A Characterization of Semisimple Plane Polynomial Automorphisms

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Abstract

It is well-known that an element of the linear group $\mathrm{GL}_n(\mathbb{C})$ is semisimple if and only if its conjugacy class is Zariski closed. The aim of this paper is to show that the same result holds for the group of complex plane polynomial automorphisms.

Keywords

Affine space, Polynomial automorphisms.

MSC

14R10.

1 Introduction

If K is any commutative ring, a polynomial endomorphism of the affine N-space $\mathbb{A}_K^N$ over K will be identified with its sequence $f = (f_1, \ldots, f_N)$ of coordinate functions $f_j \in K[X_1, \ldots, X_N]$. We define its degree by $\deg f = \max_j \deg f_j$. If $N \leq 3$, we will use the indeterminates X, Y, Z instead of X_1, X_2, X_3 and if $K = \mathbb{C}$ we will write $\mathbb{A}^N$ instead of $\mathbb{A}_{\mathbb{C}}^N$. Let $\mathcal{G}$ (resp. $\mathcal{G}(K)$) be the group of polynomial automorphisms of $\mathbb{A}^2$ (resp. $\mathbb{A}_K^2$).

In linear algebra it is a well-known result that an element of $\mathrm{GL}_n(\mathbb{C})$ has a closed conjugacy class if and only if it is semisimple (see e.g. [27, pp. 92-93], where this result is proved for any complex reductive algebraic group). This is a very useful characterization, especially from a group action viewpoint. It is a natural question to ask if a polynomial automorphism is semisimple if and only if its conjugacy class is closed in the set of polynomial automorphisms. This last statement hides two definitions: what is a semisimple polynomial automorphism and what topology does one have on the group of polynomial automorphisms?

According to [8], the usual notion of semisimplicity can be extended from the linear to the polynomial case by saying that a polynomial automorphism is semisimple if it admits a vanishing polynomial with single roots. In this paper we restrict to the dimension 2. In this case, we will show in subsection 3.5 that it is equivalent to being diagonalizable, i.e. conjugate to some diagonal automorphism (aX, bY) where $a, b \in \mathbb{C}^*$.

The study of an infinite-dimensional (algebraic) variety of polynomial automorphisms (including its topology) has been introduced in [24]. However, this paper contains some inaccuracies and this theory remains mysterious (see [25, 26, 14, 15]). Let us describe this infinite-dimensional variety in dimension 2 (the description would be analogous in dimension N). The space $\mathcal{E} := \mathbb{C}[X, Y]^2$ of polynomial endomorphisms of $\mathbb{A}^2$ is naturally an infinite-dimensional variety (see [24, 25] and subsection 2.1 for the precise definition). This roughly means that $\mathcal{E}_{\leq m} := \{f \in \mathcal{E}, \deg f \leq m\}$ is a (finite-dimensional) variety for any $m \geq 1$, which comes out from the fact that it is an affine space. Afterwards, each algebraic variety will be endowed with its Zariski topology. If $A \subseteq \mathcal{E}$, we set $A_{\leq m} := A \cap \mathcal{E}_{\leq m}$. The space $\mathcal{E}$ is endowed with the topology of the inductive limit, in which A is closed (resp. open) if and only if each $A_{\leq m}$ is closed (resp. open) in $\mathcal{E}_{\leq m}$. A subset of some topological space is called locally closed when it is the intersection of an open and a closed subset. According to subsection 2.2, $\mathcal{G}$ is locally closed in $\mathcal{E}$, so that it is naturally an infinite-dimensional algebraic variety. Furthermore, the associated topology is the induced one. By lemma 2, a subset A of $\mathcal{G}$ is closed in $\mathcal{G}$ if and only if each $A_{\leq m}$ is closed in $\mathcal{G}_{\leq m}$. The aim of this paper is to show the following result.

Main Theorem. A complex plane polynomial automorphism is semisimple if and only if its conjugacy class is closed.

Application. If f is a finite-order automorphism of the affine space $\mathbb{A}^3$, it is still unknown whether or not it is diagonalizable. Since any finite-order linear automorphism is diagonalizable, it amounts to saying that f is linearizable, i.e. conjugate to some linear automorphism. To our knowledge, even the case where f fixes the last coordinate is unsolved. In this latter case, f is traditionally seen as an element of $\mathcal{G}(\mathbb{C}[Z])$. For each $z \in \mathbb{C}$, let $f_z \in \mathcal{G}$ be the automorphism induced by f on the plane $Z = z$. Using the amalgamated structure of $\mathcal{G}(\mathbb{C}(Z))$, we know that f is conjugate in this group to some (aX, bY), where $a, b \in \mathbb{C}^*$ (see [13, 16]). This implies that f_z is generically conjugate to (aX, bY), i.e. for all values of z except perhaps finitely many. The above theorem shows us that there is no exception: for all z, f_z is conjugate to (aX, bY). This could be one step for showing that such an f is diagonalizable in the group of polynomial automorphisms of $\mathbb{A}^3$. One can even wonder if the following is true.

Question 1. *Is any finite-order automorphism belonging to $\mathcal{G}(\mathbb{C}[Z])$ diagonalizable in this group?*

We begin in section 2 with some generalities on infinite-dimensional varieties. In section 3, we study the so called locally finite plane polynomial automorphisms, i.e. the automorphisms admitting a non-zero vanishing polynomial. The principal tool is the notion of pseudo-eigenvalues (3.2). It is used for defining a trace (3.3) and the subset $\mathcal{S} \subseteq \mathcal{G}$ of automorphisms admitting a single fixed point (3.4). Our text contains four natural questions which we were not yet able to answer (questions 1, 9, 10 and 16). Finally, we study semisimple automorphisms and show that their conjugacy classes are characterized by the pseudo-eigenvalues (3.5). The proof of the main theorem is given in section 4. Subsection 4.1 is devoted to an algebraic lemma whose proof relies on a valuative criterion. Subsection 4.2 is devoted to a few topological lemmas (lemma 25 for example relies on the Brouwer fixed point theorem). Let us note that our paper uses and extends results of [5, 6, 7, 8] and that [4] is heavily used too.

2 Infinite-dimensional varieties

2.1 Definition

According to Shafarevich (see [24]), a set U is called an **infinite-dimensional (complex algebraic) variety** when it is endowed with an increasing sequence $m \mapsto U_{\leq m}$ of subsets, each one being a (finite-dimensional complex algebraic) variety, satisfying:

$$\text{(i) } U = \bigcup_m U_{\leq m}; \qquad \text{(ii) each } U_{\leq m} \text{ is closed in } U_{\leq m+1}.$$

Let us recall that each (algebraic) variety is endowed with its Zariski topology. The set U is then endowed with the inductive limit topology, for which A is closed (resp. open) in U if and only if each $A_{\leq m} := A \cap U_{\leq m}$ is closed (resp. open) in $U_{\leq m}$.

A **morphism** between infinite-dimensional varieties $U = \bigcup_m U_{\leq m}$ and $V = \bigcup_m V_{\leq m}$ is by definition a map $\varphi : U \to V$ such that for each m, there exists an integer n for which $\varphi(U_{\leq m}) \subseteq V_{\leq n}$ and such that the restricted map $\varphi : U_{\leq m} \to V_{\leq n}$ is a morphism of (finite-dimensional) varieties.

2.2 Locally closed subsets

Let us begin with a few remarks on locally closed subsets.

In any topological space, the following assertions are equivalent:

(i) A is locally closed in C;
(ii) there exists a subset B such that A is closed in B and B is open in C;
(iii) there exists a subset B such that A is open in B and B is closed in C.

Furthermore, if A is locally closed in B and B is locally closed in C, then A is locally closed in C. Finally, a subset A of some topological space is locally closed if and only if $\overline{A} \setminus A$ is closed.

If V is any subset of the infinite-dimensional variety U, let us note that V is both endowed with the **induced topology** for which a set is closed in V if and only if it is the trace of some closed set of U and the **inductive limit topology** which comes from the sequence $m \mapsto V_{\leq m} := V \cap U_{\leq m}$. The inductive limit topology is always finer than the induced topology, but they are not in general the same (see 2.3). However:

Lemma 2. *If V is locally closed in U, these topologies coincide.*

Proof. If $A \subseteq V$, let us set $A_{\leq m} := A \cap U_{\leq m} = A \cap V_{\leq m}$. We may assume that V is closed or open in U. Indeed, if V is locally closed in U, there exists a subset W of U such that V is closed in W and W is open in U. Therefore, the induced and the inductive limit topologies will coincide on W and then on V.

If V is closed in U, then A is closed in V for the induced topology if and only if A is closed in U. This means that each $A_{\leq m}$ is closed in $U_{\leq m}$. This is equivalent to each $A_{\leq m}$ being closed in $V_{\leq m}$, which means that A is closed in V for the inductive limit topology.

If V is open in U, the same proof (replacing the word "closed" by "open") shows that A is open in V for the induced topology if and only if A is open in V for the inductive limit topology. $\square$

If V is locally closed in U, let us note that each $V_{\leq m}$ is locally closed in $U_{\leq m}$ so that $V_{\leq m}$ is naturally a (finite-dimensional) variety. Therefore, the sequence $m \mapsto V_{\leq m}$ endows V with the structure of an infinite-dimensional variety. In particular, the next results endows $\mathcal{G}$ with the structure of an infinite-dimensional variety.

Lemma 3. *$\mathcal{G}$ is locally closed in $\mathcal{E}$.*

Proof. The set $\mathcal{J}$ of polynomial endomorphisms whose Jacobian determinant is a non-zero constant is locally closed in $\mathcal{E}$. Indeed, the Jacobian determinant $\text{Jac} : \mathcal{E} \to \mathbb{C}[X, Y]$ is a morphism (of infinite-dimensional varieties) and $\mathcal{J}$ is the preimage of the locally closed subset $\mathbb{C}^*$ of $\mathbb{C}[X, Y]$. It is still unknown whether $\mathcal{G} = \mathcal{J}$ or not (i.e. if the Jacobian conjecture in dimension 2 holds or not). However, it is proved in [1], using the following theorem of Gabber, that each $\mathcal{G}_{\leq m}$ is closed in $\mathcal{J}_{\leq m}$. Therefore, by lemma 2, $\mathcal{G}$ is closed in $\mathcal{J}$. Finally $\mathcal{G}$ is locally closed in $\mathcal{E}$. $\square$

Theorem (Gabber). Let f be a polynomial automorphism of $\mathbb{A}^N$. We have:

$$\deg f^{-1} \leq (\deg f)^{N-1}.$$

Remark. Contrary to what the first author claimed in [7, §0], it is not enough to check that each $\mathcal{G}_{\leq m}$ is locally closed in $\mathcal{E}_{\leq m}$ to assert that $\mathcal{G}$ is locally closed in $\mathcal{E}$. Fortunately, lemma 3 still holds! We are grateful to the referee for pointing out this subtlety and we refer to the next subsection for a counterexample.

2.3 A counterexample

In this subsection, we give an example of an infinite-dimensional variety U (defined by a sequence $m \mapsto U_{\leq m}$) admitting a subset A such that each $A_{\leq m} := A \cap U_{\leq m}$ is locally closed in $U_{\leq m}$, but such that:

(1) the inductive limit and the induced topologies of A do not coincide;
(2) A is not locally closed in U;
(3) $\overline{A} \neq \bigcup_m \overline{A_{\leq m}}$.

Let U be a vector space with a countable basis $(e_m)_{m \geq 1}$. For example, one can take $U = \mathbb{C}[X]$ and $e_m = X^{m-1}$. Since the linear subspace $U_{\leq m} := \mathrm{Span}(e_1, \ldots, e_m)$ is naturally an algebraic variety (being an affine space), U is an infinite-dimensional variety.

For each m, let us consider the line $D_m := me_1 + \mathrm{Span}(e_m)$ and $D_m^* := D_m \setminus \{me_1\}$. Let us set $A := D_2 \cup \bigcup_{m \geq 3} D_m^*$ and $A_{\leq m} := A \cap U_{\leq m}$ for each m.

One would easily check that each $A_{\leq m}$ is locally closed in $U_{\leq m}$ and that A satisfies assertions (1-3) above:

(3): it is enough to check that $\overline{A} = \bigcup_{m \geq 1} D_m$ and $\bigcup_m \overline{A_{\leq m}} = \bigcup_{m \geq 2} D_m$.

(2): it is enough to check that $\overline{A} \setminus A = D_1 \setminus \{2e_1\}$ is not closed in U.

(1): it is enough to check that $\bigcup_{m \geq 3} D_m^*$ is closed in A for the inductive limit topology but not for the induced topology.

3 Locally finite plane polynomial automorphisms

3.1 Characterization

According to [8], a polynomial endomorphism is called locally finite (LF for short) if it admits a non-zero vanishing polynomial. The class of LF plane polynomial automorphisms will be denoted by $\mathcal{LF}$. We recall that an automorphism is said to be triangularizable if it is conjugate to some triangular automorphism $(aX + p(Y), bY + c)$, where $a, b \in \mathbb{C}^*$, $c \in \mathbb{C}$ and $p \in \mathbb{C}[Y]$. Using the amalgamated structure of $\mathcal{G}$ (see [12, 17, 21]), one can show the following:

Theorem 4. *If $f \in \mathcal{G}$, the following assertions are equivalent:*
(i) f is triangularizable;
(ii) the dynamical degree $dd(f) := \lim_{n \to \infty} (\deg f^n)^{1/n}$ is equal to 1;
(iii) deg $f^2 \leq$ deg f;
(iv) $\forall\, n \in \mathbb{N}$, deg $f^n \leq$ deg f;
(v) for each $\xi \in \mathbb{C}^2$, the sequence $n \mapsto f^n(\xi)$ is a linear recurrence sequence;
(vi) f is LF.

Proof. The equivalence between (i), (ii), (v) and (vi) is explained in the final remark of [6, section I]. However, the equivalence between (i) and (ii) is proved in [4] and the equivalence between (i) and (vi) is proved in [8]. The equivalence between (i), (iii) and (iv) is proved in [5]. □

In this case, the minimal polynomial μ_f of f is defined as the (unique) monic polynomial generating the ideal $\{p \in \mathbb{C}[T],\, p(f) = 0\}$. Even if the class $\mathcal{LF}$ is invariant by conjugation, the minimal polynomial is not.

Corollary 5. *$\mathcal{LF}$ is closed in $\mathcal{G}$.*

Proof. For any $m \geq 1$, we have $\mathcal{LF}_{\leq m} = \{f \in \mathcal{G}, \forall n \in \mathbb{N}, \deg f^n \leq m\}$. The inclusion ($\subseteq$) comes from the implication (vi) $\Longrightarrow$ (iv) of the last theorem, while the reverse inclusion ($\supseteq$) comes from the implication (ii) $\Longrightarrow$ (vi). This proves that $\mathcal{LF}_{\leq m}$ is closed in $\mathcal{G}_{\leq m}$. $\square$

3.2 The pseudo-eigenvalues

Any $f \in \mathcal{LF}$ is conjugate to some triangular automorphism $t = (aX + p(Y), bY + c)$. It is explained in [4, p. 87] that the unordered pair $\{a, b\}$ is an invariant: if t has a fixed point, then a and b are equal to the two eigenvalues of the derivative at that fixed point and if t has no fixed point, then the pair $\{a, b\}$ must be equal to $\{1, \operatorname{Jac} f\}$.

Definition. a, b are called the pseudo-eigenvalues of f.

Let $< a, b > := \{a^k b^l,\, k, l \in \mathbb{N}\}$ be the submonoid of $\mathbb{C}^*$ generated by a, b and let $f^* : r \mapsto r \circ f$ be the algebra automorphism of $\mathbb{C}[X, Y]$ associated to f. The following result relates the pseudo-eigenvalues of f with the eigenvalues of f^*.

Lemma 6. *If a, b are the pseudo-eigenvalues of $f \in \mathcal{LF}$, then $< a, b >$ is the set of eigenvalues of f^*.*

Proof. We may assume that $f = (aX + p(Y), bY + c)$. Let d be the degree of $p(Y)$.

Let $M := \{X^k Y^l,\, k, l \geq 0\}$ be the set of all monomials in X, Y and let us endow M with the monomial order $\prec$ (see [3]) defined by

$$X^k Y^l \prec X^m Y^n \Longleftrightarrow k < m \text{ or } (k = m \text{ and } l < n).$$

For any $s \geq 0$, we observe that the vector space V_s generated by the $X^k Y^l$ such that $dk + l \leq s$ is stable by f^*. Let us denote by $f^*_{||V_s}$ the induced linear endomorphism of V_s.

Since $f^*(X^k Y^l) - a^k b^l X^k Y^l \in \operatorname{Span}(X^m Y^n)_{X^m Y^n \prec X^k Y^l}$ (exercise), the matrix of $f^*_{||V_s}$ in the basis $X^k Y^l$ (where the $X^k Y^l$ are taken with the order $\prec$) is upper triangular with the $a^k b^l$'s on the diagonal. The result follows from the equality $\mathbb{C}[X, Y] = \bigcup_s V_s$. $\square$

It is well-known that the eigenvalues of a linear automorphism are roots of its minimal polynomial. The same result holds for LF plane polynomial automorphisms:

Lemma 7. *The pseudo-eigenvalues are roots of the minimal polynomial.*

Our proof of this lemma (as well as forthcoming results in this paper) will use the basic language of linear recurrence sequences that we now recall (see [2] for a nice overview of this subject). If U is any complex vector space, the set of sequences $u : \mathbb{N} \to U$ is denoted by $U^{\mathbb{N}}$. For $p = \sum_k p_k\, T^k \in \mathbb{C}[T]$, we define $p(u) \in U^{\mathbb{N}}$ by the formula

$$\forall\, n \in \mathbb{N},\ \big(p(u)\big)\,(n) = \sum_k p_k\ u(n+k).$$

Let $U[\nu]$ be the set of polynomials in the indeterminate ν with coefficients in U, alias the set of polynomial maps from $\mathbb{C}$ to U. The theory of linear recurrence sequences relies on the next result (see [2]):

Theorem 8. *Let $p = \alpha \prod_{1 \le k \le c} (T - \omega_k)^{r_k}$ be the decomposition into irreducible factors of some non-zero polynomial $p \in \mathbb{C}[T]$. The equality $p(u) = 0$ holds if and only if there exist $q_1, \ldots, q_c \in U[\nu]$ with deg $q_k \le r_k - 1$ such that*

$$\forall\, n \in \mathbb{N},\ u(n) \ = \sum_{1 \le k \le c} \omega_k^n\, q_k(n).$$

We set $\mathcal{I}_u := \{p \in \mathbb{C}[T],\ p(u) = 0\}$. Since $\mathcal{I}_u$ is a vector subspace of $\mathbb{C}[T]$ which is closed under multiplication by T, it is clear that $\mathcal{I}_u$ is an ideal of $\mathbb{C}[T]$. We say that u is a linear recurrence sequence when $\mathcal{I}_u \neq \{0\}$. In this case, the minimal polynomial of u is the (unique) monic polynomial μ_u generating the ideal $\mathcal{I}_u$.

We say that u is of exponential type if the following equivalent assertions are satisfied:

(i) there exist $\omega_1, \ldots, \omega_c \in \mathbb{C}$, $q_1, \ldots, q_c \in U$ such that $\forall\, n,\ u(n) \ = \sum_{1 \le k \le c} \omega_k^n\, q_k$.

(ii) μ_u has single roots.

Remark. If $l : U \to V$ is any linear map, let us note that $v := l(u)$ is still a linear recurrence sequence and that μ_v divides μ_u.

If f is a linear endomorphism of $\mathbb{C}^N$, it is clear that its minimal polynomial is equal to the minimal polynomial of the linear recurrence sequence $n \mapsto f^n$. More generally, the same statement holds if f is a LF polynomial endomorphism of $\mathbb{C}^N$.

We now obtain

Proof of lemma 7.

By [4, lemma 6.2], any triangular automorphism is conjugate via a triangular automorphism either to an automorphism of the form

$$(aX + ap(Y), bY) \qquad (1)$$

where $a, b \neq 0$ and p satisfies the identity $p(bY) = ap(Y)$, or to an automorphism of the form

$$(aX, Y + c) \qquad (2)$$

where $a \neq 0$. However, the automorphism $(Y, X) \circ (aX, Y + c) \circ (Y, X)^{-1} = (X + c, aY)$ is of the form (1). As a conclusion, any $f \in \mathcal{LF}$ can be expressed $f = \varphi \circ t \circ \varphi^{-1}$ where $\varphi, t \in \mathcal{G}$ and t is of the form (1).

As noted in [4, p. 87], we have a simple expression for the n-fold iterate of t:

$$t^n = (a^n X + na^n p(Y), b^n Y).$$

Let us set $\psi := \varphi^{-1}$. We recall that $\varphi = (\varphi_1, \varphi_2)$ and $\psi = (\psi_1, \psi_2)$. Let (e_1, e_2) be the canonical basis of the $\mathbb{C}[X,Y]$-module $\mathbb{C}[X,Y]^2$. Since the family $\psi_1^i \psi_2^j$ for $i, j \geq 0$ is a basis of $\mathbb{C}[X,Y]$, the family $\psi_1^i \psi_2^j e_k$ is a basis of $\mathcal{E} = \mathbb{C}[X,Y]^2$. If $g \in \mathcal{E}$ and $j, k \in \{1,2\}$, let us denote by $\Pi_{j,k}(g)$ its $\psi_j e_k$-component.

Let us set $\lambda = p(0)$ and let us write $\varphi_k = \sum_{i,j} \varphi_{k,i,j} \, X^i Y^j$ for $k = 1, 2$.

An easy computation would show that:

$$\Pi_{1,k}(f^n) = \sum_{i \geq 1} i\lambda^{i-1} \varphi_{k,i,0} \, n^{i-1} a^{ni} \qquad \text{and} \qquad \Pi_{2,k}(f^n) = \sum_{i \geq 0} \lambda^i \varphi_{k,i,1} \, n^i (a^i b)^n.$$

But the matrix $\begin{bmatrix} \varphi_{1,1,0} & \varphi_{1,0,1} \\ \varphi_{2,1,0} & \varphi_{2,0,1} \end{bmatrix}$ corresponds to the linear part of φ so that it is invertible. Therefore at least one of the $\varphi_{k,1,0}$ is non-zero. By theorem 8, a is a root of the minimal polynomial of the linear recurrence sequence $n \mapsto \Pi_{1,k}(f^n)$. By the remark following this theorem, a is a root of the minimal polynomial of the linear recurrence sequence $n \mapsto f^n$. This means that $\mu_f(a) = 0$. In the same way, at least one of the $\varphi_{k,0,1}$ is non-zero showing that $\mu_f(b) = 0$. □

3.3 The trace

It is natural to set the following

Definition. If $f \in \mathcal{LF}$ has pseudo-eigenvalues $\{a, b\}$, its trace is $\operatorname{Tr} f := a + b$.

Remark. The trace is by construction an invariant of conjugation. It is well-known that the Jacobian determinant map $\operatorname{Jac} : \mathcal{G} \to \mathbb{C}^*$ also. In the locally finite case, we have of course $\operatorname{Jac} f = ab$.

Question 9. *Is the map* $\operatorname{Tr} : \mathcal{LF} \to \mathbb{C}$ *regular?*

This means that for any m the restricted map $\mathcal{LF}_{\leq m} \to \mathbb{C}$ is regular. This regularity would imply a positive answer to the following

Question 10. *Is the map* $\operatorname{Tr} : \mathcal{LF}_{\leq m} \to \mathbb{C}$ *continuous for the transcendental topology?*

Remark. This continuity would easily prove the most difficult point of our main theorem. If f, g are semisimple automorphisms such that g belongs to the closure of the conjugacy class of f, we want to show that they have the same pseudo-eigenvalues. Indeed, it is clear that $\operatorname{Jac} f = \operatorname{Jac} g$ and the above continuity would show that $\operatorname{Tr} f = \operatorname{Tr} g$.

Definition. Let $\mathcal{U}$ (resp. $\mathcal{S}$) be the set of LF polynomial automorphisms whose pseudo-eigenvalues are equal to 1 (resp. are different from 1).

Remarks. 1. By [8, theorem 2.3], $\mathcal{U}$ is the set of polynomial automorphisms f satisfying the following equivalent assertions:

(i) f is unipotent, i.e. f is annihilated by $(T-1)^d$ for some d;
(ii) f is the exponential of some locally nilpotent derivation of $\mathbb{C}[X,Y]$.

2. It is easy to check that $\mathcal{S}$ is the set of LF automorphisms admitting a single fixed point (in fact, we will see in proposition 12 below that we can get rid of the LF hypothesis).

3. Since $\mathcal{U} = \mathrm{Tr}^{-1}(\{2\}) \cap \mathrm{Jac}^{-1}(\{1\})$ and $\mathcal{S} = \{f \in \mathcal{LF},\, \mathrm{Tr}(f) \neq 1 + \mathrm{Jac}(f)\}$, the regularity of the trace map would imply directly that $\mathcal{U}$ (resp. $\mathcal{S}$) is closed (resp. open) in $\mathcal{LF}$.

Let us check that $\mathcal{U}$ is closed. If $m \geq 1$, let d be the dimension of $\mathcal{E}_{\leq m}$ and let $p(T) = (T-1)^d \in \mathbb{C}[T]$. By assertion (iv) of theorem 4, we get $\mathcal{U}_{\leq m} = \{f \in \mathcal{E}_{\leq m},\, p(f) = 0\}$. This shows that $\mathcal{U}_{\leq m}$ is closed in $\mathcal{E}_{\leq m}$ for any m, i.e. $\mathcal{U}$ is closed in $\mathcal{E}$.

We will show in the next subsection that $\mathcal{S}$ is open in $\mathcal{LF}$.

3.4 The set $\mathcal{S}$

Definition. If f, g are polynomial endomorphisms of $\mathbb{A}^2$, let us define their coincidence ideal $\Delta(f,g)$ as the ideal generated by the $f^*(p) - g^*(p)$, where p describes $\mathbb{C}[X,Y]$.

The coincidence ideal $\Delta(f, \mathrm{id})$ is called the fixed point ideal of f.

Remarks. 1. The closed points of $\mathrm{Spec}\, \mathbb{C}[X,Y]/\Delta(f,g)$ correspond to the points $\xi \in \mathbb{A}^2$ such that $f(\xi) = g(\xi)$.

2. Using the relation $f^*(uv) - g^*(uv) = f^*(u)[f^*(v) - g^*(v)] + g^*(v)[f^*(u) - g^*(u)]$, we see that if the algebra $\mathbb{C}[X,Y]$ is generated by the u_k $(1 \leq k \leq l)$, then the ideal $\Delta(f,g)$ is generated by the $f^*(u_k) - g^*(u_k)$ $(1 \leq k \leq l)$.

3. In particular, $\Delta(f,g) = \Big(f^*(X) - g^*(X), f^*(Y) - g^*(Y)\Big) = (f_1 - g_1, f_2 - g_2)$.

The computation of the set of fixed points of a triangular automorphism is easy and left to the reader. We obtain the following result (see also [4, lemma 3.8]).

Lemma 11. *If $f \in \mathcal{LF}$, the set of its fixed points is either empty, either a point of multiplicity 1 (if and only if $f \in \mathcal{S}$) or either a finite disjoint union of subvarieties isomorphic to $\mathbb{A}^1$.*

An automorphism admits exactly one fixed point with multiplicity 1 if and only if its fixed point ideal is a maximal ideal of $\mathbb{C}[X,Y]$. Using the amalgamated structure of $\mathcal{G}$, it is observed in [4] that a polynomial automorphism $f \in \mathcal{G}$ is either triangularizable (i.e. belongs to $\mathcal{LF}$) or conjugate to some cyclically reduced element g (see [23, I.1.3] or [4, p. 70] for the definition). In this latter case, the degree d of g is ≥ 2 and it is shown in [4, theorem 3.1] that $\dim \mathbb{C}[X,Y]/\Delta(g, \mathrm{id}) = d$. As a conclusion, we obtain the nice characterization of elements of $\mathcal{S}$:

Proposition 12. *If $f \in \mathcal{G}$, the following assertions are equivalent:*

(i) $f \in \mathcal{S}$*;*
(ii) f *has a unique fixed point of multiplicity* 1*;*
(iii) the fixed point ideal of f *is a maximal ideal of* $\mathbb{C}[X,Y]$*.*

The next result is taken from [8, lemma 4.1] and will be used to prove propositions 14 and 18 below.

Lemma 13. *Any triangularizable automorphism* f *can be triangularized in a "good" way with respect to the degree: there exist a triangular automorphism* t *and an automorphism* φ *such that* $f = \varphi \circ t \circ \varphi^{-1}$ *with* $\deg f = \deg t\, (\deg \varphi)^2$*.*

The vector space $\mathbb{A}^2$ will be endowed with the norm $\| (\alpha,\beta) \| = \sqrt{|\alpha|^2+|\beta|^2}$. The open (resp. closed) ball of radius $R \geq 0$ centered at a point $\xi \in \mathbb{A}^2$ will be denoted by $B_{\xi,R}$ (resp. $B'_{\xi,R}$). If $\xi = 0$, we will write B_R (resp. B'_R) instead of $B_{0,R}$ (resp. $B'_{0,R}$).

Since $\mathcal{E}$ is composed of C^∞ maps from $\mathbb{A}^2$ to $\mathbb{A}^2$, it is endowed with the C^k-topology (for each $k \geq 0$) which is the topology of uniform convergence of the k first derivatives on all compact subsets. However, $\mathcal{E}_{\leq m}$ being a finite-dimensional complex vector space, it admits a unique Hausdorff topological vector space structure. Therefore, the C^k-topology on $\mathcal{E}_{\leq m}$ is nothing else than the transcendental topology. A subset of some topological space is called constructible when it is a finite union of locally closed subsets. We finish these topological remarks by recalling that for any constructible subset in some complex algebraic variety, the (Zariski-)closure coincide with the transcendental closure (see for example [19]).

Proposition 14. $\mathcal{S}$ *is open in* $\mathcal{LF}$*.*

Proof. We want to show that $\mathcal{S}_{\leq m}$ is open in $\mathcal{LF}_{\leq m}$.

Claim. $\mathcal{S}_{\leq m}$ is constructible in $\mathcal{LF}_{\leq m}$.

Let $\mathcal{T}$ be the variety of triangular automorphisms of the form $(aX + p(Y), bY + c)$ where $a, b \in \mathbb{C} \setminus \{0,1\}$, $c \in \mathbb{C}$ and $p \in \mathbb{C}[Y]$ is a polynomial of degree $\leq m$.

The image W of the morphism $\mathcal{G}_{\leq m} \times \mathcal{T} \to \mathcal{G}$, $(\varphi, t) \mapsto \varphi \circ t \circ \varphi^{-1}$ is constructible and $\mathcal{S}_{\leq m} = W \cap \mathcal{LF}_{\leq m}$ by lemma 13 so that the claim is proved.

It is enough to show that $\mathcal{S}_{\leq m}$ is open for the transcendental topology. Let f be a given element of $\mathcal{S}_{\leq m}$ and let $\xi \in \mathbb{A}^2$ be its fixed point. The map $F := f - \mathrm{id}$ is a local diffeomorphism near ξ since $F'(\xi)$ is invertible. Let $\varepsilon, \eta > 0$ be such that $B_\eta \subseteq F(B_{\xi,\varepsilon})$ and $\forall\, x \in B_{\xi,\varepsilon}, |\det F'(x)| \geq \eta$. If g is "near" f for the C^1-topology, then $G := g - \mathrm{id}$ will be "near" F so that we will have $B_{\eta/2} \subseteq G(B_{\xi,\varepsilon})$ and $\forall\, x \in B_{\xi,\varepsilon}, |\det G'(x)| \geq \eta/2$. Therefore, g will have an isolated fixed point in $B_{\xi,\varepsilon}$. If $g \in \mathcal{LF}$, lemma 11 shows us that $g \in \mathcal{S}$. □

The next statement is given in [11, p. 49] (cf. the application of theorem 3). The result is also given for the field of rationals in [18, p. 312]. However, the proof remains unchanged for the field of complex numbers. Finally, [22, § 57] contains a similar result.

Theorem 15. *Let $K := d + (sd)^{2^n}$. If $p, p_1, \ldots, p_s \in \mathbb{C}[X_1, \ldots, X_n]$ are of degree $\leq d$ and if $p \in (p_1, \ldots, p_s)$, there exist $\lambda_1, \ldots, \lambda_s \in \mathbb{C}[X_1, \ldots, X_n]$ such that*

(i) $p = \sum_{1 \leq i \leq s} \lambda_i\, p_i$ *and* *(ii)* $\deg \lambda_i \leq K$ *for all* i.

If $f \in \mathcal{S}$, its fixed point $\xi = (\alpha, \beta) \in \mathbb{A}^2$ is implicitly defined by the equality of the ideals $(f_1 - X, f_2 - Y)$ and $(X - \alpha, Y - \beta)$. Using theorem 15, one can express more "effectively" α, β in terms of f_1, f_2. Indeed, if $m \geq 1$ and $K_m := m + (2m)^4$, then for any $f \in \mathcal{S}_{\leq m}$ there exist $\lambda_1, \ldots, \lambda_4 \in \mathbb{C}[X, Y]$ of degree $\leq K_m$ such that $X - \alpha = \lambda_1\,(f_1 - X) + \lambda_2\,(f_2 - Y)$ and $Y - \beta = \lambda_3\,(f_1 - X) + \lambda_4\,(f_2 - Y)$. Even with such "effective" results, we were not able to answer the following

Question 16. *Is the map $Fix : \mathcal{S} \to \mathbb{A}^2$ sending $f \in \mathcal{S}$ to its unique fixed point regular?*

This means that for any m the restricted map $\mathcal{S}_{\leq m} \to \mathbb{A}^2$ is regular. The proof of proposition 14 shows us at least that it is continuous for the transcendental topology.

3.5 Semisimple automorphisms

According to [8], a plane polynomial automorphism f is said to be semisimple if the following equivalent assertions are satisfied:

(i) f^* is semisimple (i.e. $\mathbb{C}[X, Y]$ admits a basis of eigenvectors);

(ii) $f \in \mathcal{LF}$ and μ_f has single roots;

(iii) f admits a vanishing polynomial with single roots.

The class of semisimple automorphisms is invariant by conjugation. Therefore, it results from proposition 18 below that (i-iii) are still equivalent to:

(iv) f is diagonalizable.

Lemma 17. *If $t = (aX + p(Y), bY + c)$ is a triangular semisimple automorphism, there exists a triangular automorphism χ of the same degree such that $t = \chi \circ (aX, bY) \circ \chi^{-1}$.*

Proof.

First step. Reduction to the case $c = 0$.

If $b = 1$, let us show that $c = 0$. The second coordinate of the nth iterate t^n is $Y + nc$. Since t is semisimple, the sequence $n \mapsto Y + nc$ must be of exponential type showing that $c = 0$.

If $b \neq 1$, set $l := (X, Y + \frac{c}{b-1})$ and replace t by $l \circ t \circ l^{-1} = (aX + p(Y - \frac{c}{b-1}), bY)$.

Second step. Reduction to the case $p = 0$.

If $\chi := (X + q(Y), Y)$, we get $\chi \circ (aX, bY) \circ \chi^{-1} = (aX + q(bY) - aq(Y), bY)$. Let us write $p = \sum_k p_k Y^k$. To show the existence of q (of the same degree as p) satisfying $q(bY) - aq(Y) = p(Y)$ it is enough to show that $a = b^k$ implies $p_k = 0$.

For any $n \geq 0$, let u_n be the Y^k-coefficient of the first component of t^n. If $a = b^k$, we get $u_{n+1} = au_n + p_k a^n$, so that $u_n = na^{n-1} p_k$. The sequence $n \mapsto u_n$ being of exponential type, we obtain $p_k = 0$. □

Combining lemmas 13 and 17, any semisimple automorphism can be written as

$$f = (\varphi \circ \chi) \circ (aX, bY) \circ (\varphi \circ \chi)^{-1} \text{ with } \deg f \;=\; \deg \chi \; (\deg \varphi)^2.$$

Since $\deg(\varphi \circ \chi) \leq \deg \varphi \, \deg \chi \leq \deg f$, we get:

Proposition 18. *Any semisimple automorphism f can be written as $f = \psi \circ (aX, bY) \circ \psi^{-1}$ where ψ is an automorphism satisfying $\deg \psi \leq \deg f$.*

Since the automorphisms (aX, bY) and (aY, bX) are conjugate, we obtain:

Corollary 19. *Two semisimple automorphisms are conjugate if and only if they have the same pseudo-eigenvalues.*

If $f \in \mathcal{G}$, let $\mathcal{C}(f) := \{\varphi \circ f \circ \varphi^{-1}, \varphi \in \mathcal{G}\}$ be its conjugacy class. Recall that $\mathcal{C}(f)$ is closed in $\mathcal{G}$ if and only if each $\mathcal{C}(f)_{\leq m}$ is closed in $\mathcal{G}_{\leq m}$.

Corollary 20. *If f is a semisimple automorphism, then $\mathcal{C}(f)_{\leq m}$ is constructible in $\mathcal{E}_{\leq m}$ (for any $m \geq 1$).*

Proof. We can assume that $f = (aX, bY)$. The image A of the map $\mathcal{G}_{\leq m} \to \mathcal{G}$, $\varphi \mapsto \varphi \circ f \circ \varphi^{-1}$ is constructible and $\mathcal{C}(f)_{\leq m} = A \cap \mathcal{G}_{\leq m}$ by proposition 18. □

Remarks. 1. This result shows us that the Zariski-closure of $\mathcal{C}(f)_{\leq m}$ coincide with its transcendental closure (see subsection 4.2).

2. One could show that $\mathcal{C}(f)_{\leq m}$ is constructible in $\mathcal{E}_{\leq m}$ for any f, but we do not need this result.

Lemma 21. *If f is semisimple, any element of $\overline{\mathcal{C}(f)_{\leq m}}$ also.*

Proof. By proposition 18, we may assume that $f = (aX, bY)$. Any element which is linearly conjugate to f is annihilated by μ_f, but for a general element of $\mathcal{C}(f)$, this is no longer true. However, we will build a polynomial p with single roots annihilating any element of $\mathcal{C}(f)_{\leq m}$. Still by proposition 18, any $g \in \mathcal{C}(f)_{\leq m}$ can be written $g = \varphi \circ f \circ \varphi^{-1}$ with $\deg \varphi \leq m$. Therefore, for any $n \geq 0$, we have $g^n = \varphi \circ (a^n X, b^n Y) \circ \varphi^{-1}$. If we set $\Omega := \{a^k b^l, \, 0 \leq k + l \leq m\}$, there exists a family of polynomial endomorphisms h_ω ($\omega \in \Omega$) such that $g^n = \sum_{\omega \in \Omega} \omega^n h_\omega$ for any n. By theorem 8, we get $p(g) = 0$, where $p(T) := \prod_{\omega \in \Omega} (T - \omega)$. The equality $p(g) = 0$ remains true if $g \in \overline{\mathcal{C}(f)_{\leq m}}$. □

4 Proof of the main theorem.

4.1 Algebraic lemma

The aim of this subsection is to prove the following result which in some sense means that the spectrum of a linear endomorphism remains unchanged at the limit (see lemma 6).

Lemma 22. *Let $f = (aX, bY) \in \mathcal{G}$. If $(\alpha X, \beta Y) \in \overline{\mathcal{C}(f)_{\leq m}}$, then $< \alpha, \beta > = < a, b >$.*

Our proof will use the valuative criterion that we give below. Even if such a criterion sounds familiar (see for example [20, chapter 2, Section 1, pp. 52-54] or [9, Section 7]), we have given a brief proof of it in [7].

Let $\mathbb{C}[[t]]$ be the algebra of complex formal power series and let $\mathbb{C}((t))$ be its quotient field. If V is a complex algebraic variety and A a complex algebra, $V(A)$ will denote the points of V with values in A, i.e. the set of morphisms $\operatorname{Spec} A \to V$. If v is a closed point of V and $\varphi \in V\Big(\mathbb{C}((t))\Big)$, we will write $v = \lim_{t \to 0} \varphi(t)$ when:

(i) the point $\varphi : \operatorname{Spec} \mathbb{C}((t)) \to V$ is a composition $\operatorname{Spec} \mathbb{C}((t)) \to \operatorname{Spec} \mathbb{C}[[t]] \to V$;

(ii) v is the point $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C}[[t]] \to V$.

For example, if $V = \mathbb{A}^1$ and $\varphi \in V\Big(\mathbb{C}((t))\Big) = \mathbb{C}((t))$, we will write $v = \lim_{t \to 0} \varphi(t)$ when $\varphi \in \mathbb{C}[[t]]$ and $v = \varphi(0)$.

Valuative criterion. Let $f : V \to W$ be a morphism of complex algebraic varieties and let w be a closed point of W. The two following assertions are equivalent:

(i) $w \in \overline{f(V)}$;

(ii) $w = \lim_{t \to 0} f(\varphi(t))$ for some $\varphi \in V\Big(\mathbb{C}((t))\Big)$.

Remark. Note the analogy with the metric case where $w \in \overline{f(V)}$ if and only if there exists a sequence $(v_n)_{n \geq 1}$ of V such that $w = \lim_{n \to +\infty} f(v_n)$.

Proof of lemma 22. Assume that $\gamma := (\alpha X, \beta Y) \in \overline{\mathcal{C}(f)_{\leq m}}$.

If $\Omega := \{a^k b^l, \, 0 \leq k+l \leq m\}$, the proof of lemma 21 tells us that $\alpha, \beta \in \Omega \subseteq < a, b >$, so that $< \alpha, \beta > \subseteq < a, b >$.

Let us prove the reverse inclusion. By proposition 18, $\mathcal{C}(f)_{\leq m}$ is included in the image of the map $\mathcal{G}_{\leq m} \to \mathcal{G}$, $\varphi \mapsto \varphi^{-1} \circ f \circ \varphi$. Using the above valuative criterion, we get the existence of $\varphi \in \mathcal{G}_{\leq m}\Big(\mathbb{C}((t))\Big)$ such that if $g := \varphi^{-1} \circ f \circ \varphi \in \mathcal{G}\Big(\mathbb{C}((t))\Big)$, then $\gamma = \lim_{t \to 0} g_t$. We have $g_t^* = \varphi_t^* \circ f^* \circ (\varphi_t^*)^{-1}$ as linear endomorphisms of the $\mathbb{C}((t))$-vector space $\mathbb{C}((t))[X, Y]$. Therefore $u_{k,l} := \varphi_t^*(X^k Y^l)$ is an eigenvector of g_t^* associated with the eigenvalue $a^k b^l$. Let $m \in \mathbb{Z}$ be such that $v_{k,l} := t^m u_{k,l}$ admits a non-zero limit $\overline{v_{k,l}}$ when t goes to zero. We have $g_t^*(v_{k,l}) = a^k b^l v_{k,l}$ and setting $t = 0$, we get $\gamma^*(\overline{v_{k,l}}) = a^k b^l \overline{v_{k,l}}$. Hence $a^k b^l$ is an eigenvalue of γ^*, so that $a^k b^l \in < \alpha, \beta >$. □

4.2 Topological lemmas

Lemma 23. *Let $f = (aX, bY) \in \mathcal{G}$. If $(\alpha X, \beta Y) \in \overline{\mathcal{C}(f)_{\leq m}}$ with $\alpha, \beta \neq 1$, then $\{\alpha, \beta\} = \{a, b\}$.*

Proof.

Claim. For any $\varepsilon > 0$ there exists a C^0-neighborhood U of $\gamma := (\alpha X, \beta Y)$ in $\mathcal{E}_{\leq m}$ such that any $g \in U$ admits a fixed point in B_ε.

Indeed, there exists an $\eta > 0$ such that $B_\eta \subseteq (\gamma - \mathrm{id})(B_\varepsilon)$, so that there exists a C^0-neighborhood U of γ such that any $g \in U$ satisfies $0 \in (g - \mathrm{id})(B_\varepsilon)$.

Let $(g_n)_{n\geq 1}$ be a sequence of $\mathcal{C}(f)_{\leq m}$ such that $\gamma = \lim_{n\to\infty} g_n$ for the C^1-topology. By the claim, there exists a sequence $(\xi_n)_{n\geq 1}$ of points of $\mathbb{A}^2$ such that $g_n(\xi_n) = \xi_n$ and $\lim_{n\to\infty} \xi_n = 0$. Therefore, we have $\gamma'(0) = \lim_{n\to\infty} g_n'(\xi_n)$ for the usual topology of $M_2(\mathbb{C})$. Since the pseudo-eigenvalues of a LF automorphism admitting a fixed point are equal to the eigenvalues of its derivative at that fixed point, we get $\operatorname{Tr} g_n'(\xi_n) = a + b$. However, we have $\operatorname{Tr} \gamma'(0) = \alpha + \beta$, so that we get $\alpha + \beta = a + b$. The equality $\alpha\beta = ab$ (obtained using the Jacobian) gives us $\{\alpha, \beta\} = \{a, b\}$. □

We will use the following convexity lemma.

Lemma 24. *If B' is a closed ball in an euclidean space, there exists a C^2-neighborhood of the identity map on the space such that for any g in this neighborhood, $g(B')$ is convex.*

Sketch of proof. Let us endow the space $E = \mathbb{R}^N$ with the usual euclidean norm $\| x \| := (\sum_j x_j^2)^{1/2}$, where $x = (x_1, \ldots, x_N)$. If l is a linear endomorphism of E, we set $|||\, l \,||| := \sup\{ \| l(x) \| \,/\, \| x \|,\ x \neq 0 \in E\}$. We may assume that B' is the closed unit ball $B' = \{x \in E, \varphi(x) \leq 1\}$, where $\varphi : E \to \mathbb{R}$ is defined by $\varphi(x) = \| x \|^2$.

Let $h : E \to E$ be a map of class C^1 satisfying $|||\, h'(x) \,||| \leq 1/2$ for any x in a convex open subset C of E. The map h is $1/2$-Lipschitzian on C:

$$\forall\, x, y \in C,\ \| h(y) - h(x) \| \leq \tfrac{1}{2}\ \| y - x \|.$$

Therefore, the map $x \mapsto x + h(x)$ defines a C^1-diffeomorphism on C.

Taking for C the open ball $B(0,3)$, we obtain the existence of a C^1-neighborhood U of the identity map on E such that for any $g \in U$, g defines a C^1-diffeomorphism on $B(0,3)$. Restricting U, we may even assume that $g(B') \subseteq B(0,2) \subseteq g(B(0,3))$, so that $g(B') = \{x \in B(0,2),\ \varphi \circ g^{-1}(x) \leq 1\}$ where g^{-1} denotes the inverse bijection of $g : B(0,3) \to g(B(0,3))$.

Let $\psi : C \to \mathbb{R}$ be a map of class C^2, where C is a convex open subset of E. It is clear that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) in the following assertions:

(i) $\forall\, x \in C$, $\psi''(x)$ is positive definite, i.e. $\forall\, u \neq 0 \in \mathbb{R}^N$, $\psi''(x)(u,u) > 0$;

(ii) ψ is strictly convex: $\forall\, \lambda \in]0,1[$, $\forall\, x \neq y \in C$, $\psi((1-\lambda)x + \lambda y) < (1-\lambda)\psi(x) + \lambda\psi(y)$;

(iii) ψ is convex;

(iv) the set $\{x \in C,\ \psi(x) \leq 1\}$ is convex.

One would easily check that there exists a C^2-neighborhood V of the identity map on $B(0,2) \subseteq E$ such that for any $g \in V$, the differential $(\varphi \circ g)''(x)$ is positive definite for each $x \in B(0,2)$.

Furthermore, one would also easily show that there exists a C^2-neighborhood $W \subseteq U$ of the identity map such that for any $g \in W$, the restriction of g^{-1} to $B(0,2)$ will belong to V. We recall that g^{-1} denotes the inverse bijection of $g : B(0,3) \to g(B(0,3))$.

It is now clear that $g(B')$ is convex when $g \in W$. $\square$

Remark. Let $B' := \{\rho e^{i\theta},\, \theta \in \mathbb{R},\, 0 \leq \rho \leq 1\}$ be the unit disc in $\mathbb{C}$. If g is "near" the identity for the C^2-topology, then we will have $g(B') = \{\rho e^{i\theta},\, \theta \in \mathbb{R},\, 0 \leq \rho \leq r(\theta)\}$ where $r : \mathbb{R} \to \mathbb{R}$ is a 2π -periodic map which is "near" the map $s \equiv 1$ for the C^2-topology. The curvature of the parametrized curve $\theta \mapsto r(\theta)e^{i\theta}$ at the point θ is well-known to be $C = \dfrac{r^2 + 2\,r'^{\,2} - r\,r''}{(r^2 + r'^{\,2})^{\frac{3}{2}}}$. If r is "near" s for the C^2-topology, it is clear that $C > 0$ at each point, showing that $g(B')$ is convex.

Lemma 25. *If f is a finite-order automorphism, $\mathcal{C}(f)$ is closed in $\mathcal{G}$.*

Proof. We may assume that $f = (aX, bY)$ where $a^q = b^q = 1$ for some $q \geq 1$ (cf. [13, 16], the introduction or proposition 18). By proposition 18 and lemma 21, it is enough to show that if $\gamma = (\alpha X, \beta Y) \in \overline{\mathcal{C}(f)_{\leq m}}$ for some m, then $\{\alpha, \beta\} = \{a, b\}$.

We begin to note that the equality $g^q = \mathrm{id}$ holds for any $g \in \mathcal{C}(f)$. Therefore, this equality also holds for any $g \in \overline{\mathcal{C}(f)}$.

Claim. For any $\varepsilon > 0$ there exists a C^2-neighborhood U of γ in $\mathcal{E}_{\leq m}$ such that if $g \in U$ and $g^q = \mathrm{id}$, then g admits a fixed point in B'_ε.

Since $\alpha^q = \beta^q = 1$, we have $\gamma(B'_\varepsilon) = B'_\varepsilon$. It is enough to take for U a C^2-neighborhood of γ such that for any $g \in U$ and any $0 \leq k < q$, $g^k(B'_\varepsilon)$ is a convex set containing the origin (such a neighborhood exists by lemma 24). Indeed, if $g \in U$ and $g^q = \mathrm{id}$, then $K := \bigcap_{0 \leq k < q} g^k(B'_\varepsilon)$ is a non-empty compact convex set such that $g(K) = K$. By Brouwer fixed point theorem, g admits a fixed point in $K \subseteq B'_\varepsilon$ and the claim is proved.

We finish the proof exactly as in lemma 23. $\square$

4.3 The proof

($\Longrightarrow$) Thanks to proposition 18 it is enough to show that if $f = (aX, bY) \in \mathcal{G}$, then $\mathcal{C}(f)$ is closed in $\mathcal{G}$. Thanks to lemma 21 it is enough to show that if $\gamma = (\alpha X, \beta Y) \in \overline{\mathcal{C}(f)_{\leq m}}$ for some m, then $\{\alpha, \beta\} = \{a, b\}$.

First case. $\alpha, \beta \neq 1$.

We conclude by lemma 23.

Second case. α or $\beta = 1$. We can assume that $\alpha = 1$.

Since $\operatorname{Jac}\gamma = \operatorname{Jac} f$, we have $\beta = ab$. But $< a, b > = < \beta >$ by lemma 22, so that there exist $k, l \geq 0$ such that $a = \beta^k$, $b = \beta^l$.

First subcase. β is not a root of unity.

The equality $\beta = ab$ gives us $\beta = \beta^{k+l}$, so that $1 = k + l$. We get $\{k, l\} = \{0, 1\}$, so that $\{a, b\} = \{1, \beta\} = \{\alpha, \beta\}$.

Second subcase. β is a root of unity.

It is clear that a, b are also roots of unity. Therefore, f is a finite-order automorphism and we conclude by lemma 25.

($\Longleftarrow$) Let f be any polynomial automorphism. We want to show that $\overline{\mathcal{C}(f)}$ contains a semisimple polynomial automorphism. It is sufficient to show that it contains a linear automorphism. Indeed, in the linear group it is well-known that any conjugacy class contains in its closure a (linear) semisimple automorphism.

First case. f is triangularizable.

We can assume that $f = (aX + p(Y), bY + c)$. If $l_t := (tX, Y)$ and $r_t := (X, tY) \in \mathcal{G}$ for $t \in \mathbb{C}^*$, we have $\lim_{t \to 0} l_t \circ f \circ (l_t)^{-1} = (aX, bY + c)$. Therefore, $u := (aX, bY + c) \in \overline{\mathcal{C}(f)}$. But $r_t \circ u \circ (r_t)^{-1} \in \overline{\mathcal{C}(f)}$ for any $t \neq 0$ and $\lim_{t \to 0} r_t \circ u \circ (r_t)^{-1} = (aX, bY)$.

Second case. f is not triangularizable.

We can assume that f is cyclically reduced of degree $d \geq 2$. By [4, theorem 3.1], f has exactly d fixed points (counting the multiplicities). In particular, it has a fixed point and by conjugating we can assume that it fixes the origin. Therefore, if $h_t := (tX, tY) \in \mathcal{G}$ for $t \neq 0$, then $\lim_{t \to 0} (h_t)^{-1} \circ f \circ h_t$ is equal to the linear part of f. □

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