BOREL SUBGROUPS OF THE PLANE CREMONA GROUP

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Abstract. It is well known that all Borel subgroups of a linear algebraic group are conjugate. Berest, Eshmatov, and Eshmatov have shown that this result also holds for the automorphism group $\mathrm{Aut}(\mathbb{A}^2)$ of the affine plane. In this paper, we describe all Borel subgroups of the complex Cremona group $\mathrm{Bir}(\mathbb{P}^2)$ up to conjugation, proving in particular that they are not necessarily conjugate. In principle, this fact answers a question of Popov. More precisely, we prove that $\mathrm{Bir}(\mathbb{P}^2)$ admits Borel subgroups of any rank $r \in \{0, 1, 2\}$ and that all Borel subgroups of rank $r \in \{1, 2\}$ are conjugate. In rank 0, there is a $1-1$ correspondence between conjugacy classes of Borel subgroups of rank 0 and hyperelliptic curves of genus $g \geq 1$. Hence, the conjugacy class of a rank 0 Borel subgroup admits two invariants: a discrete one, the genus g, and a continuous one, corresponding to the coarse moduli space of hyperelliptic curves of genus g. This moduli space is of dimension $2g-1$.

Contents

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1. Introduction

Let $\mathbb{L}$ be an algebraically closed field. The famous Lie–Kolchin theorem asserts that any closed connected solvable subgroup of $\mathrm{GL}_n(\mathbb{L})$ is triangularisable, i.e. conjugate to a subgroup all of whose elements are upper-triangular. More generally, let G be a linear algebraic group defined over $\mathbb{L}$. Define a Borel subgroup of G as a maximal subgroup among the closed connected solvable subgroups. Then, Borel has shown that any closed connected solvable subgroup of G is contained in a Borel subgroup and that all Borel subgroups of G are conjugate. By [3] (see also [16]), the same result still holds for the automorphism group $\mathrm{Aut}(\mathbb{A}^2)$ of the affine plane.

The Cremona group $\mathrm{Bir}(\mathbb{P}^n)$ is the group of all birational transformations of the n-dimensional complex projective space $\mathbb{P}^n$. From an algebraic point of view, it corresponds to the group of $\mathbb{C}$-automorphisms of the field $\mathbb{C}(x_1, \ldots, x_n)$. This group is naturally endowed with the Zariski topology introduced by Demazure [12] and Serre [22]. We describe this topology in Section 2; see in particular Definition 2.2. For more details on this subject we refer to [6]. Following Popov we define the Borel subgroups of $\mathrm{Bir}(\mathbb{P}^n)$ as the maximal closed connected solvable subgroups with respect to this topology [21]. Borel subgroups of a closed subgroup G of $\mathrm{Bir}(\mathbb{P}^n)$ are defined analogously.

An element of $\mathrm{Bir}(\mathbb{P}^n)$ is of the form

$$f\colon [x_1 : \cdots : x_{n+1}] \dashrightarrow [F_1(x_1, \ldots, x_{n+1}) : \cdots : F_{n+1}(x_1, \ldots, x_{n+1})],$$

where the $F_i \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ are homogeneous polynomials of the same degree. We then often write $f = [F_1 : \cdots : F_{n+1}]$. Using the open embedding

$$\mathbb{A}^n \hookrightarrow \mathbb{P}^n, \quad (x_1, \ldots, x_n) \mapsto [x_1 : \cdots : x_n : 1],$$

we have a natural isomorphism $\mathrm{Bir}(\mathbb{A}^n) \simeq \mathrm{Bir}(\mathbb{P}^n)$. This allows us to write f in affine coordinates as

$$f\colon (x_1, \ldots, x_n) \dashrightarrow (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)),$$

where the $f_i \in \mathbb{C}(x_1, \ldots, x_n)$ are rational functions. We then often write $f = (f_1, \ldots, f_n)$.

Let $\mathcal{B}_n \subseteq \mathrm{Bir}(\mathbb{P}^n)$ be the subgroup of birational transformations of $\mathbb{P}^n$ of the form $f = (f_1, \ldots, f_n)$, $f_i = a_i x_i + b_i$, with $a_i, b_i \in \mathbb{C}(x_{i+1}, \ldots, x_n)$ and $a_i \neq 0$. The following result is proven by Popov [21, Theorem 1].

Theorem 1.1. *The group $\mathcal{B}_n$ is a Borel subgroup of* $\mathrm{Bir}(\mathbb{P}^n)$.

In the same paper he also makes the following conjecture.

Conjecture 1.2. *For $n \geq 5$, the Cremona group* $\mathrm{Bir}(\mathbb{P}^n)$ *contains nonconjugate Borel subgroups.*

The main result of our paper is the description of all Borel subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ up to conjugation. Before giving the precise statement a few definitions are needed.

The Jonquières group Jonq is defined as the group of birational transformations of $\mathbb{P}^2$ preserving the pencil of lines passing through the point $[1 : 0 : 0] \in \mathbb{P}^2$. In affine coordinates, it is the group of birational transformations of the form

$$(x,y) \dashrightarrow \left(\frac{\alpha(y)x+\beta(y)}{\gamma(y)x+\delta(y)}, \frac{ay+b}{cy+d}\right),$$

where $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{C}(y))$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2$. Hence we have

$$\mathrm{Jonq} = \mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2 \;\supseteq\; \mathrm{Aff}_1(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1 = \mathcal{B}_2\,.$$

Here and in the rest of this paper we write PGL_2 and Aff_1 instead of $\mathrm{PGL}_2(\mathbb{C})$ and $\mathrm{Aff}_1(\mathbb{C})$. Also, all the semidirect products considered will be inner semidirect products. This means we will write $G = N \rtimes H$ when N, H are subgroups of the (abstract) group G which satisfy the three following assertions:

(1) N is a normal subgroup $N \lhd G$;
(2) $G = NH$;
(3) $N \cap H = \{1\}$ (where 1 denotes the identity element of G).

For any nonsquare element f of $\mathbb{C}(y)$ we define the subgroup

$$\mathbb{T}_f := \left\{ \left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right),\; a,b \in \mathbb{C}(y),\; (a,b) \neq (0,0) \right\}$$

of $\mathrm{PGL}_2(\mathbb{C}(y)) \subseteq \mathrm{Jonq}$ and for any coprime integers $p,q \in \mathbb{Z}$, we define the 1-dimensional torus

$$\mathbb{T}_{p,q} := \{(t^p x, t^q y),\; t \in \mathbb{C}^*\} \;\subseteq\; \mathrm{Jonq}.$$

We make the following two conventions. When talking about the genus of a complex curve, we will always mean the genus of the associated smooth projective curve. If f is a nonsquare element of $\mathbb{C}(y)$, when talking about the hyperelliptic curve associated with $x^2 = f(y)$, we will always mean the smooth projective curve whose function field is equal to $\mathbb{C}(y)[\sqrt{f}]$.

Note that we allow hyperelliptic curves of genus 0 and 1.

We can now state the principal result of our paper.

Theorem 1.3 (Main Theorem). *Up to conjugation, any Borel subgroup of* $\mathrm{Bir}(\mathbb{P}^2)$ *is one of the following groups:*

(1) $\mathcal{B}_2$*;*
(2) $\mathbb{T}_y \rtimes \mathbb{T}_{1,2}$*;*
(3) $\mathbb{T}_f$*, where f is a nonsquare element of $\mathbb{C}(y)$ such that the genus g of the hyperelliptic curve associated with $x^2 = f(y)$ satisfies $g \geq 1$.*

Moreover these three cases are mutually disjoint and in case (3) the Borel subgroups $\mathbb{T}_f$ and $\mathbb{T}_g$ are conjugate if and only if the hyperelliptic curves associated with $x^2 = f(y)$ and $x^2 = g(y)$ are isomorphic.

If G is an abstract group, we denote by $D(G)$ its derived subgroup. It is the subgroup generated by all commutators $[g,h] := ghg^{-1}h^{-1}$, $g,h \in G$. The n-th derived subgroup of G is then defined inductively by $D^0(G) := G$ and $D^n(G) := D(D^{n-1}(G))$ for $n \geq 1$. A group G is called solvable if $D^n(G) = \{1\}$ for some integer $n \geq 0$. The smallest such integer n is called the derived length of G and is denoted $\ell(G)$. Recall that the subgroup of upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$ is solvable and has derived length $\lceil \log_2(n) \rceil + 1$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to the real number x (see e.g. [25, page 16]). Also, the subgroup of upper triangular automorphisms in $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ has derived length $n+1$ (see [16, Lemma 3.2]). In contrast, we will prove the following result in Appendix A.

Proposition 1.4. *The derived length of the Borel subgroup $\mathcal{B}_n$ of $\mathrm{Bir}(\mathbb{P}^n)$ is equal to $2n$.*

As usual we let the rank $\mathrm{rk}(G)$ of a complex linear algebraic group G be the maximal dimension d of an algebraic torus $(\mathbb{C}^*)^d$ in G. Analogously, the rank $\mathrm{rk}(G)$ of a closed subgroup G of $\mathrm{Bir}(\mathbb{P}^n)$ is defined as the maximal dimension of an algebraic torus in G. The following result is proven by Białynicki-Birula [4, Corollary 2, page 180] (see also [20, Theorem 1 (i)]).

Theorem 1.5. *All algebraic tori in $\mathrm{Bir}(\mathbb{P}^n)$ are of rank $\leq n$. Moreover, all algebraic tori of a given rank $\geq n-2$ are conjugate in $\mathrm{Bir}(\mathbb{P}^n)$.*

Hence, we have $\mathrm{rk}(\mathrm{Bir}(\mathbb{P}^n)) = \mathrm{rk}(\mathcal{B}_n) = n$, and any closed subgroup G of $\mathrm{Bir}(\mathbb{P}^n)$ satisfies $\mathrm{rk}(G) \leq n$.

The derived lengths and ranks of the three kinds of Borel subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ are given in the following table (see Remark 11.4 for the computations):

Type of Borel subgroup	Derived length	Rank
$\mathcal{B}_2$	4	2
$\mathbb{T}_y \rtimes \mathbb{T}_{1,2}$	2	1
$\mathbb{T}_f$	1	0

Theorem 1.3 directly gives the following result:

Corollary 1.6. *All Borel subgroups of maximal rank 2 of $\mathrm{Bir}(\mathbb{P}^2)$ are conjugate.*

More generally the following question seems natural.

Question 1.7. *Are all Borel subgroups of maximal rank n of $\mathrm{Bir}(\mathbb{P}^n)$ conjugate?*

In view of Theorem 1.3 we believe that the following slightly strengthened version of Popov's conjecture should hold.

Conjecture 1.8. *For $n \geq 2$, the Cremona group $\mathrm{Bir}(\mathbb{P}^n)$ contains nonconjugate Borel subgroups (note that Theorem 1.3 establishes the case $n = 2$).*

Our article is organised as follows: In Section 2 we outline the construction of the Zariski topology on $\mathrm{Bir}(\mathbb{P}^n)$, following [7, §5.2]. We also establish various results to be used later on. In Section 3, we prove that any Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is conjugate to a subgroup of the Jonquières group Jonq (Theorem 3.1). This key result heavily relies on Urech's nice paper [24]. Then, in Section 4, we prove that any Borel subgroup of Jonq is conjugate to a subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ (Theorem 4.1). These two results directly imply that any Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is conjugate to a (Borel) subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ (Theorem 4.2). Surprisingly enough, we will see that a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ is not necessarily a Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ (Example B.1 of Appendix B). In Section 5, we define the $\mathbb{K}$-Borel subgroups of $\mathrm{PGL}_2(\mathbb{K})$ for any field $\mathbb{K}$ of characteristic zero in Definition 5.3 and we describe them in Theorem 5.9. This result is then used in Section 6 where all Borel subgroups of the closed subgroup $\mathrm{PGL}_2(\mathbb{C}(y))$ of $\mathrm{Bir}(\mathbb{P}^2)$ are described in Theorem 6.4. It turns out that these Borel subgroups coincide with the $\mathbb{C}(y)$-Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y))$! At the end of Section 6 we then prove that the maximal derived length of a closed connected solvable subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is 4 (Lemma 6.5), and deduce from this result that any closed connected solvable subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is contained in a Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ (Proposition 6.6). In Section 7 we study the groups $\mathbb{T}_f \subseteq \mathrm{Bir}(\mathbb{P}^2)$. In Proposition 7.15 we give different equivalent conditions characterising the fact that $\mathbb{T}_f$ and $\mathbb{T}_g$ are conjugate in $\mathrm{Bir}(\mathbb{P}^2)$ and in Proposition 7.22 we compute the neutral connected component $\mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_f)^\circ$ of the normaliser of $\mathbb{T}_f$ in Jonq. In Section 8 we show that an algebraic subgroup G of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ isomorphic to $\mathbb{C}^*$ is conjugate to $\mathbb{T}_{0,1}$ if and only if the second projection $\mathrm{pr}_2 \colon \mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1 \to \mathrm{Aff}_1$ induces an isomorphism $G \to \mathrm{pr}_2(G)$ (see Lemma 8.2). In Section 9 we show that up to conjugation the additive group $(\mathbb{C}, +)$ admits exactly two embeddings in Jonq (Proposition 9.1). The main result of Section 10 is Theorem 10.7, asserting that any Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ contains at least one Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$. Then in Theorem 10.9 we show that any Borel subgroup B' of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ actually contains a unique Borel subgroup B of $\mathrm{PGL}_2(\mathbb{C}(y))$ and that the corresponding map $B' \mapsto B$ defines a bijection from the set of Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ to the set of Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y))$. Finally in Section 11 we show that all the subgroups listed in Theorem 1.3 are actually Borel subgroups of $\mathrm{Bir}(\mathbb{P}^2)$, cf. Theorem 11.2, and in Section 12 we show that up to conjugation there are no others, cf. Theorem 12.1. These two sections contain the two following additional results: Any Borel subgroup of Jonq is a Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ (Proposition 11.1); if B is a Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$, then we have $B = \mathrm{N}_{\mathrm{Bir}(\mathbb{P}^2)}(B)^\circ$ (Proposition 12.3; this statement is an analog of the usual Borel normaliser theorem which asserts that $B = \mathrm{N}_G(B)$ when B is a Borel subgroup of a linear algebraic group G).

Our paper also contains two appendices. In Appendix A we show that the derived length of $\mathcal{B}_n$ is equal to $2n$. In Appendix B we give an example of a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ which is not a Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ – even if we have shown that any Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is conjugate to a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$!

2. The Zariski topology on $\mathrm{Bir}(\mathbb{P}^n)$

Following [12, 22], the notion of families of birational maps is defined, and used in Definition 2.2 for describing the natural Zariski topology on $\mathrm{Bir}(W)$ where W is an irreducible complex algebraic variety.

Definition 2.1. *Let A, W be irreducible complex algebraic varieties, and let f be an A-birational map of the A-variety $A \times W$, inducing an isomorphism $U \to V$, where U, V are open subsets of $A \times W$, whose projections on A are surjective.*

The birational map f is given by $(a, w) \dashrightarrow (a, p_2(f(a, w)))$, where p_2 is the second projection, and for each $\mathbb{C}$-point $a \in A$, the birational map $w \dashrightarrow p_2(f(a, w))$ corresponds to an element $f_a \in \mathrm{Bir}(W)$. The map $a \mapsto f_a$ represents a map from A (more precisely from the $\mathbb{C}$-points of A) to $\mathrm{Bir}(W)$, and will be called a morphism from A to $\mathrm{Bir}(W)$.

Definition 2.2. *A subset $F \subseteq \mathrm{Bir}(W)$ is closed in the Zariski topology if for any algebraic variety A and any morphism $A \to \mathrm{Bir}(W)$ the preimage of F is closed.*

Recall that a birational transformation f of $\mathbb{P}^n$ is given by

$$f\colon [x_1 : \cdots : x_{n+1}] \dashrightarrow [f_1(x_1, \ldots, x_{n+1}) : \cdots : f_{n+1}(x_1, \ldots, x_{n+1})],$$

where the f_i are homogeneous polynomials of the same degree. Choosing the f_i without common component, the degree of f is the degree of the f_i. If d is a positive integer, we set $\mathrm{Bir}(\mathbb{P}^n)_d := \{f \in \mathrm{Bir}(\mathbb{P}^n),\ \deg(f) \le d\}$. We will use the following result, which is [6, Proposition 2.10]:

Lemma 2.3. *A subset $F \subseteq \mathrm{Bir}(\mathbb{P}^n)$ is closed if and only if $F \cap \mathrm{Bir}(\mathbb{P}^n)_d$ is closed in $\mathrm{Bir}(\mathbb{P}^n)_d$ for any positive integer d.*

Remark 2.4. *Since $\mathrm{Bir}(\mathbb{P}^n)_d$ is closed in $\mathrm{Bir}(\mathbb{P}^n)$ (see [6, Corollary 2.8]), a subset F of $\mathrm{Bir}(\mathbb{P}^n)$ is closed if and only if there exists a positive integer D such that $F \cap \mathrm{Bir}(\mathbb{P}^n)_d$ is closed in $\mathrm{Bir}(\mathbb{P}^n)_d$ for any $d \ge D$.*

We will now describe the topology on $\mathrm{Bir}(\mathbb{P}^n)_d$. A convenient way to handle this topology is through the map $\pi_d\colon \mathfrak{Bir}(\mathbb{P}^n)_d \to \mathrm{Bir}(\mathbb{P}^n)_d$ that we introduce in the next definition and whose properties are given in Lemma 2.7 below.

Let's now fix the integer $d \ge 1$. We will henceforth use the following notation:

Definition 2.5. *Denote by $\mathfrak{Rat}(\mathbb{P}^n)_d$ the projective space associated with the complex vector space of $(n+1)$-tuples $(f_1, \ldots, f_{n+1})$ where all $f_i \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ are homogeneous polynomials of degree d. The equivalence class of $(f_1, \ldots, f_{n+1})$ will be denoted by $[f_1, \ldots, f_{n+1}]$.*

For each $f = [f_1, \ldots, f_{n+1}] \in \mathfrak{Rat}(\mathbb{P}^n)_d$, we denote by ψ_f the rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ defined by

$$[x_1 : \cdots : x_{n+1}] \dashrightarrow [f_1(x_1, \ldots, x_{n+1}) : \cdots : f_{n+1}(x_1, \ldots, x_{n+1})].$$

Writing $\mathrm{Rat}(\mathbb{P}^n)$ for the set of rational maps from $\mathbb{P}^n$ to $\mathbb{P}^n$ and setting

$$\mathrm{Rat}(\mathbb{P}^n)_d := \{h \in \mathrm{Rat}(\mathbb{P}^n),\ \deg(h) \le d\},$$

we obtain a surjective map

$$\Psi_d\colon \mathfrak{Rat}(\mathbb{P}^n)_d \to \mathrm{Rat}(\mathbb{P}^n)_d, \quad f \mapsto \psi_f.$$

This map induces a surjective map $\pi_d\colon \mathfrak{Bir}(\mathbb{P}^n)_d \to \mathrm{Bir}(\mathbb{P}^n)_d$, where $\mathfrak{Bir}(\mathbb{P}^n)_d$ is defined to be $\Psi_d^{-1}(\mathrm{Bir}(\mathbb{P}^n)_d)$.

The following result is [6, Lemma 2.4(2)].

Proposition 2.6. *The set $\mathfrak{Bir}(\mathbb{P}^n)_d$ is locally closed in the projective space $\mathfrak{Rat}(\mathbb{P}^n)_d$ and thus inherits from $\mathfrak{Rat}(\mathbb{P}^n)_d$ the structure of an algebraic variety.*

The following result, which is [6, Corollary 2.9], will be crucial for us since it provides a bridge from the "weird" topological space $\mathrm{Bir}(\mathbb{P}^n)_d$ to the "nice" topological space $\mathfrak{Bir}(\mathbb{P}^n)_d$ which is an algebraic variety.

Lemma 2.7. *The map $\pi_d\colon \mathfrak{Bir}(\mathbb{P}^n)_d \to \mathrm{Bir}(\mathbb{P}^n)_d$ is continuous and closed. In particular, it is a quotient topological map: A subset $F \subseteq \mathrm{Bir}(\mathbb{P}^n)_d$ is closed if and only if its preimage $\pi_d^{-1}(F)$ is closed.*

Remark 2.8. *Let F be a closed subset of $\mathrm{Bir}(\mathbb{P}^n)_d$ and let $\mathfrak{F} := \pi_d^{-1}(F) \subseteq \mathfrak{Bir}(\mathbb{P}^n)_d$ be its preimage via π_d. By what has been said above, $\mathfrak{F}$ is naturally a (finite dimensional, but not necessarily irreducible) variety (being closed in the variety $\mathfrak{Bir}(\mathbb{P}^n)_d$), and the closed continuous map $\pi_d\colon \mathfrak{Bir}(\mathbb{P}^n)_d \to \mathrm{Bir}(\mathbb{P}^n)_d$ induces a closed continuous map $\pi_{d,F}\colon \mathfrak{F} \to F$ whose fibres are connected and nonempty. It follows that $\pi_{d,F}$ induces a $1-1$ correspondence between the connected components of $\mathfrak{F}$ and the connected components of F. More precisely, if $\mathfrak{C}$ is a connected component of $\mathfrak{F}$, then $\pi_{d,F}(\mathfrak{C})$ is a connected component of F and conversely if C is a connected component of F, then $(\pi_{d,F})^{-1}(C)$ is a connected component of $\mathfrak{F}$. In particular, F admits finitely many connected components and these connected components are closed and open in F.*

Lemma 2.3, Remark 2.4, and Lemma 2.7 give the following useful characterisation of closed subsets of $\mathrm{Bir}(\mathbb{P}^n)$ (this criterion is a slight generalisation of [6, Corollary 2.7]).

Lemma 2.9. *A subset $F \subseteq \mathrm{Bir}(\mathbb{P}^n)$ is closed if and only if there exists a positive integer D such that $\pi_d^{-1}(F \cap \mathrm{Bir}(\mathbb{P}^n)_d) \subseteq \mathfrak{Bir}(\mathbb{P}^n)_d$ is closed for any $d \geq D$.*

Our main use of the following lemma will be the characterisation of the connectedness of a closed subset of $\mathrm{Bir}(\mathbb{P}^n)$ given in (3). The intermediate characterisation (2) will only be used in the proof of the equivalence between (1) and (3).

Lemma 2.10. *Let F be a closed subset of $\mathrm{Bir}(\mathbb{P}^n)$. Then, the three following assertions are equivalent:*

(1) F is connected;
(2) For each $\varphi, \psi \in F$, there exists a positive integer d such that φ, ψ belong to the same connected component of $F \cap \mathrm{Bir}(\mathbb{P}^n)_d$;
(3) For each $\varphi, \psi \in F$, there exists a connected (not necessarily irreducible) curve C and a morphism $\lambda\colon C \to \mathrm{Bir}(\mathbb{P}^n)$ (see Definition 2.1) whose image satisfies:

$$\varphi, \psi \in \mathrm{Im}(\lambda) \subseteq F.$$

Proof. (1) $\Longrightarrow$ (2). For each positive integer d, set $F_d := F \cap \mathrm{Bir}(\mathbb{P}^n)_d$. Assume that φ is an element of some F_{d_0}. For each $d \geq d_0$, the connected component of F_d which contains φ will be denoted by $F_{d,\varphi}$. We want to prove that $F_\varphi := \bigcup_{d \geq d_0} F_{d,\varphi}$ is equal to F. For this, it is sufficient to check that F_φ is open and closed in F, i.e. that $F_\varphi \cap F_d$ is closed and open in F_d for each positive integer d. However, since the set F_d has only finitely many connected components (see Remark 2.8) and since $F_\varphi \cap F_d$ is a union of such connected components, it follows that $F_\varphi \cap F_d$ is actually closed and open in F_d.

(2) $\Longrightarrow$ (3). Let φ, ψ be elements of F and let d be a positive integer such that φ, ψ belong to the same connected component of $F \cap \mathrm{Bir}(\mathbb{P}^n)_d$. Since each connected component of $F \cap \mathrm{Bir}(\mathbb{P}^n)_d$ is of the form $\pi_d(A)$ for some connected variety A (see Remark 2.8), we have

$$\varphi, \psi \in \pi_d(A) \subseteq F.$$

Let $a, b \in A$ be such that $\varphi = \pi_d(a)$, $\psi = \pi_d(b)$. It's enough to show that there exists a connected curve C on the variety A containing a and b. Let A' be the set of points $c \in A$ for which there exists a connected curve $C \subseteq A$ containing a and c. By [19, Lemma on page 56], for any irreducible variety V and any points $v, w \in V$, there is an irreducible curve on V containing v and w. It follows that if c belongs to A', then all the irreducible components of A which contain c are contained in A'. Since A is connected, this shows that $A' = A$ and this concludes the proof of the implication (2) $\Longrightarrow$ (3).

(3) $\Longrightarrow$ (1). This is obvious. □

Definition 2.11. *A map $\varphi\colon \mathrm{Bir}(\mathbb{P}^n) \to \mathrm{Bir}(\mathbb{P}^n)$ will be called a morphism if for each irreducible complex algebraic variety A and each morphism $\rho\colon A \to \mathrm{Bir}(\mathbb{P}^n)$ (in the sense of Definition 2.1), the composition $\varphi \circ \rho$ is also a morphism (still in the sense of Definition 2.1).*

The following result directly follows from the Definitions 2.2 and 2.11.

Lemma 2.12. *The four following assertions are satisfied.*

1. *The inverse map $\iota\colon \mathrm{Bir}(\mathbb{P}^n) \to \mathrm{Bir}(\mathbb{P}^n)$, $g \mapsto g^{-1}$, is a morphism.*
2. *Let $\pi\colon \mathrm{Bir}(\mathbb{P}^n) \times \mathrm{Bir}(\mathbb{P}^n) \to \mathrm{Bir}(\mathbb{P}^n)$ be the map that sends (g, g') onto $g \circ g'$. If $\varphi, \varphi'\colon \mathrm{Bir}(\mathbb{P}^n) \to \mathrm{Bir}(\mathbb{P}^n)$ are morphisms, then the map $\pi(\varphi, \varphi')\colon \mathrm{Bir}(\mathbb{P}^n) \to \mathrm{Bir}(\mathbb{P}^n)$, that sends g onto $\varphi(g) \circ \varphi'(g)$, is a morphism.*
3. *Any morphism $\varphi\colon \mathrm{Bir}(\mathbb{P}^n) \to \mathrm{Bir}(\mathbb{P}^n)$ is continuous.*
4. *If V, W are two connected subsets of $\mathrm{Bir}(\mathbb{P}^n)$, then their product*
$$V.W = \{v \circ w,\ v \in V,\ w \in W\} \subseteq \mathrm{Bir}(\mathbb{P}^n)$$
is connected.

Proof. (1) Let A be an algebraic variety and $\rho\colon A \to \mathrm{Bir}(\mathbb{P}^n)$ be a morphism. We want to show that $\iota \circ \rho\colon A \to \mathrm{Bir}(\mathbb{P}^n)$ is a morphism. Since ρ is a morphism, there exists an A-birational map f of the A-variety $A \times \mathbb{P}^n$, inducing an isomorphism $U \to V$, where U, V are open subsets of $A \times \mathbb{P}^n$, whose projections on A are surjective, and such that ρ is the family associated to f. This last point means that for each $\mathbb{C}$-point $a \in A$, the birational map $\rho(a)\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is the transformation $w \dashrightarrow p_2(f(a, w))$. For showing that $\iota \circ \rho$ is a morphism, it's enough to note that $\iota \circ \rho$ is the family associated to the A-birational map f^{-1}.

(2) Set $\psi := \pi(\varphi, \varphi')$. Let A be an algebraic variety and $\rho\colon A \to \mathrm{Bir}(\mathbb{P}^n)$ be a morphism. We want to show that $\psi \circ \rho\colon A \to \mathrm{Bir}(\mathbb{P}^n)$ is a morphism.

Since $\varphi \circ \rho$ is a morphism, there exists an A-birational map f of the A-variety $A \times \mathbb{P}^n$, inducing an isomorphism $\theta\colon U \to V$, where U, V are open subsets of $A \times \mathbb{P}^n$, whose projections on A are surjective, and such that $\varphi \circ \rho$ is the family associated to f.

Analogously, since $\varphi' \circ \rho$ is a morphism, there exists an A-birational map f' of the A-variety $A \times \mathbb{P}^n$, inducing an isomorphism $\theta'\colon U' \to V'$, where U', V' are open subsets of $A \times \mathbb{P}^n$, whose projections on A are surjective, and such that $\varphi' \circ \rho$ is the family associated to f'.

For showing that $\psi \circ \rho$ is a morphism, it's enough to note that $\psi \circ \rho$ is the family associated to the A-birational map $f \circ f'$. Let us just check that there exists open subsets U'', V'' of $A \times \mathbb{P}^n$, whose projections on A are surjective, and such that $f \circ f'$ induces an isomorphism $U'' \to V''$. Since U, V' are open subsets of $A \times \mathbb{P}^n$, whose projections on A are surjective, one would easily check that $U \cap V'$ is also an open subset of $A \times \mathbb{P}^n$, whose projection on A is surjective. It's now enough to set $U'' := (\theta')^{-1}(U \cap V')$ and $V'' := \theta(U \cap V')$.

(3) Let F be a closed subset of $\mathrm{Bir}(\mathbb{P}^n)$. We want to show that $\varphi^{-1}(F)$ is closed in $\mathrm{Bir}(\mathbb{P}^n)$, i.e. that for each algebraic variety A and each morphism $\rho\colon A \to \mathrm{Bir}(\mathbb{P}^n)$, the preimage $\rho^{-1}(\varphi^{-1}(F))$ is closed. Since $\varphi \circ \rho\colon A \to \mathrm{Bir}(\mathbb{P}^n)$ is a morphism (by Definition 2.11), this follows from Definition 2.2.

(4) We may assume that both V and W are nonempty. Take $w_0 \in W$. Since

$$V.W = \bigcup_{v \in V} v.W$$

where each $v.W$ is connected and intersects the fixed connected subset $V.w_0$ of $V.W$, this shows that $V.W$ is connected. $\square$

If G is a linear algebraic group, it is well-known that its derived group $D(G)$ is closed (see e.g. [18, Proposition 17.2, page 110]). It is not clear whether this result remains true for closed subgroups G of $\mathrm{Bir}(\mathbb{P}^n)$, but thanks to the following definition and to the next two lemmas, this will not be a concern for us.

Definition 2.13. *Let G be any subgroup of $\mathrm{Bir}(\mathbb{P}^n)$.*

(1) Set $\mathcal{D}(G) := \overline{D(G)}$ (the closure of the derived group of G).

(2) We then define $\mathcal{D}^k(G)$ inductively by

$$\mathcal{D}^0(G) := \overline{G}, \quad \mathcal{D}^1(G) := \mathcal{D}(G), \text{ and } \mathcal{D}^k(G) := \mathcal{D}^1(\mathcal{D}^{k-1}(G)) \text{ for } k \geq 2.$$

Lemma 2.14. *Let G be any subgroup of $\mathrm{Bir}(\mathbb{P}^n)$.*

(1) We have $D(G) \subseteq D(\overline{G}) \subseteq \mathcal{D}(G)$.

(2) We have $\mathcal{D}(G) = \mathcal{D}(\overline{G})$.

(3) If G is closed, then $\mathcal{D}(G)$ is normal in G and the quotient $G/\mathcal{D}(G)$ is abelian.

(4) If G is connected, then $D(G)$ is also connected.

(5) If G is closed and connected, then $\mathcal{D}(G)$ is also closed and connected.

Proof. (1) The inclusion $D(G) \subseteq D(\overline{G})$ being obvious, let's prove $D(\overline{G}) \subseteq \overline{D(G)}$. Fix an element h of G. By Lemma 2.12(1)-(2), the map

$$\varphi_h\colon \mathrm{Bir}(\mathbb{P}^n) \to \mathrm{Bir}(\mathbb{P}^n), \quad g \mapsto [g,h] = ghg^{-1}h^{-1}$$

is a morphism. By Lemma 2.12(3), it is in particular continuous. Since G is obviously contained in $(\varphi_h)^{-1}(\overline{D(G)})$, we get $\overline{G} \subseteq (\varphi_h)^{-1}(\overline{D(G)})$. Consequently, we have proven that

$$\forall g \in \overline{G}, \quad \forall h \in G, \quad [g,h] \in \overline{D(G)}.$$

Similarly, for each fixed element g of $\overline{G}$, the map $\psi\colon \mathrm{Bir}(\mathbb{P}^n) \to \mathrm{Bir}(\mathbb{P}^n)$, $h \mapsto [g,h]$ is continuous. Since G is included in $\psi^{-1}(\overline{D(G)})$, we get $\overline{G} \subseteq \psi^{-1}(\overline{D(G)})$ and thus

$$\forall g,h \in \overline{G}, \quad [g,h] \in \overline{D(G)}.$$

This implies the desired inclusion.

(2) By taking the closure of (1), we obtain $\mathcal{D}(G) \subseteq \mathcal{D}(\overline{G}) \subseteq \mathcal{D}(G)$ and the equality follows.

(3) Any group H such that $D(G) \subseteq H \subseteq G$ is normal in G with G/H abelian. Hence, the result is a consequence of (1).

(4) Let's begin by checking that the set $\mathcal{Com}(G)$ of commutators of G is connected. Let h be an element of G. We have seen above (in the proof of (1)) that the map

$$\varphi_h\colon \mathrm{Bir}(\mathbb{P}^n) \to \mathrm{Bir}(\mathbb{P}^n), \quad g \mapsto [g,h] = ghg^{-1}h^{-1}$$

is continuous. Hence $\varphi_h(G)$ is connected. Since all $\varphi_h(G)$, $h \in G$, are connected and contain $\mathrm{id} \in G$, it follows from the equality $\mathcal{Com}(G) = \bigcup_{h\in G} \varphi_h(G)$ that $\mathcal{Com}(G)$ is connected. It now follows from Lemma 2.12(4) that for each positive integer j, the set

$$\mathcal{Com}^j(G) := \{c_1 \dots c_j,\ c_1, \dots, c_j \in \mathcal{Com}(G)\}$$

is also connected. Therefore the increasing union $D^1(G) = \bigcup_j \mathcal{Com}^j(G)$ is connected.

(5) This is a direct consequence of the previous point. □

An induction based on Lemma 2.14(2) yields the following result:

Lemma 2.15. *Let G be a subgroup of $\mathrm{Bir}(\mathbb{P}^n)$. Then, for each nonnegative integer k, we have $\mathcal{D}^k(G) = \overline{D^k(G)}$. In particular, we have $D^k(G) = \{1\}$ if and only if $\mathcal{D}^k(G) = \{1\}$. This means that if G is solvable its derived length is also equal to the least nonnegative integer k such that $\mathcal{D}^k(G) = \{1\}$.*

Definition 2.16. *A subset A of $\mathrm{Bir}(\mathbb{P}^n)$ is bounded if there exists a constant K such that $\deg(g) \leq K$ for all $g \in A$.*

Remark 2.17. *An algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^n)$ is nothing else than a bounded closed subgroup* (*see* [6, Remark 2.20]).

Lemma 2.18. *(1) The group Jonq is closed in $\mathrm{Bir}(\mathbb{P}^2)$.*

(2) The projection $\mathrm{pr}_2\colon \mathrm{Jonq} \to \mathrm{PGL}_2$, $\left(\frac{\alpha(y)x+\beta(y)}{\gamma(y)x+\delta(y)}, \frac{ay+b}{cy+d}\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, is continuous.

(3) If A is a bounded closed subset of Jonq, then $\mathrm{pr}_2(A)$ is a constructible subset of PGL_2.

(4) If H is a closed subgroup of PGL_2, then the group $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes H$ is a closed subgroup of Jonq, and hence of $\mathrm{Bir}(\mathbb{P}^2)$. In particular, the groups $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ and $\mathrm{PGL}_2(\mathbb{C}(y)) \subseteq \mathrm{Jonq}$ are closed in Jonq.

Proof. (1) Even if the proof is already given in [7, Remark 5.22], we recall it here in preparation for the proof of (2). By Lemma 2.9, it is enough to prove that $\mathfrak{Jonq}_d = \pi_d^{-1}(\mathrm{Jonq} \cap \mathrm{Bir}(\mathbb{P}^2)_d)$ is closed in $\mathfrak{Bir}(\mathbb{P}^2)_d$ for each d. Denote by $\mathfrak{L}$ the projective space (of dimension 3) associated with the complex vector space of pairs (g_1, g_2) where $g_1, g_2 \in \mathbb{C}[y,z]$ are homogeneous polynomials of degree 1. The equivalence class of (g_1, g_2) will be denoted by $[g_1 : g_2]$. Denote by $Y \subseteq \mathfrak{Bir}(\mathbb{P}^2)_d \times \mathfrak{L}$ the closed subvariety given by elements $([f_1 : f_2 : f_3], [g_1 : g_2])$ satisfying $f_2 g_2 = f_3 g_1$. Since $\mathfrak{L}$ is a complete variety, the first projection $p_1\colon \mathfrak{Bir}(\mathbb{P}^2)_d \times \mathfrak{L} \to \mathfrak{Bir}(\mathbb{P}^2)_d$ is a closed morphism. Hence, assertion (1) follows from the equality $\mathfrak{Jonq}_d = p_1(Y)$.

(2) It is enough to prove that the restriction of pr_2 to the set $\mathrm{Jonq}_d := \mathrm{Jonq} \cap \mathrm{Bir}(\mathbb{P}^2)_d$ is continuous. This can be seen using the following commutative diagram:

$$\begin{tikzcd}
 & Y \arrow[dl, "p_1"'] \arrow[ddr, "p_2"] & \\
\mathfrak{Jonq}_d \arrow[d, "\pi_d"'] & & \\
\mathrm{Jonq}_d \arrow[rr, "\mathrm{pr}_2"] & & \mathfrak{L}
\end{tikzcd}$$

Note that PGL_2 is naturally identified with the open subset of $\mathfrak{L}$ whose elements $[ay+bz : cy+dz]$ satisfy $\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \neq 0$. Hence, it is enough to note that the horizontal map $\mathrm{pr}_2 \colon \mathrm{Jonq}_d \to \mathfrak{L}$ is continuous. We will use the fact that p_2 is continuous (being a morphism of algebraic varieties) and that $p_1 \colon Y \to \mathfrak{Jonq}_d$ and $\pi_d \colon \mathfrak{Jonq}_d \to \mathrm{Jonq}_d$ are surjective and closed (the surjectivity is obvious, the closedness comes from the fact that these two maps are restrictions to closed subsets of the closed maps $p_1 \colon \mathfrak{Bir}(\mathbb{P}^2)_d \times \mathfrak{L} \to \mathfrak{Bir}(\mathbb{P}^2)_d$ and $\pi_d \colon \mathfrak{Bir}(\mathbb{P}^2)_d \to \mathrm{Bir}(\mathbb{P}^2)_d$, see Lemma 2.7).

Take any closed subset F of $\mathfrak{L}$. We want to prove that $(\mathrm{pr}_2)^{-1}(F)$ is closed in Jonq_d. This comes from the previous remarks and the equality $(\mathrm{pr}_2)^{-1}(F) = (\pi_d \circ p_1)((p_2)^{-1}(F))$.

(3) Choose d so that we have $A \subseteq \mathrm{Jonq}_d$. Since $p_1 \colon Y \to \mathfrak{Jonq}_d$ and $\pi_d \colon \mathfrak{Jonq}_d \to \mathrm{Jonq}_d$ are surjective, we have $\mathrm{pr}_2(A) = p_2((\pi_d \circ p_1)^{-1}(A))$, and since $p_2 \colon Y \to \mathfrak{L}$ is a morphism of algebraic varieties, the closed subset $(\pi_d \circ p_1)^{-1}(A)$ of Y is sent by p_2 onto the constructible subset $\mathrm{pr}_2(A)$ of $\mathfrak{L}$. This also shows that $\mathrm{pr}_2(A)$ is a constructible subset of PGL_2.

(4) The group $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes H$ is the preimage of H by $\mathrm{pr}_2 \colon \mathrm{Jonq} \to \mathrm{PGL}_2$. Therefore, the result follows from (2). ☐

3. Any Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is conjugate to a subgroup of Jonq

The aim of this section is to prove the following result:

Theorem 3.1. *Any closed connected solvable subgroup of* $\mathrm{Bir}(\mathbb{P}^2)$ *is conjugate to a subgroup of* Jonq. *In particular, any Borel subgroup of* $\mathrm{Bir}(\mathbb{P}^2)$ *is conjugate to a Borel subgroup of* Jonq.

Recall that by definition Jonq is the group of birational transformations preserving the pencil of lines through $[1:0:0] \in \mathbb{P}^2$. An element, resp. a subgroup, of $\mathrm{Bir}(\mathbb{P}^2)$, is conjugate to an element, resp. a subgroup, of Jonq if and only if it preserves a *rational fibration*. In our text a rational fibration denotes what is often called a rational fibration with rational fibres. For the sake of clarity we include the following complete definition.

Definition 3.2. *(1) A rational fibration of* $\mathbb{P}^2$ *is a rational map* $\pi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ *whose generic fibre is birational to* $\mathbb{P}^1$*. By Tsen's lemma, this is equivalent to* *saying that the element $\pi \in \mathbb{C}(x, y)$ is the coordinate of a Cremona transformation, i.e. there exists a rational map $\pi' \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ such that the rational map $(\pi, \pi') \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is birational.*

(2) *The rational fibration $\pi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ is preserved by the Cremona transformation $\alpha \in \mathrm{Bir}(\mathbb{P}^2)$ if there exists an automorphism $\beta \in \mathrm{Aut}(\mathbb{P}^1)$ such that the following diagram is commutative:*

$$\begin{tikzcd} \mathbb{P}^2 \arrow[r, dashed, "\alpha"] \arrow[d, dashed, "\pi"'] & \mathbb{P}^2 \arrow[d, dashed, "\pi"] \\ \mathbb{P}^1 \arrow[r, "\beta"] & \mathbb{P}^1 \end{tikzcd}$$

Equivalently, there exists a Cremona transformation $\varphi = [\varphi_1 : \varphi_2 : \varphi_3] \in \mathrm{Bir}(\mathbb{P}^2)$ such that:
   (a) $\pi = [\varphi_2 : \varphi_3]$;
   (b) $\varphi\alpha\varphi^{-1} \in \mathrm{Jonq}$.

(3) *Two rational fibrations $\pi, \pi' \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ are called equivalent if there exists an automorphism $\beta \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\pi' = \beta\pi$.*

(4) *We say that the rational fibration $\pi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ is the only rational fibration preserved by $\alpha \in \mathrm{Bir}(\mathbb{P}^2)$ if it is preserved by α and if all rational fibrations preserved are equivalent to π.*

The following lemma should not come as a surprise:

Lemma 3.3. *Any countable closed subset of* $\mathrm{Bir}(\mathbb{P}^2)$ *is discrete.*

Proof. Let C be such a countable closed subset. We want to prove that any subset $C' \subseteq C$ is closed in C. By Lemma 2.3, this is equivalent to proving that $C' \cap \mathrm{Bir}(\mathbb{P}^2)_d$ is closed in $\mathrm{Bir}(\mathbb{P}^2)_d$ for any $d \geq 1$. Therefore, it is sufficient to prove that $C_d := C \cap \mathrm{Bir}(\mathbb{P}^2)_d$ is finite. Writing $C_d = \bigcup_{n\geq 1} F_n$ as an increasing union of finite subsets, we get $(\pi_d)^{-1}(C_d) = \bigcup_{n\geq 1} (\pi_d)^{-1}(F_n)$. Since $(\pi_d)^{-1}(C_d)$ and $(\pi_d)^{-1}(F_n)$, $n \geq 1$, are closed subvarieties of $\mathfrak{Bir}(\mathbb{P}^2)_d$ and since the ground field $\mathbb{C}$ is uncountable, this proves that the increasing union is stationary, i.e. $(\pi_d)^{-1}(C_d) = (\pi_d)^{-1}(F_n)$ for some n, proving that $C_d = F_n$ is finite. □

Corollary 3.4. *Any closed connected and countable subgroup of* $\mathrm{Bir}(\mathbb{P}^2)$ *is trivial.*

The following result is for example proven in [15, Proposition 9.3.1]:

Lemma 3.5. *Any torsion subgroup of a linear algebraic group is countable.*

Our proof of the next result relies on Urech's paper [24]:

Lemma 3.6. *Each solvable subgroup G of* $\mathrm{Bir}(\mathbb{P}^2)$ *satisfies one of the following assertions:*
   (1) *G is conjugate to a subgroup of* Jonq;
   (2) *G is countable;*
   (3) *G is bounded.*

Proof. By [24, Theorem 8.1, page 25], G satisfies one of the following assertions:
(a) G is conjugate to a subgroup of Jonq;

(b) G is countable;
(c) G is conjugate to a subgroup of the automorphism group of a Halphen surface;
(d) G is a subgroup of elliptic elements.

In case (c), it is well known that G is countable (see e.g. [24, Theorem 2.4, page 9]). In case (d), it follows from [24, Theorem 1.3, page 3] that G satisfies one of the following assertions (here we use that G preserves a rational fibration if and only if it is conjugate to a subgroup of Jonq):

(i) G is conjugate to a subgroup of Jonq;
(ii) G is a bounded subgroup;
(iii) G is a subgroup of torsion elements.

In case (iii), it follows from [24, Theorem 1.5, page 3] that G is isomorphic to a subgroup of $\mathrm{GL}_{48}(\mathbb{C})$. Then, G is countable by Lemma 3.5. □

We are now able to prove Theorem 3.1:

Proof of Theorem 3.1. Let G be a closed connected solvable subgroup of $\mathrm{Bir}(\mathbb{P}^2)$. By Lemma 3.6 and Corollary 3.4, we may assume that G is a bounded subgroup. Therefore, G is an algebraic group (see Remark 2.17). It follows from Enriques theorem (see [23, Theorem (2.25), page 238] and also [23, Proposition (2.18), page 233]) that any connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is contained in a maximal connected algebraic subgroup, and that, up to conjugation, any maximal connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is one of the following subgroups:

(1) The group $\mathrm{Aut}(\mathbb{P}^2) \simeq \mathrm{PGL}_3$;
(2) The connected component $\mathrm{Aut}^\circ(\mathbb{F}_n)$ of the automorphism group of the n-th Hirzebruch surface $\mathbb{F}_n$ where n is a nonnegative integer different from 1 (because $\mathbb{F}_n$ needs to be a minimal surface).

In case (1), recall that any closed connected solvable subgroup of PGL_3 is contained in a Borel subgroup of PGL_3, that such a Borel subgroup is conjugate to the subgroup of upper triangular matrices, and that this latter group is contained in Jonq. In case (2), it is enough to note that all groups $\mathrm{Aut}^\circ(\mathbb{F}_n)$ are already conjugate to subgroups of Jonq. □

4. Any Borel subgroup of Jonq is conjugate to a subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$

The aim of this short section is to prove the following easy result:

Theorem 4.1. *Any closed connected solvable subgroup of* Jonq *is conjugate to a subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ *by an element of* $\{1\} \rtimes \mathrm{PGL}_2 \subseteq \mathrm{Jonq}$. *In particular, any Borel subgroup of* Jonq *is conjugate to a Borel subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ *by an element of* $\{1\} \rtimes \mathrm{PGL}_2 \subseteq \mathrm{Jonq}$.

Proof. Let G be a closed connected solvable subgroup of Jonq. Let

$$\mathrm{pr}_2 \colon \mathrm{Jonq} = \mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2 \to \mathrm{PGL}_2$$

be the second projection. Since pr_2 is a morphism of groups, the image $\mathrm{pr}_2(G)$ is a solvable subgroup of PGL_2 and since pr_2 is continuous (Lemma 2.18(2)) $\mathrm{pr}_2(G)$ is moreover connected. It follows that $\overline{\mathrm{pr}_2(G)}$ is a closed connected solvable subgroup of PGL_2. Up to conjugation we may assume that it is contained in the subgroup of upper triangular matrices of PGL_2. ☐

Theorems 3.1 and 4.1 directly give the following result:

Theorem 4.2. *Any closed connected solvable subgroup of* $\mathrm{Bir}(\mathbb{P}^2)$ *is conjugate to a subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$. *In particular, any Borel subgroup of* $\mathrm{Bir}(\mathbb{P}^2)$ *is conjugate to a Borel subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$.

5. $\mathbb{K}$-Borel subgroups of $\mathrm{PGL}_2(\mathbb{K})$

In this section, $\mathbb{K}$ denotes a field of characteristic zero. For the sake of clarity, we begin with a classical definition and a well-known lemma.

Definition 5.1. *Let V be an affine variety defined over $\mathbb{K}$. The Zariski $\mathbb{K}$-topology on $V(\mathbb{K})$ (or for short, the $\mathbb{K}$-topology) is the topology for which a subset is closed if it is the zero set of some collection of elements of the affine algebra $\mathbb{K}[V]$.*

Lemma 5.2. *For any field extension $\mathbb{K} \subseteq \mathbb{K}'$ and any affine variety V defined over $\mathbb{K}$, the $\mathbb{K}$-topology on $V(\mathbb{K})$ coincides with the topology induced by the $\mathbb{K}'$-topology on $V(\mathbb{K}')$.*

Proof. It is clear that any $\mathbb{K}$-closed subset of $V(\mathbb{K})$ is the trace of a $\mathbb{K}'$-closed subset of $V(\mathbb{K}')$. Conversely, choose any basis $(e_i)_{i\in I}$ of $\mathbb{K}'$ over $\mathbb{K}$. We have $\mathbb{K}' = \bigoplus_{i\in I} \mathbb{K}e_i$ and $\mathbb{K}'[V] = \bigoplus_{i\in I} \mathbb{K}[V]e_i$. Hence, if $Z \subseteq V(\mathbb{K})$ is the zero set of some collection $(f_j)_{j\in J}$ of elements $f_j \in \mathbb{K}'[V]$, then it is also the zero set of the collection $(f_{ij})_{(i,j)\in I\times J}$ where each f_j is decomposed as $f_j = \sum_{i\in I} f_{ij}e_i$, $f_{ij} \in \mathbb{K}[V]$. ☐

Definition 5.3. *Let G be a linear algebraic group defined over $\mathbb{K}$. We will say that a subgroup B of $G(\mathbb{K})$ is a $\mathbb{K}$-Borel subgroup if it is a maximal closed connected solvable subgroup of $G(\mathbb{K})$ for the $\mathbb{K}$-topology.*

This notion is not to be confused with the classical notion of a Borel subgroup of G defined over $\mathbb{K}$ (or equivalently an algebraic $\mathbb{K}$-Borel subgroup of G) which we now recall: This is a $\mathbb{K}$-closed subgroup B of G such that $B(\overline{\mathbb{K}})$ is a maximal $\overline{\mathbb{K}}$-closed connected solvable subgroup of $G(\overline{\mathbb{K}})$. We will not use this classical notion at all in our text. One reason for studying $\mathbb{K}$-Borel subgroups rather than algebraic $\mathbb{K}$-Borel subgroups will become apparent in Theorem 6.4.

We will prove in Theorem 5.9 that if f is a nonsquare element of $\mathbb{K}$, then the group

$$\mathbb{T}_f := \left\{ \left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right),\ a, b \in \mathbb{K},\ (a,b) \neq (0,0) \right\}$$

is a $\mathbb{K}$-Borel subgroup of $\mathrm{PGL}_2(\mathbb{K})$.

Proposition 5.4. *If f is a nonsquare element of $\mathbb{K}$, then the group*

$$\mathbb{T}_f = \left\{ \left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right),\ a, b \in \mathbb{K},\ (a,b) \neq (0,0) \right\}$$

is abstractly isomorphic to the group $\mathbb{K}[\sqrt{f}]^*/\mathbb{K}^*$.

Proof. Let $\mathbb{K}[C]$ be the $\mathbb{K}$-subalgebra of $\mathrm{M}_2(\mathbb{K})$ spanned by C. The minimal polynomial of the matrix $C := \left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{K})$ being equal to $\mu_C(T) = T^2 - f$, we have $\mathbb{K}[C] = \{aI + bC,\ a, b \in \mathbb{K}\}$ and the map

$$\mathbb{K}[C] \to \mathbb{K}[\sqrt{f}], \quad aI + bC \mapsto a + b\sqrt{f}$$

is a $\mathbb{K}$-isomorphism of fields. This map induces the isomorphism of groups

$$\mathbb{T}_f \xrightarrow{\sim} \mathbb{K}[\sqrt{f}]^*/\mathbb{K}^*. \qquad \square$$

In the sequel we will often use the distinguished element ι_f of $\mathbb{T}_f$ that we now define.

Definition 5.5. *For a nonsquare element f of $\mathbb{K}$, we set $\iota_f := \left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right) \in \mathbb{T}_f$.*

A straightforward computation establishes the following result.

Lemma 5.6. *Let f be a nonsquare element of $\mathbb{K}$. Then ι_f is the unique involution of $\mathbb{T}_f$.*

Lemma 5.7. *Let f be a nonsquare element of $\mathbb{K}$. Then each nontrivial element of $\mathbb{T}_f$ is nontriangularisable in $\mathrm{PGL}_2(\mathbb{K})$.*

Proof. Set $A := \left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{K})$ where $a, b \in \mathbb{K}$ and $b \neq 0$. If the class $\overline{A}$ of A in $\mathrm{PGL}_2(\mathbb{K})$ was triangularisable, then A should be triangularisable in $\mathrm{GL}_2(\mathbb{K})$. However, its characteristic polynomial is $\chi_A(T) = T^2 - 2aT + (a^2 - b^2 f) \in \mathbb{K}[T]$ whose discriminant $\Delta = 4b^2 f$ is not a square. Hence χ_A does not split over $\mathbb{K}$. A contradiction. $\square$

The following result is the key lemma of this section.

Lemma 5.8. *Let H be a closed connected subgroup of $\mathrm{PGL}_2(\mathbb{K})$. Then, up to conjugation, one of the following cases occurs:*

(1) $H = \{\mathrm{id}\}$;
(2) $H = \{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right),\ a \in \mathbb{K}^*\} \simeq (\mathbb{K}^*, \times)$;
(3) $H = \{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right),\ a \in \mathbb{K}\} \simeq (\mathbb{K}, +)$;
(4) $H = \mathbb{T}_f$ *for some nonsquare element* $f \in \mathbb{K}$;
(5) $H = \mathrm{Aff}_1(\mathbb{K}) = \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right),\ a \in \mathbb{K}^*,\ b \in \mathbb{K}\}$;
(6) $H = \mathrm{PGL}_2(\mathbb{K})$.

Proof. Let $p\colon \mathrm{GL}_2(\mathbb{K}) \to \mathrm{PGL}_2(\mathbb{K})$ be the natural surjection. The map $H \mapsto p^{-1}(H)$ induces a bijection between the closed connected subgroups of $\mathrm{PGL}_2(\mathbb{K})$ and the closed connected subgroups of $\mathrm{GL}_2(\mathbb{K})$ containing the group $\mathbb{K}^*\,\mathrm{id}$. If G is a linear algebraic group defined over $\mathbb{K}$ whose Lie algebra is denoted $\mathfrak{g}$, recall that each closed connected subgroup H of G is uniquely determined by its Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$ (but that not every Lie subalgebra of $\mathfrak{g}$ corresponds to an algebraic subgroup of G; the Lie algebras corresponding to algebraic subgroups are called algebraic)(see e.g. [8, §7, page 105]). In our case, we want to describe, up to conjugation in $\mathrm{GL}_2(\mathbb{K})$, all algebraic Lie subalgebras of $\mathfrak{gl}_2$ containing $\mathbb{K}\,\mathrm{id}$. Actually, we will show that each Lie subalgebra of $\mathfrak{gl}_2$ containing $\mathbb{K}\,\mathrm{id}$ is algebraic, i.e. is the Lie algebra of some closed connected subgroup of $\mathrm{GL}_2(\mathbb{K})$ containing $\mathbb{K}^*\,\mathrm{id}$. Since $\mathfrak{gl}_2 = \mathfrak{sl}_2 \oplus \mathbb{K}\,\mathrm{id}$ as Lie algebras, our problem amounts to describing up to conjugation all Lie subalgebras $\mathfrak{h}$ of $\mathfrak{sl}_2$.

If $\dim \mathfrak{h} = 2$, let's prove that $\mathfrak{h}$ is conjugate by an element of $\mathrm{GL}_2(\mathbb{K})$ to the Lie algebra $\mathfrak{u}$ of upper triangular matrices. Write $\mathfrak{sl}_2 = \mathrm{Vect}(E, F, H)$ where

$$E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \quad H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right),$$

and recall that $[H, E] = 2E$, $[H, F] = -2F$, and $[E, F] = H$. It is clear that $\mathfrak{h}$ admits a basis A, B where A is upper triangular, i.e. of the form $A = \left(\begin{smallmatrix} a & b \\ 0 & -a \end{smallmatrix}\right)$. Up to conjugation and multiplication by an element of $\mathbb{K}^*$, we may even assume that A is either E (if $a = 0$) or H (if $a \neq 0$).

If $A = E$, then we may assume that $B = \alpha F + \beta H$ where $\alpha, \beta \in \mathbb{K}$. We have $[A, B] = \alpha H - 2\beta E$. This yields $\alpha = 0$ (because otherwise $\mathfrak{h}$ would contain E, H, F and would not be 2-dimensional) and hence $\mathfrak{h} = \mathrm{Vect}(E, H) = \mathfrak{u}$.

If $A = H$, then we may assume that $B = \alpha E + \beta F$ where $\alpha, \beta \in \mathbb{K}$. We have $[A, B] = 2\alpha E - 2\beta F$. Hence, the matrices B and $[A, B]$ are linearly dependent, i.e. $\det \left(\begin{smallmatrix} \alpha & \beta \\ 2\alpha & -2\beta \end{smallmatrix}\right) = 0$, i.e. $4\,\alpha\beta = 0$. If $\alpha = 0$, we get $\mathfrak{h} = \mathrm{Vect}(F, H)$ and finally $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mathfrak{h} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^{-1} = \mathfrak{u}$. If $\beta = 0$, we get $\mathfrak{h} = \mathrm{Vect}(E, H) = \mathfrak{u}$. We have actually proven that up to conjugation $\mathfrak{h}$ is always equal to the Lie subalgebra of upper triangular matrices. We are in case (5).

If $\dim \mathfrak{h} = 1$, then $\mathfrak{h} = \mathrm{Vect}(A)$ for some nonzero matrix A of $\mathfrak{sl}_2$. Setting $f := -\det(A)$, the characteristic polynomial of A is equal to $\chi_A = T^2 - f$. If f is a nonsquare element, then χ_A is irreducible, and up to conjugation by an element of $\mathrm{GL}_2(\mathbb{K})$, we may assume that A is the companion matrix $\left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right)$. We are in case (4). Assume now that f is a square. If $f = 0$, then, up to conjugation, we may assume that $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. We are in case (3). If $f \neq 0$, then, up to conjugation and multiplication by an element of $\mathbb{K}^*$, we may assume that $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We are in case (2). ☐

Our main result is a direct consequence of Lemma 5.8.

Theorem 5.9. *Up to conjugation, any $\mathbb{K}$-Borel subgroup of $\mathrm{PGL}_2(\mathbb{K})$ is equal to either $\mathrm{Aff}_1(\mathbb{K})$ or some $\mathbb{T}_f$, where f is a nonsquare element of $\mathbb{K}$. Moreover, any closed connected solvable subgroup of $\mathrm{PGL}_2(\mathbb{K})$ for the $\mathbb{K}$-topology is contained in a $\mathbb{K}$-Borel subgroup.*

Proposition 5.10. *If $f, g \in \mathbb{K}$ are nonsquares, then the following assertions are equivalent:*

(1) The groups $\mathbb{T}_f$ and $\mathbb{T}_g$ are conjugate;
(2) The ratio f/g is a perfect square.

Proof. (1) $\Longrightarrow$ (2). Assume that $\mathbb{T}_f$ and $\mathbb{T}_g$ are conjugate. Then Lemma 5.6 shows that $\iota_f = \left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right)$ and $\iota_g = \left(\begin{smallmatrix} 0 & g \\ 1 & 0 \end{smallmatrix}\right)$ are conjugate. Denote by $\overline{\det}\colon \mathrm{PGL}_2(\mathbb{K}) \to \mathbb{K}^*/(\mathbb{K}^*)^2$ the morphism of groups induced by the determinant morphism $\det\colon \mathrm{GL}_2(\mathbb{K}) \to \mathbb{K}^*$. The equality $\overline{\det}(\iota_f) = \overline{\det}(\iota_g)$ exactly means that f/g is a square.

(2) $\Longrightarrow$ (1). Assume that $g = \lambda^2 f$ for some nonzero element $\lambda \in \mathbb{C}(y)$. If $a, b \in \mathbb{C}(y)^*$ the equality $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right) \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} \lambda a & bg \\ b & \lambda a \end{smallmatrix}\right)$ shows that $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathbb{T}_f \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \mathbb{T}_g$. ☐

The following result shows that a non-triangularisable $\mathbb{K}$-Borel subgroup of $\mathrm{PGL}_2(\mathbb{K})$ is uniquely determined by any of its nontrivial elements.

Lemma 5.11. *If A is a nontrivial element of a non-triangularisable $\mathbb{K}$-Borel subgroup B of $\mathrm{PGL}_2(\mathbb{K})$, then B is the unique $\mathbb{K}$-Borel subgroup of $\mathrm{PGL}_2(\mathbb{K})$ containing A.*

Proof. Let $p\colon \mathrm{GL}_2(\mathbb{K}) \to \mathrm{PGL}_2(\mathbb{K})$ be the canonical surjection and let $\tilde{A}$ be an element of $\mathrm{GL}_2(\mathbb{K})$ satisfying $p(\tilde{A}) = A$. Set also $I := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{K})$. Let's begin by checking that the following equality holds

$$B = \{p(\alpha I + \beta \tilde{A}),\ \alpha, \beta \in \mathbb{K},\ (\alpha, \beta) \neq (0,0)\,\}. \tag{1}$$

Up to conjugation, we may assume that $B = \mathbb{T}_f$ for some nonsquare element f of $\mathbb{K}$. Setting $J_f := \left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{K})$, $\tilde{A}$ is necessarily of the form $\tilde{A} = \left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right)$, i.e. $\tilde{A} = aI + bJ_f$, for some $a, b \in \mathbb{K}$ with $b \neq 0$. This shows that in the $\mathbb{K}$-vector space $\mathrm{M}_2(\mathbb{K})$ we have

$$\mathrm{Vect}(I, J_f) = \mathrm{Vect}(I, \tilde{A}).$$

It follows that

$$\begin{aligned} \mathbb{T}_f &= \{p(\alpha I + \beta J_f),\ \alpha, \beta \in \mathbb{K},\ (\alpha, \beta) \neq (0,0)\,\} \\ &= \{p(\alpha I + \beta\, \tilde{A}),\ \alpha, \beta \in \mathbb{K},\ (\alpha, \beta) \neq (0,0)\,\}, \end{aligned}$$

and this proves (1).

Assume now that B' is a $\mathbb{K}$-Borel subgroup of $\mathrm{PGL}_2(\mathbb{K})$ containing $A = \left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right)$. By Lemma 5.7, B' is non-triangularisable. Hence the equality (1) also applies to B' and this shows that $B = B'$. □

Lemma 5.11 provides the following useful result.

Proposition 5.12. *Let B, B' be two $\mathbb{K}$-Borel subgroups of $\mathrm{PGL}_2(\mathbb{K})$. Assume moreover that B or B' is not triangularisable. Then, the following assertions hold.*

(1) We have $B = B'$ if and only if $B \cap B' \neq \{1\}$.

(2) If φ is an element of $\mathrm{PGL}_2(\mathbb{K})$ and A a nontrivial element of B, we have $\varphi B \varphi^{-1} = B'$ if and only if $\varphi A \varphi^{-1} \in B'$.

For later use we now compute the normalisers of $\mathbb{T}_f$ and $\mathrm{Aff}_1(\mathbb{K})$.

Lemma 5.13. *Let f be a nonsquare element of $\mathbb{K}$. Recall that we have set $\iota_f = \left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right)$. Then we have*

$$\mathrm{N}_{\mathrm{PGL}_2(\mathbb{K})}(\mathbb{T}_f) = \mathrm{Cent}_{\mathrm{PGL}_2(\mathbb{K})}(\iota_f) = \mathbb{T}_f \rtimes \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle.$$

Proof. We will prove that

$$\mathrm{N}_{\mathrm{PGL}_2(\mathbb{K})}(\mathbb{T}_f) \subseteq \mathrm{Cent}_{\mathrm{PGL}_2(\mathbb{K})}(\iota_f) \subseteq \mathbb{T}_f \rtimes \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle \subseteq \mathrm{N}_{\mathrm{PGL}_2(\mathbb{K})}(\mathbb{T}_f).$$

The inclusion $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{K})}(\mathbb{T}_f) \subseteq \mathrm{Cent}_{\mathrm{PGL}_2(\mathbb{K})}(\iota_f)$ directly follows from Lemma 5.6. Assume now that the element $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\mathrm{PGL}_2(\mathbb{K})$ centralises ι_f. We have

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \quad \text{i.e.} \quad \left(\begin{smallmatrix} b & af \\ d & cf \end{smallmatrix}\right) = \left(\begin{smallmatrix} cf & df \\ a & b \end{smallmatrix}\right).$$

This is equivalent to the existence of $\varepsilon \in \mathbb{K}^*$ such that

$$cf = \varepsilon b, \quad d = \varepsilon a, \quad a = \varepsilon d, \quad b = \varepsilon c f.$$

Since $a = \varepsilon^2 a$ and $b = \varepsilon^2 b$ where $(a, b) \neq (0,0)$, we have $\varepsilon^2 = 1$, i.e. $\varepsilon = \pm 1$, and these equations are equivalent to

$$d = \varepsilon a, \quad b = \varepsilon c f.$$

This means that we have $M = \left(\begin{smallmatrix} a & \varepsilon cf \\ c & \varepsilon a \end{smallmatrix}\right)$, i.e. $M = \left(\begin{smallmatrix} a & cf \\ c & a \end{smallmatrix}\right)\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{T}_f \rtimes \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle$. We have proven $\mathrm{Cent}_{\mathrm{PGL}_2(\mathbb{K})} \subseteq \mathbb{T}_f \rtimes \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle$.

Finally, the equality $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & -bf \\ -b & a \end{smallmatrix}\right)$ shows that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ normalises $\mathbb{T}_f$. This proves that $\mathbb{T}_f \rtimes \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle \subseteq \mathrm{N}_{\mathrm{PGL}_2(\mathbb{K})}(\mathbb{T}_f)$. □

Remark 5.14. *Recall that a Borel subgroup B of a linear algebraic group G defined over an algebraically closed field is always maximal among the solvable subgroups of G by* [18, Corollary 23.1A, page 143]. *In contrast, the $\mathbb{K}$-Borel subgroup $\mathbb{T}_f$ is contained in the larger solvable subgroup $\mathbb{T}_f \rtimes \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle$.*

Lemma 5.15. *We have*

$$\mathrm{N}_{\mathrm{PGL}_2(\mathbb{K})}(\mathrm{Aff}_1(\mathbb{K})) = \mathrm{Aff}_1(\mathbb{K}).$$

Proof. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{K})$. Then $A \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) A^{-1} \in \mathrm{Aff}_1(\mathbb{K})$ if and only if $-c^2 = 0$, i.e. if and only if $A \in \mathrm{Aff}_1(\mathbb{K})$. □

From now on we will only consider the groups $\mathrm{PGL}_2(\mathbb{K})$ and $\mathbb{T}_f$ when the base field $\mathbb{K}$ is equal to $\mathbb{C}(y)$.

6. Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y)) \subseteq \mathrm{Bir}(\mathbb{P}^2)$

The aim of this section is to prove Theorem 6.4 which describes all Borel subgroups of the closed subgroup $\mathrm{PGL}_2(\mathbb{C}(y))$ of $\mathrm{Bir}(\mathbb{P}^2)$. It turns out that these Borel subgroups coincide with the $\mathbb{C}(y)$-Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y))$ which have been defined in Definition 5.3 and described in Theorem 5.9.

The group $\mathrm{PGL}_2(\mathbb{C}(y))$ admits two natural topologies. The first one (which is the one of most interest to us) is the topology induced by the inclusion $\mathrm{PGL}_2(\mathbb{C}(y)) \subseteq \mathrm{Bir}(\mathbb{P}^2)$. The second one is the $\mathbb{C}(y)$-topology which has been defined in Definition 5.1. We will refer to the first one as the $\mathrm{Bir}(\mathbb{P}^2)$-topology and to the second as the $\mathbb{C}(y)$-topology. When not specified, the topology is always understood to be the $\mathrm{Bir}(\mathbb{P}^2)$-topology. Finally note that the $\mathbb{C}(y)$-topology will only be used in this section and in the proof of Theorem 10.7.

We begin with the following result:

Lemma 6.1. *The $\mathrm{Bir}(\mathbb{P}^2)$-topology on $\mathrm{PGL}_2(\mathbb{C}(y))$ is finer than the $\mathbb{C}(y)$-topology.*

Proof. Any $\mathbb{C}(y)$-closed set in $\mathrm{PGL}_2(\mathbb{C}(y))$ is an intersection of sets of the form

$$F_P := \left\{ \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{C}(y)) \mid P(\alpha,\beta,\gamma,\delta) = 0 \right\} \subset \mathrm{PGL}_2(\mathbb{C}(y)),$$

where $P \in \mathbb{C}[y][X_1, X_2, X_3, X_4]$ is a polynomial with coefficients in $\mathbb{C}[y]$ which is homogeneous with respect to the variables $X_1, \ldots, X_4$. Therefore it is enough to show that such a set F_P is $\mathrm{Bir}(\mathbb{P}^2)$-closed. By Lemma 2.9 we need to show that $\pi_d^{-1}(F_P \cap \mathrm{Bir}(\mathbb{P}^2)_d) \subseteq \mathfrak{Bir}(\mathbb{P}^2)_d$ is closed for any positive $d \geq 2$. Denote by $\mathfrak{M}$ the projective space associated with the complex vector space of 4-tuples $(\alpha,\beta,\gamma,\delta)$ where $\alpha,\beta,\gamma,\delta \in \mathbb{C}[y,z]$ are homogeneous polynomials of respective degrees $d-1, d, d-2, d-1$. The equivalence class of $(\alpha,\beta,\gamma,\delta)$ will be denoted by $[\alpha : \beta : \gamma : \delta]$. Denote by $Z \subseteq \mathfrak{Bir}(\mathbb{P}^2)_d \times \mathfrak{M}$ the closed subset given by elements $([f_1 : f_2 : f_3], [\alpha : \beta : \gamma : \delta])$ satisfying the two following conditions:

$$\forall\, i,j \in \{1,2,3\},\ f_i g_j = f_j g_i, \tag{2}$$

where $(g_1,g_2,g_3) := \Big(\alpha(y,z)x+\beta(y,z), y(\gamma(y,z)x+\delta(y,z)), z(\gamma(y,z)x+\delta(y,z))\Big)$;

$$P(\alpha(y,1),\beta(y,1),\gamma(y,1),\delta(y,1)) = 0. \tag{3}$$

The condition (2) means that the elements $[f_1 : f_2 : f_3]$ and $[g_1 : g_2 : g_3]$ define the same rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Note that we then have $\gamma(y,1)x+\delta(y,1) \neq 0$ and that in affine coordinates this rational map is the following birational map

$$\mathbb{A}^2 \dashrightarrow \mathbb{A}^2, \quad (x,y) \dashrightarrow \left(\frac{\alpha(y,1)x+\beta(y,1)}{\gamma(y,1)x+\delta(y,1)}, y\right).$$

The condition (3) means that the element $\left(\begin{smallmatrix} \alpha(y,1) & \beta(y,1) \\ \gamma(y,1) & \delta(y,1)\end{smallmatrix}\right)$ of $\mathrm{PGL}_2(\mathbb{C}(y))$ belongs to F_P.

We leave it as an easy exercise for the reader to check that the condition (3) is actually a closed condition on the coefficients of the polynomials $\alpha,\beta,\gamma,\delta$. Since the first projection $p_1\colon \mathfrak{Bir}(\mathbb{P}^2)_d \times \mathfrak{M} \to \mathfrak{Bir}(\mathbb{P}^2)_d$ is a closed morphism, the equality $\pi_d^{-1}(F_P \cap \mathrm{Bir}(\mathbb{P}^2)_d) = p_1(Z)$ shows that $\pi_d^{-1}(F_P \cap \mathrm{Bir}(\mathbb{P}^2)_d)$ is closed in $\mathfrak{Bir}(\mathbb{P}^2)_d$. $\square$

Here is an example showing that the $\mathrm{Bir}(\mathbb{P}^2)$-topology on $\mathrm{PGL}_2(\mathbb{C}(y))$ is strictly finer than the $\mathbb{C}(y)$-topology.

Example 6.2. *The set* $\{\left(\begin{smallmatrix} 1 & t \\ 0 & 1\end{smallmatrix}\right),\ t \in \mathbb{C}\} \subseteq \mathrm{PGL}_2(\mathbb{C}(y))$ *is closed for the* $\mathrm{Bir}(\mathbb{P}^2)$-*topology, but its* $\mathbb{C}(y)$-*closure is* $\{\left(\begin{smallmatrix} 1 & t \\ 0 & 1\end{smallmatrix}\right),\ t \in \mathbb{C}(y)\}$.

Lemma 6.3. *Any closed connected subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y))$ *for the* $\mathbb{C}(y)$-*topology is closed connected for the* $\mathrm{Bir}(\mathbb{P}^2)$-*topology.*

Proof. By Lemma 6.1, each $\mathbb{C}(y)$-closed subset of $\mathrm{PGL}_2(\mathbb{C}(y))$ is $\mathrm{Bir}(\mathbb{P}^2)$-closed. It's enough to prove that when $\mathbb{K} = \mathbb{C}(y)$, the six subgroups $H_1,\dots,H_6$ listed in the Cases (1), …, (6) of Lemma 5.8, are $\mathrm{Bir}(\mathbb{P}^2)$-connected. Since $H_1 = \{\mathrm{id}\}$, $H_5 = H_3 \rtimes H_2$, and $H_6 = \langle H_5, (H_5)^t\rangle$ (where $(H_5)^t$ denotes the transpose of H_5), it's enough to show that H_2, H_3, H_4 are $\mathrm{Bir}(\mathbb{P}^2)$-connected. Using Lemma 2.10, it's enough to show that these three subgroups satisfy the assertion (3) of that lemma, i.e. that for any $i \in \{2,3,4\}$ and any $\varphi,\psi \in H_i$, there exists a connected (not necessarily irreducible) curve C and a morphism $\lambda\colon C \to \mathrm{Bir}(\mathbb{P}^2)$ (see Definition 2.1) whose image satisfies $\varphi,\psi \in \mathrm{Im}(\lambda) \subseteq H_i$.

Let $h := \left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right)$, $a \in \mathbb{C}(y) \setminus \{0,1\}$, be a nontrivial element of H_2. Define the curve C by $C := \mathbb{A}^1$ if $a \notin \mathbb{C}$ and $C := \mathbb{A}^1 \setminus \{\frac{1}{1-a}\}$ otherwise. Then, the image of the morphism $C \to \mathrm{Bir}(\mathbb{P}^2)$, $t \mapsto \left(\begin{smallmatrix} 1-t+ta & 0 \\ 0 & 1\end{smallmatrix}\right)$ is contained in H_2, and it connects h and id. Hence H_2 is connected.

If $h := \left(\begin{smallmatrix} 1 & a \\ 0 & 1\end{smallmatrix}\right)$, $a \in \mathbb{C}(y)$, is any element of H_3, the image of the morphism $\mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^2)$, $t \mapsto \left(\begin{smallmatrix} 1 & ta \\ 0 & 1\end{smallmatrix}\right)$ is contained in H_3, and it connects h and id. Hence H_3 is connected.

Let $h := \left(\begin{smallmatrix} a & bf \\ b & a\end{smallmatrix}\right)$, $a,b \in \mathbb{C}(y)^*$, be any element of $H_4 = \mathbb{T}_f$ which is different from 1 and ι_f. The two morphisms $\mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^2)$, $t \mapsto \left(\begin{smallmatrix} a & tbf \\ tb & a\end{smallmatrix}\right)$ and $\mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^2)$,

$t \mapsto \begin{pmatrix} ta & bf \\ b & ta \end{pmatrix}$ both have their images contained in H_4. The first one connects h and id, and the second connects h and $\iota_f = \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix}$. Hence H_4 is connected. $\square$

We can now prove the main result of this section.

Theorem 6.4. *Up to conjugation, any Borel subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y))$ *is equal to either* $\mathrm{Aff}_1(\mathbb{C}(y))$ *or some* $\mathbb{T}_f$*, where* f *is a nonsquare element of* $\mathbb{C}(y)$*. Moreover, any closed connected solvable subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y))$ *is contained in a Borel subgroup.*

Proof. By Theorem 5.9, it's enough to show the two following assertions:

(1) Each closed connected solvable subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$ is contained in a $\mathbb{C}(y)$-Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$;

(2) The Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y))$ coincide with the $\mathbb{C}(y)$-Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y))$.

(1) Let H be a closed connected solvable subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$ and let $\overline{H}$ be its $\mathbb{C}(y)$-closure in $\mathrm{PGL}_2(\mathbb{C}(y))$. By Lemma 6.1, H is $\mathbb{C}(y)$-connected, and hence $\overline{H}$ is $\mathbb{C}(y)$-connected as well. The group $\overline{H}$ being a $\mathbb{C}(y)$-closed connected solvable subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$, it is contained in a $\mathbb{C}(y)$-Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$ by Theorem 5.9.

(2) We will successively prove that each Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$ is a $\mathbb{C}(y)$-Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$ and that each $\mathbb{C}(y)$-Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$ is a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$.

Let B be a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$. By (1), B is contained in a $\mathbb{C}(y)$-Borel subgroup B' of $\mathrm{PGL}_2(\mathbb{C}(y))$. But B' is a closed connected solvable subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$ (see Lemma 6.3). Hence, by maximality of B, we get $B = B'$, showing that B is actually a $\mathbb{C}(y)$-Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$.

Let now B be a $\mathbb{C}(y)$-Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$. First of all, B is a closed connected solvable subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$ (see Lemma 6.3). Secondly, let's check that B is maximal for this property. Let's assume that we have $B \subseteq H$, where H is a closed connected solvable subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$. By (1), H is contained in a $\mathbb{C}(y)$-Borel subgroup B' of $\mathrm{PGL}_2(\mathbb{C}(y))$. Hence we have $B \subseteq H \subseteq B'$ where B, B' are two $\mathbb{C}(y)$-Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y))$. By maximality of B, we get $B = B'$ from which we get $H = B$. We have actually shown that B is a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$. $\square$

We end this section by showing that any closed connected solvable subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is contained in a Borel subgroup. Actually we need a slightly stronger result, to be used later (see Proposition 6.6 below). The proof is based on the following result.

Lemma 6.5. *The maximal derived length of a closed connected solvable subgroup of* $\mathrm{Bir}(\mathbb{P}^2)$ *is* 4.

Proof. Let's begin by proving that the derived length of a closed connected solvable subgroup H of $\mathrm{Bir}(\mathbb{P}^2)$ is at most 4. By Theorem 4.2, we may assume that H is contained in $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$. But then, we have $\mathcal{D}^2 H \subseteq \mathrm{PGL}_2(\mathbb{C}(y))$ and since $\mathcal{D}^2 H$ is a closed connected solvable subgroup (Lemma 2.14(5)) its derived length is at most 2 (Theorem 6.4). We have actually proven that the derived length of H is at most 4. We conclude the proof by noting that the derived length of $\mathcal{B}_2$ is 4 (see Proposition 1.4). □

Proposition 6.6. *Let G be a closed subgroup of $\mathrm{Bir}(\mathbb{P}^2)$. Then, any closed connected solvable subgroup of G is contained in a Borel subgroup of G.*

Proof. If H is a closed connected solvable subgroup of G, denote by $\mathfrak{F}$ the set of connected solvable subgroups of G containing H. The bound given in Lemma 6.5 implies that $(\mathfrak{F}, \subseteq)$ is inductive. Indeed, if $(H_i)_{i\in I}$ is a chain in $\mathfrak{F}$, i.e. a totally ordered family of $\mathfrak{F}$, then the group $\cup_i H_i$ is connected solvable (for details, see [16, Proposition 3.10]). Therefore, by Zorn's lemma, the poset $\mathfrak{F}$ admits a maximal element B. Since $\overline{B}$ is connected and solvable, this shows that $B = \overline{B}$. Hence B is closed. Being maximal among the closed connected solvable subgroups of G, it is a Borel subgroup of G. □

7. The groups $\mathbb{T}_f$

If f is any nonsquare element of $\mathbb{C}(y)$, it follows from Theorem 6.4 that the group

$$\mathbb{T}_f = \left\{ \left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right),\ a, b \in \mathbb{C}(y),\ (a,b) \neq (0,0) \right\}$$

is a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$.

Definition 7.1. *The geometric degree of a rational function $f \in \mathbb{C}(y)$ is denoted $\mathrm{d}(f)$. If f is written α/β where $\alpha, \beta \in \mathbb{C}[y]$ are coprime, it is given by*

$$\mathrm{d}(f) = \max\{\deg(\alpha), \deg(\beta)\} \in \mathbb{Z}_{\geq 0}.$$

The geometric degree $\mathrm{d}(f)$ is the number of preimages (counted with multiplicities) of any point of $\mathbb{P}^1$ for the morphism $f \colon \mathbb{P}^1 \to \mathbb{P}^1$. It satisfies the following elementary properties:

Lemma 7.2. *For each $f, g \in \mathbb{C}(y)$ we have:*

(1) $\mathrm{d}(f+g) \leq \mathrm{d}(f) + \mathrm{d}(g)$.
(2) $\mathrm{d}(fg) \leq \mathrm{d}(f) + \mathrm{d}(g)$.
(3) $\mathrm{d}(f \circ g) = \mathrm{d}(f)\,\mathrm{d}(g)$ *(when the composition $f \circ g$ exists).*

Proof. Write $f = \alpha/\beta$ and $g = \gamma/\delta$ where $\alpha, \beta \in \mathbb{C}[y]$ are coprime and $\gamma, \delta \in \mathbb{C}[y]$ are coprime.

(1) The equality $f + g = \dfrac{\alpha\delta + \beta\gamma}{\beta\delta}$ yields

$$\begin{aligned} \mathrm{d}(f+g) &\leq \max\{\deg(\alpha\delta + \beta\gamma), \deg(\beta\delta)\} \\ &\leq \max\{\deg(\alpha\delta), \deg(\beta\gamma), \deg(\beta\delta)\} \\ &\leq \mathrm{d}(f) + \mathrm{d}(g). \end{aligned}$$

(2) This is straightforward.
(3) This is obvious from the geometric interpretation. □

Recall that a subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is called algebraic if it is closed and bounded for the degree. Also an element f of $\mathrm{Bir}(\mathbb{P}^2)$ is called algebraic if it belongs to an algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$. Equivalently this means that the sequence $n \mapsto \deg f^n$, $n \in \mathbb{Z}_{\geq 0}$, is bounded. For elements in $\mathrm{PGL}_2(\mathbb{C}(y))$, we will use the following characterisation of algebraic elements by Cerveau–Déserti [10, Theorem A].

Lemma 7.3. *Let A be an element of $\mathrm{GL}_2(\mathbb{C}(y))$ and let $\bar{A}$ be its class in $\mathrm{PGL}_2(\mathbb{C}(y))$. Then, the element $\bar{A}$ of $\mathrm{PGL}_2(\mathbb{C}(y)) \subseteq \mathrm{Bir}(\mathbb{P}^2)$ is algebraic if and only if its Baum-Bott index*

$$\mathrm{BB}(\bar{A}) := \mathrm{tr}^2(A)/\det(A)$$

belongs to $\mathbb{C}$.

We now describe the algebraic elements of $\mathrm{PGL}_2(\mathbb{C}(y))$ up to conjugation:

Lemma 7.4. *Any algebraic element of $\mathrm{PGL}_2(\mathbb{C}(y))$ is conjugate to one of the following elements:*

(1) $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where $a \in \mathbb{C}^*$;
(2) $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$;
(3) $\left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right)$, where f is a nonsquare element of $\mathbb{C}(y)$.

Proof. Let $A \in \mathrm{GL}_2(\mathbb{C}(y))$ be an element whose class $\bar{A}$ in $\mathrm{PGL}_2(\mathbb{C}(y))$ is algebraic. We then have $\mathrm{BB}(\bar{A}) \in \mathbb{C}$. If $\mathrm{BB}(\bar{A}) = 0$, we get $\mathrm{tr}(A) = 0$ and $(\bar{A})^2 = \mathrm{id}$. Since A is not a homothety, there exists a vector $u \in (\mathbb{C}(y))^2$ such that u and $v := Au$ are linearly independent. If $P = (u, v) \in \mathrm{GL}_2(\mathbb{C}(y))$ is the element whose first (resp. second) column is u (resp. v), we have $P^{-1}AP = \left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right)$ for some nonzero element f of $\mathbb{C}(y)$. If f is a square, then the class of this matrix is conjugate to $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathrm{PGL}_2(\mathbb{C}(y))$ and we are in case (1). If f is not a square, we are in case (3). Assume now that $\mathrm{BB}(\bar{A}) \in \mathbb{C}^*$. As in the proof of Lemma 7.3, up to dividing the matrix $A \in \mathrm{GL}_2(\mathbb{C}(y))$ by $\mathrm{tr}(A)$, we may assume that $\mathrm{tr}(A)$ and $\det(A) \in \mathbb{C}$. Hence, the eigenvalues of A are complex numbers, and A is conjugate to a Jordan matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$, where $a, b \in \mathbb{C}^*$. In the first case, the class of this element is equal to the class of $\left(\begin{smallmatrix} a/b & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and we are in case (1). In the second case, the matrix $\left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$ is conjugate to the matrix $\left(\begin{smallmatrix} a & a \\ 0 & a \end{smallmatrix}\right)$ whose class is equal to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathrm{PGL}_2(\mathbb{C}(y))$. We are in case (2). □

Lemma 7.5. *The group $\mathbb{T}_f \subseteq \mathrm{Bir}(\mathbb{P}^2)$ contains exactly two algebraic elements which correspond to the elements $\left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right)$ with $ab = 0$. If $b = 0$, we get the identity element of $\mathrm{Bir}(\mathbb{P}^2)$ and if $a = 0$, we get the involution*

$$\iota_f \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (x, y) \dashrightarrow (f(y)/x, y).$$

Proof. We apply the criterion for algebraicity given in Lemma 7.3. If $A := \left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right)$ with $ab \neq 0$, then we have

$$\mathrm{BB}(A) = \frac{\mathrm{tr}^2(A)}{\det(A)} = \frac{4a^2}{a^2 - b^2 f} = \frac{4}{1 - \frac{b^2 f}{a^2}}$$

and this element is nonconstant because otherwise we would have $b^2 f/a^2 \in \mathbb{C}^*$, proving that f is a square. A contradiction. □

The previous result directly implies the following one:

Lemma 7.6. *The group $\mathbb{T}_f$ contains no nontrivial connected closed algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$.*

Lemma 7.6 gives us:

Lemma 7.7. *We have $\mathrm{rk}(\mathbb{T}_f) = 0$.*

Proposition 7.8. *Let f be a nonsquare element of $\mathbb{C}(y)$. Then we have*

$$\mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_f) = \mathrm{Cent}_{\mathrm{Jonq}}(\iota_f).$$

Proof. The inclusion $\mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_f) \subseteq \mathrm{Cent}_{\mathrm{Jonq}}(\iota_f)$ directly follows from Lemma 5.6.

Let now φ be an element of $\mathrm{Cent}_{\mathrm{Jonq}}(\iota_f)$. Note that $\varphi\mathbb{T}_f\varphi^{-1}$ is a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$. For seeing it, we can write $\varphi = uv$ with $u \in \mathrm{PGL}_2(\mathbb{C}(y))$, $v \in \mathrm{PGL}_2$, and observe that $\varphi\mathbb{T}_f\varphi^{-1} = uv\mathbb{T}_f v^{-1}u^{-1} = u\mathbb{T}_{f\circ v^{-1}}u^{-1}$. Since $\varphi\mathbb{T}_f\varphi^{-1}$ and $\mathbb{T}_f$ are two Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y))$ whose intersection is nontrivial (because it contains ι_f), they are equal by Proposition 5.12. Hence we have shown that φ belongs to $\mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_f)$ and this proves the inclusion $\mathrm{Cent}_{\mathrm{Jonq}}(\iota_f) \subseteq \mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_f)$. □

For later use, we prove the following basic result.

Lemma 7.9. *Let $\alpha \in \mathrm{Jonq}$ be a Jonquières transformation. The following assertions are equivalent:*

(1) α preserves a unique rational fibration;
(2) $\forall\, \varphi \in \mathrm{Bir}(\mathbb{P}^2), \quad \varphi\alpha\varphi^{-1} \in \mathrm{Jonq} \implies \varphi \in \mathrm{Jonq}$.

Proof. Let's note that the rational fibration $\pi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ is preserved by α if and only if the rational fibration $\pi\varphi^{-1}$ is preserved by $\varphi\alpha\varphi^{-1}$.

Set $\Pi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $[x : y : z] \dashrightarrow [y : z]$. Note that Π corresponds to the fibration $y = \mathrm{const}$.

(1) $\implies$ (2) Let $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ be such that $\varphi\alpha\varphi^{-1} \in \mathrm{Jonq}$. This means that there exists $\beta \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\Pi(\varphi\alpha\varphi^{-1}) = \beta\Pi$. This is equivalent to $(\Pi\varphi)\alpha = \beta(\Pi\varphi)$. Hence $\Pi\varphi$ is preserved by α. Therefore, there exists $\beta' \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\Pi\varphi = \beta\Pi$ and $\varphi \in \mathrm{Jonq}$.

(2) $\implies$ (1) Assume that $\pi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ is a rational fibration preserved by α. By Definition 3.2(2) there exists $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ satisfying $\pi = \Pi\varphi$ and $\varphi\alpha\varphi^{-1} \in \mathrm{Jonq}$. The assumption we've made yields $\varphi \in \mathrm{Jonq}$, i.e. $\Pi\varphi = \beta\Pi$ for some $\beta \in \mathrm{Aut}(\mathbb{P}^1)$. We obtain $\pi = \beta\Pi$ which shows that π is equivalent to Π. □

Following the literature a non algebraic element of Jonq will be called a Jonquières twist.

Lemma 7.10. *Let φ be an element of $\mathrm{Bir}(\mathbb{P}^2)$. Then, the following assertions are equivalent:*

(1) We have $\varphi \in \mathrm{Jonq}$.
(2) We have $\varphi\mathbb{T}_f\varphi^{-1} \subseteq \mathrm{PGL}_2(\mathbb{C}(y))$.
(3) We have $\varphi\mathbb{T}_f\varphi^{-1} \subseteq \mathrm{Jonq}$.
(4) $\exists\, t \in \mathbb{T}_f \setminus \{\mathrm{id}, \iota_f\}$ such that $\varphi t\varphi^{-1} \in \mathrm{Jonq}$.
(5) There exists a Jonquières twist $t \in \mathrm{Jonq}$ such that $\varphi t\varphi^{-1} \in \mathrm{Jonq}$.

Proof. (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) is obvious.
(4) $\implies$ (5) is a consequence of Lemma 7.5.
(5) $\implies$ (1). By [13, Lemma 4.5] a Jonquières twist preserves a unique rational fibration. Hence the result follows from Lemma 7.9. □

Lemma 7.10 directly yields the next result:

Proposition 7.11. *The group* Jonq *is equal to its own normaliser in* $\mathrm{Bir}(\mathbb{P}^2)$.

Consider the embedding $\mathbb{C}(y)^* \hookrightarrow \mathrm{PGL}_2(\mathbb{C}(y))$, $\lambda \mapsto d_\lambda = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right) = (\lambda(y)x, y)$.

Any element $\varphi \in \mathbb{C}(y)^* \rtimes \mathrm{PGL}_2 \subseteq \mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2 = \mathrm{Jonq}$ can be (uniquely) written $\varphi = \mu \circ d_\lambda$, where $\mu = (x, \mu(y)) \in \mathrm{PGL}_2$ and $\lambda \in \mathbb{C}(y)^*$. Equivalently, we have $\varphi = (\lambda(y)x, \mu(y))$. We then have:

$$\varphi \circ \mathbb{T}_f \circ \varphi^{-1} = \mu \circ d_\lambda \circ \mathbb{T}_f \circ (d_\lambda)^{-1} \circ \mu^{-1} = \mu \circ \mathbb{T}_{\lambda^2 f} \circ \mu^{-1} = \mathbb{T}_{(\lambda^2 f)\circ \mu^{-1}}. \tag{4}$$

We summarise this computation in the following statement

Lemma 7.12. *Considering the action of* $\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2$ *on* $\mathbb{C}(y)$ *given by*

$$\forall\, \varphi = (\lambda, \mu) \in \mathbb{C}(y)^* \rtimes \mathrm{PGL}_2,\ \forall\, f \in \mathbb{C}(y), \quad \varphi.f := (\lambda^2 f) \circ \mu^{-1},$$

we have $\varphi \mathbb{T}_f \varphi^{-1} = \mathbb{T}_{\varphi.f}$ *for any nonsquare element* f *of* $\mathbb{C}(y)$.

Definition 7.13. *If* $f \in \mathbb{C}(y)$ *is a nonzero rational function, define its odd support* $\mathrm{S}_{\mathrm{odd}}(f)$ *as the support of the divisor* $\overline{\mathrm{div}(f)}$ *where* $\mathrm{div}(f) \in \mathrm{Div}(\mathbb{P}^1)$ *is the usual divisor of* f*, and* $\overline{\mathrm{div}(f)}$ *denotes its image by the canonical map* $\mathrm{Div}(\mathbb{P}^1) \to \mathrm{Div}(\mathbb{P}^1) \otimes_{\mathbb{Z}} \mathbb{Z}_2$.

Alternatively, if $\mathrm{v}_a(f)$ *is the order of vanishing of* f *at the point* a *(counted positively if* f *actually vanishes at* a *and negatively if* f *admits a pole at* a*), we have*

$$\mathrm{S}_{\mathrm{odd}}(f) = \{a \in \mathbb{P}^1,\ \mathrm{v}_a(f) \text{ is odd}\}.$$

If $f(y) = c\prod_i (y - a_i)^{n_i}$, where $c \in \mathbb{C}^*$, the a_i are distinct complex numbers, and the n_i are integers, we have

$$\mathrm{S}_{\mathrm{odd}}(f) = \{a_i,\ n_i \text{ is odd}\} \qquad \text{if } \sum_i n_i \text{ is even, and}$$

$$\mathrm{S}_{\mathrm{odd}}(f) = \{a_i,\ n_i \text{ is odd}\} \cup \{\infty\} \qquad \text{if } \sum_i n_i \text{ is odd.}$$

Note that $\mathrm{S}_{\mathrm{odd}}(f) = \emptyset$ if and only if f is a square. When f is not a square, let $\mathscr{g}$ denote the genus of the curve $x^2 = f(y)$. The following formula is a well-known consequence of the Riemann-Hurwitz formula:

$$2\mathscr{g} + 2 = |\,\mathrm{S}_{\mathrm{odd}}(f)|.$$

We will constantly use the following straightforward lemma:

Lemma 7.14. *Let* $f, g \in \mathbb{C}(y)^*$ *and let* $v \in \mathrm{PGL}_2$ *be a homography. Then, the following assertions are equivalent:*

(1) We have $\dfrac{f(v(y))}{g(y)} = \lambda^2(y)$ *for some* $\lambda \in \mathbb{C}(y)$.

(2) We have $v(\mathrm{S}_{\mathrm{odd}}(g)) = \mathrm{S}_{\mathrm{odd}}(f)$.

Proposition 7.15. *Let* f, g *be nonsquare elements of* $\mathbb{C}(y)$*. Then, the following assertions are equivalent:*

(1) $\mathbb{T}_f$ *and* $\mathbb{T}_g$ *are conjugate in* $\mathrm{Bir}(\mathbb{P}^2)$*;*
(2) $\mathbb{T}_f$ *and* $\mathbb{T}_g$ *are conjugate in* Jonq*;*
(3) $\mathbb{T}_f$ *and* $\mathbb{T}_g$ *are conjugate by an element of* $\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2$*;*
(4) f *and* g *are in the same orbit for the action of* $\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2$ *on* $\mathbb{C}(y)$*;*

(5) *The involutions ι_f and ι_g are conjugate in* $\mathrm{Bir}(\mathbb{P}^2)$*;*
(6) *The fields* $\mathbb{C}(y)[\sqrt{f}]$ *and* $\mathbb{C}(y)[\sqrt{g}]$ *are* $\mathbb{C}$*-isomorphic;*
(7) *There exists* $\mu \in \mathrm{Aut}(\mathbb{P}^1)$ *such that* $\mathrm{S}_{\mathrm{odd}}(g) = \mu\big(\mathrm{S}_{\mathrm{odd}}(f)\big)$*;*
(8) *The hyperelliptic curves associated with* $x^2 = f(y)$ *and* $x^2 = g(y)$ *are isomorphic.*

Proof. (1) $\Longrightarrow$ (2) If $\varphi\, \mathbb{T}_f\, \varphi^{-1} = \mathbb{T}_g$ for some $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$, it follows by Lemma 7.10 that $\varphi \in \mathrm{Jonq}$, since $\mathbb{T}_g \subset \mathrm{PGL}_2(\mathbb{C}(y))$.

(2) $\Longrightarrow$ (3) If $\mu\varphi\, \mathbb{T}_f\, \varphi^{-1}\mu^{-1} = \mathbb{T}_g$ for some $\mu \in \mathrm{PGL}_2$ and $\varphi = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{C}(y))$, there exists an element $\left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right) \in \mathbb{T}_f$ such that

$$\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)\left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)^{-1} = \mu^{-1}\left(\begin{smallmatrix} 0 & g \\ 1 & 0 \end{smallmatrix}\right)\mu$$

Comparing traces, we obtain $a = 0$ so that $\left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right)$. It follows that

$$\left(\begin{smallmatrix} \delta(\mu^{-1}.g) & \gamma f(\mu^{-1}.g) \\ \beta & \alpha f \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right).$$

Thus $\beta = 0$ if and only if $\gamma = 0$. If $\beta = \gamma = 0$ we have $\mu \circ \varphi \in \mathbb{C}(y)^* \rtimes \mathrm{PGL}_2$ as desired, so we may assume that β and γ are both nonzero. Then $\beta/\gamma = \gamma f(\mu^{-1}.g)/\beta$ so that $f(\mu^{-1}.g)$ is a square. This is equivalent to $\mu^{-1}.g/f$ being a square, so $\mu^{-1}.g = fh^2$ for some $h \in \mathbb{C}(y)$. It follows that $d_h \mathbb{T}_f (d_h)^{-1} = \mathbb{T}_{\mu^{-1}.g}$ and hence $\mu d_h \mathbb{T}_f (d_h)^{-1}\mu^{-1} = \mathbb{T}_g$.

(3) $\Longrightarrow$ (4) By assumption we have $\mu\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right)\mu^{-1} = \left(\begin{smallmatrix} 0 & g \\ 1 & 0 \end{smallmatrix}\right)$ for some $\mu \in \mathrm{PGL}_2$, $\lambda \in \mathbb{C}(y)^*$ and $\left(\begin{smallmatrix} a & bf \\ b & a \end{smallmatrix}\right) \in \mathbb{T}_f$. Comparing traces, we obtain $a = 0$ and it follows that $(\lambda^2 f) \circ \mu^{-1} = g$. Hence $\varphi.f = g$ with $\varphi = \mu \circ d_\lambda \in \mathbb{C}(y)^* \rtimes \mathrm{PGL}_2$.

(4) $\Longrightarrow$ (1) We have $\varphi\, \mathbb{T}_f\, \varphi^{-1} = \mathbb{T}_{\varphi.f}$ for $\varphi \in \mathbb{C}(y)^* \rtimes \mathrm{PGL}_2$ and $f \in \mathbb{C}(y)$. It is straightforward to check that the set of squares in $\mathbb{C}(y)$ is invariant for the $\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2$-action on $\mathbb{C}(y)$, so if f is not a square, then neither is $\varphi.f$.

(1) $\Longrightarrow$ (5) This follows from Lemma 5.6.

(5) $\Longrightarrow$ (6) Since the set of fixed points of the involution ι_f is the curve $x^2 = f(y)$, the conclusion follows from Lemma 7.18 below (see also Definition 7.17).

(6) $\Longrightarrow$ (7) Denote by $\pi_f \colon C_f \to \mathbb{P}^1$ the $2:1$ morphism corresponding to the inclusion $\mathbb{C}(y) \subseteq \mathbb{C}(y)[\sqrt{f}]$. Note that $\mathrm{S}_{\mathrm{odd}}(f)$ is equal to the ramification locus of π_f. We will consider three cases depending on the genus $\mathscr{g}$ of C_f.

If $\mathscr{g} \geq 2$, then it is well-known that π_f is the only $2:1$ morphism to $\mathbb{P}^1$ up to left composition by an automorphism of $\mathbb{P}^1$ (see [14, Theorem III.7.3, page 101]). If we take $\varphi \colon C_f \to C_g$ to be any isomorphism, then there exists an automorphism $\mu \in \mathrm{Aut}(\mathbb{P}^1)$ making the following diagram commute:

$$\begin{tikzcd} C_f \arrow[r, "\varphi"] \arrow[d, "\pi_f"'] & C_g \arrow[d, "\pi_g"] \\ \mathbb{P}^1 \arrow[r, "\mu"] & \mathbb{P}^1 \end{tikzcd}$$

The equality $\pi_g\varphi = \mu\pi_f$ shows that π_g and $\mu\pi_f$ have the same ramification, i.e. $\mathrm{S}_{\mathrm{odd}}(g) = \mathrm{S}_{\mathrm{odd}}(f\mu^{-1})$, i.e. $\mathrm{S}_{\mathrm{odd}}(g) = \mu\big(\mathrm{S}_{\mathrm{odd}}(f)\big)$.

If $g = 1$, it is well-known that if the two elliptic curves C_f and C_g are isomorphic, then there exists an automorphism of $\mathbb{P}^1$ sending the 4 ramification points of π_f onto the 4 ramification points of π_g (see [17, (IV, 4.4), page 318]).

If $g = 0$, then, the two ramification loci of π_f and π_g have 2 elements. It is clear that there exists an automorphism of $\mathbb{P}^1$ sending the first ramification locus onto the other.

(4) $\Longleftrightarrow$ (7) is a direct consequence of Lemma 7.14

(6) $\Longleftrightarrow$ (8) It is well known that two projective smooth curves are isomorphic if and only if their function fields are isomorphic. □

The following lemma is an easy consequence of Proposition 7.15.

Lemma 7.16. *Let $f \in \mathbb{C}(y)$ be a nonsquare rational function. Then there exists a monic squarefree polynomial $g \in \mathbb{C}[y]$ of odd degree and divisible by y such that $\mathbb{T}_f$ is conjugate to $\mathbb{T}_g$ in* Jonq.

The following definition already made in [11] is [5, Definition 2.1]:

Definition 7.17 (Normalised fixed curve: NFC)**.** *Let $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ be a nontrivial element of finite order. If no curve of positive genus is fixed (pointwise) by φ, we say that $\mathrm{NFC}(\varphi) = \emptyset$; otherwise φ fixes exactly one curve of positive genus ([1], [11]), and $\mathrm{NFC}(\varphi)$ is then the isomorphism class of the normalisation of this curve.*

The following result (proven by Bayle–Beauville in [1, Proposition 2.7]) is mentioned just after Definition 2.1 in [5]. It shows in particular that an involution φ of $\mathrm{Bir}(\mathbb{P}^2)$ is linearisable, i.e. conjugate to an automorphism of $\mathbb{P}^2$, if and only if $\mathrm{NFC}(\varphi) = \emptyset$.

Lemma 7.18. *Two involutions $\varphi_1, \varphi_2 \in \mathrm{Bir}(\mathbb{P}^2)$ are conjugate if and only if $\mathrm{NFC}(\varphi_1) = \mathrm{NFC}(\varphi_2)$.*

For later use we will now compute the neutral connected component $\mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_f)^\circ$ of the normaliser of $\mathbb{T}_f$ in Jonq (the final result is given in Proposition 7.22 below). This is how the proof goes: we begin by introducing the group $\mathcal{N}_{\mathbb{C}(y)^*\rtimes\mathrm{PGL}_2}(\mathbb{T}_f)$ in Definition 7.19; we then compute its neutral connected component $\mathcal{N}_{\mathbb{C}(y)^*\rtimes\mathrm{PGL}_2}(\mathbb{T}_f)^\circ$ in Lemma 7.20 showing in particular that it is either trivial or isomorphic to $\mathbb{C}^*$; we then prove the equality $\mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_f) = \mathbb{T}_f \rtimes \mathcal{N}_{\mathbb{C}(y)^*\rtimes\mathrm{PGL}_2}(\mathbb{T}_f)$ in Lemma 7.21, from which it will straightforwardly follow that $\mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_f)^\circ = \mathbb{T}_f \rtimes \mathcal{N}_{\mathbb{C}(y)^*\rtimes\mathrm{PGL}_2}(\mathbb{T}_f)^\circ$ in Proposition 7.22.

Note that point (3) of Lemma 7.20 is to be used only later on (in the proof of Proposition 10.1).

Definition 7.19. *Let f be a nonsquare element of $\mathbb{C}(y)$.*

(1) Let $\mathcal{N}_{\mathbb{C}(y)^*\rtimes\mathrm{PGL}_2}(\mathbb{T}_f)$ be the subgroup of elements $\varphi \in \mathbb{C}(y)^* \rtimes \mathrm{PGL}_2 \subseteq \mathrm{Jonq}$ which normalise $\mathbb{T}_f$, i.e. such that $\varphi\mathbb{T}_f\varphi^{-1} = \mathbb{T}_f$.

(2) Let $\mathrm{Stab}(\mathrm{S}_{\mathrm{odd}}(f)) := \{v \in \mathrm{PGL}_2,\ v(\mathrm{S}_{\mathrm{odd}}(f)) = \mathrm{S}_{\mathrm{odd}}(f)\}$ be the subgroup of elements $v \in \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$ which globally preserve $\mathrm{S}_{\mathrm{odd}}(f) \subseteq \mathbb{P}^1$.

(3) Let $\mathrm{Fix}(\mathrm{S}_{\mathrm{odd}}(f)) \subseteq \mathrm{PGL}_2$ be the subgroup of elements $v \in \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$ which preserve pointwise $\mathrm{S}_{\mathrm{odd}}(f) \subseteq \mathbb{P}^1$.

Lemma 7.20. *Let f be a nonsquare element of $\mathbb{C}(y)$ and let $\mathfrak{g}$ denote the genus of the curve $x^2 = f(y)$. Then, the following assertions hold:*

(1) We have the short exact sequence

$$1 \to \{(\pm x, y)\} \to \mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f) \xrightarrow{\mathrm{pr}_2} \mathrm{Stab}(\mathrm{S}_{\mathrm{odd}}(f)) \to 1. \tag{5}$$

(2) The group $\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ$ is either trivial or isomorphic to $\mathbb{C}^*$:

(a) $\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ = \{1\}$ if $\mathfrak{g} \geq 1$, i.e. $|\mathrm{S}_{\mathrm{odd}}(f)| \geq 4$;

(b) $\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ \simeq \mathbb{C}^*$ if $\mathfrak{g} = 0$, i.e. $|\mathrm{S}_{\mathrm{odd}}(f)| = 2$. For $f = y$, we have $\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_y)^\circ = \mathbb{T}_{1,2}$.

(3) In case (b) above the second projection $\mathrm{pr}_2 \colon \mathrm{Jonq} \to \mathrm{PGL}_2$ induces the short exact sequence

$$1 \to \{(\pm x, y)\} \to \mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ \xrightarrow{\mathrm{pr}_2} \mathrm{Fix}(\mathrm{S}_{\mathrm{odd}}(f)) \to 1, \tag{6}$$

where $\mathrm{Fix}(\mathrm{S}_{\mathrm{odd}}(f)) \simeq \mathbb{C}^*$.

Proof. (1) By Lemma 7.12 we have

$$\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f) = \{(\lambda x, v(y)),\ \lambda \in \mathbb{C}(y)^*,\ v \in \mathrm{PGL}_2,\ f \circ v(y) = \lambda^2 f(y)\}.$$

Hence Lemma 7.14 shows that pr_2 induces a surjection $\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f) \to \mathrm{Stab}(\mathrm{S}_{\mathrm{odd}}(f))$ and it is therefore clear that the short exact sequence (5) holds.

(a) If $\mathfrak{g} \geq 1$, i.e. $|\mathrm{S}_{\mathrm{odd}}(f)| \geq 4$, then $\mathrm{Stab}(\mathrm{S}_{\mathrm{odd}}(f))$ is finite, and the short exact sequence (5) shows that $\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)$ is finite. This proves that we have $\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ = \{1\}$.

(b) If $\mathfrak{g} = 0$, i.e. $|\mathrm{S}_{\mathrm{odd}}(f)| = 2$, then y is in the orbit of f under the action of $\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2$ on $\mathbb{C}(y)$ (see Lemma 7.12) so that we may assume $f = y$. We then get

$$\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_y) = \{(\lambda x, \lambda^2 y),\ \lambda \in \mathbb{C}^*\} \cup \{(\lambda y^{-1} x, \lambda^2 y^{-1}),\ \lambda \in \mathbb{C}^*\},$$
$$\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_y)^\circ = \{(\lambda x, \lambda^2 y),\ \lambda \in \mathbb{C}^*\} = \mathbb{T}_{1,2},$$
$$\mathrm{Fix}(\mathrm{S}_{\mathrm{odd}}(y)) = \mathrm{Fix}(\{0, \infty\}) = \{(y \mapsto \mu y),\ \mu \in \mathbb{C}^*\} = \mathbb{T}_{0,1}.$$

This shows (b).

(3) Still in case (b) the above computation shows that the short exact sequence (6) holds, and the isomorphism $\mathrm{Fix}(\mathrm{S}_{\mathrm{odd}}(f)) \simeq \mathbb{C}^*$ is clear. □

We have $\mathbb{T}_f \cap \mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f) = \{\mathrm{id}\}$ and the group $\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)$ normalises $\mathbb{T}_f$ (by definition of $\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)$), hence we have a semidirect product structure $\mathbb{T}_f \rtimes \mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)$. Note that this semidirect product structure is not induced by the semidirect product $\mathrm{Jonq} = \mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2$ which is usually considered in this paper.

Lemma 7.21. *Let $f \in \mathbb{C}(y)$ be a nonsquare element. Then the normaliser of $\mathbb{T}_f$ in* Jonq *is equal to*

$$\mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_f) = \mathbb{T}_f \rtimes \mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f).$$

Proof. The inclusion $\mathbb{T}_f \rtimes \mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f) \subseteq \mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)$ being clear, it is enough to prove $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2}(\mathbb{T}_f) \subseteq \mathbb{T}_f \rtimes \mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)$. Take $g \in \mathrm{Jonq}$ such that $g \mathbb{T}_f g^{-1} = \mathbb{T}_f$. Let's begin by proving that $\mathrm{pr}_2(g)$ belongs to $\mathrm{Stab}(\mathrm{S}_{\mathrm{odd}}(f))$ where $\mathrm{pr}_2 \colon \mathrm{Jonq} \to \mathrm{PGL}_2$ denotes the second projection. Writing $g = uv$ with $u \in$

$\mathrm{PGL}_2(\mathbb{C}(y))$, $v \in \mathrm{PGL}_2$, we have $\mathrm{pr}_2(g) = v$ and we want to prove that $v(\mathrm{S}_{\mathrm{odd}}(f)) = \mathrm{S}_{\mathrm{odd}}(f)$. Since $g\mathbb{T}_f g^{-1} = u(v\mathbb{T}_f v^{-1})u^{-1} = u(\mathbb{T}_{f\circ v^{-1}})u^{-1} = \mathbb{T}_f$, the tori $\mathbb{T}_{f\circ v^{-1}}$ and $\mathbb{T}_f$ are conjugate in $\mathrm{PGL}_2(\mathbb{C}(y))$. This proves that $f \circ v^{-1} = \lambda^2 f$ for some $\lambda \in \mathbb{C}(y)$ (by Proposition 5.10) and now Lemma 7.14 shows that $v(\mathrm{S}_{\mathrm{odd}}(f)) = \mathrm{S}_{\mathrm{odd}}(f)$.

Since $\mathrm{pr}_2(g) \in \mathrm{Stab}(\mathrm{S}_{\mathrm{odd}}(f))$, the short exact sequence (5) of Lemma 7.20 shows that there exists $g' \in \mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)$ such that $\mathrm{pr}_2(g') = \mathrm{pr}_2(g)$. By definition of $\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)$, g' normalises $\mathbb{T}_f$. Hence $g'' := g(g')^{-1}$ also normalises $\mathbb{T}_f$ and moreover it belongs to $\mathrm{PGL}_2(\mathbb{C}(y))$. This shows that g'' belongs to $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y))}(\mathbb{T}_f) = \mathbb{T}_f \rtimes \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle \subseteq \mathbb{T}_f \rtimes \mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)$ (see Lemma 5.13) and this concludes the proof. □

The following result directly follows from Lemmas 7.20 and 7.21.

Proposition 7.22. *Let f be a nonsquare element of $\mathbb{C}(y)$ and let $\mathfrak{g}$ be the genus of the curve $x^2 = f(y)$. Then we have*

$$\mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_f)^\circ = \mathbb{T}_f \rtimes \mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ.$$

Moreover the group $\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ$ is trivial if $\mathfrak{g} \geq 1$ and isomorphic to $\mathbb{C}^*$ if $\mathfrak{g} = 0$.

Spelling out the details in the two cases, we have:

(1) $\mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_f)^\circ = \mathbb{T}_f$ *if* $\mathfrak{g} \geq 1$;

(2) $\mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_y)^\circ = \mathbb{T}_y \rtimes \mathbb{T}_{1,2}$.

Remark 7.23. *Let f be a nonsquare element of $\mathbb{C}(y)$. Note that the equation $x^2 = f(y)$ defines an affine curve in $\mathbb{A}^2$. The group $\mathbb{T}_f$ fixes this curve pointwise. In the case $f = y$, the group $\mathbb{T}_y \rtimes \mathbb{T}_{1,2}$ stabilises this curve (but not pointwise).*

8. Subgroups of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ conjugate to $\mathbb{T}_{0,1}$

The following technical lemma describes rather explicitly a morphism from a factorial irreducible affine variety W to $\mathrm{Bir}(\mathbb{P}^2)$ whose image is contained in $\mathrm{PGL}_2(\mathbb{C}(y))$. We will use it in the proof of Lemma 8.2.

Lemma 8.1. *Consider the monoid*

$$\mathrm{M}_2(\mathbb{C}[y])_{\det\neq 0} := \{M \in \mathrm{M}_2(\mathbb{C}[y]),\ \det M \neq 0\}$$

and let

$$p\colon \mathrm{M}_2(\mathbb{C}[y])_{\det\neq 0} \to \mathrm{PGL}_2(\mathbb{C}(y)), \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),$$

be the natural surjective monoid morphism. Let $\iota\colon \mathrm{PGL}_2(\mathbb{C}(y)) \hookrightarrow \mathrm{Bir}(\mathbb{P}^2)$ be the natural injection. If W is a factorial irreducible affine variety and $\varphi\colon W \to \mathrm{PGL}_2(\mathbb{C}(y))$ a map such that $\iota \circ \varphi\colon W \to \mathrm{Bir}(\mathbb{P}^2)$ is a morphism in the sense of Definition 2.1, then φ admits a lifting $\widehat{\varphi}\colon\ W \to \mathrm{M}_2(\mathbb{C}[y])_{\det\neq 0}$ (i.e. a map such that $\varphi = p \circ \widehat{\varphi}$) which is a morphism of ind-varieties.[1] *This exactly means that $\widehat{\varphi}$ (or φ!) is of the form*

$$w \mapsto \begin{pmatrix} a(w,y) & b(w,y) \\ c(w,y) & d(w,y) \end{pmatrix},$$

for some $a, b, c, d \in \mathbb{C}[W][y]$ and that for any $w \in W$, the determinant $\det \begin{pmatrix} a(w,y) & b(w,y) \\ c(w,y) & d(w,y) \end{pmatrix}$ is a nonzero element of $\mathbb{C}[y]$.

[1] For the definitions of ind-varieties and morphisms of ind-varieties, see e.g. [15, §1.1].

Proof. By Definition 2.1, $\iota \circ \varphi$ corresponds to a birational map $\psi\colon W \times \mathbb{A}^2 \dashrightarrow W \times \mathbb{A}^2$, $(w,x,y) \dashrightarrow (w, f(w,x,y), y)$, i.e. a $(W \times \mathbb{A}^1_y)$-birational map of the $(W \times \mathbb{A}^1_y)$-variety $(W \times \mathbb{A}^1_y) \times \mathbb{A}^1_x$. Such a birational map corresponds to a $\mathbb{C}(W)(y)$-automorphism of the field $\mathbb{C}(W)(y)(x)$ and hence the rational function $f \in \mathbb{C}(W)(y)(x)$ is necessarily of the form $f\colon (w,x,y) \dashrightarrow \frac{ax+b}{cx+d}$, for some $a,b,c,d \in \mathbb{C}(W)(y)$ which satisfy $ad - bc \neq 0$. Since $\mathbb{C}(W)(y)$ is the field of fractions of $\mathbb{C}[W][y]$ we may assume that a,b,c,d belong to $\mathbb{C}[W][y]$ and since $\mathbb{C}[W][y]$ is factorial, we may even assume that $\gcd(a,b,c,d) = 1$ in $\mathbb{C}[W][y]$. This determines a,b,c,d uniquely up to a common factor in $\mathbb{C}[W]^*$ (the group of invertible elements of $\mathbb{C}[W]$). We could easily check that we have $\gcd(ax+b, cx+d) = 1$ in $\mathbb{C}[W][x,y]$. If h belongs to $\mathbb{C}[W][x,y]$, its zero set is defined by

$$Z(h) = \{(w,x,y) \in W \times \mathbb{A}^2,\ h(w,x,y) = 0\} \subseteq W \times \mathbb{A}^2.$$

Set

$$\begin{aligned} S_1(\psi) &= Z(ad-bc) \cup Z(cx+d) \subset W \times \mathbb{A}^2 \qquad \text{and} \\ S_2(\psi) &= Z(ad-bc) \cup Z(-cx+a) \subset W \times \mathbb{A}^2. \end{aligned}$$

By the previous remark $S_1(\psi)$ and $S_2(\psi)$ only depend on ψ and not on the choice of the functions $a,b,c,d \in \mathbb{C}[W][y]$. It is clear that ψ induces an isomorphism $U \xrightarrow{\sim} V$ where $U := (W \times \mathbb{A}^2) \setminus S_1(\psi)$ and $V := (W \times \mathbb{A}^2) \setminus S_2(\psi)$, the inverse map ψ^{-1} being given by $(w,x,y) \dashrightarrow (w, g(w,x,y), y)$ where $g := \frac{dx-b}{-cx+a}$.

We now show that $\mathrm{Exc}(\psi) = S_1(\psi)$, where the exceptional set $\mathrm{Exc}(\psi) \subset W \times \mathbb{A}^2$, by definition, is the complement of the open set consisting of all points at which ψ induces a local isomorphism. In particular $\mathrm{Exc}(\psi)$ is closed in $W \times \mathbb{A}^2$. We have

$$Z(cx+d) \cap D(ax+b) \subset \mathrm{Exc}(\psi),$$

where $D(h) \subset W \times \mathbb{A}^2$, for a function $h \in \mathbb{C}[W][x,y]$, denotes the principal open subset $(W \times \mathbb{A}^2) \setminus Z(h)$. Since $\gcd(ax+b, cx+d) = 1$, the hypersurfaces $Z(ax+b)$ and $Z(cx+d)$ have no common irreducible components and thus the intersection of $D(ax+b)$ with any irreducible component of $Z(cx+d)$ is nonempty. Hence $D(ax+b) \cap Z(cx+d)$ is dense in $Z(cx+d)$ and since $\mathrm{Exc}(\psi)$ is closed, all of $Z(cx+d)$ is contained in $\mathrm{Exc}(\psi)$. In particular, $Z(c) \cap Z(d) \subset Z(cx+d)$ is contained in $\mathrm{Exc}(\psi)$. It only remains to show that $Z(ad-bc) \setminus (Z(c) \cap Z(d))$ is contained in $\mathrm{Exc}(\psi)$. Let $(w_0,x_0,y_0) \in Z(ad-bc) \setminus (Z(c) \cap Z(d))$, so that $c(w_0,y_0)x + d(w_0,y_0) \in \mathbb{C}[x]$ is nonzero at x_0. Then ψ is defined in (w_0,x_0,y_0) but, since $(w_0,x_0,y_0) \in Z(ad-bc)$, it is constant along the line $\mathbb{A}^1 \simeq \{w_0\} \times \mathbb{A}^1_x \times \{y_0\}$. In particular ψ does not induce a local isomorphism at (w_0,x_0,y_0). We have proven that $\mathrm{Exc}(\psi) = S_1(\psi)$. Similarly $\mathrm{Exc}(\psi^{-1}) = S_2(\psi)$.

The map $\iota \circ \varphi$ being a morphism in the sense of Definition 2.1, the projection of U to W is surjective. In other words, for each $w_0 \in W$, there exists $(x_0,y_0) \in \mathbb{A}^2$ such that $(w_0,x_0,y_0) \notin Z(ad-bc) \cup Z(cx+d)$. In particular we have $\begin{pmatrix} a(w_0,y) & b(w_0,y) \\ c(w_0,y) & d(w_0,y) \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C}(y))$. Hence the correspondence $w \mapsto \begin{pmatrix} a(w,y) & b(w,y) \\ c(w,y) & d(w,y) \end{pmatrix}$ actually defines a morphism of ind-varieties $\widehat{\varphi} : W \to \mathrm{M}_2(\mathbb{C}[y])_{\det \neq 0}$ which satisfies $\varphi = p \circ \widehat{\varphi}$. $\square$

Lemma 8.2. *Let G be an algebraic subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ isomorphic to the multiplicative algebraic group $(\mathbb{C}^*, \times)$. Then, the following assertions are equivalent:*

(1) The map $\mathrm{pr}_2\colon \mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1 \to \mathrm{Aff}_1$ *induces an isomorphism* $G \to \mathrm{pr}_2(G)$*;*
(2) The groups G *and* $\mathbb{T}_{0,1}$ *are conjugate.*

Proof. (2) $\Longrightarrow$ (1) is clear. Let's prove (1) $\Longrightarrow$ (2). If pr_2 induces an isomorphism $G \to \mathrm{pr}_2(G)$, then, up to conjugation by an element of $\mathrm{Aff}_1 \subseteq \mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ we have $\mathrm{pr}_2(G) = \{\lambda y,\ \lambda \in \mathbb{C}^*\}$. Hence, there is a morphism $s\colon \mathbb{C}^* \to \mathrm{PGL}_2(\mathbb{C}(y))$, $\lambda \mapsto s_\lambda$, such that $B = \{(s_\lambda(y), \lambda y),\ \lambda \in \mathbb{C}^*\}$. Actually, by Lemma 8.1, there exist polynomials $a, b, c, d \in \mathbb{C}[\lambda^{\pm 1}][y]$ such that $s_\lambda(y) = \left(\begin{smallmatrix} a(\lambda,y) & b(\lambda,y) \\ c(\lambda,y) & d(\lambda,y)\end{smallmatrix}\right)$. Since we have $(s_{\lambda\mu}(y)x, \lambda\mu y) = (s_\lambda(y)x, \lambda y) \circ (s_\mu(y)x, \mu y)$ this yields $s_{\lambda\mu}(y) = s_\lambda(\mu y)s_\mu(y)$. For all $\lambda \in \mathbb{C}^*$ we have $a(\lambda, y)d(\lambda, y) - b(\lambda, y)c(\lambda, y) \neq 0$ in $\mathbb{C}[y]$. We may therefore choose $y_0 \in \mathbb{C} \setminus \{0\}$ such that $a(\lambda, y_0)d(\lambda, y_0) - b(\lambda, y_0)c(\lambda, y_0) \neq 0$ in $\mathbb{C}[\lambda^{\pm 1}]$. It follows that $s_y(y_0) \in \mathrm{PGL}_2(\mathbb{C}(y))$ and because $y_0 \neq 0$ we can replace y by yy_0^{-1} to obtain an element $t(y) := s_{yy_0^{-1}}(y_0)$ which belongs to $\mathrm{PGL}_2(\mathbb{C}(y))$ as well. Let $\varphi := (t(y)x, y) \in \mathrm{PGL}_2(\mathbb{C}(y)) \subseteq \mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$. Then we have $s_{\lambda yy_0^{-1}}(y_0) = s_\lambda(y)s_{yy_0^{-1}}(y_0)$, i.e. $t(\lambda y) = s_\lambda(y)t(y)$, i.e. $t(\lambda y)t(y)^{-1} = s_\lambda(y)$, i.e.

$$(s_\lambda(y), \lambda y) = \varphi \circ (x, \lambda y) \circ \varphi^{-1},$$

showing that up to conjugation we have $B = \mathbb{T}_{0,1}$. $\square$

9. Embeddings of $(\mathbb{C}, +)$ into Jonq

The aim of this section is to prove the proposition below which describes the different embeddings of the additive group $(\mathbb{C}, +)$ into Jonq up to conjugation.

Proposition 9.1. *Any algebraic subgroup of* Jonq *isomorphic to the additive algebraic group* $(\mathbb{C}, +)$ *is either conjugate to* $\mathbb{U}_1 := \{(x+c, y),\ c \in \mathbb{C}\}$ *or to* $\mathbb{U}_2 := \{(x, y+c),\ c \in \mathbb{C}\}$*.*

Our proof relies on the following characterisation of the embeddings of $(\mathbb{C}, +)$ into Jonq which are conjugate to $\mathbb{U}_2$.

Lemma 9.2. *Let* G *be an algebraic subgroup of* Jonq *isomorphic to the additive algebraic group* $(\mathbb{C}, +)$*. Then, the following assertions are equivalent:*

(1) The group $\mathrm{pr}_2(G)$ *is nontrivial (where* $\mathrm{pr}_2\colon \mathrm{Jonq} \to \mathrm{PGL}_2$ *is the second projection).*
(2) The groups G *and* $\mathbb{U}_2$ *are conjugate by an element of* $\mathrm{PGL}_2(\mathbb{C}(y)) \subseteq \mathrm{Jonq}$.

Proof. (2) $\Longrightarrow$ (1) is clear. Let's prove (1) $\Longrightarrow$ (2). If $\mathrm{pr}_2(G)$ is nontrivial, pr_2 induces an isomorphism from G onto its image $\mathbb{U}_2 \simeq (\mathbb{C}, +)$. Up to conjugation the inverse isomorphism $(\mathbb{C}, +) \to G$ is of the form $\theta\colon c \mapsto (A_c, y + c) \in \mathrm{Jonq}$, where the map $c \mapsto A_c$ is defined as a morphism $\mathbb{C} \to \mathrm{Bir}(\mathbb{P}^2)$ (see Definition 2.1) with values in $\mathrm{PGL}_2(\mathbb{C}(y))$. By Lemma 8.1, there exist polynomials $a(t, y), b(t, y), c(t, y), d(t, y) \in \mathbb{C}[t, y]$ such that

$$\forall\, t \in \mathbb{A}^1,\ A_t(y) = \left(\begin{smallmatrix} a(t,y) & b(t,y) \\ c(t,y) & d(t,y)\end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{C}(y)).$$

In particular, the determinant

$$\Delta(t,y) = \det\begin{pmatrix} a(t,y) & b(t,y) \\ c(t,y) & d(t,y)\end{pmatrix} \in \mathbb{C}[t,y]$$

is such that $\Delta(t_0,y)$ is a nonzero element of $\mathbb{C}[y]$ for each $t_0 \in \mathbb{A}^1$.

The equality $\theta(t+u) = \theta(t)\circ\theta(u)$ yields

$$A_{t+u}(y) = A_t(y+u)A_u(y). \tag{7}$$

Setting $t=-u$, we get:

$$\mathrm{I}_2 = A_0(y) = A_{-u}(y+u)A_u(y), \quad \text{i.e.}$$

$$\begin{pmatrix} a(0,y) & b(0,y) \\ c(0,y) & d(0,y)\end{pmatrix} = \begin{pmatrix} a(-u,y+u) & b(-u,y+u) \\ c(-u,y+u) & d(-u,y+u)\end{pmatrix}\begin{pmatrix} a(u,y) & b(u,y) \\ c(u,y) & d(u,y)\end{pmatrix}.$$

Choosing $y_0 \in \mathbb{C}$ such that $\Delta(0,y_0)\neq 0$ and replacing y with y_0 in the latter equality, proves that $A_u(y_0)$ defines an element of $\mathrm{PGL}_2(\mathbb{C}(u))$ (Example 9.3 below shows that the polynomial $\Delta(0,y_0)$ may actually vanish for some values of y_0). Setting $y=y_0$ in (7), we get:

$$A_{t+u}(y_0) = A_t(u+y_0)A_u(y_0),$$

i.e. $A_t(u+y_0) = A_{t+u}(y_0)A_u(y_0)^{-1}$. Replacing u with $u-y_0$ gives

$$A_t(u) = A_{u+t-y_0}(y_0)A_{u-y_0}(y_0)^{-1}.$$

Writing $B(u) := A_{u-y_0}(y_0) \in \mathrm{PGL}_2(\mathbb{C}(u))$, we get $A_t(u) = B(u+t)B(u)^{-1}$, i.e. (replacing u with y) $A_t(y) = B(y+t)B(y)^{-1}$, or equivalently

$$A_c(y) = B(y+c)B(y)^{-1}. \tag{8}$$

Setting $\varphi := (B(y),y) \in \mathrm{PGL}_2(\mathbb{C}(y)) \subseteq \mathrm{Jonq}$, the equation (8) shows that

$$\theta(c) = \varphi\circ(x,y+c)\circ\varphi^{-1}. \qquad \square$$

Example 9.3. *Set* $\tilde{A}_t(y) := \begin{pmatrix} (y+t+1)(y-1) & -t \\ 0 & (y+t-1)(y+1)\end{pmatrix} \in \mathrm{M}_2(\mathbb{C}[t,y])$ *and* $A_t(y) := [\tilde{A}_t(y)] \in \mathrm{PGL}_2(\mathbb{C}(y))$. *We have*

$$(y+u+1)(y+u-1)\tilde{A}_{t+u}(y) = \tilde{A}_t(y+u)\tilde{A}_u(y),$$

showing that the equation (7) *above is satisfied. Moreover, we have*

$$\Delta(t,y) := \det\tilde{A}_t(y) = (y+t+1)(y+t-1)(y+1)(y-1)$$

showing that $\Delta(0,y_0)\neq 0$ *if and only if* $y_0 \neq \pm 1$. *Writing* $y_0 = 0$ *and setting as in the proof of Lemma 9.2*

$$B(u) := A_{u-y_0}(y_0) = A_u(0) = \begin{pmatrix} -1-u & -u \\ 0 & -1+u\end{pmatrix} = \begin{pmatrix} 1+u & u \\ 0 & 1-u\end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C}(u)),$$

we get $B(y) = \begin{pmatrix} 1+y & y \\ 0 & 1-y\end{pmatrix}$ *and* $A_c(y) = B(y+c)B(y)^{-1}$.

Proof of Proposition 9.1. Let G be an algebraic subgroup of Jonq isomorphic to $(\mathbb{C},+)$. If $\mathrm{pr}_2(G)$ is nontrivial, then the conclusion follows from Lemma 9.2. We may therefore assume that G is contained in $\mathrm{PGL}_2(\mathbb{C}(y))$. By Lemma 7.4, we may assume (up to conjugation) that G contains the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C}(y))$ from which it follows that $G = \mathbb{U}_1$. $\square$

10. Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y))$ and of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$

In this section, we will prove that any Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$ is contained in a unique Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ and conversely that any Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ contains a unique Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$. Hence, there will be a natural bijection from the set of Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y))$ to the set of Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ (see Theorem 10.9).

The main difficulty is to prove that any Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ contains at least one Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$ and we will begin by showing this result which is Theorem 10.7 below. The proof relies on a few preliminary results.

We will use the next technical result for the proof of both Theorem 10.7 and Proposition 10.11.

Proposition 10.1. *Let f be a nonsquare element of $\mathbb{C}(y)$. Then we have*

$$\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y))\rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ = \mathbb{T}_f \rtimes \mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ \tag{9}$$

where $\mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{Aff}_1}(\mathbb{T}_f)$ is the subgroup of elements $\varphi \in \mathbb{C}(y)^* \rtimes \mathrm{Aff}_1 \subseteq \mathrm{Jonq}$ which normalise $\mathbb{T}_f$, i.e. such that $\varphi\mathbb{T}_f\varphi^{-1} = \mathbb{T}_f$. Moreover, the group $\mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ$ is either trivial or isomorphic to $\mathbb{C}^*$. More precisely, if g denotes the genus of the curve $x^2 = f(y)$, we have

(1) $\mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ = \{\mathrm{id}\}$ if $g \geq 1$ or $\infty \notin \mathrm{S}_{\mathrm{odd}}(f)$;
(2) $\mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ = \mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ \simeq \mathbb{C}^*$ if $g = 0$ and $\infty \in \mathrm{S}_{\mathrm{odd}}(f)$.

Proof. We have

$$\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y))\rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ = (\mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_f)^\circ \cap [\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1])^\circ$$

and $\mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_f)^\circ = \mathbb{T}_f \rtimes \mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ$ by Proposition 7.22. Hence we get

$$\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y))\rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ = \mathbb{T}_f \rtimes (\mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ \cap [\mathbb{C}(y)^* \rtimes \mathrm{Aff}_1])^\circ$$

and the equality $\mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ = (\mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ \cap [\mathbb{C}(y)^* \rtimes \mathrm{Aff}_1])^\circ$ establishes (9).

Let's now show (1). If $g \geq 1$ we have $\mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ = \{1\}$ by Lemma 7.20(a) and this yields $\mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ = \{\mathrm{id}\}$. So, assume now that $g = 0$ and $\infty \notin \mathrm{S}_{\mathrm{odd}}(f)$. Then, the following short exact sequence (which is equation (6) of Lemma 7.20)

$$1 \to \{(\pm x, y)\} \to \mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ \xrightarrow{\mathrm{pr}_2} \mathrm{Fix}(\mathrm{S}_{\mathrm{odd}}(f)) \to 1$$

yields

$$1 \to \{(\pm x, y)\} \to \mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ \cap [\mathbb{C}(y)^* \rtimes \mathrm{Aff}_1] \xrightarrow{\mathrm{pr}_2} \mathrm{Fix}\left(\mathrm{S}_{\mathrm{odd}}(f) \cup \{\infty\}\right) \to 1. \tag{10}$$

The hypothesis $\infty \notin \mathrm{S}_{\mathrm{odd}}(f)$ shows that $\mathrm{S}_{\mathrm{odd}}(f) \cup \{\infty\}$ admits 3 elements and this proves that $\mathrm{Fix}\left(\mathrm{S}_{\mathrm{odd}}(f) \cup \{\infty\}\right)$ is trivial. Hence (10) shows that

$$\mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{PGL}_2}(\mathbb{T}_f)^\circ \cap [\mathbb{C}(y)^* \rtimes \mathrm{Aff}_1] = \{(\pm x, y)\}$$

and finally $\mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ = \{\mathrm{id}\}$.

Let's now show (2). So assume that we have $g = 0$ and $\infty \in \mathrm{S}_{\mathrm{odd}}(f)$. We have therefore $\mathrm{Fix}(\mathrm{S}_{\mathrm{odd}}(f)) \subseteq \mathrm{Fix}(\infty) = \mathrm{Aff}_1$ and the first exact sequence above (which was equation (6) of Lemma 7.20) shows that $\mathcal{N}_{\mathbb{C}(y)^*\rtimes\mathrm{PGL}_2}(\mathbb{T}_f)^\circ \subseteq \mathbb{C}(y)^* \rtimes \mathrm{Aff}_1$. Hence we have $\mathcal{N}_{\mathbb{C}(y)^*\rtimes\mathrm{Aff}_1}(\mathbb{T}_f)^\circ = \mathcal{N}_{\mathbb{C}(y)^*\rtimes\mathrm{PGL}_2}(\mathbb{T}_f)^\circ$ and the isomorphism $\mathcal{N}_{\mathbb{C}(y)^*\rtimes\mathrm{PGL}_2}(\mathbb{T}_f)^\circ \simeq \mathbb{C}^*$ was already shown in Lemma 7.20(b). ☐

The following few results, culminating in Proposition 10.5, are also in preparation for Theorem 10.7.

Lemma 10.2. *Let G be a closed connected subgroup of $\mathrm{Bir}(\mathbb{P}^2)$, let H be a linear algebraic group isomorphic to $(\mathbb{C},+)$ or $(\mathbb{C}^*,\times)$, and let $\varphi\colon G \to H$ be a morphism of groups such that for each variety A and each morphism $\lambda\colon A \to \mathrm{Bir}(\mathbb{P}^2)$ (see Definition 2.1) with values in G, the composition $\varphi\circ\lambda\colon A \to H$ is a morphism of varieties. Then, we have $\varphi(G) = \{1\}$ or H.*

Proof. Assume that $\varphi(G) \neq \{1\}$. Then, there exist two elements g_1, g_2 of G such that $\varphi(g_1) \neq \varphi(g_2)$. By Lemma 2.10, there exists a connected (not necessarily irreducible) curve C and a morphism $\lambda\colon C \to \mathrm{Bir}(\mathbb{P}^2)$ whose image satisfies

$$g_1, g_2 \in \mathrm{Im}(\lambda) \subseteq G.$$

The image U of the morphism of algebraic varieties $\varphi\circ\lambda\colon C \to H$ is connected, constructible, and contains at least two points. Hence, it is a dense open subset of H. Moreover, U is contained in the image of φ. Recall that if V_1, V_2 are two dense open subsets of a linear algebraic group K, then we have $K = V_1.V_2$ (see e.g. [18, Lemma (7.4), page 54]). Here, we find that the image of φ contains $U.U = H$ and hence is equal to H. ☐

Remark 10.3. *If $\varphi\colon G \to H$ is a morphism of linear algebraic groups, it is well-known that $\varphi(G)$ is a closed subgroup of H. However, if G is a closed subgroup of $\mathrm{Bir}(\mathbb{P}^2)$, H a linear algebraic group, and $\varphi\colon G \to H$ a morphism of groups such that for each variety A and each morphism $\lambda\colon A \to \mathrm{Bir}(\mathbb{P}^2)$ with values in G, the composition $\varphi\circ\lambda\colon A \to H$ is a morphism of varieties, then it is in general false that $\varphi(G)$ is a closed subgroup of H. Take for G the subgroup of monomial transformations*

$$G := \{(x^a y^b, x^c y^d),\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})\} \subseteq \mathrm{Bir}(\mathbb{P}^2)$$

and consider the natural morphism $\varphi\colon G \to \mathrm{GL}_2(\mathbb{C})$, $(x^a y^b, x^c y^d) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then G is a closed subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ whose image by φ is equal to $\mathrm{GL}_2(\mathbb{Z})$, which is not closed in $\mathrm{GL}_2(\mathbb{C})$.

Lemma 10.4. *Let B be a closed connected subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$. Then, $\mathrm{pr}_2(B)$ is a closed connected subgroup of $\mathrm{Aff}_1 \subset \mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$. In particular, up to conjugation, $\mathrm{pr}_2(B)$ is equal to one of the following four subgroups of Aff_1:*

$$\{\mathrm{id}\}, \quad \mathbb{T}_{0,1} = \{(x, ay),\ a \in \mathbb{C}^*\}, \quad \mathbb{U}_2 = \{(x, y+c),\ c \in \mathbb{C}\}, \quad \textit{or} \quad \mathrm{Aff}_1.$$

Proof. Set $H := \mathrm{pr}_2(B)$ and consider the closed subgroup $\overline{H}$ of Aff_1. Up to conjugation, $\overline{H}$ is equal to $\{\mathrm{id}\}$, $\mathbb{T}_{0,1}$, $\mathbb{U}_2$, or Aff_1. In the first three cases, the conclusion follows from Lemma 10.2. Hence, we may assume that $\overline{H} = \mathrm{Aff}_1$. This shows that H is non-abelian, i.e. $D^1(H) \subseteq \mathbb{U}_2$ is nontrivial. Setting $\mathcal{D}(B) := \overline{D(B)} \subseteq B$, this implies that $\mathrm{pr}_2(\mathcal{D}(B)) \subseteq \mathbb{U}_2$ is nontrivial, and so, Lemma 10.2 yields $\mathrm{pr}_2(\mathcal{D}(B)) = \mathbb{U}_2$. In particular, we have $\mathbb{U}_2 \subseteq \mathrm{pr}_2(B)$. Set $\eta\colon \mathrm{Aff}_1 \to \mathrm{Aff}_1/\mathbb{U}_2 = \mathbb{C}^*$, $ay+b \mapsto a$. The group $\eta \circ \mathrm{pr}_2(B) \subseteq \mathbb{C}^*$ being nontrivial, Lemma 10.2 shows that $\eta \circ \mathrm{pr}_2(B) = \mathbb{C}^*$. Since H contains $\mathbb{U}_2$ and since $\eta(H) = \mathbb{C}^*$, we have $H = \mathrm{Aff}_1$. $\square$

Proposition 10.5. *There is no Borel subgroup B of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ such that $\mathrm{pr}_2|_B \colon B \to \mathrm{Aff}_1$ is injective.*

Proof. Suppose for contradiction that B is a Borel subgroup such that $\mathrm{pr}_2|_B \colon B \to \mathrm{Aff}_1$ is injective. By Lemma 10.4 and up to conjugation, we may assume that $\mathrm{pr}_2(B)$ is equal to $\{\mathrm{id}\}, \mathbb{T}_{0,1}$, $\mathbb{U}_2$ or Aff_1. The case $\{\mathrm{id}\}$ is of course excluded. Let's begin by showing that B is bounded. For each $d \geq 1$, set $B_d := \{f \in B,\ \deg f \leq d\}$. By Lemma 2.18(3), $\mathrm{pr}_2(B_d)$ is a constructible subset of PGL_2. Now, since the variety $\mathrm{pr}_2(B)$ is the increasing union of the constructible subsets $\mathrm{pr}_2(B_d)$, $d \geq 1$, there exists an integer d such that $\mathrm{pr}_2(B) = \mathrm{pr}_2(B_d)$, i.e. $B = B_d$ (see e.g. [15, Lemma 1.3.1, page 15]). Hence, we have shown that B is bounded, and it follows that B is an algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ (see Remark 2.17).

If $\mathrm{pr}_2(B) = \mathbb{U}_2$, then by Lemma 9.2 and up to conjugation we may assume that $B = \mathbb{U}_2$. But then B would be strictly contained in the closed connected solvable subgroup $\mathcal{B}_2$ of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$. A contradiction. If $\mathrm{pr}_2(B) = \mathrm{Aff}_1$, then since Aff_1 normalises $\mathbb{U}_2$, B normalises the preimage of $\mathbb{U}_2$ by the isomorphism $B \to \mathrm{Aff}_1$ induced by pr_2. Up to conjugation and by Lemma 9.2 again we may (and will) assume that this latter group is $\mathbb{U}_2$. Hence B is contained in $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y))\rtimes \mathrm{Aff}_1}(\mathbb{U}_2) = \mathrm{PGL}_2 \times \mathrm{Aff}_1$. Since B is a Borel subgroup of $\mathrm{PGL}_2 \times \mathrm{Aff}_1$, this implies that, up to conjugation, we have $B = \mathrm{Aff}_1 \times \mathrm{Aff}_1 = \{(ax + b, cy + d),\ a, b, c, d \in \mathbb{C},\ ac \neq 0\}$. This is again a contradiction since this group is strictly contained in $\mathcal{B}_2$. Finally, if $\mathrm{pr}_2(B) = \mathbb{T}_{0,1}$, we have $B = \mathbb{T}_{0,1}$ up to conjugation, by Lemma 8.2. This is again a contradiction, because this group is clearly not a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ (being strictly contained in $\mathcal{B}_2$). $\square$

Lemma 10.6. *Set $h := (x + 1, y) \in \mathrm{Bir}(\mathbb{P}^2)$.*

For any $g \in \mathrm{Jonq}$, the following assertions are equivalent:

(1) We have $ghg^{-1} \in \mathrm{Aff}_1(\mathbb{C}(y))$.
(2) We have $ghg^{-1} \in \mathcal{B}_2$.
(3) We have $g \in \mathrm{Aff}_1(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2$.

Proof. (1) $\Longrightarrow$ (2) This is obvious.

(2) $\Longrightarrow$ (3) Write $g \in \mathrm{Jonq} = \mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2$ as $g = uv$ where $u = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{C}(y))$ and $v \in \mathrm{PGL}_2$. We have

$$ghg^{-1} = uhu^{-1} = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \delta & -\beta \\ -\gamma & \alpha \end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ -\gamma^2 & * \end{smallmatrix}\right).$$

Hence, the condition (2) gives $\gamma = 0$, i.e. $u \in \mathrm{Aff}_1(\mathbb{C}(y))$, and finally $g \in \mathrm{Aff}_1(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2$.

(3) $\Longrightarrow$ (1) Write $g = uv$, where $u \in \mathrm{Aff}_1(\mathbb{C}(y))$, $v \in \mathrm{PGL}_2$, and note that $ghg^{-1} = uhu^{-1} \in \mathrm{Aff}_1(\mathbb{C}(y))$. $\square$

Theorem 10.7. *Any Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ contains at least one Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$.*

Proof. Let B be a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$.

Step 1. Let's show that $K := B \cap \mathrm{PGL}_2(\mathbb{C}(y))$ is $\mathbb{C}(y)$-closed in $\mathrm{PGL}_2(\mathbb{C}(y))$. Let $\overline{K}$ be the $\mathbb{C}(y)$-closure of K in $\mathrm{PGL}_2(\mathbb{C}(y))$. Note that $\overline{K}$ is normalised by B (since K is normalised by B). Hence $B' := B.\overline{K}$ is a subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$. Moreover, since B and $\overline{K}$ are solvable, B' is also solvable (this follows from the fact that $\overline{K}$ and $B'/\overline{K}$ are solvable; the solvability of $B'/\overline{K}$ follows from the isomorphism $B'/\overline{K} \simeq B/(B \cap \overline{K})$).

Let's now check that B' is connected with respect to the $\mathrm{Bir}(\mathbb{P}^2)$-topology. Let $(\overline{K})^\circ$ be the neutral connected component of $\overline{K}$ with respect to the $\mathbb{C}(y)$-topology on $\mathrm{PGL}_2(\mathbb{C}(y))$. By Lemma 6.3, it is $\mathrm{Bir}(\mathbb{P}^2)$-connected. Since there exist elements $k_1, \dots, k_r$ of K such that $\overline{K} = \bigcup_i k_i(\overline{K})^\circ$, it follows that $B' = B(\overline{K})^\circ$, and this is sufficient for showing that B' is $\mathrm{Bir}(\mathbb{P}^2)$-connected.

The Borel subgroup B being contained in B' which is solvable and connected, we necessarily have $B = B'$. This shows that $\overline{K}$ is contained in B. Hence we have $\overline{K} = K$ and we have actually proven that K is $\mathbb{C}(y)$-closed.

Set $H := K^\circ$ where the neutral connected component is taken with respect to the $\mathbb{C}(y)$-topology on $\mathrm{PGL}_2(\mathbb{C}(y))$.

Step 2. Let's prove that $H \neq \{\mathrm{id}\}$.

Assume by contradiction that $H = \{\mathrm{id}\}$. Then K is a finite subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$ and we have a short exact sequence

$$1 \to K \to B \to \mathrm{pr}_2(B) \to 1.$$

Note that K is nontrivial thanks to Proposition 10.5. Since the connected group B normalises the finite group K, it necessarily centralises it. Choosing a nontrivial element k of K we have $B \subsetneq \mathrm{Cent}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(k)^\circ$. By [2, Theorem 4.2.] and up to conjugation, one of the two following assertions holds:

(1) $k = \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $a \in \mathbb{C} \setminus \{0, 1\}$;

(2) $k = \left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right)$ where f is a nonsquare element of $\mathbb{C}(y)$.

In case (1), an easy computation would show that $\mathrm{Cent}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(\mathrm{k})^\circ = \mathbb{C}(y)^* \rtimes \mathrm{Aff}_1$. In case (2), Proposition 7.8 yields $\mathrm{Cent}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(k)^\circ = \mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ$ and Proposition 10.1 yields $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ = \mathbb{T}_f \rtimes \mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ$. Hence, we have

$$\mathrm{Cent}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(k)^\circ = \mathbb{T}_f \rtimes \mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ.$$

Recall that $\mathcal{N}_{\mathbb{C}(y)^* \rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ$ is commutative (being either trivial or isomorphic to $\mathbb{C}^*$). Hence $\mathrm{Cent}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(k)^\circ$ is solvable in both cases. Since it contains B and is connected (by definition) it is equal to B. However, $\mathrm{Cent}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(k)^\circ \cap \mathrm{PGL}_2(\mathbb{C}(y))$ is infinite in both cases. A contradiction.

Step 3. Since H is a nontrivial closed connected solvable subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$, we conclude by Lemma 5.8 that up to conjugation it is either $\mathbb{T}_f$ for some nonsquare element $f \in \mathbb{C}(y)$, or one of the following three groups: $\mathrm{Aff}_1(\mathbb{C}(y))$, $\{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right),\ a \in \mathbb{C}(y)^*\}$, $\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right),\ a \in \mathbb{C}(y)\}$. Since B normalises K, it also normalises $H = K^\circ$, i.e. B is contained in $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(H)$. Since B is connected we even have

$$B \subsetneq \mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(H)^\circ.$$

If $H = \{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right),\ a \in \mathbb{C}(y)\}$, we get $B \subseteq \mathcal{B}_2$ by Lemma 10.6 – a contradiction, because this would yield $B = \mathcal{B}_2$ and then $H = \mathrm{Aff}_1(\mathbb{C}(y))$. If $H = \{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right),\ a \in \mathbb{C}(y)^*\}$, we start by noting that $\{1\} \rtimes \mathrm{Aff}_1$ normalises H, and it is well known that only diagonal or anti-diagonal matrices in $\mathrm{PGL}_2(\mathbb{C}(y))$ normalise $\{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right),\ a \in \mathbb{C}(y)^*\}$. We conclude that B is contained in $\{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right),\ a \in \mathbb{C}(y)^*\} \rtimes \mathrm{Aff}_1$ – a contradiction because this would yield $B = \mathbb{C}(y)^* \rtimes \mathrm{Aff}_1$, and B is strictly contained in the closed connected solvable group $\mathcal{B}_2$. Hence, we have either $H = \mathbb{T}_f$ or $\mathrm{Aff}_1(\mathbb{C}(y))$ and this achieves the proof. □

In a linear algebraic group a closed connected solvable subgroup equal to its own normaliser is not necessarily a Borel subgroup, as shown in the following example: The diagonal group T in the group B of upper triangular matrices of size 2 is equal to its own normaliser, but it is not a Borel subgroup of B. In contrast, the following result holds:

Lemma 10.8. *If B' is a solvable subgroup of* Jonq *containing a Borel subgroup B of* $\mathrm{PGL}_2(\mathbb{C}(y))$*, then B' normalises B.*

Proof. Consider the subgroup G of Jonq generated by the subgroups $\varphi B \varphi^{-1}$, $\varphi \in B'$. As each $\varphi B \varphi^{-1}$ is contained in both $\mathrm{PGL}_2(\mathbb{C}(y))$ and B', the same holds for G. Hence, G is a solvable subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$ containing B. Since G is obviously connected (being generated by connected subgroups), we have $G = B$. □

The following interesting result provides a description of all Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$. This is the reason for including it even if it will not be used in the proof of Theorem 1.3.

Theorem 10.9. *Each Borel subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y))$ *is contained in a unique Borel subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ *and conversely each Borel subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ *contains a unique Borel subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y))$*. The corresponding bijection from the set of Borel subgroups of* $\mathrm{PGL}_2(\mathbb{C}(y))$ *to the set of Borel subgroups of* $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ *is the map*

$$B \mapsto \mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(B)^\circ$$

and its inverse is the map

$$B' \mapsto [\mathrm{PGL}_2(\mathbb{C}(y)) \cap B']^\circ.$$

Proof. Let B be a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$. If B' is a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ containing B, then we have $B' \subseteq \mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(B)$ by Lemma 10.8 and even $B' \subseteq \mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(B)^\circ$ because B' is connected. For showing that $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(B)^\circ$ is the unique Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ containing B, it remains to show that this group is solvable. Actually, we will even check that $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(B)$ is solvable. Since $D^2(\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(B))$ is contained in $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y))}(B)$ it is enough to check that this latter group is solvable. But up to conjugation in $\mathrm{PGL}_2(\mathbb{C}(y))$ we have either $B = \mathbb{T}_f$ for some nonsquare element $f \in \mathbb{C}(y)$ or $B = \mathrm{Aff}_1(\mathbb{C}(y))$ (Theorem 6.4). If $B = \mathbb{T}_f$, we have $[\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y))}(B) : B] = 2$ by Lemma 5.13 and if $B = \mathrm{Aff}_1(\mathbb{C}(y))$ we have $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y))}(B) = B$. In both cases $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y))}(B)$ is actually solvable.

Moreover, any Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ contains a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$ (see Theorem 10.7). Hence the map

$$B \mapsto \mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y))\rtimes \mathrm{Aff}_1}(B)^\circ$$

from the set of Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y))$ to the set of Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ is surjective.

Let B' be a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$. By what has just been said, there exists at least one Borel subgroup B of $\mathrm{PGL}_2(\mathbb{C}(y))$ such that $B \subseteq B'$. Since B is contained in the closed connected solvable subgroup $[\mathrm{PGL}_2(\mathbb{C}(y)) \cap B']^\circ$ of $\mathrm{PGL}_2(\mathbb{C}(y))$ this shows that we necessarily have $B = [\mathrm{PGL}_2(\mathbb{C}(y)) \cap B']^\circ$ and this concludes the proof. □

Lemma 10.10. *We have* $\mathrm{N}_{\mathrm{Jonq}}(\mathrm{Aff}_1(\mathbb{C}(y))) = \mathrm{Aff}_1(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2$.

Proof. This follows from the equality $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y))}(\mathrm{Aff}_1(\mathbb{C}(y))) = \mathrm{Aff}_1(\mathbb{C}(y))$ (see Lemma 5.15). □

If B is a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y))$, we now describe more precisely the unique Borel subgroup B' of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ which contains it. Recall that up to conjugation we have either $B = \mathrm{Aff}_1(\mathbb{C}(y))$ or $B = \mathbb{T}_f$ for some nonsquare element $f \in \mathbb{C}(y)$(Theorem 6.4). If $B = \mathrm{Aff}_1(\mathbb{C}(y))$ it follows from Lemma 10.10 that the unique Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ containing $\mathrm{Aff}_1(\mathbb{C}(y))$ is

$$\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y))\rtimes \mathrm{Aff}_1}\Big(\mathrm{Aff}_1(\mathbb{C}(y))\Big)^\circ = \mathcal{B}_2\,.$$

If $B = \mathbb{T}_f$ the situation is described in Proposition 10.11 below which is a direct consequence of Proposition 10.1.

Proposition 10.11. *Let f be a nonsquare element of $\mathbb{C}(y)$. Then the unique Borel subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ *containing* $\mathbb{T}_f$ *is*

$$\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y))\rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ = \mathbb{T}_f \rtimes \mathcal{N}_{\mathbb{C}(y)^*\rtimes \mathrm{Aff}_1}(\mathbb{T}_f)^\circ.$$

11. Some Borel subgroups of $\mathrm{Bir}(\mathbb{P}^2)$

In this section we show that the groups listed in Theorem 1.3 are Borel subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ (see Theorem 11.2). In the proof we will use the following result.

Proposition 11.1. *Let $B \subseteq$ Jonq be a subgroup. Then, the following assertions are equivalent.*

(1) B is a Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$*;*
(2) B is a Borel subgroup of Jonq.

Proof. The implication (1) $\Longrightarrow$ (2) being obvious, let's prove (2) $\Longrightarrow$ (1). Assume that B is a Borel subgroup of Jonq. Up to conjugation by an element of Jonq we may assume that B is a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ (see Theorem 4.1). Let now B' be a closed connected solvable subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ containing B. We want to show that $B = B'$. By Theorem 3.1, there exists $g \in \mathrm{Bir}(\mathbb{P}^2)$ such that $gB'g^{-1} \subseteq$ Jonq and by Theorem 10.7, B contains a Borel subgroup B'' of $\mathrm{PGL}_2(\mathbb{C}(y))$. Since B'' is conjugate in $\mathrm{PGL}_2(\mathbb{C}(y))$ to either $\mathrm{Aff}_1(\mathbb{C}(y))$ or $\mathbb{T}_f$ for some nonsquare element f of $\mathbb{C}(y)$ (see Theorem 6.4), it contains a Jonquières twist. Hence the inclusion

$gB''g^{-1} \subseteq \mathrm{Jonq}$ and Lemma 7.9 show, in combination with [13, Lemma 4.5], that $g \in \mathrm{Jonq}$. This proves that B' is actually contained in Jonq and finally the inclusion $B \subseteq B'$ implies that $B = B'$. $\square$

As announced, we now describe some Borel subgroups of $\mathrm{Bir}(\mathbb{P}^2)$. Later on, in Theorem 12.1, we will show that each Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is actually conjugate to one of them.

Theorem 11.2. *The following subgroups are Borel subgroups of* $\mathrm{Bir}(\mathbb{P}^2)$*:*

(1) $\mathcal{B}_2$*;*
(2) $\mathbb{T}_y \rtimes \mathbb{T}_{1,2}$*;*
(3) $\mathbb{T}_f$ *where* f *is a nonsquare element of* $\mathbb{C}(y)$ *such that the genus* $\mathfrak{g}$ *of the curve* $x^2 = f(y)$ *satisfies* $\mathfrak{g} \geq 1$.

Proof. By Proposition 11.1, it is enough to prove that all these groups are Borel subgroups of Jonq. By Proposition 6.6 any Borel subgroup B of $\mathrm{PGL}_2(\mathbb{C}(y))$ is contained in a Borel subgroup B' of Jonq and by Lemma 10.8 we have $B' \subseteq \mathrm{N}_{\mathrm{Jonq}}(B)^\circ$.
(1) If $B = \mathrm{Aff}_1(\mathbb{C}(y))$, this yields $B \subseteq B' \subseteq B \rtimes \mathrm{PGL}_2$ (Lemma 10.10). By Theorem 4.1 and up to conjugation by an element of $\{1\} \rtimes \mathrm{PGL}_2 \subseteq \mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2 = \mathrm{Jonq}$, we may moreover assume that we have $B' \subseteq \mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$. Using the equality
$$(B \rtimes \mathrm{PGL}_2) \cap (\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1) = B \rtimes \mathrm{Aff}_1$$
we obtain
$$B \subseteq B' \subseteq B \rtimes \mathrm{Aff}_1.$$
Since $B \rtimes \mathrm{Aff}_1$ is solvable, we get $B' = B \rtimes \mathrm{Aff}_1 = \mathrm{Aff}_1(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1 = \mathcal{B}_2$ and we have actually proven that $\mathcal{B}_2$ is a Borel subgroup of Jonq. Note that this proof is different from the one given by Popov in [21].
(2) If $B = \mathbb{T}_y$, we have $B' \subseteq \mathbb{T}_y \rtimes \mathbb{T}_{1,2}$ by Proposition 7.22(2). Since $\mathbb{T}_y \rtimes \mathbb{T}_{1,2}$ is solvable, this gives $B' = \mathbb{T}_y \rtimes \mathbb{T}_{1,2}$.
(3) If $B := \mathbb{T}_f$ with $\mathfrak{g} \geq 1$, we have $B' \subseteq \mathbb{T}_f$ by Proposition 7.22(1) and this gives $B' = \mathbb{T}_f$. $\square$

Lemma 11.3. *Let* G, H, K *be closed subgroups of* $\mathrm{Bir}(\mathbb{P}^2)$ *such that* $G = K \rtimes H$*, then we have* $\max(\mathrm{rk}\, H, \mathrm{rk}\, K) \leq \mathrm{rk}\, G \leq \mathrm{rk}\, H + \mathrm{rk}\, K$.

Proof. Since H, K are contained in G, the first inequality is obvious. Let now T be a torus in G. The short exact sequence $1 \to K \to G \xrightarrow{\pi} H \to 1$ induces the short exact sequence $1 \to T' \to T \to T'' \to 1$ where $T' := T \cap K$ and $T'' := \pi(T)$. Since $(T')^\circ$ and T'' are connected algebraic groups consisting of semisimple elements, they are tori (see [18, Exercise 21.2, page 137]). Hence, we get $\dim T = \dim T' + \dim T'' = \dim(T')^\circ + \dim T'' \leq \mathrm{rk}\, K + \mathrm{rk}\, H$ and the conclusion follows. $\square$

Remark 11.4. *The three kinds of examples given in Theorem 11.2 are non-isomorphic, hence non-conjugate, since the derived lengths of* $\mathcal{B}_2$*,* $\mathbb{T}_y \rtimes \mathbb{T}_{1,2}$*,* $\mathbb{T}_f$ *are respectively 4,2,1 (see Proposition 1.4 for the derived length of* $\mathcal{B}_2$*).*

Let's also note that the ranks of these groups are respectively 2,1,0: We know from Lemma 7.7 that $\mathrm{rk}(\mathbb{T}_f) = 0$*; Lemma 11.3 shows that* $\mathrm{rk}(\mathbb{T}_y \rtimes \mathbb{T}_{1,2}) = 1$*; the equality* $\mathrm{rk}(\mathcal{B}_2) = 2$ *is obvious.*

12. All Borel subgroups of $\mathrm{Bir}(\mathbb{P}^2)$

In Theorem 11.2 we have given some examples of Borel subgroups of $\mathrm{Bir}(\mathbb{P}^2)$. We now prove that up to conjugation there are no others.

Theorem 12.1. *Any Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is necessarily conjugate to one of the following subgroups:*

(1) $\mathcal{B}_2$;
(2) $\mathbb{T}_y \rtimes \mathbb{T}_{1,2}$;
(3) $\mathbb{T}_f$ where f is a nonsquare element of $\mathbb{C}(y)$ such that the genus g of the curve $x^2 = f(y)$ satisfies $g \geq 1$.

Proof. Let B' be a Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$. Up to conjugation, it is a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ (cf. Theorem 4.2). By Theorem 10.7 B' contains a Borel subgroup B of $\mathrm{PGL}_2(\mathbb{C}(y))$ and by Lemma 10.8 we have $B' \subseteq \mathrm{N}_{\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1}(B)^\circ$. Moreover, up to conjugation, one of the following cases occurs:

(1) $B = \mathrm{Aff}_1(\mathbb{C}(y))$;
(2) $B = \mathbb{T}_y$;
(3) $B = \mathbb{T}_f$ for some nonsquare element f of $\mathbb{C}(y)$ such that the genus g of the curve $x^2 = f(y)$ satisfies $g \geq 1$.

Then Lemma 10.10, Proposition 7.22(2), and Proposition 7.22(1) prove that B' is contained respectively in $\mathcal{B}_2$, $\mathbb{T}_y \rtimes \mathbb{T}_{1,2}$, and $\mathbb{T}_f$. Since these groups are solvable, this shows that B' is equal respectively to $\mathcal{B}_2$, $\mathbb{T}_y \rtimes \mathbb{T}_{1,2}$, and $\mathbb{T}_f$. □

Let's now give the proof of our main theorem.

Proof of Theorem 1.3. This follows from Theorems 11.2 & 12.1, Remark 11.4, and Proposition 7.15. □

Recall that a Borel subgroup of a linear algebraic group is equal to its own normaliser [18, Theorem 23.1, page 143]. We end this section by showing an analogous result in $\mathrm{Bir}(\mathbb{P}^2)$ (see Proposition 12.3 below). We begin with the following lemma.

Lemma 12.2. *The group $\mathcal{B}_2$ is equal to its own normaliser in $\mathrm{Bir}(\mathbb{P}^2)$.*

Proof. By Lemma 7.9, we have $\mathrm{N}_{\mathrm{Bir}(\mathbb{P}^2)}(\mathcal{B}_2) = \mathrm{N}_{\mathrm{Jonq}}(\mathcal{B}_2)$ (since $\mathcal{B}_2$ contains at least one Jonquières twist). By Lemma 10.6, we have $\mathrm{N}_{\mathrm{Jonq}}(\mathcal{B}_2) \subseteq \mathrm{Aff}_1(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2$. We conclude by using the second projection $\mathrm{pr}_2 \colon \mathrm{Jonq} \to \mathrm{PGL}_2$ and the equality $\mathrm{N}_{\mathrm{PGL}_2}(\mathrm{Aff}_1) = \mathrm{Aff}_1$ (besides being obvious this latter equality is also Lemma 5.15 or follows from the Borel normaliser theorem applied with the Borel subgroup Aff_1 of PGL_2). □

Proposition 12.3. *Let B be a Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$. Then we have*

$$B = \mathrm{N}_{\mathrm{Bir}(\mathbb{P}^2)}(B)^\circ.$$

Proof. Up to conjugation, one of the following cases occurs:

(1) $B = \mathcal{B}_2$;
(2) $B = \mathbb{T}_y \rtimes \mathbb{T}_{1,2}$;
(3) $B = \mathbb{T}_f$ where f is a nonsquare element of $\mathbb{C}(y)$ such that the genus g of the curve $x^2 = f(y)$ satisfies $g \geq 1$.

Note that $\mathrm{N}_{\mathrm{Bir}(\mathbb{P}^2)}(B) = \mathrm{N}_{\mathrm{Jonq}}(B)$ in each of these cases by Lemma 7.9.

In case (1), the conclusion follows from Lemma 12.2.

In case (2), we have $\mathrm{N}_{\mathrm{Jonq}}(B) \subseteq \mathrm{N}_{\mathrm{Jonq}}(\mathcal{D}(B))$ because $\mathcal{D}(B) = \overline{D(B)}$ and $D(B)$ is a characteristic subgroup of B. Since $\mathcal{D}(\mathbb{T}_y \rtimes \mathbb{T}_{1,2}) = \mathbb{T}_y$ we obtain $\mathrm{N}_{\mathrm{Jonq}}(B)^\circ \subseteq \mathrm{N}_{\mathrm{Jonq}}(\mathbb{T}_y)^\circ = \mathbb{T}_y \rtimes \mathbb{T}_{1,2}$ (see Proposition 7.22(2)) and the conclusion follows.

The case (3) follows from Proposition 7.22(1). □

Remark 12.4. *Note that the usual Borel normaliser theorem $B = \mathrm{N}_G(B)$ does not hold when $G = \mathrm{Bir}(\mathbb{P}^2)$ and $B = \mathbb{T}_f$ for some nonsquare element f of $\mathbb{C}(y)$ such that the genus g of the curve $x^2 = f(y)$ satisfies $g \geq 1$. Actually, in this case $\mathrm{N}_{\mathrm{Bir}(\mathbb{P}^2)}(\mathbb{T}_f)$ is strictly larger than $\mathbb{T}_f$ because it contains $\mathbb{T}_f \rtimes \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle$ (see Lemma 5.13).*

Appendix A. Computation of the derived length of $\mathcal{B}_n$

In this appendix we give the proof of Proposition 1.4, stated in the introduction.

Proof of Proposition 1.4. Set $U_k := D^{2n-2k}(\mathcal{B}_n)$ for $0 \leq k \leq n$ and $V_k := D(U_k)$ for $1 \leq k \leq n$.

We could easily check by induction that U_k is contained in the group of elements $f = (f_1, \ldots, f_n)$ in $\mathcal{B}_n$ satisfying $f_i = x_i$ for $i > k$ and that V_k is contained in the group of elements $f = (f_1, \ldots, f_n)$ in $\mathcal{B}_n$ satisfying $f_i = x_i$ for $i > k$ and $f_k = x_k + b_k$ with $b_k \in \mathbb{C}(x_{k+1}, \ldots, x_n)$. This implies $U_0 = \{\mathrm{id}\}$. Hence the derived length of $\mathcal{B}_n$ is at most $2n$.

Conversely, we will now prove that this derived length is at least $2n$, i.e. that $V_1 \neq \{\mathrm{id}\}$. For this, we introduce the following notation. For each $k \in \{1, \ldots, n\}$, each nonzero element $a \in \mathbb{C}(x_{k+1}, \ldots, x_n)$, and each element $b \in \mathbb{C}(x_{k+1}, \ldots, x_n)$, we define the dilatation $d(k, a) \in \mathrm{Bir}(\mathbb{P}^n)$ and the elementary transformation $e(k, b) \in \mathrm{Bir}(\mathbb{P}^n)$ by

$$d(k, a) = (g_1, \ldots, g_n) \quad \text{and} \quad e(k, b) = (h_1, \ldots, h_n),$$

where $g_k = ax_k$, $h_k = x_k + b$ and $g_i = h_i = x_i$ for $i \neq k$.

If $i \in \{1, \ldots, n-1\}$ and if G is a subgroup of $\mathrm{Bir}(\mathbb{P}^n)$, the properties (D_i) and (E_i) for G are defined in the following way:

(D_i) $\exists\, a \in \mathbb{C}(x_{i+1}) \setminus \mathbb{C}$ such that $d(i, a) \in G$;
(E_i) $\forall\, b \in \mathbb{C}(x_{i+1})$ we have $e(i, b) \in G$.

For $i = n$, the properties (D_n) and (E_n) are defined in the following slightly different way:

(D_n) $\exists\, a \in \mathbb{C} \setminus \{0, 1\}$ such that $d(n, a) \in G$;
(E_n) $\forall\, b \in \mathbb{C}$ we have $e(n, b) \in G$.

We will prove by decreasing induction on k that for each $k \in \{1, \ldots, n\}$ the two following assertions hold:

(a_k) The group U_k satisfies (E_i) and (D_i) for $i \in \{1, \ldots, k\}$.
(b_k) The group V_k satisfies (E_i) for $i \in \{1, \ldots, k\}$ and (D_i) for $i \in \{1, \ldots, k-1\}$.

We will use two obvious identities. The first one is

$$[d(i, a), e(i, b)] = e(i, b(a-1)) \tag{11}$$

where we have either $i \in \{1, \dots, n-1\}$, $a \in \mathbb{C}(x_{i+1}) \setminus \{0\}$, $b \in \mathbb{C}(x_{i+1})$, or $i = n$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$.

The second one is

$$[d(i,a), e(i+1,c)] = d\left(i, \frac{a(x_{i+1})}{a(x_{i+1}-c)}\right) \tag{12}$$

where $i \in \{1, \dots, n-1\}$, $a \in \mathbb{C}(x_{i+1}) \setminus \{0\}$, $c \in \mathbb{C}$.

The identity (11) implies that if a subgroup G of $\mathrm{Bir}(\mathbb{P}^n)$ satisfies (D_i) and (E_i), then its derived subgroup $D^1(G)$ also satisfies (E_i).

Before making use of the identity (12), let's check that if $a \in \mathbb{C}(x)$ is nonconstant and $c \in \mathbb{C}$ is nonzero, then $a(x)/a(x-c)$ is nonconstant. Otherwise, we would have $a(x) = \lambda a(x-c)$ for some $\lambda \in \mathbb{C}^*$. This implies that the union $U(a)$ of the zeros and poles of a is invariant by translation by c. Since $U(a)$ is finite, it must be empty, proving that a is constant. A contradiction.

It follows from (12) and the last observation that if a subgroup G of $\mathrm{Bir}(\mathbb{P}^n)$ satisfies (D_i) and (E_{i+1}), then $D^1(G)$ also satisfies (D_i).

We are now ready for the induction. We begin by noting that the hypothesis (a_n) is obviously satisfied. It is then enough to observe that we have $(a_k) \Longrightarrow (b_k)$ for each $k \in \{1, \dots, n\}$ and $(b_k) \Longrightarrow (a_{k-1})$ for each $k \in \{2, \dots, n\}$. □

Appendix B. More on Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$

We have seen in Theorem 10.9 that there is a natural bijection between the Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y))$ and the Borel subgroups of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$. Hence the interplay between $\mathrm{PGL}_2(\mathbb{C}(y))$ and $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ is very straightforward in this respect. Analogously the interplay between the set of conjugacy classes of Borel subgroups of Jonq and the set of conjugacy classes of Borel subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ is very clear: Any Borel subgroup of Jonq is a Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ (Proposition 11.1); any Borel subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is conjugate to a Borel subgroup of Jonq (Theorem 3.1); and finally two Borel subgroups of Jonq are conjugate in Jonq if and only if they are conjugate in $\mathrm{Bir}(\mathbb{P}^2)$ (Lemma 7.9 and Theorem 12.1). Hence, the map sending the conjugacy class of a Borel subgroup B of Jonq to its conjugacy class in $\mathrm{Bir}(\mathbb{P}^2)$ is a bijection from the set of conjugacy classes of Borel subgroups of Jonq to the set of conjugacy classes of Borel subgroups of $\mathrm{Bir}(\mathbb{P}^2)$.

The interplay between $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ and $\mathrm{Jonq} = \mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2$ is more intricate: We now give an example of a Borel subgroup B of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ which is no longer a Borel subgroup of Jonq. To put it differently, B is conjugate in Jonq to a subgroup B' of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ which is no longer a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$! This kind of situation can of course not occur for linear algebraic groups: Let H be a closed subgroup of a linear algebraic group G and let B, B' be two subgroups of H which are conjugate in G. Then, B is a Borel subgroup of H if and only if B' is a Borel subgroup of H (if B, B' are closed, it is enough to note that they have the same dimension).

Example B.1. *The subgroups $\mathbb{T}_{y(y-1)}$ and $\mathbb{T}_y$ of $\mathrm{PGL}_2(\mathbb{C}(y))$ are conjugate in* Jonq *since the curves $x^2 = y(y-1)$ and $x^2 = y$ both have genus 0. However, since the odd supports $\mathrm{S}_{\mathrm{odd}}(y(y-1))$ and $\mathrm{S}_{\mathrm{odd}}(y)$ are respectively equal to $\{0,1\}$ and $\{0,\infty\}$,* *Proposition 10.11 asserts that* $\mathbb{T}_{y(y-1)}$ *is a Borel subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ *but that* $\mathbb{T}_y$ *is not. In fact, the unique Borel subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$ *containing* $\mathbb{T}_y$ *is* $\mathbb{T}_y \rtimes \mathbb{T}_{1,2}$*; see Theorem 12.1. This also shows that even if* $\mathbb{T}_{y(y-1)}$ *is a Borel subgroup of* $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{Aff}_1$*, it is no longer a Borel subgroup of* Jonq.

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