

# EXTENSION OF AUTOMORPHISMS OF RATIONAL SMOOTH AFFINE CURVES

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ABSTRACT. We provide the existence, for every complex rational smooth affine curve  $\Gamma$ , of a linear action of  $\text{Aut}(\Gamma)$  on the affine 3-dimensional space  $\mathbb{A}^3$ , together with a  $\text{Aut}(\Gamma)$ -equivariant closed embedding of  $\Gamma$  into  $\mathbb{A}^3$ .

It is not possible to decrease the dimension of the target, the reason for this obstruction is also precisely described.

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## 1. INTRODUCTION

Throughout this article, all varieties are algebraic varieties defined over the field  $\mathbb{C}$  of complex numbers. The affine (resp. projective)  $n$ -space is denoted by  $\mathbb{A}^n$  (resp.  $\mathbb{P}^n$ ).

It is well known that any smooth affine variety  $X$  of dimension  $n$  admits a closed embedding into  $\mathbb{A}^m$ , when  $m \geq 2n + 1$  [Sri91, Theorem 1]. If moreover  $m \geq 2n + 2$ , then, by a result of Nori, Srinivas and Kaliman (see [Sri91] and [Kal91]), any two closed embeddings  $\iota, \iota': X \rightarrow \mathbb{A}^m$  are equivalent in the sense that there exists an automorphism  $f \in \text{Aut}(\mathbb{A}^m)$  such that  $\iota' = f \circ \iota$ .

In particular, if  $\iota: X \rightarrow \mathbb{A}^m$  is a closed embedding of a smooth affine variety of dimension  $n$  into some affine space of dimension  $m \geq 2n + 2$ , then it follows that every automorphism  $\varphi$  of  $X$  extends to an automorphism of the ambient space  $\mathbb{A}^m$ , since the two embeddings  $\iota \circ \varphi$  and  $\iota$  are equivalent.

However, Derksen, Kutzschebauch and Winkelmann showed in [DKW99] that it is not always possible to extend the group structure of  $\text{Aut}(X)$ , i.e. to find a closed embedding  $\iota: X \rightarrow \mathbb{A}^m$  and an action of  $\text{Aut}(X)$  on  $\mathbb{A}^m$  that restricts on  $X$  to the action of  $\text{Aut}(X)$  on it. More precisely, they proved that there does not exist, for any integer  $m$ , any injective group homomorphism from  $\text{Aut}(\mathbb{C}^* \times \mathbb{C}^*) \cong \text{GL}_2(\mathbb{Z}) \rtimes (\mathbb{C}^*)^2$  to the group  $\text{Diff}(\mathbb{R}^m)$  of diffeomorphisms of  $\mathbb{R}^m$ .

Recall that, if  $G$  is an algebraic group acting on an affine variety  $X$ , then  $X$  admits a  $G$ -equivariant closed embedding into a finite dimensional  $G$ -module (see [Bor91, Proposition 1.12, p. 56]). In particular, there exist, for every smooth affine curve  $\Gamma$ , a linear action of  $\text{Aut}(\Gamma)$  on an affine space  $\mathbb{A}^m$  and a  $\text{Aut}(\Gamma)$ -equivariant closed embedding of  $\Gamma$  into  $\mathbb{A}^m$ . A natural question is then to find the smallest possible  $m$ .

In this article, we settle the case of rational smooth affine curves. In this setting, the proof of Borel only gives the embedding dimension  $m = 2 \cdot |\text{Aut}(\Gamma)|$ , when the

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automorphism group  $\text{Aut}(\Gamma)$  is finite. However, our main result shows that it is already possible to obtain  $m = 3$ :

**Theorem 1.** *Every rational smooth affine curve  $\Gamma$  admits an  $\text{Aut}(\Gamma)$ -equivariant closed embedding into the affine space  $\mathbb{A}^3$ . Furthermore, there exist such embeddings for which the action of  $\text{Aut}(\Gamma)$  on  $\mathbb{A}^3$  is linear.*

It is easy to construct closed embeddings into the affine plane  $\mathbb{A}^2$  for all rational smooth affine curves  $\Gamma$ . But it is of course not possible in general to ask for  $\text{Aut}(\Gamma)$ -equivariant embeddings into  $\mathbb{A}^2$ . Indeed, there exist rational smooth affine curves whose automorphism groups are isomorphic to the alternating group  $\mathfrak{A}_4$ , to  $\mathfrak{A}_5$ , or to the symmetric group  $\mathfrak{S}_4$  (see Section 6) and it is well known that the group  $\mathfrak{A}_4$  has no faithful representation of dimension two. Since all finite subgroups of  $\text{Aut}(\mathbb{A}^2)$  are linearizable, it follows that we cannot embed equivariantly such a curve into the plane, even if we allow non linear actions on  $\mathbb{A}^2$ .

In fact, we establish stronger impossibility statements showing that it would be also too optimistic in general to look for closed embeddings into  $\mathbb{A}^2$  in such a way that every single automorphism of the curve extends to an automorphism of the ambient space (see Corollary 2.6).

**Theorem 2.** *There exist rational smooth affine curves  $\Gamma$  with  $\text{Aut}(\Gamma) \neq 1$  and such that for every closed embedding of  $\Gamma$  into  $\mathbb{A}^2$ , the identity on  $\Gamma$  is its only automorphism that extends to an automorphism of  $\mathbb{A}^2$ .*

Let us also emphasize that Theorem 1 cannot be generalized to all smooth affine curves. Actually, there even exist, for every natural number  $n$ , smooth affine curves  $\Gamma$  which do not admit any  $\text{Aut}(\Gamma)$ -equivariant closed embedding into  $\mathbb{A}^n$ .

To see this, recall that every finite group  $G$  is equal to the automorphism group of a smooth projective curve, and thus of an affine one [Gre63], and take a smooth affine curve  $\Gamma_n$  whose automorphism group is isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^{n+1}$ . Then,  $\Gamma_n$  does not admit any  $\text{Aut}(\Gamma_n)$ -equivariant embedding into  $\mathbb{A}^n$ , because  $(\mathbb{Z}/2\mathbb{Z})^{n+1}$  does not act faithfully on  $\mathbb{A}^n$ . Indeed, by Smith theory, the action of a finite  $p$ -group on  $\mathbb{A}^n$  has always a fixed point (see e.g. [Bre72, Th. 7.11, p 145], [PR86, p. 204], or [DKW99, Proposition 1]) and the induced tangential (linear) representation at that fixed point should be faithful too (see e.g. [DKW99, Lemma 4]).

It would however be interesting to know what happens in the case of smooth affine curves of genus 1. Sathaye proved that such curves admit closed embeddings into  $\mathbb{A}^2$  [Sat77]. Nevertheless, we do not know what is the minimal  $m$  (if it exists) such that every smooth affine curve  $\Gamma$  of genus 1 admits an  $\text{Aut}(\Gamma)$ -equivariant closed embedding into  $\mathbb{A}^m$ .

The article is organized as follows.

Section 2 concerns embeddings of rational smooth affine curves into the affine plane. We give examples of automorphisms of such curves that do not extend, and prove Theorem 2 (see Corollary 2.6).

Section 3 is devoted to the study of embeddings of smooth rational curves into  $\mathbb{A}^3$  whose images are contained in a hyperplane. We prove that they are all equivalent and thus that any two closed embeddings of a rational smooth affine curve into  $\mathbb{A}^2$  become

equivalent, when seen as embeddings in  $\mathbb{A}^3$  (Proposition 3.1). This answers a question of Bhatwadekar and Srinivas in this case.

In section 4 we realize every non-empty subset of  $\mathbb{P}^1$  that is invariant by a subgroup  $H$  of  $\text{Aut}(\mathbb{P}^1)$  as the fixed-point set of a  $H$ -equivariant endomorphism of  $\mathbb{P}^1$  (Corollary 4.4). This result is used in Section 5 to prove Theorem 1 (see Theorem 5.2). Explicit formulas are given in Section 6.

## 2. EMBEDDINGS OF RATIONAL SMOOTH AFFINE CURVES INTO THE PLANE

Let us recall that every rational smooth affine curve  $\Gamma$  is isomorphic to  $\mathbb{P}^1 \setminus \Lambda$ , where  $\Lambda$  is a finite set of  $r \geq 1$  points.

In particular, it admits a closed embedding into  $\mathbb{A}^2$ . Indeed,  $\Gamma$  can also be seen as the complement in  $\mathbb{A}^1$  of a finite number (possibly zero) of points and we can consider the closed embedding  $\tau: \Gamma \rightarrow \mathbb{A}^2$  given by  $x \mapsto (x, \frac{1}{P(x)})$ , where  $P \in \mathbb{C}[x]$  is a polynomial whose roots are exactly the removed points. Note that the image of  $\tau$  is the curve of  $\mathbb{A}^2$  defined by the equation  $P(x)y = 1$ .

Moreover, the automorphism group  $\text{Aut}(\Gamma)$  of the curve  $\Gamma = \mathbb{P}^1 \setminus \Lambda$  is equal to the group of automorphisms of  $\mathbb{P}^1$  that preserve the set  $\Lambda$ . This gives a group homomorphism from  $\text{Aut}(\Gamma)$  to the symmetric group  $\text{Sym}_r$ . Note that this homomorphism is injective if and only if  $r \geq 3$ .

If  $r$  is equal to 1 or 2, then  $\Gamma$  is isomorphic to  $\mathbb{A}^1$  or  $\mathbb{A}^1 \setminus \{0\}$ , and its automorphism group is  $\mathbb{C}^* \times \mathbb{C}$  or  $\{\pm 1\} \times \mathbb{C}^*$  respectively. If  $r \geq 3$ , then  $\text{Aut}(\Gamma)$  is a finite group.

The Abhyankar-Moh-Suzuki theorem states that all closed embeddings of  $\mathbb{A}^1$  into  $\mathbb{A}^2$  are equivalent to the one given by  $t \mapsto (t, 0)$ . This implies that every automorphism of an affine line embedded into  $\mathbb{A}^2$  extends to an automorphism of the ambient space. If  $r \geq 2$  we can on the contrary construct embeddings of the curve  $\Gamma$  which do not have this property. Actually, we can choose embeddings such that, except the identity, no automorphisms of  $\Gamma$  extend.

**Lemma 2.1.** *Let  $\Gamma = \mathbb{A}^1 \setminus \Delta$ , where  $\Delta$  is a non-empty finite set. Then, there exist infinitely many non-equivalent closed embeddings  $\iota: \Gamma \rightarrow \mathbb{A}^2$  such that the identity is the only automorphism of  $\mathbb{A}^2$  that preserves  $\iota(\Gamma)$ .*

*Proof.* We can assume that  $\Delta = \{0, a_1, \dots, a_m\}$  with  $m \geq 0$  and  $a_1, \dots, a_m \in \mathbb{C} \setminus \{0, 1\}$ .

For every  $k \geq 2$ , we denote by  $\iota_k: \Gamma \rightarrow \mathbb{A}^2$  the embedding given by  $x \mapsto \left(x, \frac{x-1}{x^k \prod_{i=1}^m (x-a_i)}\right)$ . It induces an isomorphism between  $\Gamma$  and the curve  $\iota_k(\Gamma)$  defined by the equation  $x = yx^k \prod_{i=1}^m (x - a_i) + 1$ .

We first remark that any automorphism of  $\mathbb{A}^2$  that sends  $\iota_k(\Gamma)$  onto a curve of degree at most  $\deg(\iota(\Gamma)) = k + m + 1$  is necessarily affine. If  $f: (x, y) \mapsto (f_1(x, y), f_2(x, y))$  is the inverse of such an automorphism, we get indeed:

$$\deg(f_1 - f_2(f_1)^k \prod_{i=1}^m (f_1 - a_i) - 1) = (k + m) \deg f_1 + \deg f_2 \leq k + m + 1.$$

This implies that  $\deg(f_1) = \deg(f_2) = 1$ , i.e. that  $f$  (and its inverse too) is affine. In particular, all above embeddings are non-equivalent. We now show that the identity is the only affine automorphism of  $\mathbb{A}^2$  that preserves the curve  $\iota_k(\Gamma)$ .

Any such automorphism extends to an automorphism  $\tau$  of  $\mathbb{P}^2$  preserving the line at infinity given by  $z = 0$  and the curve of equation

$$xz^{k+m} - yx^k \prod_{i=1}^m (x - a_i z) - z^{k+m+1} = 0.$$

On the line at infinity we get the two points  $[0 : 1 : 0]$  and  $[1 : 0 : 0]$ . The point  $[1 : 0 : 0]$  is smooth with tangent  $y = 0$  and the point  $[0 : 1 : 0]$  is singular with tangent cone given by  $x^k \prod_{i=1}^m (x - a_i z) = 0$ . Hence, both lines  $x = 0$  and  $y = 0$  are invariant. Therefore,  $\tau$  is given by a diagonal automorphism of the form  $[x : y : z] \mapsto [\mu x : \nu y : z]$ ,  $\mu, \nu \in \mathbb{C}^*$ . Replacing in the equation yields  $\mu = \nu = 1$ .  $\square$

The curves  $\mathbb{A}^1$  and  $\mathbb{A}^1 \setminus \{0\}$  admit closed embeddings into  $\mathbb{A}^2$  such that all their automorphisms extend to automorphisms of  $\mathbb{A}^2$ . Consider for example the curves of equations  $y = 0$  and  $xy = 1$ . However, it is no longer true for the curve  $\mathbb{A}^1 \setminus \{0, 1\}$ .

**Proposition 2.2.** *Let  $\Gamma = \mathbb{A}^1 \setminus \{0, 1\}$ . For every closed embedding  $\tau: \Gamma \rightarrow \mathbb{A}^2$ , there exists an automorphism of  $\Gamma$  that does not extend to  $\mathbb{A}^2$ .*

Before proving this statement, let us recall the following classical result (see e.g. [Fur83, Theorem 2]).

**Lemma 2.3.** *Every finite subgroup of  $\text{Aut}(\mathbb{A}^2)$  is conjugate to a subgroup of  $\text{GL}(2, \mathbb{C})$ .*

*Proof of Proposition 2.2.* Note that the group of automorphisms of  $\Gamma$  is the group  $\text{Sym}_3$  of permutations of a set of three elements, corresponding to the three points “at infinity”, i.e. the points of  $\mathbb{P}^1 \setminus \iota(\Gamma)$ , where  $\iota$  is any (open) embedding of  $\Gamma$  in  $\mathbb{P}^1$ . It is generated by the automorphisms  $\rho: x \mapsto 1/(1-x)$  and  $\sigma: x \mapsto 1-x$  and we have

$$\text{Aut}(\Gamma) = \langle \sigma, \rho \mid \sigma^2 = \rho^3 = 1, \sigma\rho\sigma^{-1} = \rho^{-1} \rangle = \text{Sym}_3.$$

Suppose for contradiction that there exists a closed embedding  $\tau: \Gamma \rightarrow \mathbb{A}^2$  for which every automorphism of  $\Gamma$  extends. Since the identity is the only automorphism of  $\mathbb{A}^2$  that restricts to the identity on a closed curve isomorphic to  $\mathbb{A}^1 \setminus \{0, 1\}$  (see Lemma 2.4 below), we would have a subgroup  $G \subset \text{Aut}(\mathbb{A}^2)$  isomorphic to  $\text{Sym}_3$  whose restriction to  $\tau(\Gamma)$  yields  $\text{Aut}(\Gamma)$ .

We now prove that this is impossible. First, recall that  $G$  is conjugate to a subgroup of  $\text{GL}(2, \mathbb{C})$  (see Lemma 2.3 above). Then, one easily checks that  $G$  is conjugate to the subgroup  $G'$  of  $\text{GL}(2, \mathbb{C})$  generated by

$$\hat{\rho}: (x, y) \mapsto (y, -x - y) \text{ and } \hat{\sigma}: (x, y) \mapsto (y, x).$$

We let  $f \in \text{Aut}(\mathbb{A}^2)$  be an automorphism such that  $fGf^{-1} = G'$  and we consider the embedding  $\hat{\tau} = f \circ \tau$  of  $\Gamma$  in  $\mathbb{A}^2$ . The automorphism group of  $\Gamma$  extends then to  $G'$  for this embedding.

Remark that the set  $\{\omega \mid \omega^2 - \omega + 1 = 0\} \subset \Gamma$ , which is the set of fixed points of  $\rho$ , is an orbit of size 2 of  $\text{Aut}(\Gamma)$ . But one checks that  $G' \subset \text{GL}(2, \mathbb{C})$  does not have any orbit of size 2 in the affine plane  $\mathbb{A}^2$ . This gives a contradiction.  $\square$

**Lemma 2.4.** *The set of fixed points of a plane polynomial automorphism is either a finite set of points (possibly empty), a finite disjoint union of subvarieties isomorphic to  $\mathbb{A}^1$ , or the whole plane.*

*Proof.* Using the amalgamated structure of  $\text{Aut}(\mathbb{A}^2)$ , it is observed in [FM89] that a plane polynomial automorphism is conjugate either to a triangular automorphism  $(x, y) \mapsto (ax + p(y), by + c)$  with  $a, b, c \in \mathbb{C}$  and  $p(y) \in \mathbb{C}[y]$ , or to some cyclically reduced element (see [Ser80, I.1.3] or [FM89, p. 70] for the definition of a cyclically reduced element). In the first case, an obvious computation shows that the set of fixed points is either empty, a point, a finite disjoint union of subvarieties isomorphic to  $\mathbb{A}^1$ , or the whole plane. In the second case, by [FM89, Theorem 3.1], the set of fixed points is a non-empty finite set of points.  $\square$

Using tools of birational geometry, we can actually specify the statement of Proposition 2.2. Indeed, Theorem 2.5 below shows that there is no closed embedding of the curve  $\mathbb{A}^1 \setminus \{0, 1\}$  into  $\mathbb{A}^2$  such that the automorphism  $\rho: x \mapsto 1/(1-x)$  extends to an automorphism of the affine plane.

Before we state this result, let us recall that any automorphism  $f$  of  $\mathbb{P}^1$  of finite order  $n > 1$  is conjugate to  $[x : y] \mapsto [x : \xi y]$ , where  $\xi$  is a primitive  $n$ -th root of unity. In particular, it has the following properties:

- (1) the automorphism  $f$  fixes exactly two points of  $\mathbb{P}^1$ ;
- (2) all other orbits under the action of  $f$  have size  $n$ .

Thus, the following holds for every automorphism  $g \in \text{Aut}(\Gamma)$  of order  $n > 1$  of a rational smooth curve  $\Gamma$ .

- (1) The automorphism  $g$  fixes 0, 1 or 2 points of  $\Gamma$ ;
- (2) all other orbits under the action of  $g$  have size  $n$ .

**Theorem 2.5.** *Let  $\Gamma$  be a rational smooth affine curve and let  $g \in \text{Aut}(\Gamma)$  be an automorphism.*

- (1) *If  $g$  fixes at most one point of  $\Gamma$ , there is a closed embedding  $\tau: \Gamma \rightarrow \mathbb{A}^2$  such that  $g$  extends to an automorphism of  $\mathbb{A}^2$ .*
- (2) *If  $g$  is of finite order  $n > 1$  with  $n$  odd and if it fixes exactly two points of  $\Gamma$ , then there is no closed embedding  $\tau: \Gamma \rightarrow \mathbb{A}^2$  such that  $g$  extends to an automorphism of  $\mathbb{A}^2$ .*

*Proof.* (1) Let  $P \in \mathbb{C}[x]$  be a non-zero polynomial such that  $\Gamma$  is isomorphic to  $\mathbb{A}^1 \setminus \{x \in \mathbb{A}^1 \mid P(x) = 0\}$ . Let  $g \in \text{Aut}(\Gamma)$  be an automorphism that fixes at most one point of  $\Gamma$ . Let us denote also by  $g$  its extension as an automorphism of  $\mathbb{P}^1$ . We can assume that  $g$  fixes the point of  $\mathbb{P}^1$  at infinity, so that it is of the form  $x \mapsto ax + b$ , for some  $a \in \mathbb{C}^*$  and  $b \in \mathbb{C}$ . Moreover  $P(ax + b) = \mu P(x)$  for some  $\mu \in \mathbb{C}^*$ .

When we embed  $\Gamma$  into  $\mathbb{A}^2$  via the map  $x \mapsto (x, \frac{1}{P(x)})$ , the automorphism  $g$  extends to  $(x, y) \mapsto (ax + b, \mu^{-1}y)$ .

(2) Let  $g \in \text{Aut}(\Gamma)$  be of finite order  $n > 1$  with  $n$  odd such that it fixes 2 points of  $\Gamma$ . Suppose, for contradiction, that there exists a closed embedding  $\tau: \Gamma \rightarrow \mathbb{A}^2$  for which  $g$  extends to an automorphism  $h$  of  $\mathbb{A}^2$ . Since  $g$  has finite order  $n$ , the automorphism  $h^n \in \text{Aut}(\mathbb{A}^2)$  fixes pointwise the curve  $\tau(\Gamma)$ . Because  $g$  fixes two points of  $\Gamma$ ,  $\tau(\Gamma)$  is not isomorphic to  $\mathbb{A}^1$ , hence  $h^n$  is trivial by Lemma 2.4.

Recall that every automorphism of  $\mathbb{A}^2$  of finite order is conjugate to a linear one (Lemma 2.3). Thus, there exists an automorphism  $f \in \text{Aut}(\mathbb{A}^2)$  such that  $\hat{h} = f \circ h \circ f^{-1}$

is linear. Moreover, the automorphism  $g \in \text{Aut}(\Gamma)$  extends to  $\hat{h}$ , when we consider the embedding  $\hat{\tau} = f \circ \tau: \Gamma \rightarrow \mathbb{A}^2$ .

The linear automorphism  $\hat{h}$  extends to an automorphism of  $\mathbb{P}^2$ , and the closure of  $\hat{\tau}(\Gamma)$  in  $\mathbb{P}^2$  is a projective rational curve  $C$ , having all its singular points on the line  $L = \mathbb{P}^2 \setminus \mathbb{A}^2$ .

If  $C$  is smooth, it is isomorphic to  $\mathbb{P}^1$ . Hence, it is a conic or a line, and thus intersects  $L$  into 1 or 2 points, which contradicts the fact that  $g$  acts on  $C$  with order  $n > 2$  and with no fixed point at infinity. This implies that  $C$  is singular.

Denote by  $\eta_1: X_1 \rightarrow \mathbb{P}^2$  the blow-up of the points of  $\mathbb{P}^2$  that are singular points of  $C$ , and write  $C_1 \subset X_1$  the strict transform of  $C$  in  $X_1$ . If  $C_1$  is singular, we denote by  $\eta_2: X_2 \rightarrow X_1$  the blow-up of the points of  $X_1$  that are singular points of  $C_1$ , and write  $C_2$  the strict transform of  $C_1$  in  $X_2$ . We continue like this until we end with a smooth curve  $C_m \subset X_m$  such that the intersection of  $C_m$  with all curves contracted by  $\eta_1 \eta_2 \dots \eta_m$  is transversal. Note that  $C_m$  is isomorphic to  $\mathbb{P}^1$ . For  $i = 1, \dots, m$ , the lift of  $\hat{h}$  yields an automorphism  $h_i$  of  $X_i$  which preserves the curve  $C_i$ . It also preserves the pull-back of  $\mathbb{A}^2$  in  $X_i$ , which is again isomorphic to  $\mathbb{A}^2$ .

For  $i = 1, \dots, m$ , we denote by  $\mathcal{B}_i \subset C_i$  the (finite) set of points not lying in  $\mathbb{A}^2$ . Each point  $p \in \mathcal{B}_i$  has a multiplicity  $m(p)$  as a point of  $C_i$ . This multiplicity is a positive integer and it is equal to 1 if and only if  $C_i$  is smooth at this point  $p$ . Denote by  $\mathcal{B}_0$  the set of points of  $C_0 = C \subset X_0 = \mathbb{P}^2$  not lying in  $\mathbb{A}^2$  and let us use the same notation as above for the multiplicities of the points of  $\mathcal{B}_0$ .

Writing  $d$  the degree of  $C \subset \mathbb{P}^2$ , the geometric genus of  $C$  can be computed with the following classical formula. (Note that it is equal to 0, since  $C$  is rational.)

$$(\star) \quad 0 = \frac{(d-1)(d-2)}{2} - \sum_{i=0}^m \sum_{p \in \mathcal{B}_i} \frac{m(p) \cdot (m(p) - 1)}{2}.$$

Let us now prove the following assertion by descending induction on  $j \leq m$ .

- Let  $j \in \{1, \dots, m\}$  and let  $J \subset \mathcal{B}_j$  be an orbit under the action of  $h_j$ .
- ( $\diamond$ ) Then  $m(p) = m(p')$  for all  $p, p' \in J$ , and the integer  $\sum_{p \in J} m(p)$  is a multiple of  $n$ .

For  $j = m$ , the assertion ( $\diamond$ ) holds for all orbits  $J \subset \mathcal{B}_m$ . Indeed,  $C_m$  is isomorphic to  $\mathbb{P}^1$  and the action of  $h_m$  on  $\mathcal{B}_m \subset C_m$  is fixed-point-free, so all orbits have size  $n$  and all multiplicities are equal to 1.

Then, we can prove ( $\diamond$ ) for  $j < m$ , assuming it holds for every integer  $k$  with  $j+1 \leq k \leq m$ . For this, let  $J \subset \mathcal{B}_j$  be an orbit under the action of  $h_j$  and let us denote by  $m_J$  the multiplicity  $m(p)$  of a point  $p \in J$ . Note that this multiplicity does not depend of the choice of  $p$ , since  $h_j$  acts transitively on  $J$ .

If  $m_J = 1$ , all points of  $J$  are smooth, and so the pull-back by  $\eta_{j+1}$  of  $J$  consists of  $|J|$  points of multiplicity  $m_j = 1$ . This implies  $\sum_{p \in J} m(p) \in n\mathbb{N}$ , by induction hypothesis.

If  $m_J > 1$ , then all points of  $J$  are singular points of the curve  $C_j$  and are thus blown-up by  $\eta_{j+1}: X_{j+1} \rightarrow X_j$ . The number  $m_J$  is the multiplicity of the curve  $C_j$  at the point  $p \in J$ . Denoting by  $E_p \subset X_{j+1}$  the curve contracted by  $\eta_{j+1}$  onto  $p$ , the number  $m_J$  is the intersection number  $E_p \cdot C_{j+1}$ . This latter is equal to the sum of  $m_q(E_p) \cdot m_q(C_{j+1})$ ,

where  $q$  runs through all points infinitely near to  $p$ , and where  $m_q(E_p)$  and  $m_q(C_{j+1})$  are the multiplicities of the strict transforms of  $E_p$  and  $C_{j+1}$  at  $q$ , respectively. Note that  $m_q(E_p)$  is equal to 0 or 1.

Therefore, the sum  $\sum_{p \in J} m_J$  is equal to a sum of multiplicities of orbits in  $\mathcal{B}_k$  for  $k \geq j+1$ . By induction hypothesis, it is a multiple of  $n$ . This achieves to prove  $(\diamond)$ .

In order to finish the proof, we will show how Equation  $(\star)$  and Assertion  $(\diamond)$  imply that the integers  $\frac{(d-1)(d-2)}{2}$  and  $d$  are both multiple of  $n$ . Since the greatest common divisor of  $d$  and  $\frac{(d-1)(d-2)}{2}$  is 1 or 2, this will contradict the assumption  $n > 2$ .

To show that  $n$  divides  $\frac{(d-1)(d-2)}{2}$ , we decompose the sum of  $(\star)$  according to orbits

$$\frac{(d-1)(d-2)}{2} = \sum_{j=0}^m \sum_{J \subset \mathcal{B}_j} \sum_{p \in J} \frac{m(p) \cdot (m(p) - 1)}{2}.$$

By Assertion  $(\diamond)$ , the multiplicities  $m(p)$  are all equal among the same orbit  $J$ , so  $\sum_{p \in J} m(p) \cdot (m(p) - 1)$  is a multiple of  $\sum_{p \in J} m(p)$ , which is a multiple of  $n$  by  $(\diamond)$ . Since  $n$  is odd,  $\sum_{p \in J} \frac{m(p) \cdot (m(p) - 1)}{2}$  is also a multiple of  $n$ , and so is  $\frac{(d-1)(d-2)}{2}$ .

It remains to show that  $d$  is also a multiple of  $n$ . For this, we observe that the intersection number  $d = L \cdot C$  is the sum of multiplicities of all points of  $C$  that belong to  $L$ , as proper or infinitely near points. Since  $L$  is invariant under the extension of the affine automorphism  $\hat{h}$ , the union of these points decomposes into orbits of  $h_j$  and the sum is then a multiple of  $n$  by Assertion  $(\diamond)$ .  $\square$

**Corollary 2.6.** *There exist rational smooth affine curves  $\Gamma$  with  $\text{Aut}(\Gamma) \neq 1$  and such that for every closed embedding of  $\Gamma$  in  $\mathbb{A}^2$ , the identity on  $\Gamma$  is its only automorphism which extends to an automorphism of  $\mathbb{A}^2$ .*

*Proof.* Let  $\omega = e^{2i\pi/3}$  and  $a_1 = 1$ . Let  $a_2, \dots, a_k$  be complex numbers algebraically independent over  $\mathbb{Q}$ . We consider the curve  $\Gamma = \mathbb{P}^1 \setminus \Lambda$ , where  $\Lambda$  is the following set of  $3k$  points

$$\Lambda = \{[a_i \omega^j : 1] \mid i = 1, \dots, k, j = 0, \dots, 2\}.$$

The map  $h: [x : y] \mapsto [x : \omega y]$  is obviously an automorphism of  $\Gamma$ . We will now prove that it generates the whole automorphism group  $\text{Aut}(\Gamma)$  if  $k \geq 3$ . This will conclude the proof, since  $h$  and  $h^2$  do not extend to automorphisms of  $\mathbb{A}^2$  by Theorem 2.5.

Let  $g \in \text{Aut}(\Gamma)$  be an automorphism of  $\Gamma$ . It extends to an automorphism of  $\mathbb{P}^1$  that preserves the set  $\Lambda$ . Let us denote this latter also by  $g$ .

Consider the 4-tuple  $V = ([1 : 1], [\omega : 1], [\omega^2 : 1], [a_2 : 1])$ . Since  $a_2, \dots, a_k$  are algebraically independent over  $\mathbb{Q}$ , the image of  $V$  by  $g$  is a 4-tuple of points contained in the set

$$S = \{[1 : 1], [\omega : 1], [\omega^2 : 1], [a_2 : 1], [a_2 \omega : 1], [a_2 \omega^2 : 1]\}.$$

Indeed, the cross-ratio of  $g(V)$  must be equal to the cross-ratio of  $V$ , i.e. to  $\omega(\omega - a_2)/(a_2 - 1)$ .

The same argument with the 4-tuple  $([1 : 1], [\omega : 1], [\omega^2 : 1], [a_3 : 1])$  allows us to conclude that  $g$  preserves the set  $\{[1 : 1], [\omega : 1], [\omega^2 : 1]\}$ . Therefore,  $g$  is either a power of  $h$ , or it is one of the maps  $\varphi_i: [x : y] \mapsto [y : x\omega^i]$  with  $i = 0 \dots 2$ .

Finally, note that  $g$  cannot be one of the  $\varphi_i$ 's, since  $\varphi_i$  sends the point  $[a_2 : 1]$  onto  $[\frac{1}{a_2\omega^i} : 1]$ , which does not belong to the set  $S$ .  $\square$

*Remark 2.7.* The proof of Corollary 2.6 shows that if  $k \geq 3$  and if the set  $\Lambda \subset \mathbb{P}^1$  is general among all sets of distinct  $3k$  points invariant by the map  $[x : y] \mapsto [x : \omega y]$ , then for all closed embeddings of the curve  $\Gamma = \mathbb{P}^1 \setminus \Lambda$  into  $\mathbb{A}^2$ , the identity is the only automorphism of  $\Gamma$  that extends to an automorphism of  $\mathbb{A}^2$ .

On the contrary, when  $k \leq 2$ , every such curve  $\Gamma$  admits an automorphism of order 2 and Proposition 2.8 below implies then that this latter extends to an automorphism of  $\mathbb{A}^2$  for a well-chosen closed embedding of  $\Gamma$  into  $\mathbb{A}^2$ .

**Proposition 2.8.** *Let  $\Gamma$  be a rational smooth affine curve and let  $\sigma \in \text{Aut}(\Gamma)$  be an automorphism of  $\Gamma$  of order 2. There exists a closed embedding of  $\Gamma$  in  $\mathbb{A}^2$  and an automorphism  $\hat{\sigma} \in \text{Aut}(\mathbb{A}^2)$  of order 2 whose restriction to  $\Gamma$  yields  $\sigma$ .*

*Proof.* Let  $\Gamma = \mathbb{P}^1 \setminus \Lambda$ , where  $\Lambda$  is a finite set of points. Let us denote by  $\sigma$  the extension of the automorphism  $\sigma \in \text{Aut}(\Gamma)$  as an automorphism of  $\mathbb{P}^1$ . If it fixes at most one point of  $\Lambda$ , the result follows from Theorem 2.5.

We can thus assume that the two points of  $\mathbb{P}^1$  fixed by (the extension of)  $\sigma$  belong to  $\Gamma$ . Let  $p$  be a point of  $\Lambda$ . Its orbit  $\{p, \sigma(p)\}$  is then contained in  $\Lambda$ . Let  $C$  be the curve  $C = \mathbb{P}^1 \setminus \{p, \sigma(p)\}$ . Note that  $C$  is isomorphic to  $\mathbb{A}^1 \setminus \{0\}$  and that  $\sigma$  restricts to an automorphism of  $C$ . Remark that all automorphisms of  $\mathbb{A}^1 \setminus \{0\}$  of order 2 with two fixed points are conjugate to the automorphism  $x \mapsto x^{-1} \in \text{Aut}(\text{Spec}(\mathbb{C}[x, x^{-1}]))$ . Therefore, there is a closed embedding  $\tau: C \rightarrow \mathbb{A}^2$  whose image is the curve defined by the equation

$$y^2 - 1 = x^2$$

and such that the automorphism  $\sigma \in \text{Aut}(C)$  extends to the automorphism  $\hat{\sigma}: (x, y) \mapsto (-x, y)$ . Moreover, the curve  $\tau(\Gamma)$  is then equal to a set of points of  $\tau(C)$  satisfying that  $\prod_{i=1}^n (y - a_i) \neq 0$ , for some  $n \geq 0$  and distinct  $a_1, \dots, a_n \in \mathbb{C} \setminus \{\pm 1\}$ .

Let  $Y \subset \mathbb{A}^2$  be the closed curve defined by the equation

$$y^2 - 1 = x^2 \cdot \left( \prod_{i=1}^n (y - a_i) \right)^2.$$

Consider finally the birational transformation of  $\mathbb{A}^2$  defined by

$$(x, y) \dashrightarrow \left( \frac{x}{\prod_{i=1}^n (y - a_i)}, y \right),$$

which restricts to an isomorphism between  $\tau(\Gamma)$  and  $Y$ . Since it commutes with the automorphism  $(x, y) \mapsto (-x, y)$ , this yields the result.  $\square$

### 3. PLANAR EMBEDDINGS IN THE SPACE

The following question of Bhatwadekar and Srinivas is asked at the end of [Sri91]: are any two embeddings of a smooth affine curve in  $\mathbb{A}^2$  equivalent, when considered as embeddings in  $\mathbb{A}^3$ ?

The next result answers positively for the case of rational smooth affine curves.



**Proposition 3.1.** *Let  $\Gamma$  be a rational smooth affine curve.*

- (1) *If  $\tau_1, \tau_2: \Gamma \rightarrow \mathbb{A}^3$  are two closed embeddings whose images are contained in a hyperplane (planar embeddings in the space), there exists an automorphism  $\alpha \in \text{Aut}(\mathbb{A}^3)$  such that  $\tau_2 = \alpha \circ \tau_1$ , i.e. any two planar embeddings in the space are equivalent.*
- (2) *In particular, fixing a planar embedding  $\Gamma \rightarrow \mathbb{A}^3$ , every automorphism of  $\Gamma$  extends to  $\mathbb{A}^3$ .*

*Proof.* Let  $\Gamma = \mathbb{A}^1 \setminus \{x \in \mathbb{A}^1 \mid P(x) = 0\}$ , where  $P \in \mathbb{C}[x]$  is a polynomial with simple roots. Note that the coordinate ring of  $\Gamma$  is  $\mathbb{C}[\Gamma] = \mathbb{C}[x, \frac{1}{P(x)}]$  and recall that the map  $x \mapsto (x, \frac{1}{P(x)})$  defines a closed embedding of  $\Gamma$  in  $\mathbb{A}^2$ . To prove the proposition, it suffices to prove that any planar embedding is equivalent to the one given by  $x \mapsto (x, \frac{1}{P(x)}, 0)$ .

Let  $\tau: \Gamma \rightarrow \mathbb{A}^3$  be a planar embedding of  $\Gamma$ . We can compose  $\tau$  with an affine automorphism  $f_1$  of  $\mathbb{A}^3$  and get an embedding  $\tau_2 = f_1 \circ \tau: \Gamma \rightarrow \mathbb{A}^3$  of the form  $x \mapsto (0, Q(x), R(x))$ , where  $Q, R \in \mathbb{C}(x)$  are rational functions without poles on  $\Gamma$ .

Since  $\tau_2$  is a closed embedding of the curve  $\Gamma$ , the equality  $\mathbb{C}[x, \frac{1}{P(x)}] = \mathbb{C}[Q(x), R(x)]$  holds. In particular, there exists a polynomial  $A \in \mathbb{C}[X, Y]$  such that  $A(Q(x), R(x)) = x$ . Now, we compose  $\tau_2$  with the automorphism of  $\mathbb{A}^3$  defined by  $f_2(X, Y, Z) = (X + A(Y, Z), Y, Z)$  and obtain the embedding  $\tau_3: \Gamma \rightarrow \mathbb{A}^3$  given by

$$\tau_3: x \mapsto (x, Q(x), R(x)).$$

Because of the equality  $\mathbb{C}[x, \frac{1}{P(x)}] = \mathbb{C}[Q(x), R(x)]$ , all zeros of  $P(x)$  are poles of  $aQ(x) + bR(x)$  for general complex numbers  $a, b \in \mathbb{C}$ . We can thus compose  $\tau_3$  with a linear automorphism of the form  $(X, Y, Z) \mapsto (X, aY + bZ, Z)$  and get an embedding  $\tau_4: \Gamma \rightarrow \mathbb{A}^3$  of the form

$$\tau_4: x \mapsto \left( x, \frac{Q_1(x)}{Q_2(x)}, \frac{R_1(x)}{R_2(x)} \right),$$

where  $Q_1, Q_2, R_1, R_2 \in \mathbb{C}[x]$  are polynomials such that  $Q_1$  and  $Q_2$  (resp.  $R_1$  and  $R_2$ ) have no common factor, and such that  $P(x)$  divides  $Q_2(x)$ .

In particular, there exist two polynomials  $U, V \in \mathbb{C}[x]$  such that  $UQ_1 + VP = 1$ . It follows

$$\frac{1}{P} = \frac{UQ_1 + VP}{P} = U \frac{Q_1}{P} + V = SU \frac{Q_1}{Q_2} + V,$$

where  $S \in \mathbb{C}[x]$  satisfies  $PS = Q_2$ .

This implies  $\mathbb{C}[x, \frac{1}{P}] \subset \mathbb{C}[x, \frac{Q_1}{Q_2}]$  and thus

$$\mathbb{C}[x, \frac{Q_1}{Q_2}, \frac{R_1}{R_2}] = \mathbb{C}[x, \frac{1}{P}] = \mathbb{C}[x, \frac{Q_1}{Q_2}].$$

Therefore, there exist polynomials  $B, C \in \mathbb{C}[X, Y]$  such that  $B(x, \frac{Q_1(x)}{Q_2(x)}) = \frac{1}{P(x)} - \frac{R_1(x)}{R_2(x)}$  and  $C(x, \frac{1}{P(x)}) = \frac{Q_1(x)}{Q_2(x)}$ . Finally, we consider the automorphisms of  $\mathbb{A}^3$  defined by  $f_4(X, Y, Z) = (X, Y, Z + B(X, Y))$  and  $f_5(X, Y, Z) = (X, Z, Y - C(X, Z))$ . One checks that  $f_5 \circ f_4 \circ \tau_4: \Gamma \rightarrow \mathbb{A}^3$  is the desired embedding  $x \mapsto (x, \frac{1}{P(x)}, 0)$ .  $\square$

Note that the proof above is constructive. In particular, a planar embedding of a smooth rational curve  $\Gamma$  in  $\mathbb{A}^3$  and an automorphism  $\varphi$  of  $\Gamma$  being given, it allows us to construct an explicit automorphism of  $\mathbb{A}^3$  which extends  $\varphi$ .

*Example 3.2.* Let  $\Gamma$  be the curve  $\Gamma = \mathbb{A}^1 \setminus \{0, 1\}$  and let  $\rho \in \text{Aut}(\Gamma)$  be the automorphism of  $\Gamma$  defined by  $\rho(x) = 1/(1-x)$ . We saw in Section 2 that there is no closed embedding of  $\Gamma$  into  $\mathbb{A}^2$  such that  $\rho$  extends to an automorphism of  $\mathbb{A}^2$ . However, it extends to an automorphism of  $\mathbb{A}^3$ , when we consider the embedding  $\tau: \Gamma \rightarrow \mathbb{A}^3$  defined by  $x \mapsto (x, 1/x(x-1), 0)$ .

Following the proof of Proposition 3.1, we let  $f_1, f_2, \dots, f_5$  be the automorphisms of  $\mathbb{A}^3$  defined by  $f_1(X, Y, Z) = (Z, Y, X)$ ,  $f_2(X, Y, Z) = (X+Y+2-YZ^2, Y, Z)$ ,  $f_3(X, Y, Z) = (X, aY+bZ, Z)$ ,  $f_4(X, Y, Z) = (X, Y, Z - \frac{1}{ab}[(b+(a-b)X)(Y-aX+2a) - (a-b)^2](1+X))$  and  $f_5(X, Y, Z) = (X, Z, Y - aX + 2a + aZ + (b-a)XZ)$ , where  $a, b \in \mathbb{C}$  are general complex numbers.

Setting  $F = f_5 \circ f_4 \circ \dots \circ f_1$ , one checks  $F \circ \tau \circ \rho = \tau$ . This implies that  $F^{-1}$  is an extension of the automorphism  $\rho \in \text{Aut}(\Gamma)$ .

*Remark 3.3.* To our knowledge, there is no known example of a smooth affine curve admitting two non-equivalent embeddings into  $\mathbb{A}^3$ . Paradoxically, we do not know any smooth affine curve such that all its embeddings into  $\mathbb{A}^3$  are equivalent!

The case of the affine line is of particular interest. On one hand, all closed embeddings of  $\mathbb{A}^1$  into  $\mathbb{A}^2$  are equivalent by the famous Abhyankar-Moh-Suzuki theorem. On the other hand, all closed embeddings of  $\mathbb{A}^1$  into  $\mathbb{A}^n$  with  $n \geq 4$  are also equivalent (see [Sri91] or [Kal91]).

#### 4. ACTIONS OF $\text{SL}(2, \mathbb{C})$ ON $\text{End}(\mathbb{A}^2)$ AND OF $\text{PGL}(2, \mathbb{C})$ ON $\mathbb{P}^1$

The aim of this section is to construct, for every non-empty subset  $\Lambda$  of  $\mathbb{P}^1$  that is invariant by a subgroup  $H$  of  $\text{Aut}(\mathbb{P}^1)$ , a  $H$ -equivariant endomorphism of  $\mathbb{P}^1$  whose fixed-point set is equal to the set  $\Lambda$  (Corollary 4.4). We will use this result later on to construct embeddings of every rational smooth affine curve into  $\mathbb{A}^3$  in such a way that the whole automorphism group of the curve extends to a subgroup of  $\text{Aut}(\mathbb{A}^3)$ .

For the rest of the paper we will consider the following actions of the group  $\text{SL}(2, \mathbb{C})$  on  $\mathcal{O}(\mathbb{A}^2) = \mathbb{C}[x, y]$  and  $\text{End}(\mathbb{A}^2) = \mathbb{C}[x, y] \times \mathbb{C}[x, y]$ .

$$\begin{aligned} \text{SL}(2, \mathbb{C}) \times \mathcal{O}(\mathbb{A}^2) &\rightarrow \mathcal{O}(\mathbb{A}^2) \\ (g, P) &\mapsto g \cdot P := P \circ g^{-1} \end{aligned}$$

and

$$\begin{aligned} \text{SL}(2, \mathbb{C}) \times \text{End}(\mathbb{A}^2) &\rightarrow \text{End}(\mathbb{A}^2) \\ (g, F) &\mapsto g \cdot F := g \circ F \circ g^{-1}. \end{aligned}$$

Note that these actions come from the natural action of  $\text{SL}(2, \mathbb{C})$  on  $\mathbb{A}^2$ . Indeed, denote by  $V$  the space  $\mathbb{A}^2$  as a complex vector space of dimension 2 and identify the set of the linear forms on it as the dual space  $V^*$ . The action of  $\text{SL}(V)$  on  $V$  yields actions on  $V^*$ , on the symmetric algebra  $S(V^*)$  and on  $S(V^*) \otimes V$ . The natural isomorphisms between  $S(V^*)$  and  $\mathbb{C}[x, y] = \mathcal{O}(\mathbb{A}^2)$ , and between  $S(V^*) \otimes V$  and  $\mathbb{C}[x, y] \times \mathbb{C}[x, y] = \text{End}(\mathbb{A}^2)$ , lead then to the  $\text{SL}(2, \mathbb{C})$ -actions that we defined above.

**Lemma 4.1.** *The map  $\rho: \text{End}(\mathbb{A}^2) \rightarrow \mathcal{O}(\mathbb{A}^2)$  defined by*

$$\begin{aligned} \mathbb{C}[x, y] \times \mathbb{C}[x, y] &\rightarrow \mathbb{C}[x, y] \\ (f_1, f_2) &\mapsto f_1y - f_2x \end{aligned}$$

is  $\text{SL}(2, \mathbb{C})$ -equivariant, when we consider the actions defined above.

*Proof.* The result could of course be checked by direct computations, but let us mention that it also follows from the fact that  $\rho$  corresponds to the morphism  $S(V^*) \otimes V \rightarrow S(V^*)$  given by the composition  $\tau_2 \circ \tau_1$ , where  $\tau_1$  and  $\tau_2$  are the two following homomorphisms of  $\text{SL}(V)$ -modules.

$$\begin{aligned} \tau_1: S(V^*) \otimes V &\rightarrow S(V^*) \otimes V \otimes V^* \otimes V \\ p \otimes v &\mapsto p \otimes v \otimes \text{id}, \end{aligned}$$

where  $\text{id}$  denotes the identity element seen as an element of  $V^* \otimes V = \text{Hom}(V, V)$ , and

$$\begin{aligned} \tau_2: S(V^*) \otimes V \otimes V^* \otimes V &\rightarrow S(V^*) \\ p \otimes v_1 \otimes v_2^* \otimes v_3 &\mapsto \det(v_1, v_3)(pv_2^*). \end{aligned}$$

□

**Lemma 4.2.** *Let  $G \subset \text{SL}(2, \mathbb{C})$  be a finite subgroup of  $\text{SL}(2, \mathbb{C})$  and let  $P \in \mathbb{C}[x, y]$ . The following conditions are equivalent:*

- (1) *The polynomial  $P$  satisfies  $P(0, 0) = 0$  and is fixed by  $G$ .*
- (2) *There exists an endomorphism  $F = (f_1, f_2)$  of  $\mathbb{A}^2$  that is fixed by  $G$  and such that  $\rho(F) = f_1y - f_2x = P$ .*

*Proof.* Let  $E_P \subset \text{End}(\mathbb{A}^2)$  be the set

$$E_P = \rho^{-1}(P) = \{(f_1, f_2) \in \mathbb{C}[x, y] \times \mathbb{C}[x, y] \mid f_1y - f_2x = P\}.$$

This defines an affine subset of the  $\mathbb{C}$ -vector space  $\text{End}(\mathbb{A}^2)$ , since the endomorphism  $(\lambda f_1 + (1 - \lambda)f_3, \lambda f_2 + (1 - \lambda)f_4)$  belongs to  $E_P$ , for any  $(f_1, f_2), (f_3, f_4) \in E_P$  and any  $\lambda \in \mathbb{C}$ . Moreover,  $E_P$  is non-empty if and only if  $P(0, 0) = 0$ .

If  $F \in \text{End}(\mathbb{A}^2)$  is fixed by  $G$  and belongs to  $E_P$ , then

$$g \cdot P = g \cdot \rho(F) = \rho(g \cdot F) = \rho(F) = P$$

hold for any  $g \in G$ . This shows (2)  $\Rightarrow$  (1).

If  $P$  is fixed by  $G$ , then the set  $E_P$  is invariant by  $G$ , since

$$\rho(g \cdot F) = g \cdot \rho(F) = g \cdot P = P$$

hold for any  $F \in E_P$  and  $g \in G$ .

Therefore, if  $F$  belongs to  $E_P$ , then  $\frac{1}{|G|} \sum_{g \in G} g \cdot F$  is an element of  $E_P$  that is fixed by  $G$ . This shows (1)  $\Rightarrow$  (2) and concludes the proof. □

**Proposition 4.3.** *Let  $H \subset \text{PGL}(2, \mathbb{C}) = \text{Aut}(\mathbb{P}^1)$  be a finite subgroup and set  $G = \pi^{-1}(H)$ , where  $\pi: \text{SL}(2, \mathbb{C}) \rightarrow \text{PGL}(2, \mathbb{C})$  is the canonical surjective map. Let  $\Lambda \subset \mathbb{P}^1$  be a non-empty  $H$ -invariant finite subset.*

- (1) *There exist homogeneous polynomials  $f_1, f_2 \in \mathbb{C}[x, y]$  of the same degree such that  $(f_1, f_2)$  is an endomorphism of  $\mathbb{A}^2$  fixed by  $G$  and such that*

$$\Lambda = \{[x : y] \in \mathbb{P}^1 \mid f_1(x, y)y - f_2(x, y)x = 0\}.$$

(2) *The morphism  $\delta: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  defined by*

$$[x : y] \mapsto [f_1(x, y) : f_2(x, y)]$$

*is  $H$ -equivariant, for all pairs  $(f_1, f_2)$  given by the statement (1) above.*

(3) *There exist polynomials  $f_1, f_2$  satisfying the statement (1) and also the extra property*

$$\Lambda = \{q \in \mathbb{P}^1 \mid \delta(q) = q\}.$$

*This latter holds moreover for all pairs  $(f_1, f_2)$  given by the statement (1), in the case where the set  $\Lambda$  consists of exactly one orbit of  $H$ .*

*Proof.* (1) We let  $p \in \mathbb{C}[x, y]$  be the (unique up a nonzero constant) square-free homogeneous polynomial whose roots correspond to the points of  $\Lambda$ . Because  $\Lambda$  is invariant by  $H$ , there exists a character  $\chi: G \rightarrow \mathbb{C}^*$  such that

$$p \circ g = \chi(g)p,$$

for all  $g \in G$ . Since  $G$  is finite, there exists a positive integer  $d$  such that the polynomial  $P = p^d$  is fixed by  $G$ .

By Lemma 4.2, there exists an endomorphism  $(f_1, f_2) \in \mathbb{C}[x, y] \times \mathbb{C}[x, y]$  of  $\mathbb{A}^2$  that is fixed by  $G$  and such that  $f_1 y - f_2 x = P$ . Since  $P$  is homogeneous and since the action of  $G$  on  $\text{End}(\mathbb{A}^2)$  is linear and preserves the filtration by degrees, we can assume that  $f_1$  and  $f_2$  are homogeneous of the same degree. This proves (1).

Statement (2) follows directly from the fact that the endomorphism  $(f_1, f_2)$  is fixed by  $G$ .

(3) Since  $\delta$  is  $H$ -equivariant, its fixed-point set is invariant by  $H$ . Let us denote it by  $\Omega_\delta$  and write  $f_1 = \alpha \tilde{f}_1$  and  $f_2 = \alpha \tilde{f}_2$ , where  $\alpha, \tilde{f}_1, \tilde{f}_2$  are homogeneous polynomials such that  $\tilde{f}_1$  and  $\tilde{f}_2$  have no common root in  $\mathbb{P}^1$ . Then,  $\delta([x : y]) = [\tilde{f}_1(x, y) : \tilde{f}_2(x, y)]$  holds for all  $[x : y] \in \mathbb{P}^1$ . The set  $\Omega_\delta = \{q \in \mathbb{P}^1 \mid \delta(q) = q\}$  is thus the zero set of  $\tilde{f}_1 y - \tilde{f}_2 x$ . In particular, it is non-empty. Moreover, the equalities  $P = f_1 y - f_2 x = \alpha(\tilde{f}_1 y - \tilde{f}_2 x)$  imply that  $\Omega_\delta$  is contained in  $\Lambda$ .

If  $\Lambda$  consists of exactly one orbit of  $H$ , then  $\Omega_\delta = \Lambda$  follows from the fact that  $\Omega_\delta$  is invariant by  $H$ .

Let us now consider the general case, where  $\Lambda$  consists of  $r > 1$  orbits of  $H$  and write  $\Lambda = \bigcup_{i=1}^r \Lambda_i$ , where  $\Lambda_1, \dots, \Lambda_r$  are disjoint orbits of  $H$ . For each  $i$ , there exist, by the previous argument, homogeneous polynomials  $f_{i,1}, f_{i,2}$  of the same degree such that the zero set of  $P_i = f_{i,1} y - f_{i,2} x$  is equal to  $\Lambda_i$  and such that the pair  $(f_{i,1}, f_{i,2})$  defines an endomorphism of  $\mathbb{A}^2$  which is fixed by  $G$ .

Set

$$g_1 = \frac{1}{r} \sum_{i=1}^r \left( f_{i,1} \prod_{j \neq i} P_j \right) \quad \text{and} \quad g_2 = \frac{1}{r} \sum_{i=1}^r \left( f_{i,2} \prod_{j \neq i} P_j \right).$$

Note that  $g_1$  and  $g_2$  are homogeneous of the same degree and satisfy the equality  $g_1 y - g_2 x = \prod_{i=1}^r P_i$ . Moreover, the endomorphism  $(g_1, g_2) \in \text{End}(\mathbb{A}^2)$  is fixed by  $G$ . In other words, it satisfies the statement (1) of the lemma.

We will now show that the set  $\Omega_{\tilde{\delta}}$  of fixed points of the morphism  $\tilde{\delta}: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  defined by  $\tilde{\delta}([x : y]) = [g_1(x, y) : g_2(x, y)]$  is equal to  $\Lambda$ , which will conclude the proof. Note that it is contained in  $\Lambda$  and invariant under the action of  $H$ , since  $\tilde{\delta}$  is  $H$ -equivariant.

Let us write  $g_1 = \beta \tilde{g}_1$  and  $g_2 = \beta \tilde{g}_2$ , where  $\beta, \tilde{g}_1, \tilde{g}_2$  are homogeneous and  $\tilde{g}_1, \tilde{g}_2$  have no common root in  $\mathbb{P}^1$ . Note that the set  $\Omega_{\tilde{\beta}}$  is equal to the zero set of the homogeneous polynomial  $\tilde{g}_1 y - \tilde{g}_2 x$ .

We claim that none of the  $P_i$  divides  $\beta$ . Indeed, otherwise such a  $P_i$  would divide both  $g_1$  and  $g_2$  and thus also  $f_{i,1} \prod_{j \neq i} P_j$  and  $f_{i,2} \prod_{j \neq i} P_j$ . Since  $P_i$  has no common root with any of the  $P_j$ , this would imply that  $P_i$  divides  $f_{i,1}$  and  $f_{i,2}$ . This is impossible, since  $P_i = f_{i,1} y - f_{i,2} x$ , hence  $P_i$  has degree bigger than  $f_{i,1}$  and  $f_{i,2}$ .

Therefore, it follows from the equalities

$$\prod_{i=1}^r P_i = g_1 y - g_2 x = \beta(\tilde{g}_1 y - \tilde{g}_2 x)$$

that, for every index  $i$ , at least one point of  $\Lambda_i$  is contained in  $\Omega_{\tilde{\beta}}$ . This latter set being invariant by  $H$  and  $\Lambda_i$  being an orbit under the action of  $H$ , we get that the whole set  $\Lambda_i$  is contained in  $\Omega_{\tilde{\beta}}$ , for each  $i = 1 \dots r$ . This achieves the proof.  $\square$

**Corollary 4.4.** *Let  $H \subset \mathrm{PGL}(2, \mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1)$  be a finite subgroup and let  $\Lambda \subset \mathbb{P}^1$  be a finite subset. The following conditions are equivalent:*

- (1) *The set  $\Lambda$  is non-empty and invariant by  $H$ .*
- (2) *There exists a  $H$ -equivariant morphism  $\delta: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  such that*

$$\Lambda = \{q \in \mathbb{P}^1 \mid \delta(q) = q\}.$$

*Proof.* The implication (1)  $\Rightarrow$  (2) follows directly from Proposition 4.3. Let us prove the other one.

Let  $\delta: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be a  $H$ -equivariant morphism whose fixed-point set is equal to  $\Lambda$ . The set  $\Lambda$  is then invariant under the action of  $H$ , since  $\delta(h(q)) = h(\delta(q)) = h(q)$  hold for all  $h \in H$  and all  $q \in \Lambda$ .

Furthermore, let  $f_1, f_2 \in \mathbb{C}[x, y]$  be two homogeneous polynomials of the same degree and without common root in  $\mathbb{P}^1$  such that  $\delta([x : y]) = [f_1(x, y) : f_2(x, y)]$  for all points  $[x : y] \in \mathbb{P}^1$ . Since  $\Lambda$  is the zero set of  $f_1 y - f_2 x$ , it is clearly non-empty.  $\square$

## 5. EQUIVARIANT EMBEDDINGS INTO THE AFFINE THREE-SPACE

Let us recall that the following morphism

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\hookrightarrow \mathbb{P}^3 \\ ([y_0 : y_1], [z_0 : z_1]) &\mapsto [y_0 z_0 : y_0 z_1 : y_1 z_0 : y_1 z_1] \end{aligned}$$

is a classical closed embedding of  $\mathbb{P}^1 \times \mathbb{P}^1$  into  $\mathbb{P}^3$  and that it induces an isomorphism between  $\mathbb{P}^1 \times \mathbb{P}^1$  and the quadric in  $\mathbb{P}^3$  defined by the equation  $x_0 x_3 = x_1 x_2$ . Moreover, since this embedding is canonical (it is given by the linear system  $|-\frac{1}{2}K_{\mathbb{P}^1 \times \mathbb{P}^1}|$ ), every automorphism of  $\mathbb{P}^1 \times \mathbb{P}^1$  extends to a unique automorphism of  $\mathbb{P}^3$ .

Identifying  $\mathbb{A}^3$  with the complement in  $\mathbb{P}^3$  of the hyperplane defined by the equation  $x_1 = x_2$ , we obtain a closed embedding  $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta \hookrightarrow \mathbb{A}^3$ , where  $\Delta$  denotes the diagonal curve  $\Delta = \{(q, q) \mid q \in \mathbb{P}^1\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ .

Consider the diagonal action of  $\mathrm{PGL}(2, \mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1)$  on  $\mathbb{P}^1 \times \mathbb{P}^1$ . Note that each automorphism of  $\mathbb{P}^1 \times \mathbb{P}^1$  coming from this action extends to an automorphism of  $\mathbb{P}^3$  which preserves the plane of equation  $x_1 = x_2$ . This yields an action of  $\mathrm{PGL}(2, \mathbb{C})$  on

$\mathbb{A}^3$  for which the closed embedding  $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta \hookrightarrow \mathbb{A}^3$ , that we defined above, becomes  $\mathrm{PGL}(2, \mathbb{C})$ -equivariant.

After a change of coordinates in  $\mathbb{A}^3$ , we obtain a  $\mathrm{PGL}(2, \mathbb{C})$ -equivariant embedding of  $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$  into  $\mathbb{A}^3$ , where the action of  $\mathrm{PGL}(2, \mathbb{C})$  on  $\mathbb{A}^3$  is linear.

**Lemma 5.1.** *The morphism*

$$\begin{aligned} \iota: \quad (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta &\hookrightarrow \mathbb{A}^3 \\ ([y_0 : y_1], [z_0 : z_1]) &\mapsto \left( \frac{y_0 z_1 + y_1 z_0}{y_0 z_1 - y_1 z_0}, \frac{2y_0 z_0}{y_0 z_1 - y_1 z_0}, \frac{2y_1 z_1}{y_0 z_1 - y_1 z_0} \right) \end{aligned}$$

is a closed embedding whose image is the hypersurface of  $\mathbb{A}^3$  defined by the equation  $yz = x^2 - 1$ .

Moreover,  $\iota$  is  $\mathrm{PGL}(2, \mathbb{C})$ -equivariant, when we consider the actions of  $\mathrm{PGL}(2, \mathbb{C})$  on  $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$  and  $\mathbb{A}^3$  defined by

$$\begin{aligned} \mathrm{PGL}(2, \mathbb{C}) \times (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta &\rightarrow (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta \\ \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ([y_0 : y_1], [z_0 : z_1]) \right) &\mapsto ([ay_0 + by_1 : cy_0 + dy_1], [az_0 + bz_1 : cz_0 + dz_1]) \end{aligned}$$

and

$$\begin{aligned} \mathrm{PGL}(2, \mathbb{C}) \times \mathbb{A}^3 &\rightarrow \mathbb{A}^3 \\ \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) &\mapsto \frac{1}{ad - bc} \begin{pmatrix} ad + bc & ac & bd \\ 2ab & a^2 & b^2 \\ 2cd & c^2 & d^2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{aligned}$$

*Proof.* Let  $Q$  denotes the quadric hypersurface of  $\mathbb{A}^3$  defined by the equation  $yz = x^2 - 1$ . One checks that the morphism  $\iota$  induces an isomorphism between  $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$  and  $Q$  whose inverse morphism is given by

$$\begin{aligned} Q &\rightarrow (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta \\ (x, y, z) &\mapsto \begin{cases} ([x + 1 : z], [y : x + 1]) & \text{if } x \neq -1, \\ ([y : x - 1], [x - 1 : z]) & \text{if } x \neq 1. \end{cases} \end{aligned}$$

It is also straightforward to check that  $\iota$  is  $\mathrm{PGL}(2, \mathbb{C})$ -equivariant for the given actions.  $\square$

Combining the latter lemma with the results of the previous section, we finally get  $\mathrm{Aut}(\Gamma)$ -equivariant embeddings of every smooth affine rational curve  $\Gamma$  into  $\mathbb{A}^3$ .

**Theorem 5.2.** *For every rational smooth affine curve  $\Gamma$ , there exist a linear action of  $\mathrm{Aut}(\Gamma)$  on  $\mathbb{A}^3$  and a closed embedding  $\tau: \Gamma \hookrightarrow \mathbb{A}^3$  which is  $\mathrm{Aut}(\Gamma)$ -equivariant for this action.*

*Proof.* If  $\Gamma = \mathbb{A}^1$ , it suffices to consider the embedding  $\tau: \mathbb{A}^1 \rightarrow \mathbb{A}^3$  defined by  $\tau(t) = (t, 0, 0)$ , and to let  $\mathrm{Aut}(\Gamma) = \{x \mapsto ax + b \mid a \in \mathbb{C}^*, b \in \mathbb{C}\}$  act on  $\mathbb{A}^3$  via the maps  $(x, y, z) \mapsto (ax + b(y + 1), y, z)$ .

If  $\Gamma = \mathbb{C}^*$ , we consider the embedding  $\tau: \Gamma \rightarrow \mathbb{A}^3$  defined by  $\tau(t) = (t, 1/t, 0)$ . Its image is the curve in  $\mathbb{A}^3$  defined by the equations  $z = 0$  and  $xy = 1$ . Recall that the automorphism group of  $\Gamma$  is  $\mathrm{Aut}(\Gamma) = \{\varphi_\lambda: x \mapsto \lambda x \mid \lambda \in \mathbb{C}^*\} \cup \{\psi_\lambda: x \mapsto \lambda x^{-1} \mid \lambda \in \mathbb{C}^*\}$ . The embedding  $\tau$  becomes  $\mathrm{Aut}(\Gamma)$ -equivariant, when we let  $\mathrm{Aut}(\Gamma)$  act on  $\mathbb{A}^3$  via the maps  $\Phi_\lambda: (x, y, z) \mapsto (\lambda x, \lambda^{-1} y, z)$  and  $\Psi_\lambda: (x, y, z) \mapsto (\lambda y, \lambda^{-1} x, z)$ .

If  $\Gamma$  is equal to  $\mathbb{P}^1 \setminus \Lambda$ , where  $\Lambda$  is a finite set of at least 3 points, then its automorphism group  $H = \text{Aut}(\Gamma)$  is the finite subgroup of  $\text{PGL}(2, \mathbb{C}) = \text{Aut}(\mathbb{P}^1)$  that preserves the set  $\Lambda$ . Applying Corollary 4.4, let  $\delta: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be a  $H$ -equivariant morphism such that  $\Lambda = \{q \in \mathbb{P}^1 \mid \delta(q) = q\}$ . This allows us to define a closed embedding  $\hat{\tau}: \Gamma \rightarrow (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$  by letting  $\hat{\tau}(q) = (q, \delta(q))$  for all  $q \in \Gamma = \mathbb{P}^1 \setminus \Lambda$ . The morphism  $\hat{\tau}$  is moreover  $H$ -equivariant, when  $H$  acts diagonally on  $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$ .

Composing  $\hat{\tau}$  with the  $\text{PGL}(2, \mathbb{C})$ -equivariant closed embedding  $\iota: (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta \hookrightarrow \mathbb{A}^3$  that we defined in Lemma 5.1, we obtain a closed embedding  $\tau: \Gamma \rightarrow \mathbb{A}^3$  which is  $H$ -equivariant, as desired.  $\square$

## 6. EXPLICIT FORMULAS FOR THE EQUIVARIANT EMBEDDINGS INTO $\mathbb{A}^3$

The proof of Theorem 5.2 is constructive and already contains explicit  $\text{Aut}(\Gamma)$ -equivariant embeddings into  $\mathbb{A}^3$  for the curves  $\Gamma = \mathbb{A}^1$  and  $\Gamma = \mathbb{A}^1 \setminus \{0\}$ . Let us now describe the construction for the other cases, i.e., when the automorphism group  $\text{Aut}(\Gamma)$  is finite.

We consider the curves  $\Gamma = \mathbb{P}^1 \setminus \Lambda$ , where  $\Lambda$  is a set of at least 3 points of  $\mathbb{P}^1$ . Let us denote by  $H$  the subgroup of  $\text{Aut}(\mathbb{P}^1) = \text{PGL}(2, \mathbb{C})$  that restricts to  $\text{Aut}(\Gamma)$ , and denote as before by  $G$  its pull-back in  $\text{SL}(2, \mathbb{C})$ , which is a finite group of order  $2|H|$ . The set  $\Lambda$  decomposes into  $r$  orbits  $\Lambda = \bigcup_{i=1}^r \Lambda_i$  of  $H$ . An orbit  $\Lambda_i$  of  $H$  is given by the zero set of a homogeneous polynomial  $p_i \in \mathbb{C}[x, y]$ . Some power  $P_i = p_i^{d_i}$  of  $p_i$  is invariant by the action of  $G$  on  $\mathbb{P}^1$  defined in Section 4. For each  $i$ , Lemma 4.2 yields the existence of a  $G$ -invariant pair  $(f_{i,1}, f_{i,2}) \in \text{End}(\mathbb{A}^2)$  which satisfy  $f_{i,1}y - f_{i,2}x = P_i$ . The  $H$ -equivariant morphism  $\delta: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  given by Proposition 4.3 (or Corollary 4.4) is thus  $\delta: [x : y] \dashrightarrow [f_1(x, y) : f_2(x, y)]$ , where

$$f_1 = \frac{1}{r} \left( \prod_{i=1}^r P_i \right) \sum_{i=1}^r \frac{f_{i,1}}{P_i} \quad \text{and} \quad f_2 = \frac{1}{r} \left( \prod_{i=1}^r P_i \right) \sum_{i=1}^r \frac{f_{i,2}}{P_i}.$$

Moreover,  $(f_1, f_2)$  is invariant by  $G$  and satisfies  $f_1y - f_2x = \prod_{i=1}^r P_i$ .

Following the proof of Theorem 5.2, we define a closed embedding  $\Gamma = \mathbb{P}^1 \setminus \Lambda \rightarrow (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$  by  $[x : y] \mapsto ([x : y], [f_1 : f_2])$ . We compose then this latter with the embedding  $\iota: (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta \rightarrow \mathbb{A}^3$  defined by Lemma 5.1, and obtain the following  $\text{Aut}(\Gamma) = H$ -equivariant closed embedding of  $\Gamma$  into  $\mathbb{A}^3$ .

$$\begin{aligned} \Gamma = \mathbb{P}^1 \setminus \Lambda &\rightarrow \mathbb{A}^3 \\ [x : y] &\mapsto \left( \frac{1}{r} \sum_{i=1}^r \frac{xf_{i,2} + yf_{i,1}}{xf_{i,2} - yf_{i,1}}, \frac{1}{r} \sum_{i=1}^r \frac{2xf_{i,1}}{xf_{i,2} - yf_{i,1}}, \frac{1}{r} \sum_{i=1}^r \frac{2yf_{i,2}}{xf_{i,2} - yf_{i,1}} \right). \end{aligned}$$

So it suffices to determine the polynomials  $f_{i,1}$  and  $f_{i,2}$ , which depend on  $H$  and  $\Lambda$ , to get explicit embeddings.

Recall that any finite subgroup of  $\text{Aut}(\mathbb{P}^1) = \text{PGL}(2, \mathbb{C})$  is isomorphic to  $\mathbb{Z}/n\mathbb{Z}$  (the cyclic group of order  $n$ ),  $D_{2n}$  (the dihedral group of order  $2n$ ),  $\mathfrak{A}_4$  (the tetrahedral group),  $\mathfrak{S}_4$  (the octahedral or cubic group) or  $\mathfrak{A}_5$  (the icosahedral or dodecahedral group) and that there is only one conjugacy class for each of these groups (see e.g. [Beau10]).

1) In the cyclic case, we can assume that  $H \subset \text{PGL}(2, \mathbb{C})$  is generated by  $[x : y] \mapsto [\xi_n x : y]$ , where  $\xi_n$  is a primitive  $n$ -th root of unity. Its pullback  $G \subset \text{SL}(2, \mathbb{C})$  is then

generated by  $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ , where  $\zeta$  is a primitive  $2n$ -th root of unity. An orbit  $\Lambda_i$  of  $H$  is given by the zero set of a polynomial  $p_i = a_i x^n + b_i y^n$  for some  $(a_i, b_i) \in \mathbb{C}^2 \setminus \{(0, 0)\}$  (the cases where  $a_i = 0$  or  $b_i = 0$  provide a fixed point with multiplicity  $n$ ). We thus get

$$P_i = (p_i)^2 \in \mathcal{O}(\mathbb{A}^2)^G$$

and

$$(f_{i,1}, f_{i,2}) = (b_i y^{n-1}(a_i x^n + b_i y^n), -a_i x^{n-1}(a_i x^n + b_i y^n)) \in \text{End}(\mathbb{A}^2)^G$$

which satisfy  $f_{i,1}y - f_{i,2}x = P_i$  (note that the  $f_{i,1}$  and  $f_{i,2}$  are here not unique, and could also be chosen without common factor). The corresponding embedding is

$$\begin{aligned} \Gamma = \mathbb{P}^1 \setminus \Lambda &\rightarrow \mathbb{A}^3 \\ [x : y] &\mapsto \left( \frac{1}{r} \sum_{i=1}^r \frac{a_i x^n - b_i y^n}{a_i x^n + b_i y^n}, \frac{1}{r} \sum_{i=1}^r \frac{-2b_i x y^{n-1}}{a_i x^n + b_i y^n}, \frac{1}{r} \sum_{i=1}^r \frac{2a_i x^{n-1} y}{a_i x^n + b_i y^n} \right). \end{aligned}$$

2) In the dihedral case, we can assume that  $H$  is generated by the maps  $[x : y] \mapsto [\xi_n x : y]$  and  $[x : y] \mapsto [y : x]$ . So  $G$  is generated by  $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$  and  $\begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}$ , where  $\mathbf{i}$  denotes the imaginary unit  $\sqrt{-1}$ .

An orbit  $\Lambda_i$  of  $H$  is given by the zero set of  $p_i = a_i(x^{2n} + y^{2n}) + 2b_i x^n y^n$  for some  $(a_i, b_i) \in \mathbb{C}^2 \setminus \{(0, 0)\}$  and we thus get

$$P_i = (p_i)^2 \in \mathcal{O}(\mathbb{A}^2)^G$$

and

$$(f_{i,1}, f_{i,2}) = (y^{n-1}(b_i x^n + a_i y^n)p_i, -x^{n-1}(a_i x^n + b_i y^n)p_i) \in \text{End}(\mathbb{A}^2)^G$$

which satisfy  $f_{i,1}y - f_{i,2}x = P_i$  (note that  $P_i = p_i$  is also possible if  $n$  is even, and that as before the polynomials  $f_{i,1}, f_{i,2}$  are not unique, and could also be chosen without common factor). This leads to the embedding  $\Gamma = \mathbb{P}^1 \setminus \Lambda \rightarrow \mathbb{A}^3$  which sends a point  $[x : y] \in \Gamma$  onto

$$\left( \frac{1}{r} \sum_{i=1}^r \frac{a_i(x^{2n} - y^{2n})}{a_i(x^{2n} + y^{2n}) + 2b_i x^n y^n}, \frac{1}{r} \sum_{i=1}^r \frac{-2x y^{n-1}(b_i x^n + a_i y^n)}{a_i(x^{2n} + y^{2n}) + 2b_i x^n y^n}, \frac{1}{r} \sum_{i=1}^r \frac{2x^{n-1} y(a_i x^n + b_i y^n)}{a_i(x^{2n} + y^{2n}) + 2b_i x^n y^n} \right).$$

3) In the case of the tetrahedral group, we can assume that  $H \cong \mathfrak{A}_4$  is generated by the maps  $[x : y] \mapsto [\mathbf{i}(x + y) : x - y]$  and  $[x : y] \mapsto [x : -y]$ . This implies that  $G$  is generated by  $\frac{1}{2} \begin{pmatrix} \mathbf{i} - 1 & \mathbf{i} - 1 \\ \mathbf{i} + 1 & -\mathbf{i} - 1 \end{pmatrix}$  and  $\begin{pmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix}$ . An orbit  $\Lambda_i$  of  $H$  is given by the zero set of

$$p_i = 6a_i(x^5 y - x y^5)^2 + b_i(x^4 + y^4)(x^8 + y^8 - 34x^4 y^4),$$

for some  $(a_i, b_i) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ . We thus get

$$\begin{aligned} P_i &= p_i \in \mathcal{O}(\mathbb{A}^2)^G \\ f_{i,1} &= a_i(x^{10}y - 6x^6y^5 + 5x^2y^9) + b_i(-11x^8y^3 - 22x^4y^7 + y^{11}) \\ f_{i,2} &= -a_i(5x^9y^2 - 6x^5y^6 + xy^{10}) - b_i(x^{11} - 22x^7y^4 - 11x^3y^8) \end{aligned}$$



which satisfy  $(f_{i,1}, f_{i,2}) \in \text{End}(\mathbb{A}^2)^G$  and  $f_{i,1}y - f_{i,2}x = P_i$  as before. This gives the embedding  $\Gamma = \mathbb{P}^1 \setminus \Lambda \rightarrow \mathbb{A}^3$  defined by

$$[x : y] \mapsto \left( \begin{array}{l} \frac{1}{r} \sum_{i=1}^r \frac{4a_i x^2 y^2 (x^4 + y^4)(x^4 - y^4) + b_i (x^{12} - 11x^8 y^4 + 11x^4 y^8 - y^{12})}{6a_i (x^5 y - x y^5)^2 + b_i (x^4 + y^4)(x^8 + y^8 - 34x^4 y^4)} \\ \frac{1}{r} \sum_{i=1}^r \frac{-2x(a_i (x^{10} y - 6x^6 y^5 + 5x^2 y^9) + b_i (-11x^8 y^3 - 22x^4 y^7 + y^{11}))}{6a_i (x^5 y - x y^5)^2 + b_i (x^4 + y^4)(x^8 + y^8 - 34x^4 y^4)} \\ \frac{1}{r} \sum_{i=1}^r \frac{2y(a_i (5x^9 y^2 - 6x^5 y^6 + x y^{10}) + b_i (x^{11} - 22x^7 y^4 - 11x^3 y^8))}{6a_i (x^5 y - x y^5)^2 + b_i (x^4 + y^4)(x^8 + y^8 - 34x^4 y^4)} \end{array} \right) \in \mathbb{A}^3.$$

It is also possible to describe similarly the other cases ( $\mathfrak{S}_4$  and  $\mathfrak{A}_5$ ), but the formulas are even more intricate.

### REFERENCES

- [Beau10] A. Beauville, *Finite subgroups of  $\text{PGL}_2(K)$* , Vector bundles and complex geometry, 23–29, Contemp. Math. **522**, Amer. Math. Soc., Providence, RI, 2010.
- [Bor91] A. Borel, *Linear algebraic groups, Second edition*, Graduate Texts in Mathematics **126**, Springer-Verlag, New York, 1991, 288 pp., ISBN 0-387-97370-2
- [Bre72] Bredon, G E *Introduction to compact transformation groups*, Académie Press, 1972
- [DKW99] H. Derksen, F. Kutzschebauch, J. Winkelmann, *Subvarieties of  $\mathbb{C}^n$  with non-extendable automorphisms*, J. Reine Angew. Math. **508** (1999), 213–235.
- [FM89] S. Friedland, J. Milnor, *Dynamical properties of plane polynomial automorphisms*, Ergod. Th & Dyn. Syst. **9** (1989), 67–99.
- [Fur83] M. Furushima, *Finite groups of polynomial automorphisms in  $\mathbb{C}^n$* , Tohoku Math. J. (2) **35** (1983), no 3, 415–424.
- [Gre63] L. Greenberg, *Maximal Fuchsian groups*, Bull. Amer. Math. Soc. **69** (1963), 569–573.
- [Kal91] S. Kaliman, *Extensions of isomorphisms between affine algebraic subvarieties of  $k^n$  to automorphisms of  $k^n$* , Proc. Amer. Math. Soc. **113**, no. 2 (1991), 325–334.
- [PR86] T. Petrie, J. D. Randall, *Finite-order algebraic automorphisms of affine varieties*, Comment. Math. Helv. 61 (1986), no. 2, 203–221.
- [Sat77] A. Sathaye, *On planar curves*, Amer. J. Math. **99** (1977), no. 5, 1105–1135.
- [Ser80] J.-P. Serre, *Trees*, Springer Verlag, Berlin, Heidelberg, New York, 1980.
- [Sri91] V. Srinivas, *On the embedding dimension of an affine variety*. Math. Ann. **289** (1991) 125–132.

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