Jet Groups.

Jean-Philippe FURTER,
Dpt. of Math., Univ. of La Rochelle,
av. M. Crépeau, 17 000 La Rochelle, FRANCE
email: jpfurter@univ-lr.fr

Abstract.

Let $N \geq 2$. We describe some classes of subgroups of the n-jet group of local analytic diffeomorphisms of $(\mathbb{C}^N, 0)$. We apply the results to show that if G is any group of algebraic automorphisms strictly containing the affine group, then any algebraic automorphism can be approximated at the origin and at any order by an element of G. We also show that any algebraic or analytic automorphism can be interpolated at any order and at any finite set of points, by a tame one.

Keywords.

Affine space, Automorphisms, Linear algebraic groups.

Mathematics Subject Classification (2000).

14R10, 20G05, 20G20.

INTRODUCTION.

The first aim of this paper is to generalize a result of [4] asserting that any algebraic automorphism of $\mathbb{C}^N$ can be approximated at the origin and at any order by a tame one. Indeed, we will show the two following results:

Theorem A. If G is any group of algebraic automorphisms of $\mathbb{C}^N$ which is strictly larger than the affine group, then any algebraic automorphism can be approximated at the origin and at any order by an element of G.

Theorem B. Any algebraic (resp. analytic) automorphism of $\mathbb{C}^N$ can be approximated at any finite set of points and at any order by a tame algebraic (resp. analytic) automorphism.

Our second aim is to establish preliminary results to be used in [17]. Since this last paper is devoted to the study of embeddings of finite union of fat points in $\mathbb{C}^N$, it is no wonder that the next result will be useful.

Theorem C. Let $n \geq 1$, let $u^{[1]}, \ldots, u^{[m]}$ be distinct points of $\mathbb{C}^N$ and let $f^{[1]}, \ldots, f^{[m]}$ be analytic (resp. algebraic) automorphisms of $\mathbb{C}^N$. There exists an analytic (resp. algebraic) tame automorphism f such that the n-jets of f and $f^{[k]}$ coincide at each $u^{[k]}$ if and only if the $f^{[k]}(u^{[k]})$ are distinct (resp. the $f^{[k]}(u^{[k]})$ are distinct and the Jacobians of the $f^{[k]}$ are equal).

Let us note that th. B is a direct consequence of th. C. Let $J_n(A)$ $\Big($resp. $J_n(\widetilde{A})\Big)$ be the group of n-jets at the origin of algebraic (resp. analytic) automorphisms of $\mathbb{C}^N$ fixing the origin. These two jet groups play an important part in the last quoted paper. A basic description of them is needed. Let $J_n(E)$ be the monoid of n-jets at the origin of algebraic (or analytic) endomorphisms of $\mathbb{C}^N$ fixing the origin and let GL be the linear group. It is shown in propositions 3.2 and 3.3 below that $J_n(A) = \{f \in J_n(E),\ \mathrm{Jac}\, f \in \mathbb{C}^*\}$ and $J_n(\widetilde{A}) = \{f \in J_n(E),\ J_1 f \in GL\}$.

Finally, our third aim is to describe the subgroups of $J_n(A)$ containing GL and the subgroups of $J_n(\widetilde{A})$ containing $J_n(A)$. Indeed, these desciptions turn out to be very nice and we found they were interesting for themselves. We will also use them to prove theorems A and C. If $K \subset \mathbb{N}$, let A_K be the set of algebraic automorphisms of $\mathbb{C}^N$ which can be written as finite sums of k-homogeneous endomorphisms, $k \in K$. If S is a submonoid of $(\mathbb{N}, +)$, is is shown in prop. 3.1 below that A_{1+S} is a group of algebraic automorphisms fixing the origin (we call monoidal such a group). Let $J_n(A_{1+S})$ be the associated n-jet group. In fact, we will prove:

Theorem D. Any subgroup of $J_n(A)$ containing GL is equal to some $J_n(A_{1+S})$ where S is a submonoid of $\mathbb{N}$.

If $1 \leq k \leq n$, let $J_{n,k} : J_n(\widetilde{A}) \to J_k(\widetilde{A})$ be the natural group-morphism associating to a n-jet its restricted k-jet. Since $J_k(A)$ is a subgroup of $J_k(\widetilde{A})$, its inverse image $J_{n,k}^{-1}(J_k(A))$ is a subgroup of $J_n(\widetilde{A})$. Actually, we will show:

Theorem E. Any subgroup of $J_n(\widetilde{A})$ containing $J_n(A)$ is equal to some $J_{n,k}^{-1}(J_k(A))$ where $1 \leq k \leq n$.

Our paper is divided into six sections. Sections I and II are devoted to establish preliminary results on the vector space E of algebraic endomorphisms of $\mathbb{C}^N$ considered either as a GL-module or as a Lie algebra. In section III, we introduce the notations and tools that we use in section IV (resp. section V) to prove th. D, E, A (resp. th. C). Finally, in section VI, we apply some of the previous notions to variables (or coordinates) and recover a result of [13].

I. THE SPACE E AS A GL-MODULE.

In this section, we will give the decomposition into irreducible submodules of the GL-module E. Let $(e_1, \ldots, e_N)$ be the canonical basis of $V := \mathbb{C}^N$ and let $(x_1, \ldots, x_N)$ be the dual basis of V^*. We will identify any element f of E to the N-uple of its coordinate functions $f = (f_1, \ldots, f_N)$ where each f_L belongs to the ring $R = \mathbb{C}[x_1, \ldots, x_N]$ of regular functions on $\mathbb{A}^N$. Let $SV^* \simeq R$ be the symmetric algebra of V^*. The isomorphism $E \to SV^* \otimes V$, $\displaystyle\sum_L f_L e_L \mapsto \sum_L f_L \otimes e_L$ will be the main thread. Since V is naturally a

GL-module, so is E, the action being the following: $GL \times E \to E$, $(g, f) \mapsto g \circ f \circ g^{-1}$. For $m \geq 0$, let $R_m \subset R$ be the space of homogeneous polynomials of degree m and let $E_m := (R_m)^N \subset E$. It is clear that E_m is a GL-submodule of E. If $f \in E$ and $r \in R$, let us set $\nabla f = \sum_L \frac{\partial f_L}{\partial x_L}$ and $\Delta\, r = r\,\mathrm{id}$, where $\mathrm{id} := (x_1, \ldots, x_N)$ is the identity element of E. We will show that E_m splits into two irreducible submodules $E_m = E_m^0 \oplus E_m^1$, where $E_m^0 := \{f \in E_m, \nabla f = 0\}$ and $E_m^1 := \{\Delta\, r, r \in R_{m-1}\}$.

We begin to give the direct sum decomposition $E = E^0 \oplus E^1$. It is closely linked with the maps of contraction $c : SV^* \otimes V \to SV^*$ and multiplication $m : SV^* \to SV^* \otimes V$. The contraction map is defined by its restriction

$$\begin{array}{ccccc} S^{m+1}V^* & \otimes & V & \to & S^m V^* \\ v_1^* \ldots v_{m+1}^* & \otimes & v & \mapsto & \sum_i < v, v_i^* > v_1^* \ldots \widehat{v_i^*} \ldots v_{m+1}^*. \end{array}$$

The induced map on E is the operator $\nabla : E \to R$.

The map m is the multiplication by the identity element $\mathrm{id} \in V^* \otimes V = Hom(V, V)$. Its restriction $S^m V^* \to S^{m+1} V^* \otimes V$ is the composition of the two maps:

$$\begin{array}{ccccc} S^m V^* & \to & S^m V^* & \otimes & V^* \otimes V \\ t & \mapsto & t & \otimes & \mathrm{id} \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} S^m V^* & \otimes & V^* & \otimes & V & \to & S^{m+1} V^* & \otimes & V \\ t & \otimes & u & \otimes & v & \mapsto & tu & \otimes & v. \end{array}$$

The induced map on $R = SV^*$ is the operator $\Delta : R \to E$.

Lemma 1.1. The maps $\nabla : E \to R$ and $\Delta : R \to E$ are GL-morphisms.

Proof. Since ∇ and Δ correspond to the natural maps c and m, the checking is straightforward. □

Lemma 1.1 shows us that $E^0 := \mathrm{Ker}\ \nabla$ and $E^1 := \mathrm{Im}\ \Delta$ are GL-submodules of E.

Lemma 1.2. $E = E^0 \oplus E^1$.

Proof. One could easily check that $\frac{1}{N+m}\Delta$ is a section of $\nabla : E_{m+1} \to R_m$. The split short exact sequence: $0 \to E_{m+1}^0 \to E_{m+1} \underset{\frac{1}{N+m}\Delta}{\overset{\nabla}{\rightleftarrows}} R_m \to 0$ shows us that $E_{m+1} = E_{m+1}^0 \oplus E_{m+1}^1$. □

It is clear that the direct sums $E = \underset{m \geq 0}{\oplus} E_m$ and $E = \underset{n=0,1}{\oplus} E^n$ are compatible, i.e. $E = \bigoplus_{(m,n) \in \mathbb{N} \times \{0,1\}} E_m^n$, where $E_m^n := E_m \cap E^n$. In fact, we have the following result, where SL denotes the special linear group of $\mathbb{C}^N$:

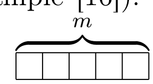
Theorem 1.1. (i) The GL-representations E_m^n, $(m, n) \in \mathbb{N} \times \{0, 1\}$, are irreducible and pairwise non isomorphic;

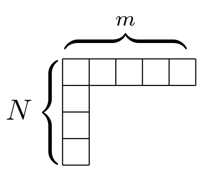
(ii) If $N \geq 3$, the restricted SL-representations are still pairwise non isomorphic;

(iii) If $N = 2$, the restricted SL-representations E^0_m, $m \in \mathbb{N}$, are still pairwise non isomorphic, but the restricted SL-representations E^0_m and E^1_{m+2} are now isomorphic.

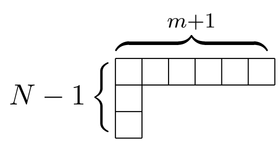
Remarks. 1. According to their definition of irreducible representations, some readers may prefer to except the case $(m, n) = (0, 1)$ where E^1_0 is the null space.

2. A GL-representation is irreducible if and only if its restricted SL-representation is. Furthermore, it is well known that two irreducible GL-representations are isomorphic if and only if their restrictions to SL and to $\mathbb{C}^* \subset GL$ are isomorphic.

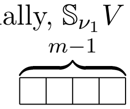
Proof. Let us begin to show that the restricted SL-representations are irreducible. The case $m = 0$ being obvious since $E^0_0 \simeq V$ and $E^1_0 = \{0\}$, let us assume that $m \geq 1$. It is sufficient to show that $E^*_m \simeq S^m V \otimes V^* \simeq S^m V \otimes \bigwedge^{N-1} V$ is the direct sum of exactly two irreducible representations. If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition of an integer $d \geq 1$ ($d = \lambda_1 + \ldots + \lambda_r$ with $\lambda_1 \geq \ldots \geq \lambda_r \geq 1$), let us denote by $\mathbb{S}_\lambda$ the Schur functor associated to λ (see for example [16]). Since $S^m V \simeq \mathbb{S}_\lambda V$, where $\lambda = (m)$ is represented by the Young diagram:  , the Littlewood-Richardson rule shows us that $E^*_m \simeq \mathbb{S}_{\nu_1} V \oplus \mathbb{S}_{\nu_2} V$ where $\nu_1 = (m, \underbrace{1, \ldots, 1}_{N-1})$ and $\nu_2 = (m+1, \underbrace{1, \ldots, 1}_{N-2})$ are represented by the hooks:



and



so that we have shown the irreducibility of the E^n_m.

Finally, $\mathbb{S}_{\nu_1} V \simeq \mathbb{S}_{\mu_1} V$ where $\mu_1 = (m-1)$ is represented by the single row with $m-1$ boxes:  , so that μ_1 and ν_2 are the reduced partitions such that $\left(E^0_m\right)^* \simeq \mathbb{S}_{\mu_1} V$ and $\left(E^1_m\right)^* \simeq \mathbb{S}_{\nu_2} V$. This shows (ii) and (iii).

It remains to show that if $N = 2$ the GL-representations E^0_m and E^1_{m+2} are non isomorphic. But this is clear since their restriction to the subgroup $\mathbb{C}^*$ of GL are non isomorphic. Indeed, $\lambda \in \mathbb{C}^*$ acts on E^0_m as the dilatation of ratio λ^{m-1} and on E^1_{m+2} as the dilatation of ratio λ^{m+1}. $\square$

If $N = 2$, the next result provides us a SL-isomorphism between E^0_m and R_{m+1}:

Proposition. If $N = 2$, the map $\begin{array}{rccc} \alpha : & R & \to & E^0 \\ & r & \mapsto & \left(\frac{\partial r}{\partial x_2}, -\frac{\partial r}{\partial x_1}\right) \end{array}$ is a SL-morphism.

Proof. If $g \in GL$, an easy computation shows that $\alpha(g.r) = (\det g)^{-1}\ g.\alpha(r)$ which proves that α is a SL-morphism (but not a GL-morphism !). $\square$

II. THE SPACE E AS A LIE ALGEBRA.

Let $\mathrm{Der}\, R$ be the set of complex linear derivations of R. The isomorphism $E \to \mathrm{Der}\, R$, $\sum_L f_L e_L \mapsto \sum_L f_L \frac{\partial}{\partial x_L}$ allows us to pull back the usual Lie bracket defined on $\mathrm{Der}\, R$ by $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$. For any $f, g \in E$, we get $[f, g] = g' \times f - f' \times g$, where we agree that $f' = f'(x) = \left(\frac{\partial f_L}{\partial x_M}\right)_{1 \leq L, M \leq N}$ is the Jacobian matrix of f. It is important to stress that even if f is usually written as the line-vector $f = (f_1, \ldots, f_N)$ in the literature (and in this paper !), it should actually be thought of as the column-vector $f = {}^t(f_1, \ldots, f_N)$ in order to grasp the last formula. If $\mathfrak{a}$, $\mathfrak{b}$ are additive subgroups of a Lie algebra $\mathfrak{g}$, the subgroup generated by all brackets $[a, b]$, $(a, b) \in \mathfrak{a} \times \mathfrak{b}$ is denoted by $[\mathfrak{a}, \mathfrak{b}]$. If $\mathfrak{a}$, $\mathfrak{b}$ are subspaces, then $[\mathfrak{a}, \mathfrak{b}]$ also. In th. 2.1 below, we will compute $[E^n_m, E^q_p]$.

Remarks. 1. We have $[E_m, E_n] \subset E_{m+n-1}$. If we set $F_m = E_{m+1}$, then $E = \underset{m \geq -1}{\oplus} F_m$ is a graded Lie algebra.

2. We may have defined the Lie bracket on $E \simeq SV^* \otimes V$ by using the contraction map $c : SV^* \otimes V \to SV^*$. We let the reader check that the Lie bracket is then given by the following map:

$$\begin{array}{ccccccccc} (S^{m+1}V^* & \otimes & V) & \otimes & (S^{n+1}V^* & \otimes & V) & \to & S^{m+n+1}V^* \otimes V \\ (t & \otimes & u) & \otimes & (z & \otimes & v) & \mapsto & t\, c(z \otimes u) \otimes v - z\, c(t \otimes v) \otimes u. \end{array}$$

This allows us to see that $\forall\, g \in GL$, $\forall\, u, v \in E$, $g.[u, v] = [g.u, g.v]$. As a consequence, if $\mathfrak{a}$, b are GL-submodules of E, then $[\mathfrak{a}, \mathfrak{b}]$ also.

3. The Lie algebra structure on E contains in some sense the GL-module structure. Indeed, let us denote by $\mathfrak{g}l = E_1$ the Lie algebra of GL.

Lemma 2.1. The Lie algebra representation: $\begin{array}{ccccc} \mathfrak{g}l & \times & E & \to & E \\ (g & , & f) & \mapsto & [f, g] \end{array}$ is the one associated with the Lie group representation: $\begin{array}{ccccc} GL & \times & E & \to & E \\ (g & , & f) & \mapsto & g \circ f \circ g^{-1}. \end{array}$

Proof. If $f \in E$, one could easily check that the differential at the point id of the map $GL \to E$, $g \mapsto g \circ f \circ g^{-1}$ is the map $\mathfrak{g}l \to E$, $g \mapsto [f, g]$. $\square$

If $\mathfrak{s}l$ (resp. $\mathfrak{c}$) denotes the Lie subalgebra of $\mathfrak{g}l$ corresponding to the subgroup SL (resp. $\mathbb{C}^*$) of GL, then the decomposition $E_1 = E^0_1 \oplus E^1_1$ is the same as the classical Levy decomposition $\mathfrak{g}l = \mathfrak{s}l \oplus \mathfrak{c}$. Furthermore, if W is a representation of a reductive Lie algebra $\mathfrak{g}$, it is well known that $\mathfrak{g}.W$ is equal to the sum of the non trivial irreducible subrepresentations of W (indeed, if we assume in addition that W is irreducible, it is clear that $\mathfrak{g}.W = \{0\}$ if W is trivial and that $\mathfrak{g}.W = W$ otherwise). If $m \geq 2$ and $n = 0, 1$, we know that:

• the $\mathfrak{sl}$-representation E_m^n is irreducible and non trivial;

• the $\mathfrak{c}$-representation E_m^n corresponds to the $\mathbb{C}^*$-representation E_m^n where $\lambda \in \mathbb{C}^*$ acts on E_m^n as the dilatation of ratio λ^{m-1}.

As a result:

Corollary 2.1. $[E_1^0, E_1^0] = [E_1^0, E_1] = [E_1, E_1] = E_1^0$;
$[E_1^1, E_1^0] = [E_1^1, E_1^1] = [E_1^1, E_1] = \{0\}$;
$[E_1^0, E_m^n] = [E_1^1, E_m^n] = [E_1, E_m^n] = E_m^n$ for $m \geq 2$ and $n = 0, 1$.

The next result will show that E^0 and E^1 are Lie subalgebra of E.

Lemma 2.2. (i) $\forall\ f, g \in E,\ \nabla\left([f,g]\right) = (\nabla g)' \times f - (\nabla f)' \times g$;
(ii) $\forall\ (r,s) \in R_m \times R_n,\ [r\,\mathrm{id}, s\,\mathrm{id}] = (n-m)\ rs\,\mathrm{id}$.

Proof. An easy computation would show that $\nabla(g' \times f) = (\nabla g)' \times f\ +\ \mathrm{Tr}(g' \times f')$ and (i) follows. We could also check that $(s\,\mathrm{id})' \times (r\,\mathrm{id}) = (n+1)\ rs\,\mathrm{id}$ and (ii) follows. □

Corollary 2.2. E^0 and E^1 are Lie subalgebra of E.

Even if $E = E^0 \oplus E^1$ as vector spaces, the sum is not direct as Lie algebra since $[E^0, E^1] \neq \{0\}$. In fact, E^0 and E^1 are even not Lie ideals of E:

Theorem 2.1. Let $m, n \geq 1$.

(i) $[E_m^0, E_n^0] = E_{m+n-1}^0$;

(ii) $[E_m^1, E_n^1] = E_{m+n-1}^1$ if $m \neq n$;
$= \{0\}$ if $m = n$;

(iii) $[E_m, E_n] = E_{m+n-1}$ if m or $n \geq 2$;
$= E_1^0$ if $m = n = 1$;

(iv) $[E_m^0, E_n^1] = E_{m+n-1}$ if $m, n \geq 2$;
$= E_n^1$ if $m = 1$, $n \geq 2$;
$= E_m^0$ if $m \geq 2$, $n = 1$;
$= \{0\}$ if $m = n = 1$.

Proof. We recall that $[x^\alpha e_L, x^\beta e_M] = \frac{\partial}{\partial x_M}(x^\alpha) x^\beta e_L - \frac{\partial}{\partial x_L}(x^\beta) x^\alpha e_M$.

(i) Since $[E_m^0, E_n^0]$ is a submodule of the irreducible GL-module E_{m+n-1}^0, it is sufficient to show that $[E_m^0, E_n^0] \neq \{0\}$. Indeed, $x_2^m e_1 \in E_m^0$, $x_1^n e_2 \in E_n^0$ and $[x_2^m e_1, x_1^n e_2] = m x_1^n x_2^{m-1} e_1 - n x_1^{m-1} x_2^n\ e_2 \neq 0$.

(ii) Point (ii) of lemma 2.2 shows us that $[E_m^1, E_n^1] = \{0\}$ if and only if $m = n$. When $m \neq n$, since $[E_m^1, E_n^1]$ is a submodule of the irreducible GL-module E_{m+n-1}^1, we can conclude to the equality.

(iv) We can assume that $m,n \geq 2$. Let us set $u = [x_2^m e_1,\, x_1^{n-1}\,\mathrm{id}] \in [E_m^0, E_n^1]$.

An easy computation would show that $u = (m-1)x_1^{n-1}x_2^m e_1 - (n-1)x_1^{n-2}x_2^m\,\mathrm{id}$, so that $u \notin E_{m+n-1}^1$. We would also get $\nabla u = -(n-1)(N+n-1)x_1^{n-2}x_2^m \neq 0$, so that $u \notin E_{m+n-1}^0$. Since $[E_m^0, E_n^1]$ is a GL-submodule of E_{m+n-1}, we must have $[E_m^0, E_n^1] = E_{m+n-1}$.

(iii) It is a consequence of (i), (ii) and (iv). □

III. NOTATIONS AND PRELIMINARY RESULTS.

1. Jets.

Let $\mathbb{A}^N$ be the vector space $V = \mathbb{C}^N$ when it is seen as an affine space. Let $\widetilde{E}$ be the space of analytic endomorphisms of $\mathbb{A}^N$. We will identify any element f of $\widetilde{E}$ to the N-uple of its coordinate functions $f = (f_1, \ldots, f_N)$ where each f_L belongs to the ring $\widetilde{R}$ of analytic functions on $\mathbb{A}^N$. If $r \in \widetilde{R}$ and $a \in \mathbb{A}^N$, we will distinguish between the (classical) n-jet of r at a: $\mathfrak{J}_{n,a}\, r := \sum\limits_{0 \leq k \leq n} \dfrac{1}{k!}\, D_a^k r \,.\, x^k$ and the centered n-jet of r at a: $J_{n,a}\, r := \sum\limits_{1 \leq k \leq n} \dfrac{1}{k!}\, D_a^k r \,.\, x^k$. Of course, $D_a^k r$ denotes the k-th differential of r at the point a and we recall that $D_a^k r \,.\, x^k = \sum\limits_{\substack{\alpha \in \mathbb{N}^N \\ |\alpha| = k}} \binom{k}{\alpha} \dfrac{\partial^k r}{\partial x^\alpha}(a)\, x^\alpha$, where $\binom{k}{\alpha} = \dfrac{k!}{\alpha!} = \dfrac{k!}{\alpha_1! \ldots \alpha_N!}$.

The classical and centered n-jets are related by the formula $\mathfrak{J}_{n,a}\, r = r(a) + J_{n,a}\, r$.

By the same way, if $f \in \widetilde{E}$ and $a \in \mathbb{A}^N$, we will denote by

$$\mathfrak{J}_{n,a}\, f := \sum_{0 \leq k \leq n} \frac{1}{k!}\, D_a^k f \,.\, x^k \qquad (\text{resp. } J_{n,a}\, f := \sum_{1 \leq k \leq n} \frac{1}{k!}\, D_a^k f \,.\, x^k)$$

the classical (resp. centered) n-jet of f at the point a. If $f = (f_1, \ldots, f_N)$, we could also have set $\mathfrak{J}_{n,a}\, f = \Big(\mathfrak{J}_{n,a}\, f_1, \ldots, \mathfrak{J}_{n,a}\, f_N\Big)$ and $J_{n,a}\, f = \Big(J_{n,a}\, f_1, \ldots, J_{n,a}\, f_N\Big)$.

If a is the origin of the affine space $\mathbb{A}^N$, we will simply write $\mathfrak{J}_n$ (resp. J_n) instead of $\mathfrak{J}_{n,0}$ (resp. $J_{n,0}$). We will denote by $\mathfrak{J}_n(R)$ (resp. $J_n(R)$) the space of classical (resp. centered) n-jets of polynomials in N indeterminates and by $\mathfrak{J}_n(E)$ (resp. $J_n(E)$) the space of classical (resp. centered) n-jets of polynomial endomorphisms of $\mathbb{A}^N$.

Observe that $\mathfrak{J}_n(R)$ (resp. $J_n(R)$, resp. $\mathfrak{J}_n(E)$, resp. $J_n(E)$) are naturally isomorphic to $R_{\,\leq n} := \bigoplus\limits_{k \leq n} R_k$ (resp. $R_{1\,\leq\,.\,\leq\,n} := \bigoplus\limits_{1 \leq k \leq n} R_k$, resp. $E_{\,\leq n} := \bigoplus\limits_{k \leq n} E_k$, resp. $E_{1\,\leq\,.\,\leq\,n} := \bigoplus\limits_{1 \leq k \leq n} E_k$).

Remark. The algebraic and analytic n-jet spaces are naturally isomorphic so that we will write $\mathfrak{J}_n(R) = \mathfrak{J}_n(\widetilde{R})$, $J_n(R) = J_n(\widetilde{R})$, $\mathfrak{J}_n(E) = \mathfrak{J}_n(\widetilde{E})$, $J_n(E) = J_n(\widetilde{E})$.

Furthermore, one could easily check that the Jacobian map $\mathrm{Jac} : E \to R$ (or Jac :

$\widetilde{E} \to \widetilde{R}$) induces a map $\mathfrak{J}_n(E) = \mathfrak{J}_n(\widetilde{E}) \to \mathfrak{J}_{n-1}(R) = \mathfrak{J}_{n-1}(\widetilde{R})$ that we will still call Jac. Therefore, the Jacobian of an endomorphism n-jet is naturally a $(n-1)$-jet.

Finally, let us recall that $J_n(E) = J_n(\widetilde{E})$ is naturally a monoid and that the formula $D_a(f \circ g) = D_{g(a)}(f) \circ D_a(g)$ for differentials is generalized by the formula $J_{n,a}(f \circ g) = J_{n,g(a)}(f) \circ J_{n,a}(g)$ for centered n-jets (the latter formula generalizes the former since $D_a(g)$ is identified with $J_{1,a}(g)$). Let $J_n(E)^*$ be the group of invertible centered n-jets. If f belongs to a graded object, let $f_{(k)}$ be its k-homogeneous component. We recall that $j \in J_n(E)$ is invertible if and only if $j_{(1)} \in GL$. This is still equivalent to saying that $\operatorname{Jac} j$ is an invertible element of $\mathfrak{J}_{n-1}(R)$ or to saying that $(\operatorname{Jac} j)(0) \neq 0$.

Let A (resp. $\widetilde{A}$) be the group of algebraic (resp. analytic) automorphisms of $\mathbb{C}^N$. If $G \leq \widetilde{A}$, let us agree, that $J_n(G)$ denotes the group of n-jets at the origin of the elements of G fixing the origin.

2. Monoidal groups.

For $K \subset \mathbb{N}$, we set $E_K := \underset{k \in K}{\oplus} E_k \subset E$ and $A_K := A \cap E_K$. We will give conditions for A_K to be a group. Our first lemma is obtained by an easy computation.

Lemma 3.1. If S is any submonoid of $\mathbb{N}$, then E_{1+S} is a monoid (for the composition).

Proof. If $K \subset \mathbb{N}$, we set $R_K := \underset{k \in K}{\oplus} R_k$. Since $E_{1+S} = (R_{1+S})^N$, it is enough to show that $\prod_{1 \leq L \leq N} f_L^{\alpha_L} \in R_{1+S}$ for any $x^\alpha \in R_{1+S}$ and any $f_1, \ldots, f_N \in R_{1+S}$. By decomposing each f_L as a sum of homogeneous polynomials, it is sufficient to show that $\prod_{1 \leq j \leq r} g_j^{\beta_j} \in R_{1+S}$ for any homogeneous polynomials $g_1, \ldots, g_r \in R_{1+S}$ and for any $\beta \in \mathbb{N}^r$ such that $|\beta| = \sum_j \beta_j \in 1+S$. Let $s_j \in S$ be such that $\deg g_j = 1 + s_j$. We have

$$\deg \prod_j g_j^{\beta_j} = \sum_j \beta_j(1+s_j) = \sum_j \beta_j + \sum_j \beta_j s_j \in 1+S+S \subset 1+S. \qquad \square$$

It is well known that any nonempty finite subset of a group which is stable by composition is a subgroup. Our second lemma is the generalization of this result for algebraic groups.

Lemma 3.2. Any nonempty closed subset of an algebraic group which is stable by composition is a subgroup.

Proof. Let G be the algebraic group and H the subset. For any $h \in H$, the map $m_h : H \to H$, $k \mapsto hk$ being an injective endomorphism, it is surjective (see prop. 17.9.6 p. 80 in [19] for the original idea, but the precise result is proven in [5], [9], [6], [11] or

[27]), so that $1 \in H$ and $h^{-1} \in H$. □

Our last lemma is an obvious consequence of the first two.

Lemma 3.3. If S is any submonoid of $\mathbb{N}$, then $J_n(A_{1+S})$ is a subgroup of $J_n(A)$.

Proposition 3.1. If S is any submonoid of $\mathbb{N}$, then A_{1+S} is a subgroup of A.

Proof. It easily follows from lemma 3.3. Indeed, if $f \in A_{1+S}$, we want to show that $f^{-1} \in A_{1+S}$. However, for any $n \geq 1$, $J_n(f) \in J_n(A_{1+S})$, so that $J_n(f^{-1}) = J_n(f)^{-1} \in J_n(A_{1+S})$. This is sufficient for showing that $f^{-1} \in A_{1+S}$. □

Example. If $S = \mathbb{N}$, then $A_{1+\mathbb{N}}$ is the group of automorphisms fixing the origin. If $S = 2\mathbb{N}$, then $A_{1+2\mathbb{N}}$ is the group of odd automorphisms, i.e. automorphisms f satisfying $f(-x) = -f(x)$. More generally, if $n \geq 2$ and $\omega_n := e^{\frac{2\pi i}{n}}$, then $A_{1+n\mathbb{N}}$ is the group of automorphisms f satisfying $f(\omega_n x) = \omega_n f(x)$.

3. A useful lemma in representation theory.

Lemma 3.4. Let G be a connected reductive complex algebraic group and let W be a finite dimensional G-reprensentation which does not contain the trivial representation. Then, any G-stable subgroup of $(W, +)$ is a vector subspace.

Proof. We will argue by induction on dim W. If dim $W = 0$, there is nothing to prove. Let us now assume that dim $W > 0$ and that F is a G-stable subgroup of W. Of course, we may assume that $\text{Span}(F) = W$. Let $\mathbb{T}$ be a maximal torus of G. If $\mathbb{T}^*$ is the character group of $\mathbb{T}$ (the set of algebraic group-morphisms $\chi : \mathbb{T} \to \mathbb{C}^*$) and $\mathbb{T}_{\mathbb{Q}} := \{t \in \mathbb{T}, \ \forall\, \chi \in \mathbb{T}^*, \ \chi(t) \in \mathbb{Q}\}$ is the subgroup of rational points of $\mathbb{T}$, it is a classical fact that $\mathbb{T}^* \simeq \mathbb{Z}^m$ and $\mathbb{T}_{\mathbb{Q}} \simeq (\mathbb{Q}^*)^m$ (as groups). If $\chi \in \mathbb{T}^*$, $W_\chi := \{u \in W, \ \forall\, t \in \mathbb{T}, \ t.u = \chi(t)\, u\}$ will denote the eigenspace of W associated to the eigenvalue χ. Since $W = \bigoplus_{\chi \in \mathbb{T}^*} W_\chi$, any $u \in W$ can be uniquely written $u = \sum_{\chi \in \mathbb{T}^*} u_\chi$, $u_\chi \in W_\chi$. By representation theory, W is a non trivial $\mathbb{T}$-module. Hence, there exists a non trivial $\psi \in \mathbb{T}^*$ for which $W_\psi \neq \{0\}$.

<u>Main claim.</u> $F \cap W_\psi \neq \{0\}$.

Since $\text{Span}(F) = W$, there exists $u \in F$ with $u_\psi \neq 0$. Let $u = \sum_{k=1}^{n} u_{\chi_k}$ be the decomposition of u in sum of eigenvectors where $\chi_1, \ldots, \chi_n$ are distinct and $\chi_1 = \psi$. The maps $\chi_{k|\mathbb{T}_{\mathbb{Q}}}$, $1 \leq k \leq n$, are still distinct ($\mathbb{T}_{\mathbb{Q}}$ being a dense subset of $\mathbb{T}$). We now use the fact that if G is any group and K any field, then the set $\text{Hom}(G, K^*)$ of all group-morphisms $G \to K^*$ is a linearly independant subset of K^G the space of all K-valued functions on G (see lemma 16.1 of [20]). Therefore, there exist $t_1, \ldots, t_n \in \mathbb{T}_{\mathbb{Q}}$ such that the

$n \times n$ matrix $M := (\chi_k(t_l))_{1\leq k,l\leq n}$ is invertible. Let $r = {}^t(r_1, \ldots, r_n) \in \mathbb{Q}^n$ be such that $M.r = {}^t(1, 0, \ldots, 0)$ and let μ be a nonzero integer such that $\mu r_1, \ldots, \mu r_n$ are integers (we can just take for μ the least common multiple of the denominators of the r_k). Let us check that $v := \mu \sum_{k=1}^{n} r_k\ t_k.u \in F \cap W_\psi$. Indeed $v = \sum_{k=1}^{n} (\mu r_k) t_k.u \in F$ and

$$v = \mu \sum_{k=1}^{n} r_k\ t_k. \left(\sum_{l=1}^{n} u_{\chi_l} \right) = \mu \sum_{k=1}^{n} r_k \sum_{l=1}^{n} \chi_l(t_k) u_{\chi_l} = \mu \sum_{l=1}^{n} \left(\sum_{k=1}^{n} r_k \chi_l(t_k) \right) u_{\chi_l}$$

$= \mu\ u_{\chi_1} = \mu\ u_\psi \in W_\psi \setminus \{0\}$ and the claim is proven.

Let us now show that F contains the G-subrepresentation $W_1 := \underset{g\in G}{Span}\, g.v$.

We have $\forall\ t \in \mathbb{T},\ t.v = \psi(t)v \in F$ and $\psi : \mathbb{T} \to \mathbb{C}^*$ is onto since non trivial. Therefore $\lambda v \in F$ for any $\lambda \in \mathbb{C}$. Any $w \in W_1$ can be written $w = \sum_{k=1}^{r} \lambda_k\ g_k.v$, where $\lambda_k \in \mathbb{C}$, $g_k \in G$. The equality $w = \sum_{k=1}^{r} g_k.(\lambda_k v)$ shows us that $w \in F$.

If W_2 is a G-subrepresentation of W such that $W = W_1 \oplus W_2$, it is clear that W_2 does not contain the trivial G-representation and that $W_2 \cap F$ is a G-stable subgroup of W_2. Therefore, by induction hypothesis, it is a subspace of W_2. It is easy to show that the subspace $W_1 \oplus (W_2 \cap F)$ is equal to F. □

Corollary 3.1. If $m \geq 2$, the SL-stable subgroups of E_m are $\{0\}, E_m, E_m^0$ and E_m^1.

Proof. The SL-modules E_m^0 and E_m^1 are irreducible, non trivial and non isomorphic. □

4. Initial groups.

Let $\mathbb{N}_{<n} := \{0, 1, \ldots, n-1\}$. The relation $G_1 \leq G_2$ will mean that G_1 is a subgroup of G_2. If $G \leq J_n(E)^*$ and $k \in \mathbb{N}_{<n}$, we define the k-th initial group of G by $H_k(G) := \{f_{(k+1)},\ f \in G,\ J_k f = J_k(\mathrm{id})\}$. Definitions like that can be traced back to the theory of automorphisms of free and relatively free groups (see [3] and [7] where automorphisms which are identical modulo some conditions are studied). It turns out that $H_0(G) = J_1(G)$ is a multiplicative subgroup of GL, whereas $H_k(G)$ is an additive subgroup of E_{k+1} for $k \geq 1$. Indeed, if $u_2, u_2 \in E_{k+1}$ and $j_m = \mathrm{id} + u_m \in J_{k+1}(E)^*$ for $m = 1, 2$, then $j_1 \circ j_2^{\pm 1} = \mathrm{id} + u_1 \pm u_2$. Furthermore, let $f, g \in \widetilde{A}$ be such that $J_m f = \mathrm{id} + u$, $u \in E_m$ and $J_n g = \mathrm{id} + v$, $v \in E_n$. If $[f, g] := f \circ g \circ f^{-1} \circ g^{-1} \in \widetilde{A}$, it is shown in [4] that $J_{m+n-1}[f, g] = \mathrm{id} - [u, v]$. Therefore, we get:

Lemma 3.5. If $k, l > 0$ are such that $k + l < n$, then $[H_k(G), H_l(G)] \subset H_{k+l}(G)$.

The groups $H_k(G)$ look like the initial ideals used in Gröbner bases theory. They satisfy an analogous fundamental property (see lemma 15.5 in [14]):

Lemma 3.6. If $G_1 \leq G_2 \leq J_n(E)^*$, then $G_1 = G_2 \Longleftrightarrow \forall\, k \in \mathbb{N}_{<n},\, H_k(G_1) = H_k(G_2)$.

Proof. If $G_1 \neq G_2$, let k be the biggest integer such that there exists $f \in G_2 \setminus G_1$ with $J_k f = J_k(\mathrm{id})$. Since $H_k(G_1) = H_k(G_2)$, we may write $J_{k+1} f = J_{k+1} g$ with $g \in G_1$. But then $f \circ g^{-1} \in G_2 \setminus G_1$ and $J_{k+1}\, f \circ g^{-1} = J_{k+1}(\mathrm{id})$; a contradiction. □

In the sequel, we will always assume that G is SL-invariant, so that $H_k(G)$ too. Similar action of the general linear group (including the modules and the Lie algebras introduced in sections I and II) was already actively used in [10]. More details may be found in the survey [12]. Using cor. 3.1, we get $H_k(G) = \{0\}$, E_{k+1}, E^0_{k+1} or E^1_{k+1} for $k \geq 1$. This incites us to set $H^l_k(G) := H_k(G) \cap E^l_{k+1}$ for $l = 0, 1$ and $1 \leq k < n$. It is clear that $H^l_k(G) = \{0\}$ or E^l_{k+1} and that $H_k(G) = \bigoplus_{l=0,1} H^l_k(G)$. Therefore, the $H_k(G)$ for $k \geq 1$ are encoded by the sets $\mathcal{I}_l(G) := \{0\} \cup \{k,\, 1 \leq k \leq n-1 \text{ and } H^l_k(G) \neq \{0\}\}$. Lemma 3.5 and th. 2.1 show that the $\mathcal{I}_l(G)$ satisfy the following properties:

Lemma 3.7. If $k, l \geq 0$ are such that $k + l < n$, then $k, l \in \mathcal{I}_0(G) \Longrightarrow k + l \in \mathcal{I}_0(G)$ and $k \in \mathcal{I}_0(G)$, $l \in \mathcal{I}_1(G) \Longrightarrow k + l \in \mathcal{I}_1(G)$.

Corollary. If $G \leq J_n(E)^*$ is SL-invariant, then:
(i) $\mathcal{I}_0(G) = \mathbb{N}_{<n} \Longleftrightarrow 1 \in \mathcal{I}_0(G)$;
(ii) $\mathcal{I}_0(G) = \mathcal{I}_1(G) = \mathbb{N}_{<n} \Longleftrightarrow 1 \in \mathcal{I}_0(G) \cap \mathcal{I}_1(G)$.

We finish this subsection by applying the previous results to the subgroups $J_n(A)$ and $J_n(\widetilde{A})$ of $J_n(E)^*$. Let us recall that the group T of algebraic tame automorphisms is the subgroup of A generated by the affine automorphisms and by the elementary automorphisms $\mathrm{id} + p(x) e_L$, where p is a polynomial independant of x_L.

Proposition 3.2. $J_n(A) = J_n(T) = \{f \in J_n(E), \mathrm{Jac}\, f \in \mathbb{C}^*\}$.

Proof. If $G_1 := J_n(A)$, $G_2 = J_n(T)$ and $G_3 := \{f \in J_n(E), \mathrm{Jac}\, f \in \mathbb{C}^*\}$, it is clear that $G_2 \leq G_1 \leq G_3$, so it is enough to show that $G_2 = G_3$. If $1 \leq k < n$ and $u \in H_k(G_3) \subset E_{k+1}$, then $f := \mathrm{id} + u \in J_{k+1}(E)$ must satisfy $\mathrm{Jac}\, f = 1$. However, $\mathrm{Jac}(\mathrm{id} + u) = 1 + \nabla u$, so that $\nabla u = 0$ and $u \in E^0_{k+1}$. This shows that $H^1_k(G_3) = \{0\}$ and $\mathcal{I}_1(G_3) = \{0\}$. But $\mathrm{id} + x_2^2 e_1 \in G_2$, so that $x_2^2 e_1 \in H^0_1(G_2)$ and $1 \in \mathcal{I}_0(G_2)$. We get $\mathcal{I}_0(G_2) = \mathbb{N}_{<n}$. Finally, it is clear that $\mathcal{I}_0(G_2) = \mathcal{I}_0(G_3) = \mathbb{N}_{<n}$, that $\mathcal{I}_1(G_2) = \mathcal{I}_1(G_3) = \{0\}$ and that $H_0(G_2) = H_0(G_3) = GL$, so that $G_2 = G_3$ by lemma 3.6. □

In some sense, at the level of n-jets, the equality $J_n(A) = \{f \in J_n(E), \mathrm{Jac}\, f \in \mathbb{C}^*\}$ solves the Jacobian problem (see [23], [8] and [15]) and the equality $J_n(A) = J_n(T)$ solves the tameness problem for algebraic automorphisms (see [22], [24], [25], [28] and [29]).

Let us recall that the group $\widetilde{T}$ of analytic tame automorphims is the subgroup of $\widetilde{A}$ generated by the affine automorphisms and by the overshears $(x', p(x')+q(x')x_N)$, where $x' = (x_1, \ldots, x_{N-1})$, $p, q : \mathbb{C}^{N-1} \to \mathbb{C}$ are analytic and q does not vanish (or similar ones obtained by permuting the variables).

Proposition 3.3. $J_n(\widetilde{A}) = J_n(\widetilde{T}) = J_n(E)^*$.

Proof. If $G_1 := J_n(\widetilde{A})$, $G_2 = J_n(\widetilde{T})$ and $G_3 := J_n(E)^*$, it is clear that $G_2 \le G_1 \le G_3$, so it is enough to show that $G_2 = G_3$. Since $f := (e^{x_2}x_1, x_2, \ldots, x_N) \in \widetilde{T}$, we get $x_1x_2\, e_1 \in H_1(G_2)$, so that $H_1(G_2) = E_2$, $1 \in \mathcal{I}_0(G_2) \cap \mathcal{I}_1(G_2)$ and $\mathcal{I}_0(G_2) = \mathcal{I}_1(G_2) = \mathbb{N}_{<n}$. Finally, it is clear that $\mathcal{I}_0(G_2) = \mathcal{I}_0(G_3) = \mathbb{N}_{<n}$, that $\mathcal{I}_1(G_2) = \mathcal{I}_1(G_3) = \mathbb{N}_{<n}$ and that $H_0(G_2) = H_0(G_3) = GL$, so that $G_2 = G_3$ by lemma 3.6. □

As in the algebraic case, the equality $J_n(\widetilde{A}) = J_n(\widetilde{T})$ solves the tameness problem for analytic automorphisms (see [3] and [2]).

IV. PROOFS OF THEOREMS D, E, A.

The submonoid generated by $I \subset \mathbb{N}$ will be denoted by $< I >$. Let $\mathcal{T}$ be the set of subsets I of $\mathbb{N}_{<n}$ such that $I =< I > \cap \mathbb{N}_{<n}$ (i.e. I is the trace of some submonoid of $\mathbb{N}$). Lemma 3.7 shows us that if $G \le J_n(E)^*$ is SL-invariant, then $\mathcal{I}_0(G) \in \mathcal{T}$. Let $J_n(A)_{id} := \{f \in J_n(A),\ J_1 f = \mathrm{id}\}$ and let $\mathcal{S}$ be the set of subgroups of $J_n(A)_{id}$ which are SL-invariant. If $I \in \mathcal{T}$, we set $\mathcal{G}(I) := J_n(A_{1+<I>}) \cap J_n(A)_{id}$ and we recall that $\mathcal{G}(I)$ is a group by prop. 3.1.

Lemma 4.1. (i) If $G_1 \le G_2$ belong to $\mathcal{S}$, then $G_1 = G_2 \Longleftrightarrow \mathcal{I}_0(G_1) = \mathcal{I}_0(G_2)$.

(ii) If $I \in \mathcal{T}$, then $\mathcal{G}(I) \in \mathcal{S}$ and $\mathcal{G}(I)$ is the subgroup generated by the $g \circ f^{[k]} \circ g^{-1}$, $g \in SL$, $k \in I$, where $f^{[k]} := \mathrm{id} + x_2^{k+1} e_1$. If $J \subset \mathbb{N}_{<n}$ satisfies $I =< J > \cap \mathbb{N}_{<n}$, then $\mathcal{G}(I)$ is also generated by the $g \circ f^{[k]} \circ g^{-1}$, $g \in SL$, $k \in J$.

Proof. (i). Since $H_0(G) = \{\mathrm{id}\}$ and $\mathcal{I}_1(G) = \{0\}$ for $G \in \mathcal{S}$, it is a direct consequence of lemma 3.6.
(ii). The fact that $\mathcal{G}(I) \in \mathcal{S}$ is obvious. Let $G_1 :=< g \circ f^{[k]} \circ g^{-1}, g \in SL, k \in J >$, $G_2 := \mathcal{G}(I)$ and let us show that $G_1 = G_2$ by applying the last point.

We clearly have $G_1 \le G_2$ and $\mathcal{I}_0(G_2) = I$. It remains to show that $\mathcal{I}_0(G_1) = I$. The relation $G_1 \le G_2$ implies $\mathcal{I}_0(G_1) \subset \mathcal{I}_0(G_2) = I$. On the converse, since $J \subset \mathcal{I}_0(G_1)$ and $\mathcal{I}_0(G_1) \in \mathcal{T}$, we have $< J > \cap \mathbb{N}_{<n} = I \subset \mathcal{I}_0(G_1)$. □

If $I \in \mathcal{T}$, it is clear that $\mathcal{I}_0(\mathcal{G}(I)) = I$. It turns out that if $G \in \mathcal{S}$, the equality $\mathcal{G}(\mathcal{I}_0(G)) = G$ is also true, but does not look so clear for us. Indeed, if $I = \mathcal{I}_0(G)$,

$G_1 = G$ and $G_2 = \mathcal{G}(\mathcal{I}_0(G))$, it is clear that $G_1, G_2 \in \mathcal{S}$ and that $\mathcal{I}_0(G_1) = \mathcal{I}_0(G_2) = I$. Unfortunately, we cannot apply right now point (i) of lemma 4.1, since we do not know yet that $G_1 \leq G_2$ or $G_2 \leq G_1$.

Theorem 4.1. *The map $\mathcal{I}_0 : \mathcal{S} \to \mathcal{T}$, $G \mapsto \mathcal{I}_0(G)$ is bijective with inverse the map $I \mapsto \mathcal{G}(I)$.*

Proof. The main point is to show that for any $G \in \mathcal{S}$ we have $G = \mathcal{G}(\mathcal{I}_0(G))$. If we set $I = \mathcal{I}_0(G)$, it is sufficient to show that $G = \mathcal{G}(I)$, i.e. (by (ii) of lemma 4.1) $G =< g \circ f^{[k]} \circ g^{-1},\ g \in SL,\ k \in I >$. We argue by induction on n.

If $n = 1$, it is obvious. If $n \geq 2$ and if $J = \mathcal{I}_0(J_{n-1}G)$, then by induction hypothesis we have $J_{n-1}G =< g \circ f^{[k]} \circ g^{-1},\ g \in SL,\ k \in J >$.

For each such k, there exists $u^{[k]} \in G$ such that $J_{n-1}f^{[k]} = J_{n-1}u^{[k]}$. Therefore, if we set $h^{[k]} := (f^{[k]})^{-1} \circ u^{[k]} \in J_n(A)$, then $J_{n-1}h^{[k]} = \mathrm{id}$ and $f^{[k]} \circ h^{[k]} \in G$.

First case. $n - 1 \in I$, i.e. $I = J \cup \{n-1\}$.

This implies $H_{n-1}(G) = E_n^0$, so that for any $u \in E_n^0$, id$+u \in G$. Therefore, $f^{[n-1]} \in G$ and $\forall\ k \in J$, $f^{[k]}$ and $h^{[k]} \in G$. It is clear that $G = \mathcal{G}(I)$.

Second case. $n - 1 \notin I$, i.e. $I = J$.

This implies $H_{n-1}(G) = \{0\}$ and $n-1 \notin< J >$. It is enough to show that $h^{[k]} = \mathrm{id}$ for each $k \in J$. This comes from the assertion stated below. First and foremost, let $\mathcal{G}(J)_{\mathbb{Q}}$ denote the subgroup of rational points of $\mathcal{G}(J)$. We recall that $\mathcal{G}(\{0, n-1\})$ denotes the subgroup of $J_n(A)_{id}$ whose elements are of the form $h = \mathrm{id} + h_{(n)}$, $h_{(n)} \in E_n^0$. Let us note that $\mathcal{G}(\{0, n-1\})$ is included into the center of $J_n(A)_{id}$ so that it will commute with $\mathcal{G}(J)$.

Assertion. $\forall\, f \in \mathcal{G}(J)_{\mathbb{Q}}$, $\forall\, h \in \mathcal{G}(\{0, n-1\})$, $f \circ h \in G \Longrightarrow h = \mathrm{id}$.

If the assertion is false, let k be the biggest integer such that there exists a counterexample (f, h) with $J_k f = \mathrm{id}$. Since $h_{(n)} \neq 0$ and since E_n^0 is an irreducible SL-module, there exist $g_1, \ldots, g_r \in SL_{\mathbb{Q}}$ such that the $g_i \circ h_{(n)} \circ g_i^{-1}$, $1 \leq i \leq r$, constitue a basis of the complex vector space E_n^0. Indeed, if W is an irreducible SL-module of dimension r and if $w \in W$ is nonzero, there exist $g_1, \ldots, g_r \in SL_{\mathbb{Q}}$ such that the $g_i.w$, $1 \leq i \leq r$, constitute a basis of the complex vector space W: the map $\varphi :\ (SL)^r \to \bigwedge^d W$, $(g_i)_{1 \leq i \leq r} \mapsto \bigwedge_i g_i.w$ being nonzero, it has to be nonzero on $(SL_{\mathbb{Q}})^r$ since $SL_{\mathbb{Q}}$ is (Zariski) dense in SL. However, $\dim\ E_{k+1}^0 < \dim\ E_n^0$, so that the $g_i \circ f_{(k+1)} \circ g_i^{-1}$, $1 \leq i \leq r$, are linearly dependant over $\mathbb{C}$. Since they belong to $\left(E_{k+1}^0\right)_{\mathbb{Q}}$, they are even linearly dependant over $\mathbb{Q}$, showing the existence of integers m_i, non all zero, such that $\sum_{1 \leq i \leq r} m_i\ \ g_i \circ f_{(k+1)} \circ g_i^{-1} = 0$. If $a_1, \ldots, a_r \in J_n(A)$, let us agree that $\prod_{i=1}^{r} a_i$ denotes the composition $a_1 \circ a_2 \circ \ldots \circ a_r$ in that order. If we set

$\widetilde{f} := \prod_{i=1}^{r} g_i \circ f^{m_i} \circ g_i^{-1} \in \mathcal{G}(J)_{\mathbb{Q}}$ and $\widetilde{h} := \prod_{i=1}^{r} g_i \circ h^{m_i} \circ g_i^{-1} \in \mathcal{G}(\{0, n-1\})$, then

- $\widetilde{f} \circ \widetilde{h} = \prod_{i=1}^{r} g_i \circ (f \circ h)^{m_i} \circ g_i^{-1} \in G$;
- $(\widetilde{f})_{(k+1)} = \sum_{i=1}^{r} m_i \, g_i \circ f_{(k+1)} \circ g_i^{-1} = 0$, so that $J_{k+1}\widetilde{f} = \mathrm{id}$;
- $(\widetilde{h})_{(n)} = \sum_{i=1}^{r} m_i \, g_i \circ h_{(n)} \circ g_i^{-1} \neq 0$, so that $\widetilde{h} \neq \mathrm{id}$. This is a contradiction. □

Corollary 4.1. If $G \in \mathcal{S}$, then $\mathcal{I}_0(G) = \{k \in \mathbb{N}_{<n},\ \exists\, f \in G,\ f_{(k+1)} \neq 0\}$.

Remark. We could give a more simple proof of the theorem using cor. 4.1. Unfortunately, we were not able to prove it without using the theorem.

Theorem 4.2. Any group G such that $SL \leq G \leq J_n(A)$ is equal to some $\mathcal{G}(I) \rtimes K$, where $SL \leq K \leq GL$.

Proof. It is sufficient to show that $H_0(G) \leq G$. If $l \in H_0(G)$, let us show by contradiction that n is the biggest integer k for which there exists some $f \in G$ satisfying $J_k f = l$.

If we had $k < n$, then $f = l + f_{(k+1)} + \ldots$ where $f_{(k+1)} \neq 0$.

We begin to show that $H_k(G \cap J_n(A)_{id}) \neq \{0\}$.

Since $f = l \circ (\mathrm{id} + l^{-1} \circ f_{(k+1)} + \ldots)$, $a := l^{-1} \circ f_{(k+1)} \in E^0_{k+1}$ and since a is a nonzero element of the irreducible non trivial SL-representation E^0_{k+1}, there exists $u \in SL$ such that $a \neq u \circ a \circ u^{-1}$. If $g := f \circ u \circ f^{-1}$, then $g_{(k+1)} = l \circ (a - u \circ a \circ u^{-1}) \circ u \circ l^{-1} \neq 0$, while $g \in G$ and $\mathrm{Jac}\, g = 1$. Therefore $h := g_{(1)}^{-1} \circ g \in G \cap J_n(A)_{id}$ and $h_{(k+1)} \neq 0$, so that $H_k(G \cap J_n(A)_{id}) \neq \{0\}$.

Since $H_k(G \cap J_n(A)_{id}) = E^0_{k+1}$, there exists $\widetilde{h} \in G \cap J_n(A)_{id}$ such that $J_{k+1}\, \widetilde{h} = \mathrm{id} - a$. Therefore $\widetilde{f} := f \circ \widetilde{h} \in G$ and $J_{k+1}\, \widetilde{f} = l$, a contradiction. □

Remark. We recall that any group K such that $SL \leq K \leq GL$ is equal to some $\det^{-1} \widetilde{K}$ where $\det : GL \to \mathbb{C}^*$ and $\widetilde{K} \leq \mathbb{C}^*$.

Corollary 4.2. Any group G such that $GL \leq G \leq J_n(A)$ is equal to some $\mathcal{G}(I) \rtimes GL$, i.e. $J_n(A_{1+<I>})$.

Let $J_n(A)_1 := \{f \in J_n(A),\ \mathrm{Jac}\, f = 1\}$.

Corollary 4.3. Any group G such that $SL \leq G \leq J_n(A)_1$ is equal to some $\mathcal{G}(I) \rtimes SL$, i.e. $J_n(A_{1+<I>}) \cap J_n(A)_1$.

Corollary 4.4. If $n \geq 2$ and $j \in J_n(A)_1$, the following assertions are equivalent:

(i) $< SL, j >= J_n(A)_1$;
(ii) $j_{(2)} \neq 0$.

Proof. If we set $G := < SL, j >$, then $G = J_n(A)_1$ if and only if $\mathcal{I}_0(G) = \mathbb{N}_{<n}$, which is still equivalent to $1 \in \mathcal{I}_0(G)$, i.e. $H_1(G) \neq \{0\}$, i.e. $j_{(2)} \neq 0$. □

Proof of th. E. Assume that $J_n(A) \leq G \leq J_n(\widetilde{A})$. Since $1 \in \mathcal{I}_0(G)$, lemma 3.7 implies that $\mathcal{I}_1(G)$ is equal to some $\{0\} \cup \{k, k+1, \dots, n-1\}$ where $1 \leq k \leq n$.

Applying lemma 3.6 with $G_1 :=< J_n(A), g^{[k]} >$, where $g^{[k]} := (1+x_1^k)\mathrm{id} \in J_n(\widetilde{A})$, and $G_2 := J_{n,k}^{-1}(J_k(A))$, one could show as in lemma 4.1 that $J_{n,k}^{-1}(J_k(A)) =< J_n(A), g^{[k]} >$. Remark: if $k = n$, we agree that $g^{[n]} = \mathrm{id}$. Using these preliminaries, let us show that $G =< J_n(A), g^{[k]} >$. As above, the proof is by induction on n. The case $n = 1$ being obvious, we may assume that $n \geq 2$.

First case. $k < n$, i.e. $n - 1 \in \mathcal{I}_1(G)$.

By induction hypothesis, the groups G and $H :=< J_n(A), g^{[k]} >$ coincide at the level of $n-1$ jets, i.e. $J_{n-1}G = J_{n-1}H$. However, since G and H both contain the group $\{\mathrm{id} + u,\ u \in E_n\} \leq J_n(\widetilde{A})$, it is clear that the last equality can be lifted up at the level of n-jets to show that $G = H$.

Second case. $k = n$, i.e. $\mathcal{I}_1(G) = \{0\}$.

Since $J_n(A) \leq G$ and $\mathcal{I}_1(J_n(A)) = \mathcal{I}_l(G) = \{0\}$, we get $J_n(A) = G$ by lemma 3.6. □

Corollary 4.5. If $n \geq 2$ and $j \in J_n(E)^*$, the following assertions are equivalent:
(i) $< J_n(A), j >= J_n(E)^*$;
(ii) $J_2 j \notin J_2(A)$.

Proof. If we set $G :=< J_n(A), j >$, then $G = J_n(E)^*$ if and only if $\mathcal{I}_1(G) = \mathbb{N}_{<n}$, which is still equivalent to $1 \in \mathcal{I}_1(G)$, i.e. $H_1^1(G) \neq \{0\}$, i.e. $J_2 j \notin J_2(A)$. □

Proof of th. A. Let $f \in G$ be a polynomial automorphism which is not affine. Let us show that there exists $a \in \mathbb{A}^N$ such that if we set $g := f \circ \tau_a$ (where $\tau_a = \mathrm{id} + a$ is the translation of vector a), then the quadratic part $g_{(2)}$ of g is nonzero. Since there exists a component p of f such that $\deg p \geq 2$, it is sufficient to show that there exists $a \in \mathbb{C}^N$ such that $q(x) := p(a + x)$ satisfies $q_{(2)} \neq 0$. But it is clear that there exist integers L, M and $a \in \mathbb{C}^N$ such that $\dfrac{\partial^2 p}{\partial x_L \partial x_M}(a) \neq 0$. By Taylor formula, we get $q(x) = p(a+x) = \sum_{\alpha \in \mathbb{N}^N} \dfrac{\partial^\alpha p}{\partial x^\alpha}(a) \dfrac{x^\alpha}{\alpha!}$, so that $q_{(2)} \neq 0$. By replacing g (where $g \in G$ satisfies $g_{(2)} \neq 0$) by $h \circ g$ (where h is a well chosen affine map), we may assume moreover that $g(0) = 0$ and that $\mathrm{Jac}\, g = 1$. By cor. 4.4, we have $< SL, J_n(g) >= J_n(A)_1$. □

V. PROOF OF THEOREM C.

1. The Algebraic case.

We have seen in prop. 3.2 above that for any $j \in J_n(E)$ whose Jacobian is a nonzero constant there exists a tame automorphism f such that $j = J_n f$. The following generalization is the algebraic case of th. C:

Theorem 5.1 (interpolation of n-jets by an algebraic tame automorphism). Let $n \geq 1$, let $u^{[1]}, \dots, u^{[m]}$ be distinct points of $\mathbb{A}^N$ and let $j^{[1]}, \dots, j^{[m]} \in \mathfrak{J}_n(E)$ be n-jets whose Jacobians are nonzero constants. The two following assertions are equivalent:

(i) $\exists\, f \in T,\ \mathfrak{J}_{n,u^{[k]}}\, f = j^{[k]},\ 1 \leq k \leq m;$ (ii) $\begin{cases} 1.\ \text{the points } j^{[k]}(0)_{1 \leq k \leq m} \text{ are distinct;} \\ 2.\ \exists\, \lambda \in \mathbb{C}^*,\ \text{Jac}\, j^{[k]} = \lambda,\ 1 \leq k \leq m. \end{cases}$

Proof. (i) $\Longrightarrow$ (ii). We have $f(u^{[k]}) = j^{[k]}(0)$, so that (1) comes from the injectivity of f. Since f is a polynomial automorphism, $\text{Jac}\, f \equiv \lambda \in \mathbb{C}^*$, i.e. $\forall\, a \in \mathbb{A}^N,\ \det\ f'(a) = \lambda$ and we get $\text{Jac}\, j^{[k]} = \det\ (j^{[k]})'(0) = \det\ f'(u^{[k]}) = \lambda$.

(ii) $\Longrightarrow$ (i). It is enough to prove that given: $u^{[1]}, \dots, u^{[m]}$ distinct points of $\mathbb{A}^N$; $v^{[1]}, \dots, v^{[m]}$ distinct points of $\mathbb{A}^N$; $\lambda \in \mathbb{C}^*$; $j^{[1]}, \dots, j^{[m]}$ centered n-jets of $J_n(E)$ such that $\text{Jac}\, j^{[k]} = \lambda$ (for $1 \leq k \leq m$) there exists $f \in T$ such that $f(u^{[k]}) = v^{[k]}$ and $J_{n,u^{[k]}}\, f = j^{[k]}$ (for $1 \leq k \leq m$).

Let G be the group of tame automorphisms f such that $f(u^{[k]}) = u^{[k]}$ (for $1 \leq k \leq m$) and such that $\text{Jac}\ f = 1$ and let $J := J_n(A)_1$. Using lemma 5.1 below, it is sufficient to show that the group-morphism $\varphi\ :\ G \to J^m,\ f \mapsto \Big(J_{n,u^{[k]}}\, f\Big)_{1 \leq k \leq m}$ is onto. This is a direct consequence of lemma 5.2 below. □

Lemma 5.1. If $u^{[1]}, \dots, u^{[m]}$ and $v^{[1]}, \dots, v^{[m]}$ are two families of m pairwise distinct points of $\mathbb{A}^N$ and if $\lambda \in \mathbb{C}^*$, then there exists a tame automorphism f with Jacobian equal to λ such that $f(u^{[k]}) = v^{[k]}$ for $1 \leq k \leq m$.

Proof. It is proven as a watermark in [21] that T acts m-transitively on $\mathbb{A}^N$. It is also a consequence of th. 2 of [30] asserting that if X_1, X_2 are smooth closed algebraic subsets of $\mathbb{A}^N$ of dimension d with $N \geq 2d + 2$, then any isomorphism from X_1 to X_2 can be extended into a tame automorphism of $\mathbb{A}^N$ (see also § 5.3 of [15] for an overview). Therefore, if we set $w^{[k]} := k\, e_N \in \mathbb{A}^N$ (for $1 \leq k \leq m$), there exist $g, h \in T$ such that $g(u^{[k]}) = w^{[k]}$ and $h(w^{[k]}) = v^{[k]}$ (for $1 \leq k \leq m$). If we set $\mu := \lambda/(\text{Jac}\, g\ \times\ \text{Jac}\, h) \in \mathbb{C}^*$ and $d_\mu := (\mu x_1, x_2, \dots, x_N) \in T$, then $f := h \circ d_\mu \circ g$ satisfies the required conditions. □

Lemma 5.2. If $u^{[0]}, \dots, u^{[m]}$ are $m + 1$ pairwise distinct points of $\mathbb{A}^N$, let G_0 be the group of tame automorphisms f satisfying $f(u^{[k]}) = u^{[k]}$ for $0 \leq k \leq m$, $J_{n,u^{[k]}}\, f = \text{id}$

for $1 \le k \le m$ and $\operatorname{Jac} f = 1$. As above, let $J := J_n(A)_1$. Then, the group-morphism $\psi : G_0 \to J$, $f \mapsto J_{n,u^{[0]}} f$ is onto.

Proof. Let us set $u = (\underbrace{1, \ldots, 1}_{N}) \in \mathbb{A}^N$. Since there exists a tame automorphism sending $u^{[k]}$ on $k\,u$ (for $0 \le k \le m$), we may assume that $u^{[k]} = k\,u$ (for $0 \le k \le m$). Using cor. 4.4, it is sufficient to show that: (i) $\mathrm{id} + x_2^2\, e_1 \in \operatorname{Im} \psi$ and (ii) $SL \subset \operatorname{Im} \psi$.

Proof of (i). Let $p(\xi) \in \mathbb{C}[\xi]$ be such that $p(\xi) \equiv \xi^2 \bmod \xi^{n+1}$ and $p(k+\xi) \equiv 0 \bmod \xi^{n+1}$, $1 \le k \le m$. Then $f := \mathrm{id} + p(x_2)e_1 \in G_0$ and $\psi(f) = J_n\, f = \mathrm{id} + x_2^2 e_1$.

Proof of (ii). We know that SL is generated by the elementary transvections $t_{\alpha,L,M} := \mathrm{id} + \alpha\, x_M\, e_L$ (where $\alpha \in \mathbb{C}$ and $L \ne M \in \{1, \ldots, N\}$). It is enough to show that $t_{\alpha,L,M} \in \operatorname{Im} \psi$. Let $p(\xi) \in \mathbb{C}[\xi]$ be such that $p(\xi) \equiv \alpha\xi \bmod \xi^{n+1}$ and $p(k+\xi) \equiv 0 \bmod \xi^{n+1}$, $1 \le k \le m$. Then $f := \mathrm{id} + p(x_M)e_L \in G_0$ and $\psi(f) = J_n\, f = t_{\alpha,L,M}$. ☐

2. The Analytic case.

We have seen in prop. 3.3 above that for any $j \in J_n(E)^*$, there exists a tame analytic automorphism f such that $j = J_n\, f$. The following generalization is the analytic case of th. C:

Theorem 5.2 (interpolation of n-jets by an analytic tame automorphism). Let $n \ge 1$, let $u^{[1]}, \ldots, u^{[m]}$ be distinct points of $\mathbb{A}^N$, let $v^{[1]}, \ldots, v^{[m]}$ be points of $\mathbb{A}^N$ and let $j^{[1]}, \ldots, j^{[m]} \in J_n(E)^*$ be invertible centered n-jets. The two following assertions are equivalent:

(i) $\exists\, f \in \widetilde{T}$, $\mathfrak{J}_{n,u^{[k]}}\, f = v^{[k]} + j^{[k]}$, $1 \le k \le m$; (ii) the points $\left(v^{[k]}\right)_{1 \le k \le m}$ are distinct.

Proof. We follow the same path as in the algebraic case. The implication (i) $\Longrightarrow$ (ii) is obvious and (ii) $\Longrightarrow$ (i) is a consequence of the following lemma. ☐

Lemma 5.3. If $u^{[0]}, \ldots, u^{[m]}$ are $m+1$ distinct points of $\mathbb{A}^N$, let $\widetilde{G}$ be the group of tame analytic automorphisms f such that $f(u^{[k]}) = u^{[k]}$, $0 \le k \le m$ and $J_{n,u^{[k]}}\, f = \mathrm{id}$, $1 \le k \le m$. Let $\widetilde{J} := J_n(E)^*$ be the group of invertible centered n-jets. Then, the group-morphism $\widetilde{\psi} : \widetilde{G} \to \widetilde{J}$, $f \mapsto J_{n,u^{[0]}}\, f$ is onto.

Proof. We may assume that $u^{[k]} = k\,u$ $(0 \le k \le m)$ where $u = (\underbrace{1, \ldots, 1}_{N}) \in \mathbb{A}^N$. Using cor. 4.5, it is enough to show that: (i) $J_n(A) \subset \operatorname{Im} \widetilde{\psi}$ and (ii) $\mathrm{id} + x_1 x_2\, e_1 \in \operatorname{Im} \widetilde{\psi}$.

Proof of (i). We already know that $J_n(A)_1 \subset \psi(G) \subset \widetilde{\psi}(\widetilde{G})$. Therefore, it is sufficient to show that for any $\lambda \in \mathbb{C}^*$, $d_\lambda := (\lambda x_1, x_2, \ldots, x_N) \in \operatorname{Im} \widetilde{\psi}$. Let us choose $\mu \in \mathbb{C}$ such that $e^\mu = \lambda$ and let us choose $p(\xi) \in \mathbb{C}[\xi]$ such that $p(\xi) \equiv \mu \bmod \xi^{n+1}$ and $p(k+\xi) \equiv$

$0 \bmod \xi^{n+1}$, $1 \le k \le m$. Then $f := (e^{p(x_2)}x_1, x_2, \ldots, x_N) \in \widetilde{G}$ and $\widetilde{\psi}(f) = J_n\, f = d_\lambda$.

Proof of (ii). Let $p(\xi) \in \mathbb{C}[\xi]$ be such that $p(\xi) \equiv \ln(1+\xi) \bmod \xi^{n+1}$ or equivalently $p(\xi) \equiv \sum_{1 \le k \le n} (-1)^{k+1} \dfrac{\xi^k}{k} \bmod \xi^{n+1}$ and $p(k+\xi) \equiv 0 \bmod \xi^{n+1}$, $1 \le k \le m$. Then $f := (e^{p(x_2)}x_1, x_2, \ldots, x_N) \in \widetilde{G}$ and $\widetilde{\psi}(f) = J_n\, f = \mathrm{id} + x_1 x_2\, e_1$. □

VI. CONSEQUENCES ON VARIABLES.

We recall that $f_1 \in R$ is called a variable, if there exist $f_2, \ldots, f_N \in R$ such that $(f_1, \ldots, f_N)$ is an algebraic automorphism.

Theorem 6.1. If $n \ge 1$ and $j_L \in \mathfrak{J}_n(R)$ for $1 \le L \le N-1$, the following assertions are equivalent:

(i) the linear parts $\mathcal{L}(j_L)$ of the j_L, $1 \le L \le N-1$ are linearly independant;

(ii) there exists $j_N \in \mathfrak{J}_n(R)$ such that $(j_1, \ldots, j_N) \in \mathfrak{J}_n(A)$;

(iii) there exists $j_N \in \mathfrak{J}_n(R)$ such that $\mathrm{Jac}(j_1, \ldots, j_N) = 1 \in \mathfrak{J}_{n-1}(R)$.

Furthermore, if these assertions are satisfied and if we choose any linear form $l \in V^*$ such that $\mathcal{L}(j_1), \ldots, \mathcal{L}(j_{N-1}), l$ is a basis of V^*, then there exists a unique $p \in \mathfrak{J}_{n-1}(R)$ such that $j_N := lp \in \mathfrak{J}_n(R)$ satisfies $\mathrm{Jac}(j_1, \ldots, j_N) = 1 \in \mathfrak{J}_{n-1}(R)$.

Proof. (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) is obvious. Let us now choose l such that $\mathcal{L}(j_1), \ldots, \mathcal{L}(j_{N-1}), l$ is a basis of V^*. Let us show that there exists a unique $p \in \mathfrak{J}_{n-1}(R)$ such that $j_N := lp$ satisfies $\mathrm{Jac}(j_1, \ldots, j_N) = 1 \in \mathfrak{J}_{n-1}(R)$. If $\varphi : \mathfrak{J}_{n-1}(R) \to \mathfrak{J}_{n-1}(R)$ is the finite dimensional linear endomorphism defined by $\varphi(p) = \mathrm{Jac}(j_1, \ldots, j_{N-1}, lp)$, it is sufficient to show that φ is an automorphism, which is equivalent to saying that $\mathrm{Ker}\ \varphi = \{0\}$. If $p \ne 0 \in \mathrm{Ker}\ \varphi$, let $h \ne 0$ be the homogeneous part of smallest degree of p. Let $l_1, \ldots, l_{N-1}$ be the linear parts of $j_1, \ldots, j_{N-1}$. The equality $\mathrm{Jac}(j_1, \ldots, j_{N-1}, lp) = 0$ implies $\mathrm{Jac}(l_1, \ldots, l_{N-1}, lh) = 0$ which is absurd by the following lemma. □

Lemma 6.1. If $l_1, \ldots, l_N$ is a basis of V^*, then the map $\psi :\ h \mapsto \mathrm{Jac}(l_1, \ldots, l_{N-1}, l_N h)$ is a linear automorphism of R.

Proof. Injectivity. We know that $h \in \mathrm{Ker}\ \psi \Longleftrightarrow$ the family $l_1, \ldots, l_{N-1}, l_N h$ is algebraically dependant over $\mathbb{C}$ (see [26], [18] or [15]). Therefore, we may assume that $l_L = x_L$ for all L, so that $\psi(h) = 0 \Longleftrightarrow \dfrac{\partial(x_N h)}{\partial x_N} = 0 \Longleftrightarrow x_N h \in \mathbb{C}[x_1, \ldots, x_{N-1}] \Longleftrightarrow h = 0$.

Surjectivity. For any $n \ge 0$, ψ induces a linear endomorphism of the finite dimensional subspace $R_{\le n}$ which is injective hence surjective. □

The next result on variables, already proven in [13], is an easy consequence of th. 6.1.

Theorem 6.2. If $n \geq 1$, then $j \in \mathfrak{J}_n(R)$ is the n-jet of a variable if and only if $j_{(1)} \neq 0$.

References

[1] E. Andersén, Volume-preserving automorphisms of $\mathbb{C}^n$, Complex Variables Theory Appl., 14, n° 1-4 (1990), 223-235.

[2] E. Andersén, L. Lempert, On the group of holomorphic automorphisms of $\mathbb{C}^n$, Invent. Math., 110, n° 2 (1992), 371-388.

[3] S. Andreadakis, On the automorphisms of free groups and free nilpotent groups, Proc. London Math. Soc. (3), 15 (1965), 239-268.

[4] D. J. Anick, Limits of Tame Automorphisms of $k[x_1, \ldots, x_N]$, J. of Algebra, 82 (1983), 459-468.

[5] J. Ax, The elementary theory of finite fields, Ann. of Math. (2) 88 (1968), 239-271.

[6] J. Ax, Injective endomorphisms of varieties and schemes, Pacific J. Math 31 (1969), 1-7.

[7] S. Bachmuth, Automorphisms of a class of metabelian groups, Trans. Amer. Math. Soc. 127 (1967), 284-293.

[8] H. Bass, E. Connell and D. Wright, The Jacobian Conjecture: Reduction of Degree and Formal Expansion of the Inverse, Bull. of the A.M.S., 7 (1982), 287-330.

[9] A. Borel, Injective endomorphisms of algebraic varieties, Arch. Math. (Basel) 20 (1969), 531-537.

[10] R.M. Bryant, V. Drensky, Dense Subgroups of the automorphism groups of free algebras, Canad. J. Math. 45 (1993), no. 6, 1135-1154.

[11] S. Cynk, R. Rusek, Injective endomorphisms of algebraic and analytic sets, Ann. Pol. Math. 56, n°1 (1991), 29-35.

[12] V. Drensky, Endomorphisms and automorphisms of relatively free algebras, in "Recent developments in the theory of algebras with polynomial identities (Palermo, 1992)", Rend. Circ. Mat. Palermo (2) Suppl. No. 31 (1993), 97-132.

[13] V. Drensky, V. Shpilrain, J-T Yu, On the density of the set of generators of a polynomial algebra, Proc. of the Amer. Math. Soc., 128, n° 12 (2000), 3465-3469.

[14] D. Eisenbud, Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics, 150, Springer -Verlag, New York, 1995.

[15] A. van den Essen, Polynomial Automorphisms and the Jacobian Conjecture, Progress in Math. (Boston Mass.) 190, Birkhäuser, Basel, 2000.

[16] W. Fulton and J. Harris, Representation Theory, a first course, Graduate texts in Mathematics 129, Springer Verlag, 1991.

[17] J.-P. Furter, Fat Points Embeddings, submitted, available at http://www.univ-lr.fr/labo/lmca/publications/06-09/06-09.pdf.

[18] P. Gordan, Über biquadratische Gleichungen, Math. Annalen 29 (1987), 318-326.

[19] A. Grothendieck, EGA 4, Inst. Hautes Études Sci. Publ. Math. n°32, 1967.

[20] J. E. Humphreys, Linear Algebraic Groups (2nd ed.), Graduate texts in Mathematics 21, Springer Verlag, 1981.

[21] Z. Jelonek, The Extension of Regular and Rational Embeddings, Math. Ann. 277 (1987), 113-120.

[22] H.W.E. Jung, Über ganze birationale Transformationen der Ebene, J. Reine Angew. Math., 184 (1942), 161-174.

[23] O. H. Keller, Ganze Cremona Transformationen, Monats. Math. Physk 47 (1939), 299-306.

[24] W. van der Kulk, On polynomial rings in two variables, Nieuw Archief voor Wiskunde, 3, n° 1 (1953), 33-41.

[25] M. Nagata, On Automorphism Group of k[x,y], Lecture Notes in Mathematics, Vol. 5, Kyoto University, Kinokuniya Bookstore, Tokyo, 1972.

[26] O. Perron, Algebra I, Die Grundlagen, Walter de Gruyter & Co., Berlin, 1951.

[27] W. Rudin, Injective polynomial maps are automorphisms, Amer. Math. Monthly 102, n°6 (1995), 540-543.

[28] I.P. Shestakov, U.U. Umirbaev, Poisson brackets and two-generated subalgebras of rings of polynomials, J. Amer. Math. Soc 17 (2004), 181-196.

[29] I.P. Shestakov, U.U. Umirbaev, The tame and the wild automorphisms of polynomial rings in three variables, J. Amer. Math. Soc 17 (2004), 197-227.

[30] V. Srinivas, On the embedding dimension of an affine variety, Math. Ann. 289 (1991), 125-132.