On the degree of iterates of automorphisms of the affine plane

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Abstract : For a polynomial automorphism f of $\mathbb{A}^2_{\mathbb{C}}$, we set $\tau = (\deg f^2)/(\deg f)$. We prove that $\tau \leq 1$ if and only if f is triangularizable. In this situation, we show (by using a deep result from number theory known as the theorem of Skolem-Mahler-Lech) that the sequence $(\deg f^n)_{n \in \mathbb{N}}$ is periodic for large n. In the opposite case, we prove that τ is an integer ($\tau \geq 2$) and that the sequence $(\deg f^n)_{n \in \mathbb{N}}$ is a geometric progression of ratio τ. In particular, if f is any automorphism, we obtain the rationality of the formal series $\sum_{n=0}^{\infty} (\deg f^n) T^n$.

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Introduction. Let f be an automorphism of $\mathbb{A}^2_{\mathbb{C}}$. By a result of Friedland and Milnor, either f is triangularizable (i.e. conjugate to a triangular automorphism), or f is conjugate to an automorphism g such that $\deg g \geq 2$ and $\deg g^n = (\deg g)^n$ for each nonnegative integer n (see [F-M]). Their proof uses the description of the group of automorphisms of $\mathbb{A}^2_{\mathbb{C}}$ as an amalgamated product (see [vdK] and I.2) and some general arguments about amalgamated products (see [Ser]).

We can easily deduce from this result that there exists a (unique) nonzero positive integer c such that the sequence $(\log(\deg f^n) - n \log c)_{n \in \mathbb{N}}$ is bounded. Furthermore, f is triangularizable if and only if $c = 1$.

The purpose of this paper is to study the exact values of the sequence $(\deg f^n)_{n \in \mathbb{N}}$ and in particular to establish the rationality of the formal series $\sum_{n=0}^{\infty}(\deg f^n)T^n$.

In part I, we introduce the notion of an amalgamated product with degree mapping ; an example is the group of automorphisms of $\mathbb{A}^2_{\mathbb{C}}$ with the usual degree. Then we obtain the following result (which is an easy consequence of Propositions 4 and 5). Let g be an element of an amalgamated product with degree mapping G and let us set $\tau = (\deg g^2)/(\deg g)$. Then, either $\tau \leq 1$ and the sequence $(\deg g^n)_{n\in\mathbb{N}}$ is bounded, or τ is an integer greater than or equal to 2 and the sequence $(\deg g^n)_{n\in\mathbb{N}}$ is a geometric progression of ratio τ. In particular, we obtain a very simple criterion for an automorphism f of $\mathbb{A}^2_{\mathbb{C}}$ to be triangularizable, i.e. f will be triangularizable if and only if $\deg f^2 \leq \deg f$.

Part II is devoted to the iteration of f when f is triangularizable. In II.1, we briefly present the theory of linear recurrence sequences. We apply it in II.2 to show that the coefficients of the iterate of f are linear recurrence sequences. In II.3, using the fact that if $u(n)_{n\in\mathbb{N}}$ is a linear recurrence sequence, then those n for which $u(n) = 0$ form a finite union of arithmetic progressions after a certain stage (theorem of Skolem-Mahler-Lech), we show that the sequence $(\deg f^n)_{n\in\mathbb{N}}$ is periodic for large n.

I. Amalgamated product with degree mapping.

1. Definition.

Let G be a group and let G_1, G_2 be subgroups of G. We suppose that G is the amalgamated product of G_1 and G_2 over $G_3 := G_1 \cap G_2$ (which we denote by $G = G_1 *_{G_3} G_2$). This means that

i. if g is any element of G, then there exist a nonnegative integer l and two sequences $(\alpha_i)_{1\leq i\leq l+1}$ in G_2 and $(\gamma_i)_{1\leq i\leq l}$ in G_1 such that

$$\begin{cases} g = \alpha_1\gamma_1\alpha_2\gamma_2\cdots\alpha_l\gamma_l\alpha_{l+1} \\ \forall\ i \in \{1,\ldots,l\},\ \gamma_i \notin G_2 \\ \forall\ i \in \{2,\ldots,l\},\ \alpha_i \notin G_1 \end{cases}$$

and that

ii. the above expression is unique in the following sense : if m is a nonnegative integer and $(\alpha'_i)_{1\leq i\leq m+1}$ (resp. $(\gamma'_i)_{1\leq i\leq m}$) is a sequence of G_2 (resp. G_1) such that

$$\begin{cases} g = \alpha'_1\gamma'_1\alpha'_2\gamma'_2\cdots\alpha'_m\gamma'_m\alpha'_{m+1} \\ \forall\ i \in \{1,\dots,m\},\ \gamma'_i \notin G_2 \\ \forall\ i \in \{2,\dots,m\},\ \alpha'_i \notin G_1 \end{cases}$$

then $l = m$ and there exist $(\beta_i)_{1\leq i\leq l}$, $(\delta_i)_{1\leq i\leq l}$ in G_3 such that

$$\begin{cases} \alpha'_1 = \alpha_1\beta_1^{-1} \\ \forall\ i \in \{2,\dots,l\},\ \alpha'_i = \delta_{i-1}\alpha_i\beta_i^{-1} \\ \alpha'_{l+1} = \delta_l\alpha_{l+1} \end{cases}$$

and $\forall\ i \in \{1,\dots,l\},\ \gamma'_i = \beta_i\gamma_i\delta_i^{-1}$.

The expression $g = \alpha_1\gamma_1\alpha_2\gamma_2\cdots\alpha_l\gamma_l\alpha_{l+1}$ will be called a reduced expression of g. Observe that l is independant of the reduced expression. We define the **length** $l(g)$ of g by $l(g) = l$. This differs from the usual convention (see [Ser]) where the length of g is $2l+1$ (resp. $2l$, resp. $2l-1$) if α_1, α_{l+1} do not belong to G_3 (resp. if exactly one among α_1, α_{l+1} belongs to G_3, resp. if both α_1, α_{l+1} belong to G_3).

Definition. We will say that $(G, G_1, G_2, \deg)$ is an amalgamated product with degree mapping if we are given a mapping $\deg :\ G_1 \to \mathbb{N}_{>0}$ satisfying both following properties :

i. let g be an element of G_1, then g belongs to G_3 if and only if $\deg g\ = 1$;

ii. if g, h belong to G_1, then $\deg(gh)\ \leq\ \max\{\deg g, \deg h\}$.

Using the degree mapping $\deg :\ G_1 \to \mathbb{N}_{>0}$, we can define the **degree** of any element g in G. Suppose that $g = \alpha_1\gamma_1\alpha_2\gamma_2\cdots\alpha_l\gamma_l\alpha_{l+1}$ is a reduced expression of g. We set $\deg g\ =\ \prod_{i=1}^{l}\deg\gamma_i$. It is easy to check that $\deg g$ does not depend on the reduced expression of g we used.

2. Fundamental example.

Let k be a field. An automorphism f of the k-variety $\mathbb{A}^2_k\ =$ Spec $k[X,Y]$ is identified with its sequence $f = (f_1, f_2)$ of coordinate functions $f_i \in k[X,Y]$ $(i = 1,2)$. We set $\deg f\ = \max\{\deg f_1,\ \deg f_2\}$ and we define :

GA, the group of automorphisms of $\mathbb{A}_k^2$;

BA, the subgroup of triangular or "de Jonquières" automorphisms, i.e. automorphisms of the shape $(aX + P(Y), bY + c)$ where a, b are in k^*, c is in k and $P(Y)$ is any element in $k[Y]$;

Af, the subgroup of affine automorphisms, i.e. automorphisms of the shape $(aX+bY+c, dX+eY+f)$ where a, b, c, d, e, f are elements of k such that $ae - bd$ is in k^*.

By W. van der Kulk ([vdK]), we know that GA = BA $*_{\mathrm{B}}$ Af where B = BA $\cap$ Af. The two following assertions are easily checked :

i. let f be an element of BA, then f belongs to B if and only if $\deg f = 1$;

ii. if f, g belong to BA, then $\deg(fg) \leq \max\{\deg f, \deg g\}$,

so that (GA, BA, Af, deg) is an amalgamated product with degree mapping.

We could also check that the extension of the mapping deg : BA $\to \mathbb{N}_{>0}$ to the mapping deg : GA $\to \mathbb{N}_{>0}$ (as explained in I.1.) coincides with the usual degree mapping on GA (as defined in this section) (see Proposition 1.9 of [Wri] or Theorem 2.1 of [F-M]).

3. The multidegree of an element of an amalgamated product with degree mapping.

From now on, $(G, G_1, G_2, \deg)$ will denote an amalgamated product with degree mapping.

If g is an element of G, we define the **multidegree** $d(g)$ of g by $d(g) = (\deg \gamma_1, \deg \gamma_2, \ldots, \deg \gamma_l)$ where $g = \alpha_1\gamma_1\alpha_2\gamma_2 \cdots \alpha_l\gamma_l\alpha_{l+1}$ is a reduced expression of g. It is easy to check that $d(g)$ does not depend on the reduced expression of g that we use. The multidegree belongs to D, the set of all finite sequences of integers greater than or equal to 2. If d,e belong to D, we denote by de the concatenation of d and e. This clearly endows D with the structure of a monoid (where the unit is the empty sequence).

As an illustration of the previous definitions we will prove the :

Proposition 1. If g, h belong to G, then $\deg(gh) \leq (\deg g)(\deg h)$ and we have equality if and only if $d(gh) = d(g)d(h)$.

Proof. Let $g = \alpha_1\gamma_1\alpha_2\gamma_2\cdots\alpha_l\gamma_l\alpha_{l+1}$ be a reduced expression of g. We begin by proving that $\deg(gh) \le (\deg g)(\deg h)$ when h belongs to $G_1 \cup G_2$.

If h belongs to G_2, it is clear that $\deg(gh) = \deg g$. If h belongs to G_1, either α_{l+1} belongs to G_3 in which case

$$\begin{aligned}
\deg(gh) &= \prod_{i=1}^{l-1}(\deg \gamma_i) \,.\, \deg(\gamma_l\alpha_{l+1}h) \\
&\le \prod_{i=1}^{l-1}(\deg \gamma_i) \,.\, \max\{\deg(\gamma_l, \alpha_{l+1}), \deg h\} \\
&\le \prod_{i=1}^{l}(\deg \gamma_i) \,.\, (\deg h) \\
&\le (\deg g)(\deg h),
\end{aligned}$$

or α_{l+1} does not belong to G_3 and we have $\deg(gh) = (\deg g)(\deg h)$.

If h is any element of G, let $h = \beta_1\delta_1\beta_2\delta_2\cdots\beta_m\delta_m\beta_{m+1}$ be a reduced expression of g. By applying repeatedly our preliminary result, we get

$$\begin{aligned}
\deg(gh) &\le (\deg g)(\deg \beta_1)(\deg \delta_1)\ldots(\deg \delta_m)(\deg \beta_{m+1}) \\
&\le (\deg g)(\deg h).
\end{aligned}$$

If $\alpha_{l+1}\beta_1$ does not belong to G_3, then we have $d(gh) = d(g)d(h)$ whence $\deg(gh) = (\deg g)(\deg h)$.

On the other hand, if $\alpha_{l+1}\beta_1$ belongs to G_3, then we have $gh = g_1g_2g_3$ where

$$\begin{cases}
g_1 = \alpha_1\gamma_1\ldots\alpha_{l-1}\gamma_{l-1}\alpha_l \\
g_2 = \gamma_l\alpha_{l+1}\beta_1\delta_1 \\
g_3 = \beta_2\delta_2\ldots\beta_m\delta_m\beta_{m+1}
\end{cases}$$

and the relations

$$\begin{cases}
\deg g_1 = (\deg \gamma_1)(\deg \gamma_2)\ldots(\deg \gamma_{l-1}) \\
\deg g_2 \le \max\{\deg \gamma_l,\ \deg \delta_1\} < (\deg \gamma_l)(\deg \delta_1) \\
\deg g_3 = (\deg \delta_2)(\deg \delta_3)\ldots(\deg \delta_m)
\end{cases}$$

imply that

$$\deg(gh) \le (\deg g_1)(\deg g_2)(\deg g_3) < (\deg g)(\deg h).$$

□

Remark. The same arguments show that $l(gh) \leq l(g) + l(h)$ and that the three following assertions are equivalent :

i. $d(gh) = d(g)d(h)$;

ii. $\deg(gh) = (\deg g)(\deg h)$;

iii. $l(gh) = l(g) + l(h)$.

4. Reduced sequences.

Proposition 2. Let $(g_1,\ g_2, \ldots , g_n)$ be a sequence of elements of G. The four following assertions are equivalent :

i. $d(g_1g_2 \ldots g_n) = d(g_1)d(g_2) \ldots d(g_n)$;

ii. $l(g_1g_2 \ldots g_n) = l(g_1) + l(g_2) + \ldots + l(g_n)$;

iii. $\deg(g_1g_2 \ldots g_n) = \deg(g_1).\deg(g_2). \ldots .\deg(g_n)$;

iv. for all i in $\{1, \ldots , n-1\}$, $d(g_ig_{i+1}) = d(g_i)d(g_{i+1})$.

Proof. The implications i. $\Longrightarrow$ ii. and i. $\Longrightarrow$ iii. are clear.

Let $d^1 = (d^1_1,\ d^1_2, \ldots , d^1_{l_1}), \ldots ,\ d^n = (d^n_1,\ d^n_2, \ldots , d^n_{l_n})$ be the multidegrees of $g_1,\ g_2, \ldots , g_n$.

Suppose that

$$\begin{cases} g_1 = \alpha^1_1\gamma^1_1\alpha^1_2\gamma^1_2 \cdots \alpha^1_{l_1}\gamma^1_{l_1}\alpha^1_{l_1+1} \\ \vdots \\ g_n = \alpha^n_1\gamma^n_1\alpha^n_2\gamma^n_2 \cdots \alpha^n_{l_n}\gamma^n_{l_n}\alpha^n_{l_n+1} \end{cases}$$

are reduced expressions of the g_i. The assertion i. is not satisfied if and only if there exists an integer i in $\{1, \ldots , n-1\}$ such that $\alpha^i_{l_i+1}\alpha^{i+1}_1$ belongs to G_3. This latter assertion is equivalent to saying that there exists an integer i in $\{1, \ldots , n-1\}$ such that $d(g_ig_{i+1}) \neq d(g_i)d(g_{i+1})$, i.e. such that iv. is not satisfied. We have proven that i. $\Longleftrightarrow$ iv.

Let us now suppose that i. is not satisfied (ie. there exists an integer i in $\{1, \ldots , n-1\}$ such that $\alpha^i_{l_i+1}\alpha^{i+1}_1$ belongs to G_3) and let us prove that ii. and iii. are not satisfied.

We have

$$g_ig_{i+1} = \alpha^i_1\gamma^i_1\alpha^i_2\gamma^i_2 \cdots \alpha^i_{l_i-1}\gamma^i_{l_i-1}\alpha^i_{l_i}\gamma\alpha^{i+1}_2\gamma^{i+1}_2\alpha^{i+1}_3 \cdots \alpha^{i+1}_{l_{i+1}}\gamma^{i+1}_{l_{i+1}}\alpha^{i+1}_{l_{i+1}+1}$$

where $\gamma = \gamma^i_{l_i}\alpha^i_{l_i+1}\alpha^{i+1}_1\gamma^{i+1}_1$ belongs to G_1.

This shows that $l(g_ig_{i+1}) < l(g_i) + l(g_{i+1})$.

We now obtain directly $l(g_1g_2 \ldots g_n) < l(g_1) + l(g_2) + \ldots + l(g_n)$.

Using Proposition 1, we have $\deg(g_i g_{i+1}) < (\deg g_i)(\deg g_{i+1})$ and we obtain $\deg(g_1 g_2 \dots g_n) < \deg(g_1).\deg(g_2).\dots.\deg(g_n)$. □

We introduce the notion of reduced sequences of G :

Definition. The sequence $(g_1,\ g_2, \dots , g_n)$ of G is called reduced if it satisfies one of the equivalent assertions of Proposition 2.

Remark. By Proposition 2, we know that the sequence $(g_1, \dots , g_n)$ is reduced if and only if the sequences $(g_1,\ g_2), (g_2,\ g_3), \dots , (g_{n-1},\ g_n)$ are reduced.

Proposition 3. Let g be an element of G. The seven following assertions are equivalent :
i. the sequence (g, g) is reduced ;
ii. $d(g^2) = d(g)^2$;
iii. $l(g^2) = 2l(g)$;
iv. $\deg g^2 = (\deg g)^2$;
v. for all positive integers n, $d(g^n) = d(g)^n$;
vi. for all positive integers n, $l(g^n) = nl(g)$;
vii. for all positive integers n, $\deg g^n = (\deg g)^n$.

Proof. The equivalence of the first four assertions is clear. The equivalence of i. with the last three assertions comes from the above remark. Indeed, the sequence $(g,\ g)$ is reduced if and only if the sequence $(\underbrace{g,\ g, \dots , g}_{n})$ is reduced $(n \geq 2)$. □

5. The sequence $(\deg g^n)_{n \in \mathbb{N}}$ when g belongs to an amalgamated product with degree mapping.

Let g be an element of G. The two following propositions give us rather precise information on the sequence $u_n = \deg g^n$.

Proposition 4. Let $g = \alpha_1 \gamma_1 \alpha_2 \gamma_2 \cdots \alpha_l \gamma_l \alpha_{l+1}$ be a reduced expression of g. For $1 \leq i \leq l$, let b_i (resp. e_i) be the product of the first (resp. last) i terms of the sequence $(\alpha_1,\ \gamma_1,\ \alpha_2, \dots , \alpha_l,\ \gamma_l,\ \alpha_{l+1})$ and let us set $v_i = e_i b_i$. Then g is conjugate to an element of $G_1 \cup G_2$ if and only if v_l belongs to G_3.

On the opposite case, let us set $m = \inf \{i,\ v_i \notin G_3\}$ and $\rho(g) =$

$\deg(e_{2l-2m+3}b_{2m-2})$ which is an integer bigger than or equal to 2. Then we have for all nonnegative integers n,

$$\deg\ g^n = (\deg\ g).\rho(g)^{n-1}.$$

Proof. If $v_l = e_l b_l$ belongs to G_3, let β be the $(l+1)$-th term of the sequence $(\alpha_1,\ \gamma_1,\ \alpha_2, \dots, \alpha_l,\ \gamma_l,\ \alpha_{l+1})$. If l is even, then β belongs to G_1 and if l is odd, then β belongs to G_2. The expression $g = b_l \beta e_l = b_l(\beta v_l)b_l^{-1}$ shows us that g is conjugate to βv_l which belongs to $G_1 \cup G_2$.

If v_l does not belong to G_3, then we will distinguish two cases according to the parity of m.

If $m = 2p+1$, then v_{2p+1} belongs to G_2 but not to G_3. This ensures us that the following expression is reduced :

$$h := \alpha_m \gamma_m \alpha_{m+1} \dots \alpha_{l-p-2} \gamma_{l-p-2} v_{2p+1} \gamma_{p+1} \alpha_{p+2} \gamma_{p+2} \dots \alpha_{m-1} \gamma_{m-1}.$$

We have $h = e_{2l-2m+3}b_{2m-2}$ so that if n is any nonnegative integer, then we obtain $g^n = b_{2m-2}h^{n-1}e_{2l-2m+3}$.

It is straightforward that the sequences (h, h) and $(b_{2m-2},\ h,\ e_{2l-2m+3})$ are reduced. We now get

$$\deg\ g^n \ = \ (\deg\ b_{2m-2})(\deg\ h)^{n-1}(\deg\ e_{2l-2m+3}).$$

Of course, $\deg\ g = (\deg\ b_{2m-2})(\deg\ e_{2l-2m+3})$ so that we can conclude.

If $m = 2p$, then v_{2p} belongs to G_1 but not to G_3. By the same arguments as before, this ensures us that the following expression is reduced :

$$h := \alpha_m \gamma_m \alpha_{m+1} \gamma_{m+1} \dots \alpha_{l-p} \gamma_{l-p} \alpha_{l-p+1} v_{2p} \alpha_{p+1} \gamma_{p+1} \dots \alpha_{m-1} \gamma_{m-1}.$$

We have $h = e_{2l-2m+3}b_{2m-2}$ so that if n is any nonnegative integer, then we obtain $g^n = b_{2m-2}h^{n-1}e_{2l-2m+3}$. We can conclude as in the previous case. □

Proposition 5. The three following assertions are equivalent :

i. g is conjugate to an element of $G_1 \cup G_2$;

ii. there exists an integer $n \geq 2$ such that $\deg\ g^n \ \leq \deg\ g$;

iii. for each nonnegative integer n, we have deg $g^n \leq$ deg g.

Proof. The implication iii. $\Longrightarrow$ ii. is clear.

The implication ii. $\Longrightarrow$ i. is clear by Proposition 4.

Let us prove i. $\Longrightarrow$ iii. Suppose that g is conjugate to an element of
$G_1 \cup G_2$. Let h be an element of G of minimal length such that $\beta := h^{-1}gh$ belongs to $G_1 \cup G_2$. Let $h = \alpha_1\gamma_1\alpha_2\gamma_2\cdots\alpha_l\gamma_l\alpha_{l+1}$ be a reduced expression of h. We consider three disjoint cases :

1. If β belongs to G_2 and $\alpha_{l+1}\beta\alpha_{l+1}^{-1}$ belongs to G_3, then we claim that $l = 0$. Otherwise, if we set $\beta' = \gamma_l\alpha_{l+1}\beta\alpha_{l+1}^{-1}\gamma_l^{-1}$ and $h' = \alpha_1\gamma_1\alpha_2\gamma_2\cdots\alpha_l$, then β' would belong to G_1 and we would have $g = h'\beta'h'^{-1}$ with $l(h') < l$ which is absurd. It is now easy to check that :

$$\deg g = \deg g^n = 1.$$

2. If β belongs to G_2 and $\alpha_{l+1}\beta\alpha_{l+1}^{-1}$ does not belong to G_3, then let us set $\beta' = \alpha_{l+1}\beta\alpha_{l+1}^{-1}$. The expression

$$g = \alpha_1\gamma_1\alpha_2\gamma_2\cdots\alpha_l\gamma_l\beta'\gamma_l^{-1}\alpha_l^{-1}\cdots\alpha_2^{-1}\gamma_1^{-1}\alpha_1^{-1}$$

is reduced and we obtain deg g = (deg h)(deg h^{-1}) whereas the expression

$$g^n = \alpha_1\gamma_1\alpha_2\gamma_2\cdots\alpha_l\gamma_l\beta'^n\gamma_l^{-1}\alpha_l^{-1}\cdots\gamma_2^{-1}\alpha_2^{-1}\gamma_1^{-1}\alpha_1^{-1}$$

shows us that deg $g^n \leq$ (deg h)(deg h^{-1}).

3. If β belongs to G_1 but not to G_2, then we have deg g = (deg h)(deg h^{-1})(deg β) and the expression $g^n = h\beta^n h^{-1}$ shows us that

$$\deg g^n \leq (\deg h)(\deg h^{-1})(\deg \beta^n) \leq (\deg h)(\deg h^{-1})(\deg \beta).$$

□

II. The iteration of a triangularizable automorphism.

The langage of linear recurrence sequences will be a very useful tool to handle the iteration of a triangularizable automorphism. In section 1, we recall some classical facts on linear recurrence sequences. We can find all these results in the paper [C-M-P].

1. Linear recurrence sequences.

Definition. A linear recurrence sequence with constant coefficients (l.r.s. for short) is a complex sequence $u(n)_{n\in\mathbb{N}}$ such that for each positive integer n

$$u(n+k) = v_{k-1}.u(n+k-1)+v_{k-2}.u(n+k-2)+\ldots+v_1.u(n+1)+v_0.u(n)$$

where the v_i are complex numbers not dependent on n and k.

Theorem 1. A sequence $u(n)_{n\in\mathbb{N}}$ is a l.r.s. if and only if there exist polynomials $P_1, \ldots, P_r$ in $\mathbb{C}[T]$ and complex numbers $\omega_1, \ldots, \omega_r$ such that for each positive integer n

$$u(n) = P_1(n)\omega_1^n + \ldots + P_r(n)\omega_r^n.$$

We will use in section 2 the following two lemmas.

Lemma 1. If u and v are l.r.s., then $u+v$ and uv are also.

Lemma 2. If u and v are complex sequences such that for each positive integer n

$$v(n) = a_k.u(n+k) + a_{k-1}.u(n+k-1) + \ldots + a_1.u(n+1) + a_0.u(n)$$

where the a_i are complex numbers not all zero, then u is a l.r.s. if and only if v is a l.r.s.

2. Triangular automorphisms and l.r.s.

Let $t = (aX + \sum_{i=0}^{d} \mu_i Y^i,\ bY + c)$ be a triangular automorphism (a, b are nonzero complex numbers, c and the μ_i are complex numbers). By a straightforward induction, we see that for each positive integer n

$$t^n = (a^n X + \sum_{i=0}^{d} \lambda_i(n) Y^i,\ b^n Y + c(n))$$

where $c(n)$ and the $\lambda_i(n)$ are complex numbers.

Lemma 3. The sequences $c(n)_{n\in\mathbb{N}}$ and $\lambda_i(n)_{n\in\mathbb{N}}$ are l.r.s.

Proof. The equality $t^{n+1} = t(t^n)$ gives us

$$\begin{cases} \sum_{i=0}^{d} \lambda_i(n+1)Y^i = a\sum_{i=0}^{d} \lambda_i(n)Y^i + \sum_{i=0}^{d} \mu_i(b^nY + c(n))^i & (1) \\ c(n+1) = b.c(n) + c & (2) \end{cases}$$

By (2) and Lemma 2, the sequence $(n \mapsto c(n))$ is a l.r.s.

By expanding the terms $(b^nY + c(n))^i$ (for $0 \leq i \leq d$) and by using Lemma 1, the equation (1) shows us that the sequences $(n \mapsto \lambda_i(n+1) - a\lambda_i(n))$ are l.r.s. (for $0 \leq i \leq d$). We conclude by Lemma 2. □

Let g, h be endomorphisms of $\mathbb{A}^2_{\mathbb{C}}$. If we set $M = (\deg\, g)(\deg\, h)(\deg\, t)$, then for each nonnegative integer n we have $\deg(gt^nh) \leq M$. Thus, there exist sequences $A_{i,j}$, $B_{i,j}$ (for i, j nonegative integers with $i + j \leq M$) such that

$$gt^nh = (\sum_{i+j\leq M} A_{i,j}(n)X^iY^j,\ \sum_{i+j\leq M} B_{i,j}(n)X^iY^j)$$

Lemma 4. The sequences $A_{i,j}$ and $B_{i,j}$ are l.r.s.

Proof. Using Lemma 3, it is clear that the sequences $A_{i,j}$ and $B_{i,j}$ are obtained from l.r.s. by additions and multiplications. We conclude by Lemma 1. □

3. The theorem of Skolem-Mahler-Lech.

Theorem 2 (see [S-T] for a statement and [Lech] for a proof). If $(n \mapsto u(n))$ is a l.r.s., then the sequence $(n \mapsto \delta_0^{u(n)})$ (where δ_j^i is the Kronecker symbol, i.e. $\delta_j^i = 1$ if $i = j$ and 0 otherwise) is periodic for large n.

Using this result, we can deduce the

Proposition 6. The sequence $\deg(gt^n h)_{n\in\mathbb{N}}$ is periodic for large n.

Proof. For $i+j \leq M$, let us define sequences $u_{i,j}$ and $v_{i,j}$ by $u_{i,j}(n) = \delta_0^{A_{i,j}(n)}$ and $v_{i,j}(n) = \delta_0^{B_{i,j}(n)}$. By Theorem 2, each of these sequences is periodic for large n. Hence, there exists a positive integer T such that for large n we have $u_{i,j}(n+T) = u_{i,j}(n)$ and $v_{i,j}(n+T) = v_{i,j}(n)$ (for all i, j with $i + j \leq M$). The formula

$$\deg(gt^n h) = \max\{i + j \text{ such that } u_{i,j}(n) \neq 0 \text{ or } v_{i,j}(n) \neq 0\}$$

gives us the result. □

4. Conclusion.

If f is an automorphism of $\mathbb{A}^2_{\mathbb{C}}$, let us set $\tau = (\deg\ f^2)/(\deg\ f)$.

If $\tau > 1$, then we saw (in Propositions 4 and 5) that τ is an integer and that for any nonnegative integer n we have

$$\deg\ f^n = (\deg\ f).\tau^{n-1}.$$

If $\tau \leq 1$, then (by Proposition 5) f is conjugate to an element of Af or BA. However, each element of Af is conjugate to an element of BA so that we can suppose that $f = gtg^{-1}$ where g, t are automorphisms, t being triangular. By proposition 6, the sequence $(\deg\ f^n)_{n\in\mathbb{N}}$ is periodic for large n.

Finally, in both cases, we see that the series $\displaystyle\sum_{n=0}^{\infty}(\deg\ f^n)T^n$ is rational.

Question. Does this result still hold if we suppose only that f is an endomorphism of $\mathbb{A}^2_{\mathbb{C}}$?

Remark. If f is a triangularizable automorphism, let us define $u : \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ by $u(n) = \deg\ f^n$. We have the relations $u(n+p) \leq u(n)u(p)$ (by Proposition 1) and $u(np) \leq u(p)$ (by Proposition 5. iii. applied to f^n) for all positive integers n and p. One can look for a more direct proof of Proposition 6 and ask whether these conditions imply the periodicity of $u(n)$ for large n.

The answer is positive if u takes the value 1. Indeed, if $u(T) = 1$ for some T, then for each positive integer n we have $u(n+T) \leq u(n)$.

Therefore, for each p, the sequence $(n \to u(nT+p))$ will be constant for large n and we see that $u(n+T) = u(n)$ for large n.

In general, the answer is negative as shown by the sequence u defined by $u(n) = 3$ if $n = 1$ or n is a prime and $u(n) = 2$ otherwise.

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