) is a consequence of exponential moment and the Hölder inequality. For (2.4.49), please see [BL85, V, Thm.2.5] and [BQ16, Prop 11.10, Lem.13.5] for more details. This inequality (2.4.49) is a consequence of the fact that the action of $G$ on $X$ is contracting. The third inequality (2.4.50) is called the Lasota-Yorke inequality. The proof is classic and we include a proof in the appendix for completeness.

We reduce Proposition 2.4.22 to Proposition 2.4.24. The reduction is standard, using Proposition 2.4.23. Please see [Dol98] and [Nau18] for more details. We also include a proof in the appendix for completeness. For $f$ in $C^\gamma(X)$, we define another norm $|f|_{\gamma,\theta} = |f|_\infty + c_\gamma(f)/|\theta|^\gamma$ for $\theta \neq 0$.

**Proposition 2.4.24.** *For every $\gamma > 0$ small enough, for $|\theta|$ large enough and $|\varpi|$ small enough, there exist $\epsilon_2$, $C_2 > 0$ such that for $f$ in $C^\gamma(X)$ and $|f|_{\gamma,\theta} \leq 1$, we have*

$$\int \left|P^{[C_2 \ln|\theta|]}_{\varpi+i\theta} f\right|^2 \mathrm{d}\nu \leq e^{-\epsilon_2 \ln|\theta|}. \tag{2.4.51}$$

Now we will distinguish two cases. **We claim that the case of $\mathbb{P}V$ is a corollary of the case of $\mathscr{P}$ up to a constant.** Recall that the stationary measure on $\mathbb{P}V$ is written as $\nu_V$. Let $f$ be a function in $C^\gamma(\mathbb{P}V)$ and $|f|_{\gamma,\theta} \leq 1$. The estimate only depends on the value of $f$ on the support of the stationary measure $\nu_V$. By Lemma 2.2.43, the stationary measure on $\mathbb{P}V$ is the pushforward measure of the stationary measure $\nu$ on $\mathscr{P}$. Hence we can define the function $\tilde{f}$ on $\mathscr{P}$ by

$$\tilde{f}(\eta) = f(V_{\chi,\eta}),$$

where $\chi$ is the highest weight of $V$. Then by $\sigma_V(g, V_{\chi,\eta}) = \chi\sigma(g,\eta)$ (see (2.2.2)),

$$\int \left|P^{[C_2 \ln|\theta|]}_{\varpi+i\theta} f\right|^2 \mathrm{d}\nu_V = \int \left|P^{[C_2 \ln|\theta|]}_{(\varpi+i\theta)\chi} \tilde{f}\right|^2 \mathrm{d}\nu.$$

We will verify that $\tilde{f}$ satisfies $|\tilde{f}|_{\gamma,\theta} \ll 1$. By (2.2.16), for two distinct points $\eta, \eta'$ in $\mathscr{P}$ we have

$$\frac{|\tilde{f}(\eta)-\tilde{f}(\eta')|}{d(\eta,\eta')^{\gamma}} = \frac{|\tilde{f}(\eta)-\tilde{f}(\eta')|}{d(V_{\chi,\eta},V_{\chi,\eta'})^{\gamma}}\frac{d(V_{\chi,\eta},V_{\chi,\eta'})^{\gamma}}{d(\eta,\eta')^{\gamma}} \ll \frac{|f(V_{\chi,\eta})-f(V_{\chi,\eta'})|}{d(V_{\chi,\eta},V_{\chi,\eta'})^{\gamma}} = |f|_{\gamma}.$$

Hence with some change of constant, we can deduce the case of $\mathbb{P}V$ from the case of $\mathscr{P}$.
We only need to prove Proposition 2.4.24 for the case of $\mathscr{P}$.

*From Fourier decay to Proposition 2.4.24.* We need to reduce (2.4.51) to Fourier decay (Theorem 2.1.7). Let

$$n = [C_2 \log|\theta|] \text{ and } \delta = e^{-\epsilon n} \tag{2.4.52}$$

(with $C_2 \geq \max_{\alpha\in\Pi}\{1/\alpha\sigma_\mu\}+1$ and $\epsilon > 0$ to be determined later), and let $G_{n,\epsilon,\alpha}$ be the subset of $G\times G$, defined as the set of couples which satisfy Lemma 2.4.18 (1) with $V = V_\alpha$. Let

$$G_{n,\epsilon} = \{g \in G \,|\, \|\kappa(g) - n\sigma_\mu\| \leq n\epsilon\}^2 \bigcap_{\alpha\in\Pi} G_{n,\epsilon,\alpha} \subset G\times G.$$

For $|f|_{\gamma,\theta} \leq 1$, let

$$A_{g,h} := \int_X e^{z\sigma(g,\eta)+\bar{z}\sigma(h,\eta)} f(g\eta)\bar{f}(h\eta)\mathrm{d}\nu(\eta).$$

Then

$$\begin{aligned}\int |P_z^n f|^2 \mathrm{d}\nu &= \int e^{z\sigma(g,\eta)+\bar{z}\sigma(h,\eta)} f(g\eta)\bar{f}(h\eta)\mathrm{d}\nu(\eta)\mathrm{d}\mu^{*n}(g)\mathrm{d}\mu^{*n}(h)\\ &= \int_{G_{n,\epsilon}} A_{g,h}\mathrm{d}\mu^{*n}(g)\mathrm{d}\mu^{*n}(h) + \int_{G_{n,\epsilon}^c} A_{g,h}\mathrm{d}\mu^{*n}(g)\mathrm{d}\mu^{*n}(h).\end{aligned} \tag{2.4.53}$$

**We first compute the term with $(g,h)$ outside of $G_{n,\epsilon}$,** where the behaviour is singular. By the Cauchy-Schwarz inequality,

$$\left|\int_{G_{n,\epsilon}^c} A_{g,h}\mathrm{d}\mu^{*n}(g)\mathrm{d}\mu^{*n}(h)\right|^2 \leq \mu(G_{n,\epsilon}^c)\int |A_{g,h}|^2\mathrm{d}\mu^{*n}(g)\mathrm{d}\mu^{*n}(h). \tag{2.4.54}$$

By large deviation principle (Proposition 2.2.46, Proposition 2.2.47), the set $G_{n,\epsilon}^c$ has exponentially small $\mu^{*2n}$ measure, that is

$$\mu(G_{n,\epsilon}^c) \ll_\epsilon \delta^c. \tag{2.4.55}$$

By $\|f\|_\infty \leq 1$ and (2.4.48), we have

$$\int |A_{g,h}|^2\mathrm{d}\mu^{*n}(g)\mathrm{d}\mu^{*n}(h) \leq |P_{2\varpi}^n \mathbb{1}|_\infty^2 \leq C^{4n\varpi}. \tag{2.4.56}$$

When $|\varpi|$ is small enough depending on $\epsilon$, by (2.4.54), (2.4.55) and (2.4.56)

$$\int_{G^c_{n,\epsilon}} A_{g,h}\mathrm{d}\mu^{*n}(g)\mathrm{d}\mu^{*n}(h) \ll_\epsilon \delta^{c/2} \leq |\theta|^{-c\epsilon/(2C_2)}. \tag{2.4.57}$$

**We compute the major term, that is $(g,h)$ in $G_{n,\epsilon}$.** We want to use Theorem 2.1.7 to control this part with $\varphi = \theta(\sigma(g,\eta) - \sigma(h,\eta))$ and a suitable $r$. For applying Theorem 2.1.7, we need that $\varphi$ is $(C,r)$ good, which will be accomplished by multiplying smooth cutoffs. The most important is G2 assumption (2.4.2), which will be verified with the help of Lemma 2.4.18. Hence we want that $r$ vanishes when $\eta$ does not satisfy Lemma 2.4.18 (2).

Let $X_{g,h,\alpha}$ be the subset of $\mathscr{P}$, defined as the set of elements which satisfy Lemma 2.4.18 (2) with $V = V_\alpha$. Let $X_{g,h} = \bigcap_{\alpha\in\Pi} X_{g,h,\alpha}$. Recall that $\tau$ be a smooth function on $\mathbb{R}$ such that $\tau|_{[0,\infty)} = 1$, $\tau$ takes values in $[0,1]$, $\mathrm{supp}\tau \subset [-1,\infty)$ and $|\tau'| \leq 2$. For $\delta > 0$, set $\tau_\delta(x) = \tau(x/\delta)$ for $x \in \mathbb{R}$. Let

$$\varphi(\eta) = |\theta|^{-1}\theta(\sigma(g,\eta) - \sigma(h,\eta)) = |\theta|^{-1}\sum_{\alpha\in\Pi}\theta_\alpha(\sigma_\alpha(g,\eta) - \sigma_\alpha(h,\eta)) \tag{2.4.58}$$

and

$$r(\eta) = f(g\eta)\bar{f}(h\eta)e^{\varpi(\sigma(g,\eta)+\sigma(h,\eta))}\prod_{\alpha\in\Pi}\tau_\alpha, \tag{2.4.59}$$

where

$$\tau_\alpha(\eta) = \tau_\delta(4\delta_\alpha(\eta,\zeta_g^m) - 4\delta)\tau_\delta(4\delta_\alpha(\eta,\zeta_h^m) - 4\delta)\tau_\delta(4\delta(V_{4\chi_\alpha-\alpha,\eta}, F_{\rho_\alpha g,\rho_\alpha h}) - 4\delta),$$

where $\delta_\alpha$ is defined to be

$$\delta_\alpha(\eta,\zeta_g^m) = \delta(V_{\alpha,\eta}, y^m_{\rho_\alpha(g)}).$$

The choice of $\tau_\alpha$ is sophisticated. We only need to keep in mind that they come from Lemma 2.4.18. Then $e^{i|\theta|\varphi}r(\eta)$ equals $e^{z\sigma(g,\eta)+\bar{z}\sigma(h,\eta)}f(g\eta)f(h\eta)$ on $X_{g,h}$.

**Lemma 2.4.25.** *Let $\epsilon_0, \epsilon_1$ be given by Theorem 2.1.7. Let $(g,h)$ be in $G_{n,\epsilon}$. With $\epsilon$ small enough depending on $\epsilon_0$ and $|\varpi|$ small enough depending on $\epsilon$ and $\epsilon_1$, for $\varphi, r$ defined in (2.4.58) and (2.4.59) we have that $\varphi$ is $(|\theta|^{\epsilon_0}, r)$ good and $c_\gamma(r) \leq |\theta|^{\epsilon_0}$, $|r|_\infty \leq |\theta|^{\epsilon_1/2}$.*

By Lemma 2.4.25, we can fix a value of $\epsilon$ and the functions $\varphi$ and $r|\theta|^{-\epsilon_1/2}$ satisfy the condition in Theorem 2.1.7. (Theorem 2.1.7 still holds when $r$ is a complex function) Hence Theorem 2.1.7 implies

$$\left|\int e^{i|\theta|\varphi(\eta)}r(\eta)\mathrm{d}\nu(\eta)\right| \leq |\theta|^{-\epsilon_1/2}. \tag{2.4.60}$$

The difference between $A_{g,h}$ and $\int e^{i|\theta|\varphi(\eta)}r(\eta)\mathrm{d}\nu(\eta)$ is bounded by

$$\nu(X^c_{g,h}) \leq \sum_{\alpha\in\Pi}\nu(X^c_{g,h,\alpha}). \tag{2.4.61}$$

Using the regularity of stationary measure (2.2.34) with $V = W_\alpha$, the irreducible sub-representation of $\wedge^2 Sym^2 V_\alpha$ with the highest weight, we have

$$\nu\{\eta \in \mathscr{P} | \delta(V_{4\chi_\alpha-\alpha,\eta}, F_{\rho_\alpha g,\rho_\alpha h}) < \delta\} \ll_\epsilon e^{-c\epsilon n}. \tag{2.4.62}$$

Using the regularity of stationary measure (2.2.34) with $V = V_\alpha$, we obtain

$$\nu\{\eta \in \mathscr{P} | V_{\alpha,\eta} \in B_h^m(\delta) \cup B_g^m(\delta)\} \ll_\epsilon e^{-c\epsilon n}. \tag{2.4.63}$$

Hence by (2.4.61)-(2.4.63), we have

$$\nu(X_{g,h}^c) \ll_\epsilon e^{-c\epsilon n} = |\theta|^{-c\epsilon/C_2}. \tag{2.4.64}$$

For $(g,h)$ in $G_{n,\epsilon}$, by (2.4.60) and (2.4.64)

$$A_{g,h} \ll |\theta|^{-\epsilon_1/2} + |\theta|^{-c\epsilon/C_2}.$$

Combined with (2.4.53) and (2.4.57), the proof is complete by setting $\epsilon_2 = \min\{\frac{\epsilon_1}{2}, \frac{c\epsilon}{4C_2}\}$. $\square$

It remains to prove Lemma 2.4.25.

*Proof of Lemma 2.4.25.* **We first verify that $\varphi$ is $(|\theta|^{\epsilon_0}, r)$ good.** Since $\epsilon$ will be taken small enough, we can suppose $|\theta|^{-\epsilon_0} \leq \delta/4$. Let $J$ be the $|\theta|^{-\epsilon_0}$ neighbourhood of supp$r$. Then for $\eta \in J$, we have $\delta_\alpha(\eta, \zeta_g^m) \geq \delta/2$ for $\alpha$ in $\Pi$.

The function $\varphi$ is a sum of functions. Each function is the lift of a function on $\mathbb{P}V_\alpha$ for some simple root $\alpha$. We write $\varphi = \sum_{\alpha\in\Pi} \varphi_\alpha$ where $\varphi_\alpha(\eta) = |\theta|^{-1}\theta_\alpha(\sigma_\alpha(g,\eta) - \sigma_\alpha(h,\eta))$. By Lemma 2.2.32, that is $\partial_{\alpha'}\varphi_\alpha = 0$ for $\alpha' \neq \alpha$, in order to verify $(|\theta|^{\epsilon_0}, r)$ good condition, it is enough to verify G1-G3 assumptions (2.4.1)-(2.4.3) for $\varphi_\alpha$ and the G4 assumption (2.4.4) for $\varphi$. Since G1-G3 are linear, we can forget the coefficients $|\theta|^{-1}\theta_\alpha$ in $\varphi_\alpha$.

Now, we verify G1-G3 assumptions and we fix a simple root $\alpha$ and consider $\varphi = \varphi_\alpha = \sigma_\alpha(g,\cdot) - \sigma_\alpha(h,\cdot)$. Recall that $v_\alpha = \sup_{\eta\in \mathrm{supp} r} |\partial_\alpha\varphi(\eta)|$. Since $J$ satisfies the hypothesis of Lemma 2.4.17 with $V = V_\alpha$, we have

$$v_\alpha, Lip_{\mathscr{P}_0}(\partial_\alpha\varphi|_{\pi^{-1}J}) < \delta^{-O(1)}. \tag{2.4.65}$$

Since $(g,h) \in G_{n,\epsilon}$ satisfies Lemma 2.4.18(1) and the support of $r$ satisfies Lemma 2.4.18(2), for $\eta$ in the support of $r$, by Lemma 2.4.18,

$$|\partial_\alpha\varphi(\eta)| > \delta^{O(1)} \geq \delta^{O(1)} v_\alpha$$

which is G2 assumption (2.4.2). This also implies

$$v_\alpha > \delta^{O(1)}, \tag{2.4.66}$$

G4 assumption (2.4.4). By (2.4.65), we have G3 assumption (2.4.3). Let $\varphi_1$ be a function on $\mathbb{P}V_\alpha$ such that $\varphi_1(V_{\alpha,\eta}) = \varphi(\eta)$. Since $J$ satisfies hypothesis of Lemma 2.2.8, this Lemma implies

$$\frac{|\varphi(\eta) - \varphi(\eta')|}{d_\alpha(\eta,\eta')} = \frac{|\varphi_1(V_{\alpha,\eta}) - \varphi_1(V_{\alpha,\eta'})|}{d(V_{\alpha,\eta}, V_{\alpha,\eta'})} \leq |Lip_{\mathbb{P}V_\alpha}\varphi_1| < \delta^{-O(1)} \leq \delta^{-O(1)} v_\alpha,$$

which is G1 assumption (2.4.1).

For general $\varphi$, it remains to verify G4 assumption (2.4.4). There exists a simple root $\alpha$ such that $|\theta_\alpha| = |\theta|$. Since $\varphi_\alpha$ satisfies G4 assumption and $|\partial_\alpha \varphi| = |\partial_\alpha \varphi_\alpha|$ by Lemma 2.2.32, the function $\varphi$ also satisfies G4 assumption.

**Finally, we verify the term $c_\gamma(r)$ and $|r|_\infty$.**

**Lemma 2.4.26.** *For $0 < \gamma \le 1$, let $f, \tau$ be two $\gamma$-Hölder functions on a compact metric space $X$. Then*

$$c_\gamma(\tau f) \le c_\gamma(\tau)\|f|_{\mathrm{supp}\tau}\|_\infty + |\tau|_\infty c_\gamma(f|_{\mathrm{supp}\tau}).$$

The proof of Lemma 2.4.26 is elementary. Recall that

$$r(\eta) = f(g\eta)\bar{f}(h\eta)e^{\varpi(\sigma(g,\eta)+\sigma(h,\eta))}\prod_{\alpha\in\Pi}\tau_\alpha.$$

For the infinity norm, due to $(g,h) \in G_{n,\epsilon}$, we have

$$|r| \le e^{|\varpi|(\|\kappa(g)\|+\|\kappa(h\|)} \le e^{|\varpi|(2\|\sigma_\mu\|+\epsilon)n} \le |\theta|^{|\varpi|C_2(2\|\sigma_\mu\|+\epsilon)}.$$

Take $|\varpi|$ small enough, then $|r|_\infty \le |\theta|^{\epsilon_1/2}$.

For the term $c_\gamma(r)$, we only need to verify that each term in the formula of $r$ has a bounded $c_\gamma$ value. Due to Lemma 2.4.26, we only need to verify the $c_\gamma$ value on $X_{g,h}$.

- Since the action of $g$ on $X_{g,h}$ is contracting, by Lemma 2.2.13, we have

  $$c_\gamma(f(g\cdot)|_{X_{g,h}}) \le c_\gamma(f)(Lip\ g|_{X_{g,h}})^\gamma \le (|\theta|\beta\delta^{-2})^\gamma.$$

  Due to (2.4.52), we have $\log\beta = -n\min_{\alpha\in\Pi}\alpha\sigma_\mu < -n/C_2 \le -\log|\theta|$. Therefore $c_\gamma(f(g\cdot)|_{X_{g,h}}) \le \delta^{-O(1)}$.

- Due to

  $$|e^a - e^b| \le \max\{e^a, e^b\}|a-b|^\gamma$$

  for all $a, b$ in $\mathbb{R}$ and $0 \le \gamma \le 1$, by Lemma 2.2.13,

  $$c_\gamma(e^{\varpi\sigma(g,\cdot)}|_{X_{g,h}}) \le e^{|\varpi|\|\kappa(g)\|}(Lip\varpi\sigma(g,\cdot)|_{X_{g,h}})^\gamma \le e^{|\varpi|(\|\sigma_\mu\|+\epsilon)n+\epsilon\gamma n}|\varpi|^\gamma.$$

  Hence when $|\varpi|$ is small enough depending on $\sigma_\mu$, we obtain $c_\gamma(e^{\varpi\sigma(g,\cdot)}|_{X_{g,h}}) \le \delta^{-O(1)}$.

- In $c_\gamma(\tau_\alpha)$, the only term we need to be careful about is $\tau_\delta(4\delta(V_{4\chi_\alpha-\alpha,\eta}, F_{\rho_\alpha g,\rho_\alpha h}) - 4\delta)$. By Lemma 2.2.14, we have $d(V_{4\chi_\alpha-\alpha,\eta}, V_{4\chi_\alpha-\alpha,\eta'}) \ll d(\eta,\eta')$. Hence the $c_\gamma$ value of this term is also bounded by $\delta^{-O(1)}$.

The proof is complete. $\square$

### 2.4.5 Exponential error term

In this section, we will prove that the speed of convergence in the renewal theorem is exponential using our result on the spectral gap. (Proposition 2.4.22) Recall $X = \mathbb{P}V$ and we have defined a renewal operator $R$ as follows: For a positive bounded Borel function $f$ on $X \times \mathbb{R}$, a point $x$ in $X$ and a real number $t$, we set

$$Rf(x,t) = \sum_{n=0}^{+\infty} \int_G f(gx, \sigma(g,x) - t)\mathrm{d}\mu^{*n}(g).$$

Recall $P_z$ is the transfer operator defined by $P_z f(x) = \int_G e^{z\sigma_V(g,x)} f(gx)\mathrm{d}\mu(g)$. Using the analytical Fredholm theorem, we summarize the property of $P_z$.

**Proposition 2.4.27.** *With the same assumption as in Theorem 2.1.4, for any $\gamma > 0$ small enough, there exists $\eta > 0$ such that when $|\Re z| < \eta$, the transfer operator $P_z$ is a bounded operator on $C^\gamma(X)$ and depends analytically on $z$. Moreover there exists an analytic operator $U(z)$ on a neighbourhood of $|\Re z| < \eta$ such that the following holds for $|\Re z| < \eta$*

$$(I - P_z)^{-1} = \frac{1}{\sigma_{V,\mu} z} N_0 + U(z),$$

*where $N_0$ is the operator defined by $N_0 f = \int_X f \mathrm{d}\nu_V$. There exists $C > 0$ such that for $|\Re z| \leq \eta$*

$$\|U(z)\|_{C^\gamma \to C^\gamma} \leq C(1 + |\Im z|)^{2\gamma}. \tag{2.4.67}$$

This is generalization of Proposition 1.4.1 in Chapter 1 and [Boy16, Theorem 4.1], and the proof is exactly the same. The main difference is that the spectral radius of $P_z$ is bounded below 1 in a strip of imaginary line (except at 0), due to Proposition 2.4.22. From this we have the analytic continuation of $U(z)$ to the strip and the bound of the operator norm of $U(z)$.

Now, we give the precise statement and the proof of Theorem 2.1.4.

**Proposition 2.4.28.** *With the same assumption as in Theorem 2.1.4, there exists $\epsilon > 0$ such that for $f \in C_c^\infty(\mathbb{R})$, we have*

$$Rf(x,t) = \frac{1}{\sigma_{V,\mu}} \int_{-t}^{\infty} f(u)\mathrm{d}u + e^{-\epsilon|t|} O(e^{\epsilon|\mathrm{supp} f|}(|f''|_{L^1} + |f|_{L^1})).$$

*Proof.* By the same computation as in Proposition 1.4.5 in Chapter 1 and [Boy16, Prop. 4.14], we have

$$Rf(x,t) = \frac{1}{\sigma_{V,\mu}} \int_{-t}^{\infty} f(u)\mathrm{d}u + \lim_{s \to 0^+} \frac{1}{2\pi} \int e^{-it\xi} \hat{f}(\xi) U(s + i\xi) \mathbb{1}(x) \mathrm{d}\xi,$$

where $\hat{f}$ is the Fourier transform of $f$ given by $\hat{\xi} = \int e^{i\xi u} f(u)\mathrm{d}u$. Hence, we only need to control the error term.

By Proposition 2.4.27, we know that $U(z)$ is analytical on $\{z \in \mathbb{C} || \Re z| \leq \eta\}$ and uniformly bounded by $(1+|\Im z|)^{2\gamma}$. Since $f$ is a compactly supported smooth function, the Fourier transform $\hat{f}$ is an analytic function on $\mathbb{C}$. By $|\hat{f}(i\epsilon+\xi)| \leq e^{\epsilon|\mathrm{supp} f|}\frac{1}{|\xi|^2}|f''|_{L^1}$, and $|\hat{f}(i\epsilon+\xi)| \leq e^{\epsilon|\mathrm{supp} f|}|f|_{L^1}$ for $\epsilon, \xi$ in $\mathbb{R}$, we have

$$|\hat{f}(i\epsilon+\xi)| \leq e^{\epsilon|\mathrm{supp} f|}\frac{2}{1+|\xi|^2}(|f''|_{L^1}+|f|_{L^1}). \tag{2.4.68}$$

By (2.4.67), (2.4.68) and the dominant convergence theorem, we have

$$\lim_{s\to 0^+}\frac{1}{2\pi}\int e^{-it\xi}\hat{f}(\xi)U(s+i\xi)\mathbb{1}(x)\mathrm{d}\xi = \frac{1}{2\pi}\int e^{-it\xi}\hat{f}(\xi)U(i\xi)\mathbb{1}(x)\mathrm{d}\xi. \tag{2.4.69}$$

**Lemma 2.4.29.** *[RS75, Thm.IX14] If $T$ is in $\mathcal{S}'(\mathbb{R})$, tempered distributions, the distribution $T$ has analytic continuation to $|\Im\xi| < a$ and $\sup_{|b|<a}\int|T(ib+y)|\mathrm{d}y < \infty$, then $\check{T}$ is a continuous function. For all $b < a$, let $C_b = \max\int|T(\pm ib+y)|\mathrm{d}y$. We have*

$$|\check{T}(t)| \leq C_b e^{-b|t|}.$$

Using Lemma 2.4.29 with $T(\xi) = \hat{f}(\xi)U(i\xi)\mathbb{1}(x)$, we have

$$\left|\int \hat{f}(\xi)U(i\xi)\mathbb{1}(x)e^{-it\xi}\mathrm{d}\xi\right| = |\check{T}(t)| \leq e^{-\epsilon|t|}\max|T(\pm i\epsilon+\xi)|_{L^1(\xi)} \tag{2.4.70}$$

By (2.4.68), we have

$$\begin{aligned}\max|T(\pm i\epsilon+\xi)|_{L^1(\xi)} &\leq e^{\epsilon|\mathrm{supp} f|}\int\frac{2}{1+|\xi|^2}(|f''|_{L^1}+|f|_{L^1})|U(\mp\epsilon+i\xi)\mathbb{1}(x)|\mathrm{d}\xi \\ &\ll_\gamma e^{\epsilon|\mathrm{supp} f|}(|f''|_{L^1}+|f|_{L^1}).\end{aligned} \tag{2.4.71}$$

Combining (2.4.69), (2.4.70) and (2.4.71), we have the result. □

## 2.5 Appendix

### 2.5.1 Two classic proofs in Section 2.4.4

In order to simply the notation, we abbreviate $\varpi, \theta$ to $a, b$.

*Proof of* (2.4.50). We need an idea of Guivarc'h

**Definition 2.5.1.** *We call the action of $G$ on $X$ is $(\mu,\gamma)$ contracting, if there exist $C > 0, \rho < 1$ such that for all $x \neq x'$ in $X$*

$$\int\left(\frac{d(gx,gx')}{d(x,x')}\right)^\gamma \mathrm{d}\mu^{*n}(g) \leq C\rho^n. \tag{2.5.1}$$

This was defined in [BQ16, Definition 11.1] and was verified for the action on the flag variety in [BQ16, Lemma 13.5]. For the projective space $\mathbb{P}V$, the same proof also works.

For the $\gamma$ norm, let $x, y$ in $X$ and $g$ in $G$

$$e^{z\sigma(g,x)}f(gx) - e^{z\sigma(g,y)}f(gy) = (e^{z\sigma(g,x)} - e^{z\sigma(g,y)})f(gx) + e^{z\sigma(g,y)}(f(gx) - f(gy))$$

Let $A_n = |\int_G \frac{e^{z\sigma(g,y)}(f(gx)-f(gy))}{d(x,y)^\gamma}\mathrm{d}\mu^{*n}(g)|$ and $B_n = |\int_G \frac{(e^{z\sigma(g,x)}-e^{z\sigma(g,y)})f(gx)}{d(x,y)^\gamma}\mathrm{d}\mu^{*n}(g)|$. By Cauchy-Schwarz's inequality

$$\begin{aligned} A_n &\leq c_\gamma(f)\int_G e^{a\sigma(g,y)}\frac{d(gx,gy)^\gamma}{d(x,y)^\gamma}\mathrm{d}\mu^{*n}(g) \\ &\leq c_\gamma(f)\left(\int_G e^{2a\sigma(g,y)}\mathrm{d}\mu^{*n}(g)\right)^{1/2}\left(\int_G\left(\frac{d(gx,gy)}{d(x,y)}\right)^{2\gamma}\mathrm{d}\mu^{*n}(g)\right)^{1/2} \end{aligned}$$

One term is controlled by (2.4.48), the other term is due to $(\mu,\gamma)$ contraction (2.5.1). Therefore when $a$ small enough, there exists $\rho_1 < 1$ such that $A_n \leq C_1\rho_1^n c_\gamma(f)$, where $C_1 > 0$.

Since

$$|e^c - e^d| \leq (2\max(e^{\Re c}, e^{\Re d}))^{1-\gamma}(\max(e^{\Re c}, e^{\Re d})|c-d|)^\gamma$$

for $c, d$ in $\mathbb{C}$, we have

$$\frac{|e^{z\sigma(g,x)} - e^{z\sigma(g,y)}|}{d(x,y)^\gamma} \leq (2e^{|a|\kappa(g)})^{1-\gamma}(e^{|a|\kappa(g)}|z|Lip(\sigma(g,\cdot)))^\gamma \leq 2e^{|a|\kappa(g)+\gamma\kappa_0(g)}|b|^\gamma,$$

where $\kappa_0(g)$ is the Lipschitz norm of $\sigma(g,\cdot)$ and $\kappa_0(g) \leq C\|\kappa(g)\|$ by [BQ16, Lemma 13.1]. Then by the hypothesis of finite exponential moment and Hölder's inequality, we have

$$B_n \leq |b|^\gamma |f|_\infty C_1^{(|a|+\gamma)n}$$

(we take the same constant $C_1$). Therefore

$$c_\gamma(P_z^n f) \leq C_1\rho_1^n c_\gamma(f) + |b|^\gamma C_1^{1+(|a|+\gamma)n}|f|_\infty \tag{2.5.2}$$

We want the term $C_1^{(|a|+\gamma)n}$ does not depend on $\gamma$. Fix $n$ large enough such that $C_1\rho_1^n = \rho_2 < 1$. For natural number $N$, iterate (2.5.2) $N$ times and use (2.4.48). We have

$$\begin{aligned} c_\gamma(P_z^{nN}f) &\leq \rho_2 c_\gamma(P_z^{n(N-1)}f) + |b|^\gamma C_1^{1+(|a|+\gamma)n}|P_z^{n(N-1)}f|_\infty \\ &\leq \rho_2 c_\gamma(P_z^{n(N-1)}f) + |b|^\gamma C_1^{1+(|a|+\gamma)n}|f|_\infty C_1^{|a|n(N-1)} \\ &\leq c_\gamma(f)\rho_2^N + |b|^\gamma C_1^{1+(|a|+\gamma)n}|f|_\infty \frac{C_1^{|a|nN}}{1-\rho_2 C_1^{-|a|n}} \leq c_\gamma(f)\rho_2^N + O_n(|b|^\gamma C_1^{|a|nN})|f|_\infty \end{aligned} \tag{2.5.3}$$

Given $m \in \mathbb{N}$, we can write $m = nN + r$ with $r \in [0, n-1]$. Therefore by (2.5.3) (2.5.2)

$$\begin{aligned} c_\gamma(P_z^m f) = c_\gamma(P^{nN+r} f) &\leq \rho_2^N c_\gamma(P_z^r f) + O_n(|b|^\gamma C^{|a|nN})|P_z^r f|_\infty \\ &\leq \rho_2^N (C_1 \rho_1^r c_\gamma(f) + |b|^\gamma C_1^{(1+(a+\gamma)r)} |f|_\infty) + O_n(|b|^\gamma C_1^{|a|m})|f|_\infty. \end{aligned}$$

By setting $\rho = \rho_2^{1/n}$ and choosing $C$ large enough, we have (2.4.50). $\square$

*From Proposition 2.4.24 to Proposition 2.4.22.* We set $N = [C_1 \ln |b|]$, by the Cauchy-Schwarz's inequality and (2.4.48), using (2.4.49) for $P^{mN}$, (2.4.51) for $P_z^N f$ and (2.4.50) for $P_z^N$

$$\begin{aligned} |P_z^{(m+1)N} f|_\infty^2 &\leq C^{|a|mN} |P^{mN} |P_z^N f|^2|_\infty \leq C^{|a|mN} \Big( \int |P_z^N f|^2 \mathrm{d}\nu + \rho^{mN} |P_z^N f|_{C^\gamma}^2 \Big) \\ &\leq C^{|a|mN} \left( e^{-\epsilon_2 N/C_1} + \rho^{mN} (C^{1+|a|N}(1+|b|^\gamma) + C\rho^N |b|^\gamma)^2 \right) \end{aligned} \tag{2.5.4}$$

So we can choose $m$ large such that $\rho^{mN}|b|^{2\gamma} = \rho^{mC \ln |b|} |b|^{2\gamma} < 1$. This $m$ is only depend on $\gamma, C$ and $\rho$. By continuity of $a$ we obtain the equality for infinity norm. That is when $m$ is large enough and $a$ is small enough depending on $m$ we have $|P_z^{(m+1)N} f|_\infty^2 \ll |b|^{-\epsilon_3}$, where $\epsilon_3 > 0$

For $\gamma$ norm, we use (2.4.50) for $(P_z^N, P_z^{(m+1)N} f)$ and $(P_z^{(m+1)N}, f)$

$$\begin{aligned} c_\gamma(P_z^{(m+2)N} f)/|b|^\gamma &\leq C^{|a|N} |P_z^{(m+1)N} f|_\infty + \rho^N c_\gamma(P_z^{(m+1)N} f)/|b|^\gamma \\ &\leq C^{|a|N} |P_z^{(m+1)N} f|_\infty + \rho^N (C^{1+|a|mN} |b|^\gamma + \rho^{mN} |b|^\gamma)/|b|^\gamma \end{aligned}$$

Then, when $|b|$ is large enough and $a$ is small enough, we have

$$|P_z^{(m+2)N} f|_{\gamma,b} \leq |b|^{-\epsilon_4} \tag{2.5.5}$$

(where we should use (2.5.4) with $m$ replaced by $m+1$).

Let $N_1 = (m+2)N = (m+2)C_1 \ln |b|$. Given $n$, we can write $n = dN_1 + r$ with $0 \leq r < N_1$. By (2.5.5), (2.4.48), (2.4.50)

$$|P_z^n f|_{\gamma,b} \leq |b|^{-\epsilon_4 d} |P_z^r f|_{\gamma,b} \leq |b|^{-\epsilon_4 d} C^{1+|a|r} \leq C|b|^{\epsilon_4} \rho^n,$$

where $\rho = |b|^{-\epsilon_4/N_1} C^{|a|} = e^{-\frac{\epsilon_4}{(m+2)C_1}} C^{|a|}$. The result follows by taking $|a|$ small enough. $\square$

### 2.5.2 Equivalence of distances

**Definition 2.5.2.** *Let $(X, d)$ be a metric space. Let $d'$ be another metric on $X$. We say that $d, d'$ are equivalent metrics if there exist $c, C > 0$ such that for all $x_1, x_2$ in $X$*

$$cd(x_1, x_2) \leq d'(x_1, x_2) \leq Cd(x_1, x_2).$$

Recall that $\mathscr{P}_0$ is the homogeneous space $G/A_eN$, on which the compact group $K$ acts simply transitively. We will define three distances on $\mathscr{P}_0$. Due to the fact that $\mathscr{P}_0$ is homeomorphic to $K$, a distance on $\mathscr{P}_0$ is also a distance on $K$ and we will continue our argument on $K$. Let $k, k'$ be two points in $K$.

- $d_0(k, k') = \sup_{\alpha\in\Pi} \|kv_\alpha - k'v_\alpha\|/\sqrt{2}$, where $v_\alpha$ is a unit vector in $V_\alpha$ with highest weight. This is also the distance induced by the embedding of $\mathscr{P}_0$ to $\Pi_{\alpha\in\Pi}\mathbb{S}V_\alpha$.
- $d_1(k, k') = \|k-k'\|$, where $\|\cdot\|$ is a $K$ invariant norm on the space of $(m+1)\times(m+1)$ square matrices $M_{m+1}(\mathbb{R}) \supset K$.
- $d_2(k, k')$ is the distance induced by the bi-invariant Riemannian metric on $K$.

We can easily verify that they are distances.

**Lemma 2.5.3.** *The three distances $d_0, d_1$ and $d_2$ on $\mathscr{P}_0$ are equivalent.*

*Proof.* First we observe that the three distances are left $K$ invariant. It is sufficient to prove the equivalence for $k'$ equal to the identity $e$.

Fix $\epsilon$ small depending on $K$. Let $B_\epsilon$ be the neighbourhood of $e$ given by $\{k \in K | d_1(k, e) < \epsilon\}$. Then $B_\epsilon^c$ is a compact subset of $K$. Consider the function $f_{i,j}(k) = \frac{d_i(k,e)}{d_j(k,e)}$ for $k \in B_\epsilon^c$ and $i, j \in \{0, 1, 2\}$. Then $f_{i,j}$ is a positive continuous function $B_\epsilon^c$. The compactness of $B_\epsilon^c$ implies that it has positive minimum on $B_\epsilon^c$. Hence there exists $c_{i,j} > 0$ such that for $k$ outside of $B_\epsilon$

$$d_i(k, e) \geq c_{i,j} d_j(k, e).$$

Finally, we only need to consider a small neighbourhood of the identity. We take $\epsilon$ small such that the exponential map at $e$ is bi-Lipschitz. Suppose that $k = \exp(tZ)$ with $Z$ a unit vector in $\mathfrak{k}$ and $t > 0$. Then

$$d_1(k, e) = \|e - \exp(tZ)\| \asymp t = d_2(k, e).$$

Due to $d_0(k, e) = \max_{\alpha\in\Pi} \|kv_\alpha - v_\alpha\|/\sqrt{2} \ll \|k - e\| = d_1(k, e)$, it remains to prove that $d_0$ is not small. We can decompose $Z$ as

$$Z = \sum_{\alpha\in R^+} c_\alpha K_\alpha.$$

There exists $\alpha_o \in R^+$ such that $c_{\alpha_o} \gg 1$. Since $v_\alpha$ is a vector of highest weight, for a positive root $\beta$ let

$$A_{\alpha,\beta} := \mathrm{d}\rho_\alpha(K_\beta)v_\alpha = \mathrm{d}\rho_\alpha(Y_\beta)v_\alpha.$$

Consider the representation of $\mathfrak{s}_\beta = \{Y_\beta, X_\beta, H_\beta\} \simeq \mathfrak{sl}_2$. Due to the classification of the representations of $\mathfrak{sl}_2$,

**Lemma 2.5.4.** *The vector $A_{\alpha,\beta}$ is non zero if and only if $\chi_\alpha(H_\beta) > 0$.*

Now take $\beta = \alpha_o$. Since $\{\chi_\alpha, \alpha \in \Pi\}$ is a basis of $\mathfrak{a}^*$, there exists a simple root $\alpha$ such that $\chi_\alpha(H_{\alpha_o}) \neq 0$. Hence by the fact that vectors of different weights are orthogonal, we have

$$\|kv_\alpha - v_\alpha\| = \|\exp(tZ)v_\alpha - v_\alpha\| \asymp t\|\mathrm{d}\rho_\alpha(Z)v_\alpha\| \geq tc_{\alpha_o}\|\mathrm{d}\rho_\alpha Y_{\alpha_o} v_\alpha\| \gg t.$$

Then we have $d_0(k,e) \gg d_2(k,e)$. The proof is finished. $\square$

Recall the definition of the sign function $\mathrm{m}$ of Section 2.2.5.

**Lemma 2.5.5.** *Let $z = kz_o, z' = k'z_o$ be two points in $\mathscr{P}_0$, then*

$$\sqrt{2}d_0(z,z') \geq d(\pi(z), \pi(z')).$$

*We have*

$$\mathrm{m}(z,z') = e \Longleftrightarrow d_0(z,z') < 1$$

*If* $\mathrm{m}(z,z') = e$, *then*

$$d(\pi(z), \pi(z')) \geq d_0(z,z').$$

*Proof.* Suppose that the angle between $kv_\alpha$ and $k'v_\alpha$ is $\theta \in [0,\pi)$, then $\|kv_\alpha - k'v_\alpha\| = 2\sin\frac{\theta}{2}$ and $d(V_{\alpha,k\eta_o}, V_{\alpha,k'\eta_o}) = \|kv_\alpha \wedge k'v_\alpha\| = \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \leq 2\sin\frac{\theta}{2}$, which implies the first inequality.

The assumption $d_0(z,z') \leq 1$ is equivalent to that for every simple root $\alpha$, the angle $\theta$ is less than $\pi/2$, which is equivalent to $\mathrm{m}(z,z') = e$ due to Lemma 2.2.19.

If $m(z,z') = e$, then for every simple root $\alpha$, the angle $\theta$ is less than $\pi/2$. Hence $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \geq \sqrt{2}\sin\frac{\theta}{2}$, which implies the result. $\square$

**Corollary 2.5.6.** *The $K$-invariant Riemannian distance on $\mathscr{P}$ is equivalent to the distance defined in* (2.2.12).

*Proof.* By $\mathscr{P} = \mathscr{P}_0/M$ and since the group $M$ is a subgroup of $K$ which preserves the distance, let $d_2$ also be the quotient Riemannian distance on $\mathscr{P}$. By the same argument of the proof as in Lemma 2.5.3, it is sufficient to prove on a small neighbourhood of $\eta_0$. For any two points $\eta$, $\eta'$ in this small neighbourhood, we can find $z$, $z'$ in $\mathscr{P}_0$ such that $\pi(z) = \eta$, $\pi(z') = \eta'$ and $d_2(z,z') = d_2(\eta,\eta')$. Due to $d_2(z,z')$ small, we see that $d_0(z,z')$ is less than 1. Hence by Lemma 2.5.5, we have $\mathrm{m}(z,z') = e$ and then

$$d(\eta,\eta') \asymp d_0(z,z').$$

By Lemma 2.5.3, we have $d_0(z,z') \asymp d_2(z,z') = d_2(\eta,\eta')$. The proof is complete. $\square$

Here we give a proof of G1 assumption (2.4.1) in the proof of Theorem 2.1.1 (Section 2.4.3).

Recall that $V$ is an irreducible representation of $G$ with a norm and with highest weight $\chi$, and $v_0, u_0$ are two unit vectors in $V$ and $\theta_\alpha = \|q_{2\chi-\alpha}(v_0 \wedge u_0)\|$ for simple root $\alpha$. Recall that $\varphi(\eta) = \frac{\langle u_0, v\rangle}{\langle v_0, v\rangle}$ for a nonzero vector $v$ in $V_{\chi,\eta}$ and $\eta \in \mathscr{P}$. By (2.4.42), we only need to verify that if $d(\eta,\eta')\leq \xi^{-\epsilon_0}$ and $\eta,\eta'$ satisfies that $|\langle v_0,v\rangle|\geq \|v\|/C_0$ for $v$ in $V_{\chi,\eta}$ and $V_{\chi,\eta'}$, then

$$|\varphi(\eta)-\varphi(\eta')|\leq \xi^{\epsilon_0/2}\sum_{\alpha\in\Pi}\theta_\alpha d(V_{\alpha,\eta},V_{\alpha,\eta'}),$$

for $\xi$ large enough. Replacing (2.4.43) by the following lemma, we conclude that G1 assumption is always verified if $\xi$ is large enough.

**Lemma 2.5.7.** *Let $C_0,C_1>0$ and let $\eta,\eta'$ be two points in $\mathscr{P}$ such that $d(\eta,\eta')\leq 1/(C_1C_0)$ and $|\langle v_0,v\rangle|\geq \|v\|/C_0$ for $v$ in $V_{\chi,\eta}$ and $V_{\chi,\eta'}$. Then with $C_1$ large enough depending on the norm, we have*

$$|\varphi(\eta)-\varphi(\eta')|\leq CC_0^m\sum_{\alpha\in\Pi}\theta_\alpha d(V_{\alpha,\eta},V_{\alpha,\eta'}),$$

*where $C$ only depends on the group $G$ and the norm on $V$.*

*Proof.* The main idea is to take derivative on $\mathscr{P}$, and prove that in every direction the result is true. We will first prove the directions given by positive roots.

The structure of $Sym^2(\wedge^2V)$ gives us a formula, that is for $v_1,v_2,w_1,w_2,w_3$ in $V$

$$\langle v_1\wedge v_2,w_1\wedge w_2\rangle=\langle v_1\wedge v_2,w_3\wedge\frac{\langle v_1,w_2\rangle w_1-\langle v_1,w_1\rangle w_2}{\langle v_1,w_3\rangle}\rangle. \tag{2.5.6}$$

In order to simply the notation, we write $Y_1,\cdots,Y_m$ for $Y_{\alpha_1},\cdots,Y_{\alpha_m}$. The structure of Lie algebra gives us that for a vector $v$ in $V$

$$v\wedge Y_1\cdots Y_kv=Y_1\cdots Y_{k-1}(v\wedge Y_kv)-\sum_I Y_Iv\wedge Y_{I^c}v, \tag{2.5.7}$$

where $I=\{j_1,\cdots,j_l\}$ is a nonempty subset of $\{1,\cdots,k-1\}$, $I^c$ is the complement of $I$ in $\{1,\cdots,k\}$ and $Y_I=Y_{j_1}\cdots Y_{j_l}$ with $j_1<\cdots<j_l$.

Let $e_1$ be the unit vector in $V$ with highest weight. We claim that if $|\langle v_0,e_1\rangle|\geq 1/C_0$, then for $J\subset\{1,\cdots,m\}$, we have

$$|\langle v_0\wedge u_0,e_1\wedge Y_Je_1\rangle|\leq CC_0^{|J|}\sum_{i\in J}\theta_{\alpha_i}. \tag{2.5.8}$$

We make an induction on $k=|J|$. By symmetry, it is sufficient to prove the claim for $Y_J=Y_1\cdots Y_k$. For $k=1$, due to $e_1\wedge Y_1e_1\in q_{2\chi-\alpha_1}(\wedge^2V)$, we have

$$|\langle v_0\wedge u_0,e_1\wedge Y_1e_1\rangle|\leq\|q_{2\chi-\alpha_1}(v_0\wedge u_0)\|=\theta_{\alpha_1}.$$

Suppose that (2.5.8) holds for all the integer less than $k-1$. Then by (2.5.7),

$$\begin{aligned}\langle v_0\wedge u_0,e_1\wedge Y_1\cdots Y_ke_1\rangle=&\langle v_0\wedge u_0,Y_1\cdots Y_{k-1}(e_1\wedge Y_ke_1)\rangle\\&-\sum_I\langle v_0\wedge u_0,Y_Ie_1\wedge Y_{I^c}e_1\rangle\end{aligned}$$

Due to $Y_1 \cdots Y_{k-1}(e_1 \wedge Y_k e_1) \in q_{2\chi-\alpha_k}(\wedge^2 V)$, the first term is controlled by $\theta_{\alpha_k}$. The other term, due to $I \neq \emptyset$, using (2.5.6) with $w_3 = e_1$, we have

$$\begin{aligned}|\langle v_0 \wedge u_0, Y_I e_1 \wedge Y_{I^c} e_1\rangle| &= |\langle v_0 \wedge u_0, e_1 \wedge \frac{\langle v_0, Y_{I^c} e_1\rangle Y_I e_1 - \langle e_1, Y_I e_1\rangle Y_{I^c} e_1}{\langle v_0, e_1\rangle}\rangle| \\ &\leq C_0 \left(|\langle v_0 \wedge u_0, e_1 \wedge Y_I e_1\rangle + \langle v_0 \wedge u_0, e_1 \wedge Y_{I^c} e_1\rangle|\right)\end{aligned}$$

Since the length of $I$ and $I^c$ are less than $k$, by the hypothesis of induction, we have the claim for $k$.

The choice of $Y_\beta$ for a positive root $\beta$ is fixed in Section 2.2 and we have $Y_{\alpha_1+\cdots+\alpha_k} = C_{\alpha_1,\cdots,\alpha_k}[Y_{\alpha_1}, [Y_{\alpha_2}, \cdots, [Y_{\alpha_{k-1}}, Y_{\alpha_k}]\cdots]]$ with a constant $C_{\alpha_1,\cdots,\alpha_k}$. By the claim,

**Lemma 2.5.8.** *Let $\beta$ be a positive root. If $|\langle v_0, e_1\rangle| \geq 1/C_0$, then*

$$|\langle v_0 \wedge u_0, e_1 \wedge Y_\beta e_1\rangle| \leq C C_0^m \sum_{\alpha\in\Pi, \alpha\leq\beta} \theta_\alpha,$$

*where $C$ only depends on $G$ and the norm on $V$. In particular, for $Z = \sum_{\beta\in R^+} c_\beta K_\beta$*

$$|\langle v_0 \wedge u_0, e_1 \wedge Z e_1\rangle| \leq C C_0^m \sum_{\alpha\in\Pi} \theta_\alpha \sum_{\beta\geq\alpha, \beta\in R^+} |c_\beta|.$$

This is almost the directional derivative of G1. For $\eta = k\eta_0, \eta' = k'\eta_0$ in $\mathscr{P}$, we can find a unit vector $Z$ in the Lie algebra $\mathfrak{k}$ such that $k' = k\exp(tZ)$ with $t \ll d(\eta, \eta')$. Let $\gamma(s) = k\exp(sZ)\eta_0$ for $0 \leq s \leq t$. Then by the Newton-Leibniz formula,

$$|\varphi(\eta) - \varphi(\eta')| \leq \int_0^t |\partial_s \varphi(\gamma(s))| \mathrm{d}s.$$

Let $k_s = k\exp(sZ)$. By the same computation of Lemma 2.4.14, we have

$$\partial_s\varphi(\gamma(s)) = \partial_Z\varphi(\gamma(s)) = \frac{\langle v_0 \wedge u_0, k_s e_1 \wedge k_s Z e_1\rangle}{\langle v_0, k_s e_1\rangle} = \frac{\langle k_s^{-1} v_0 \wedge k_s^{-1} u_0, e_1 \wedge Z e_1\rangle}{\langle k_s^{-1} v_0, e_1\rangle}.$$

Due to $\|k_s e_1 - e_1\| \ll d(\gamma(s), \gamma(0)) \leq d(\eta, \eta') \leq 1/(C_0 C_1)$, with $C_1$ large enough, we have

$$|\langle k_s^{-1} v_0, e_1\rangle| = |\langle v_0, k_s e_1\rangle| \geq |\langle v_0, e_1\rangle| - \|k_s e_1 - e_1\| \geq 1/(2C_0)$$

Due to $\|q_{2\chi-\alpha}(k_s^{-1}(v_0 \wedge u_0))\| = \|q_{2\chi-\alpha}(v_0 \wedge u_0)\| = \theta_\alpha$, by Lemma 2.5.8, we have

$$|\partial_s\varphi(\gamma(s))| \leq C C_0^m \sum_{\alpha\in\Pi} \theta_\alpha \sum_{\beta\geq\alpha, \beta\in R^+} |c_\beta|.$$

Hence

$$|\varphi(\eta) - \varphi(\eta')| \leq C C_0^m t \sum_{\alpha\in\Pi} \theta_\alpha \sum_{\beta\geq\alpha, \beta\in R^+} |c_\beta|. \tag{2.5.9}$$

For $Z$ in $\mathfrak{k}$, let $Z_\alpha = \sum_{\beta\geq\alpha,\beta\in R^+} c_\beta K_\beta$. Due to $t$ small, we only need to consider distance is a small neighbourhood. Hence by the fact that vectors of different weights are orthogonal, we have ($Y_\beta v_\alpha$ is nonzero for every positive root $\beta \geq \alpha$ due to Lemma 2.5.4)

$$d(V_{\alpha,\eta}, V_{\alpha,\eta'}) = d(v_\alpha, \exp(tZ)v_\alpha) \asymp \|\exp(tZ)v_\alpha - v_\alpha\| \asymp t\|Zv_\alpha\| = t\|Z_\alpha v_\alpha\| \asymp t\|Z_\alpha\|.$$

Therefore, combined with (2.5.9)

$$\sum_{\alpha\in\Pi} \theta_\alpha d(V_{\alpha,\eta}, V_{\alpha,\eta'}) \geq \sum_{\alpha\in\Pi} t\theta_\alpha \|Z_\alpha\| \geq t\sum_{\alpha\in\Pi}\theta_\alpha \sum_{\beta\geq\alpha,\beta\in R^+} |c_\beta| \geq \frac{1}{CC_0^m}|\varphi(\eta)-\varphi(\eta')|.$$

The proof is complete. $\square$

# Chapter 3

# Discretized Sum-product and Fourier decay in $\mathbb{R}^n$

## 3.1 Introduction

The purpose of this manuscript is to generalize a result of Bourgain to $\mathbb{R}^n$. This result deals with the Fourier decay of the multiplicative convolution of Borel probability measures on $\mathbb{R}$.

If $E$ is a metric space, we write $B_E(x,r)$ for a close ball centered at $x$ of radius $r$. Vectors in $\mathbb{R}^n$ are seen as column vectors. The product structure on $\mathbb{R}^n$ is given by coordinate, that is for $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in $\mathbb{R}^n$, the product is defined to be $xy = (x_1 y_1, \cdots, x_n y_n)$. For a Borel probability measure on $\mathbb{R}^n$, let $\mu_k$ be the $k$-times multiplicative convolution of $\mu$.

**Theorem 3.1.1.** *Given $\kappa_0 > 0$, there exist $\epsilon, \epsilon_1 > 0$ and $k \in \mathbb{N}$ such that the following holds for $\delta > 0$ small enough. Let $\mu$ be a probability measure on $[1/2, 1]^n \subset \mathbb{R}^n$ which satisfies $(\delta, \kappa_0, \epsilon)$ projective non concentration assumption, that is*

$$\forall \rho \geq \delta, \quad \sup_{a \in \mathbb{R}, v \in \mathbb{S}^{n-1}} (\pi_v)_* \mu(B_{\mathbb{R}}(a,\rho)) = \sup_{a,v} \mu\{x | \langle v, x \rangle \in B_{\mathbb{R}}(a,\rho)\} \leq \delta^{-\epsilon} \rho^{\kappa_0}. \tag{3.1.1}$$

*Then for all $\xi \in \mathbb{R}^n$ with $\|\xi\| \in [\delta^{-1}/2, \delta^{-1}]$,*

$$|\hat{\mu}_k(\xi)| = \left| \int \exp(2i\pi \langle \xi, x^1 \cdots x^k \rangle) \mathrm{d}\mu(x^1) \cdots \mathrm{d}\mu(x^k) \right| \leq \delta^{\epsilon_1}. \tag{3.1.2}$$

**Remark 3.1.2.** *We cannot have a sharper result like $\epsilon_1 \geq n/2$, because here we only use the product structure. In $\mathbb{R}^*$, there exist Borel subgroups which have fractional dimension. (See [EV66] for example) For a measure supported on a fractional Borel subgroup, the decay rate of Fourier transform is controlled by the Hausdorff dimension of the Borel subgroup. Hence, fractional Borel subgroups are obstacles for large decay rate of Fourier transform.*

*If we continue to exploit the additive structure, that is to say replacing $\mu_k$ by $\nu = (\mu_k)^{*r}$, the $r$-times additive convolution of $\mu_k$, then the Fourier transform of $\nu$ can have arbitrary large decay rate.*

*The Fourier transform detects the additive structure. But our measure $\mu_k$ has the multiplicative structure. The decay of Fourier transform means that the additive and multiplicative structures are hard to coexist, the sum-product philosophy.*

The projective non concentration means the projection of the measure $\mu$ on every one dimensional linear subspace $\mathbb{R}v$ satisfies a non concentration assumption (the case of $\mathbb{R}$).

The case $n = 1$ is due to Bourgain [Bou10, Lemma 8.43]. The main ingredient of the proof of Fourier decay is the discretized sum-product estimates in $\mathbb{R}^n$. The sum-product estimate roughly says that if the set does not concentrate in small balls, then under addition or multiplication the size of the set will become robustly larger than the initial set.

For $\delta > 0$ and a bounded set $A$ in a metric space $E$, let $\mathcal{N}_\delta(A)$ be the minimal number of closed balls of radius $\delta$ needed to cover $A$. In a metric space, we say that a set $A$ is $\rho$ **away from** a set $B$ if $A$ is not contained in the $\rho$ neighborhood of $B$, that is there exists $x$ in $A$ such that $d(x, B) \geq \rho$. In $(\mathbb{R}^*)^n$, we note *id* the identity element $(1, \cdots, 1) \in (\mathbb{R}^*)^n$. In $\mathbb{R}^n$, we will consider maximal proper unitary subalgebras, such subalgebras are given by $\{x \in \mathbb{R}^n | x_i = x_j\}$ for $1 \leq i < j \leq n$. We say that $A$ is $\rho$ away from proper unitary subalgebras of $\mathbb{R}^n$ if $A$ is $\rho$ away from any maximal proper unitary subalgebra of $\mathbb{R}^n$.

Now we state the discretized sum-product estimates on $\mathbb{R}^n$, which is the main ingredient of the proof of Theorem 3.1.1.

**Theorem 3.1.3.** *We will consider the action of $(\mathbb{R}^*)^n$ on $V = \mathbb{R}^n$. The action is given by $gv = (g_1 v_1, \ldots, g_n v_n)$ for $g$ in $(\mathbb{R}^*)^n$ and $v$ in $V$. There exists a neighborhood $U$ of the identity in $(\mathbb{R}^*)^n$ such that the following holds. Given $\kappa > 0, \sigma \in (0, n)$, there exists $\epsilon > 0$ such that for all $\delta > 0$ sufficiently small, if $A \subset U$ and $X \subset B_V(0, \delta^{-\epsilon})$ satisfy the following $(\delta, \kappa, \sigma, \epsilon)$ assumption:*

*(i)For $j = 1, \ldots, n$*

$$\forall \rho \geq \delta,\ \mathcal{N}_\rho(\pi_j(A)) \geq \delta^\epsilon \rho^{-\kappa},$$

*where $\pi_j$ denotes the projection into $j$-th coordinate,*

*(ii) $A$ is $\delta^\epsilon$ away from proper unitary subalgebras of $\mathbb{R}^n$,*

*(iii)For $j = 1, \ldots, n$*

$$\forall \rho \geq \delta,\ \mathcal{N}_\rho(\pi_j(X)) \geq \delta^\epsilon \rho^{-\kappa},$$

*(iv)$\mathcal{N}_\delta(X) \leq \delta^{-(n-\sigma)-\epsilon}$.*

*Then*

$$\mathcal{N}_\delta(X + X) + \sup_{a \in A} \mathcal{N}_\delta(X + aX) \geq \delta^{-\epsilon} \mathcal{N}_\delta(X).$$

**Remark 3.1.4.** *The case $n = 1$ is due to Bourgain. Compared with [BG12, Prop.1], our situation does have invariant subspace under the action. Hence we put more regularity on the projection into coordinate subspaces.*

**Remark 3.1.5.** *Roughly speaking, (i) and (iii) mean that the projections of $A, X$ into coordinate subspaces are non concentrate. Assumption (ii) is reasonable since it prevents $A$ from being trapped in a subalgebra.*

*Compared with the projective non concentration in Theorem 3.1.1, the assumption here is weaker. In multiplicative convolution, we need additionally that $\mu$ is not trapped in any affine subspace.*

From the discretized sum-product theorem to the Fourier decay of multiplicative convolution can be found in [Bou10]. The analogue result for finite fields is established in [BGK06]. See also [Gre09], where he gave a really clear treatment of the sum-product phenomenon in $\mathbb{F}_p$. The proof of Theorem 3.1.1 from Theorem 3.1.3 will be given in Section 3.3.

### Notation

We will make use of some classic notation: For two real valued functions $A$ and $B$, we write $A = O(B), A \ll B$ or $B \gg A$ if there exists constant $C > 0$ such that $|A| \leq CB$, where $C$ only depends on the ambient space. We also write $A \sim B$ if $B \ll A \ll B$.

We write $A = O_r(B), A \ll_r B$, $B \gg_r A$ and $A \sim_r B$ if the constant $C$ depends on an extra parameter $r > 0$.

## 3.2 Discretized sum-product estimates in $\mathbb{R}^n$

The non concentration assumption in Theorem 3.1.1 is a little different from that in [Bou10], but the two assumptions are equivalent up to constants.

**Lemma 3.2.1.** *Let $1 > \delta > 0$. Let $\nu$ be a Borel probability measure on $\mathbb{R}$. We have two non concentration assumptions.*

*(1)* $(\delta, \kappa_1, \epsilon_1)$ *For $\forall \rho \geq \delta$, we have $\nu(B(a,\rho)) \leq \delta^{-\epsilon_1}\rho^{\kappa_1}$.*

*(2)* $(\delta, \kappa, \epsilon)$ *For $\rho \in [\delta, \delta^{\epsilon}]$, we have $\nu(B(a,\rho)) \leq \rho^{\kappa}$.*

*Then* $(2)(\delta, \kappa, \epsilon)$ *implies* $(1)(\delta, \min \kappa, 1, \epsilon)$ *and if $\kappa_1 > 2\epsilon_1$, we have that* $(1)(\delta, \kappa_1, \epsilon_1)$ *implies* $(2)(\delta, \kappa_1/2, 2\epsilon_1/\kappa_1)$.

*Proof.* (2) $\Rightarrow$ (1) For $\rho < \delta^{\epsilon}$, it is obvious. For $\rho > \delta^{\epsilon}$, we use the trivial bound

$$\nu(B(a,\rho)) \leq 1 \leq \delta^{-\epsilon + \epsilon \min\{\kappa, 1\}}.$$

Hence (2) implies that (1) holds for $(\epsilon_1, \kappa_1) = (\epsilon, \min\{\kappa, 1\})$.

(1) $\Rightarrow$ (2) We want to find $(\epsilon, \kappa)$ such that (2) holds. Let $\rho = \delta^t$. That means

$$\epsilon_1 - t\kappa_1 \leq -t\kappa \text{ for } t \in [\epsilon, 1].$$

Due to $\kappa_1 > 2\epsilon_1$, we can take $(\epsilon, \kappa) = (2\epsilon_1/\kappa_1, \kappa_1/2)$. □

The assumption (2) in Lemma 3.2.1 is the original definition of Bourgain. This assumption roughly says that the measure $\nu$ has dimension $\kappa$ at scale $\delta$ to scale $\delta^\epsilon$. The assumption (1) is more convenient to be proved. The smaller the parameter $\epsilon_1$ is, the more regularity the measure $\nu$ has.

Let $A$ be a bounded subset of $\mathbb{R}^n$. Let $\langle A\rangle_s$ be the set of elements which are obtained by taking sum or multiplication of elements in $A$ at most $s$ times.

**Lemma 3.2.2.** *Let $A$ be a subset of $B_{\mathbb{R}^n}(0,K)$. If*

$$\mathcal{N}_\delta(A+A)+\mathcal{N}_\delta(A+A\cdot A)\leq K\mathcal{N}_\delta(A),$$

*then for any integer $s$*

$$\mathcal{N}_\delta(\langle A\rangle_s)\leq K^{O_s(1)}\mathcal{N}_\delta(A).$$

(See [He, Lemma 11] and [Bre11, Lemma 4.5] for more details) This lemma tells us that instead of proving that $A+A$ or $A+A\cdot A$ is large, it is sufficient to prove that $\langle A\rangle_s$ is substantially large.

Our result on the discretized sum-product estimates relies on a result of He and de Saxcé. They study sum-product phenomenon in finite dimensional linear representations of Lie groups. We will state the version we need, their theorem is much more general.

**Definition 3.2.3.** *Recall that we consider the action of $(\mathbb{R}^*)^n$ on $V=\mathbb{R}^n$ given by multiplication in each coordinate. Let $W$ be a linear subspace of $V$ such that $W$ is not a submodule, that is there exists $g$ in $(\mathbb{R}^*)^n$ such that $gW\not\subset W$. Then we call $Stab_{(\mathbb{R}^*)^n}(W)$ a **proper stabilizer**.*

Let $A$ be a subset of $(\mathbb{R}^*)^n$ and let $X$ be a subset of $\mathbb{R}^n$. For $s\geq 1$, we define $\langle A,X\rangle_s$ to be the set of elements which can be obtained as sums, differences and products of at most $s$ elements of $A$ and $X$. For example, we have $\langle A,X\rangle_s=\{\pm g_{1,1}\cdots g_{1,i_1}v_1\pm\cdots\pm g_{l,1}\cdots g_{l,i_l}v_l|\ i_1,\cdots,i_l,l\in\mathbb{N},\ i_1+\cdots i_l+l\leq s\}$

**Proposition 3.2.4.** *[HdS18, Thm.2.3] Recall that we consider the action of $(\mathbb{R}^*)^n$ on $V=\mathbb{R}^n$ given by multiplication in each coordinate. There exists a neighborhood $U$ of the identity in $(\mathbb{R}^*)^n$ such that the following holds. Given $\epsilon_0,\kappa>0$, there exist $s\geq 1$ and $\epsilon>0$ such that for all $\delta>0$ sufficiently small, if $A\subset U$ and $X\subset B_V(0,1)$ satisfy the following $(\delta,\kappa,\epsilon)$ assumption:*

*(i) For $j=1,\ldots,n$*

$$\forall\rho\geq\delta,\ \mathcal{N}_\rho(\pi_j(A))\geq\delta^\epsilon\rho^{-\kappa},$$

*where $\pi_j$ denotes the projection into $j$-th coordinate,*

*(ii) $A$ is $\delta^\epsilon$ away from proper stabilizers,*

*(iii) $X$ is $\delta^\epsilon$ away from coordinate subspaces.*

*Then,*

$$B_V(0,\delta^{\epsilon_0})\subset\langle A,X\rangle_s+B_V(0,\delta).$$

We will use the ring structure of $\mathbb{R}^n$. Recall that for a subset $A$ of $(\mathbb{R}^*)^n$, which is also a subset of $\mathbb{R}^n$, we define $\langle A\rangle_s$ as $\langle A, X\rangle_s$ with $X = A$. As a corollary of Proposition 3.2.4, we have

**Proposition 3.2.5.** *There exists a neighborhood $U$ of the identity in $(\mathbb{R}^*)^n$ such that the following holds. Given $\kappa > 0, \epsilon_0 > 0$, there exist $\epsilon > 0$ and $s > 0$ such that, for $\delta$ sufficiently small, if $A$ is a subset of $U$ satisfies the following $(\delta, \kappa, \epsilon)$ assumption:*

*(i)For $j = 1, \ldots, n$*

$$\forall \rho \geq \delta,\ \mathcal{N}_\rho(\pi_j(A)) \geq \delta^\epsilon \rho^{-\kappa},$$

*where $\pi_j$ denotes the projection into $j$-th coordinate,*

*(ii)$A$ is $\delta^\epsilon$ away from maximal proper unitary subalgebras.*

*Then we have*

$$B_{\mathbb{R}^n}(0, \delta^{\epsilon_0}) \subset \langle A\rangle_s + B_{\mathbb{R}^n}(0, \delta).$$

*Proof.* Take $X = A - A$. We can shrink $U$ to ensure that $X \subset U - U \subset B_{\mathbb{R}^n}(0, 1)$. Then we claim that $A, X$ satisfies $(\delta, \kappa, 2\epsilon/\kappa)$ assumption of Proposition 3.2.4.

Assumption (i) of Proposition 3.2.4 is the same as Assumption (i) of this proposition. For assumption (iii) of Proposition 3.2.4, take $\rho = \delta^{2\epsilon/\kappa}$. Then

$$\mathcal{N}_\rho(\pi_j(X)) \geq \mathcal{N}_\rho(\pi_j(A)) \geq \delta^\epsilon \rho^{-\kappa} = \delta^{-\epsilon} > 1.$$

Hence, $X$ is $\delta^{2\epsilon/\kappa}$ away from coordinate subspaces. The assumption(iii) in Proposition 3.2.4 is satisfied.

It remains to verify Assumption (ii) of Proposition 3.2.4. We need to change the point of view. The set $G = (\mathbb{R}^*)^n \subset \mathbb{R}^n$ is seen as subsets of $Aut(V) \subset End(V)$, the automorphism group and the endomorphism ring of $V$. The main point is that in the case of $\mathbb{R}^n$, proper stabilizers are contained in the subalgebras. In other words, let $W$ be a subspace of $V$ which is not a $G$-submodule. Then the proper stabilizer satisfies

$$Stab_G(W) = G \cap Stab_{\mathbb{R}^n}(W) = G \cap \{a \in \mathbb{R}^n | a \cdot W \subset W\}.$$

By definition, $Stab_G W$ is a proper subgroup of $G$. The fact that $Stab_{\mathbb{R}^n}(W)$ is a unitary subalgebra of $\mathbb{R}^n$ implies that $Stab_{\mathbb{R}^n}(W)$ must be a proper unitary subalgebra of $\mathbb{R}^n$. Hence, the assumption (ii) of Proposition 3.2.4 is automatically satisfied.

Applying Proposition 3.2.4 with $\kappa, \epsilon_0$ implies that there exists $s_1$ such that

$$B_{\mathbb{R}^n}(0, \delta^{\epsilon_0}) \subset \langle A, X\rangle_{s_1} + B_{\mathbb{R}^n}(0, \delta),$$

when $\epsilon$ is small enough. The observation that

$$\langle A, X\rangle_{s_1} = \langle A, A - A\rangle_{s_1} \subset \langle A\rangle_{2s_1}$$

implies the result. □

As a byproduct, using Lemma 3.2.2, we have the following version of discretized sum-product estimates in $\mathbb{R}^n$.

**Proposition 3.2.6.** *There exists a neighborhood $U$ of the identity in $(\mathbb{R}^*)^n$ such that the following holds. Given $\kappa > 0, \sigma \in (0,n)$, there exists $\epsilon > 0$ such that for all $\delta > 0$ sufficiently small, if $A \subset U$ satisfies the following:*

*(i) For $j = 1, \ldots, n$*

$$\forall \rho \geq \delta,\ \mathcal{N}_\rho(\pi_j(A)) \geq \delta^\epsilon \rho^{-\kappa},$$

*where $\pi_j$ denotes the projection into $j$-th coordinate,*

*(ii) $A$ is $\delta^\epsilon$ away from proper unitary subalgebras of $\mathbb{R}^n$,*

*(iii) $\mathcal{N}_\delta(A) \leq \delta^{-\sigma-\epsilon}$.*

*Then*

$$\mathcal{N}_\delta(A + A) + \mathcal{N}_\delta(A + A \cdot A) \geq \delta^{-\epsilon}\mathcal{N}_\delta(A).$$

We deduce Proposition 3.2.6 from Lemma 3.2.2 and Proposition 3.2.5. The proof is exactly the same as the proof of [He, Theorem 2]. We include its proof for completeness.

*Proof of Proposition 3.2.6.* Suppose that the result fails. For every $\epsilon > 0$ there exists $A$ satisfying the hypothesis of Proposition 3.2.6 but

$$\mathcal{N}_\delta(A + A) + \mathcal{N}_\delta(A + A \cdot A) < \delta^{-\epsilon}\mathcal{N}_\delta(A).$$

We will reach a contradiction when $\epsilon$ is small enough depending only on $\kappa, \sigma$ and $\mathbb{R}^n$.

Then by Lemma 3.2.2 and assumption (ii) of Proposition 3.2.6, for every integer $s$, we have

$$\mathcal{N}_\delta(\langle A \rangle_s) \leq \delta^{-O_s(\epsilon)}\mathcal{N}_\delta(A) \leq \delta^{-O_s(\epsilon)-\sigma}. \tag{3.2.1}$$

On the other hand, $A$ also satisfies the assumptions of Proposition 3.2.5. Given $\epsilon_0 > 0$, there exist $\epsilon_1 > 0$ and integer $s$ such that if $\epsilon \leq \epsilon_1$, then

$$B_{\mathbb{R}^n}(0, \delta^{\epsilon_0}) \subset \langle A \rangle_s + B_{\mathbb{R}^n}(0, \delta).$$

Therefore

$$\mathcal{N}_\delta(\langle A \rangle_s) \geq \mathcal{N}_\delta(B_{\mathbb{R}^n}(0, \delta^{\epsilon_0})) = \delta^{n(-1+\epsilon_0)} \tag{3.2.2}$$

If we take $\epsilon_0$ sufficiently small such that $n(1 - \epsilon_0) > \sigma$, and take $\epsilon$ sufficiently small such that

$$n(1 - \epsilon_0) > O_s(\epsilon) + \sigma,$$

then (3.2.1) contradicts (3.2.2). □

This version is not sufficient to imply the decrease of Fourier transform of multiplicative convolution of measures. We will introduce more tools of additive combinatorics to obtain a stronger form of discretized sum-product estimates.

### 3.2.1 Basics of discretized sets

Before proving our results, we recall some elementary and known results in the discretized setting. Let $\delta > 0$ be the scale. Let $K \geq 2$ be a roughness constant. Two quantities bounded by a polynomial of $K$ is considered as equivalent.

**Lemma 3.2.7.** *Let $f$ be a $K$-Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $A$ be a bounded subset of $\mathbb{R}^n$. We have*

$$\mathcal{N}_\delta(fA) \ll K^n \mathcal{N}_\delta(A). \tag{3.2.3}$$

**Definition 3.2.8.** *For a bounded subset $A$ of $\mathbb{R}^n$, we denote by $A^{(\delta)}$ the $\delta$-neighborhood of $A$, given by*

$$A^{(\delta)} = \{x \in \mathbb{R}^n | d(x, A) \leq \delta\}.$$

**Lemma 3.2.9.** *Let $A$ be a bounded subset of $\mathbb{R}^n$. Let $\tilde{A}$ be a maximal $\delta$-separated subset of $A$, that is different elements of $\tilde{A}$ have distance at least $\delta$ and $\tilde{A}$ is maximal for inclusion. Then*

$$\mathcal{N}_\delta(A) \sim |A^{(\delta)}|\delta^{-n} \sim \#\tilde{A}, \tag{3.2.4}$$

*where $|A|$ denotes the volume of $A$ and $\#\tilde{A}$ denotes the number of elements of $\tilde{A}$.*

**Definition 3.2.10** (Ruzsa distance)**.** *Let $A, B$ be two bounded subsets of $\mathbb{R}^n$. We define the Ruzsa distance of $A, B$ at scale $\delta$ by*

$$d_\delta(A, B) = \frac{1}{2} \log \frac{\mathcal{N}_\delta(A - B)^2}{\mathcal{N}_\delta(A)\mathcal{N}_\delta(B)}.$$

*This is not a real distance. It measures the additive structure of $A$ and $B$.*

**Lemma 3.2.11** (Ruzsa triangular inequality)**.** *Let $A, B, C$ be three bounded subsets of $\mathbb{R}^n$. Then*

$$\mathcal{N}_\delta(B)\mathcal{N}_\delta(A - C) \ll \mathcal{N}_\delta(A - B)\mathcal{N}_\delta(B - C). \tag{3.2.5}$$

The above inequality (3.2.5) is roughly a triangular inequality for the Ruzsa distance $d_\delta$.

**Lemma 3.2.12** (Plünnecke-Ruzsa inequality)**.** *Let $A, B$ be two bounded subsets of $\mathbb{R}^n$. If $\mathcal{N}_\delta(A + B) \leq K\mathcal{N}_\delta(B)$, then for $k, l$ in $\mathbb{N}$ we have*

$$\mathcal{N}_\delta(kA - lA) \leq O(K)^{k+l}\mathcal{N}_\delta(B).$$

In [He17], He explains how to deduce the discretized version from the discrete version of the above two lemmas. For the discrete version, please see [TV06]. The main ingredient of proof is the Ruzsa covering lemma.

**Definition 3.2.13.** *Let $A, B$ be two bounded subsets of $\mathbb{R}^n$. We define the doubling constant of $A$ at scale $\delta$ by*

$$\sigma_\delta[A] := \frac{\mathcal{N}_\delta(A + A)}{\mathcal{N}_\delta(A)} = \exp(d_\delta(A, -A)).$$

*We write $A \approx_K B$ if $\mathcal{N}_\delta(A + B) \leq K\mathcal{N}_\delta(A)^{1/2}\mathcal{N}_\delta(B)^{1/2}$, which is equivalent to that the Ruzsa distance is small, that is $d_\delta(A, -B) \leq \log K$.*

**Lemma 3.2.14** (Ruzsa calculus)**.** *Let $A, B, C$ be three bounded subsets of $\mathbb{R}^n$. Then*

*(1) If $A \approx_K B$, then $A \approx_{K^{O(1)}} -B$, $K^{-O(1)}\mathcal{N}_\delta(B) \le \mathcal{N}_\delta(A) \le K^{O(1)}\mathcal{N}_\delta(B)$ and $\sigma_\delta[A], \sigma_\delta[B] \le K^{O(1)}$.*

*(2) If $A \approx_K B$ and $B \approx_K C$, then $A \approx_{K^{O(1)}} C$.*

*(3) If $\sigma_\delta[A], \sigma_\delta[B] \le K$ and $\mathcal{N}_\delta(A^{(\delta)} \cap B^{(\delta)}) \ge K^{-1}\mathcal{N}_\delta(A)^{1/2}\mathcal{N}_\delta(B)^{1/2}$, then $A \approx_{K^{O(1)}} B$.*

The proofs are direct applications of the Ruzsa triangular inequality and the Plünnecke-Ruzsa inequality. For the discrete version, please see [TV06] and the second note of Green in [Gre]. The first and second statements says that the Ruzsa distance is symmetric and transitive. The Ruzsa calculus will be used to prove Proposition 3.3.9 (Additive-Multiplicative Balog-Szemerédi-Gowers theorem).

### The additive energy: the discrete case

We first introduce the additive energy in the discrete case. Let $A, B$ be two finite sets in an abelian group $G$. We define the additive energy $\omega(+, A \times B)$ as the number of the quadruplet $(a, b, a', b')$ in $A \times B \times A \times B$ such that $a + b = a' + b'$, that is

$$\omega(+, A \times B) = \#\{(a, b, a', b') \in A \times B \times A \times B | a + b = a' + b'\}.$$

We also have a formulation with $\ell^2$ norm

$$\omega(+, A \times B) = \|\mathbb{1}_A * \mathbb{1}_B\|_2^2, \tag{3.2.6}$$

where the measure in defining $\ell^2$ norm is the counting measure. From the definition, by Young's inequality, we have

$$\omega(+, A \times B) \le |A|^{3/2}|B|^{3/2}, \tag{3.2.7}$$

where $|A|$ denotes the number of elements in $A$. The additive energy is important because it reflects the additive structure of $A$ and $B$. If $|A + B| \le K|A|^{1/2}|B|^{1/2}$, then by the Cauchy-Schwarz inequality,

$$\omega(+, A \times B) \ge \frac{|A|^2|B|^2}{|A + B|} \ge K^{-1}|A|^{3/2}|B|^{3/2}, \tag{3.2.8}$$

which is robustly large with respect to the optimal value of $\omega(+, A \times B)$ (3.2.7). (See [TV06] and [Gre09] for more details).

### The additive energy: the continuous case

We now define the discretized version of the additive energy. On a Cartesian product $X \times Y$ of metric spaces, we use the distance defined by

$$d((x, y), (x', y')) = \sqrt{d_X^2(x, x') + d_Y^2(y, y')},$$

where $x, x'$ are in $X$ and $y, y'$ are in $Y$.

**Definition 3.2.15** (Energy of a map). *Let $X, Y$ be two metric spaces, and let $\varphi$ be a Lipschitz map from $X$ to $Y$. For a subset $C$ of $X$, the energy of $\varphi$ at scale $\delta$ is defined by*

$$\omega_\delta(\varphi, C) = \mathcal{N}_\delta(\{(a, a') \in C \times C | d(\varphi(a), \varphi(a')) \leq \delta\}). \tag{3.2.9}$$

**Lemma 3.2.16.** *Let $\varphi$ be a $K$-Lipschitz map from $\mathbb{R}^m$ to $\mathbb{R}^n$, and let $C$ be a bounded subset of $\mathbb{R}^m$. Then*

*(i) We have*

$$\mathcal{N}_\delta(C)^2 \gg \omega_\delta(\varphi, C) \gg_{n,m} \frac{\mathcal{N}_\delta(C)^2}{\mathcal{N}_\delta(\varphi(C))}. \tag{3.2.10}$$

*(ii) Let $\tilde{C}$ be a maximal $\delta-$separated subset of $C$. Then*

$$\omega_\delta(\varphi, C) \ll \#\{(a, a') \in \tilde{C}^2 | d(\varphi(a), \varphi(a')) \leq (1 + 2K)\delta\}. \tag{3.2.11}$$

(See [He, Lemma 12] for more details) When $m = 2n$, $C = A \times B \subset \mathbb{R}^{2n}$ with $A, B$ in $\mathbb{R}^n$ and $\varphi(a, b) = a + b$, we call $\omega_\delta(+, A \times B)$ the additive energy of $A, B$ at scale $\delta$. We have a formulation with $L^2$ norm (see [BISG17, Appendix A.1] for example. This is also the discretized version of (3.2.6).) We have an inequality

$$\omega_\delta(+, A \times B) \gg \delta^{-3n} \|\mathbb{1}_A * \mathbb{1}_B\|_2^2. \tag{3.2.12}$$

Lemma 3.2.16 (i) implies that

$$\omega_\delta(+, A \times B) \gg \frac{\mathcal{N}_\delta(A \times B)^2}{\mathcal{N}_\delta(A + B)} \geq \frac{\mathcal{N}_\delta(A)^2 \mathcal{N}_\delta(B)^2}{\mathcal{N}_\delta(A + B)}. \tag{3.2.13}$$

If $A \approx_K B$, that is $\mathcal{N}_\delta(A + B) \leq K \mathcal{N}_\delta(A)^{1/2} \mathcal{N}_\delta(B)^{1/2}$, then (3.2.13) implies

$$\omega_\delta(+, A \times B) \gg K^{-1} \mathcal{N}_\delta(A)^{3/2} \mathcal{N}_\delta(B)^{3/2}. \tag{3.2.14}$$

This means that when two sets $A, B$ have additive structure then the additive energy is relatively large.

The additive energy is powerful when combined with the following proposition, a partial converse to (3.2.14), which says that if two sets have relatively large additive energy, then there exist large subsets which have additive structure.

**Proposition 3.2.17** (Balog-Szemerédi-Gowers). *[Tao08, Theorem 6.10] Let $A, B$ be two bounded subsets of $\mathbb{R}^n$ such that*

$$\omega_\delta(+, A \times B) \geq K^{-1} \mathcal{N}_\delta(A)^{3/2} \mathcal{N}_\delta(B)^{3/2}.$$

*Then there exist subsets $A', B'$ of $A, B$ such that*

$$\mathcal{N}_\delta(A') \gg_n K^{-O(1)} \mathcal{N}_\delta(A), \ \mathcal{N}_\delta(B') \gg_n K^{-O(1)} \mathcal{N}_\delta(B)$$

*and*

$$\mathcal{N}_\delta(A' + B') \ll_n K^{O(1)} \mathcal{N}_\delta(A')^{1/2} \mathcal{N}_\delta(B')^{1/2}.$$

### 3.2.2 Sum-product estimates in $\mathbb{R}^n$

We first state the discrete version of the growth under a ball.

**Lemma 3.2.18.** *[Gre09, Lemma 3.1] Let $p$ be a prime number. If $X$ is a subset of $\mathbb{F}_p$, then*

$$\sup_{a\in\mathbb{F}_p}|X+aX|\geq\frac{1}{2}\min\{|X|^2,p\}.$$

The proof is by calculating the additive energy in two ways. Suppose that the result does not hold, then the additive energy $\omega(+,X\times aX)$ is large for every $a$ in $\mathbb{F}_p$. But the sum of the additive energy $\omega(+,X\times aX)$ with respect to $a$ in $\mathbb{F}_p$ is small, which gives the contradiction.

The continuous version uses a Fubini type argument to study the growth under a ball in $(\mathbb{R}^*)^n$. Recall that $\mathrm{id}=(1,\cdots,1)$ is the identity in $(\mathbb{R}^*)^n$.

**Lemma 3.2.19.** *Given $\kappa>0,\sigma\in(0,n)$, there exists $\epsilon>0$ such that for $\delta$ sufficiently small the following holds. Let $X$ be a bounded subset of $\mathbb{R}^n$ such that for $j=1,\dots n$*

$$\forall\rho>\delta,\ \mathcal{N}_\rho(\pi_j(X))\geq\delta^\epsilon\rho^{-\kappa}$$

*and $\mathcal{N}_\delta(X)\leq\delta^{-\sigma-\epsilon}$. Then*

$$\sup_{a\in B_{\mathbb{R}^n}(\mathrm{id},1/2)}\mathcal{N}_\delta(X+aX)\geq\delta^{-\epsilon}\mathcal{N}_\delta(X).$$

**Remark 3.2.20.** *We follow closely the proof of [He, Theorem 3]. To prove the stronger version, we need another lemma, which is a reducible version of [He, Prop.29]. The proof is essentially the same as the irreducible version, with the estimate of small balls replaced by thin cylinders.*

*Proof of Lemma 3.2.19.* Assume that $\mathcal{N}_\delta(X+X)<\delta^{-\epsilon}\mathcal{N}_\delta(X)$, if not the proof is finished. For $\rho>\delta$ and $j=1,\cdots,n$, we have

$$\mathcal{N}_\delta(X+X)\geq\mathcal{N}_\rho(\pi_j(X))\max_{b\in\mathbb{R}}\mathcal{N}_\delta(X\cap\pi_j^{-1}B_{\mathbb{R}}(b,\rho)).\tag{3.2.15}$$

This can be proved by the following standard argument. Choose a maximal subset $\{c_i\}$ of $X$ such that $\pi_j(c_i)$ is $2\rho$-separated. Fix $b$ in $\mathbb{R}$. Choose a maximal $\delta$-separated subset $\{d_k\}$ of $X\cap\pi_j^{-1}B_{\mathbb{R}}(b,\rho)$. If $(i,k)\neq(i',k')$, then

$$d(c_i+d_k,c_{i'}+d_{k'})\geq\delta.$$

Hence $\{c_i+d_k\}_{i,k}$ is a $\delta$-separated subset of $X+X$ and (3.2.15) follows.

For all $b$ in $\mathbb{R}$, by (3.2.15) and hypothesis

$$\mathcal{N}_\delta(X\cap\pi_j^{-1}B_{\mathbb{R}}(b,\rho))\leq\frac{\mathcal{N}_\delta(X+X)}{\mathcal{N}_\rho(\pi_j(X))}\leq\frac{\delta^{-\epsilon}\mathcal{N}_\delta(X)}{\delta^\epsilon\rho^{-\kappa}}=\delta^{-2\epsilon}\rho^\kappa\mathcal{N}_\delta(X).\tag{3.2.16}$$

Let $\mu$ be the normalized Lebesgue measure on $B_{\mathbb{R}^n}(\mathrm{id}, 1/2)$ with total mass 1, and let $a$ be a random variable following the law of $\mu$. Define $\varphi_a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\varphi_a(x, y) = x + ay.$$

By Lemma 3.2.16 (i),

$$\mathcal{N}_\delta(\varphi_a(X \times X)) \gg \frac{\mathcal{N}_\delta(X \times X)^2}{\omega_\delta(\varphi_a, X \times X)},$$

which is also

$$\mathcal{N}_\delta(X + aX) \gg \frac{\mathcal{N}_\delta(X)^4}{\omega_\delta(\varphi_a, X \times X)}.$$

By the Jensen inequality on the function $t \mapsto \frac{1}{t}$ from $\mathbb{R}^+$ to $\mathbb{R}^+$,

$$\mathbb{E}(\mathcal{N}_\delta(X + aX)) \gg \frac{\mathcal{N}_\delta(X)^4}{\mathbb{E}(\omega_\delta(\varphi_a, X \times X))}. \tag{3.2.17}$$

Therefore it is sufficient to give a bound that $\mathbb{E}(\omega_\delta(\varphi_a, X \times X)) \ll \delta^\epsilon \mathcal{N}_\delta(X)^3$.

By Lemma 3.2.16 (ii), letting $\tilde{X}$ be a maximal $\delta$-separated subset of $X$, we have

$$\begin{aligned}\mathbb{E}(\omega_\delta(\varphi_a, X \times X)) &\ll \mathbb{E}(\#\{(x, x', y, y') \in \tilde{X}^4 | \|(x - x') + a(y - y')\| \leq 5\delta\}) \\ &= \sum_{x,x',y,y' \in \tilde{X}} \mathbb{P}\{\|a(y - y') + (x - x')\| \leq 5\delta\},\end{aligned} \tag{3.2.18}$$

where $a$ is contained in $B_{\mathbb{R}^n}(\mathrm{id}, 1/2)$ and $K = 2$. Let $\rho$ be a parameter to be fixed later. We distinguish two cases

- If $\min_j |y_j - y'_j| \geq \rho$, then

$$\mathbb{P}\{\|a(y - y') + (x - x')\| \leq 5\delta\} \ll \delta^n \rho^{-n}. \tag{3.2.19}$$

- Otherwise, the number of pairs $(y, y')$ such that $\min_j |y_j - y'_j| < \rho$ can be bounded using (3.2.16) and (3.2.4)

$$\#\{(y, y') \in \tilde{X}^2 | \min_j |y_j - y'_j| < \rho\} \leq \#\tilde{X}(\sum_j \max_{b \in \mathbb{R}} \#\{\tilde{X} \cap \pi_j^{-1} B_{\mathbb{R}}(b, \rho)\}) \ll \delta^{-2\epsilon} \rho^\kappa \mathcal{N}_\delta(X)^2. \tag{3.2.20}$$

Moreover, we have for all $x, y, y' \in \tilde{X}$,

$$\sum_{x' \in \tilde{X}} \mathbb{P}\{\|a(y - y') + (x - x')\| \leq 5\delta\} \ll 1, \tag{3.2.21}$$

since for every event, there exists a finite number of $x'$ which satisfies the assumption.

Therefore combining (4.4.3) (3.2.19) (3.2.20) and (3.2.21), and taking $\rho = \delta^{\frac{n-\sigma}{n+\kappa}}$,

$$\begin{aligned}\mathbb{E}(\omega_\delta(\varphi_a, X \times X)) &\ll \mathcal{N}_\delta(X)^4 \delta^n \rho^{-n} + \mathcal{N}_\delta(X)^3 \delta^{-2\epsilon} \rho^\kappa \\ &\ll \mathcal{N}_\delta(X)^3 (\delta^{n-\sigma-\epsilon} \rho^{-n} + \delta^{-2\epsilon} \rho^\kappa) \\ &\ll \mathcal{N}_\delta(X)^3 \delta^{-2\epsilon + \frac{\kappa(n-\sigma)}{n+\kappa}}.\end{aligned}$$

When $\epsilon$ is sufficiently small, we have $\mathbb{E}(\omega_\delta(\varphi_a, X \times X)) \ll \mathcal{N}_\delta(X)^3 \delta^\epsilon$, which finishes the proof. □

Before proving Theorem 3.1.3, we need to introduce $S_\delta$, the set of "good elements". Let $A$ be a bounded subset of $\mathbb{R}^n$. Let

$$S_\delta(A, K) = \{a \in B_{End(\mathbb{R}^n)}(0, K) | \mathcal{N}_\delta(A + aA) \leq K \mathcal{N}_\delta(A)\}.$$

The following lemma says that $S_\delta(A, K)$ has a "ring structure".

**Lemma 3.2.21.** *Let $A \subset B(0, K)$ be a subset of $\mathbb{R}^n$.*

*(i) If $a$ is in $S_\delta(A, K)$ and $\|a - b\| \leq K\delta$, then $b$ belongs to $S_\delta(A, K^{O(1)})$.*

*(ii) If $\mathrm{id}, a, b$ are in $S_\delta(A, K)$, then $a - b, a + b, ab$ belong to $S_\delta(A, K^{O(1)})$.*

*(iii) Suppose that $a$ is invertible. If $a^{-1}$ is in $B(0, K)$ and $a$ is in $S_\delta(A, K)$, then $a^{-1}$ belongs to $S_\delta(A, K^{O(1)})$.*

(See [He, Lemma 30] and [BKT04, Proposition 3.3] for more details)

*Proof of Theorem 3.1.3.* The idea is to use Proposition 3.2.5 to force $A$ to grow to a fat ball. Then Lemma 3.2.19 implies the growth of regularity under the action of a fat ball.

Assume that the result fails. That is for every $\epsilon > 0$, there exist $A, X$ satisfying the assumptions of Theorem 3.1.3 such that

$$A \subset S_\delta(X, \delta^{-\epsilon}). \tag{3.2.22}$$

We will reach a contradiction when $\epsilon$ is small enough depending on $\kappa, \sigma$.

By Proposition 3.2.5, for every $\epsilon_0 > 0$, there exist $s \in \mathbb{N}$ and $\epsilon_1 > 0$ depending only on $\epsilon_0$ and $\kappa$, such that if $\epsilon < \epsilon_1$ then

$$B_{\mathbb{R}^n}(0, \delta^{\epsilon_0}) \subset \langle A \rangle_s + B_{\mathbb{R}^n}(0, \delta). \tag{3.2.23}$$

By Lemma 3.2.21 (ii) with $K = \delta^{-\epsilon}$ and (3.2.22), we have

$$\langle A \rangle_s \subset S_\delta(X, \delta^{-O(s)\epsilon}). \tag{3.2.24}$$

By Lemma 3.2.21 (i) with $K = \delta^{-O(s)\epsilon}$ and (3.2.24), (3.2.23)

$$B_{\mathbb{R}^n}(0, \delta^{\epsilon_0}) \subset S_\delta(X, \delta^{-O(s)\epsilon}). \tag{3.2.25}$$

By Lemma 3.2.21 (iii) with $K = \delta^{-O(s)\epsilon-\epsilon_0}$, $a = \frac{\delta^{\epsilon_0}}{2}\mathrm{id}$ and (3.2.25), we have

$$2\delta^{-\epsilon_0}\mathrm{id} = a^{-1} \in S_\delta(X, \delta^{-O(s)\epsilon-O(\epsilon_0)}). \tag{3.2.26}$$

Again by Lemma 3.2.21 (ii), using product and (3.2.25), (3.2.26), we obtain

$$B_{\mathbb{R}^n}(\mathrm{id}, 1/2) \subset B_{\mathbb{R}^n}(0,2) = 2\delta^{-\epsilon_0}\mathrm{id} \cdot B_{\mathbb{R}^n}(0, \delta^{\epsilon_0}) \subset S_\delta(X, \delta^{-O(s)\epsilon-O(\epsilon_0)}). \tag{3.2.27}$$

By Lemma 3.2.19, there exists $\epsilon_2 > 0$ depending only on $\sigma$ and $\kappa$, such that when $\epsilon < \epsilon_2$

$$\sup_{a\in B_{\mathbb{R}^n}(\mathrm{id},1/2)} \mathcal{N}_\delta(X + aX) \geq \delta^{-\epsilon_2}\mathcal{N}_\delta(X). \tag{3.2.28}$$

Taking $\epsilon_0$ sufficiently small, and then taking $\epsilon$ sufficiently small such that $O(s)\epsilon + O(\epsilon_0) < \epsilon_2$, we get a contradiction from (3.2.27) (3.2.28)

$$\delta^{-O(s)\epsilon-O(\epsilon_0)}\mathcal{N}_\delta(X) \geq \sup_{a\in B_{\mathbb{R}^n}(\mathrm{id},1/2)} \mathcal{N}_\delta(X + aX) \geq \delta^{-\epsilon_2}\mathcal{N}_\delta(X).$$

The proof is complete. □

## 3.3 Application to multiplicative convolution of measures

**Notation**: For a measure $\mu$ on $\mathbb{R}^n$, let $\mu^-$ be the symmetry of $\mu$, that is $\mu^-(E) = \mu(-E)$ for any Borel set $E$ of $\mathbb{R}^n$. Let $\mu^{(r)}$ be the $r$-times additive convolution of $\mu$. Recall that $\mu_k$ is the $k$-times multiplicative convolution of $\mu$. For an element $x$ in $\mathbb{R}^n$, we write $x_j$ for its $j$-th coordinate, $j = 1, 2, \cdots, n$. We use the norm induced by the standard scalar product on $\mathbb{R}^n$, that is to say for $x \in \mathbb{R}^n$, $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$. All vectors $x, \xi$ in $\mathbb{R}^n$ are column vectors, and $\langle\cdot,\cdot\rangle$ is the inner product. For $y$ in $\mathbb{R}^n$ and measure $\nu$ on $\mathbb{R}^n$, let $(m_y)_*\nu$ be the pushforward measure of $\mu$ by the multiplication action of $y$, that is $(m_y)_*\nu(E) = \nu(y^{-1}E)$. In order to simplify the notation, we abbreviate $B_{\mathbb{R}^n}(0, R)$ to $B(0, R)$. For a function $f$ on $\mathbb{R}^n$, we write $\|f\|_p$, $p = 1, 2, \infty$, for its $L^p$ norm on $\mathbb{R}^n$.

Let $P_\delta = \frac{\mathbb{1}_{B(0,\delta)}}{|B(0,\delta)|}$, where $|\cdot|$ is the Lebesgue measure of a Borel set in $\mathbb{R}^n$. Let $\nu_\delta = \nu * P_\delta$, which is an approximation of $\nu$ at scale $\delta$.

### 3.3.1 $L^2$-flattening

**Lemma 3.3.1** ($L^2$-flattening). *Given $\sigma_1$, $\kappa > 0$, there exists $\epsilon = \epsilon(\sigma_1, \kappa) > 0$ such that the following holds for $\delta$ small enough. Let $\nu$ be a symmetric Borel probability measure on $[-\delta^{-\epsilon}, \delta^{-\epsilon}]^n \subset \mathbb{R}^n$. Assume that*

$$\|\nu_\delta\|_2^2 \geq \delta^{-\sigma_1}$$

*and $\nu$ satisfies $(\delta, \kappa, \epsilon)$ projective non concentration assumption, that is*

$$\forall \rho \geq \delta, \quad \sup_{a\in\mathbb{R}, v\in\mathbb{S}^{n-1}} (\pi_v)_*\nu(B_{\mathbb{R}}(a, \rho)) = \sup_{a,v} \nu\{x | \langle v, x\rangle \in B_{\mathbb{R}}(a, \rho)\} \leq \delta^{-\epsilon}\rho^\kappa. \tag{3.3.1}$$

*Then*

$$\int \|\nu_\delta * (m_y)_*\nu_\delta\|_2^2 \mathrm{d}\nu(y) \leq \delta^\epsilon \|\nu_\delta\|_2^2. \tag{3.3.2}$$

**Remark 3.3.2.** *The first assumption that the $L^2$ norm is not small means that the measure is not too smooth. Because if the measure is already smooth, then the convolution can not make the measure more smooth. This assumption should be compared with the assumption (iv) in Theorem 3.1.3, where we need that the covering number of the set is not too large.*

*By definition and (3.3.1), $\|\nu_\delta\|_2^2 \leq \|\nu_\delta\|_\infty \|\nu_\delta\|_1 \leq \delta^{\kappa-\epsilon-n}$. Hence $\kappa+\sigma_1 \leq \epsilon+n$, that is the non concentration assumption gives a upper bound of $L^2$ norm. Another explication of the $L^2$ norm is in Lemma 3.3.14.*

**Remark 3.3.3.** *The non concentration assumption here is stronger than the non concentration in Theorem 3.1.3. This is because we need to make multiplication in the proof. The projective non concentration assumption is stable under multiplication and addition. But the non concentration assumption in Theorem 3.1.3 is not.*

*The hypothesis of projective non concentration can be weakened to (i) non concentration on coordinate subspaces and (ii) away from linear subspaces. Please see Remark 3.3.11. But the assumption needed in Theorem 3.1.1 is projective non concentration. Hence we write the same assumption here for simplicity. The step where we really need a projective non concentration is explained in Remark 3.3.18.*

**Remark 3.3.4.** *When $n$ equals 1, this is due to Bourgain [Bou03] [Bou10]. It roughly says that under multiplicative and additive convolution the Hölder regularity of a measure will increase, that is given $\kappa > 0$ there exists $\epsilon > 0$ such that if for all $x$ in $\mathbb{R}$ and $r > 0$, we have $\nu(B(x,r)) \leq r^\kappa$, then $\nu * \nu_2(B(x,r)) \leq r^{\kappa+\epsilon}$. With this observation, Bourgain gave a quantitative proof of the Erdös-Volkmann ring conjecture [Bou03, Section 4].*

Instead of using the original approach in [Bou03] [Bou10], we will follow the approach used for proving $L^2$-flattening in the case of simple Lie groups, using dyadic decomposition to simplify the argument, developed by Bourgain and Gamburd (see [BG08], [BdS16], [BISG17] for example). We introduce an approximation by dyadic level sets.

**Definition 3.3.5.** *Let $\{D_i\}_{i\in I}$ be a family of subsets of $\mathbb{R}^n$. We call $\{D_i\}_{i\in I}$ an essentially disjoint union, if each point $x$ in $\mathbb{R}^n$ is covered by at most $C$ different $D_i$, where $C$ is a fixed constant only depending on $\mathbb{R}^n$.*

**Lemma 3.3.6.** *[LS15][BISG17, Lemma A.4] Let $\nu$ be a Borel probability measure on $\mathbb{R}^n$. Let $\mathcal{C}$ be a maximal $\delta$-separated set of $\mathbb{R}^n$. Let $\mathcal{C}_0 = \{x \in \mathcal{C} | 0 < \nu_{2\delta}(x) \leq 1\}$ and $\mathcal{C}_i = \{x \in \mathcal{C} | 2^{i-1} < \nu_{2\delta}(x) \leq 2^i\}$ for $i \geq 1$. For $i \geq 0$, let $X_i = \cup_{x\in\mathcal{C}_i} B(x,\delta)$. Then $X_i$ is empty if $i \geq O(\log \frac{1}{\delta})$, and we have*

*(1) $\nu_\delta \ll \sum_{i\geq 0} 2^i \mathbb{1}_{X_i}$ and $\sum_{i>0} 2^i \mathbb{1}_{X_i} \ll \nu_{3\delta}$.*

*(2) $X_i$ is an essentially disjoint union of balls of radius $\delta$, for each $i \geq 0$.*

**Lemma 3.3.7.** *[BISG17, Lemma A.5] Let $a > 0$ and $\nu$ be a Borel probability measure on $\mathbb{R}^n$. Then*

$$\|\nu_{a\delta}\|_2 \ll_a \|\nu_\delta\|_2.$$

We also need the following inequality, which is an inverse Chebyshev's inequality. Its proof is elementary.

**Lemma 3.3.8.** *Let $K > 0$. Let $\nu$ be a probability measure on a measure space $X$. Let $f$ be a nonnegative function on $X$. If $|f(x)| \leq K \int_X f \mathrm{d}\nu$ on the support of $\nu$, then*

$$\nu\left\{x \in X \middle| f(x) \geq \frac{1}{2}\int_X f \mathrm{d}\nu\right\} \geq \frac{1}{2K}.$$

Here is the main idea of the proof of $L^2$-flattening: Suppose that (3.3.2) fails. By (3.2.12), we can obtain two sets with large additive energy from the convolution of its character function. Hence we can find some sets in the support of $\nu_\delta$ with large additive energy. Together with Balog-Szemerédi-Gowers theorem (Proposition 3.2.17), this produces two sets which violate sum-product estimates (Theorem 3.1.3).

*Proof of $L^2$-flattening (Lemma 3.3.1).* We follow closely the proof of [BdS16, Lemma 2.5]. Proof by contradiction: Assume that the result fails. Then for every $\epsilon > 0$, there exist $\delta$ small and a measure $\nu$ satisfying

$$\int \|\nu_\delta * (m_y)_*\nu_\delta\|_2^2 \mathrm{d}\nu(y) > \delta^\epsilon \|\nu_\delta\|_2^2. \tag{3.3.3}$$

We will reach a contradiction for $\epsilon$ sufficiently small.

Lemma 3.3.6, (3.3.3) and Cauchy-Schwarz's inequality imply

$$\delta^\epsilon\|\nu_\delta\|_2^2 \ll \int \|\sum_{i,j} 2^i \mathbb{1}_{X_i} * 2^j \mathbb{1}_{yX_j}\|_2^2 \mathrm{d}\nu(y) \ll (\log\delta)^2 \sum_{i,j}\int \|2^i\mathbb{1}_{X_i} * 2^j \mathbb{1}_{yX_j}\|_2^2 \mathrm{d}\nu(y).$$

There must exist $i,j$ such that

$$Q := \int \|2^i \mathbb{1}_{X_i} * 2^j \mathbb{1}_{yX_j}\|_2^2 \mathrm{d}\nu(y) \gg \frac{\delta^\epsilon}{(\log\delta)^4}\|\nu_\delta\|_2^2 \gg \delta^{O(\epsilon)}\|\nu_\delta\|_2^2 \geq \delta^{O(\epsilon)-\sigma}. \tag{3.3.4}$$

With the same argument as in [BISG17, Appendix A.2], we can conclude that $i,j > 0$. If $i = 0$, since $\mathrm{supp}\nu \subset [-\delta^{-\epsilon}, \delta^{-\epsilon}]^n$, we have a bound on volume, that is $|X_0| \leq \delta^{-O(\epsilon)}$. If $j > 0$, by Lemma 3.3.6, then $\|2^j\mathbb{1}_{X_j}\|_1 \ll \|\nu_{3\delta}\|_1 = 1$. Therefore, for $j \geq 0$ and $\|y\| \leq \delta^{-\epsilon}$, by Young's inequality

$$\|\mathbb{1}_{X_0} * 2^j\mathbb{1}_{yX_j}\|_2 \leq \|\mathbb{1}_{X_0}\|_2\|2^j\mathbb{1}_{yX_j}\|_1 \leq \delta^{-O(\epsilon)},$$

which contradicts to (3.3.4) if $\epsilon$ is sufficiently small with respect to $\sigma$. Similarly, we obtain $j > 0$.

Therefore, Lemma 3.3.6 implies

$$\begin{aligned} 2^i|X_i| &= \|2^i\mathbb{1}_{X_i}\|_1 \ll \|\nu_{3\delta}\|_1 = 1, \\ 2^{2i}|X_i| &= \|2^i\mathbb{1}_{X_i}\|_2^2 \ll \|\nu_{3\delta}\|_2^2 \ll \|\nu_\delta\|_2^2, \text{ and similarly for } j, \end{aligned} \tag{3.3.5}$$

where the last inequality is due to Lemma 3.3.7. Hence by Young's inequality, for every $y$ in the support of $\nu$

$$\|2^i\mathbb{1}_{X_i} * 2^j\mathbb{1}_{yX_j}\|_2 \leq \|2^i\mathbb{1}_{X_i}\|_1\|2^j\mathbb{1}_{yX_j}\|_2 = 2^i|X_i|\|2^j\mathbb{1}_{X_j}\|_2|\det y|^{1/2} \ll \delta^{-O(\epsilon)}\|\nu_\delta\|_2, \tag{3.3.6}$$

where $\det y$ is the determinant of $y$ seen as an endomorphism of $\mathbb{R}^n$, that is $\det y = y_1 \cdots y_n$.

Then we take a set $B$ such that for every $y$ in $B$ we have that $\|2^i\mathbb{1}_{X_i} * 2^j\mathbb{1}_{yX_j}\|_2^2$ is relatively large. Let

$$B = \{y \in \mathbb{R}^n | \|2^i\mathbb{1}_{X_i} * 2^j\mathbb{1}_{yX_j}\|_2^2 \geq Q/2\}. \tag{3.3.7}$$

Using Lemma 3.3.8 with $f(y) = \|2^i\mathbb{1}_{X_i} * 2^j\mathbb{1}_{yX_j}\|_2^2$ and (3.3.4), (3.3.6) we have

$$\nu(B) \geq \frac{Q}{2\sup_{y\in\operatorname{supp}\nu} f(y)} \gg \delta^{O(\epsilon)}. \tag{3.3.8}$$

We verify that $X_i, X_j$ and $B$ satisfy some natural assumptions. Take $y$ in $B$. By (3.3.4) and Young's inequality, we have

$$\delta^{O(\epsilon)}\|\nu_\delta\|_2 \ll \|2^i\mathbb{1}_{X_i} * 2^j\mathbb{1}_{yX_j}\|_2 \leq \|2^i\mathbb{1}_{X_i}\|_2\|2^j\mathbb{1}_{yX_j}\|_1 = 2^j|X_j|\|2^i\mathbb{1}_{X_i}\|_2|\det y|. \tag{3.3.9}$$

By (3.3.5), the inequality (3.3.9) gives

$$|\det y| \gg \delta^{O(\epsilon)}, \text{ for } y \in B. \tag{3.3.10}$$

By $\|2^j\mathbb{1}_{X_j}\|_2 \ll \|\nu_\delta\|_2$ , $|\det y| \leq \delta^{-O(\epsilon)}$ and (3.3.5), the inequality (3.3.9) implies

$$2^j|X_j| = \delta^{O(\epsilon)}, \text{ and similarly } 2^i|X_i| = \delta^{O(\epsilon)}. \tag{3.3.11}$$

Next, (3.3.5) and (3.3.9) also imply

$$\delta^{O(\epsilon)}\|\nu_\delta\|_2 \ll 2^j|X_j|\|2^i\mathbb{1}_{X_i}\|_2|\det y| \ll \delta^{-O(\epsilon)}2^i|X_i|^{1/2} \leq \delta^{-O(\epsilon)}2^{i/2}.$$

We have

$$2^i \gg \delta^{O(\epsilon)}\|\nu_\delta\|_2^2 \geq \delta^{-\sigma_1+O(\epsilon)}. \tag{3.3.12}$$

Since $X_i$ is an essentially disjoint union of $\delta$ balls, we have $\mathcal{N}_\delta(X_i) \sim \frac{|X_i|}{\delta^n}$ and $\mathcal{N}_\delta(X_i \cap \pi_l^{-1}B_{\mathbb{R}}(a,\rho)) \ll \delta^{-n}|X_i \cap \pi_l^{-1}B_{\mathbb{R}}(a,2\rho)|$ for every $\rho \geq \delta$ and $l = 1, \cdots, n$ . By (3.3.11) and (3.3.12) we have

$$\mathcal{N}_\delta(X_i) \sim \frac{|X_i|}{\delta^n} = \delta^{O(\epsilon)}2^{-i}\delta^{-n} \ll \delta^{-n+\sigma_1-O(\epsilon)}. \tag{3.3.13}$$

By Lemma 3.3.6(1), the projective non concentration and (3.3.13) for $\rho \geq \delta$, $a \in \mathbb{R}$ and $l = 1, \cdots, n$,

$$\begin{aligned}\mathcal{N}_\delta(X_i \cap \pi_l^{-1}B_{\mathbb{R}}(a,\rho)) &\ll \delta^{-n}|X_i \cap \pi_l^{-1}B_{\mathbb{R}}(a,2\rho)| \leq \delta^{-n}2^{-i}\nu_{3\delta}(\pi_l^{-1}B_{\mathbb{R}}(a,2\rho)) \\ &\ll \mathcal{N}_\delta(X_i)\delta^{-O(\epsilon)}\rho^\kappa.\end{aligned} \tag{3.3.14}$$

This means that $X_i$ inherits non concentration from $\nu$.

We calculate additive energy. By (3.2.12) we have

$$\omega_\delta(+, X_i \times yX_j) \gg \delta^{-3n} \|\mathbb{1}_{X_i} * \mathbb{1}_{yX_j}\|_2^2.$$

Then for every $y$ in $B$, by (3.3.7), (3.3.5), (3.3.11) and (3.3.13)

$$\begin{aligned}\omega_\delta(+, X_i \times yX_j) &\gg \delta^{-3n+O(\epsilon)} \|\nu_\delta\|_2^2 2^{-2i-2j} \\ &\gg \delta^{-3n+O(\epsilon)} 2^{-i-j} |X_i|^{1/2} |X_j|^{1/2} \gg \delta^{-3n+O(\epsilon)} |X_i|^{3/2} |X_j|^{3/2} \\ &\gg \delta^{O(\epsilon)} \mathcal{N}_\delta(X_i)^{3/2} \mathcal{N}_\delta(X_j)^{3/2}.\end{aligned}$$

We can use the following proposition, which is a uniform version of the Balog-Szemerédi-Gowers theorem, inspired by the version on finite field $\mathbb{F}_p$ due to Bourgain.

**Proposition 3.3.9** (Additive-Multiplicative Balog-Szemerédi-Gowers theorem)**.** *Let $K > 2$ be the roughness constant, let $X, X', B$ be bounded subsets of $\mathbb{R}^n$ in $B(0, K)$, with $B^{-1}$ bounded by $K$ (if $b \in B$ then $|b_j| \geq 1/K$ for $j = 1, \dots n$), and let $\mu$ be a Borel probability measure on $B$. If for every $b \in B$ we have*

$$\omega_\delta(+, X \times bX') \geq \frac{1}{K} \mathcal{N}_\delta(X)^{3/2} \mathcal{N}_\delta(X')^{3/2}.$$

*Then there exist $X_o \subset X$, $b_o \in B$ and $B_1 \subset B \cap B(b_o, 1/K^2)$ such that $\mathcal{N}_\delta(X_o) \geq K^{-O(1)} \mathcal{N}_\delta(X)$, $\mu(B_1) \geq K^{-O(1)}$ and for every $b \in b_o^{-1} B_1$*

$$\mathcal{N}_\delta(X_o + bX_o) \leq K^{O(1)} \mathcal{N}_\delta(X_o).$$

Take $K = \delta^{O(\epsilon)}$, $\mu = \frac{1}{\nu(B)} \nu|_B$, $X = X_i$ and $X' = X_j$. By (3.3.10), the set $B$ satisfies the assumption in Proposition 3.3.9. Take $B(1, 2r) \subset U$ as in Theorem 3.1.3 with the group $G = (\mathbb{R}^*)^n$, $V = \mathbb{R}^n$. Proposition 3.3.9 implies that for $\delta$ small enough that $\delta^\epsilon \leq r$ there exist $C_1 > 0$,

$$X_o \subset X_i \text{ and } B_1 \subset B \cap B(b_o, \delta^\epsilon r)$$

such that

$$\mathcal{N}_\delta(X_o) \geq \delta^{C_1 \epsilon} \mathcal{N}_\delta(X_i), \tag{3.3.15}$$

$$\mu(B_1) \geq \delta^{C_1 \epsilon}, \tag{3.3.16}$$

and for $b \in b_o^{-1} B_1$

$$\mathcal{N}_\delta(X_o + bX_o) \leq \delta^{-C_1 \epsilon} \mathcal{N}_\delta(X_o). \tag{3.3.17}$$

**Lemma 3.3.10.** *There exists $C_2 > 0$. These sets $b_o^{-1} B_1, X_o$ satisfy the $(\delta, \kappa, \sigma_1, C_2 \epsilon)$ assumption of Theorem 3.1.3 when $\delta$ is small enough.*

*Proof.* By Proposition 3.3.9, the set $X_o$ satisfies $X_o \subset X_i \subset \operatorname{supp} \nu^{(4\delta)} \subset B(0, \delta^{-O(\epsilon)})$, and $B_1$ satisfies $b_o^{-1} B_1 \subset b_o^{-1} B(b_o, \delta^\epsilon r) \subset U$.

Let

$$\nu_1 = \frac{1}{\nu(B_1)} (\nu|_{B_1}).$$

By (3.3.8) and (3.3.16)

$$\nu(B_1) = \nu(B)\mu(B_1) \gg \delta^{O(\epsilon)}.$$

Hence for any Borel measurable set $E$, we have

$$\nu_1(E) \leq \delta^{-O(\epsilon)}\nu(E). \tag{3.3.18}$$

Assumption (i) (non concentration): By (3.3.18) and projective non concentration

$$\forall \rho > \delta, \ \sup_{a\in\mathbb{R}} \nu_1(\pi_j^{-1}B_{\mathbb{R}}(a,\rho)) \ll \delta^{-O(\epsilon)} \sup_{a\in\mathbb{R}} \nu(\pi_j^{-1}B_{\mathbb{R}}(a,\rho)) \leq \delta^{-O(\epsilon)}\rho^{\kappa},$$

Therefore by $\|b_o^{-1}\| \leq \delta^{-O(\epsilon)}$ and Lemma 3.2.7,

$$\mathcal{N}_\rho(\pi_j(b_o^{-1}B_1)) \geq \delta^{O(\epsilon)}\mathcal{N}_\rho(\pi_j(B_1)) \geq \frac{\nu_1(B_1)}{\sup_{a\in\mathbb{R}} \nu_1(\pi_j^{-1}B_{\mathbb{R}}(a,\rho))} \geq \delta^{O(\epsilon)}\rho^{-\kappa}. \tag{3.3.19}$$

Assumption (ii) (away from proper unitary subalgebras): All the maximal unitary subalgebras of $\mathbb{R}^n$ have a form $\{x \in \mathbb{R}^n | x_i = x_j\}$ with $i \neq j$. Let $f_{ij}(x) = x_i - x_j$ for $x \in \mathbb{R}^n$. By (3.3.10) we know that $|(b_o)_i|, |(b_o)_j| \geq \delta^{O(\epsilon)}$. By (3.3.18),

$$\nu_1\{x | f_{ij}(b_o^{-1}x) \in B_{\mathbb{R}}(0,\rho)\} \leq \delta^{-O(\epsilon)}\nu\{x | f_{ij}(b_o^{-1}x) \in B_{\mathbb{R}}(0,\rho)\}.$$

This is an estimate of being away from linear subspace. If we take the vector $w$ with its $i$-th, $j$-th coordinate equal to $(b_o)_i^{-1}$, $-(b_o)_j^{-1}$, and other coordinates equal to zero, and let $v = w/\|w\|$, then

$$f_{ij}(b_o^{-1}x) = \langle w, x\rangle.$$

Hence projective non concentration (3.3.1) for $v$ implies that

$$\nu\{x | f_{ij}(b_o^{-1}x) \in B_{\mathbb{R}}(0,\rho)\} \leq \nu(\pi_v^{-1}B_{\mathbb{R}}(0,\delta^{-O(\epsilon)}\rho)) \leq \delta^{-O(\epsilon)}\rho^{\kappa}.$$

Hence $b_o^{-1}B_1$ is $\delta^{O(\epsilon)}$ away from proper subalgebra.

Assumption (iii) (non concentration of $X_o$): By (3.3.15) and (3.3.14) we have for $\rho \geq \delta$ and $j = 1, \cdots, n$,

$$\mathcal{N}_\rho(\pi_j(X_o)) \geq \frac{\mathcal{N}_\delta(X_o)}{\sup_{a\in\mathbb{R}} \mathcal{N}_\delta(X_o \cap \pi_j^{-1}B_{\mathbb{R}}(a,\rho))} \gg \delta^{O(\epsilon)} \frac{\mathcal{N}_\delta(X_i)}{\sup_{a\in\mathbb{R}} \mathcal{N}_\delta(X_i \cap \pi_j^{-1}B_{\mathbb{R}}(a,\rho))} \gg \delta^{O(\epsilon)}\rho^{-\kappa}.$$

Assumption (iv): By (3.3.13),

$$\mathcal{N}_\delta(X_o) \ll \mathcal{N}_\delta(X_i) \ll \delta^{-n+\sigma_1-O(\epsilon)}.$$

When $\delta$ is small enough such that $\delta^\epsilon \leq 1/2$, the inequalities with Landau notation can be replaced by $\geq$ or $\leq$ with augmenting $O(\epsilon)$. $\square$

The end of the proof of the $L^2$-flattening lemma: Let $C_1\epsilon$ and $C_2\epsilon$ be given in (3.3.17) and Lemma 3.3.10, respectively. Suppose that $C_2 \geq C_1$ (we can always augment $C_2$ in Lemma 3.3.10. The larger $C_2$ is, the easier the assumption is). Applying Theorem 3.1.3 with $A = b_o^{-1}B_1$ and $X = X_o$, when $\epsilon$ is sufficiently, we have

$$\mathcal{N}_\delta(X_o + X_o) + \sup_{b\in b_o^{-1}B_1} \mathcal{N}_\delta(X_o + bX_o) \geq \delta^{-C_2\epsilon}\mathcal{N}_\delta(X_o).$$

Due to $C_2 \geq C_1$, we have $\delta^{-C_2\epsilon}\mathcal{N}_\delta(X_o) \geq \delta^{-C_1\epsilon}\mathcal{N}_\delta(X_o)$, which contradicts (3.3.17). The proof is complete. □

**Remark 3.3.11.** *The only place where we need a stronger non concentration than non concentration on coordinate subspaces is in the proof of Lemma 3.3.10, when we check assumption (ii) of Theorem 3.1.3. In this step, we need a property of being away from a linear subspace.*

It remains to prove Proposition 3.3.9. We first state a similar version on $\mathbb{F}_p$

**Proposition 3.3.12.** *[Bou09, Thm.C] [Gre09, Prop. 4.1] Let $K > 1$. Let $A \subset \mathbb{F}_p$ and $B \subset \mathbb{F}_p^*$ be two sets. If for all $b$ in $B$, we have $\omega(+, A \times bA) \geq K^{-1}|A|^{3/2}|B|^{3/2}$. Then there exist $x$ in $B$ and $A' \subset A$, $B' \subset x^{-1}B$ with $|A'| \geq K^{-O(1)}|A|$ and $|B'| \geq K^{-O(1)}|B|$ such that for all $b' \in B'$,*

$$|A' + b'A'| \leq K^{O(1)}|A'|.$$

The main point is to find $A'$ which is uniform for $b$. This is accomplished by using the pigeonhole principle. For more details, please see [Gre09, Prop. 4.1] or the following proof.

*Proof of Proposition 3.3.9.* We follow closely the proof of [Gre09, Proposition 4.1]. Since $B$ and $B^{-1}$ are bounded by $K$, if we multiply a set by an element in $B$, then Lemma 3.2.7 implies that we only lose some power on $K$, which does not change the result. That means for $b$ in $B$ and a subset $X$ of $\mathbb{R}^n$, we have

$$K^{-O(1)}\mathcal{N}_\delta(bX) \leq \mathcal{N}_\delta(X) \leq K^{O(1)}\mathcal{N}_\delta(bX)$$

Hence, we will not write the comparison of $\mathcal{N}_\delta(A)$ with $\mathcal{N}_\delta(bA)$ for bounded set $A$. They have the same size.

For every $b \in B$, using additive Balog-Szemerédi-Gowers theorem (Proposition 3.2.17), we have $X_b \times X'_b \subset X \times X'$ such that

$$\mathcal{N}_\delta(X_b + bX'_b) \leq K^{O(1)}\mathcal{N}_\delta(X)^{1/2}\mathcal{N}_\delta(X')^{1/2} \tag{3.3.20}$$

and

$$\mathcal{N}_\delta(X_b) \geq K^{-O(1)}\mathcal{N}_\delta(X),\ \mathcal{N}_\delta(X'_b) \geq K^{-O(1)}\mathcal{N}_\delta(X'). \tag{3.3.21}$$

The result we need is a uniform version, independent of $b$. For this purpose, we want to find an element $b_o$ in $B$ and a portion of $B$ such that the intersection of $X_{b_o}$, $X_b$ is large for $b$ in this portion.

**Lemma 3.3.13.** *Let $\mu$ be a probability measure on a set $B \subset B_{\mathbb{R}^n}(K)$. Let $S$ be a compact set of $\mathbb{R}^n$. Assume that for every $b$ in $B$, there exists $S_b \subset S$ such that*

$$|S_b| \geq K^{-1}|S|.$$

*Then there exists $b_o$ in $B$ and $B_1 \subset B \cap B(b_o, 1/K^2)$ such that $\mu(B_1) \geq K^{-O(1)}$, and for every $b$ in $B_1$*

$$|S_b \cap S_{b_o}| \geq K^{-O(1)}|S|. \tag{3.3.22}$$

*Proof.* We cover $B$ with $O(K^{2n})$ balls of radius $1/K^2$, written as $C_1, \ldots, C_j$. We claim that: There exists $i$ such that

$$\int_{C_i^2} |S_b \cap S_{b'}| \mathrm{d}\mu(b)\mathrm{d}\mu(b') \gg K^{-O(1)}|S|. \tag{3.3.23}$$

By hypothesis, we have

$$\int_B \int_S \mathbb{1}_{S_b}(x)\mathrm{d}x\mathrm{d}\mu(b) = \int_B |S_b|\mathrm{d}\mu(b) \geq K^{-1}|S|. \tag{3.3.24}$$

By Cauchy-Schwarz's inequality

$$K^{2n} \sum_i \left( \int_{C_i} \mathbb{1}_{S_b}(x)\mathrm{d}\mu(b) \right)^2 \gg \left( \int_B \mathbb{1}_{S_b}(x)\mathrm{d}\mu(b) \right)^2. \tag{3.3.25}$$

By Cauchy-Schwarz's inequality and (3.3.24)

$$\int_S \left( \int_B \mathbb{1}_{S_b}(x)\mathrm{d}\mu(b) \right)^2 \mathrm{d}x \geq \left( \int_S \int_B \mathbb{1}_{S_b}(x)\mathrm{d}\mu(b)\mathrm{d}x \right)^2 /|S| \geq K^{-O(1)}|S|. \tag{3.3.26}$$

Rewrite the left hand side of (3.3.25) and integrate it with respect to the Lebesgue measure on $S$. Combined with (3.3.26) we have

$$\sum_i \int_{C_i^2} |S_b \cap S_{b'}|\mathrm{d}\mu(b)\mathrm{d}\mu(b') \gg K^{-O(1)}|S|.$$

The claim (3.3.23) follows.

By Lemma 3.3.8, we can find $C'$, a subset of $C_i^2$, such that $\mu \otimes \mu(C') \gg K^{-O(1)}$ and for all $(b, b') \in C'$

$$|S_b \cap S_{b'}| \gg K^{-O(1)}|S|. \tag{3.3.27}$$

By Fubini's theorem, we can find a $b_o$ such that $\mu\{b \in C_i | (b_o, b) \in C'\} \gg K^{-O(1)}$. We let $B_1 = \{b \in C_i | (b_o, b) \in C'\}$, then this set satisfies the measure assumption. $\square$

The $\delta$ neighborhood of a set behaves well under intersection. In order to simplify the notation, abbreviate $X^{(\delta)}, X'^{(\delta)}, X_b^{(\delta)}, X_b'^{(\delta)}$ to $Y, Y', Y_b, Y_b'$. By (3.2.4) we have

$$\mathcal{N}_\delta(X) \sim |Y|\delta^{-n}. \tag{3.3.28}$$

Due to (3.3.20) and (3.3.21), we have $\mathcal{N}_\delta(X)^{1/2}\mathcal{N}_\delta(X')^{1/2} \geq K^{-O(1)}\mathcal{N}_\delta(X_b + bX'_b) \geq K^{-O(1)}\mathcal{N}_\delta(X_b) \geq K^{-O(1)}\mathcal{N}_\delta(X)$, which implies

$$\mathcal{N}_\delta(X) \sim_{K^{O(1)}} \mathcal{N}_\delta(X').$$

Hence

$$|Y| \sim \delta^n \mathcal{N}_\delta(X) \sim_{K^{O(1)}} \delta^n \mathcal{N}_\delta(X') \sim |Y'|. \tag{3.3.29}$$

Let

$$S = Y \times Y' \text{ and } S_b = Y_b \times Y'_b \text{ for } b \in B. \tag{3.3.30}$$

By (3.3.21), we have $|Y_b| \geq \delta^{O(\epsilon)}|Y|$ and $|Y'_b| \geq \delta^{O(\epsilon)}|Y'|$. Hence, we can use Lemma 3.3.13 with $K = \delta^{-O(\epsilon)}$ to obtain $\mu$, $B_1$ with desired property. Next, we want to find $X_o$. Due to

$$\delta^{O(\epsilon)}|Y||Y'| = \delta^{O(\epsilon)}|S| \leq |S_b \cap S_{b_o}| = |Y_b \cap Y_{b_o}||Y'_b \cap Y'_{b_o}|,$$

together with (3.3.29), we obtain

$$|Y_b \cap Y_{b_o}|, |Y'_b \cap Y'_{b_o}| \geq \delta^{O(\epsilon)}|Y|. \tag{3.3.31}$$

The proof concludes by Ruzsa calculus. By Lemma 3.2.14(1) and (3.3.20), we have

$$\sigma_\delta[X_{b_o}], \sigma_\delta[X_b], \sigma_\delta[X'_{b_o}], \sigma_\delta[X'_b] \leq K^{O(1)}.$$

By (3.3.31) and (3.3.28), we have

$$|X_{b_o}^{(\delta)} \cap X_b^{(\delta)}|, |X_{b_o}'^{(\delta)} \cap X_b'^{(\delta)}| \geq K^{-O(1)}|X^{(\delta)}| \geq K^{-O(1)}\delta^n \mathcal{N}_\delta(X).$$

By Lemma 3.2.14(3), we have $X_{b_o} \approx_{K^{O(1)}} X_b$ and $X'_{b_o} \approx_{K^{O(1)}} X'_b$, the latter implies $bX'_{b_o} \approx_{K^{O(1)}} bX'_b$. Therefore by Lemma 3.2.14(2)

$$X_{b_o} \approx_{K^{O(1)}} X_b \approx_{K^{O(1)}} bX'_b \approx_{K^{O(1)}} bX'_{b_o} = \frac{b}{b_o} b_o X'_{b_o} \approx_{K^{O(1)}} \frac{b}{b_o} X_{b_o}.$$

We get $X_{b_o} \approx_{K^{O(1)}} \frac{b}{b_o} X_{b_o}$. Let $X_o = X_{b_o} \subset X$. The proof is complete. □

### 3.3.2 Proof of the Fourier decay of multiplicative convolutions

Using $L^2$-flattening (Lemma 3.3.1), we give a proof of Theorem 3.1.1. The strategy is to apply $L^2$-flattening to

$$\nu = \frac{1}{2}\left((\mu_{2k} * \mu_{2k}^-)^{(r)} + (\mu_k * \mu_k^-)^{(r)}\right).$$

We need a lemma which explains the connection of $\|\nu_\delta\|_2$ and the Fourier transform of $\nu$

**Lemma 3.3.14.** *Let $\delta > 0$, $C > 1$ and let $\delta_1 = 2\delta/C$. Let $\nu$ be a Borel probability measure on $\mathbb{R}^n$ with support in $B(0, C)$. We have*

$$\|\nu_{\delta_1}\|_2^2 \sim_C \int_{B(0,2\delta^{-1})} |\hat{\nu}(\xi)|^2 \mathrm{d}\xi, \tag{3.3.32}$$

$$\int \|\nu_{\delta_1} * (m_y)_* \nu_{\delta_1}\|_2^2 \mathrm{d}\nu(y) \gg \int_{B(0,2\delta^{-1})} \int |\hat{\nu}(\xi)|^2 |\hat{\nu}(y\xi)|^2 \mathrm{d}\nu(y) \mathrm{d}\xi. \tag{3.3.33}$$

The proof of Lemma 3.3.14 will be given at the end of this section.

Recall that $\mu_k$ is the $k$-times multiplicative convolution of $\mu$. We have

$$\left|\int \exp(2i\pi\langle \xi, x^1\cdots x^k\rangle)\mathrm{d}\mu(x^1)\cdots\mathrm{d}\mu(x^k)\right| = |\hat{\mu}_k(\xi)|.$$

For $k, r \in \mathbb{N}$, let $\sigma_{k,r}$ be the real number defined by

$$\sigma_{k,r} = \frac{\log\int_{\xi\in B(0,2\delta^{-1})}|\hat{\mu}_k(\xi)|^{2r}\mathrm{d}\xi}{|\log\delta|} \sim \frac{\|(\mu_{k,r})_\delta\|_2^2}{|\log\delta|}, \tag{3.3.34}$$

where $\mu_{k,r} = (\mu_k * \mu_k^-)^{*r}$.

The remainder of the proof is to control $\sigma_{k,r}$, divided into two steps. We first prove that if $\sigma_{k,r}$ is not sufficiently small, then $L^2$-flattening (Lemma 3.3.1) reduces the value of $\sigma_{k,r}$. When $\sigma_{k,r}$ is sufficiently small, the Hölder regularity of $\mu$ enables us to finish the proof. This can be understood that if a measure $\mu$ satisfies non concentration assumption, then after sufficient multiplicative and additive convolutions, the sum-product phenomenon implies that $\mu_{k,r}$ is much more smooth.

*Proof of Theorem 3.1.1.* Let

$$\kappa_1 = \kappa_0/4,\ \epsilon = \min\{\epsilon(\kappa_1/2,\kappa_0),\kappa_0\}/2, \tag{3.3.35}$$

where $\epsilon(\kappa_1/2,\kappa_0)$ is given in $L^2$-flattening (Lemma 3.3.1).

**Reducing the value**: We have a consequence of $L^2$-flattening (Lemma 3.3.1), whose proof will be given later.

**Lemma 3.3.15.** *Under the assumption of Theorem 3.1.1, if $\sigma_{k,r}\geq\kappa_1$, then for $\delta$ small enough depending on $k, r$, we have*

$$\sigma_{2k,r'} \leq \sigma_{k,r} - \epsilon,$$

*where $r' = 8r^2 + 4r$.*

**Sufficient regularity**: We have a higher dimensional version of [Bou10, Theorem 7], which says that if two measures have sufficient Hölder regularity, then the multiplicative convolution of these two measures has power decay in its Fourier transform.

**Lemma 3.3.16.** *Let $\alpha > \beta > 0$ and $\delta > 0$. Let $\mu$ be a measure on $B(0,1)$ such that for $j = 1,\ldots,n$*

$$\sup_a (\pi_j)_*\mu(B_{\mathbb{R}}(a,\delta)) \leq \delta^\alpha. \tag{3.3.36}$$

*Let $K > 2$ be a parameter. Let $\nu$ be a compactly supported measure on $B(0,K)$ such that*

$$\int_{B(0,2\delta^{-1})}|\hat{\nu}(\xi)|\mathrm{d}\xi \leq \delta^{-\beta}. \tag{3.3.37}$$

*Then for $\|\xi\|\in[\delta^{-1}/2,\delta^{-1}]$*

$$\int|\hat{\nu}(x\xi)|\mathrm{d}\mu(x) \ll_{K,n} \delta^{\frac{\alpha-\beta}{n+2}}.$$

The proof of Lemma 3.3.16 is classic and will be given at the end of Section 3.3.2 for completeness.

If $\sigma_{1,1} \geq \kappa_1$, iterating Lemma 3.3.15 several times implies that $\sigma_{k,r} < \kappa_1$, where $k, r$ only depend on $\kappa_1$.

We will now apply Lemma 3.3.16 to a well-chosen measure. Take $(\mu_k * \mu_k^-)^{(r)}$ as $\nu$, $\alpha = \kappa_0 - \epsilon$, $\beta = \kappa_1$ and $\tau = \frac{\alpha-\beta}{n+2}$. For $\|\xi\| \in [\delta^{-1}/2, \delta^{-1}]$, by Hölder's inequality and Lemma 3.3.16,

$$|\hat{\mu}_{k+1}(\xi)|^{2r} = \left|\int \hat{\mu}_k(x\xi)\mathrm{d}\mu(x)\right|^{2r} \leq \int |\hat{\mu}_k(x\xi)|^{2r}\mathrm{d}\mu(x) = \int |\hat{\nu}(x\xi)|\mathrm{d}\mu(x) \leq_{k,n} \delta^{\tau}.$$

When $\delta$ is small enough, this yields (3.1.2) with

$$\epsilon_1 = \frac{\tau}{4r} = \frac{\kappa_0 - \epsilon - \kappa_1}{4(n+2)r} \geq \frac{\kappa_0/2 - \kappa_1}{4(n+2)r} \geq \frac{\kappa_1}{4(n+2)r},$$

where the last two inequalities are due to (3.3.35) and $r$ only depends on $\kappa_0$. □

Now we will prove Lemma 3.3.15, where we use the $L^2$-flattening (Lemma 3.3.1).

*Proof of Lemma 3.3.15.* Fix $k, r$ and set

$$\nu = \frac{1}{2}\left((\mu_{2k} * \mu_{2k}^-)^{(r)} + (\mu_k * \mu_k^-)^{(r)}\right). \tag{3.3.38}$$

This is the key construction of this proof. The measure $\nu$ is the bridge to connect $\mu_{2k}$ and $\mu_k$. We summarize the properties of $\nu$ in the following lemma.

**Lemma 3.3.17.** *The measure $\nu$ satisfies $(\delta/r, \kappa_0, 2\epsilon)$ projective non concentration assumption when $\delta$ is sufficient small depending on $k, r$.*

*Proof.* Projective non concentration property is invariant under addition. That is if a probability measure $m$ satisfies projective non concentration, then $m * m'$ also satisfies projective non concentration for any probability measure $m'$. The reason is the following calculation. By Fubini's theorem, we have

$$(\pi_v)_*(m * m')(B_{\mathbb{R}}(a,\rho)) \leq \sup_{b\in\mathbb{R}}(\pi_v)_* m(B_{\mathbb{R}}(b,\rho)).$$

Hence we can drop the additive convolution, and for $\rho \geq \delta_1$, we have

$$\sup_{a\in\mathbb{R}, v\in\mathbb{S}^{n-1}}(\pi_v)_*\nu(B_{\mathbb{R}}(a,\rho)) \leq \frac{1}{2}\sup_{a,v}(\pi_v)_*\mu_k(B_{\mathbb{R}}(a,\rho)) + \frac{1}{2}\sup_{a,v}(\pi_v)_*\mu_{2k}(B_{\mathbb{R}}(a,\rho)). \tag{3.3.39}$$

The property that the support of $\mu$ is contained in $[1/2, 1]^n$ and the projective non concentration of $\mu$ imply the left hand side of (3.3.39) is less than

$$\sup_{a,v}(\pi_v)_*\mu(B_{\mathbb{R}}(a, 4^k\rho)) \leq \delta^{-\epsilon}(\max\{4^k\rho, r\rho\})^{\kappa_0} \leq \delta_1^{-2\epsilon}\rho^{\kappa_0}, \tag{3.3.40}$$

where we have used $r\rho \geq r\delta_1 = \delta$ for projective non concentration and the last inequality holds for $\delta$ small enough depending on $k, r$. Then (3.3.1) follows from (3.3.39) and (3.3.40). The measure $\nu$ satisfies non concentration with $(\kappa_0, 2\epsilon)$ at scale $\delta_1$. □

**Remark 3.3.18.** *This is a step where we really need projective non concentration.*

**Lemma 3.3.19.** *Let $C > 0$ and $r \in \mathbb{N}$. Let $\mu$ be a probability measure on $[1/C, 1]^n \subset \mathbb{R}^n$. Let $\nu$ be defined by*

$$\nu = \frac{1}{2}\left((\mu_2 * \mu_2^-)^{(r)} + (\mu * \mu^-)^{(r)}\right).$$

*We have*

$$\int_{B(0,2\delta^{-1})} |\hat{\nu}(\xi)|^2 \mathrm{d}\xi \sim_C \int_{B(0,2\delta^{-1})} |\hat{\mu}(\xi)|^{4r} \mathrm{d}\xi, \tag{3.3.41}$$

*and*

$$\int_{B(0,2\delta^{-1})} |\hat{\mu}_2(\xi)|^{r'} \mathrm{d}\xi \ll \int_{B(0,2\delta^{-1})} \int |\hat{\nu}(\xi)|^2 |\hat{\nu}(y\xi)|^2 \mathrm{d}\nu(y) \mathrm{d}\xi, \tag{3.3.42}$$

*where $r' = 8r^2 + 4r$.*

The proof is an elementary computation, using Fourier transform and the Hölder inequality.

*Proof.* The lower bound part of (3.3.41) is trivial, which is due to the definition of $\nu$.

For a measure $m$ on $\mathbb{R}$ and $r \in \mathbb{N}$, we have a formula

$$|\hat{m}(\xi)|^{4r} = |\widehat{(m * m^-)^{(2r)}}(\xi)|. \tag{3.3.43}$$

By the multiplicative structure of $\mathbb{R}^n$, we have

$$|\hat{\mu}_2(\xi)| = \left|\int e^{2i\pi\langle \xi, xy\rangle} \mathrm{d}\mu(x)\mathrm{d}\mu(y)\right| = \left|\int \hat{\mu}(y\xi)\mathrm{d}\mu(y)\right|. \tag{3.3.44}$$

By the Hölder inequality,

$$|\hat{\mu}_2(\xi)|^{4r} \leq \int |\hat{\mu}(y\xi)|^{4r} \mathrm{d}\mu(y).$$

Integrating $\xi$ on $B(0, 2\delta^{-1})$, we have

$$\int_{B(0,2\delta^{-1})} |\hat{\mu}_2(\xi)|^{4r} \mathrm{d}\xi \leq \int_{y\in\mathbb{R}^n, \xi\in B(0,2\delta^{-1})} |\hat{\mu}(y\xi)|^{4r} \mathrm{d}\mu(y)\mathrm{d}\xi.$$

Due to $\mathrm{supp}\mu \subset [1/C, 1]^n$, we have

$$\begin{aligned}\int_{B(0,2\delta^{-1})} \int |\hat{\mu}(y\xi)|^{4r} \mathrm{d}\mu(y)\mathrm{d}\xi &\leq C^n \int \int_{B(0,2\delta^{-1})} |\hat{\mu}(y\xi)|^{4r} \mathrm{d}(y\xi)\mathrm{d}\mu(y) \\ &= C^n \int_{B(0,2\delta^{-1})} |\hat{\mu}(\xi)|^{4r} \mathrm{d}\xi,\end{aligned}$$

which implies that

$$\int_{B(0,2\delta^{-1})} |\hat{\mu}_2(\xi)|^{4r} \mathrm{d}\xi \leq_C \int_{B(0,2\delta^{-1})} |\hat{\mu}(\xi)|^{4r} \mathrm{d}\xi.$$

Therefore (3.3.41) follows from

$$\int_{B(0,2\delta^{-1})} |\hat{\nu}(\xi)|^2 \mathrm{d}\xi = \frac{1}{4}\int_{B(0,2\delta^{-1})} \left(|\hat{\mu}(\xi)|^{2r} + |\hat{\mu}_2(\xi)|^{2r}\right)^2 \mathrm{d}\xi \leq_C \int_{B(0,2\delta^{-1})} |\hat{\mu}(\xi)|^{4r}\mathrm{d}\xi.$$

By (3.3.44), Hölder's inequality and (3.3.43)

$$\begin{aligned} |\hat{\mu}_2(\xi)|^{8r^2} &= \left|\int \hat{\mu}(x\xi)\mathrm{d}\mu(x)\right|^{8r^2} \leq \left(\int |\hat{\mu}(x\xi)|^{2r}\mathrm{d}\mu(x)\right)^{4r} \\ &= \left|\int \widehat{(\mu * \mu^-)}^{(r)}(x\xi)\mathrm{d}\mu(x)\right|^{4r}. \end{aligned}$$

By the Plancherel theorem and Hölder's inequality, the above inequality becomes

$$|\hat{\mu}_2(\xi)|^{8r^2} \leq \left|\int \hat{\mu}(y\xi)\mathrm{d}(\mu*\mu^-)^{(r)}(y)\right|^{4r} \leq \int |\hat{\mu}(y\xi)|^{4r}\mathrm{d}(\mu*\mu^-)^{(r)}(y). \tag{3.3.45}$$

Let

$$A_r = \int_{y\in\mathbb{R}^n,\xi\in B(0,2\delta^{-1})} |\hat{\mu}(\xi)|^{4r}|\hat{\mu}(y\xi)|^{4r}\mathrm{d}(\mu*\mu^-)^{(r)}(y)\mathrm{d}\xi.$$

Therefore, by (3.3.45) and (3.3.43)

$$\int_{B(0,2\delta^{-1})} |\hat{\mu}_2(\xi)|^{8r^2+4r}\mathrm{d}\xi \leq \int_{B(0,2\delta^{-1})} |\hat{\mu}_2(\xi)|^{4r}\int |\hat{\mu}(y\xi)|^{4r}\mathrm{d}(\mu*\mu^-)^{(r)}(y)\mathrm{d}\xi = A_r. \tag{3.3.46}$$

By (3.3.38)

$$A_r \ll \int_{y\in\mathbb{R}^n,\xi\in B(0,2\delta^{-1})} |\hat{\nu}(\xi)|^2|\hat{\nu}(y\xi)|^2\mathrm{d}\nu(y)\mathrm{d}\xi.$$

Combined with (3.3.46), we obtain (3.3.42). □

Lemma 3.3.17 and Lemma 3.3.19 enable us to decrease the parameter $\sigma_{k,r}$ by $L^2$-flattening (Lemma 3.3.1).

We return to the proof of Lemma 3.3.15. By (3.3.41) and the hypothesis $\sigma_{k,2r}\geq\kappa_1$, we have

$$\int_{B(0,2\delta^{-1})} |\hat{\nu}(\xi)|^2\mathrm{d}\xi \gg_k \delta^{-\kappa_1}. \tag{3.3.47}$$

Due to $\mathrm{supp}\nu\in[-r,r]^n$ and (3.3.47), taking $C=r$ in Lemma 3.3.14, we have

$$\|\nu_{\delta_1}\|_2^2 \gg_{r,k} \delta^{-\kappa_1} = r^{-\kappa_1}\delta_1^{-\kappa_1}.$$

When $\delta$ is small enough depending on $k, r, \kappa_1$, we have

$$\|\nu_{\delta_1}\|_2^2 \geq \delta_1^{-\kappa_1/2} \text{ and } \mathrm{supp}\nu\subset[-r,r]^n\subset[-\delta_1^{-2\epsilon},\delta_1^{-2\epsilon}]^n. \tag{3.3.48}$$

Lemma 3.3.17 implies that $\nu$ satisfies assumption of $L^2$-flattening lemma with $\sigma_1 = \kappa_1/2, \kappa = \kappa_0$ at scale $\delta_1$. Also notice that (3.3.35) implies $2\epsilon \le \epsilon(\kappa_1/2, \kappa_0)$. Then $L^2$-flattening (Lemma 3.3.1) implies

$$\int \|\nu_{\delta_1} * (m_y)_*\nu_{\delta_1}\|_2^2 \mathrm{d}\nu(y) \le \delta_1^{2\epsilon}\|\nu_{\delta_1}\|_2^2. \tag{3.3.49}$$

Using Lemma 3.3.17, we obtain

$$\int_{B(0,2\delta^{-1})} \int |\hat{\nu}(\xi)|^2 |\hat{\nu}(y\xi)|^2 \mathrm{d}\nu(y)\mathrm{d}\xi \le_r \delta_1^{2\epsilon} \int_{B(0,2\delta^{-1})} |\hat{\nu}(\xi)|^2 \mathrm{d}\xi. \tag{3.3.50}$$

Using Lemma 3.3.19 with $\mu = \mu_k$ and $C = 2^k$, by (3.3.50), we have

$$\int_{B(0,2\delta^{-1})} |\hat{\mu}_{2k}(\xi)|^{r'} \mathrm{d}\xi \ll_{r,k} \delta_1^{2\epsilon} \int_{B(0,2\delta^{-1})} |\hat{\mu}_k(\xi)|^{4r} \mathrm{d}\xi = \delta^{-\sigma_{k,2r}} \delta_1^{2\epsilon} \ll_r \delta^{2\epsilon - \sigma_{k,2r}}.$$

Therefore we have

$$\sigma_{2k,r'} \le \sigma_{k,2r} - 2\epsilon + C_{k,r}/\log \delta^{-1},$$

with some constant $C_{k,r} > 0$. For $\delta$ small enough, it follows that $\delta_{2k,r'} \le \sigma_{k,2r} - \epsilon$. □

It remains to prove Lemma 3.3.14 and Lemma 3.3.16.

*Proof of Lemma 3.3.14.* Recall that $\delta = 2C\delta_1$. We observe that the Fourier transform of $P_\delta$ satisfies

$$\widehat{P_\delta}(\xi) = \int P_\delta(x) e^{i\langle \xi, x\rangle} \mathrm{d}x = \int P_1(x/\delta)\delta^{-n} e^{i\langle \xi, x\rangle} \mathrm{d}x = \widehat{P_1}(\delta\xi).$$

Due to $\widehat{P_1}(\xi) = \Re \int_{B(0,1)} e^{i\langle \xi, x\rangle} \mathrm{d}x |B(0,1)|^{-1} = \int_{B(0,1)} \cos(\langle \xi, x\rangle) \mathrm{d}x |B(0,1)|^{-1}$, we see that

$$\widehat{P_1} \text{ is positive for } \xi \in B(0,1). \tag{3.3.51}$$

We are going to prove (3.3.32). By (3.3.51), we have $\widehat{P_1}(\delta_1\xi) \gg 1$ for $\xi$ in $B(0, 1/\delta_1)$, which implies

$$\begin{aligned} \|\nu_{\delta_1}\|_2^2 &= \int |\hat{\nu}(\xi)|^2 |\widehat{P_{\delta_1}}(\xi)|^2 \mathrm{d}\xi = \int |\hat{\nu}(\xi)|^2 |\widehat{P_1}(\delta_1\xi)|^2 \mathrm{d}\xi \\ &\gg \int_{B(0,1/\delta_1)} |\hat{\nu}(\xi)|^2 \mathrm{d}\xi \ge \int_{B(0,2\delta^{-1})} |\hat{\nu}(\xi)|^2 \mathrm{d}\xi. \end{aligned} \tag{3.3.52}$$

For the other direction of (3.3.32), let $\delta_2 = 2\delta = 4C\delta_1$. Due to $1/\delta_2 + 1/\delta_2 = 1/\delta$, we have $P_{1/\delta} \gg P_{1/\delta_2} * P_{1/\delta_2}$, which implies

$$\begin{aligned} \int_{B(0,2\delta^{-1})} |\hat{\nu}(\xi)|^2 \mathrm{d}\xi &\gg \int |\hat{\nu}(\xi)|^2 |P_{1/\delta}(\xi)|^2 \delta^{-2n} \mathrm{d}\xi \gg \int |\hat{\nu}(\xi)|^2 |P_{1/\delta_2} * P_{1/\delta_2}(\xi)|^2 \delta^{-2n} \mathrm{d}\xi \\ &= \delta^{-2n} \int |\nu * \widehat{P}^2_{1/\delta_2}(x)|^2 \mathrm{d}x. \end{aligned} \tag{3.3.53}$$

By (3.3.51), we have $\widehat{P}^2_{1/\delta_2}(x) = \widehat{P}^2_1(x/\delta_2) \gg \mathbb{1}_{B(0,\delta_2)}(x)$. Combined with $|B(0,\delta_1)| \gg_C \delta^{-n}$, this implies

$$\nu * \widehat{P}^2_{1/\delta_2}(x)\delta^{-n} \geq \nu * P_{\delta_1}(x) = \nu_{\delta_1}(x).$$

Together with (3.3.53), we have the other direction of (3.3.32).

The second inequality (3.3.33) follows from the same argument. By Parseval's formula

$$\begin{aligned}\int \|\nu_{\delta_1} * (m_y)_*\nu_{\delta_1}\|_2^2 \mathrm{d}\nu(y) &= \int\int |\hat{\nu}_{\delta_1}(\xi)|^2 |\widehat{(m_y)_*\nu_{\delta_1}}(\xi)|^2 \mathrm{d}\xi \mathrm{d}\nu(y) \\ &= \int\int |\hat{\nu}(\xi)|^2 |\hat{\nu}(y\xi)|^2 |\widehat{P}_{\delta_1}(y)^2||\widehat{P}_{\delta_1}(y\xi)|^2 \mathrm{d}\nu(y)\mathrm{d}\xi \qquad (3.3.54) \\ &= \int\int |\hat{\nu}(\xi)|^2 |\hat{\nu}(y\xi)|^2 |\widehat{P}_1(\delta_1 y)^2||\widehat{P}_1(\delta_1 y\xi)|^2 \mathrm{d}\nu(y)\mathrm{d}\xi.\end{aligned}$$

For $y \in B(0,C)$ and $\xi \in B(0, 2\delta^{-1})$, we have $\|\delta_1 y\xi\| \leq 1$. By (3.3.51), the inequality (3.3.54) implies (3.3.33). □

*Proof of Lemma 3.3.16.* Let $R = \delta^{-1}$. Consider $H_{R,t} = \{\xi \in B(0,R) | |\hat{\nu}(\xi)| \geq t\}$, where $0 < t < 1$ will be fixed later. Since $\nu$ is supported on $B(0,K)$, the function $|\hat{\nu}|$ is $K$ Lipschitz. We have

$$H_{R,t} + B\left(0, \frac{t}{2K}\right) \subset H_{R+1,\frac{t}{2}}.$$

Hence by (3.2.4)

$$\mathcal{N}_t(H_{R,t}) \ll_K |H_{R,t}^{(\frac{t}{2K})}| t^{-n} \leq |H_{R+1,\frac{t}{2}}| t^{-n}.$$

By the definition of $H_{R,t}$, Chebyshev's inequality and (3.3.37),

$$\mathcal{N}_t(H_{R,t}) \ll t^{-n-1} \int_{B(0,R+1)} |\hat{\nu}(\xi)| \mathrm{d}\xi \ll R^{\beta} t^{-n-1}. \qquad (3.3.55)$$

From now on, suppose that $\|\xi\| \in [R/2, R]$. Let $H^{\xi}_{R,t} = \{x \in \mathbb{R}^n | x\xi \in H_{R,t}\}$. Then due to $\|\xi\| \leq R$, we have $\|x\xi\| \leq R$ for $x \in \mathrm{supp}\mu \subset B(0,1)$, and

$$\int |\hat{\nu}(x\xi)| \mathrm{d}\mu(x) \leq t + \mu(H^{\xi}_{R,t}). \qquad (3.3.56)$$

We cover $H_{R,t}$ with balls of radius $t$ and we also get a cover of $H^{\xi}_{R,t}$ by $B^{\xi}(y,t) = \{x \in \mathbb{R}^n | x\xi \in B(y,t)\}$. Due to $\|\xi\| \geq R/2$, there is at least one $j \in \{1, \ldots, n\}$ such that $|\xi_j| \geq R/(2n)$. Therefore, we can replace $B^{\xi}(y,t)$ by a cylinder $\pi_j^{-1} B_{\mathbb{R}}(y, 2n/R)$ and we obtain

$$\mu B^{\xi}(y,t) = \mu\{x \in \mathbb{R}^n | x\xi \in B(y,t)\} \ll \sup_{y \in \mathbb{R}, j=1,\ldots,n} (\pi_j)_*\mu\{x | x \in B_{\mathbb{R}}(y, 2n/R)\}.$$

The above inequality combined with the hypothesis (3.3.36) implies

$$\mu B^{\xi}(y,t) \ll R^{-\alpha}. \qquad (3.3.57)$$

Therefore by (3.3.55) and (3.3.57)

$$\mu(H_{R,t}^{\xi}) \leq \mathcal{N}_t(H_{R,t}) \max \mu B^{\xi}(y,t) \ll_K R^{\beta-\alpha} t^{-(n+1)}. \tag{3.3.58}$$

If we take $t = R^{-\frac{\alpha-\beta}{n+2}}$, then the result follows from (3.3.56) and (3.3.58). $\square$

## 3.4 Appendix

The main purpose of the Appendix is to give a version of Theorem 3.1.1 (Proposition 3.4.4) for its application to the random product of matrices.

In the application, we need to vary the measure. Using the same idea as in [BD17, Propostion 3.2], we have a version for several different measures (Proposition 3.4.2). The measures appearing in the random product of matrices are not compactly supported, hence we will relax the assumption on support in Proposition 3.4.4.

**Proposition 3.4.1.** *Fix $\kappa > 0$. Then there exist $k \in \mathbb{N}, \epsilon > 0$ depending only on $\kappa_1$ such that the following holds for $\tau$ large enough. Let $\lambda$ be a Borel probability measure on $[\frac{1}{2}, 1]^n \subset \mathbb{R}^n$. Assume that for all $\rho \in [\tau^{-1}, \tau^{-\epsilon}]$ and*

$$\sup_{a\in\mathbb{R}, v\in\mathbb{S}^{n-1}} (\pi_v)_*\lambda(B_{\mathbb{R}}(a,\rho)) = \sup_{a,v} \lambda\{x | \langle v, x\rangle \in B_{\mathbb{R}}(a,\rho)\} \leq \rho^{\kappa}. \tag{3.4.1}$$

*Then for $\xi$ in $\mathbb{R}^n$ with $\|\xi\| \in [\tau/2, \tau]$*

$$\left|\int \exp(2i\pi\langle \xi, x_1 \cdots x_k\rangle) \mathrm{d}\lambda(x_1) \ldots \mathrm{d}\lambda(x_k)\right| \leq \tau^{-\epsilon}.$$

*Proof.* By Lemma 3.2.1, Theorem 3.1.1 implies the result. $\square$

We state a version with different measures.

**Proposition 3.4.2.** *Fix $\kappa > 0$. Then there exist $k \in \mathbb{N}, \epsilon > 0$ depending only on $\kappa_1$ such that the following holds for $\tau$ large enough. Let $\lambda_1, \ldots \lambda_k$ be Borel measures on $[\frac{1}{2}, 1]^n \subset \mathbb{R}^n$ with total mass less than 1. Assume that for all $\rho \in [\tau^{-1}, \tau^{-\epsilon}]$ and $j = 1, \ldots, k$*

$$\sup_{a\in\mathbb{R}, v\in\mathbb{S}^{n-1}} (\pi_v)_*\lambda_j(B_{\mathbb{R}}(a,\rho)) = \sup_{a,v} \lambda_j\{x | \langle v, x\rangle \in B_{\mathbb{R}}(a,\rho)\} \leq \rho^{\kappa}. \tag{3.4.2}$$

*Then for $\xi$ in $\mathbb{R}^n$ with $\|\xi\| \in [\tau/2, \tau]$*

$$\left|\int \exp(2i\pi\langle \xi, x_1 \cdots x_k\rangle) \mathrm{d}\lambda_1(x_1) \ldots \mathrm{d}\lambda_k(x_k)\right| \leq \tau^{-\epsilon}.$$

*Proof.* The proof is the same as the argument in [BD17, Propostion 3.2]. For completeness, we give a ketch here.

We first verify that if the mass of the measure $\lambda$ is less than 1, the result also holds. Let $\epsilon_2$ be given by Proposition 3.4.1 when the regular exponent equals $\kappa/2$. We distinguish two cases

- If $\lambda(\mathbb{R}^d) \geq \tau^{-\epsilon_2\kappa/2}$, then replace $\lambda$ by $\lambda' = \lambda/\lambda(\mathbb{R}^d)$. For $\rho \in [\tau^{-1}, \tau^{-\epsilon_2}]$, we have

$$\sup_{a\in\mathbb{R}, v\in\mathbb{S}^{n-1}} (\pi_v)_*\lambda'(B_{\mathbb{R}}(a,\rho)) \leq \tau^{\epsilon_2\kappa/2} \sup_{a\in\mathbb{R}, v\in\mathbb{S}^{n-1}} (\pi_v)_*\lambda(B_{\mathbb{R}}(a,\rho)) \leq \tau^{\epsilon_2\kappa/2}\rho^{\kappa}$$

  Due to $\rho \leq \tau^{-\epsilon_2}$, we have $\tau^{\epsilon_2\kappa/2}\rho^\kappa \leq \rho^{\kappa/2}$. The measure $\lambda'$ satisfies non concentration with $\kappa/2$. By Proposition 3.4.1, we have the result.

- If $\lambda(\mathbb{R}^d) < \tau^{-\epsilon_2\kappa/2}$, then we have

$$\int \exp(2i\pi\langle \xi, x_1\cdots x_k\rangle)\mathrm{d}\lambda(x_1)\ldots\mathrm{d}\lambda(x_k)| \leq \tau^{-k\epsilon_2\kappa/2}.$$

Hence we can take $\epsilon = \min\{\epsilon_2, k\epsilon_2\kappa/2\}$.

Then we want to prove that the result holds for different measures. For $z \in \mathbb{R}^k$, let $\lambda_z = \sum_{1\leq j\leq k} z_j\lambda_j$. Let

$$G(\lambda_1, \cdots, \lambda_k) = \int \exp(2i\pi\langle \xi, x_1\cdots x_k\rangle)\mathrm{d}\lambda_1(x_1)\ldots\mathrm{d}\lambda_k(x_k)$$

and

$$F(z) = F(z_1, \cdots, z_k) = G(\lambda_z, \cdots, \lambda_z).$$

Then $F(z)$ is polynomial of $k$ variables of degree $k$, and $k!G(\lambda_1, \cdots, \lambda_k)$ is the coefficient of $z_1\cdots z_k$ in $F(z)$. For $z \in \mathbb{R}^k_{\geq 0}$, we have

$$|F(z)| \leq |z|^k\tau^{-\epsilon}, \text{ where } |z| = \sum_{1\leq j\leq k} |z_j|,$$

by using the result of the first part with $\lambda = \frac{1}{|z|}\lambda_z$.

**Lemma 3.4.3.** *Let $F$ be a polynomial of $k$ variables of degree less than $n$. Let $h(F)$ be the maximum of the absolutely value of the coefficients in $F$. Then*

$$h(F) \leq O_{k,n} \sup_{z\in\{0,\cdots,n\}^k\subset\mathbb{R}^k} \{|F(z)|\}.$$

In this lemma, we define two norms on the space of polynomials of $k$ variable of degree less than $n$. The inequality is due to the equivalence of norms on finite dimensional vector space. Hence

$$|G(\lambda_1, \cdots, \lambda_k)| \ll_k h(F) \ll_k \tau^{-\epsilon}.$$

The proof is complete. □

Now we will give another version of Fourier decay of multiplicative convolution, which releases the assumption on the support of $\lambda_j$.

**Proposition 3.4.4.** *Fix $\kappa_0 > 0$. Let $C_0 > 0$. Then there exist $\epsilon_2$ and $k \in \mathbb{N}$ depending only on $\kappa_0$ such that the following holds for $\tau$ large enough depending on $C_0, \kappa_0$. Let $\lambda_1, \dots \lambda_k$ be Borel measures on $\mathbb{R}^n$ supported in $([-\tau^{\epsilon_3}, -\tau^{-\epsilon_3}] \cup [\tau^{-\epsilon_3}, \tau^{\epsilon_3}])^n$ with total mass less than 1, where $\epsilon_3 = \min\{\epsilon_2, \epsilon_2\kappa_0, 1\}/10k$. Assume that for all $\rho \in [\tau^{-2}, \tau^{-\epsilon_2}]$ and $j = 1, \dots, k$*

$$\sup_{a\in\mathbb{R}, v\in\mathbb{S}^{n-1}} (\pi_v)_*\lambda_j(B_{\mathbb{R}}(a,\rho)) = \sup_{a,v} \lambda_j\{x | \langle v, x\rangle \in B_{\mathbb{R}}(a,\rho)\} \le C_0\rho^{\kappa_0}. \tag{3.4.3}$$

*Then for all $\varsigma \in \mathbb{R}^n$, $\|\varsigma\| \in [\tau^{3/4}, \tau^{5/4}]$ we have*

$$\left| \int \exp(2i\pi\langle \varsigma, x_1 \cdots x_k\rangle) \mathrm{d}\lambda_1(x_1)\cdots \mathrm{d}\lambda_k(x_k)\right| \le \tau^{-\epsilon_2}. \tag{3.4.4}$$

**Remark 3.4.5.** *The proof is tedious, but the idea is clear. If the non concentration assumption is valid in some large range, then there is some place to rescale a little the measure and the result still holds. We only need to find some exponent $\epsilon_3$ carefully.*

*Proof.* It is sufficient to prove the case that $\mathrm{supp}\lambda_j \in [\tau^{-\epsilon_3}, \tau^{\epsilon_3}]^n$. Because we can divide each measure into $\lambda_j = \sum_{m\in(\mathbb{Z}/2\mathbb{Z})^n} \lambda_j^m$, where $\lambda_j^m$ is the unique part of $\lambda_j$ whose support is in the same orthant as $m$ and we identify $(\mathbb{Z}/2\mathbb{Z})^n$ with $\{-1,1\}^n \in \mathbb{R}^n$. Then

$$\begin{aligned} &\left| \int \exp(2i\pi\langle \varsigma, x_1 \cdots x_k\rangle) \mathrm{d}\lambda_1^{m_1}(x_1)\cdots \mathrm{d}\lambda_k^{m_k}(x_k)\right| \\ =& \left| \int \exp(2i\pi(\langle \varsigma m_1\cdots m_k, x_1 \cdots x_k\rangle) \mathrm{d}\lambda_1^{m_1}(m_1x_1)\cdots \mathrm{d}\lambda_k^{m_k}(m_kx_k)\right|. \end{aligned}$$

We know that the support of measure $(m_j)_*\lambda_j$ is in $[\tau^{-\epsilon_3}, \tau^{\epsilon_3}]^n$. Hence by the result of the case $\mathrm{supp}\lambda_j$ positive, we have the result with a constant $2^{nk}$.

Let $\epsilon$ as in Proposition 3.4.2 with $\kappa = \kappa_0/2$, and let $\epsilon_2 = \epsilon/4$.

Divide $[\tau^{\epsilon_3}, \tau^{-\epsilon_3}]^n$ into $[2^l, 2^{l+1}] = [2^{l_1}, 2^{l_1+1}] \times \cdots [2^{l_n}, 2^{l_n+1}]$ with $l \in \mathbb{Z}^n$. We rescale the measure in each interval to $[1/2, 1]^n$. Let $\lambda^l(A) = \lambda|_{[2^{l-1}, 2^l]}(2^l A)$. For $\rho \in [\tau^{3/2}, \tau^{\epsilon_2/2}]$ we have

$$(\pi_v)_*\lambda^l(B_{\mathbb{R}}(a,\rho)) \le (\pi_{v'})_*\lambda(\|2^l v\| B_{\mathbb{R}}(a,\rho)), \tag{3.4.5}$$

where $v' = 2^l v/\|2^l v\|$. The inequality $\|2^l v\| \in [\tau^{-\epsilon_3}, \tau^{\epsilon_3}]$ implies that $\|2^l v\|\rho \in [\tau^{-3/2-\epsilon_3}, \tau^{-\epsilon_2/2+\epsilon_3}] \subset [\tau^{-2}, \tau^{-\epsilon_2/4}]$. Due to $\rho^{-1/2} \ge \tau^{\epsilon_2/4} \ge \tau^{\epsilon_3} \ge \|2^l v\|$ for $\rho \in [\tau^{-3/2}, \tau^{-\epsilon_2/2}]$, by (3.4.5) we have

$$(\pi_v)_*\lambda^l(B_{\mathbb{R}}(a,\rho)) \le C_0(\|2^l v\|\rho)^{\kappa_0} \le \rho^{\kappa_0/2}, \tag{3.4.6}$$

for $\tau$ large enough depending on $C_0$.

Summing up over $\|l\| \le \epsilon_3 \log_2 \tau$, we have

$$\begin{aligned}
&\left|\int \exp(2i\pi\langle \varsigma, x_1 \cdots x_k\rangle) \mathrm{d}\lambda_1(x_1) \cdots \mathrm{d}\lambda_k(x_k)\right| \\
&\le \sum_{l^j \in \mathbb{Z}^n, \|l^j\| \le \epsilon_3 \log \tau} \left|\int \exp(2i\pi\langle \varsigma, x_1 \cdots x_k\rangle) \mathrm{d}\lambda_1^{l^1}(2^{-l^1} x_1) \cdots \mathrm{d}\lambda_k^{l^k}(2^{-l^k} x_k)\right| \\
&= \sum_{l^j \in \mathbb{Z}^n, \|l^j\| \le \epsilon_3 \log \tau} \left|\int \exp(2i\pi\langle \varsigma, 2^{l^1+\cdots+l^k} y_1 \cdots y_k\rangle) \mathrm{d}\lambda_1^{l^1}(y_1) \cdots \mathrm{d}\lambda_k^{l^k}(y_k)\right|.
\end{aligned}$$

Let $\tau_1 = \|\varsigma 2^{l^1+\cdots+l^k}\|$, then $\tau_1 \in [\tau^{3/4-k\epsilon_3}, \tau^{5/4+k\epsilon_3}]$. Then we have $[\tau_1^{-1}, \tau_1^{-\epsilon_2}] \subset [\tau^{-3/2}, \tau^{-\epsilon_2/2}]$. The assumption of Proposition 3.4.2 is verified by (3.4.6) with $\tau$ replaced by $\tau_1$. Therefore

$$\begin{aligned}
&\sum_{l^j} \left|\int \exp(2i\pi\langle \varsigma, 2^{l^1+\cdots+l^k} y_1 \cdots y_k\rangle) \mathrm{d}\lambda_1^{l^1}(y_1) \cdots \mathrm{d}\lambda_k^{l^k}(y_k)\right| \le \sum_{l^j \in \mathbb{Z}^n, \|l^j\| \le \epsilon_3 \log \tau} \|\varsigma 2^{l^1+\cdots+l^k}\|^{-\epsilon_2} \\
&\le (2\epsilon_3 \log_2 \tau)^{kn} (\tau^{3/4-k\epsilon_3})^{-\epsilon} \le \tau^{-\epsilon/4},
\end{aligned}$$

when $\tau$ is large enough depending on $k, n, \epsilon$. The proof is complete. □

# Chapter 4

# Finiteness of Small Eigenvalues of Geometrically Finite Rank one Locally Symmetric Manifolds

Let $M$ be a geometrically finite rank one locally symmetric manifolds. We prove that the spectrum of the Laplace operator on $M$ is finite in a small interval which is optimal.

## 4.1 Introduction

Let $M$ be a complete Riemannian manifold. The Laplace operator $\Delta$ acts on the compactly supported smooth functions and admits an extension to an unbounded self-adjoint operator on $L^2(M)$. The study of the spectrum of the Laplace operator on geometrically finite hyperbolic manifolds was started by Lax and Phillips in [LP82a]. The motivation is to give an exponential error term in the asymptotic distribution of orbits for discrete subgroups of the group of isometries of $\tilde{M}$. They proved that on a geometrically finite real hyperbolic manifold of dimension $n$, the intersection of the interval $(-(n-1)^2/4, 0]$ and the spectrum consists of at most finitely many eigenvalues. Geometrical finiteness means that the quotient manifold has a fundamental domain which is a finitely sided polyhedron.

For general cases (see [Ham04]), Hamenstädt proved that on geometrically finite rank one locally symmetric manifolds $\Gamma\backslash X$, where $X = \mathbb{H}^n_{\mathbb{F}}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{H}^n_{\mathbb{F}} = \mathbb{H}^2_{\mathbb{O}}$, the intersection of the interval $[-\delta(X)^2/4 + \chi, 0]$ and the spectrum also consists of at most finitely many eigenvalues, where $\chi$ is any positive number and the exponent of growth $\delta(X)$ is $(n+1)\dim_{\mathbb{R}} \mathbb{F} - 2$. In this case, the applications to counting problems were given in [Kim15]. Inspired by the method in [LP82a], we generalize the result of Lax and Phillips to the case of rank one locally symmetric manifolds.

**Theorem 4.1.1.** *Let $M = \Gamma\backslash X$ be a geometrically finite rank one locally symmetric manifold. Then the intersection of the spectrum of the Laplace operator and the critical* *interval* $(-\delta(X)^2/4, 0]$ *consists of finitely many eigenvalues of finite multiplicities.*

**Remark 4.1.2.** *1. If $M$ has infinite volume, the result is optimal in the following sense. Under the same assumption as in Theorem 4.1.1, the interval $(-\infty, -\delta(X)^2/4]$ is contained in the essential spectrum of $\Delta$. This may be proved as in [Bor07, Prop.7.2]. Together with our result, this implies that the essential spectrum of $\Delta$ is exactly $(-\infty, -\delta(X)^2/4]$ when the volume is infinite.*

*2. In the convex cocompact complex hyperbolic case, the meromorphic extension of the resolvent is already known from [MM87]. The finiteness of the spectrum in the critical interval is a consequence of the meromorphic extension, as in [Bor07, Prop.7.3]. The main idea is that an eigenvalue of the Laplace operator in the critical interval corresponds to a pole of the resolvent in $[0, \delta(X)/2)$.*

*3. In the convex cocompact quaternion hyperbolic case, the finiteness of the discrete spectrum is due to [BO00]. Later, they published a paper [BO08] to treat the geometrically finite rank one case. But they made an additional assumption for quaternion case and they didn't treat the Cayley hyperbolic case. Our result is new in these two cases.*

Unless otherwise stated we assume that the manifolds, the Riemannian metrics and the functions are $C^\infty$ smooth.

## 4.2 Estimates for the spectrum on Riemannian manifolds

### 4.2.1 Barta's trick

Here we will use Barta's trick, a way to estimate the bottom of the spectrum of the Laplace operator by one good function (see [RT15, Lem.3.3]). Recall that for every complete Riemannian manifold $M$, the bottom of the spectrum of the Laplace operator $\lambda_0(M)$ equals the infimum of the Rayleigh quotients $\mathcal{R}(f) = \frac{\int \|\nabla f\|^2 \mathrm{d}vol}{\int f^2 \mathrm{d}vol}$, over all nonzero smooth functions on $M$ with compact support. Hence it is natural to consider the quantity $\int (\|\nabla f\|^2 - \lambda f^2)\mathrm{d}vol$.

**Lemma 4.2.1.** *Let $u, \varphi$ be two real smooth functions on a Riemannian manifold $M$, the support of $u$ being compact. We have*

$$\int \|\nabla(\varphi u)\|^2 \mathrm{d}vol = \int \varphi^2 \|\nabla u\|^2 \mathrm{d}vol - \int u^2 \varphi \Delta \varphi \mathrm{d}vol. \tag{4.2.1}$$

Please see [RT15, Lemma 3.3]

**Proposition 4.2.2.** *Let $M$ be a Riemannian manifold. Let $f$ be a real smooth function on $M$ with compact support. Assume that for some $\lambda > 0$ there exists a smooth function $\varphi$ such that $\varphi > 0$ and $-\Delta\varphi \geq \lambda\varphi$. Then*

$$\int \|\nabla f\|^2 \mathrm{d}vol - \lambda \int f^2 \mathrm{d}vol \geq \int \varphi^2 \|\nabla(f/\varphi)\|^2 \mathrm{d}vol.$$

*Proof.* Applying Lemma 4.2.1 with $u = f/\varphi$ implies

$$\begin{aligned}\int \|\nabla f\|^2 \mathrm{d}vol &= \int \varphi^2 \|\nabla(f/\varphi)\|^2 \mathrm{d}vol - \int (f/\varphi)^2 \varphi \Delta\varphi \mathrm{d}vol \\ &\geq \int \varphi^2 \|\nabla(f/\varphi)\|^2 \mathrm{d}vol + \lambda \int f^2 \mathrm{d}vol.\end{aligned}$$

The proof is complete. □

**Remark 4.2.3.** *Compared with [RT15, Proposition 3.2], we keep the last term* $\int \varphi^2 \|\nabla(f/\varphi)\|^2$. *This term is important and will be exploited in Section 4.2.2.*

The following proposition says that for a Riemannian manifold with quasi-warped product structure, the derivative of the volume density along the vertical geodesic can give us a good control of the bottom of the spectrum.

**Proposition 4.2.4.** *Let* $M = L \times \mathbb{R}$ *be a Riemannian manifold with the metric given by*

$$g = \mathrm{d}t^2 + g(x,t), \tag{4.2.2}$$

*where* $g(x,t)$ *is a metric on* $L \times \{t\}$. *We call this a quasi-warped product metric.*

*The Riemannian volume element can be written as* $\omega = \mathrm{d}t \times h(x,t)\mathrm{d}x$, *where* $\mathrm{d}x$ *is the volume element on* $L \times \{0\}$ *and* $h$ *is a density function from* $L \times \mathbb{R}$ *to* $\mathbb{R}_{\geq 0}$ . *Assume that for some* $\lambda > 0$ *and a nonnegative function* $c$ *on* $\mathbb{R}$, *we have*

$$\partial_t h \geq 2\lambda(1 - c(t))h. \tag{4.2.3}$$

*Then for* $f \in C_c^\infty(M)$, *we have*

$$\int \|\nabla f\|^2 \mathrm{d}vol - \lambda^2 \int (1 - 2c(t)) f^2 \mathrm{d}vol \geq \int e^{-2\lambda t} \|\nabla(f e^{\lambda t})\|^2 \mathrm{d}vol. \tag{4.2.4}$$

**Remark 4.2.5.** *The terminology quasi-warped product is a generalization of warped product in Riemannian geometry. Let* $(M_1, g_1), (M_2, g_2)$ *be two Riemannian manifolds. The warped product* $M_1 \times_f M_2$ *is the product manifold* $M_1 \times M_2$ *equipped with the warped product metric given by*

$$g = g_1 + f g_2, \text{ where } f \text{ is a positive function on } M_1.$$

*Proof.* Let $\varphi(x,t) = e^{-\lambda t}$. We will prove that $-\Delta\varphi \geq \lambda^2(1 - 2c(t))\varphi$ on $L \times \mathbb{R}$. It is sufficient to prove this inequality locally.

Take a local chart on $L$ by $\phi : \mathbb{R}^n \supset U \to L$. Then $\tilde{\phi} = (\phi, Id) : U \times \mathbb{R} \to L \times \mathbb{R}$ is a local chart on $M$. We write elements in $U \times \mathbb{R}$ by $(x_1, \ldots, x_n, x_{n+1})$ with $x_{n+1} = t$. By the definition of $\Delta$, we have

$$\Delta\varphi = \frac{1}{\sqrt{|\det(g)|}} \sum_{1 \leq i,j \leq n+1} \partial_i(\sqrt{|\det(g)|} g^{ij} \partial_j \varphi),$$

where $g^{ij}$ is the inverse matrix of the matrix of metric. By definition, we have

$$h(x,t) = \sqrt{|\det(g_{x,t})|}.$$

Since $\varphi$ is a function only depending on $t$ and the formula of metric implies $g^{tj} = 0$ when $j \neq t$, we have

$$\Delta\varphi = \frac{1}{\sqrt{|\det(g)|}}\partial_t(\sqrt{|\det(g)|}\partial_t\varphi) = \frac{1}{h}\partial_t(h\partial_t\varphi) = \partial_{tt}\varphi + \frac{\partial_t h\partial_t\varphi}{h}. \tag{4.2.5}$$

Due to $\partial_t h \geq 2\lambda(1-c(t))h$ and $\partial_t\varphi = -\lambda\varphi < 0$, we have

$$-\Delta\varphi \geq -\partial_{tt}\varphi - 2\lambda(1-c(t))\partial_t\varphi = \lambda^2(1-2c(t))\varphi.$$

Applying Lemma 4.2.1 with this $\varphi$ and $u = f/\varphi$, we obtain the result. □

### 4.2.2 The Lax-Phillips inequality

Compared with [Ham04], where she fully used the information from the spectrum of the Laplace operator on each component and the negative curvature, the key new input in our article is the observation that the particular form of the Laplace operator gives us more information. A version on $\mathbb{R}$ is as follows:

**Proposition 4.2.6.** *Let $\Delta = \partial_{tt}$ be the standard Laplace operator on $\mathbb{R}$. For every $C > 0$, the positive spectrum of $\Delta + Ce^{-|t|}$ on $L^2(\mathbb{R})$ is finite.*

This is a classical result in spectral theory (see [Dav96, Thm 8.5.1] for a similar version, which says that the positive spectrum is at most countable). In Davies' book, the result comes from a compact perturbation. Another way to prove this type of result is to use the following observation, because the exponential function $e^{-|t|}$ has a very rapid decay. We will give a proof of Proposition 4.2.6 in Section 4.3.

**Lemma 4.2.7.** *For every $C_0 > 0$, there exist a compact interval $U$ and $C_1 > 0$ such that for every real smooth, compactly supported function $f$ on $\mathbb{R}$ we have*

$$\int \|\nabla f\|^2 + C_1\int_U f^2 \geq C_0\int e^{-|t|}f(t)^2\mathrm{d}t. \tag{4.2.6}$$

We call (4.2.6) the Lax-Phillips inequality (LPI). This is a consequence of the following elementary lemma.

**Lemma 4.2.8.** *If $g$ is a real smooth bounded function on $\mathbb{R}$, then for all $r$ in $\mathbb{R}_{\geq 0}$ we have*

$$\int_r^\infty g(t)^2e^{-t}\mathrm{d}t \leq 2\left(g(r)^2e^{-r} + \int_r^\infty g'(t)^2(1+t-r)e^{-t}\mathrm{d}t\right). \tag{4.2.7}$$

*Proof.* By the Newton-Leibniz formula,

$$g(t) = g(r) + \int_r^t g'(s)\mathrm{d}s. \tag{4.2.8}$$

By the Cauchy-Schwarz inequality,

$$g(t)^2 \le 2g(r)^2 + 2\left(\int_r^t g'(s)\mathrm{d}s\right)^2 \le 2g(r)^2 + 2(t-r)\int_r^t g'(s)^2\mathrm{d}s. \tag{4.2.9}$$

Combining (4.2.8) and (4.2.9), we have

$$\begin{aligned}\int_r^\infty g(t)^2 e^{-t}\mathrm{d}t &\le \int_r^\infty \left(2g(r)^2 + 2(t-r)\int_r^t g'(s)^2\mathrm{d}s\right) e^{-t}\mathrm{d}t \\ &= 2\left(g(r)^2 e^{-r} + \int_r^\infty g'(s)^2(1+s-r)e^{-s}\mathrm{d}s\right).\end{aligned}$$

The proof is complete. □

*Proof of LPI (Lemma 4.2.7).* We divide $\mathbb{R}$ into $\mathbb{R}_{\ge 0}$ and $\mathbb{R}_{<0}$. By symmetry, we only need to prove the inequality on $\mathbb{R}_{\ge 0}$. Let $T$ be a large number such that $e^T \ge C_0$. Then by monotonicity, we have $(1+t-T)e^{-t} \le 1/C_0$ for all $t \ge T$. Integrating (4.2.7) on $[T, T+1]$ with $g = f$, we have

$$\int_{T+1}^\infty f(t)^2 e^{-t}\mathrm{d}t \le \int_T^{T+1} f(t)^2 e^{-t}\mathrm{d}t + \frac{1}{C_0}\int_T^\infty f'(t)^2\mathrm{d}t.$$

The result follows by taking $U = [0, T+1]$ and $C_1 = 2C_0$. □

This idea is already utilized in Theorem 3.3 of [LP82a]. The error term $\int e^{-|x|}f^2$ is not artificial, which will appear naturally when we estimate the spectrum on the complement of the convex core.

**Corollary 4.2.9.** *Under the same assumption as in Proposition 4.2.4, suppose that $c(t) = e^{-t}$. If there exists $C_0 > 0$ such that $e^{2\lambda t}/C_0 \le h(t) \le C_0 e^{2\lambda t}$ for $t \in \mathbb{R}$, then for every $C > 0$, there exist a compact interval $I$ and a positive constant $C_1$ such that the following holds. For every smooth, compactly supported function $f$ on $L \times \mathbb{R}$ and every compact subset $K$ of $L$, we have*

$$\int (\|\nabla f\|^2 - \lambda^2 f^2)\mathrm{d}vol + C_1\int_{K\times I} f^2\mathrm{d}vol \ge C\int_{K\times\mathbb{R}} e^{-|t|}f^2\mathrm{d}vol.$$

*Proof.* Let $f_1 = fe^{\lambda t}$. Due to the quasi-warped product structure of the Riemannian metric,

$$\|\nabla f_1\|^2 = \|\nabla' f_1\|^2 + \partial_t f_1^2 \ge \partial_t f_1^2, \tag{4.2.10}$$

where $\nabla'$ is the gradient on $L \times \{t\}$. Substituting (4.2.10) into LPI (Lemma 4.2.7), with the constant $C_2 = C_0^2(C + 2\lambda^2)$, implies that there exist $I$ compact, $C_1 > 0$ such that

$$\int_{\mathbb{R}} \|\nabla f_1(x,t)\|^2 \mathrm{d}t + C_1 \int_I f_1(x,t)^2 \mathrm{d}t \geq C_2 \int_{\mathbb{R}} e^{-|t|} f_1^2(x,t)\mathrm{d}t, \text{ everywhere on } L. \quad (4.2.11)$$

Integrating (4.2.11) over $K$ with respect to $\mathrm{d}x$, we get

$$\int_{K\times\mathbb{R}} \|\nabla f_1\|^2 \mathrm{d}x\mathrm{d}t + C_1 \int_{K\times I} f_1^2 \mathrm{d}x\mathrm{d}t \geq (C + 2\lambda^2) \int_{K\times\mathbb{R}} e^{-|t|} f_1^2 \mathrm{d}x\mathrm{d}t.$$

Using $\mathrm{d}vol = h(x,t)\mathrm{d}t\mathrm{d}x$ and $e^{2\lambda t}/C_0 \leq h(t) \leq C_0 e^{2\lambda t}$, we obtain

$$\int_{K\times\mathbb{R}} \|\nabla f_1\|^2 e^{-2\lambda t} \mathrm{d}vol + C_1 \int_{K\times I} f^2 \mathrm{d}vol \geq (C + 2\lambda^2) \int_{K\times\mathbb{R}} e^{-|t|} f^2 \mathrm{d}vol.$$

This formula together with Proposition 4.2.4 implies the result. □

## 4.3 Finiteness of the spectrum

**Proposition 4.3.1.** *Let $M$ be a complete Riemannian manifold. If for some smooth bounded function $c(x) \geq 0$, there exists a compact subset $U$ with smooth boundary and $\epsilon, C_U > 0$ such that the following holds. For any compact subset $V$ there is $\epsilon_V > 0$ such that for all complex valued function $f \in C_c^\infty(M)$ we have*

$$\int (\|\nabla f\|^2 - c(x)|f|^2)\mathrm{d}vol + C_U \int_U |f|^2 \mathrm{d}vol \geq \epsilon \int_U \|\nabla f\|^2 \mathrm{d}vol + \epsilon_V \int_V |f|^2 \mathrm{d}vol. \quad (4.3.1)$$

*Then the positive spectrum of the operator $T = \Delta + c(x)$ consists of at most finitely many eigenvalues of finite multiplicities.*

*Proof of Proposition 4.2.6 from Proposition 4.3.1.* From Lemma 4.2.7, we have

$$\int \left( \|\nabla f\|^2 - \frac{C_0}{4} e^{-|t|} f^2 \right) \mathrm{d}t + \frac{1}{2} C_1 \int_U f^2 \geq \frac{1}{2} \int \|\nabla f\|^2 + \frac{C_0}{4} \int e^{-|t|} f(t)^2 \mathrm{d}t.$$

Take $C_0 = 4C$ and $c(x) = Ce^{-|x|}$. Then $c(x)$ satisfies (4.3.1). Therefore, Proposition 4.2.6 follows from Proposition 4.3.1. □

This is a “baby case" of the main result of this manuscript, whose proof will also follow from Proposition 4.3.1. We will establish (4.3.1) for geometrically finite rank one locally symmetric manifolds in Section 4.5.

It remains to prove Proposition 4.3.1. Recall some results in spectral theory:

**Definition 4.3.2.** *Let $H$ be a complex Hilbert space, $T$ be a linear operator with the domain of definition $D$, which is dense in $H$. We call $T$ self-adjoint if its adjoint $T^*$ equals $T$ on $D$ and the domain of definition of $T^*$ satisfies $D^* = D$, where $D^*$ is defined as*

$$D^* = \{f \in H | \exists C_f > 0 \text{ such that } \forall g \in D, \ |(f, Tg)| \leq C_f |g|_H\}.$$

*We call $T$ positive if for every nonzero $f$ in $D$, we have $(Tf, f) > 0$.*

**Proposition 4.3.3.** *Let $M$ be a complete Riemannian manifold. Define the space $H_0^1(M)$ as the completion of $C_c^\infty(M)$ under the norm*

$$\|f\|_{H^1} = \|f\|_{L^2} + \|\nabla f\|_{L^2}.$$

*Let the domain of the Laplace operator be*

$$D = \{f \in H_0^1(M) |\ \Delta f \in L^2(M)\}.$$

*Then $\Delta : D \subset L^2 \to L^2$ is a self-adjoint operator.*

(See [Tay10b, Sec.8.2] for more details.)

**Proposition 4.3.4.** *If $T$ is a self-adjoint operator on a Hilbert space $H$, then there is a decomposition*

$$H = H_+ \oplus H_- \oplus \ker T$$

*such that $T$ preserves the decomposition, $T$ is self-adjoint on $H_+$, $H_-$ and $T$ is positive, negative on $H_+, H_-$, respectively.*

*If there exists $\lambda > 0$ such that $\lambda Id - T$ is positive, then $H_+$ in the above decomposition is actually in $D$, the domain of definition.*

(See [Lax02, 32 Thm1] for more details.) For $f \in C_c^\infty(M)$, define

$$K(f) = C_U \int_U |f|^2 \mathrm{d}vol,\ E(f) = \int (\|\nabla f\|^2 - c(x)|f|^2)\mathrm{d}vol,$$

and $F(f) = E(f) + K(f)$. Inequality (4.3.1) implies that $F$ is a positive definite quadratic form on $C_c^\infty(M)$. We define a Hilbert space $\mathcal{H}$ as the completion of $C_c^\infty(M)$ with respect to the norm $|\cdot|_F$, written as $\mathcal{H} = \overline{C_c^\infty(M)_F}$.

For an open subset $V$ of $M$, we define $H^1(V)$ as the completion of $C^\infty(V)$ with respect to the norm $\|f\|^2_{H^1(V)} = \int_V |f|^2 + \int_V \|\nabla f\|^2$.

**Proposition 4.3.5.** *With the same assumption as in Proposition 4.3.1, there exists a subspace $\mathcal{H}_1$ of finite codimension in $\mathcal{H}$, on which $E$ is positive definite.*

*Proof.* Because of inequality (4.3.1), we can define a bounded restriction map from $\mathcal{H}$ to $H^1(U)$ and compose it with the injection $\iota$ from $H^1(U)$ to $L^2(U)$, that is

$$S : \mathcal{H} \to H^1(U) \xrightarrow{\iota} L^2(U).$$

Let $S^*$ be the adjoint of $S$, then $S^*S : \mathcal{H} \to \mathcal{H}$ is a self-adjoint operator. For $f \in \mathcal{H}$, we have

$$F(S^*Sf, f) = \int_U |f|^2 \mathrm{d}vol = \frac{1}{C_U} K(f).$$

By the Rellich theorem [Tay10a, Chapter 4, Proposition 4.4] and the smoothness of the boundary of $U$, the injection $\iota$ is compact. Therefore $S^*S$ is a compact self-adjoint operator. The set of eigenvalues has a unique accumulation point 0. In a subspace of finite codimension in $\mathcal{H}$, we have $|K(f)| = C_U|F(S^*Sf, f)| \leq \frac{1}{2}|F(f)|$. Therefore

$$E(f) = F(f) - K(f) \geq \frac{1}{2}F(f).$$

The proof is complete. □

*Proof of Proposition 4.3.1.* By definition,

$$\|f\|_F \leq \|f\|_{L^2} + \|\nabla f\|_{L^2} = \|f\|_{H^1}.$$

Thus we can extend the injection $C_c^\infty(M) \to \mathcal{H}$ to an application $j : H_0^1(M) \to \mathcal{H}$. By (4.3.1), for any compact subset $V$ of $M$, we have $F(f) \geq \epsilon_V \int_V |f|^2$ for $f$ in $C_c^\infty(M)$. Therefore $j$ is injective and can be seen as an inclusion.

Since $\Delta$ is self-adjoint and $c$ is bounded, the operator $T = \Delta + c(x)$ is also self-adjoint. Since $(\|c\|_\infty + 1)Id - T$ is positive, by using Proposition 4.3.4 with $T$ and $H = L^2(M)$, we have $H_+ \subset D \subset H_0^1(M) \subset \mathcal{H}$. For a nonzero element $u$ in $H_+$, we have

$$E(u) = \int_M (\|\nabla u\|^2 - c(x)|u|^2)\mathrm{d}vol = \int_M -(Tu)\bar{u}\mathrm{d}vol = -(Tu, u) < 0.$$

Proposition 4.3.5 implies $\mathcal{H}_1 \cap H_+ = \{0\}$, therefore $H_+$ is of finite dimension. Due to Proposition 4.3.4, the operator $T$ is self-adjoint on $H_+$, hence the positive spectrum of $T$ is finite and each element is an eigenvalue of finite multiplicity. □

## 4.4 Rank one locally symmetric manifolds

We study the spectrum on cusps and complement of convex sets in this section, which will be used in Section 4.5 to obtain global result.

### 4.4.1 Real rank one globally symmetric spaces

Real rank one globally symmetric spaces are usually classified into four families: $\mathbb{H}^n_{\mathbb{R}} = SO_o(1,n)/SO(n)$, $\mathbb{H}^n_{\mathbb{C}} = PU(1,n)/U(n)$, $\mathbb{H}^n_{\mathbb{H}} = PSp(1,n)/(Sp(1) \times Sp(n))$ and an exceptional one, the Cayley hyperbolic plane $\mathbb{H}^2_{\mathbb{O}} = F_4/Spin(9)$. For the first three types, we have a uniform treatment by projective models, in which the metric and the curvature can be computed explicitly (see for example [Mos73], [Pan89], [Qui06]). But the Cayley hyperbolic plane needs different model.

The space $\mathbb{H}^n_{\mathbb{F}}$ is a Riemannian manifold with pinched curvature between $-1$ and $-4$. The ideal boundary of the symmetric space $\mathbb{H}^n_{\mathbb{F}}$, denoted by $\partial\mathbb{H}^n_{\mathbb{F}}$, is the set of equivalent classes of geodesic rays. From now on, abbreviate $\mathbb{H}^n_{\mathbb{F}}$, $\partial\mathbb{H}^n_{\mathbb{F}}$, $\mathbb{H}^n_{\mathbb{F}} \cup \partial\mathbb{H}^n_{\mathbb{F}}$ to $X$, $X_I$, $X_c$. Equip $X_c$ with the topology such that $X_c$ becomes a compact manifold with boundary. The space $X$ is homeomorphic to an open ball of dimension $n \dim_{\mathbb{R}} \mathbb{F}$, and $X_c$

is homeomorphic to a closed ball of the same dimension. Let $\xi$ be a point in $X_I$, and let $x, y$ be two points in $X$. We define the Busemann function by

$$b_\xi(x,y) = \lim_{t\to+\infty}(d(x,\gamma(t)) - d(y,\gamma(t))),$$

where $d$ is the distance induced by the Riemannian metric, and $\gamma(t)$ is a geodesic ray asymptotic to $\xi$. Let $o$ be a fixed reference point in $X$. The level sets of $b_\xi(\cdot, o)$ are called horospheres based at $\xi$. A horoball based at $\xi$ is a set $b_\xi(\cdot,o)^{-1}(-\infty, r]$.

We introduce the horospherical model for $X$ following [Pan89, Section 9]. Fix a point $\infty$ of $X_I$. Let $G = KAN$ be the Iwasawa decomposition with $A$ and $N$ fixing $\infty$, and $K$, a maximal compact subgroup, fixing $o$. The group $A$ is isomorphic to $\mathbb{R}$ and $N$ is a simply connected nilpotent Lie group. Let $M = K \cap Stab_\infty(G)$. Here $M$ is also the maximal subgroup of $K$ which commutes with $A$. Let $\mathcal{N}$ be the Lie algebra of $N$. The group $A$ normalizes $N$, and the conjugation action of $A$ induces an automorphism on $\mathcal{N}$. We make a choice of a particular generator of $A$ such that $\mathcal{N}$ admits a decomposition

$$\mathcal{N} = V^1 \oplus V^2,$$

where $V^j$ equals to $Ker(\mathrm{Ad}(a_t) - e^{jt})$ for $t \neq 0$ and $a_t$ is an element in $A$. Moreover, $V^2$ is the centre of $\mathcal{N}$ and $[V^1, V^1] = V^2$. We can identify $V^1, V^2$ with $\mathbb{F}^{n-1}, \Im\mathbb{F}$, and the Lie algebra structure is given by

$$[(x_1,\ldots,x_{n-1}),(y_1,\ldots,y_{n-1})] = \Im\sum_{j=1}^{j=n-1}\bar{x}_j y_j.$$

Starting from $V^1 \oplus V^2$, we obtain two left invariant distributions $W^1, W^2$ on $N$. Let $\alpha$ be the endomorphism on $\mathcal{N}$, which is the differential of $\mathrm{Ad}(a_t)$ at $t = 0$, mapping vector $v$ in $V^j$ to $jv$. Let $e^\alpha$ be the induced automorphism on $N$. Let $c(t)$ be the geodesic ray starting from $o$ to $\infty$, that is $c(t) = a_t o$. We have a diffeomorphism from $N \times A$ to $X$ given by $(\nu, t) \to \nu \cdot c(t)$, and the action of $A$ reads as $a_t(\nu, s) = (e^{t\alpha}\nu, t + s)$. Let $m = (n-1)\dim_{\mathbb{R}}\mathbb{F}$ and $q = \dim_{\mathbb{R}}\mathbb{F} - 1$. The hyperbolic metric on $X$ can be written as

$$g = g_t \oplus \mathrm{d}t^2, \tag{4.4.1}$$

where these left invariant metrics $g_t$ on $N$ have, under the distribution $W^1, W^2$, a matrix of the form

$$\begin{pmatrix} e^{2t}Id_m & 0 \\ 0 & e^{4t}Id_q \end{pmatrix}.$$

We also need to calculate the Riemann curvature tensor. Let $\mathbf{n}$ be the unit tangent vector at $o$ to the geodesic $c(t)$. If $v \in V^1$, then $\mathbf{n}, v$ are tangent to a totally real $2-$plane of curvature $-1$. If $v \in V^2$, then $\mathbf{n}, v$ are tangent to a $\mathbb{F}-$line of curvature $-4$. Therefore

$$R(\mathbf{n}, v_j, \mathbf{n}) = -j^2 v_j \text{ for } v_j \in V^j \quad j = 1, 2. \tag{4.4.2}$$

We introduce a half space model. As in the horospherical model, with $o$ in $X$ and $\infty$ in $X_I$ fixed, let $y = e^{-t}$. The coordinate map is replaced by a map from $N \times \mathbb{R}_{>0}$ to

$X$, that is $(\nu, y) \to \nu \cdot c(\log y)$. We also give a differential structure to $X_c$. Locally the compactification is obtained by adding the hyperplane $\{y = 0\}$. Locally, this also gives a differential structure to the manifold with boundary. If we fix another point in $X_I$, and give a differential structure by the same procedure, then by the compatibility of the differential structures, we have a differential structure on $X_c$.

### 4.4.2 Discrete subgroups

Let $G$ be the group of isometries of the symmetric space $X$, and let $\Gamma$ be a discrete subgroup of $G$. The group $\Gamma$ may have torsion. In the geometrically finite case, the group $\Gamma$ is always finitely generated [Bow95, Prop5.5.1]. We can use a result of Selberg [Sel60], passing to a normal subgroup $\Gamma'$ which is of finite index in $\Gamma$ and has no torsion. Since the spectrum of the Laplace operator on a finite covering space contains the spectrum of the original one, we suppose that $\Gamma$ is without torsion.

Fix a point $x$ in $X$, and let $\overline{\Gamma \cdot x}$ be the closure of the orbit $\Gamma \cdot x$ in $X_c$. The limit set of the group $\Gamma$ is defined by

$$\Lambda(\Gamma) = X_I \cap \overline{\Gamma \cdot x}.$$

This definition of the limit set is independent of the choice of $x$. Let $\Omega = X_I - \Lambda(\Gamma)$. Since the actions of $\Gamma$ on $X \cup \Omega$ and $X$ are properly discontinuous [Bow95, Prop3.2.6], we set

$$M_c(\Gamma) = \Gamma\backslash(X \cup \Omega),\ M(\Gamma) = \Gamma\backslash X.$$

These are a manifold with boundary and a rank one locally symmetric manifold, respectively.

In order to study the spectrum, we define an energy form, which will be used throughout the paper. Let

$$E(f) = \int_{M(\Gamma)} (\|\nabla f\|^2 - \ell^2 f^2) \mathrm{d}vol \tag{4.4.3}$$

for $f \in C_c^\infty(M(\Gamma))$, where $\Gamma$ is a discrete subgroup of $G$ and $\ell = \delta(X)/2 = ((n+1)\dim_{\mathbb{R}} \mathbb{F} - 2)/2$ is half the exponent of growth.

### 4.4.3 Cusps

**Definition 4.4.1** (Parabolic subgroup)**.** *Let $\Pi$ be a subgroup of $G$. We call $\Pi$ parabolic if the set of fixed points of $\Pi$ in $X_c$ consists of a unique point $\xi$ in $X_I$, and $\Pi$ preserves setwise every horosphere based at $\xi$.*

Let $\Pi$ be a discrete parabolic subgroup with fixed point $\infty$, a point in $X_I$. We use the horospherical model $N \times A \to X$ introduced in Section 4.4.1. Then $\Pi$ is a subgroup of $M \ltimes N$. Recall that $M$ is the subgroup of $K$ which preserves setwise every horosphere based at $\infty$ and the metric $g_t$ on $N$. The part $N$ of $M \ltimes N$ acts as translation on $N$ and $M$ acts as rotation. Let $\Pi\backslash N$ be the quotient space of $N$ under the action of $\Pi$. The quotient manifold satisfies

$$M(\Pi) \simeq (\Pi\backslash N) \times \mathbb{R} \text{ as a topologic space.}$$

We will apply Proposition 4.2.4 to $M(\Pi)$. Since $\Pi$ preserves the metric $g_t$, we have a quotient metric on $\Pi\backslash N$. The formula (4.4.1) implies that the metric on $(\Pi\backslash N) \times \mathbb{R}$ is a quasi-warped product metric, and the volume element is equal to $e^{2\ell t}\mathrm{d}t\mathrm{d}\eta$, where $\mathrm{d}\eta$ is the volume element on $(\Pi\backslash N) \times \{0\}$. So the function $h$ in Proposition 4.2.4 equals $e^{2\ell t}$, and $h$ satisfies (4.2.3) with $\lambda = \ell$ and $c(t) = 0$. Applying Proposition 4.2.4 and Corollary 4.2.9, we have

**Lemma 4.4.2.** *Let $E$ be as in (4.4.3). For a function $f \in C_c^\infty((\Pi\backslash N) \times \mathbb{R})$, we have*

$$E(f) = \int \|\nabla(e^{\ell t} f)\|^2 e^{-2\ell t}\mathrm{d}vol. \tag{4.4.4}$$

*Given $C > 0$, there exist a compact interval $I \subset \mathbb{R}$ and a constant $C_1 > 0$ such that for all compact set $K$ in $\Pi\backslash N$ we have*

$$E(f) + C_1 \int_{K\times I} f^2\mathrm{d}vol \geq C \int_{K\times\mathbb{R}_{\geq 0}} e^{-t} f^2\mathrm{d}vol. \tag{4.4.5}$$

### 4.4.4 Convex subsets and the normal exponential map

We need a lemma which says that in the normal exponential coordinate, the Riemannian manifold has a quasi-warped product metric. This lemma is similar to the Gauss lemma, where the hypersurface $S$ degenerates to a point.

**Lemma 4.4.3.** *Let $S$ be a smooth hypersurface of a Riemannian manifold $M$. Let $\exp^\perp : S\times\mathbb{R}_{\geq 0} \to M$ be the normal exponential map given by $\exp^\perp(x,t) = \exp_x(t\mathbf{n}(x))$. Assume $\exp^\perp$ is an embedding. Then for every $x$ in $S$ and $s \geq 0$, the curve $\gamma_x : t \to \exp_x(t\mathbf{n}(x))$ is normal to the hypersurfaces $\exp^\perp(S \times \{s\})$.*

**Definition 4.4.4.** *Let $M$ be a complete Riemannian manifold, and let $D$ be a closed subset of $M$. We call $D$ geodesically convex if the preimage $\tilde{D}$ of $D$ in the universal cover $\tilde{M}$ is convex, that is for any two points $x, y$ in $\tilde{D}$ there exists a unique minimizing geodesic contained in $\tilde{D}$ which connects $x, y$.*

Let $M$ be a rank one locally symmetric manifold such that its universal cover has exponent of growth $2\ell$. Let $D$ be a geodesically convex closed subset of $M$ with non empty interior and with smooth boundary. Let $S = \partial D$, and let $\exp^\perp : S \times \mathbb{R}_{\geq 0}$ be the outer normal exponential map given by $\exp^\perp(x,t) = \exp_x(t\mathbf{n}(x))$. Assume that $\exp^\perp$ is a diffeomorphism from $S \times \mathbb{R}_{\geq 0}$ to $M - \mathring{D}$. By Lemma 4.4.3, the metric can be written as in (4.2.2). Let $h$ be the density function defined as in Proposition 4.2.4.

**Lemma 4.4.5.** *With the above assumption, if we have an upper bound on the second fundamental form on $S$, then there exists $C_1 > 0$ depending on the bound such that the density function $h$ satisfies the following inequalities for every $x$ in $S$, and every $t \geq 0$:*

$$\partial_t h(x,t) \geq 2\ell(\tanh t)h(x,t), \quad e^{2\ell t}/C_1 \leq h(x,t) \leq C_1 e^{2\ell t}. \tag{4.4.6}$$

**Remark 4.4.6.** *The first inequality in (4.4.6) has already been used in [Ham04, Lemma 2.3] without proof. For completeness, a proof is given here.*

**Remark 4.4.7.** *This lemma is a consequence of the negative curvature and the convexity. We do not have a better inequality $\partial_t h(t) \geq 2\ell h(t)$. For example, in $\mathbb{H}^2$, let $D$ be the $r-$neighbourhood of a geodesic, then we can compute explicitly that $h(x,t) = \cosh(t+r)/\cosh r$.*

The proofs of these two lemmas, which use the standard computations for Jacobi fields, will be given later. By Lemma 4.4.5, we have

$$\partial_t h \geq 2\ell(\tanh t)h \geq 2\ell(1-2e^{-t})h. \tag{4.4.7}$$

Using Proposition 4.2.4 and Corollary 4.2.9 with $S \times \mathbb{R}_{\geq 0}$, (Proposition 4.2.4 deals with manifolds $L \times \mathbb{R}$, but the proof of $L \times \mathbb{R}_{\geq 0}$ is exactly the same) we have

**Lemma 4.4.8.** *Let $E$ be as in (4.4.3). With the same assumptions as in Lemma 4.4.5, for a function $f \in C_c^\infty(\exp^\perp(S \times \mathbb{R}_{\geq 0}))$, we have*

$$E(f) \geq \int (\|\nabla(e^{\ell t})f)\|^2 e^{-2\ell t} - 4\ell^2 e^{-t} f^2) \mathrm{d}vol. \tag{4.4.8}$$

*Given $C > 0$, there exist a compact set $I \subset \mathbb{R}_{\geq 0}$ and a constant $C_1 > 0$ such that for all compact set $K$ in $S$*

$$E(f) + C_1 \int_{\exp^\perp(K \times I)} f^2 \mathrm{d}vol \geq C \int_{\exp^\perp(K \times \mathbb{R}_{\geq 0})} e^{-t} f^2 \mathrm{d}vol. \tag{4.4.9}$$

It remains to prove Lemma 4.4.3 and Lemma 4.4.5.

*Proof of Lemma 4.4.5.* The main idea is to compute the density function $h$ by the second fundamental form. The second fundamental form of $S$ at $x$ is defined to be the symmetric form $\mathrm{II}_S : T_x S \times T_x S \to \mathbb{R}$,

$$\mathrm{II}_S(v,u) = g(D_v \mathbf{n}(x), u),$$

where $v, u$ are two vectors in the tangent space $T_x S$. By the convexity of $S$, the second fundamental form $\mathrm{II}_S$ is positive definite at every $x$ in $S$.

Fix a point $x$ in $S$. By (4.4.2), starting from the outer unit normal vector $\mathbf{n}(x)$, we can find an orthonormal basis $\{\mathbf{n}(x), (Y_j)_{1 \leq j \leq m}, (Y_k)_{m+1 \leq k \leq m+q}\}$ of $T_x M$ such that

$$R(\mathbf{n}(x), Y_j, \mathbf{n}(x)) = -Y_j, \;\; R(\mathbf{n}(x), Y_k, \mathbf{n}(x)) = -4Y_k, \;\text{ where } m = (n-1)\dim_{\mathbb{R}} \mathbb{F}, q = \dim_{\mathbb{R}} \mathbb{F} - 1.$$

Let $B$ be the matrix representation of $\mathrm{II}_S$ with the basis $\{(Y_j)_{1 \leq j \leq m}, (Y_k)_{m+1 \leq k \leq m+q}\}$ of $T_x S$.

**Lemma 4.4.9.** *With the same assumption as in Lemma 4.4.5, we have*

$$h(x,t) = \det\left(\begin{pmatrix} \cosh t Id_m & 0 \\ 0 & \cosh 2t Id_q \end{pmatrix} + B \begin{pmatrix} \sinh t Id_m & 0 \\ 0 & \frac{1}{2}\sinh 2t Id_q \end{pmatrix}\right). \tag{4.4.10}$$

*Proof.* There exists a local chart on $S$ defined by $(\phi,\, U)$ $\phi : \mathbb{R}^{m+q} \supset U \to S$ such that $\phi(0) = x$ and $\frac{\partial}{\partial u_i}\phi(0) = Y_i$. This particular choice of local chart implies that $\mathrm{d}u$ is the volume element at $x$. Let

$$\tilde{\phi}(u,t) = \exp_{\phi(u)}(t\mathbf{n}(\phi(u))$$

and $\tilde{U} = U \times \mathbb{R}_{\geq 0}$. Then $(\tilde{\phi}, \tilde{U})$ is a local chart of $M$. For every fixed $u$, the curve $t \to \tilde{\phi}(u,t)$ is a geodesic starting from $\phi(u)$ with tangent vector $\mathbf{n}(\phi(u))$.

Fix all $u_w$ to $0$ except $w = i$. Then the map $H : \mathbb{R}^2 \to M$ defined by $(u_i, t) \mapsto \tilde{\phi}(u_i, t)$, is a variation of geodesic. Let $J_i(t)$ be the Jacobi field defined by $J_i(t) = \frac{\partial}{\partial u_i} H(0,t)$. The volume element, in a local chart, can be written as $\sqrt{\det\left(g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_w}\right)\right)}\mathrm{d}x$. In our case, the volume element at $\tilde{\phi}(0,t)$ is $\sqrt{\det(g(J_i(t), J_w(t)))}\mathrm{d}u\mathrm{d}t$. Therefore by definition, we have $h(x,t) = \sqrt{\det(g(J_i(t), J_w(t)))}$.

Let $Y_i(t)$, $\mathbf{n}(t)$ be the images of $Y_i$ and $\mathbf{n}(x)$ under the parallel transport along the geodesic $t \mapsto \tilde{\phi}(0,t)$ with $t \geq 0$. They also form an orthonormal basis. By Lemma 4.4.3, the vectors $J_i(t)$ are orthogonal to $\mathbf{n}(t)$. We decompose $J_i(t)$ with respect to $Y_i(t)$, that is $J_i(t) = \Sigma_w a_{iw}(t) Y_w(t)$, and write $A(t) = (a_{iw}(t))_{1 \leq i,w \leq m+q}$. Then $A(0) = Id_{m+q}$ and the matrix $A'(0)$ equals $B$, the matrix representation of the second fundamental form. This is because by Schwarz's theorem, we have

$$a'_{iw}(0) = \partial_t g(J_i(t), Y_w(t))|_{t=0} = g\left(\frac{D}{\mathrm{d}t} J_i(0), Y_w\right) = g\left(D_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial t}, Y_w\right) = g(D_{Y_i}\mathbf{n}(x), Y_w).$$

Because $J_i$ are Jacobi fields, they satisfy the Jacobi equation:

$$\frac{D^2}{\mathrm{d}t^2} J_i(t) + R(\mathbf{n}(t), J_i(t))\mathbf{n}(t) = 0. \tag{4.4.11}$$

The map $R(\mathbf{n}(t), \cdot)\mathbf{n}(t)$ is a linear map on the orthogonal complement of $\mathbf{n}(t)$ in the tangent space $T_{\tilde{\phi}(0,t)}M$. By [Hel79, IV, Thm1.3], in locally symmetric spaces the curvature tensor $R$ is invariant under parallel transport. Hence in our choice of the orthonormal basis $\{Y_i(t)\}_{1 \leq i \leq m+q}$, the linear map $R(\mathbf{n}(t), \cdot)\mathbf{n}(t)$ can be represented by the matrix $diag\{Id_m, 4Id_q\}$. From (4.4.11), we have

$$A''(t) = A(t)\begin{pmatrix} Id_m & 0 \\ 0 & 4Id_q \end{pmatrix}.$$

The solution is determined by the initial conditions. Due to $A(0) = Id$ and $A'(0) = B$, it is

$$A(t) = \begin{pmatrix} \cosh t Id_m & 0 \\ 0 & \cosh 2t Id_q \end{pmatrix} + B\begin{pmatrix} \sinh t Id_m & 0 \\ 0 & \frac{1}{2}\sinh 2t Id_q \end{pmatrix}.$$

Hence $h(x,t) = \det(g(J_i(t), J_w(t)))^{1/2} = \det(A(t))$, which implies the result. $\square$

For computing the determinant, we need a lemma.

**Lemma 4.4.10.** *Let $D$ be a diagonal matrix with nonnegative entries, let $B$ be a symmetric positive semidefinite matrix, and let* $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ *be a block partition of $B$ such that $B_{11}, B_{22}$ are square matrices. Then for all $\lambda_1, \lambda_2 > 0$ we have*

$$\det\left(D + \begin{pmatrix} \lambda_1 B_{11} & \lambda_2 B_{12} \\ \lambda_1 B_{21} & \lambda_2 B_{22} \end{pmatrix}\right) \geq \det(D).$$

*Proof.* It is elementary that the sum of two symmetric positive semidefinite matrices has determinant no less than the determinant of each one. We only need to transform our matrix to a symmetric matrix. We have

$$\det\left(D + \begin{pmatrix} \lambda_1 B_{11} & \lambda_2 B_{12} \\ \lambda_1 B_{21} & \lambda_2 B_{22} \end{pmatrix}\right) = \det\left((D \begin{pmatrix} \lambda_1^{-1} Id_m & 0 \\ 0 & \lambda_2^{-1} Id_q \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}) \begin{pmatrix} \lambda_1 Id_m & 0 \\ 0 & \lambda_2 Id_q \end{pmatrix}\right).$$

Since $D$ is diagonal, the first matrix in the right-hand side is again symmetric. We have

$$\det\left(D + \begin{pmatrix} \lambda_1 B_{11} & \lambda_2 B_{12} \\ \lambda_1 B_{21} & \lambda_2 B_{22} \end{pmatrix}\right) \geq \det(D).$$

The proof is complete. □

Return to the proof of Lemma 4.4.5. Let $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ be the block partition of $B$ such that $B_{11}, B_{22}$ are square matrices of order $m$, $q$. By (4.4.10)

$$h(x,t) = (\cosh t)^m (\cosh 2t)^q \det\left(Id_{m+q} + \begin{pmatrix} \tanh t B_{11} & \frac{\tanh 2t}{2} B_{12} \\ \tanh t B_{21} & \frac{\tanh 2t}{2} B_{22} \end{pmatrix}\right). \tag{4.4.12}$$

Since $B$ is positive semidefinite, using Lemma 4.4.10, we have $h(t) \geq \cosh^m t \cosh^q 2t \geq e^{2\ell t}/C_1$. The upper bound of $h(t)$ is due to the upper bound on the second fundamental form, that is to say $B$ is bounded.

The derivative in the scalar part of (4.4.12), $\cosh^m t \cosh^q 2t$, gives us

$$(m \tanh t + 2q \tanh 2t) h(t) \geq (m + 2q) \tanh t \, h(t) = 2\ell \tanh t \, h(t).$$

It remains to prove the positivity of the derivative of the determinant part of (4.4.12), which is the sum of derivatives in every column. Since all the terms are similar, we need only to show that the derivative in the first column is non negative. The derivative of $\tanh t$ is $1/\cosh^2 t$, and we multiply the first column with $\tanh t \cosh^2 t$ to recover the original column. The determinant of the derivative of the first column becomes

$$\frac{1}{\tanh t \cosh^2 t} \det\left(\begin{pmatrix} 0 & 0 \\ 0 & Id_{m+q-1} \end{pmatrix} + \begin{pmatrix} \tanh t B_{11} & \frac{\tanh 2t}{2} B_{12} \\ \tanh t B_{21} & \frac{\tanh 2t}{2} B_{22} \end{pmatrix}\right),$$

which is nonnegative by Lemma 4.4.10. □

It remains to prove Lemma 4.4.3.

*Proof of Lemma 4.4.3.* Use the same notation as in the proof of Lemma 4.4.9. Let

$$J(t) = \frac{\partial}{\partial u_i} H(0,t) = \frac{\partial}{\partial u_i} \tilde{\phi}(u,t)|_{u=0}.$$

Then $J(t)$ is a variation of geodesic, and it is a Jacobi field, which is determined by its value and derivative at 0. We have $J(0) = \frac{\partial}{\partial u_i}\tilde{\phi}(u,0)|_{u=0} = \frac{\partial}{\partial u_i}\phi(0)$, which is a tangent vector of $S$ at $x$. Hence $J(0)$ is normal to $\mathbf{n}(x)$. For the derivative, by Schwarz's theorem we have

$$J'(0) = \frac{D}{\mathrm{d}t} J(t)|_{t=0} = D_{\frac{\partial}{\partial t}} \frac{\partial}{\partial u_i} = D_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial t}. \tag{4.4.13}$$

Therefore

$$g(J'(0), \mathbf{n}(x)) = g\left(D_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \frac{1}{2}\frac{\partial}{\partial u_i}\left\|\frac{\partial}{\partial t}\right\|^2 = 0.$$

Hence $J(t)$ is normal to the tangent vector of the geodesic. This is true for all $i$, and the result follows. □

## 4.5 Geometrically finite manifolds

We return to the study of the spectrum of the whole manifold. We give a topological decomposition of geometrically finite manifolds. In order to describe it, we use standard cusp regions. For a subset $S$ of $M_c$, we use $\mathring{S}$ to denote its topological interior, that is $\mathring{S} = S - \bar{S^c} \cap S$. The reader who is only interested in convex cocompact manifolds (geometrically finite manifolds without cusps) can skip Section 4.5.1 and go directly to Section 4.5.2. Our goal in this section is to obtain the Lax-Phillips inequality, then Theorem 4.1.1 follows by Proposition 4.3.1.

### 4.5.1 Standard cusp regions

The manifold with boundary $M_c$ may have some cusps. To analyse the structure of cusps, we introduce the concept of a topological end.

**Definition 4.5.1.** *Let $T$ be a differential manifold. An end $e$ is a function which assigns every compact subset $K$ of $T$ a non empty connected component of $T \backslash K$, such that if $K \subset K'$, then $e(K) \supset e(K')$.*

*Let $K_1 \subset K_2 \subset \ldots$ be an ascending sequence of compact subsets whose interiors cover $T$. We call $e(K_i)$ a system of neighbourhoods for the end $e$.*

*A neighbourhood for the end $e$ is an open subset $U$ such that $U \supset e(K_n)$ for some $n$.*

**Proposition 4.5.2.** *[Bow95, Proposition 4.4] If $\Pi$ is a discrete parabolic subgroup of $G$, then $M_c(\Pi)$ has precisely one topological end. Moreover, we can find a system of neighbourhoods for the end consisting of geodesically convex submanifolds of $M_c(\Pi)$.*

Following [Bow95], we define a *standard cusp region* (with boundary). This definition is not explicitly written, but it is implicitly defined in [Bow95, Sec.5.1]. For more details on the real hyperbolic case, we refer to [Bow93, Sec.3.1]

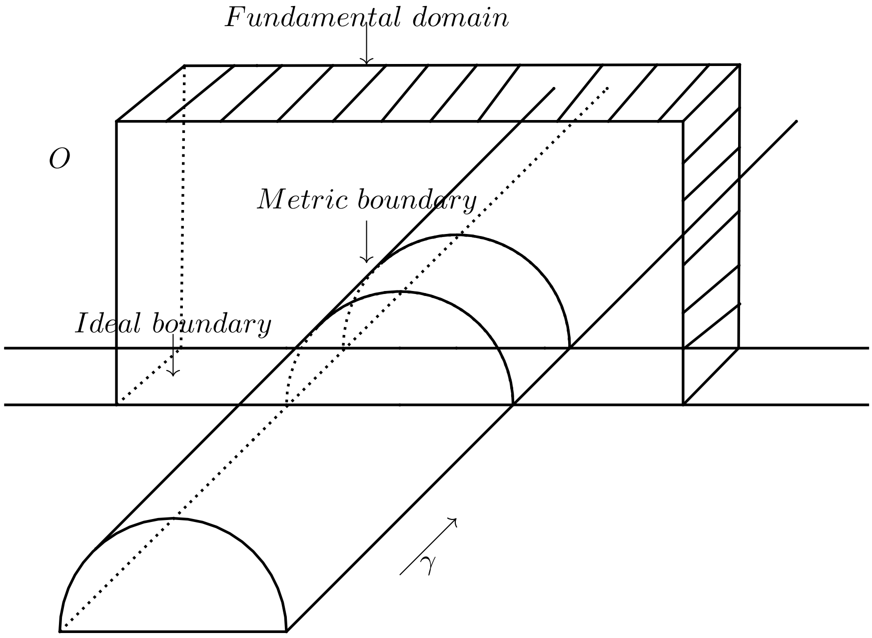


Figure 4.5.1: A standard cusp region

**Definition 4.5.3.** *Let $\Gamma$ be a discrete subgroup of $G$. An end of of $M_c(\Gamma)$ is called a cusp. A standard cusp region of $M_c(\Gamma)$ is a closed subset $O$ which is isometric to a closed subset $E$ of $M_c(\Pi)$, where $\Pi$ is a discrete parabolic subgroup of $\Gamma$ and the interior of $E$ is supposed to be a geodesically convex submanifold and a neighbourhood for the end of $M_c(\Pi)$.*

We have a corollary of Proposition 4.5.2.

**Proposition 4.5.4.** *Let $\Gamma$ be a discrete subgroup of $G$, and let $O$ be a standard cusp region of $M_c(\Gamma)$. There exists a smaller standard cusp region $O'$ such that $O'$ is contained in the interior of $O$.*

*Proof.* By Definition 4.5.3, the cusp region $O$ is isometric to $E$ and the complement of the interior of $E$ is a compact subset $K$ of $M_c(\Pi)$. Denote by $f$ the isometric map from $O$ to $E$. By Proposition 4.5.2, $M_c(\Pi)$ has only one end and we have a larger compact subset $K_1$, whose interior contains $K$ and whose complement is geodesically convex. Let $E_1 = \overline{K_1^c}$ and let $O'$ be the corresponding subset of $O$, that is $O' = f^{-1}(E_1)$. □

For a symmetric space of real rank one, we use the half space model introduced at the end of Section 4.4.1, that is a diffeomorphism from $N \times \mathbb{R}_{>0}$ to $X$. Recall that $\Pi$

is a parabolic subgroup of $G$ which fixes $\infty$. It is a subgroup of $M \ltimes N$, which acts isometrically on $N$ equipped with the metric $g_t$. We have

$$M_c(\Pi) \simeq (\Pi\backslash N) \times \mathbb{R}_{\geq 0} \text{ as topological spaces}$$

Then our standard cusp region is isometric to the complement of an open relatively compact subset in $M_c(\Pi)$, with additional convex condition on the interior. (see figure 4.5.1.)

**Definition 4.5.5** (Maximal rank)**.** *Let $\Pi$ be a discrete parabolic subgroup of $G$. We call $\Pi$ a subgroup of maximal rank if the quotient $\Pi\backslash N$ is compact.*

*A cusp region is said to be maximal rank, if the corresponding discrete parabolic subgroup is of maximal rank*

**Remark 4.5.6.** *We explain our definition as follows. The rank of a nilpotent group is defined to the sum of the rank of its central series.*

*For real hyperbolic case, $\Pi$ is a discrete subgroup of $Isom(\mathbb{R}^{n-1}) = O(n-1) \ltimes \mathbb{R}^{n-1}$. By Bieberbach's theorem (see for example [Bow93, Theorem 2.2.5]), the group $\Pi$ is virtually abelian. The rank of $\Pi$ is defined to be the rank of its maximal normal abelian subgroup. Hence when $\Pi$ attends maximal rank, the quotient space is compact.*

*For rank one symmetric spaces, by Margulis' Lemma, the discrete parabolic subgroup $\Pi$ is virtually nilpotent. As in [CI99, Lemma 3.4], for a virtually nilpotent discrete subgroup $\Pi < M \ltimes N$, we can find a subgroup $\Pi_1 < \Pi$ of finite index which is nilpotent, and there exists a subgroup $\Pi_2 < N$ which is isomorphic to $\Pi_1$ and satisfies $\Pi_1 \cdot x = \Pi_2 \cdot x$ for some $x$ in $N$. This means that the $\Pi_1$, $\Pi_2$-orbits of $x$ are the same in $N$. Let $N_2$ be the Zariski closure of $\Pi_2$. Then the rank of $\Pi$ is the same as the rank of $N_2$. When $\Pi$ attends maximal rank, $N_2$ coincides with $N$. Then $\Pi_2$ is a cocompact subgroup of $N$ because $N$ is nilpotent. Due to $\Pi_2 \cdot x = \Pi_1 \cdot x$, every point in $N$ has a bounded distance to the orbit $\Pi_1 \cdot x$. Hence $\Pi_1\backslash N$ is compact, so is $\Pi\backslash N$.*

**Proposition 4.5.7** (Maximal rank)**.** *Let $\Gamma$ be a discrete subgroup of $G$. Let $O$ be a standard cusp region of $M_c(\Gamma)$ with maximal rank. Then we can find a smaller cusp region $O_1$, which is isomorphic to the quotient of a horoball by a discrete parabolic subgroup of $\Gamma$.*

*Proof.* By Definition 4.5.3, there is a discrete subgroup $\Pi$ of $\Gamma$, such that $O$ is isometric to a closed subset $E$ of $M_c(\Pi)$, whose interior is a neighbourhood for the end of $M_c(\Pi)$. By Definition 4.5.2 and Proposition 4.5.2, the complement of $E$ in $M_c(\Pi)$ is relatively compact. Under the half space model, we can suppose that $E^c \subset (\Pi\backslash N) \times [0, 1]$. Let $B$ be the horoball, which is homeomorphic to $N \times \mathbb{R}_{\geq 1}$. Then the quotient $\Pi\backslash B$ is a subset of $E$. Due to maximal rank, the quotient $\Pi\backslash N$ is compact. Hence

$$M_c(\Pi) - \Pi\backslash \mathring{B} \simeq (\Pi\backslash N) \times [0, 1]$$

is compact. The quotient $\Pi\backslash B$ is geodesically convex. Let $O_1$ be the preimage of $\Pi\backslash B$ in $O$ under the isometric map from $O$ to $E$. The proof is complete. □

In later proof, for cusps of maximal rank, we will always take the quotient of horoball as a standard cusp region.

### 4.5.2 A good partition of unity

**Definition 4.5.8** (Geometrical finiteness)**.** *A discrete subgroup $\Gamma$ in $G$ is called geometrically finite, if $M_c(\Gamma)$ is the union of a compact set and a finite number of standard cusp regions $O_i$ for $1 \le i \le k$, that is to say $M_c(\Gamma) - \bigcup_{1\le i\le k} \mathring{O}_i$ is compact.*

This definition is not explicitly written in [Bow95], but is given in the discussion after [Bow95, Def. F1]. (See also [Bow93, Def.(GF1)] for the real hyperbolic case. In [Bow93], Bowditch explained the equivalence of the definition in the introduction and Definition 4.5.8 for the real hyperbolic case.) By [Bow95, Lemma 6.2], if $\Gamma$ is geometrically finite, then there exist standard cusp regions $O_i$ for $1 \le i \le k$, such that $O_i$ are pairwise disjoint. For the purpose of the exposition, we can limit our consideration to the case that there is at most one cusp; the results hold and the methods of proof work for general cases.

For a real number $r > 0$, we define the $r$-neighbourhood of a set $Q$ in $X$ by $N_r(Q) = \{x \in X | d(x, Q) \le r\}$. Let $W_r = N_r(\mathrm{hull}(\Lambda(\Gamma)) \cap X)$ be the $r$-neighbourhood of the convex hull of the limit set $\Lambda(\Gamma)$. Let $C(M)$ be the convex core defined by $C(M) = \Gamma\backslash(\mathrm{hull}(\Lambda(\Gamma)) \cap X)$. Let $C_r(M) = \Gamma\backslash W_r$ be the $r$-neighbourhood of the convex core. One problem here is that the boundary of $C_r(M)$ may not be $C^\infty$-smooth, but is only $C^{1,1}$-smooth (see for instance [Wal76] or [Fed59]). To overcome this difficulty, we use a result of [PP12, Prop.6]. (In the statement of Proposition 6 in [PP12], they do not have a $\Gamma$-invariant condition. But if we start from a $\Gamma$-invariant set, their method automatically gives us a $\Gamma$-invariant set.) We can find a closed convex subset $W'$ with $C^\infty$ smooth boundary such that $W_1 \subset W' \subset W_{3/2}$ and $W'$ is also $\Gamma$-invariant. Let $D = \Gamma\backslash W'$. Then $D \subset C_2(\mathring{M})$.

Let $O$ be a standard cusp region of the unique cusp in $M_c$. We have a smaller standard cusp region $O'$ such that $O' \subset \mathring{O}$.

We can cover the geometrically finite manifold with three open sets

$$M_c = \mathring{O} \cup (D^c - O') \cup (C_2(\mathring{M}) - O'), \tag{4.5.1}$$

where $D^c$ is the complement in $M_c$. For the simplicity of the notation, we write

$$M_1 = \mathring{O}, M_2 = D^c - O' \text{ and } M_3 = C_2(\mathring{M}) - O'.$$

Since $M_c$ inherits the differential structure from $X \cup \Omega$, the covering is about a differential manifold with boundary. We can find a smooth partition of unity subordinate to this cover, written as $\{\bar{\varphi}_1,\ \bar{\varphi}_2,\ \bar{\varphi}_3\}$, which is smooth on the boundary. Here $M_1, M_2$ may intersect the ideal boundary $M_I = M_c - M$.

Our covering has the advantage that $M_3$ is compact, and $M_1, M_2$ have quasi-warped product Riemannian structure.

**Proposition 4.5.9.** *The set $M_3$ is relatively compact in $M$.*

*Proof.* By Definition 4.5.8, we have that $M_c - \mathring{O}'$ is compact in $M_c$. Therefore $C_2(M) - \mathring{O}'$ is compact in $M_c$. It is also contained in $M$, hence it is a compact subset of $M$. □

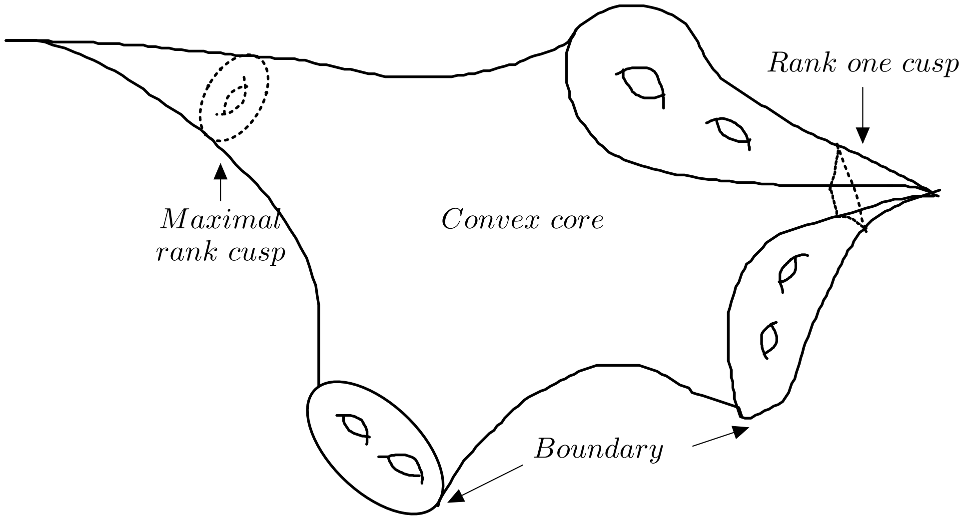


Figure 4.5.2: Convex core

### 4.5.3 The energy form

We keep the assumptions on $M$ of Sec.4.5.2, that $M$ is a geometrically finite locally symmetric manifold of real rank one with at most one cusp. Recall that $E$ is the energy form defined in (4.4.3).

**Lemma 4.5.10.** *Let $f \in C_c^\infty(M)$, and let $\varphi$ be a smooth function. We have*

$$E(\varphi f) = \int_M \varphi^2(\|\nabla f\|^2 - \ell^2 f^2)\mathrm{d}vol - \int_M f^2 \varphi \Delta\varphi \mathrm{d}vol.$$

This is a direct consequence of Lemma 4.2.1. The following proposition says that in order to calculate the energy form on the entire manifold, it is sufficient to calculate the energy forms on $M_1, M_2, M_3$ and an error term. We want to separate the energy formula, but we need a partition of unity such that the square root of the partition function is also smooth. The exact choice of the partition function is not important and we take $\theta = \frac{\pi}{2}\bar{\varphi}_1, \vartheta = \frac{\pi}{2}\bar{\varphi}_2$ and $\varphi_1 = \sin\theta,\ \varphi_2 = \sin\vartheta$.

**Proposition 4.5.11.** *For $f \in C_c^\infty(M)$, we have*

$$\begin{aligned} E(f) = &\int_{M_3} \big((1-\varphi_1^2-\varphi_2^2)(\|\nabla f\|^2 - \ell^2 f^2) + (\varphi_1\Delta\varphi_1 + \varphi_2\Delta\varphi_2)f^2\big)\,\mathrm{d}vol \\ &+ E(\varphi_1 f) + E(\varphi_2 f) - \int_{M_3^c} \|\nabla\theta\|^2 f^2 \mathrm{d}vol. \end{aligned} \tag{4.5.2}$$

*Proof.* By Lemma 4.5.10, we have

$$E(\varphi_2 f) + E(\varphi_1 f) = \int (\varphi_2^2 + \varphi_1^2)(\|\nabla f\|^2 - \ell^2 f^2) - \int (\varphi_2 \Delta \varphi_2 + \varphi_1 \Delta \varphi_1) f^2.$$

Since $\theta + \vartheta = \pi/2$ outside of $M_3$, we have for $x \in M_3^c$

$$\begin{aligned}\varphi_2 \Delta \varphi_2 + \varphi_1 \Delta \varphi_1 &= \sin\theta \Delta \sin\theta + \sin\vartheta \Delta \sin\vartheta \\ &= \frac{1}{2}(\Delta(\sin^2\theta + \cos^2\theta) - \|\nabla \sin\theta\|^2 - \|\nabla \cos\theta\|^2) = -\|\nabla\theta\|^2.\end{aligned}$$

The proof is complete. □

We want to prove that $E(\varphi_1 f)$ and $E(\varphi_2 f)$ are positive after adding an integral over a compact subset, and to give an estimate of the error term $\int_{M_3^c} \|\nabla\theta\|^2 f^2$.

### 4.5.4 Positivity of the energy form

In this part, we take into account the topology of the whole manifold, together with the results in standard cusp regions and the complement of convex subsets, to prove the positivity.

**Proposition 4.5.12** (the Lax-Phillips inequality)**.** *Let $M$ be a geometrically finite locally symmetric manifold of real rank one. There exist a relatively compact open set $U$ in $M$ with smooth boundary and a constant $C_U > 0$ such that the following holds. For any compact set $V$ in $M$ there exists $\epsilon_V > 0$ such that for all complex valued function $f \in C_c^\infty(M)$ we have*

$$E(f) + C_U \int_U |f|^2 \mathrm{d}vol \geq \frac{1}{4}\int_U \|\nabla f\|^2 \mathrm{d}vol + \epsilon_V \int_V |f|^2 \mathrm{d}vol. \tag{4.5.3}$$

*Proof of Theorem 4.1.1.* Our main theorem of this manuscript follows from Proposition 4.5.12 and 4.3.1 with $c(x) = \ell^2$. □

It remains to prove Proposition 4.5.12.

**Remark 4.5.13.** *For the simplicity of the exposition, here we will only prove the case that $M$ has only maximal rank cusps ($M$ may have no cusp). For the general case, please see the appendix. The idea of the proof is the same, but the appearance of non-maximal cusps will add some technical difficulties.*

*It is sufficient to prove this inequality for real valued functions. The complex version is immediate by separating $f = f_1 + i f_2$ with $f_1, f_2$ real valued and using the real version for each component $f_1, f_2$.*

Let $\tilde\rho$ be the nearest point retraction from $X$ to $W'$. We can extend this map continuously to $X \cup \Omega$, such that if $\xi$ is in $\Omega$, then $\tilde\rho(\xi)$ is the first point of contact of

$W'$ with an expanding family of horoballs based at $\xi$. (See [Bow95, Lem.2.2.4] for more details.) This retraction descends to a map

$$\rho : M_c \to D,$$

which is also continuous by the openness of the covering map.

In the cusp region, recall that we will use the half space model $(\Pi\backslash N) \times \mathbb{R}_{\geq 0}$ for $M_c(\Pi)$. We write $M_c(\Pi) = (\Pi\backslash N) \times \mathbb{R}_{\geq 0}$, which means the equality holds under the coordinate map. Others equalities are similar. By definition, $O'$ is isometric to a subset of $M_c(\Pi)$, where $\Pi$ is a discrete parabolic subgroup. Hence we can identify the two sets. By Definition 4.5.1 and Proposition 4.5.2, the complement of the cusp region $O'$ in $M_c(\Pi)$ is relatively compact. Hence we can suppose that $(O')^c$ is contained in $(\Pi\backslash N) \times [0, 1]$. (Otherwise, we can change the coordinate.) Let $H$ be the quotient of a horoball based at $\infty$, defined by $H = (\Pi\backslash N) \times [1, \infty) \subset O'$. We define $proj_H$ to be the nearest point retraction from $M_c(\Pi)$ to $H$. For a point $(\eta, y) \in M_c(\Pi)$ with $y < 1$, the map is given by $proj_H(\eta, y) = (\eta, 1)$.

**Lemma 4.5.14.** *Let $K = \rho(M_2)$ and let $L = proj_H(O - O')$. Then $K, L$ are relatively compact in $M$.*

*When $O$ is of maximal rank, we can take $O = (\Pi\backslash N) \times \mathbb{R}_{\geq c_1}$ and $O' = (\Pi\backslash N) \times \mathbb{R}_{\geq c_2}$ for some $0 < c_1 < c_2 \leq 1$ and we have $L = (\Pi\backslash N) \times \{1\}$.*

*Proof.* Since $M_c$ has only one cusp, by Definition 4.5.8, the sets $M_2 = D^c - O'$ and $O - O'$ are relatively compact in $M_c$. The continuity of $\rho$ and $proj_H$ implies that $\rho(M_2)$ and $proj_H(O - O')$ are relatively compact in $D$ and $H$. The latter two sets are in $M$, hence we have the result.

The last assertion is due to Proposition 4.5.7. □

For $x$ in $M$, let

$$t_1(x) = d(x, H), t_2(x) = d(x, D). \tag{4.5.4}$$

Later we will see that $t_1, t_2$ are the geometric descriptions of the coordinate $t$ in cusps and the complement of convex set. Recall that $M_1, M_2$ are subsets of $M_c$, which may intersect the ideal boundary $M_I$.

**Lemma 4.5.15.** *In the standard cusp region, there exist a compact set $U_1$ in $M_1 \cap M$ and a constant $C_1 > 0$ such that the following holds. For any compact set $V$ in $M_1 \cap M$ there exists $\epsilon_V > 0$ such that for all $f \in C_c^\infty(M_1 \cap M)$ we have*

$$E(f) + C_1 \int_{U_1} f^2 \mathrm{d}vol \geq \frac{1}{2} \int \|\nabla(e^{\ell t_1} f)\|^2 e^{-2\ell t_1} \mathrm{d}vol + \epsilon_V \int_V f^2 \mathrm{d}vol. \tag{4.5.5}$$

*Proof.* Recall that we identify $M_1 = O'$ as a subset of $(\Pi\backslash N) \times \mathbb{R}_{\geq 0}$. Due to the choice of $H$, we have $O - O' \subset H^c$, and the coordinate $y$ of a point $x$ under the horospherical model satisfies $y = e^{-t_1(x)}$ for $x \in H^c$. We have that $t_1(x)$ equals the coordinate $t$ in horospherical model when $x \in H^c$. Using Lemma 4.4.2 with compact set $L$ and constant

$C = 2$, and adding (4.4.4) and (4.4.5), we obtain inequality (4.5.5) with the last term replaced by $\int_{L\times\mathbb{R}_{\geq 0}} e^{-t_1} f^2 \mathrm{d}vol$. Since $O$ is of maximal rank, by Lemma 4.5.14 we have $L \times \mathbb{R}_{\geq 0} \simeq M_c(\Pi) \supset M_1$. The proof is complete. □

**Lemma 4.5.16.** *In the complement of the convex core, there exist a compact set $U_2$ in $M_2 \cap M$ and a constant $C_2 > 0$ such that the following holds. For any compact set $V$ in $M_2 \cap M$ there exists $\epsilon_V > 0$ such that for all $f \in C_c^\infty(M_2 \cap M)$ we have*

$$E(f) + C_2 \int_{U_2} f^2 \mathrm{d}vol \geq \frac{1}{2} \int \|\nabla(e^{\ell t_2} f)\|^2 e^{-2\ell t_2} \mathrm{d}vol + \epsilon_V \int_V f^2 \mathrm{d}vol. \tag{4.5.6}$$

*Proof.* We first verify that $D$ satisfies the conditions in Lemma 4.4.5.

Recall that $D = \Gamma\backslash W'$, where $W'$ is a convex subset of $X$ with smooth boundary. By convexity, the normal exponential map, given by $\exp_x(t\mathbf{n}(x))$, is a diffeomorphism from $\partial W' \times \mathbb{R}_{\geq 0}$ onto $X - \mathring{W}'$, and satisfies $t = d(\exp_x(t\mathbf{n}(x)), W')$. With the help the nearest point retraction $\tilde{\rho}$, the inverse of the normal exponential map from $X - \mathring{W}'$ to $\partial W' \times \mathbb{R}_{\geq 0}$ is given by

$$x \mapsto (\tilde{\rho}(x), d(x, W')).$$

Descend to the quotient space. Let $S$ be the boundary of $D$. Then the normal exponential map $\exp^\perp : (x,t) \mapsto \exp_x(t\mathbf{n}(x))$ from $S \times \mathbb{R}_{\geq 0}$ to $M - \mathring{D}$ is again a diffeomorphism, and

$$t = d(\exp^\perp(x,t), S) = t_2(\exp^\perp(x,t)).$$

The upper bound of the second fundamental form is due to [PP12, Thm.1, Prop.6], that is the obtained convex set $W'$ has bounded second fundamental form on its boundary.

Applying Lemma 4.4.8 with the set $K = \rho(M_2)$, defined in Lemma 4.5.14, and the constant $C = 2(4\ell^2 + 1)$, there exists a bounded interval $I \in \mathbb{R}_{\geq 0}$ such that (4.4.9) holds for $U_2 = K \times I$. Adding (4.4.8) and (4.4.9) implies the result. □

*Proof of Proposition 4.5.12.* In view of (4.5.2), the main problem is the term $\int_{M_3^c} \|\nabla\theta\|^2 f^2 \mathrm{d}vol$. The support of $\|\nabla\theta\|$ is contained in $O - O'$, which may not be compact. But with the hypothesis that the manifold has only maximal rank cusps, the region $O - O'$ is already relatively compact in $M$ due to Lemma 4.5.14. Because $O$ does not intersect the ideal boundary and the complement of $O'$ is relatively compact. Hence $\|\nabla\theta\|^2$ is a bounded function supported on $O - O'$.

The compact set $V$ is replaced by $V \cap M_1$ and $V \cap M_2$ in Lemma 4.5.15 and Lemma 4.5.16. Using Lemma 4.5.15 and 4.5.16, we obtain $U_1$, $U_2$. Since $O - O', M_3$, $U_1$, $U_2$ are relatively compact in $M$, we can find a relatively compact open set $U \subset M$ with smooth boundary, which contains the four sets. By (4.5.2), (4.5.5) and (4.5.6), there exists a constant $C_4$ large enough such that

$$\begin{aligned} E(f) + C_4 \int_U f^2 \mathrm{d}vol \geq \frac{1}{2} \int \left( \|\nabla(\varphi_1 f e^{\ell t_1})\|^2 e^{-2\ell t_1} + \|\nabla(\varphi_2 f e^{\ell t_2})\|^2 e^{-2\ell t_2} \right) \mathrm{d}vol \\ + \int_{M_3} (1 - \varphi_1^2 - \varphi_2^2) \|\nabla f\|^2 \mathrm{d}vol + \epsilon_V \int_V f^2 \mathrm{d}vol. \end{aligned}$$

We restrict our computation on $U \cap \mathrm{supp}\varphi_2 \subset M_2$

$$\|\nabla(\varphi_2 f e^{\ell t_2})\|^2 \geq \frac{1}{2}(\varphi_2 e^{\ell t_2})^2 \|\nabla f\|^2 - f^2 \|\nabla(\varphi_2 e^{\ell t_2})\|^2 \geq e^{2\ell t_2}\left(\frac{1}{2}\varphi_2^2\|\nabla f\|^2 - C_{\varphi_2} f^2\right),$$

where we use the estimate $\sup_{x\in U}\|\nabla\varphi_2(x)\| < \infty$, thanks to the relative compactness of $U$. In the standard cusp region we have the same estimate. Therefore taking $C_U$ large enough, we have (4.5.3). $\square$

## 4.6 Appendix

As stated in Section 4.5.4, we will give a proof of Proposition 4.5.12 without the assumption that $M$ has only maximal rank cusps.

### 4.6.1 Compactification and estimate at infinite

Let $g$ be a Riemannian metric on a manifold $M$. We define the musical isomorphism as follows (see [GHL04] for more details). For a vector $X$ in $T_xM$, let $X^b$ be the unique 1-form such that $X^b(v) = g(X, v)$ for every $v \in T_xM$. This isomorphism gives a dual tensor field (the symmetric covariant 2-tensor fields) $g^*$ of $g$, and $(\nabla f)^b = \mathrm{d}f$.

We will consider the compactification of a Riemannian manifold $M = L \times (0, 1]$ with metric given by $g = g_1(x, y)/y^2$, where $g_1(x, y)$ is a positive definite symmetric bilinear form on $T_{(x,y)}M$. Now we add $y = 0$ to obtain a differential manifold with boundary, called $\bar{M}$. By definition, we have (using local coordinate vectors $(\frac{\partial}{\partial x_i})_{1\leq i\leq n}$ and $(\mathrm{d}x_i)_{1\leq i\leq n}$, $g^*$ can be written as the inverse matrix of $g$)

$$g^* = (g_1/y^2)^* = y^2 g_1^*. \tag{4.6.1}$$

Suppose that $g_1^*$ can be smoothly extended to $y = 0$, but $g_1^*(x, 0)$ may degenerate to a positive semidefinite form, that means $g_1(x, y)$ may blow up when $y \to 0$.

**Lemma 4.6.1.** *Assume that the Riemannian metric on $L \times (0, 1]$ satisfies the above condition. Let $f$ be a smooth function on $L \times [0, 1]$. For every compact subset $U$ of $L$, there exists a constant $C > 0$ such that for any $(x, y) \in U \times (0, 1]$ we have*

$$\|\nabla f(x, y)\|^2 \leq Cy^2.$$

*Proof.* By definition and (4.6.1), we have

$$\|\nabla f\|^2 = g^*(\mathrm{d}f, \mathrm{d}f) = y^2 g_1^*(\mathrm{d}f, \mathrm{d}f).$$

By the smoothness of $f$ and $g_1^*$ on the boundary and the compactness, there exists a constant $C > 0$ such that $\|\nabla f\|^2 \leq Cy^2$. $\square$

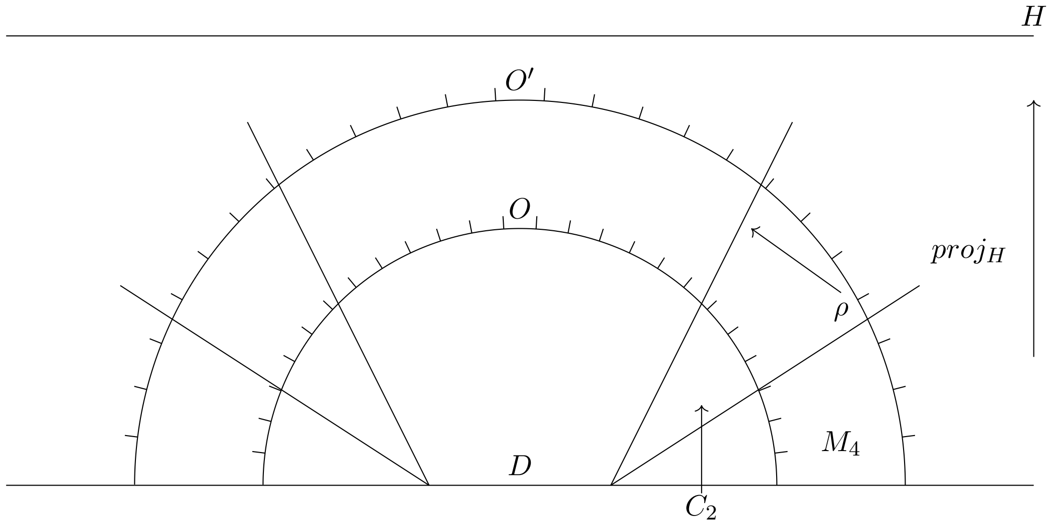


Figure 4.6.1: Projection in standard cusp region

### 4.6.2 Manifolds with non maximal rank cusp

We will give a proof of Proposition 4.5.12 with non-maximal rank cusps. In Section 4.5.4, the assumption that $M$ has only maximal rank cusps is only used in Lemma 4.5.15 and in the proof of Proposition 4.5.12. The proof works similarly, and we will strengthen Lemma 4.5.15, 4.5.16 and give a control of $\int_{M_3^c}\|\nabla\theta\|^2 f^2 \mathrm{d}vol$.

If $O$ is a standard cusp region of non-maximal rank, then the region $(O-O')\cap M$ is not relatively compact in $M$. Because the cusp region $O$ intersects the ideal boundary $M_I = M_c - M$. To overcome this difficulty, we use the compactness in $M_c$. The fact that the partition of unity is smooth not only on $M$ but also on $M_c$ is the key point to apply Lemma 4.6.1.

Recall that $M$ is a geometrically finite rank one locally symmetric manifold. We have a covering $M_c = M_1 \cup M_2 \cup M_3$, where $M_1$ is the cusp region and $M_2$ is a subset of the complement of the convex core. Recall that $D = \Gamma\backslash W'$ is a neighbourhood of the convex core and $H$ is a subset of $M_1$, which is the quotient of a horoball. For $x$ in $M$, in (4.5.4) we have defined

$$t_1(x) = d(x,H), t_2(x) = d(x,D).$$

Recall that $\varphi_1, \varphi_2$ are two smooth functions supported on $M_1, M_2$, respectively, and $\varphi_1^2+\varphi_2^2 = 1$ outside of $M_3$. Let $M_4 = M_1 \cap M_2 - M_3$. The following lemma is the key additional ingredient for the general case.

**Lemma 4.6.2.** *For every smooth function $f$ on $M_4$, there exists $C_f > 0$, such that for all $x \in M_4 \cap M$*

$$\|\nabla f(x)\|^2 \leq C_f \left( e^{-t_1(x)}\varphi_1(x)^2 + e^{-t_2(x)}\varphi_2(x)^2 \right). \tag{4.6.2}$$

*Proof.* By (4.4.1), letting $y = e^{-t}$, the metric in the half space model is given by $g = g_y \oplus \frac{\mathrm{d}y^2}{y^2}$ where $g_y = \begin{pmatrix} \frac{1}{y^2} Id_m & 0 \\ 0 & \frac{1}{y^4} Id_q \end{pmatrix}$ under some distributions $W^1$ and $W^2$ on $N$, which satisfies the condition in Lemma 4.6.1. Due to $M_4 \subset M_1 \subset M_c(\Pi) = (\Pi \backslash N) \times \mathbb{R}_{\geq 0}$, where $\Pi$ is a discrete parabolic subgroup of $\Gamma$ which preserves the metric $g_y$ on $N$ for every $y$ in $\mathbb{R}_{\geq 0}$, the quotient metric on $M_4$ also satisfies the condition in Lemma 4.6.1. Moreover, by $M_4 \subset M_1 \cap M_2 \subset O - O'$, we have that $M_4$ is contained in $L \times [0,1]$, where $L = proj_H(O - O')$ is relatively compact by Lemma 4.5.14. Therefore, using Lemma 4.6.1 with $U = L$ there exists $C'_f > 0$ such that

$$\|\nabla f(\cdot, y)\|^2 \leq C'_f y^2. \tag{4.6.3}$$

Due to $M_4 = M_1 \cap M_2 - M_3$ and the definition of $\varphi_1, \varphi_2$, we have $1 = \varphi_1^2(x) + \varphi_2^2(x)$ for $x$ in $M_4$, and the right-hand side of the above inequality equals $C'_f y^2(|\varphi_1|^2 + |\varphi_2|^2)$.

We restrict our attention to $x$ in $M_4 \cap M$. By the argument of the proof of Lemma 4.5.15, we have

$$y = e^{-t_1(x)}. \tag{4.6.4}$$

By definition of $K, L$, the nearest points of $x$ in $H$ and $D$ are contained in $K$ and $L$. Therefore by compactness of $K, L$ (Lemma 4.5.14), we have

$$|t_1(x) - t_2(x)| = |d(x, H) - d(x, D)| \leq \sup\{d(x_1, x_2) | x_1 \in K, x_2 \in L\} = C_{K,L}. \tag{4.6.5}$$

Hence $e^{-t_1(x)} \leq e^{C_{K,L}} e^{-t_2(x)}$ for $x$ in $M_4 \cap M$. Therefore on $M_4 \cap M$, by (4.6.3),(4.6.4) and (4.6.5)

$$\|\nabla f(x)\|^2 \leq C'_f e^{-2t_1(x)}(\varphi_1^2(x) + \varphi_2^2(x)) \leq C'_f (e^{-2t_1(x)} \varphi_1^2(x) + e^{2C_{K,L}} e^{-2t_2(x)} \varphi_2^2(x)).$$

The proof is complete due $t_1, t_2 \geq 0$. □

We state our strengthened version of Lemma 4.5.15 and 4.5.16.

**Lemma* 4.5.15.** *In the standard cusp region, for every $C > 0$ there exist a compact set $U_1$ in $M_1 \cap M$ and a constant $C_1 > 0$ such that the following holds. For any compact set $V$ in $M_1 \cap M$ there exists $\epsilon_V > 0$ such that for all $f \in C_c^\infty(M_1 \cap M)$ we have*

$$E(f) + C_1 \int_{U_1} f^2 \mathrm{d}vol \geq \frac{1}{2} \int \|\nabla(e^{\ell t_1} f)\|^2 e^{-2\ell t_1} \mathrm{d}vol + \epsilon_V \int_V f^2 \mathrm{d}vol + C \int_{M_4} e^{-t_1} f^2 \mathrm{d}vol. \tag{4.6.6}$$

*Proof.* By Lemma 4.5.14, we have $M_4 \subset L \times \mathbb{R}_{\geq 0}$. By the same argument as in the proof of Lemma 4.5.15, using Proposition 4.4.2, we have the above inequality with $\int_V f^2$ replaced by $\int_{L \times \mathbb{R}_{\geq 0}} e^{-t_1} f^2$. The desired term is due to Poincaré's inequality.

**Lemma 4.6.3.** *For relatively compact sets $V_1, V_2$ in a Riemannian manifold $M$, where $V_1$ is connected open and $V_2$ is a subset of $V_1$ with nonempty interior, there exists a positive constant $\epsilon$, such that for all $g \in C_c^\infty(M)$ we have*

$$\int_{V_1} \|\nabla g\|^2 \mathrm{d}vol + \int_{V_2} g^2 \mathrm{d}vol \geq \epsilon \int_{V_1} g^2 \mathrm{d}vol. \tag{4.6.7}$$

Applying (4.6.7) with $g = e^{\ell t_1} f$, $V_2 = L \times [1,2]$ and $V_1$ a connected open set containing $V_2 \cup V$ implies that

$$\int_{V_1} \|\nabla(e^{lt_1} f)\|^2 e^{-2lt_1} \mathrm{d}vol + \int_{V_2} e^{-t_1} f^2 \mathrm{d}vol \geq \epsilon_1 \left( \int_{V_1} \|\nabla g\|^2 \mathrm{d}vol + \int_{V_2} g^2 \mathrm{d}vol \right)$$
$$\geq \epsilon_2 \int_{V_1} g^2 \mathrm{d}vol = \epsilon_2 \int_{V_1} f^2 e^{2lt_1} \mathrm{d}vol \geq \epsilon_3 \int_V f^2 \mathrm{d}vol.$$

The proof is complete. □

**Lemma* 4.5.16.** *In the complement of the convex core, for every $C > 0$ there exist a compact set $U_2$ in $M_2 \cap M$ and a constant $C_2 > 0$ such that the following holds. For any compact set $V$ in $M_2 \cap M$ there exists $\epsilon_V > 0$ such that for all $f \in C_c^\infty(M_2 \cap M)$ we have*

$$E(f) + C_2 \int_{U_2} f^2 \mathrm{d}vol \geq \frac{1}{2} \int \|\nabla(e^{\ell t_2} f)\|^2 e^{-2\ell t_2} \mathrm{d}vol + \epsilon_V \int_V f^2 \mathrm{d}vol + C \int_{M_4} e^{-t_2} f^2 \mathrm{d}vol. \tag{4.6.8}$$

*Proof.* By Lemma 4.5.14, we have $M_4 \subset M_2 \subset \exp^\perp(K \times \mathbb{R}_{\geq 0})$. Then the proof is exactly the same as the proof of Lemma 4.5.16, using Lemma 4.4.8 with the constant $2(4\ell^2 + 1) + C$. □

*Proof of Proposition 4.5.12.* Applying Lemma 4.6.2 with $\theta = \frac{\pi}{2}\bar{\varphi}_3$, we get a constant $C_\theta$ such that (4.6.2) holds for $\theta$. Using Lemma* 4.5.15 and 4.5.16 with $C = C_\theta$, we obtain $U_1$, $U_2$. Then follow the same argument as in the proof of the special case of Proposition 4.5.12. The proof is complete. □

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## Harmonic analysis of stationary measures


**Abstract:**

Let $\mu$ be a Borel probability measure on $\mathrm{SL}_n(\mathbb{R})$, whose support generates a Zariski dense subgroup. Let $V$ be a finite dimensional irreducible linear representation of $\mathrm{SL}_n(\mathbb{R})$. A theorem of Furstenberg says that there exists a unique $\mu$-stationary probability measure on $\mathbb{P}V$ and we are interested in the Fourier decay of the stationary measure. The main result of the thesis is that the Fourier transform of the stationary measure has a power decay. From this result, we obtain a spectral gap of the transfer operator, whose properties allow us to establish an exponential error term for the renewal theorem in the context of products of random matrices. A key technical ingredient for the proof is a Fourier decay of multiplicative convolutions of measures on $\mathbb{R}^n$, which is a generalisation of Bourgain's theorem on dimension 1. We establish this result by using a sum-product estimate due to He-de Saxcé.

In the last part, we generalize a result of Lax-Phillips and a result of Hamenstädt on the finiteness of small eigenvalues of the Laplace operator on geometrically finite hyperbolic manifolds.



Key words : *stationary measures, harmonic analysis, Lie groups, sum-product estimates, Fourier decay, renewal theorem.*


## Analyse harmonique des mesures stationnaires


**Résumé:**

Soit $\mu$ une mesure de probabilité borélienne sur $\mathrm{SL}_n(\mathbb{R})$ tel que le sous-groupe engendré par le support de $\mu$ est Zariski dense. Soit $V$ une représentation irréductible de dimension finie de $\mathrm{SL}_n(\mathbb{R})$. D'après un théorème de Furstenberg, il existe une unique mesure $\mu$-stationnaire sur $\mathbb{P}V$ et nous nous somme intéressés à la décroissance de Fourier de cette mesure. Le résultat principal de cette thèse est que la transformée de Fourier de la mesure stationnaire a une décroissance polynomiale. À partir de ce résultat, nous obtenons un trou spectral de l'opérateur de transfert, dont les propriétés nous permettent d'établir un terme d'erreur exponentiel pour le théorème de renouvellement dans le cadre des produits de matrices aléatoires. L'ingrédient essentiel est une propriété de décroissance de Fourier des convolutions multiplicatives de mesures sur $\mathbb{R}^n$, qui est une généralisation d'un théorème de Bourgain en dimension 1. Nous établissons cet ingrédient en utilisant un estimée somme-produit de He et de Saxcé.

Dans la dernière partie, nous généralisons un résultat de Lax et Phillips et un résultat de Hamenstädt sur la finitude des petites valeurs propres de l'opérateur de Laplace sur les variétés hyperboliques géométriquement finies.



Mots clefs : *mesures stationnaires, analyse harmonique, groupes de Lie, estimées sommes-produits, décroissance de Fourier, théorème de renouvellement.*


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