

ON THE MAXIMALITY OF THE TRIANGULAR SUBGROUP

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ABSTRACT. We prove that the subgroup of triangular automorphisms of the complex affine n -space is maximal among all solvable subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ for every n . In particular, it is a Borel subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$, when the latter is viewed as an ind-group. In dimension two, we prove that the triangular subgroup is a maximal closed subgroup and that nevertheless, it is not maximal among all subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$. Given an automorphism f of $\mathbb{A}_{\mathbb{C}}^2$, we study the question whether the group generated by f and the triangular subgroup is equal to the whole group $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$.

1. INTRODUCTION

The main purpose of this paper is to study the Jonquières subgroup \mathcal{B}_n of the group $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ of polynomial automorphisms of the complex affine n -space, i.e. its subgroup of triangular automorphisms. We will settle the titular question by providing three different answers, depending on to which properties the maximality condition is referring to.

Theorem 1. (1) *For every $n \geq 2$, the subgroup \mathcal{B}_n is maximal among all solvable subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$.*

(2) *The subgroup \mathcal{B}_2 is maximal among the closed subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$.*

(3) *The subgroup \mathcal{B}_2 is not maximal among all subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$.*

Recall that $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ is naturally an ind-group, i.e. an infinite dimensional algebraic group. It is thus equipped with the usual ind-topology (see Section 2 for the definitions). In particular, since \mathcal{B}_n is a closed connected solvable subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$, the first statement of Theorem 1 can be interpreted as follows:

Corollary 2. *The group \mathcal{B}_n is a Borel subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$.*

This generalizes a remark of Berest, Eshmatov and Eshmatov [BEE16] stating that triangular automorphisms of $\mathbb{A}_{\mathbb{C}}^2$, of Jacobian determinant 1, form a Borel subgroup (i.e. a maximal connected solvable subgroup) of the group $\text{SAut}(\mathbb{A}_{\mathbb{C}}^2)$ of polynomial automorphisms of $\mathbb{A}_{\mathbb{C}}^2$ of Jacobian determinant 1. Actually, the proofs in [BEE16] also imply Corollary 2 in the case $n = 2$. Nevertheless, since they are based on results of Lamy [Lam01], which use the Jung-van der Kulk-Nagata structure theorem for $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$, these arguments are specific to the dimension 2 and cannot be generalized to higher dimensions.

The Jonquières subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ is thus a good analogue of the subgroup of invertible upper triangular matrices, which is a Borel subgroup of the classical linear algebraic group $\text{GL}_n(\mathbb{C})$. Moreover, Berest, Eshmatov and Eshmatov strengthen this

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analogy when $n = 2$ by proving that \mathcal{B}_2 is, up to conjugacy, the only Borel subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$. On the other hand, it is well known that there exist, if $n \geq 3$, algebraic additive group actions on $\mathbb{A}_{\mathbb{C}}^n$ that cannot be triangularized [Bas84, Pop87]. Therefore, we ask the following problem.

Problem 3. *Show that Borel subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ are not all conjugate ($n \geq 3$).*

This problem turns out to be closely related to the question of the boundedness of the derived length of solvable subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$. We give such a bound when $n = 2$. More precisely, the maximal derived length of a solvable subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ is equal to 5 (see Proposition 3.14). As a consequence, we prove that the group $\text{Aut}_z(\mathbb{A}_{\mathbb{C}}^3)$ of automorphisms of \mathbb{A}^3 fixing the last coordinate admits non-conjugate Borel subgroups (see Corollary 3.22). Note that such a phenomenon has already been pointed out in [BEE16].

The paper is organized as follows. Section 1 is the present introduction. In Section 2, we recall the definitions of ind-varieties and ind-groups given by Shafarevich and explain how the automorphism group of the affine n -space may be endowed with the structure of an ind-group.

In Section 3, we prove the first two statements of Theorem 1 and discuss the question, whether the ind-group $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ does admit non-conjugate Borel subgroups. We then study the group of all automorphisms of $\mathbb{A}_{\mathbb{C}}^3$ fixing the last variable, proving that it admits non-conjugate Borel subgroups. In the last part of Section 3, we give examples of maximal closed subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$.

Finally, we consider $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ as an “abstract” group in Section 4. We show that triangular automorphisms do not form a maximal subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$. More precisely, after defining the affine length of an automorphism in Definition 4.1, we prove the following statement:

Theorem 4. *For any field \mathbf{k} , the two following assertions hold.*

- (1) *If the affine length of an automorphism $f \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ is at least 1 (i.e. f is not triangular) and at most 4, then the group generated by \mathcal{B}_2 and f satisfies*

$$\langle \mathcal{B}_2, f \rangle = \text{Aut}(\mathbb{A}_{\mathbf{k}}^2).$$

- (2) *There exists an automorphism $f \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ of affine length 5 such that the group $\langle \mathcal{B}_2, f \rangle$ is strictly included into $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$.*

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2. PRELIMINARIES: THE IND-GROUP OF POLYNOMIAL AUTOMORPHISMS

In [Sha66, Sha81], Shafarevich introduced the notions of ind-varieties and ind-groups, and explained how to endow the group of polynomial automorphisms of the affine n -space with the structure of an ind-group. Since these two papers are well-known to contain several inaccuracies, we now recall the definitions from Shafarevich and describe the ind-group structure of the automorphism group of the affine n -space.

For simplicity, we assume in this section that \mathbf{k} is an algebraically closed field.

2.1. Ind-varieties and ind-groups. We first define the category of infinite dimensional algebraic varieties (*ind-varieties* for short).

Definition 2.1 (Shafarevich, 1966).

- (1) An ind-variety V (over \mathbf{k}) is a set together with an ascending filtration $V_{\leq 0} \subseteq V_{\leq 1} \subseteq V_{\leq 2} \subseteq \cdots \subseteq V$ such that the following holds:
 - (a) $V = \bigcup_d V_{\leq d}$.
 - (b) Each $V_{\leq d}$ has the structure of an algebraic variety (over \mathbf{k}).
 - (c) Each $V_{\leq d}$ is Zariski closed in $V_{\leq d+1}$.
- (2) A morphism of ind-varieties (or ind-morphism) is a map $\varphi: V \rightarrow W$ between two ind-varieties $V = \bigcup_d V_{\leq d}$ and $W = \bigcup_d W_{\leq d}$ such that there exists, for every d , an e for which $\varphi(V_{\leq d}) \subseteq W_{\leq e}$ and such that the induced map $V_{\leq d} \rightarrow W_{\leq e}$ is a morphism of varieties (over \mathbf{k}).

In particular, every ind-variety V is naturally equipped with the so-called ind-topology in which a subset $S \subseteq V$ is closed if and only if every subset $S_{\leq d} := S \cap V_{\leq d}$ is Zariski-closed in $V_{\leq d}$.

We remark that the product $V \times W$ of two ind-varieties $V = \bigcup_d V_{\leq d}$ and $W = \bigcup_d W_{\leq d}$ has the structure of an ind-variety for the filtration $V \times W = \bigcup_d V_{\leq d} \times W_{\leq d}$.

Definition 2.2. An *ind-group* is a group G which is an ind-variety such that the multiplication $G \times G \rightarrow G$ and inversion $G \rightarrow G$ maps are morphisms of ind-varieties.

If G is an abstract group, we denote by $D(G) = D^1(G)$ its (first) derived subgroup. It is the subgroup generated by all commutators $[g, h] := ghg^{-1}h^{-1}$, $g, h \in G$. The n -th derived subgroup of G is then defined inductively by $D^n(G) = D^1(D^{n-1}(G))$ for $n \geq 1$, where by definition $D^0(G) = G$. A group G is called *solvable* if $D^n(G) = \{1\}$ for some integer $n \geq 0$. Furthermore, the smallest such integer n is called the *derived length* of G .

For later use, we state (and prove) the following results which are well-known for algebraic groups and which extend straightforwardly to ind-groups.

Lemma 2.3. *Let H be a subgroup of an ind-group G . Then, the following assertions hold.*

- (1) *The closure \overline{H} of H is again a subgroup of G .*
- (2) *We have $D(\overline{H}) \subseteq \overline{D(H)}$.*
- (3) *If H is solvable, then \overline{H} is solvable too.*

Proof. (1). The proof for algebraic groups given in [Hum75, Proposition 7.4A, page 54] directly applies to ind-groups. This proof being very short, we give it here. Inversion being a homeomorphism, we get $(\overline{H})^{-1} = \overline{H^{-1}} = \overline{H}$. Similarly, left translation by an element x of H being a homeomorphism, we get $x\overline{H} = \overline{xH} = \overline{H}$, i.e. $H\overline{H} \subseteq \overline{H}$. In turn, right translation by an element x of \overline{H} being a homeomorphism, we get $\overline{H}x = \overline{Hx} \subseteq \overline{H\overline{H}} \subseteq \overline{H} = \overline{H}$. This says that \overline{H} is a subgroup.

(2). Fix an element y of H . The map $\varphi: G \rightarrow G$, $x \mapsto [x, y] = xyx^{-1}y^{-1}$ being an ind-morphism, it is in particular continuous. Since H is obviously contained in $\varphi^{-1}(\overline{D(H)})$, we get $\overline{H} \subseteq \varphi^{-1}(\overline{D(H)})$. Consequently, we have proven that

$$\forall x \in \overline{H}, \forall y \in H, [x, y] \in \overline{D(H)}.$$

In turn (and analogously), for each fixed element x of \overline{H} , the map $\psi: G \rightarrow G$, $y \mapsto [x, y]$ is continuous. Since H is included into $\psi^{-1}(\overline{D(H)})$, we get $\overline{H} \subseteq \psi^{-1}(\overline{D(H)})$ and thus

$$\forall x, y \in \overline{H}, [x, y] \in \overline{D(H)}.$$

This implies the desired inclusion.

(3). If H is solvable, it admits a sequence of subgroups such that

$$H = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_n = \{1\} \quad \text{and} \quad D(H_i) \subseteq H_{i+1} \text{ for each } i.$$

This yields $\overline{H} = \overline{H_0} \supseteq \overline{H_1} \supseteq \cdots \supseteq \overline{H_n} = \{1\}$ and by (2) we get $D(\overline{H_i}) \subseteq \overline{D(H_i)} \subseteq \overline{H_{i+1}}$ for each i . \square

2.2. Automorphisms of the affine n -space. As usual, given an endomorphism $f \in \text{End}(\mathbb{A}_{\mathbf{k}}^n)$, we denote by f^* the corresponding endomorphism of the algebra of regular functions $\mathcal{O}(\mathbb{A}_{\mathbf{k}}^n) = \mathbf{k}[x_1, \dots, x_n]$. Note that every endomorphism $f \in \text{End}(\mathbb{A}_{\mathbf{k}}^n)$ is uniquely determined by the polynomials $f_i = f^*(x_i)$, $1 \leq i \leq n$.

In the sequel, we identify the set $\mathcal{E}_n(\mathbf{k}) := \text{End}(\mathbb{A}_{\mathbf{k}}^n)$ with $(\mathbf{k}[x_1, \dots, x_n])^n$. We thus simply denote by $f = (f_1, \dots, f_n)$ the element of $\mathcal{E}_n(\mathbf{k})$ whose corresponding endomorphism f^* is given by

$$f^*: \mathcal{O}(\mathbb{A}_{\mathbf{k}}^n) \rightarrow \mathcal{O}(\mathbb{A}_{\mathbf{k}}^n), \quad P(x_1, \dots, x_n) \mapsto P \circ f = P(f_1, \dots, f_n).$$

The composition $g \circ f$ of two endomorphisms $f = (f_1, \dots, f_n)$ and $g = (g_1, \dots, g_n)$ is equal to

$$g \circ f = (g_1(f_1, \dots, f_n), \dots, g_n(f_1, \dots, f_n)).$$

Note that for each nonnegative integer d , the following set is naturally an affine space (and therefore an algebraic variety!)

$$\mathbf{k}[x_1, \dots, x_n]_{\leq d} := \{P \in \mathbf{k}[x_1, \dots, x_n], \deg P \leq d\}.$$

If $f = (f_1, \dots, f_n) \in \mathcal{E}_n(\mathbf{k})$, we set $\deg f := \max_i \{\deg f_i\}$ and define

$$\mathcal{E}_n(\mathbf{k})_{\leq d} := \{f \in \mathcal{E}_n(\mathbf{k}), \deg f \leq d\}.$$

The equality $\mathcal{E}_n(\mathbf{k})_{\leq d} = (\mathbf{k}[x_1, \dots, x_n]_{\leq d})^n$ shows that $\mathcal{E}_n(\mathbf{k})_{\leq d}$ is naturally an affine space. Moreover, the filtration $\mathcal{E}_n(\mathbf{k}) = \bigcup_d \mathcal{E}_n(\mathbf{k})_{\leq d}$ defines a structure of ind-variety on $\mathcal{E}_n(\mathbf{k})$.

We denote by $\mathcal{G}_n(\mathbf{k}) = \text{Aut}(\mathbb{A}_{\mathbf{k}}^n)$ the automorphism group of $\mathbb{A}_{\mathbf{k}}^n$. The next result allows us to endow $\mathcal{G}_n(\mathbf{k})$ with the structure of an ind-variety.

Lemma 2.4. *Denote by $\mathcal{C}_n(\mathbf{k})$, resp. $\mathcal{J}_n(\mathbf{k})$, the set of elements f in $\mathcal{E}_n(\mathbf{k})$ whose Jacobian determinant $\text{Jac}(f)$ is a constant, resp. a nonzero constant. Then, the following assertions hold:*

- (1) *The set $\mathcal{C}_n(\mathbf{k})$ is closed in $\mathcal{E}_n(\mathbf{k})$.*
- (2) *The set $\mathcal{J}_n(\mathbf{k})$ is open in $\mathcal{C}_n(\mathbf{k})$.*
- (3) *The set $\mathcal{G}_n(\mathbf{k})$ is closed in $\mathcal{J}_n(\mathbf{k})$.*

Proof. (1). Since $\deg(\text{Jac}(f)) \leq n(\deg(f) - 1)$, the map $\text{Jac}: \mathcal{E}_n(\mathbf{k}) \rightarrow \mathbf{k}[x_1, \dots, x_n]$ is an ind-morphism. By definition, $\mathcal{C}_n(\mathbf{k})$ is the preimage of the set \mathbf{k} which is closed in $\mathbf{k}[x_1, \dots, x_n]$.

(2). The Jacobian morphism induces a morphism $\varphi: \mathcal{C}_n(\mathbf{k}) \rightarrow \mathbf{k}$, $f \mapsto \text{Jac}(f)$. By definition, $\mathcal{J}_n(\mathbf{k})$ is the preimage of the set \mathbf{k}^* which is open in \mathbf{k} .

(3). Set $\mathcal{J}_{n,0} := \{f \in \mathcal{J}_n(\mathbf{k}), f(0) = 0\}$. Every element $f \in \mathcal{J}_{n,0}$ admits a formal inverse for the composition (see e.g. [vdE00, Theorem 1.1.2]), i.e. a formal power series $g = \sum_{d \geq 1} g_d$, where each $g_d = (g_{d,1}, \dots, g_{d,n})$ is a d -homogeneous element of $\mathcal{E}_n(\mathbf{k})$, meaning that $g_{d,1}, \dots, g_{d,n}$ are d -homogeneous polynomials in $\mathbf{k}[x_1, \dots, x_n]$ such that

$$f \circ g = g \circ f = (x_1, \dots, x_n) \quad (\text{as formal power series}).$$

Furthermore, for each d , the map $\psi_d: \mathcal{J}_{n,0} \rightarrow \mathcal{E}_n(\mathbf{k})$ sending f onto g_d is a morphism because each coefficient of every component of g_d can be expressed as a polynomial in the coefficients of the components of f and in the inverse $(\text{Jac } f)^{-1}$ of the polynomial $\text{Jac } f$. Recall furthermore (see [BCW82, Theorem 1.5]) that every automorphism $f \in \mathcal{G}_n(\mathbf{k})$ satisfies

$$(5) \quad \deg(f^{-1}) \leq (\deg f)^{n-1}.$$

Therefore, an element $f \in \mathcal{J}_n(\mathbf{k})_{\leq d}$ is an automorphism if and if $\tilde{f} := f - f(0)$ is an automorphism. This amounts to saying that f is an automorphism if and only if $\psi_e(\tilde{f}) = 0$ for all integers $e > d^{n-1}$. These conditions being closed, we have proven that $\mathcal{G}_n(\mathbf{k})_{\leq d}$ is closed in $\mathcal{J}_n(\mathbf{k})_{\leq d}$ for each d , i.e. that $\mathcal{G}_n(\mathbf{k})$ is closed in $\mathcal{J}_n(\mathbf{k})$. Note that when the field \mathbf{k} has characteristic zero, the Jacobian conjecture (see for example [BCW82, vdE00]) asserts that the equality $\mathcal{G}_n(\mathbf{k}) = \mathcal{J}_n(\mathbf{k})$ actually holds. \square

Since the multiplication $\mathcal{G}_n(\mathbf{k}) \times \mathcal{G}_n(\mathbf{k}) \rightarrow \mathcal{G}_n(\mathbf{k})$ and inversion $\mathcal{G}_n(\mathbf{k}) \rightarrow \mathcal{G}_n(\mathbf{k})$ maps are morphisms (for the inversion, this again relies on the fundamental inequality (5)), we obtain that $\mathcal{G}_n(\mathbf{k})$ is an ind-group.

3. BOREL SUBGROUPS

Throughout this section, we work over the field $\mathbf{k} = \mathbb{C}$ of complex numbers.

Note that the affine subgroup

$$\mathcal{A}_n = \{f = (f_1, \dots, f_n) \in \mathcal{G}_n(\mathbb{C}) \mid \deg(f_i) = 1 \text{ for all } i = 1 \dots n\}$$

and the Jonquières (or triangular) subgroup

$$\begin{aligned} \mathcal{B}_n &= \{f = (f_1, \dots, f_n) \in \mathcal{G}_n(\mathbb{C}) \mid \forall i, f_i = a_i x_i + p_i, a_i \in \mathbb{C}^*, p_i \in \mathbb{C}[x_{i+1}, \dots, x_n]\} \\ &= \{f = (f_1, \dots, f_n) \in \mathcal{G}_n(\mathbb{C}) \mid \forall i, f_i \in \mathbb{C}[x_i, \dots, x_n]\} \end{aligned}$$

are both closed in $\mathcal{G}_n(\mathbb{C})$.

It is well known that the group $\mathcal{G}_n(\mathbb{C})$ is connected (see e.g. [Sha81, proof of Lemma 4], [Kal92, Proposition 2] or [Pop14, Theorem 6]). The same is true for \mathcal{B}_n .

Lemma 3.1. *The groups $\mathcal{G}_n(\mathbb{C}) = \text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ and \mathcal{B}_n are connected.*

Proof. We say that a variety V is curve-connected if for all points $x, y \in V$, there exists a morphism $\varphi: C \rightarrow V$, where C is a connected curve (not necessarily irreducible) such that x and y both belong to the image of φ . The same definition applies to ind-varieties.

We prove that $\mathcal{G}_n(\mathbb{C})$ and \mathcal{B}_n are curve-connected. Let f be an element in $\mathcal{G}_n(\mathbb{C})$. We first consider the morphism $\alpha: \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathcal{G}_n(\mathbb{C})$ defined by

$$\alpha(t) = f - tf(0, \dots, 0)$$

which is contained in \mathcal{B}_n if f is triangular. Note that $\alpha(0) = f$ and that the automorphism $\tilde{f} := \alpha(1)$ fixes the origin of $\mathbb{A}_{\mathbb{C}}^n$.

Therefore the morphism $\beta: \mathbb{A}_{\mathbb{C}}^1 \setminus \{0\} \rightarrow \mathcal{G}_n(\mathbb{C})$, $t \mapsto (t^{-1} \cdot \text{id}_{\mathbb{A}_{\mathbb{C}}^n}) \circ \tilde{f} \circ (t \cdot \text{id}_{\mathbb{A}_{\mathbb{C}}^n})$ extends to a morphism $\beta: \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathcal{G}_n(\mathbb{C})$ (with values in \mathcal{B}_n if f , thus \tilde{f} , is triangular) such that $\beta(1) = \tilde{f}$ and such that $\beta(0)$ is a linear map, namely the linear part of \tilde{f} . This concludes the proof since $\text{GL}_n(\mathbb{C})$ (resp. the set of all invertible upper triangular matrices) is curve-connected. \square

Recall that the subgroup of upper triangular matrices in $\text{GL}_n(\mathbb{C})$ is solvable and has derived length $\lceil \log_2(n) \rceil + 1$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to the real number x (see e.g. [Weh73, page 16]). In contrast, we have the following result.

Lemma 3.2. *The group \mathcal{B}_n is solvable of derived length $n + 1$.*

Proof. For each integer $k \in \{0, \dots, n\}$, denote by U_k the subgroup of \mathcal{B}_n whose elements are of the form $f = (f_1, \dots, f_n)$ where $f_i = x_i$ for all $i > k$ and $f_i = x_i + p_i$ with $p_i \in \mathbb{C}[x_{i+1}, \dots, x_n]$ for all $i \leq k$. We will prove $D(\mathcal{B}_n) = U_n$ and $D^j(U_n) = U_{n-j}$ for all $j \in \{0, \dots, n\}$.

For this, we consider the dilatation $d(j, \lambda_j)$ and the elementary automorphism $e(j, q_j)$ which are defined for every integer $j \in \{1, \dots, n\}$, every nonzero constant $\lambda_j \in \mathbb{C}^*$ and every polynomial $q_j \in \mathbb{C}[x_{j+1}, \dots, x_n]$ by

$$d(j, \lambda_j) = (g_1, \dots, g_n) \quad \text{and} \quad e(j, q_j) = (h_1, \dots, h_n),$$

where $g_j = \lambda_j x_j$, $h_j = x_j + q_j$ and $g_i = h_i = x_i$ for $i \neq j$. Note that an element $f \in U_k$ as above is equal to

$$f = e(k, p_k) \circ \dots \circ e(2, p_2) \circ e(1, p_1).$$

In particular, this tells us that U_k is generated by the elements $e(j, q_j)$, $j \leq k$, $q_j \in \mathbb{C}[x_{j+1}, \dots, x_n]$.

The inclusion $D(\mathcal{B}_n) \subseteq U_n$ is straightforward and left to the reader. The converse inclusion $U_n \subseteq D(\mathcal{B}_n)$ follows from the equality

$$[e(j, q_j), d(j, \lambda_j)] = e(j, (1 - \lambda_j)q_j).$$

Finally, we prove $D^j(U_n) = U_{n-j}$ by proving that the equality $D(U_{k+1}) = U_k$ holds for all $k \in \{0, \dots, n-1\}$. The inclusion $D(U_{k+1}) \subseteq U_k$ is straightforward and left to the reader. To prove the converse inclusion, let us introduce the map $\Delta_i: \mathbb{C}[x_i, \dots, x_n] \rightarrow \mathbb{C}[x_i, \dots, x_n]$, $q \mapsto q(x_i, \dots, x_n) - q(x_i - 1, x_{i+1}, \dots, x_n)$. Note that Δ_i is surjective and that

$$[e(j, q_j), e(j+1, 1)] = e(j, \Delta_{j+1}(q_j))$$

for all $j \in \{1, \dots, n-1\}$ and all $q_j \in \mathbb{C}[x_{j+1}, \dots, x_n]$. This implies $U_k \subseteq D(U_{k+1})$ and concludes the proof. \square

3.1. Triangular automorphisms form a Borel subgroup. In this section, we prove the first two statements of the theorem 1 from the introduction. For this, we need the following result.

Proposition 3.3. *Let $n \geq 2$ be an integer. If a closed subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ strictly contains \mathcal{B}_n , then it also contains at least one linear automorphism that is not triangular.*

Proof. Let H be a closed subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ strictly containing \mathcal{B}_n . We first prove that H contains an automorphism whose linear part is not triangular. Let $f = (f_1, \dots, f_n)$ be an element in $H \setminus \mathcal{B}_n$. Then, there exists at least one component f_i of f that depends on an indeterminate x_j with $j < i$, i.e. such that $\frac{\partial f_i}{\partial x_j} \neq 0$. Now, choose $c = (c_1, \dots, c_n) \in \mathbb{A}_{\mathbb{C}}^n$ such that $\frac{\partial f_i}{\partial x_j}(c) \neq 0$ and consider the translation $t_c := (x_1 + c_1, \dots, x_n + c_n) \in \mathcal{B}_n$. Since

$$f_i(x + c) = f_i(c) + \sum_k \frac{\partial f_i}{\partial x_k}(c)x_k + (\text{terms of higher order}),$$

the linear part l of $f \circ t_c$ is not triangular because it corresponds to the (non-triangular) invertible matrix $\left(\frac{\partial f_i}{\partial x_k}(c)\right)_{ik}$. Composing on the left hand side by another translation t' , we obtain an element $g := t' \circ f \circ t \in H$ which fixes the origin of $\mathbb{A}_{\mathbb{C}}^n$ and whose linear part is again l .

For every $\varepsilon \in \mathbb{C}^*$, set $h_\varepsilon := (\varepsilon x_1, \dots, \varepsilon x_n) \in \mathcal{B}_n$. We can finally conclude by noting that

$$\lim_{\varepsilon \rightarrow 0} h_\varepsilon^{-1} \circ g \circ h_\varepsilon = l \in H,$$

where the limit means that the ind-morphism $\varphi: \mathbb{C}^* \rightarrow \text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$, $\varepsilon \mapsto h_\varepsilon^{-1} \circ g \circ h_\varepsilon$ extends to a morphism $\psi: \mathbb{C} \rightarrow \text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ such that $\psi(0) = l$. Since we have $\psi(\varepsilon) \in H$ for each $\varepsilon \in \mathbb{C}^*$, it is clear that $\psi(0)$ must also belong to H . Indeed, note that the set $\{\varepsilon \in \mathbb{C}, \psi(\varepsilon) \in H\}$ is Zariski-closed in \mathbb{C} . \square

Proposition 3.4. *Let $n \geq 2$ be an integer. Then, the Jonquières group \mathcal{B}_n is maximal among all solvable subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$.*

Proof. Suppose by contradiction that there exists a solvable subgroup H of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ that strictly contains \mathcal{B}_n . Up to replacing H by its closure \overline{H} (see Lemma 2.3), we may assume that H is closed. By Proposition 3.3, the group $H \cap \mathcal{A}_n$ strictly contains $\mathcal{B}_n \cap \mathcal{A}_n$. But since $\mathcal{B}_n \cap \mathcal{A}_n$ is a Borel subgroup of \mathcal{A}_n , this prove that $H \cap \mathcal{A}_n$ is not solvable, thus that H itself is not solvable. Notice that we have used the fact that every Borel subgroup of a connected linear algebraic group is a maximal solvable subgroup. Indeed, every parabolic subgroup (i.e. a subgroup containing a Borel subgroup) of a connected linear algebraic group is necessarily closed and connected. See e.g. [Hum75, Corollary B of Theorem (23.1), page 143]. \square

In dimension two, we establish another maximality property of the triangular subgroup which is actually stronger than the above one (see Remark 3.7 below).

Proposition 3.5. *The Jonquières group \mathcal{B}_2 is maximal among the closed subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$.*

Proof. Let H be a closed subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ strictly containing \mathcal{B}_2 . By Proposition 3.3 above, H contains a linear automorphism which is not triangular. This implies that H contains all linear automorphisms, hence \mathcal{A}_2 , and it is therefore equal to $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$. Recall indeed that the subgroup $B_2 = \mathcal{B}_2 \cap \text{GL}_2(\mathbb{C})$ of invertible upper triangular matrices is a maximal subgroup of $\text{GL}_2(\mathbb{C})$, since the Bruhat decomposition expresses $\text{GL}_2(\mathbb{C})$ as the disjoint union of two double cosets of B_2 , which are namely B_2 and $B_2 \circ f \circ B_2$, where f is any element of $\text{GL}_2(\mathbb{C}) \setminus B_2$. \square

Remark 3.6. Proposition 3.5 can not be generalized to higher dimension, since \mathcal{B}_n is strictly contained into the (closed) subgroup of automorphisms of the form $f = (f_1, \dots, f_n)$ such that $f_n = a_n x_n + b_n$ for some $a_n, b_n \in \mathbb{C}$ with $a_n \neq 0$.

Remark 3.7. Proposition 3.5 implies Proposition 3.4 for $n = 2$. Indeed, suppose that \mathcal{B}_2 is strictly included into some solvable subgroup H of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$. Up to replacing H by \overline{H} (see Lemma 2.3), we may further assume that H is closed. By Proposition 3.5, we would thus get that $H = \text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$. But this is a contradiction because the group $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ is obviously not solvable, since it contains the linear group $\text{GL}(2, \mathbb{C})$ which is not solvable.

By Proposition 3.4, we can say that the triangular group \mathcal{B}_n is a Borel subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$. This was already observed, in the case $n = 2$ only, by Berest, Eshmatov and Eshmatov in the nice paper [BEE16] in which they obtained the following strong results. (In [BEE16], these results are stated for the group $\text{SAut}(\mathbb{A}_{\mathbb{C}}^2)$ of polynomial automorphisms of $\mathbb{A}_{\mathbb{C}}^2$ of Jacobian determinant 1, but all the proofs remain valid for $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$.)

Theorem 3.8 ([BEE16]). (1) *All Borel subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ are conjugate to \mathcal{B}_2 .*
 (2) *Every connected solvable subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ is conjugate to a subgroup of \mathcal{B}_2 .*

Recall that there exist, for every $n \geq 3$, connected solvable subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ that are not conjugate to subgroups of \mathcal{B}_n [Bas84, Pop87]. Hence, the second statement of the above theorem does not hold for $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$, $n \geq 3$. Similarly, we believe that not all Borel subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ are conjugate to \mathcal{B}_n if $n \geq 3$. This would be clearly the case, if we knew that the following question has a positive answer.

Question 3.9. *Is every connected solvable subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$, $n \geq 3$, contained into a maximal connected solvable subgroup?*

The natural strategy to attack the above question would be to apply Zorn's lemma, as we do in the proof of the following general proposition.

Proposition 3.10. *Let G be a group endowed with a topology. Suppose that there exists an integer $c > 0$ such that every solvable subgroup of G is of derived length at most c . Then, every solvable (resp. connected solvable) subgroup of G is contained into a maximal solvable (resp. maximal connected solvable) subgroup.*

Proof. Let H be a solvable (resp. connected solvable) subgroup of G . Denote by \mathcal{F} the set of solvable (resp. connected solvable) subgroups of G that contain H . Our hypothesis, on the existence of the bound c , implies that the poset (\mathcal{F}, \subseteq) is inductive. Indeed, if $(H_i)_{i \in I}$ is a chain in \mathcal{F} , i.e. a totally ordered family of \mathcal{F} , then the group $\bigcup_i H_i$ is solvable, because we have that

$$D^j\left(\bigcup_i H_i\right) = \bigcup_i D^j(H_i)$$

for each integer $j \geq 0$. Moreover, if all H_i are connected, then so is their union. Thus, \mathcal{F} is inductive and we can conclude by Zorn's lemma. \square

Remark 3.11. Proposition 3.10 does not require any compatibility conditions between the group structure and the topology on G . Let us moreover recall that an algebraic group (and all the more an ind-group) is in general not a topological group.

We are now left with another concrete question.

Definition 3.12. Let G be a group. We set

$$\psi(G) := \sup\{l(H) \mid H \text{ is a solvable subgroup of } G\} \in \mathbb{N} \cup \{+\infty\},$$

where $l(H)$ denotes the derived length of H .

Question 3.13. *Is $\psi(\text{Aut}(\mathbb{A}_{\mathbb{C}}^n))$ finite?*

Recall that $\psi(\text{GL}(n, \mathbb{C}))$ is finite. This classical result has been first established in 1937 by Zassenhaus [Zas37, Satz 7] (see also [Mal56]). More recently, Martelo and Ribón have proved in [MR14] that $\psi((\mathcal{O}_{\text{ana}}(\mathbb{C}^n), 0)) < +\infty$, where $(\mathcal{O}_{\text{ana}}(\mathbb{C}^n), 0)$ denotes the group of germs of analytic diffeomorphisms defined in a neighbourhood of the origin of \mathbb{C}^n .

Our next result answers Question 3.13 in the case $n = 2$.

Proposition 3.14. *We have $\psi(\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)) = 5$.*

Proof. The proof relies on a precise description of all subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$, due to Lamy, that we will recall below. Using this description, the equality $\psi(\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)) = 5$ directly follows from the equality $\psi(\mathcal{A}_2) = 5$ that we will establish in the next section (see Proposition 3.16). The description of all subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ given by Lamy uses the amalgamated structure of this group, generally known as the theorem of Jung, van der Kulk and Nagata: The group $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ is the amalgamated product of its subgroups \mathcal{A}_2 and \mathcal{B}_2 over their intersection

$$\text{Aut}(\mathbb{A}_{\mathbb{C}}^2) = \mathcal{A}_2 *_{\mathcal{A}_2 \cap \mathcal{B}_2} \mathcal{B}_2.$$

In the discussion below, we will use the Bass-Serre tree associated to this amalgamated structure. We refer the reader to [Ser03] for details on Bass-Serre trees in full generality and to [Lam01] for details on the particular tree associated to the above amalgamated structure. That latter tree is the tree whose vertices are the left cosets $g \circ \mathcal{A}_2$ and $h \circ \mathcal{B}_2$, $g, h \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$. Two vertices $g \circ \mathcal{A}_2$ and $h \circ \mathcal{B}_2$ are related by an edge if and only if there exists an element $k \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ such that $g \circ \mathcal{A}_2 = k \circ \mathcal{A}_2$ and $h \circ \mathcal{B}_2 = k \circ \mathcal{B}_2$, i.e. if and only if $g^{-1} \circ h \in \mathcal{A}_2 \circ \mathcal{B}_2$. The group $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ acts on the Bass-Serre tree by left translation: For all $g, h \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$, we set $g.(h \circ \mathcal{A}_2) = (g \circ h) \circ \mathcal{A}_2$ and $g.(h \circ \mathcal{B}_2) = (g \circ h) \circ \mathcal{B}_2$. Each element of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ satisfies one property of the following alternative:

- (1) It is triangularizable, i.e. conjugate to an element of \mathcal{B}_2 . This is the case where the automorphism fixes at least one point on the Bass-Serre tree.
- (2) It is a Hénon automorphism, i.e. it is conjugate to an element of the form

$$g = a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k,$$

where $k \geq 1$, each a_i belongs to $\mathcal{A}_2 \setminus \mathcal{B}_2$ and each b_i belongs to $\mathcal{B}_2 \setminus \mathcal{A}_2$. This is the case where the automorphism acts without fixed points, but preserves a (unique) geodesic of the Bass-Serre tree on which it acts as a translation of length $2k$.

Furthermore, according to [Lam01, Theorem 2.4], every subgroup H of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ satisfies one and only one of the following assertions:

- (1) It is conjugate to a subgroup of \mathcal{A}_2 or of \mathcal{B}_2 .

- (2) Every element of H is triangularizable and H is not conjugate to a subgroup of \mathcal{A}_2 or of \mathcal{B}_2 . In that case, H is Abelian.
- (3) The group H contains some Hénon automorphisms (i.e. non triangularizable automorphisms) and all those have the same geodesic on the Bass-Serre tree. The group H is then solvable.
- (4) The group H contains two Hénon automorphisms having different geodesics. Then, H contains a free group with two generators.

Let H be now a solvable subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$. If we are in case (1), then we may assume that H is a subgroup of \mathcal{A}_2 or of \mathcal{B}_2 . Since $\psi(\mathcal{A}_2) = 5$ and $\psi(\mathcal{B}_2) = 3$ (the group \mathcal{B}_2 being solvable of derived length 3), this settles this case. In case (2), H is Abelian hence of derived length at most 1. In case (3), there exists a geodesic Γ which is globally fixed by every element of H . Therefore, we may assume without restriction that

$$H = \{f \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^2), f(\Gamma) = \Gamma\}.$$

Note that $D^2(H)$ is included into the group K that fixes pointwise the geodesic Γ . Up to conjugation, we may assume that Γ contains the vertex \mathcal{B}_2 , i.e. that K is included into \mathcal{B}_2 . By [Lam01, Proposition 3.3], each element of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ fixing an unbounded set of the Bass-Serre tree has finite order. If $f, g \in K$, their commutator is of the form $(x + p(y), y + c)$. This latter automorphism being of finite order, it must be equal to the identity, showing that K is Abelian. Therefore, we get $D^3(H) = \{1\}$.

Finally, we cannot be in case (4), because a free group with two generators is not solvable. \square

From Propositions 3.10 and 3.14, we get at once the following result, which also follows from Theorem 3.8 above.

Corollary 3.15. *Every solvable connected subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ is contained into a Borel subgroup.*

3.2. Proof of the equality $\psi(\mathcal{A}_2) = 5$. Recall that Newman [New72] has computed the exact value $\psi(\text{GL}(n, \mathbb{C}))$ for all n . It turns out that $\psi(\text{GL}(n, \mathbb{C}))$ is equivalent to $5 \log_3(n)$ as n goes to infinity (see [Weh73, Theorem 3.10]). Let us give a few particular values for $\psi(\text{GL}(n, \mathbb{C}))$ taken from [New72].

n	1	2	3	4	5	6	7	8	9	10	18	26	34	66	74
$\psi(\text{GL}(n, \mathbb{C}))$	1	4	5	6	7	7	7	8	9	10	11	12	13	14	15

We now consider the affine group \mathcal{A}_n . On the one hand, observe that \mathcal{A}_n is isomorphic to a subgroup of $\text{GL}(n+1, \mathbb{C})$. Hence, $\psi(\mathcal{A}_n) \leq \psi(\text{GL}(n+1, \mathbb{C}))$. On the other hand, we have the short exact sequence

$$1 \rightarrow \mathbb{C}^n \rightarrow \mathcal{A}_n \xrightarrow{L} \text{GL}_n(\mathbb{C}) \rightarrow 1,$$

where $L: \mathcal{A}_n \rightarrow \text{GL}(n, \mathbb{C})$ is the natural morphism sending an affine transformation to its linear part. Thus, if H is a solvable subgroup of \mathcal{A}_n , we have a short exact sequence

$$1 \rightarrow H \cap (\mathbb{C}^n) \rightarrow H \xrightarrow{L} L(H) \rightarrow 1.$$

Since $L(H)$ is solvable of derived length at most $\psi(\mathrm{GL}_n(\mathbb{C}))$ and since $H \cap (\mathbb{C}^n)$ is Abelian, this implies that $l(H) \leq \psi(\mathrm{GL}_n(\mathbb{C})) + 1$. Therefore, we have proved the general formula

$$\psi(\mathrm{GL}_n(\mathbb{C})) \leq \psi(\mathcal{A}_n) \leq \min\{\psi(\mathrm{GL}(n, \mathbb{C})) + 1, \psi(\mathrm{GL}(n + 1, \mathbb{C}))\}.$$

For $n = 2$, this yields $\psi(\mathcal{A}_2) = 4$ or 5 . We shall now prove that \mathcal{A}_2 contains solvable subgroups of derived length 5 (see Lemma 3.19 below), hence the following desired result.

Proposition 3.16. *The maximal derived length of a solvable subgroup of the affine group \mathcal{A}_2 is 5 , i.e. we have $\psi(\mathcal{A}_2) = 5$.*

As explained above, it still remains to provide an example of a solvable subgroup of \mathcal{A}_2 of derived length 5 . In that purpose, recall that the group $\mathrm{PSL}(2, \mathbb{C})$ contains a subgroup isomorphic to the symmetric group S_4 and that all such subgroups are conjugate (see for example [Bea10]).

Definition 3.17. The *binary octahedral group* $2\mathrm{O}$ is the pre-image of the symmetric group S_4 by the $(2 : 1)$ -cover $\mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{PSL}(2, \mathbb{C})$.

The following result is also well-known.

Lemma 3.18. *The derived length of the binary octahedral group $G = 2\mathrm{O}$ is 4 .*

Proof. Using the short exact sequence

$$0 \rightarrow \{\pm I\} \rightarrow G \xrightarrow{\pi} S_4 \rightarrow 0,$$

we get $\pi(D^2G) = D^2(\pi(G)) = D^2(S_4) = V_4$, where $V_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ is the Klein group. One could also easily check that $\pi^{-1}(V_4)$ is isomorphic to the quaternion group Q_8 . The equality $\pi(D^2G) = V_4$ is then sufficient for showing that $D^2G = \pi^{-1}(V_4)$. Indeed, if D^2G was a strict subgroup of $\pi^{-1}(V_4) \simeq Q_8$, it would be cyclic, hence $\pi(D^2G) = V_4$ would be cyclic too. A contradiction. Since $D^2G \simeq Q_8$ has derived length 2 , this shows us that the derived length of G is $2 + 2 = 4$. \square

Lemma 3.19. *Consider the pre-image $L^{-1}(G) \simeq G \times \mathbb{C}^2$ of the binary octahedral group $G := 2\mathrm{O} \subseteq \mathrm{SL}(2, \mathbb{C})$ by the natural morphism $L: \mathcal{A}_2 \rightarrow \mathrm{GL}(2, \mathbb{C})$ sending an affine transformation onto its linear part. Then, the derived length of $L^{-1}(G)$ is equal to 5 .*

Proof. By Lemma 3.18, the derived length of G is 4 . The short exact sequence

$$1 \rightarrow \mathbb{C}^2 \rightarrow G \times \mathbb{C}^2 \rightarrow G \rightarrow 1$$

implies that the derived length of $G \times \mathbb{C}^2$ is at most $4 + 1 = 5$. Moreover, the strictly decreasing sequence $G = D^0(G) > D^1(G) > D^2(G) > D^3(G) > D^4(G) = 1$ shows that the group $D^2(G)$ is non-Abelian and in particular non-cyclic. By Lemma 3.20 below, we thus have $D^i(G \times \mathbb{C}^2) = D^i(G) \times \mathbb{C}^2$ for every $i \leq 3$. But since $D^3(G)$ is non-trivial, the group $D^3(G \times \mathbb{C}^2) = D^3(G) \times \mathbb{C}^2$ strictly contains the subgroup $(\mathbb{C}^2, +)$ of translations and cannot be Abelian, because the group \mathbb{C}^2 is its own centralizer in \mathcal{A}_2 . Finally, we get $D^4(G \times \mathbb{C}^2) \neq 1$, proving that the derived length of $G \times \mathbb{C}^2$ is indeed 5 . \square

Lemma 3.20. *Let H be a finite non-cyclic subgroup of $\mathrm{GL}(2, \mathbb{C})$. Then the derived subgroup of $L^{-1}(H) = H \times \mathbb{C}^2 \subseteq \mathcal{A}_2$ is the group $D(H) \times \mathbb{C}^2$.*

Proof. Set $K := D(H \ltimes \mathbb{C}^2) \cap \mathbb{C}^2$. Note that K contains the commutator $[\text{id} + v, h]$ for all $v \in \mathbb{C}^2$, $h \in H$, i.e. it contains all elements $h \cdot v - v$. It is enough to show that these vectors generate \mathbb{C}^2 . Indeed, it would then imply that there exist h_1, v_1, h_2, v_2 such that the vectors $h_1 \cdot v_1 - v_1$ and $h_2 \cdot v_2 - v_2$ are linearly independent. But then, K would also contain the vectors $h_1 \cdot (\lambda_1 v_1) - (\lambda_1 v_1) + h_2 \cdot (\lambda_2 v_2) - \lambda_2 v_2$ for any $\lambda_1, \lambda_2 \in \mathbb{C}$, proving that $K = \mathbb{C}^2$. Therefore, let us assume by contradiction that there exists a non-zero vector $w \in \mathbb{C}^2$ such that $h \cdot v - v$ is a multiple of w for all $h \in H$, $v \in \mathbb{C}^2$. Take $w' \in \mathbb{C}^2$ such that (w, w') is a basis of \mathbb{C}^2 . In this basis, any element of H admits a matrix of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

Therefore, by the theory of representations of finite group, we may assume, up to conjugation, that each element of H admits a matrix of the form

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

This would imply that H is isomorphic to a finite subgroup of \mathbb{C}^* , hence that it is cyclic. A contradiction. \square

3.3. An ind-group with nonconjugate Borel subgroups. In this section, we consider the subgroup $\text{Aut}_z(\mathbb{A}_{\mathbb{C}}^3)$ of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^3)$ of all automorphisms $f = (f_1, f_2, z)$ fixing the last coordinate of $\mathbb{A}_{\mathbb{C}}^3 = \text{Spec}(\mathbb{C}[x, y, z])$. Since it is clearly a closed subgroup, it is also an ind-group. Note that $\text{Aut}_z(\mathbb{A}_{\mathbb{C}}^3)$ is naturally isomorphic to a subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}(z)}^2)$. In its turn, the field $\mathbb{C}(z)$ can be embedded into the field \mathbb{C} , so that the group $\text{Aut}(\mathbb{A}_{\mathbb{C}(z)}^2)$ is isomorphic to a subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$. Therefore, by Proposition 3.14, we get

$$\psi(\text{Aut}_z(\mathbb{A}_{\mathbb{C}}^3)) \leq \psi(\text{Aut}(\mathbb{A}_{\mathbb{C}(z)}^2)) \leq \psi(\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)) = 5.$$

Recall moreover that $\text{Aut}_z(\mathbb{A}_{\mathbb{C}}^3)$ contains nontriangularizable additive group actions [Bas84]. Let us briefly describe the example given by Bass. Consider the following locally nilpotent derivation of $\mathbb{C}[x, y, z]$:

$$\Delta = -2y\partial_x + z\partial_y.$$

Then, the derivation $(xz + y^2)\Delta$ is again locally nilpotent. We associate it with the morphism

$$(\mathbb{C}, +) \rightarrow \text{Aut}_{\mathbb{C}}(\mathbb{C}[x, y, z]), \quad t \mapsto \exp(t(xz + y^2)\Delta).$$

The automorphism of $\mathbb{A}_{\mathbb{C}}^3$ corresponding to $\exp(t(xz + y^2)\Delta)$ is given by

$$f_t := (x - 2ty(xz + y^2) - t^2z(xz + y^2)^2, y + tz(xz + y^2), z) \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^3).$$

For $t = 1$, we get the famous Nagata automorphism. Note that the fixed point set of the corresponding $(\mathbb{C}, +)$ -action on $\mathbb{A}_{\mathbb{C}}^3$ is the hypersurface $\{xz + y^2 = 0\}$ which has an isolated singularity at the origin. On the other hand, the fixed point set of a triangular $(\mathbb{C}, +)$ -action on $\mathbb{A}_{\mathbb{C}}^3$

$$t \mapsto g_t = \exp(t(a(y, z)\partial_x + b(z)\partial_y)) \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^3)$$

is the set $\{a(y, z) = b(z) = 0\}$, which is isomorphic to a cylinder $\mathbb{A}_{\mathbb{C}}^1 \times Z$ for some variety Z . This implies that the $(\mathbb{C}, +)$ -action $t \mapsto f_t$ is not triangularizable.

By Proposition 3.10, it follows that $\text{Aut}_z(\mathbb{A}_{\mathbb{C}}^3)$ contains Borel subgroups that are not conjugate to a subgroup of the group

$$\mathcal{B}_z = \{(f_1, f_2, z) \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^3) \mid f_1 \in \mathbb{C}[x, y, z], f_2 \in \mathbb{C}[y, z]\}$$

of triangular automorphisms of $\text{Aut}_z(\mathbb{A}_{\mathbb{C}}^3)$.

Proposition 3.21. *The group \mathcal{B}_z is a Borel subgroup of $\text{Aut}_z(\mathbb{A}_{\mathbb{C}}^3)$.*

Proof. With the same proof as for Lemma 3.1, we obtain easily that \mathcal{B}_z is connected. It is also solvable, since it can be seen as a subgroup of the Jonquières subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}(z)}^2)$, which is solvable.

Now, we simply follow the proof of Proposition 3.3. Let $H \subset \text{Aut}_z(\mathbb{A}_{\mathbb{C}}^3)$ be a closed subgroup containing strictly \mathcal{B}_z and take an element f in $H \setminus \mathcal{B}_z$, i.e. an element $f = (f_1, f_2, z)$ with $f_2 \in \mathbb{C}[x, y, z] \setminus \mathbb{C}[y, z]$. Arguing as before, we can find suitable translations $t_c = (x + c_1, y + c_2, z)$ and $t_{c'} = (x + c'_1, y + c'_2, z)$ such that the automorphism $g = t_c \circ f \circ t_{c'}$ fixes the point $(0, 0, 0)$ and is of the form $g = (g_1, g_2, z)$ with $g_2 = xc(z) + yd(z) + h(x, y, z)$ for some $c(z), d(z) \in \mathbb{C}[z]$, $c(z) \neq 0$, and some polynomial $h(x, y, z)$ belonging to the ideal (x^2, xy, y^2) of $\mathbb{C}[x, y, z]$.

Conjugating this g by the automorphism $(tx, ty, z) \in H$, $t \neq 0$, and taking the limit when t goes to 0, we obtain an element of the form $(a(z)x + b(z)y, c(z)x + d(z)y, z)$ with $c(z) \neq 0$ in H . By Lemma 3.23 below, this implies that the group H is not solvable. \square

Corollary 3.22. *The ind-group $\text{Aut}_z(\mathbb{A}_{\mathbb{C}}^3)$ contains non-conjugate Borel subgroups.*

In the course of the proof of Proposition 3.21, we have used the following lemma that we prove now.

Lemma 3.23. *The subgroup $B_2(\mathbb{C}[z])$ of upper triangular matrices of $\text{GL}_2(\mathbb{C}[z])$ is a maximal solvable subgroup.*

Proof. For every $\alpha \in \mathbb{C}$, denote by $\text{ev}_{\alpha}: \text{GL}_2(\mathbb{C}[z]) \rightarrow \text{GL}_2(\mathbb{C})$ the evaluation map that associates to an element $M(z) \in \text{GL}_2(\mathbb{C}[z])$ the constant matrix $M(\alpha)$ obtained by replacing z by α . Let H be a subgroup of $\text{GL}_2(\mathbb{C}[z])$ strictly containing the group $B_2(\mathbb{C}[z])$. By definition, H contains a non-triangular matrix, i.e. a matrix of the form

$$M = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}, \text{ with } c \neq 0.$$

Choose a complex number α such that $c(\alpha) \neq 0$. Then, the group $\text{ev}_{\alpha}(H)$ contains the upper triangular constant matrices $B_2(\mathbb{C})$ and a non-triangular matrix. Therefore, $\text{ev}_{\alpha}(H) = \text{GL}_2(\mathbb{C})$ and H is not solvable. \square

Remark 3.24. By Nagao's theorem (see [Nag59] or e.g. [Ser03, Chapter II, no 1.6]), we have an amalgamated product structure

$$\text{GL}_2(\mathbb{C}[z]) = \text{GL}_2(\mathbb{C}) *_{B_2(\mathbb{C})} B_2(\mathbb{C}[z]).$$

However, contrarily to the case of $\text{Aut}(\mathbb{A}^2)$, the group $B_2(\mathbb{C}[z])$ is not a maximal closed subgroup. Indeed, for every complex number α , this group is strictly included into the group $\text{ev}_{\alpha}^{-1}(B_2(\mathbb{C}))$.

3.4. Maximal closed subgroups. In this section, we mainly focus on the following question.

Question 3.25. *What are the maximal closed subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$?*

First of all, it is easy to observe that, since the action of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ on $\mathbb{A}_{\mathbb{C}}^n$ is infinite transitive, i.e. m -transitive for all integers $m \geq 1$, the stabilizers of a finite number of points are examples of maximal closed subgroups.

Proposition 3.26. *For every finite subset Δ of $\mathbb{A}_{\mathbb{C}}^n$, $n \geq 2$, the group*

$$\text{Stab}(\Delta) = \{f \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^n), f(\Delta) = \Delta\}$$

is a maximal subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$. Furthermore, it is closed.

Proof. Let $\Delta = \{a_1, \dots, a_k\}$ be a finite subset of $\mathbb{A}_{\mathbb{C}}^n$. Let $f \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^n) \setminus \text{Stab}(\Delta)$. We will prove that $\langle \text{Stab}(\Delta), f \rangle = \text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$, where $\langle \text{Stab}(\Delta), f \rangle$ denotes the subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ that is generated by $\text{Stab}(\Delta)$ and f . We will use repetitively the well-known fact that $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ acts $2k$ -transitively on $\mathbb{A}_{\mathbb{C}}^n$.

We first observe that $\langle \text{Stab}(\Delta), f \rangle$ contains an element g such that $g(\Delta) \cap \Delta = \emptyset$. To see this, denote by $m := |\Delta \cap f(\Delta)|$ the cardinality of the set $\Delta \cap f(\Delta)$. Up to composing it by an element of $\text{Stab}(\Delta)$, we can suppose that f fixes the points a_1, \dots, a_m and maps a_{m+1}, \dots, a_k outside Δ . If $m \geq 1$, then we consider an element $\alpha \in \text{Stab}(\Delta)$ that maps the point a_m onto a_{m+1} and sends all points $f(a_{m+1}), \dots, f(a_k)$ outside the set $f^{-1}(\Delta)$. Remark that $g = f \circ \alpha \circ f$ is an element of $\langle \text{Stab}(\Delta), f \rangle$ with $|\Delta \cap g(\Delta)| < m$. By descending induction on m , we can further suppose that $|\Delta \cap g(\Delta)| = 0$ as desired.

Now, consider any $\varphi \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$. Let us prove that φ belongs to the subgroup $\langle \text{Stab}(\Delta), g \rangle$. Take an element $\beta \in \text{Stab}(\Delta)$ such that $\beta(\varphi(\Delta)) \cap g^{-1}(\Delta) = \emptyset$. Then, $g(\beta(\varphi(\Delta))) \cap \Delta = \emptyset$ and we can find an element $\gamma \in \text{Stab}(\Delta)$ such that $(\gamma \circ g \circ \beta \circ \varphi)(a_i) = g(a_i)$ for all i . We have $\varphi = \beta^{-1} \circ g^{-1} \circ \gamma^{-1} \circ g \circ \delta \in \langle \text{Stab}(\Delta), g \rangle$, where $\delta := g^{-1} \circ (\gamma \circ g \circ \beta \circ \varphi)$ is an element of $\text{Stab}(\Delta)$, proving that $\langle \text{Stab}(\Delta), g \rangle$ is equal to the whole group $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$. Therefore, the group $\text{Stab}(\Delta)$ is actually maximal in $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$. Finally, note that for each point $a \in \mathbb{A}_{\mathbb{C}}^n$ the evaluation map $\text{ev}_a: \text{Aut}(\mathbb{A}_{\mathbb{C}}^n) \rightarrow \mathbb{A}_{\mathbb{C}}^n, f \mapsto f(a)$ is an ind-morphism. Since Δ is a closed subset of $\mathbb{A}_{\mathbb{C}}^n$ the equality

$$\text{Stab}(\Delta) = \bigcap_i (\text{ev}_{a_i})^{-1}(\Delta)$$

implies that $\text{Stab}(\Delta)$ is closed in $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$. □

Besides the above examples and the triangular subgroup \mathcal{B}_2 , the only other maximal closed subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ that we are aware of is the affine subgroup \mathcal{A}_2 . The fact that \mathcal{A}_2 is maximal among all closed subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ is a particular case of the following recent result of Edo [Edo16]. (We recall that the so-called *tame subgroup* of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ is its subgroup generated by \mathcal{A}_n and \mathcal{B}_n .)

Theorem 3.27 ([Edo16]). *If a closed subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$, $n \geq 2$, contains strictly the affine subgroup \mathcal{A}_n , then it also contains the whole tame subgroup, hence its closure. In particular, for $n = 2$, the affine group \mathcal{A}_2 is maximal among the closed subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$.*

Remark 3.28. Note that Theorem 3.27 does not allow us to settle the question of the (non) maximality of \mathcal{A}_n among the closed subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$ when $n \geq 3$. Indeed, on the one hand, it was recently shown that, in dimension 3, the tame subgroup is not closed (see [EP15]). But, on the other hand, it is still unknown whether it is dense in $\text{Aut}(\mathbb{A}_{\mathbb{C}}^3)$ or not. For $n \geq 4$, the three questions, whether the tame subgroup is closed, whether it is dense, or even whether it is a strict subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^n)$, are all open.

Let us finally remark that the affine group \mathcal{A}_2 is not a maximal among all abstract subgroups of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$. Indeed, using the amalgamated structure

$$\text{Aut}(\mathbb{A}_{\mathbb{C}}^2) = \mathcal{A}_2 *_{\mathcal{A}_2 \cap \mathcal{B}_2} \mathcal{B}_2$$

and following [FM89], we can define the multidegree (or polydegree) of any automorphism $f \in \text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ in the following way. If f admits an expression

$$f = a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k \circ a_{k+1},$$

where each a_i belongs to \mathcal{A}_2 , each b_i belongs to \mathcal{B}_2 and $a_i \notin \mathcal{B}_2$ for $2 \leq i \leq k$, $b_i \notin \mathcal{A}_2$ for $1 \leq i \leq k$, the multidegree of f is defined as the finite sequence (possibly empty) of integers at least equal to 2:

$$\text{mdeg}(f) = (\deg b_1, \deg b_2, \dots, \deg b_k).$$

Then, the subgroup $M_r := \langle \mathcal{A}_2, (\mathcal{B}_2)_{\leq r} \rangle \subseteq \text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ coincides with the set of automorphisms whose multidegree is of the form (d_1, \dots, d_k) for some k with $d_1, \dots, d_k \leq r$. We thus have a strictly increasing sequence of subgroups

$$\mathcal{A}_2 = M_1 < M_2 < \cdots < M_d < \cdots,$$

showing in particular that \mathcal{A}_2 is not a maximal abstract subgroup.

4. NON-MAXIMALITY OF THE JONQUIÈRES SUBGROUP IN DIMENSION 2

Throughout this section, we work over an arbitrary ground field \mathbf{k} .

Recall that by the famous Jung-van der Kulk-Nagata theorem [Jun42, vdK53, Nag72], the group $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$, of algebraic automorphisms of the affine plane, is the amalgamated free product of its affine subgroup

$$A = \{(ax + by + c, a'x + b'y + c') \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2) \mid a, b, c, a', b', c' \in \mathbf{k}\}$$

and its Jonquières subgroup

$$B := \{(ax + p(y), b'y + c') \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2) \mid a, b', c' \in \mathbf{k}, p(y) \in \mathbf{k}[y]\}$$

above their intersection. Therefore, every element $f \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ admits a *reduced expression* as a product of the form

$$(*) \quad f = t_1 \circ a_1 \circ t_2 \circ \cdots \circ a_n \circ t_{n+1},$$

where a_1, \dots, a_n belong to $A \setminus A \cap B$, and t_1, \dots, t_{n+1} belong to B with $t_2, \dots, t_n \notin A \cap B$.

Definition 4.1. The number n of affine non-triangular automorphisms appearing in such an expression for f is unique. We call it the *affine length* of f and denote it by $\ell_A(f)$.

Remark 4.2. Instead of counting affine elements to define the length of an automorphism of \mathbb{A}^2 , one can of course also consider the Jonquières elements and define the triangular length $\ell_B(f)$ of every $f \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$. Actually, this is the triangular length, that one usually uses in the literature. Let us in particular recall that this length map $\ell_B : \text{Aut}(\mathbb{A}_{\mathbb{C}}^2) \rightarrow \mathbb{N}$ is lower semicontinuous [Fur02], when considering $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ as an ind-group. Since

$$\ell_A(f) = \max_{b_1, b_2 \in B} \ell_B(b_1 \circ f \circ b_2) - 1$$

for every $f \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ and since the supremum of arbitrarily many lower semicontinuous maps is lower semicontinuous, we infer that ℓ_A has also this property.

Proposition 4.3. *The affine length map $\ell_A : \text{Aut}(\mathbb{A}_{\mathbb{C}}^2) \rightarrow \mathbb{N}$ is lower semicontinuous.*

The next result shows that the Jonquières subgroup is not a maximal subgroup of $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$.

Proposition 4.4. *Let $p \in \mathbf{k}[y]$ be a polynomial that fulfils the following property:*

$$(WG) \quad \forall \alpha, \beta, \gamma \in \mathbf{k}, \deg[p(y) - \alpha p(\beta y + \gamma)] \leq 1 \implies \alpha = \beta = 1 \text{ and } \gamma = 0,$$

and consider the following elements of $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$:

$$\sigma = (y, x), \quad t = (-x + p(y), y), \quad f = (\sigma \circ t)^2 \circ \sigma \circ (t \circ \sigma)^2.$$

Then, the subgroup generated by B and f is a strict subgroup of $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$, i.e. $\langle B, f \rangle \neq \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$.

Remark 4.5. Polynomials satisfying the above property (WG) are called *weakly general* in [FL10], where a stronger notion of a general polynomial is also given (see [FL10, Definition 15, page 585]). In particular, by [FL10, Example 65, page 608], the polynomial $q = y^5 + y^4$ is weakly general if \mathbf{k} is a field of characteristic zero.

Moreover, the polynomial $q = y^{2p} - y^{2p-1}$ is weakly general if $\text{char}(\mathbf{k}) = p > 0$. This follows directly from the fact that the coefficients of y^{2p} , y^{2p-1} and y^{2p-2} in the polynomial $q(y) - \alpha q(\beta y + \gamma)$ are equal to $1 - \alpha\beta^{2p}$, $1 - \alpha\beta^{2p-1}$ and $-\alpha\beta^{2p-2}\gamma$, respectively.

Proof of Proposition 4.4. Remark that σ and t , hence f , are involutions. Therefore, every element $g \in \langle B, f \rangle$ can be written as

$$g = b_1 \circ f \circ b_2 \circ f \circ \cdots \circ b_k \circ f \circ b_{k+1},$$

where the elements b_i belong to B and where we can assume without restriction that b_2, \dots, b_k are different from the identity (otherwise, the expression for g could be shortened using that $f^2 = \text{id}$).

In order to prove the proposition, it is enough to show that no element g as above is of affine-length equal to 1. Note that $\ell_A(g) = 0$ if $k = 0$ and that $\ell_A(g) = \ell_A(f) = 5$ if $k = 1$. It remains to consider the case where $k \geq 2$.

For this, let us define four subgroups B_0, \dots, B_3 of B by

$$\begin{aligned} B_0 &= B, \\ B_1 &= A \cap B = \{(ax + by + c, b'y + c') \mid a, b, c, b', c' \in \mathbf{k}, a, b' \neq 0\}, \\ B_2 &= (A \cap B) \cap [\sigma \circ (A \cap B) \circ \sigma] = \{(ax + c, b'y + c') \mid a, c, b', c' \in \mathbf{k}, a, b' \neq 0\}, \\ B_3 &= \{(x, y + c') \mid c' \in \mathbf{k}\}. \end{aligned}$$

Note that $B = B_0 \supseteq B_1 \supseteq B_2 \supseteq B_3$. We will now give a reduced expression of $u_i := (t \circ \sigma)^2 \circ b_i \circ (\sigma \circ t)^2$ for each $i \in \{2, \dots, k\}$. We do it by considering successively the four following cases:

1. $b_i \in B_0 \setminus B_1$; 2. $b_i \in B_1 \setminus B_2$; 3. $b_i \in B_2 \setminus B_3$; 4. $b_i \in B_3 \setminus \{\text{id}\}$.

Case 1: $b_i \in B_0 \setminus B_1$.

Since $b_i \in B \setminus A$, the element u_i admits the following reduced expression

$$u_i = (t \circ \sigma)^2 \circ b_i \circ (\sigma \circ t)^2.$$

Case 2: $b_i \in B_1 \setminus B_2$.

Since $\widehat{b}_i := \sigma \circ b_i \circ \sigma \in A \setminus B$, the element u_i has the following reduced expression

$$u_i = t \circ \sigma \circ t \circ \widehat{b}_i \circ t \circ \sigma \circ t.$$

Case 3: $b_i \in B_2 \setminus B_3$.

Let us check that $\overline{b}_i := t \circ \sigma \circ b_i \circ \sigma \circ t \in B \setminus A$. We are in the case where $b_i = (ax + c, b'y + c')$ with $(a, c, b') \neq (1, 0, 1)$. A direct calculation gives that

$$\overline{b}_i = (b'x + p(ay + c) - b'p(y) - c', ay + c).$$

By the assumption made on p , we have that $\deg[p(ay + c) - b'p(y)] \geq 2$, hence that $\overline{b}_i \in B \setminus A$. Therefore u_i admits the following reduced expression

$$u_i = t \circ \sigma \circ \overline{b}_i \circ \sigma \circ t.$$

Case 4: $b_i \in B_3 \setminus \{\text{id}\}$.

Let us check that $\widetilde{b}_i := (t \circ \sigma)^2 \circ b_i \circ (\sigma \circ t)^2 \in B \setminus A$. We are in the case where $b_i = (x, y + c')$ with $c' \in \mathbb{C}^*$. Using the computation in case 3 with $(a, c, b') = (1, 0, 1)$, we then obtain that

$$\widetilde{b}_i = t \circ \sigma \circ (x - c', y) \circ \sigma \circ t = t \circ (x, y - c') \circ t = (x + p(y - c') - p(y), y - c') \in B \setminus A.$$

Therefore, the element u_i has the following reduced expression

$$u_i = \widetilde{b}_i.$$

Finally we obtain a reduced expression for an element $g \in \langle B, f \rangle$ from the above study of cases, since we can express

$$\begin{aligned} g &= b_1 \circ f \circ b_2 \circ f \circ \dots \circ b_k \circ f \circ b_{k+1} \\ &= b_1 \circ (\sigma \circ t)^2 \circ \sigma \circ u_2 \circ \sigma \circ \dots \circ \sigma \circ u_k \circ \sigma \circ (t \circ \sigma)^2 \circ b_{k+1}. \end{aligned}$$

In particular, observe that $\ell_A(g) \geq 6$ if $k \geq 2$. This concludes the proof. \square

Note that the element f such that $\langle B, f \rangle \neq \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$, that we constructed in Proposition 4.4, is of affine-length $\ell_A(f) = 5$. Our next result shows that 5 is precisely the minimal length for elements $f \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2) \setminus B$ with that property.

Proposition 4.6. *Suppose that $f \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ is an automorphism of affine length ℓ with $1 \leq \ell \leq 4$. Then, the subgroup generated by B and f is equal to the whole group $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$, i.e. $\langle B, f \rangle = \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$.*

In order to prove the above proposition, it is useful to remark that we can impose extra conditions on the elements $t_1, \dots, t_{n+1}, a_1, \dots, a_n$ appearing in a reduced expression (*) of an automorphism $f \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$. We do it in Proposition 4.10 below. First, we need to introduce some notations.

Notation 4.7. In the sequel, we will denote, as in the proof of Proposition 4.4, by σ the involution

$$\sigma = (y, x) \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$$

and by B_2 the subgroup

$$B_2 = \{(ax + c, b'y + c') \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2) \mid a, c, b', c' \in \mathbf{k}\} \subset A \cap B.$$

Moreover, we denote by I the subset

$$I = \{(-x + p(y), y) \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2) \mid p(y) \in \mathbf{k}[y], \deg p(y) \geq 2\} \subset B \setminus A \cap B.$$

Note that the elements of I are all involutions.

Lemma 4.8. *The followings hold:*

- (1) $B_2 \circ \sigma = \sigma \circ B_2$.
- (2) $B \setminus A \cap B = I \circ B_2 = B_2 \circ I = B_2 \circ I \circ B_2$.
- (3) $A \setminus A \cap B \subset (A \cap B) \circ \sigma \circ (A \cap B)$.

Remark 4.9. In particular, Assertion (3) implies that the group generated by σ and all triangular automorphisms is equal to the whole $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$, i.e. $\langle B, \sigma \rangle = \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$.

Proof. The first assertion is an easy consequence of the following equalities:

$$(ax + c, b'y + c') \circ \sigma = (ay + c, b'x + c') = \sigma \circ (b'x + c', ay + c).$$

Let us now prove the second assertion. It is easy to check that $I \circ B_2 = B_2 \circ I = B_2 \circ I \circ B_2 \subset B \setminus A \cap B$. On the other hand, let $f = (ax + p(y), b'y + c')$ be an element of $B \setminus A \cap B$. Then f belongs to $I \circ B_2$, since we can write

$$f = (-x + p(\frac{y - c'}{b'}), y) \circ (-ax, b'y + c').$$

It remains to prove the last assertion. For this, it suffices to write, given an element $f = (ax + by + c, a'x + b'y + c')$ of $A \setminus A \cap B$ with $a' \neq 0$, that

$$f = (ax + by + c, a'x + b'y + c') = (x + \frac{a}{a'}y + c, y + c') \circ \sigma \circ (a'x + b'y, \frac{ba' - ab'}{a'}y).$$

□

Proposition 4.10. *Let $f \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ be an automorphism of affine length $\ell = n + 1$ with $n \geq 0$. Then there exist triangular automorphisms $\tau_1, \tau_2 \in B$ and triangular involutions $i_1, \dots, i_n \in I$ such that*

$$(**) \quad f = \tau_1 \circ \sigma \circ i_1 \circ \sigma \circ \dots \circ \sigma \circ i_n \circ \sigma \circ \tau_2.$$

In particular, the inverse of f is given by

$$f^{-1} = \tau_2^{-1} \circ \sigma \circ i_n \circ \sigma \circ \dots \circ \sigma \circ i_1 \circ \sigma \circ \tau_1^{-1}.$$

Proof. Let f be an automorphism of affine length $\ell = n + 1$. By definition,

$$f = t_1 \circ a_1 \circ t_2 \circ \cdots \circ a_n \circ t_{n+1},$$

for some $a_1, \dots, a_n \in A \setminus A \cap B$, $t_1, t_{n+1} \in B$ and $t_2, \dots, t_n \in B \setminus A \cap B$. Using Assertion (3) of Lemma 4.8, we may replace every a_i by σ . The proposition then follows from Assertions (1) and (2) of Lemma 4.8. \square

We can now proceed to the proof of Proposition 4.6.

Proof of Proposition 4.6. Case $\ell = 1$. Let $f \in B$ with $\ell_A(f) = 1$. By Proposition 4.10, we can write $f = \tau_1 \circ \sigma \circ \tau_2$ for some $\tau_1, \tau_2 \in B$. Thus, $\langle B, f \rangle = \langle B, \sigma \rangle = \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ follows from Remark 4.9.

The proofs for affine length $\ell = 2, 3, 4$ will be based on explicit computations. In particular, it will be useful to observe that all $i = (-x + p(y), y) \in I$ satisfy that

$$(6) \quad i \circ (x + 1, y) \circ i = (x - 1, y),$$

$$(7) \quad \sigma \circ i \circ (x + 1, y) \circ i \circ \sigma = (x, y - 1)$$

and

$$(8) \quad i \circ (x, y - 1) \circ i \circ (-x, y + 1) = (-x + (p(y) - p(y + 1)), y).$$

Case $\ell = 2$. Let $f \in B$ with $\ell_A(f) = 2$. By Proposition 4.10, we can suppose that $f = \sigma \circ i \circ \sigma$ for some involution $i = (-x + p(y), y) \in I$. Consider the elements $b_1 = \sigma \circ (x, y - 1) \circ \sigma$ and $b_2 = \sigma \circ (-x, y + 1) \circ \sigma$ of B_2 . Since

$$f \circ b_1 \circ f \circ b_2 = \sigma \circ i \circ (x, y - 1) \circ i \circ (-x, y + 1) \circ \sigma,$$

it follows from Equality (8) above that the automorphism $\sigma \circ (-x + (p(y) - p(y + 1)), y) \circ \sigma$ belongs to $\langle B, f \rangle$. By induction, we thus obtain an element in $\langle B, f \rangle$ of the form $\sigma \circ (-x + q(y), y) \circ \sigma$ with $\deg(q) = 1$. This element is in fact an element of $A \setminus A \cap B$ and has therefore affine length 1. This implies that $\langle B, f \rangle = \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$.

Case $\ell = 3$. Let $f \in B$ with $\ell_A(f) = 3$. By Proposition 4.10, we can suppose that $f = \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma$ for some $i_1 = (-x + p_1(y), y), i_2 = (-x + p_2(y), y) \in I$. We first use Equality (7), which implies that

$$(9) \quad \sigma \circ i_2 \circ \sigma \circ b \circ \sigma \circ i_2 \circ \sigma = (x, y - 1),$$

where b denotes the element $b = \sigma \circ (x + 1, y) \circ \sigma \in B_2$. Hence, denoting by b' the element $b' = \sigma \circ (-x, y + 1) \circ \sigma$ in B_2 and using Equalities (8) and (9), we obtain that

$$\begin{aligned} f \circ b \circ f^{-1} \circ b' &= \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \circ b \circ \sigma \circ i_2 \circ \sigma \circ i_1 \circ \sigma \circ b' \\ &= \sigma \circ i_1 \circ (x, y - 1) \circ i_1 \circ \sigma \circ b' \\ &= \sigma \circ i_1 \circ (x, y - 1) \circ i_1 \circ (-x, y + 1) \circ \sigma \\ &= \sigma \circ (-x + (p_1(y) - p_1(y + 1)), y) \circ \sigma \end{aligned}$$

is an element of affine length 2 (or 1 in the case where $\deg(p_1) = 2$), which belongs to $\langle B, f \rangle$. Consequently, $\langle B, f \rangle = \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$.

Case $\ell = 4$. Let $f \in B$ with $\ell_A(f) = 4$. By Proposition 4.10, we can suppose that $f = \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \circ i_3 \circ \sigma$ for some $i_j = (-x + p_j(y), y) \in I$, $j = 1, 2, 3$. Letting $b = \sigma \circ (x + 1, y) \circ \sigma$ as above, one get that

$$\begin{aligned}
f \circ b \circ f^{-1} &= \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \circ i_3 \circ \sigma \circ b \circ \sigma \circ i_3 \circ \sigma \circ i_2 \circ \sigma \circ i_1 \circ \sigma \\
&= \sigma \circ i_1 \circ \sigma \circ i_2 \circ (x, y - 1) \circ i_2 \circ \sigma \circ i_1 \circ \sigma \\
&= \sigma \circ i_1 \circ \sigma \circ i_2 \circ (x, y - 1) \circ i_2 \circ (-x, y + 1) \circ (-x, y - 1) \circ \sigma \circ i_1 \circ \sigma \\
&= \sigma \circ i_1 \circ \sigma \circ i'_2 \circ (-x, y - 1) \circ \sigma \circ i_1 \circ \sigma \\
&= \sigma \circ i_1 \circ \sigma \circ i'_2 \circ \sigma \circ (x - 1, -y) \circ i_1 \circ \sigma \\
&= \sigma \circ i_1 \circ \sigma \circ i'_2 \circ \sigma \circ i'_1 \circ (x + 1, -y) \circ \sigma \\
&= \sigma \circ i_1 \circ \sigma \circ i'_2 \circ \sigma \circ i'_1 \circ \sigma \circ (-x, y + 1),
\end{aligned}$$

where $i'_2 = (-x + p'_2(y), y)$ and $i'_1 = (-x + p'_1(y), y)$ for the polynomials $p'_2(y) = p_2(y) - p_2(y + 1)$ and $p'_1(y) = p_1(-y)$, respectively. In particular, $\langle B, f \rangle$ contains the element $\sigma \circ i_1 \circ \sigma \circ i'_2 \circ \sigma \circ i'_1 \circ \sigma$. Since $\deg(p'_2) = \deg(p_2) - 1$, we obtain by induction an element in $\langle B, f \rangle$ of the form $\sigma \circ i_1 \circ \sigma \circ \tilde{i}_2 \circ \sigma \circ i'_1 \circ \sigma$ with $\tilde{i}_2 = (-x + \tilde{p}_2(y), y)$ and $\deg(\tilde{p}_2) = 1$. Since $\sigma \circ \tilde{i}_2 \circ \sigma$ is an element of $A \setminus A \cap B$, the above $\sigma \circ i_1 \circ \sigma \circ \tilde{i}_2 \circ \sigma \circ i'_1 \circ \sigma$ is an automorphism of affine length 3, and the proposition follows. \square

To conclude, let us emphasize that, as pointed to us by S. Lamy, our results concerning the non-maximality of B are related to those of [FL10] about the existence of normal subgroups for the group $\text{SAut}(\mathbb{A}_{\mathbb{C}}^2)$ of automorphisms of the complex affine plane whose Jacobian determinant is equal to 1. Indeed, the subgroup $\langle B, f \rangle$, generated by B and a given automorphism f , is contained into the subgroup $B \circ \langle f \rangle_N = \{h \circ g \mid h \in B, g \in \langle f \rangle_N\}$, where $\langle f \rangle_N$ denotes the normal subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ that is generated by f .

Combined with Proposition 4.6, the above observation gives us a short proof of the following result.

Theorem 4.11 ([FL10, Theorem 1]). *If $f \in \text{SAut}(\mathbb{A}_{\mathbb{C}}^2)$ is of affine length at most 4 and $f \neq \text{id}$, then the normal subgroup $\langle f \rangle_N$ generated by f in $\text{SAut}(\mathbb{A}_{\mathbb{C}}^2)$ is equal to the whole group $\text{SAut}(\mathbb{A}_{\mathbb{C}}^2)$.*

Proof. The case where f is a triangular automorphism being easy to treat (see [FL10, Lemma 30, p. 590]), suppose that $f \in \text{SAut}(\mathbb{A}_{\mathbb{C}}^2)$ is of affine length at most 4 and at least 1. By Proposition 4.6, we have $\langle B, f \rangle = \text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$. Since the group $B \circ \langle f \rangle_N$ contains B and f , we get $B \circ \langle f \rangle_N = \text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$. In particular, the element $(-y, x)$ can be written as $(-y, x) = b \circ g$ for some $b \in B$ and $g \in \langle f \rangle_N$. Consequently, $\langle f \rangle_N$ contains the element $g = b^{-1} \circ (-y, x)$ which is of affine length 1.

Remark that the Jacobian determinant of b is equal to 1. Therefore, we can write $b^{-1} = (ax + P(y), a^{-1}y + c)$ for some $a \in \mathbb{C}^*$, $c \in \mathbb{C}$ and $P(y) \in \mathbb{C}[y]$. Thus, g is given by

$$g = (-ay + P(x), a^{-1}x + c).$$

Next, we consider the translation $\tau = (x + 1, y)$ and compute the commutator $[\tau, g] = \tau \circ g \circ \tau^{-1} \circ g^{-1}$, which is an element of $\langle f \rangle_N$. Since

$$\begin{aligned}
[\tau, g] &= (x + 1, y) \circ (-ay + P(x), a^{-1}x + c) \circ (x + 1, y) \circ (ay - ac, -a^{-1}x + a^{-1}P(ay - ac)) \\
&= (x - P(ay - ac) + P(ay - ac - 1) + 1, y - a^{-1})
\end{aligned}$$

is a triangular automorphism different from the identity, the theorem follows directly from [FL10, Lemma 30, p. 590]. \square

On the other hand, we can retrieve the fact that the Jonquières subgroup is not a maximal subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ as a corollary of [FL10, Theorem 2]. Indeed, the latter produces elements $f \in \text{SAut}(\mathbb{A}_{\mathbb{C}}^2)$ of affine length $\ell_A(f) = 7$ such that $\langle f \rangle_N \neq \text{SAut}(\mathbb{A}_{\mathbb{C}}^2)$. In particular, by [FL10, Theorem 1] above, the identity is the only automorphism of affine length smaller than or equal to 4 contained in $\langle f \rangle_N$. Therefore, since $\langle B, f \rangle \subset B \circ \langle f \rangle_N$, the subgroup $\langle B, f \rangle$ does not contain any non-triangular automorphism of affine length ≤ 4 . Consequently, $\langle B, f \rangle$ is a strict subgroup of $\text{Aut}(\mathbb{A}_{\mathbb{C}}^2)$.

REFERENCES

- [Bas84] Hyman Bass, *A nontriangular action of \mathbf{G}_a on \mathbf{A}^3* , J. Pure Appl. Algebra **33** (1984), no. 1, 1–5.
- [BCW82] Hyman Bass, Edwin H. Connell, and David Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 2, 287–330.
- [Bea10] Arnaud Beauville, *Finite subgroups of $\text{PGL}_2(K)$* , Vector bundles and complex geometry, Contemp. Math., vol. 522, Amer. Math. Soc., Providence, RI, 2010, pp. 23–29.
- [BEE16] Yuri Berest, Alimjon Eshmatov, and Farkhod Eshmatov, *Dixmier groups and Borel subgroups*, Adv. Math. **286** (2016), 387–429.
- [Edo16] Éric Edo, *Closed subgroups of the polynomial automorphism group containing the affine subgroup*, Transform. Groups, posted on 2016, DOI 10.1007/s00031-016-9412-7.
- [EP15] Éric Edo and Pierre-Marie Poloni, *On the closure of the tame automorphism group of affine three-space*, Int. Math. Res. Not. IMRN **19** (2015), 9736–9750.
- [vdE00] Arno van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progress in Mathematics, vol. 190, Birkhäuser Verlag, Basel, 2000.
- [FM89] Shmuel Friedland and John Milnor, *Dynamical properties of plane polynomial automorphisms*, Ergodic Theory Dynam. Systems **9** (1989), no. 1, 67–99.
- [Fur02] Jean-Philippe Furter, *On the length of polynomial automorphisms of the affine plane*, Math. Ann. **322** (2002), no. 2, 401–411.
- [FL10] Jean-Philippe Furter and Stéphane Lamy, *Normal subgroup generated by a plane polynomial automorphism*, Transform. Groups **15** (2010), no. 3, 577–610.
- [Hum75] James E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 21.
- [Jun42] Heinrich W. E. Jung, *Über ganze birationale Transformationen der Ebene*, J. Reine Angew. Math. **184** (1942), 161–174 (German).
- [Kal92] Shulim Kaliman, *Isotopic embeddings of affine algebraic varieties into \mathbf{C}^n* , The Madison Symposium on Complex Analysis (Madison, WI, 1991), Contemp. Math., vol. 137, Amer. Math. Soc., Providence, RI, 1992, pp. 291–295.
- [vdK53] W. van der Kulk, *On polynomial rings in two variables*, Nieuw Arch. Wiskunde (3) **1** (1953), 33–41.
- [Lam01] Stéphane Lamy, *L’alternative de Tits pour $\text{Aut}[\mathbf{C}^2]$* , J. Algebra **239** (2001), no. 2, 413–437 (French, with French summary).
- [Mal56] Anatoliĭ Ivanovich Mal’cev, *On certain classes of infinite soluble groups*, Amer. Math. Soc. Translations (2) **2** (1956), 1–21.
- [MR14] Mitchael Martelo and Javier Ribón, *Derived length of solvable groups of local diffeomorphisms*, Math. Ann. **358** (2014), no. 3-4, 701–728.
- [Nag59] Hirosi Nagao, *On $\text{GL}(2, K[x])$* , J. Inst. Polytech. Osaka City Univ. Ser. A **10** (1959), 117–121.
- [Nag72] Masayoshi Nagata, *On automorphism group of $k[x, y]$* , Kinokuniya Book-Store Co., Ltd., Tokyo, 1972. Department of Mathematics, Kyoto University, Lectures in Mathematics, No. 5.

- [New72] Mike F. Newman, *The soluble length of soluble linear groups*, Math. Z. **126** (1972), 59–70.
- [Pop87] Vladimir L. Popov, *On actions of \mathbf{G}_a on \mathbf{A}^n* , Algebraic groups Utrecht 1986, Lecture Notes in Math., vol. 1271, Springer, Berlin, 1987, pp. 237–242.
- [Pop14] ———, *On infinite dimensional algebraic transformation groups*, Transform. Groups **19** (2014), no. 2, 549–568.
- [Ser03] Jean-Pierre Serre, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.
- [Sha66] Igor R. Shafarevich, *On some infinite-dimensional groups*, Rend. Mat. e Appl. (5) **25** (1966), no. 1-2, 208–212.
- [Sha81] ———, *On some infinite-dimensional groups. II*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 1, 214–226, 240 (Russian).
- [Weh73] B. A. F. Wehrfritz, *Infinite linear groups. An account of the group-theoretic properties of infinite groups of matrices*, Springer-Verlag, New York-Heidelberg, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 76.
- [Zas37] Hans Zassenhaus, *Beweis eines satzes über diskrete gruppen*, Abh. Math. Sem. Univ. Hamburg **12** (1937), no. 1, 289–312.

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