On the Length of Polynomial Automorphisms of the Affine Plane

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Abstract : The automorphism group of the affine plane is a mysterious and challenging object. Although we know that it is an amalgamated product of two well known subgroups, many questions are still unsolved. Moreover, the group has the structure of an infinite-dimensional algebraic group. But the interactions between these two structures are not yet clear. In this paper, we study the length of an element (defined using the amalgamated structure) with respect to the algebraic structure. If the ground field is of characteristic zero, we prove that the length is a lower semicontinuous function on the group.

1. Introduction.

a. Notations.

If K is a field, $\mathcal{E}nd$ will denote the monoid of K-endomorphisms of $\mathbb{A}^2 := \text{Spec } K[X,Y]$ and $\mathcal{GA}$ the group of K-automorphisms of $\mathbb{A}^2$. An element f in $\mathcal{E}nd$ will be identified with its sequence $f = (f_1, f_2)$ of coordinate functions $f_j \in K[X,Y]$. We define the degree of f by $\deg f = \max\limits_{1 \le j \le 2} \deg f_j$.

Let

$\mathcal{A}f := \{(aX + bY + c, dX + eY + f),\ a,b,c,d,e,f \in K,\ ae - bd \neq 0\}$

be the subgroup of affine automorphisms and

$\mathcal{BA} := \{(aX + P(Y), bY + c),\ a,b,c \in K, P \in K[Y],\ ab \neq 0\}$

be the subgroup of triangular automorphisms ($\mathcal{BA}$ may be viewed as a Borel subgroup of $\mathcal{GA}$).

If $f \in \mathcal{GA}$, by [J] and [K], one can write $f = \alpha_1.\gamma_1.\ldots.\alpha_k.\gamma_k.\alpha_{k+1}$ where the α_j (resp. γ_j) belong to $\mathcal{A}f$ (resp. $\mathcal{BA}$). By contracting such an expression, one might as well suppose that it is reduced, i.e. $\forall j,\ \gamma_j \notin \mathcal{A}f$ and $\forall j,\ 2 \le j \le k,\ \alpha_j \notin \mathcal{BA}$. It follows from the amalgamated structure of $\mathcal{GA}$ that if $f = \alpha'_1.\gamma'_1.\ldots.\alpha'_l.\gamma'_l.\alpha'_{l+1}$ is another reduced expression of f, then $k = l$ and there exist $(\beta_j)_{1 \le j \le k}$, $(\gamma_j)_{1 \le j \le k}$ in $\mathcal{A}f \cap \mathcal{BA}$ such that $\alpha'_1 = \alpha_1.\beta_1^{-1}$, $\alpha'_j = \delta_{j-1}.\alpha_j.\beta_j^{-1}$ (for $2 \le j \le k$), $\alpha'_{k+1} = \delta_k.\alpha_{k+1}$ and $\gamma'_j = \beta_j.\gamma_j.\delta_j^{-1}$ (for

$1 \leq j \leq k$). Therefore, following [F-M], we define the multidegree of f by $d(f) := (\deg \gamma_1, \ldots, \deg \gamma_k)$.

Let $n \geq 0$ be an integer. We set $\mathcal{E}nd_{\leq n} := \{f \in \mathcal{E}nd,\ \deg f \leq n\}$ and $\mathcal{GA}_{\leq n} := \{f \in \mathcal{GA},\ \deg f \leq n\}$. The set $\mathcal{E}nd_{\leq n}$ is a K-affine space and can therefore be given the structure of a K-algebraic variety. It is well known that $\mathcal{GA}_{\leq n}$ is locally closed in $\mathcal{E}nd_{\leq n}$ (see [B-C-W]), so that $\mathcal{GA}_{\leq n}$ can also be given the structure of a K-algebraic variety.

Finally, the equalities $\mathcal{E}nd = \bigcup_n \mathcal{E}nd_{\leq n}$ and $\mathcal{GA} = \bigcup_n \mathcal{GA}_{\leq n}$ yield structure of infinite-dimensional algebraic varieties on $\mathcal{E}nd$ and $\mathcal{GA}$ (see [S]).

Let us recall that a set X with a fixed sequence of subsets X_n, each of which has a structure of a finite-dimensional algebraic variety, is called an infinite-dimensional algebraic variety if the following conditions are satisfied :

1) $X = \bigcup_n X_n$;

2) X_n is a closed algebraic subvariety of X_{n+1}.

Each of the X_n will be considered with its Zariski topology and we endow X with the topology of the inductive limit, in which a set $Z \subset X$ is closed if and only if $Z \cap X_n$ is closed in X_n for all n.

Let $X = \bigcup_n X_n$ and $Y = \bigcup_n Y_n$ be two infinite-dimensional algebraic varieties. A morphism between X and Y is a map $f :\ X \to Y$ such that for each n there exists m satisfying $f(X_n) \subset Y_m$ and such that the restriction $f :\ X_n \to Y_m$ is a morphism of finite-dimensional varieties. Any morphism is easily seen to be continuous.

b. A conjecture.

We will denote by D the set of multidegrees. From the definition, we see that D is the set of finite sequences of integers greater than or equal to 2 (including the empty sequence). If $d = (d_1, \ldots, d_k)$ is a multidegree, then $\mathcal{GA}_d \subset \mathcal{GA}$ will denote the set of automorphisms whose multidegree is equal to d. We obviously have $\mathcal{GA} = \coprod_{d \in D} \mathcal{GA}_d$. We believe that the following statement is true :

Conjecture. If $K = \mathbb{C}$ and $d = (d_1, \ldots, d_k)$ is a multidegree, then

$\overline{\mathcal{GA}_d} = \coprod_{d' \preceq d} \mathcal{GA}_{d'}$ where $\preceq$ is the partial order induced on D by the relations :

(i) $\forall d \in D,\ \emptyset \preceq d$;

(ii) If $d = (d_1, \dots, d_k)$, $e = (e_1, \dots, e_k) \in D$ and if $\forall j,\ d_j \leq e_j$, then $d \preceq e$;

(iii) If $d = (d_1, \dots, d_k) \in D$ and if $1 \leq j \leq j+1 \leq k$, then $d_{(j,j+1)} \preceq d$, where we set $d_{(j,j+1)} := (d_1, \dots, d_{j-1}, d_j + d_{j+1} - 1, d_{j+2}, \dots, d_k)$.

Remarks. 1. Note the analogy with the Bruhat decomposition of reductive complex connected linear algebraic groups. If G is such a group with Borel subgroup B and Weyl group W, then we have $G = \coprod_{w \in W} BwB$ (disjoint union). Furthermore, $\overline{BwB} = \coprod_{w' \leq w} Bw'B$ where $\leq$ is the Bruhat order on W (see [Sp]).

2. If the multidegree of an automorphism is $d = (d_1, \dots, d_k)$, then its degree is $d_1 d_2 \dots d_k$ (see Proposition 1.9 of [W] or Theorem 2.1 of [F-M]). Hence, as soon as $n \geq d_1 \dots d_k$, we have $\mathcal{GA}_d \subset \mathcal{GA}_{\leq n}$ and $\overline{\mathcal{GA}_d}$ is the usual Zariski closure of $\mathcal{GA}_d$ in $\mathcal{GA}_{\leq n}$.

3. It is shown in [F-M] that $\mathcal{GA}_d$ is an analytic variety which is biholomorphic to $\mathbb{C}^{d_1 + \dots + d_k + 6}$.

4. The condition (iii) may seem mysterious. To motivate it, let us consider the family of automorphisms of $\mathcal{GA}_{\leq 4}$ induced by the Nagata automorphism (see [N]) $N :\ \mathbb{C} \to \mathcal{GA}_{\leq 4}$, $Z \mapsto N_Z$ where

$$N_Z := (X - 2Y(XZ + Y^2) - Z(XZ + Y^2)^2, Y + Z(XZ + Y^2)).$$

If $Z \neq 0$, we have $N_Z = \gamma_Z . \alpha_Z . \delta_Z$ where $\gamma_Z = (X - Z^{-1}Y^2, Y) \in \mathcal{BA} \setminus \mathcal{A}f$, $\alpha_Z = (X, Y + Z^2 X) \in \mathcal{A}f \setminus \mathcal{BA}$ and $\delta_Z = (X + Z^{-1}Y^2, Y) \in \mathcal{BA} \setminus \mathcal{A}f$, so that the multidegree of N_Z is $(2,2)$. However, $N_0 = (X + 2Y^3, Y)$ so that the multidegree of N_0 is (3). By using this example and Proposition 12 of [F], one can show the above conjecture for $d = (2,2)$.

5. In order to study the Jacobian Conjecture, an analysis of the irreducible components of $\mathcal{GA}_{\leq n}$ seems interesting (see [B-C-W]). Let us first notice that $\mathcal{GA}_{\leq n} = \coprod_{d \in D_n} \mathcal{GA}_d$ where D_n denotes the set of multidegrees $(d_1, \dots, d_k)$ such that $d_1 \dots d_k \leq n$. But $\overline{\mathcal{GA}_d}$ is an irreducible algebraic variety (see [F]). As a result, by admitting the above conjecture, the irreducible components of $\mathcal{GA}_{\leq n}$ would precisely be the $\overline{\mathcal{GA}_d}$ where d is any maximal element of D_n for $\preceq$.

This statement is another formulation of the conjecture we gave in [F] on page 620. It asserts that the irreducible components of $\mathcal{GA}_{\leq n}$ are the

$\overline{\mathcal{GA}_d}$ where d is any maximal element of D_n for the order $\leq$ induced by the relations :

(i) $\forall d \in D,\ \emptyset \leq d$;

(ii) If $d = (d_1, \ldots, d_k),\ e = (e_1, \ldots, e_k) \in D$ and if $\forall j,\ d_j \leq e_j$, then $d \leq e$;

(iii)' If $d = (d_1, \ldots, d_k) \in D$ and if $1 \leq j \leq k$, then $d_{\hat{j}} \leq d$, where we set $d_{\hat{j}} := (d_1, \ldots, d_{j-1}, d_{j+1}, \ldots, d_k)$.

Indeed, it is easily checked that the maximal elements of D_n for $\preceq$ and $\leq$ coincide.

c. The length of an automorphism.

If the multidegree of $f \in \mathcal{GA}$ is $(d_1, \ldots, d_k)$, we will say that its length is k and write $l(f) = k$. It is clear that the length of f is the minimal number of triangular automorphisms we need to write f as a composition of affine and triangular automorphisms. Let us denote by $\mathcal{GA}^{\leq k}$ the set of automorphisms whose length is less than or equal to k. The aim of this paper is to show the following result (which is a particular case of the above conjecture) :

Theorem 1. If the characteristic of K is zero and if $k \geq 0$ is an integer, then $\mathcal{GA}^{\leq k}$ is closed in $\mathcal{GA}$. Equivalently, the map $l : \ \mathcal{GA} \to \mathbb{Z}$ is lower semicontinuous.

Remark. This is equivalent to saying that the map $l : \ \mathcal{GA}_{\leq n} \to \mathbb{Z}$ is lower semicontinuous for each $n \geq 0$. If $K = \mathbb{C}$, this implies that this map is lower semicontinuous when $\mathcal{GA}_{\leq n}$ is endowed with the transcendental topology. The above example, based on Nagata automorphism, helps to understand why this statement is not completely obvious (at least for us). The point is that the reduced expression of N_Z when $Z \neq 0$ does not pass to the limit when Z goes to 0.

d. The length of a variable.

Let us set $\mathcal{P} := K[X, Y]$.

We have $\mathcal{P} = \bigcup_n \mathcal{P}_{\leq n}$ where $\mathcal{P}_{\leq n} := \{P \in \mathcal{P}, \quad \deg P \leq n\}$. This endows $\mathcal{P}$ with the structure of an infinite-dimensional algebraic variety.

Let P be an element of $\mathcal{P}$. We say that P is a variable if P is the component of an automorphism of $\mathbb{A}^2$. In this case, let us recall that $K[X, Y]/(P)$

is K-isomorphic to $K[T]$ where T is an indeterminate. Therefore, P is an irreducible element of $K[X,Y]$.

Let us denote by $\mathcal{V}$ the set of variables (of $\mathcal{P}$). If $p_1 : \ \mathcal{E}nd \to \mathcal{P}$ is the first projection (defined by $p_1(P,Q)=P$), then $\mathcal{V}=p_1(\mathcal{GA})$.

If $k \geq 0$ is an integer, we set $\mathcal{V}^{\leq k} := p_1(\mathcal{GA}^{\leq k})$. In other words, an element of $\mathcal{P}$ belongs to $\mathcal{V}^{\leq k}$ if and only if it is the component of an automorphism f of $\mathbb{A}^2$ satisfying $l(f) \leq k$.

Finally, if P is a variable, we set $l(P) := \min\{k, \ P \in \mathcal{V}^{\leq k}\}$.

By introducing in section 2 the multidegree of a variable, we will show the following elementary but fundamental result :

Theorem 2. If $f=(f_1,f_2) \in \mathcal{GA}$, then $l(f)=\max(l(f_1),l(f_2))$.

Since $l : \ \mathcal{GA} \to \mathbb{Z}$ is equal to $\max(l \circ p_1, l \circ p_2)$ where p_1 (resp. p_2) : $\mathcal{GA} \to \mathcal{V}$ is the first (resp. second) projection and since the supremum of any set of lower semicontinuous maps is lower semicontinuous, Theorem 1 will be a consequence of Theorem 2 and of the following result :

Theorem 3. We assume that the characteristic of K is zero. If $k \geq 0$, then $\mathcal{V}^{\leq k}$ is closed in $\mathcal{V}$. Equivalently, the map $l : \ \mathcal{V} \to \mathbb{Z}$ is lower semicontinuous.

We shall prove this result in section 3.

2. The multidegree of a variable.

a. Preliminary remarks.

The next result, which is well known in characteristic zero, holds in any characteristic :

Lemma 1. Let $\gamma=(\gamma_1,\gamma_2) \in \mathcal{GA}$. If $\gamma_2 = Y$, then $\gamma \in \mathcal{BA}$.

Proof. Since $\gamma_1 \in K[X,Y]$, there exists a unique polynomial $P(Y) \in K[Y]$ such that $\gamma_1 - P(Y)$ is divisible by X. But $(\gamma_1 - P(Y), Y) \in \mathcal{GA}$, hence $\gamma_1 - P(Y)$ is a variable, hence $\gamma_1 - P(Y)$ is irreducible, hence there exists $a \in K^*$ such that $\gamma_1 - P(Y) = aX$. $\square$

Definition 1. Let $\sim$ be the equivalence relation defined on $\mathcal{V}$ by $P_1 \sim P_2$ if there exist a in K^* and b in K such that $P_1 = aP_2 + b$.

If $P \in \mathcal{V}$, let us denote by $\overline{P}$ its equivalence class (for $\sim$).

Also, let us denote by $\overline{\mathcal{V}}$ the set of equivalence classes (for $\sim$) of $\mathcal{V}$.

Remark. If $\overline{P_1} = \overline{P_2}$, then deg $P_1 =$ deg P_2. Furthermore, if $\overline{Q_1} = \overline{Q_2}$, then $(P_1, Q_1) \in \mathcal{GA}$ if and only if $(P_2, Q_2) \in \mathcal{GA}$. Thus, we can define the relation $\succ$ on $\overline{\mathcal{V}}$ in the following manner :

Definition 2. We set $\overline{P} \succ \overline{Q}$ if and only if deg $P >$ deg Q and $(P, Q) \in \mathcal{GA}$. If $\overline{P} \succ \overline{Q}$, we will say that $\overline{Q}$ is a predecessor of $\overline{P}$.

Lemma 2. If $\overline{P} \in \overline{\mathcal{V}}$ and if deg $P \geq 2$, then $\overline{P}$ admits a unique predecessor.

Proof. Existence. By hypothesis, there exists $R \in \mathcal{V}$ such that $f := (R, P)$ is an element of $\mathcal{GA}$. By [K], there exists $S(Y) \in K[Y]$ such that if we set $\gamma := (X + S(Y), Y) \in \mathcal{BA}$ and if we denote by Q the first component of the automorphism $(R + S(P), P) = \gamma.f$, then deg Q $<$ deg P. Therefore, $\overline{Q}$ is a predecessor of $\overline{P}$.
Unicity. Let $Q_1, Q_2 \in \mathcal{V}$ be such that $\overline{P} \succ \overline{Q_1}$ and $\overline{P} \succ \overline{Q_2}$. We want to show that $\overline{Q_1} = \overline{Q_2}$.

Let $\gamma = (\gamma_1, \gamma_2) \in \mathcal{GA}$ be such that $(Q_2, P) = \gamma.(Q_1, P)$. We have $\gamma_2 = Y$ so that $\gamma \in \mathcal{BA}$ by Lemma 1. Hence, there exist $a \in K^*$ and $R(Y) \in K[Y]$ such that $\gamma_1 = aX + R(Y)$, whence $Q_2 = aQ_1 + R(P)$.

Hence deg$(Q_2 - aQ_1)$ is divisible by deg P.

But, by hypothesis, deg$(Q_2 - aQ_1) <$ deg P, so that $Q_2 - aQ_1 \in K$. $\square$

b. Composition sequence and multidegree of a variable.

Definition 3. Let P be a variable. We say that $(P_0, P_1, \ldots, P_k)$ is a composition sequence of P if $P_0 = P$, deg $P_k = 1$ and if $\overline{P_0} \succ \overline{P_1} \succ \ldots \succ \overline{P_k}$.

By Lemma 2, we see that any variable admits a composition sequence and that such a sequence is unique modulo the equivalence relation $\sim$.

Definition 4. Let P be a variable and let $(P_0, \ldots, P_k)$ be a composition sequence of P. We define the multidegree of P by

$$d(P) := \left(\frac{\deg P_0}{\deg P_1}, \frac{\deg P_1}{\deg P_2}, \ldots, \frac{\deg P_{k-1}}{\deg P_k}\right)$$

and the altitude of P by $a(P) := k$.

Remark. The condition $\overline{P_j} \succ \overline{P_{j+1}}$ shows that $\frac{\deg\ P_j}{\deg\ P_{j+1}}$ is actually an integer greater than or equal to 2.

The following statement relates the multidegree of a variable and the multidegree of an automorphism :

Lemma 3. Let P be a variable and let $(P_0, \ldots, P_k)$ be a composition sequence of P. If $k \geq 1$, then the multidegree of the automorphism (P_0, P_1) is equal to the multidegree of the variable $P = P_0$.

Proof. Let us set $\sigma := (Y, X) \in \mathcal{A}f$. If $0 \leq j \leq k-2$, let $\gamma_{j+1} \in \mathcal{GA}$ be such that $(P_j, P_{j+1}) = \gamma_{j+1}.\sigma.(P_{j+1}, P_{j+2})$. Also, let $\alpha \in \mathcal{A}f$ be such that its last coordinate is P_k (it is possible because deg $P_k = 1$) and let $\gamma_k \in \mathcal{GA}$ be such that $(P_{k-1}, P_k) = \gamma_k.\alpha$.

It is easy to see that each γ_j admits Y as its last coordinate, so that $\gamma_j \in \mathcal{BA}$. Furthermore, we observe that deg $\gamma_j = \frac{\deg\ P_{j-1}}{\deg\ P_j} \geq 2$.

By noting that $\sigma \in \mathcal{A}f \setminus \mathcal{BA}$ and that each $\gamma_j \in \mathcal{BA} \setminus \mathcal{A}f$, we conclude that the following expression is reduced $(P_0, P_1) = \gamma_1.\sigma.\gamma_2.\sigma.\ldots.\sigma.\gamma_k.\alpha$.

Therefore, $d(P_0, P_1) = d(P_0) = (\deg\ \gamma_1, \ldots, \deg\ \gamma_k)$. □

c. Proof of Theorem 2.

Lemma 4. Let $f = (f_1, f_2) \in \mathcal{GA}$.

(i) If deg $f_1 >$ deg f_2, then $l(f) = a(f_1) = a(f_2) + 1$;

(ii) If deg $f_1 =$ deg f_2, then $l(f) = a(f_1) = a(f_2)$.

Proof. The statement (i) is a direct consequence of Lemma 3, since in that case we can complete the sequence (f_1, f_2) in order to obtain a composition sequence of f_1.

Let us assume that deg $f_1 =$ deg f_2. If deg $f = 1$, we have clearly $l(f) = a(f_1) = a(f_2) = 0$.

Otherwise, there exists $\lambda \in K$ such that

$$\deg(f_2 - \lambda f_1) = \deg(f_1 - \lambda^{-1} f_2) < \deg f$$

and the statement (ii) follows from the statement (i) by noting that the automorphisms (f_1, f_2), $(f_1, f_2 - \lambda f_1)$ and $(f_2, f_1 - \lambda^{-1} f_2)$ all have the same length. □

Corollary. If P is a variable, then $l(P) = a(P)$.

Proof. Let $(P_0, \ldots, P_k)$ be a composition sequence of P.

If $k = 0$, we clearly have $l(P) = a(P) = 0$.

If $k \geq 1$, then (P_0, P_1) is an automorphism whose length is k by Lemma 3. By Lemma 4, any automorphism of the shape (P_0, Q) has a length greater than or equal to k, so that $l(P) = a(P) = k$. □

In view of this Corollary and of Lemma 4, Theorem 2 is now obvious.

We end this section by a result which will be used in section 3. Its proof is obvious and is left to the reader.

Lemma 5. If $P \in \mathcal{V}^{\leq k+1}$, then there exists $Q \in \mathcal{V}^{\leq k}$ such that $(P, Q) \in \mathcal{GA}$ and deg $Q \leq$ deg P.

3. Computation of the closure of $\mathcal{V}^{\leq k}$ in $\mathcal{P}$.

In this section, we assume that the characteristic of K is zero. The closure of $\mathcal{V}^{\leq k}$ in $\mathcal{P} = K[X, Y]$ will be denoted by $\overline{\mathcal{V}^{\leq k}}$.

a. Description of $\overline{\mathcal{V}^{\leq k}}$.

Define $\mathcal{W}_k \subset \mathcal{P}$ by

$$\mathcal{W}_k := \{P \in \mathcal{P},\ \exists (Q, R) \in K[T] \times \mathcal{V}^{\leq k-1}, P = Q(R)\}$$

for $k \geq 1$ and by $\mathcal{W}_0 := K$ for $k = 0$.

Theorem 4. If $k \geq 0$, then $\overline{\mathcal{V}^{\leq k}} = \mathcal{V}^{\leq k} \cup \mathcal{W}_k$.

Let us first show that this result implies Theorem 3. In fact we will only use the inclusion $\overline{\mathcal{V}^{\leq k}} \subset \mathcal{V}^{\leq k} \cup \mathcal{W}_k$ to prove that $\mathcal{V}^{\leq k}$ is closed in $\mathcal{V}$.
The closure of $\mathcal{V}^{\leq k}$ in $\mathcal{V}$ is equal to $\mathcal{V} \cap \overline{\mathcal{V}^{\leq k}}$ so that Theorem 3 is equivalent to the inclusion $\mathcal{V} \cap \mathcal{W}_k \subset \mathcal{V}^{\leq k}$. Let us prove it.

This is obvious if $k = 0$, because $\mathcal{V} \cap \mathcal{W}_0 = \emptyset$.

If $k \geq 1$, it is sufficient to show that $\mathcal{V} \cap \mathcal{W}_k \subset \mathcal{V}^{\leq k-1}$.

Let $P \in \mathcal{V} \cap \mathcal{W}_k$. There exists $(Q, R) \in K[T] \times \mathcal{V}^{\leq k-1}$ such that $P = Q(R)$. By writing $Q(T) = TQ_1(T) + b$ where $Q_1(T) \in K[T]$ and $b \in K$, we obtain $P - b = RQ_1(R)$. However, P is a variable, so that $P - b$ is also a variable, so that $P - b$ is irreducible, so that $Q_1(R) \in K^*$. If we set $a := Q_1(R)$, then we have $P = aR + b$ and we are done.

To prove Theorem 4, we will successively show that $\mathcal{W}_k \subset \overline{\mathcal{V}^{\leq k}}$ (subsection b) and that $\mathcal{V}^{\leq k} \cup \mathcal{W}_k$ is closed in $\mathcal{P}$ (subsection c).

b. Proof of the inclusion $\mathcal{W}_k \subset \overline{\mathcal{V}^{\leq k}}$.

First case. $k = 0$.

Let $a \in K$. We define $h : \mathbb{A}^1 \to \mathcal{P}$ by $h(\varepsilon) := \varepsilon X + a$. If $\varepsilon \neq 0$, we have $h(\varepsilon) \in \mathcal{V}^{\leq 0}$, so that $h(0) = a \in \overline{\mathcal{V}^{\leq 0}}$.

Second case. $k \geq 1$.

Assume that $(Q, R) \in K[T] \times \mathcal{V}^{\leq k-1}$.

There exists $S \in \mathcal{V}$ such that $(R, S) \in \mathcal{GA}$.

We define $h : \mathbb{A}^1 \to \mathcal{P}$ by $h(\varepsilon) := \varepsilon S + Q(R)$. If $\varepsilon \neq 0$, we have $(R, \varepsilon S + Q(R)) \in \mathcal{GA}$ and $R \in \mathcal{V}^{\leq k-1}$, so that $h(\varepsilon) \in \mathcal{V}^{\leq k}$ by Lemma 4. Hence $h(0) = Q(R) \in \overline{\mathcal{V}^{\leq k}}$.

c. Proof of the closed nature of $\mathcal{V}^{\leq k} \cup \mathcal{W}_k$.

First step : preliminary reduction.

Let $\Pi := \{P \in \mathcal{P},\ P(0,0) = 0\}$. As a subset of $\mathcal{P}$, Π is endowed with the induced topology. Let us set $\Pi_{\leq n} := \{P \in \Pi,\ \deg P \leq n\} = \mathcal{P}_{\leq n} \cap \Pi$. The subset $Z \subset \Pi$ is closed if and only if $Z \cap \Pi_{\leq n} \subset \Pi_{\leq n}$ is closed for each n.

Since the sets $\mathcal{V}^{\leq k}$ and $\mathcal{W}_k$ are invariant by any translation $P \mapsto P + c$ where $c \in K$, it is sufficient to show that $V^{\leq k} \cup W_k$ is closed in Π, where we set $V^{\leq k} := \mathcal{V}^{\leq k} \cap \Pi$ and $W_k := \mathcal{W}_k \cap \Pi$.

Second step : reduction to a projective problem.

Let us denote by $\mathbb{P}$ the set of lines of the K-vector space Π.
If $\mathbb{P}_{\leq n}$ denotes the set of lines of $\Pi_{\leq n}$, then the equality $\mathbb{P} = \bigcup_n \mathbb{P}_{\leq n}$ endows $\mathbb{P}$ with the structure of an infinite-dimensional algebraic variety.

By definition, $C \subset \Pi$ is called a cone if $0 \in C$ and if $\forall\ \lambda \in K,\ \forall\ u \in C,\ \lambda u \in C$.

Let us recall that there exists a natural correspondence between the cones of Π and the subsets of $\mathbb{P}$. Furthermore, a cone of Π is closed if and only if the corresponding subset of $\mathbb{P}$ is closed. Let us denote by $F_k \subset \mathbb{P}$ the subset corresponding to the cone $(V^{\leq k} \cup W_k) \subset \Pi$. We just want to show that $F_k \subset \mathbb{P}$ is closed.

We argue by induction on k.
For $k = 0$, this is clear, because $V^{\leq 0} \cup W_0 = \Pi_{\leq 1}$, so that $F_1 = \mathbb{P}_{\leq 1}$. Before

proving the induction step, let us introduce the Jacobian variety.

Third step : the Jacobian variety.

If $(P,Q) \in \mathcal{E}nd$, its Jacobian is $[P,Q] := \frac{\partial P}{\partial X}\frac{\partial Q}{\partial Y} - \frac{\partial P}{\partial Y}\frac{\partial Q}{\partial X}$. Furthermore, the kernel of the derivation $D := X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y}$ of $K[X,Y]$ is equal to K, since the characteristic is zero. Therefore, if we set $< P,Q > := D([P,Q])$, which is the same as $< P,Q > := [DP,Q] + [P,DQ] - 2[P,Q]$, then $< P,Q >= 0$ if and only if $[P,Q]$ belongs to K.

Finally, the map $\Pi \times \Pi \to K[X,Y]$ which sends (P,Q) to $< P,Q >$ is bilinear. As a result, the equality $< P,Q >= 0$ defines a closed subset $J \subset \mathbb{P} \times \mathbb{P}$ which we will call the Jacobian variety.

Here, we endow $\mathbb{P} \times \mathbb{P}$ with the structure of an infinite-dimensional algebraic variety thanks to the equality $\mathbb{P} \times \mathbb{P} = \bigcup_n (\mathbb{P}_{\leq n} \times \mathbb{P}_{\leq n})$.

Fourth step : the induction.

We assume that $F_k \subset \mathbb{P}$ is closed and we want to show that $F_{k+1} \subset \mathbb{P}$ is closed. We will denote by p_1 (resp. p_2): $\mathbb{P} \times \mathbb{P} \to \mathbb{P}$ the first (resp. second) projection. The main idea is to establish that $F_{k+1} = p_1\left(J \cap p_2^{-1}(F_k)\right)$.

Remarks. 1. Unfortunately, this last equality is not sufficient to prove that $F_{k+1} \subset \mathbb{P}$ is closed. Indeed, by the fundamental theorem of elimination theory, for each $n \geq 0$ the map $p_1 : \mathbb{P}_{\leq n} \times \mathbb{P}_{\leq n} \to \mathbb{P}_{\leq n}$ is closed, but the map $p_1 : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ is no longer closed.

To see this, consider a sequence $(x_n)_{n\geq 1}$ of distinct points of $\mathbb{P}_{\leq 1} \subset \mathbb{P}$ the union of which is not $\mathbb{P}_{\leq 1}$ and a sequence $(y_n)_{n\geq 1}$ of points of $\mathbb{P}$ such that y_n belongs to $\mathbb{P}_{\leq n+1} \setminus \mathbb{P}_{\leq n}$ for each n. Then, the union $Z \subset \mathbb{P} \times \mathbb{P}$ of the points $(x_n, y_n) \in \mathbb{P} \times \mathbb{P}$ is a closed subset, while $p_1(Z) \subset \mathbb{P}$ is not closed (because $p_1(Z)$ is an infinite subset of the projective line $\mathbb{P}_{\leq 1}$ which is not $\mathbb{P}_{\leq 1}$).

2. However, it is clear that if $Z \subset \mathbb{P} \times \mathbb{P}$ is closed and such that $p_1(Z) \cap \mathbb{P}_{\leq n} = p_1\left(Z \cap (\mathbb{P}_{\leq n} \times \mathbb{P}_{\leq n})\right)$ for each n, then $p_1(Z) \subset \mathbb{P}$ is closed.

Let us observe that $\left(J \cap p_2^{-1}(F_k)\right) \subset \mathbb{P}\times\mathbb{P}$ is closed. Therefore the conclusion will follow from our last statement :

Lemma 6. If $k \geq 0$ is an integer and if $Z_k := \left(J \cap p_2^{-1}(F_k)\right) \subset \mathbb{P} \times \mathbb{P}$, then $F_{k+1} = p_1(Z_k)$. Moreover $F_{k+1} \cap \mathbb{P}_{\leq n} = p_1(Z_k \cap (\mathbb{P}_{\leq n} \times \mathbb{P}_{\leq n}))$ for all $n \geq 1$.

Proof. We begin by showing that $p_1(Z_k) \subset F_{k+1}$. This amounts to proving that if $P \in \Pi$ is such that there exists a nonzero element Q of $V^{\leq k} \cup W_k$

satisfying $< P, Q >= 0$, then P belongs to $V^{\leq k+1} \cup W_{k+1}$.

First case. $Q \in V^{\leq k}$

First subcase. $[P, Q] \in K^*$.

Since Q is a variable, the condition $[P, Q] \in K^*$ is well known to imply that (P, Q) is an automorphism (since the characteristic is zero). We also have $Q \in V^{\leq k}$, so that $P \in V^{\leq k+1}$ by Lemma 4.

Second subcase. $[P, Q] = 0$.

By [No], this implies that there exist $R, S \in K[T]$ and $L \in K[X, Y]$ such that $P = R(L)$ and $Q = S(L)$ (here, we use again the zero characteristic). Since Q is a variable, we saw in subsection 3.a that the equality $Q = S(L)$ implies that there exist $a \in K^*$ and $b \in K$ such that $Q = aL + b$. Therefore, P can be expressed as a polynomial in Q. This shows that $P \in W_{k+1}$.

Second case. $Q \in W_k$.

Because Q is nonzero, we necessarily have $k \geq 1$. By the definition of W_k, there exist $R \in K[T]$ and S in $\mathcal{V}^{\leq k-1}$ such that $Q = R(S)$. We can suppose without restriction that S belongs to $V^{\leq k-1}$ and because Q is nonzero, R is also nonzero.

The equality $< P, Q >= 0$ means that $[P, Q] \in K$, which can be written $[P, R(S)] = [P, S]R'(S) \in K$, which implies that $[P, S] \in K$, which implies that $< P, S >= 0$. This takes us back to the first case.

Now, we must show that $F_{k+1} \cap \mathbb{P}_{\leq n} \subset p_1(Z_k \cap (\mathbb{P}_{\leq n} \times \mathbb{P}_{\leq n}))$ for all $n \geq 1$. Equivalently, we must prove that if P is a nonzero element of $V^{\leq k+1} \cup W_{k+1}$, then there exists a nonzero element Q of $V^{\leq k} \cup W_k$ satisfying $< P, Q >= 0$ and $\deg Q \leq \deg P$.

First case. $P \in V^{\leq k+1}$.

By Lemma 5, there exists $Q \in \mathcal{V}^{\leq k}$ such that $(P, Q) \in \mathcal{GA}$ and $\deg Q \leq \deg P$. Using a translation, there is no restriction to assume that $Q \in V^{\leq k}$. Moreover, it is well known that the condition $(P, Q) \in \mathcal{GA}$ implies the condition $[P, Q] \in K^*$ (the converse being the Jacobian Conjecture), so that $< P, Q >= 0$.

Second case. $P \in W_{k+1}$.

There exist R in $K[T]$ and Q in $V^{\leq k}$ such that $P = R(Q)$. We clearly have $\deg Q \leq \deg P$ and $[P, Q] = [R(Q), Q] = R'(Q)[Q, Q] = 0$. □

Remark. Thanks to the description of $\overline{\mathcal{V}^{\leq k}}$, one may easily show that the closure of $\mathcal{V}$ in $\mathcal{P}$ is equal to $\mathcal{V} \cup \mathcal{W}$ where we set

$$\mathcal{W} := \{P \in \mathcal{P},\ \exists (Q, R) \in K[T] \times \mathcal{V},\ P = Q(R)\}.$$

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