On the variety of automorphisms
of the affine plane

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Abstract : The main subject of our study is $\mathrm{GA}_{2,n}$, the variety of automorphisms of the affine plane of degree bounded by a positive integer n. After precising some definitions and notations in section 1, we give in section 2 an algorithm to decide whether an endomorphism of the affine plane over an integral domain is a tame automorphism. Then, by applying this algorithm to the Nagata automorphism, we recover easily the known results on it. In section 6, we compute the number of irreducible components of $\mathrm{GA}_{2,n}$ when $n \leq 9$ and we show that $\mathrm{GA}_{2,n}$ is reducible when $n \geq 4$. Our proofs are based on a precise decomposition theorem for automorphisms given in section 3 and a characterization of length one automorphisms given in section 5. Finally, in section 7, we give some details on the case $n = 4$.

1. Definitions and notations.

Let k be an integral domain. $k[X,Y]$ is the polynomial algebra in the indeterminates X, Y (endowed with the degree function and the valuation function at the origin, which are denoted by deg and val) and $\mathbb{A}_k^2 = \mathrm{Spec}\ k[X,Y]$ is the affine plane over k. A k-endomorphism f of $\mathbb{A}_k^2$ will be identified with its sequence $f = (f_1, f_2)$ of coordinate functions $f_i \in k[X,Y]$ $(i = 1, 2)$. If g is a k-endomorphism of $\mathbb{A}_k^2$, we agree that fg will denote the k-endomorphism of $\mathbb{A}_k^2$ obtained by the composition of f and g.
We define :

the degree of f by $\deg f = \max \{\deg f_1, \deg f_2\}$;

the bidegree of f by $\mathrm{bideg}\,(f) = (\deg f_1, \deg f_2)$;

the total degree of f by $\mathrm{tdeg}\,(f) = \deg f_1 + \deg f_2$;

the Jacobian of f by $\mathrm{Jac}(f) = \frac{\partial f_1}{\partial X}\frac{\partial f_2}{\partial Y} - \frac{\partial f_1}{\partial Y}\frac{\partial f_2}{\partial X}$

and we introduce the following groups :

$\mathrm{GA}_2(k)$, the group of all k-automorphisms of $\mathbb{A}_k^2$;

$\mathrm{Af}_2(k)$, the subgroup of $\mathrm{GA}_2(k)$ of affine automorphisms, i.e. automorphisms of the shape $(aX + bY + c, dX + eY + f)$ where a, b, c, d, e, f are elements of k satisfying $ae - bd \in k^*$;

$\mathrm{GL}_2(k)$, the subgroup of $\mathrm{Af}_2(k)$ of linear automorphisms ;

$\mathrm{BA}_2(k)$, the subgroup of $\mathrm{GA}_2(k)$ of triangular or "de Jonquières" automorphisms, i.e. automorphisms of the shape $(aX + P(Y), bY + c)$ where a, b are units of k, c is an element of k and $P(Y)$ is an element of $k[Y]$;

$\mathrm{TA}_2(k)$, the subgroup of $\mathrm{GA}_2(k)$ of tame automorphisms, i.e. the subgroup of $\mathrm{GA}_2(k)$ generated by $\mathrm{Af}_2(k)$ and $\mathrm{BA}_2(k)$.

2. An algorithm to decide whether an endomorphism is a tame automorphism.

The following proposition is the key of the announced algorithm : if $f \in \mathrm{TA}_2(k)$ satisfies tdeg $f \geq 3$, it ensures the existence of $\alpha \in \mathrm{GA}_2(k)$ of a very special shape such that tdeg $(\alpha^{-1}f) <$ tdeg f.

Proposition 1. *Let $f = (f_1, f_2) \in TA_2(k)$ with bideg$(f) = (d_1, d_2)$. Denote by g_i the homogeneous component of degree d_i of f_i for $i = 1, 2$. Then $d_1|d_2$ or $d_2|d_1$. If we suppose moreover that deg $(f) > 1$, then we have :*

i) if $d_1 < d_2$, then there exists $\lambda \in k$ such that $g_2 = \lambda . g_1^{d_2/d_1}$;

ii) if $d_2 < d_1$, then there exists $\lambda \in k$ such that $g_1 = \lambda . g_2^{d_1/d_2}$;

iii) if $d_1 = d_2$, then there exists $(\alpha, h) \in Af_2(k) \times TA_2(k)$ such that $f = \alpha h$ and deg $h_1 >$deg h_2 where $h = (h_1, h_2)$.

Remarks.

1. Let us assume that deg $(f) > 1$. We set $\alpha = (X, Y + \lambda X^{d_2/d_1})$ in case i), $\alpha = (X + \lambda Y^{d_1/d_2}, Y)$ in case ii) and if we are in case iii), we keep α as explained there. Then, it is clear that we have tdeg $(\alpha^{-1}f) <$ tdeg f.

2. If $d_1 = d_2$, in general, there does not exist $\lambda \in k$ such that $g_2 = \lambda . g_1$ or $g_1 = \lambda . g_2$, as shown by the following example :

$$f = ((1 - Z)X + ZY + (1 - Z)Y^2, -ZX + (1 + Z)Y - ZY^2)$$

It can be checked that $f = \alpha\gamma$ where $\alpha = ((1 - Z)X + ZY, -ZX + (1 + Z)Y) \in \mathrm{Af}_2(\mathbb{C}[Z])$ and $\gamma \in (X + Y^2, Y) \in \mathrm{BA}_2(\mathbb{C}[Z])$ so that $f \in \mathrm{TA}_2(\mathbb{C}[Z])$. However, bideg $f = (2, 2)$, the homogeneous component of degree 2 of f is $((1 - Z)Y^2, -ZY^2)$ and there exists no $\lambda \in \mathbb{C}[Z]$ such that $(1 - Z)Y^2 = \lambda.(-ZY^2)$ or $-ZY^2 = \lambda.(1 - Z)Y^2$.

3. Nevertheless, still in the case $d_1 = d_2$, if $k = \mathbb{C}$ and $\deg f > 1$, the equality Jac $f \in \mathbb{C}^*$ implies that $\mathrm{Jac}(g_1, g_2) = 0$ which is well known to imply that g_1 and g_2 are proportional polynomials.

The following group-theoretical lemma will be used in the proof of Proposition 1.

Lemma 1. *If a group G is generated by two subgroups H and K, then each element g of G can be written*

$$g = h_1.k_1.h_2.k_2.\cdots.h_l.k_l.h_{l+1}$$

where l is a non negative integer, the h_i (resp. k_i) are elements of H (resp. K) satisfying the additional condition ($\forall\ 2 \leq i \leq l, h_i \notin K$ and $\forall\ 1 \leq i \leq l, k_i \notin H$).

Proof. If g is an element of G, we have

$$g = h_1.k_1.h_2.k_2.\cdots.h_l.k_l.h_{l+1}$$

where l is a non negative integer and the h_i (resp. k_i) are elements of H (resp. K). If the additional condition ($\forall\ 2 \leq i \leq l, h_i \notin K$ and $\forall\ 1 \leq i \leq l, k_i \notin H$) is not satisfied, then we can obtain an expression for g of the same shape but where l is replaced by $l-1$. Indeed, if for example, $h_{i_0} \in K$ ($2 \leq i_0 \leq l$), then we have

$$g = h_1.k_1.h_2.k_2.\cdots h_{i_0-1}.(k_{i_0-1}.h_{i_0}.k_{i_0}).h_{i_0+1}.\cdots.h_l.k_l.h_{l+1}$$

where $k_{i_0-1}.h_{i_0}.k_{i_0} \in K$. After a finite number of such reductions, we will necessarily obtain the desired expression for g. □

Proof of Proposition 1. By Lemma 1 applied with $G = \mathrm{TA}_2(k)$, $H = \mathrm{Af}_2(k)$, $K = \mathrm{BA}_2(k)$ and $g = f$, we obtain the existence of a non-negative integer l and of $(\alpha, \beta) \in \left(\mathrm{Af}_2(k)\right)^2$, $(\alpha_i)_{1 \leq i \leq l-1} \in \left(\mathrm{Af}_2(k) \setminus \mathrm{BA}_2(k)\right)^{l-1}$, $(\gamma_i)_{1 \leq i \leq l} \in \left(\mathrm{BA}_2(k) \setminus \mathrm{Af}_2(k)\right)^l$ such that

$$f = \alpha\gamma_1\alpha_1\gamma_2\cdots\alpha_{l-2}\gamma_{l-1}\alpha_{l-1}\gamma_l\beta.$$

We show by a decreasing induction on i, beginning with $i = l$ and finishing with $i = 1$ that

$$\mathrm{bideg}(\gamma_i\alpha_i\cdots\gamma_{l-1}\alpha_{l-1}\gamma_l\beta) = (\prod_{j=i}^{l} \deg\,\gamma_j,\ \prod_{j=i+1}^{l} \deg\,\gamma_j).$$

The conclusion is clear for $i = l$. Let us suppose it is true for a given i with $2 \leq i \leq l$. Let us write $\alpha_{i-1} = (a_{i-1}X + b_{i-1}Y + c_{i-1}, d_{i-1}X + e_{i-1}Y + f_{i-1})$. The relations $\alpha_{i-1} \notin \mathrm{BA}_2(k)$ and $\gamma_{i-1} \notin \mathrm{Af}_2(k)$ imply that $d_{i-1} \neq 0$ and $\deg(\gamma_{i-1}) \geq 2$. We now obtain bideg $(\alpha_{i-1}\gamma_i\alpha_i \cdots \gamma_{l-1}\alpha_{l-1}\gamma_l\beta) = (p_i, \prod_{j=i}^{l} \deg \gamma_j)$ with $p_i \leq \prod_{j=i}^{l} \deg \gamma_j$ and bideg $(\gamma_{i-1}\alpha_{i-1} \cdots \gamma_{l-1}\alpha_{l-1}\gamma_l\beta) = (\prod_{j=i-1}^{l} \deg \gamma_j, \prod_{j=i}^{l} \deg \gamma_j)$.

It is now proven that

$$\mathrm{bideg}(\gamma_1\alpha_1 \cdots \gamma_{l-1}\alpha_{l-1}\gamma_l\beta) = (\prod_{j=1}^{l} \deg \gamma_j, \prod_{j=2}^{l} \deg \gamma_j).$$

Let us write $\alpha = (aX + bY + c, dX + eY + f)$. To conclude, we just have to check that we are in case i) or ii) or iii) according as $a = 0$ or $d = 0$ or $ad \neq 0$. □

We deduce at once from this last proposition an algorithm to decide if a k-endomorphism of $\mathbb{A}_k^2$ belongs to $\mathrm{TA}_2(k)$:

Algorithm :

1. Enter a k-endomorphism f of A_k^2
2. Let $(d_1, d_2) = \mathrm{bideg}(f_1, f_2)$
3. If $d_1 = d_2 = 1$, then goto 8
4. If $d_1 \neq d_2$, then goto 6
5. If there exists $\alpha \in \mathrm{Af}_2(k)$ such that $\mathrm{tdeg}(\alpha f) < \mathrm{tdeg}\,(f)$, then replace f by αf and goto 2, else STOP : $f \notin \mathrm{TA}_2(k)$
6. If $d_2 < d_1$, then replace $f = (f_1, f_2)$ by (f_2, f_1).
7. If ($d_1 | d_2$ and there exists $\lambda \in k$ such that $g_2 = \lambda g_1^{d_2/d_1}$), then replace f by $(X, Y - \lambda X^{d_2/d_1})f$ and goto 2, else STOP : $f \notin \mathrm{TA}_2(k)$
8. If $\mathrm{Jac}(f) \in k^*$, then STOP : $f \in \mathrm{TA}_2(k)$, else STOP : $f \notin \mathrm{TA}_2(k)$.

Application :

Let us consider the Nagata automorphism

$$\phi = \big(X - 2Y(XZ + Y^2) - Z(XZ + Y^2)^2, Y + Z(XZ + Y^2)\big) \in \mathrm{GA}_2(k)$$

for $k = \mathbb{C}[Z]$ and $k = \mathbb{C}[Z, Z^{-1}]$.

By expanding ϕ, we get

$$\phi = (X - 2ZXY - Z^3X^2 - 2Y^3 - 2Z^2XY^2 - ZY^4, Y + Z^2X + ZY^2)$$

so that the homogeneous components of highest degree of ϕ are $-ZY^4$ and ZY^2.

1) If $k = \mathbb{C}[Z]$, $(ZY^2)^2$ does not divide $-ZY^4$ and so $\phi \notin \mathrm{TA}_2(\mathbb{C}[Z])$.

2) If $k = \mathbb{C}[Z, Z^{-1}]$, the equation $-ZY^4 = -Z^{-1}(ZY^2)^2$ allows us to obtain :

$$\phi = (X - Z^{-1}Y^2, Y)(X + Z^{-1}Y^2, Y + Z^2X + ZY^2)$$

and we easily see that :

$$(X + Z^{-1}Y^2, Y + Z^2X + ZY^2) = (X, Y + Z^2X)(X + Z^{-1}Y^2, Y)$$

whence $\phi = (X - Z^{-1}Y^2, Y)(X, Y + Z^2X)(X + Z^{-1}Y^2, Y)$
and $\phi \in \mathrm{TA}_2(\mathbb{C}[Z, Z^{-1}])$.

3. A precise decomposition theorem for automorphisms of the affine plane.

From now on, we set $k = \mathbb{C}$ and we agree to note GA_2 instead of $\mathrm{GA}_2(\mathbb{C})$, Af_2 instead of $\mathrm{Af}_2(\mathbb{C})$, etc ...

Let us recall that by a famous result of H.W.E. Jung and W. van der Kulk ([vdK]), we have GA_2=TA_2.

We give here a precise decomposition theorem for elements of GA_2 : any such element has a unique expression of a given shape (see Theorem 1). This result allows us (see section 6, Proposition 10) to compute the dimension of the algebraic variety $\mathrm{GA}_{2,n}$.

Before stating the theorem, we need the following definitions :

T_2 is the subgroup of GA_2 of automorphisms of the shape $(X+P(Y), Y)$ where $P(Y)$ is an element of $\mathbb{C}[Y]$ satisfying $\mathrm{val}(P) \geq 1$;

UA_2 is the subgroup of T_2 of elements of the shape $(X+P(Y), Y)$ where $P(Y)$ is any element of $\mathbb{C}[Y]$ satisfying $\mathrm{val}(P) \geq 2$;

for all $(a, b) \in \mathbb{C}^2$, we set $\tau_{(a,b)} = (X + a, Y + b) \in \mathrm{GA}_2$ and for all $c \in \mathbb{C}$, we set $\sigma_c = (Y, X + cY) \in \mathrm{GA}_2$. We note $\sigma = \sigma_0 = (Y, X)$.

Theorem 1. **(Precise Decomposition Theorem)** *Let* $f = (f_1, f_2) \in GA_2 \setminus Af_2$*, then :*

i) if $deg(f_1) > deg(f_2)$*,*

$$\exists!\ (a,b) \in \mathbb{C}^2, \exists!\ l \in \mathbb{N}, \exists!\ (\gamma_i)_{1\le i\le l} \in \big(T_2\backslash GL_2\big)^{l-1}\times\big(UA_2\backslash GL_2\big), \exists!\ \beta \in GL_2$$

such that $f = \tau_{(a,b)} \circ \gamma_1 \circ \sigma \circ \gamma_2 \circ \sigma \circ \cdots \circ \sigma \circ \gamma_{l-1} \circ \sigma \circ \gamma_l \circ \beta$*.*

ii) if $deg(f_1) \le deg(f_2)$*,*

$$\exists!\ (a,b,c) \in \mathbb{C}^3, \exists!\ l \in \mathbb{N}, \exists!\ (\gamma_i)_{1\le i\le l} \in \big(T_2\backslash GL_2\big)^{l-1}\times\big(UA_2\backslash GL_2\big), \exists!\ \beta \in GL_2$$

such that $f = \tau_{(a,b)} \circ \sigma_c \circ \gamma_1 \circ \sigma \circ \gamma_2 \circ \sigma \circ \cdots \circ \sigma \circ \gamma_{l-1} \circ \sigma \circ \gamma_l \circ \beta$*.*

To prove this theorem, we will use the following three decomposition lemmas :

Lemma 2. *Let* $f = (f_1, f_2) \in GA_2$ *satisfy* $deg(f_1) > deg(f_2) > 1$*, then* $\exists!\ (\gamma, p) \in \big(T_2 \setminus GL_2\big) \times GA_2$ *such that* $f = \gamma \circ p$ *and* $p = (p_1, p_2)$ *satisfies* $deg(p_1) < deg(p_2) = deg(f_2)$*.*

Proof. Existence : We begin by proving that for each automorphism $f = (f_1, f_2) \in \mathrm{GA}_2$ satisfying $\deg(f_1) \ge \deg(f_2) > 1$ there exists $(\gamma, p) \in \mathrm{T}_2 \times \mathrm{GA}_2$ such that $f = \gamma \circ p$ and $p = (p_1, p_2)$ satisfies $\deg(p_1) < \deg(f_1)$. Let g_i denotes the homogeneous component of degree deg f_i of f_i. From Proposition 1 and Remark 3 following it, we deduce that $d_2|d_1$ and that there exists $\lambda \in \mathbb{C}$ such that $g_1 = \lambda.g_2^{\frac{d_1}{d_2}}$. We can now just write $f = (X + \lambda Y^{\frac{d_1}{d_2}}, Y) \circ p$ where $p \in \mathrm{GA}_2$ and it is easy to check that $p = (p_1, p_2)$ satisfies $\deg(p_1) < \deg(f_1)$.

From this fact, we deduce by an immediate induction on deg f_1 that for each automorphism $f = (f_1, f_2) \in \mathrm{GA}_2$ satisfying $\deg(f_1) \ge \deg(f_2) > 1$, there exists $(\gamma, p) \in \mathrm{T}_2 \times \mathrm{GA}_2$ such that $f = \gamma \circ p$ and $p = (p_1, p_2)$ satisfies $\deg(p_1) < \deg(p_2) = \deg f_2$. It terminates the proof of the existence.

Unicity : Let $\gamma, \delta, p, q \in \mathrm{GA}_2$. We suppose that $\gamma = (X + R(Y), Y), \delta = (X + S(Y), Y)$ where $R(Y)$ and $S(Y)$ are elements of $\mathbb{C}[Y]$ such that val $(R) \ge 1$ and val $(S) \ge 1$. We also suppose that $p = (p_1, p_2)$ and $q = (q_1, q_2)$ satisfy deg $p_1 <$ deg p_2 and deg $q_1 <$ deg q_2. Then, the equality $\gamma \circ p = \delta \circ q$ implies that $\big(p_1 + R(p_2), p_2\big) = \big(q_1 + S(q_2), q_2\big)$ from which we deduce firstly that $p_2 = q_2$ and secondly that $p_1 - q_1 = S(p_2) - R(p_2)$. In the last equality, the left hand term is of degree strictly less than deg p_2 which implies that $S - R = 0$ and finally $\gamma = \delta$ and $p = q$. $\square$

Lemma 3. *Let $f = (f_1, f_2) \in GA_2$ satisfy $f(0,0) = (0,0)$ and deg$(f_1) >$ deg$(f_2) = 1$, then $\exists!\ (\gamma, \beta) \in (UA_2 \setminus GL_2) \times GL_2$ such that $f = \gamma \circ \beta$.*

Proof. Existence : Let $f = (f_1, f_2) \in \mathrm{GA}_2$ satisfy $f(0,0) = (0,0)$ and $\deg(f_1) > \deg(f_2) = 1$. We define β as the linear part of f. Then, if $\beta = (\beta_1, \beta_2)$, we have $\mathbb{C}[\beta_1, \beta_2] = \mathbb{C}[X, Y]$ and so f writes down in the following shape $f = (\beta_1 + g_1(\beta_1, \beta_2), \beta_2)$ where $g_1 \in \mathbb{C}[X, Y]$ and $\mathrm{val}(\mathrm{g}_1) \geq 2$. Let us set $\gamma = (X + g_1, Y)$. On the one hand, we have $f = \gamma \circ \beta$ and on the other hand, γ being an automorphism, we have necessarily $g_1 \in \mathbb{C}[Y]$ and finally $\gamma \in \mathrm{UA}_2 \setminus \mathrm{GL}_2$.

Unicity : If $f = \gamma \circ \beta$ where $(\gamma, \beta) \in (\mathrm{UA}_2 \setminus \mathrm{GL}_2) \times \mathrm{GL}_2$, then β is necessarily equal to the linear part of f and this gives us the uniqueness. $\square$

Lemma 4. *Let $f = (f_1, f_2) \in GA_2$ satisfy $f(0,0) = (0,0)$ and deg$(f_1) >$ deg(f_2), then*

$$\exists!\ l \in \mathbb{N}^*, \exists!\ (\gamma_i)_{1 \leq i \leq l} \in \left(T_2 \setminus GL_2\right)^{l-1} \times \left(UA_2 \setminus GL_2\right), \exists!\ \beta \in GL_2$$

such that $f = \gamma_1 \circ \sigma \circ \gamma_2 \circ \sigma \circ \cdots \circ \sigma \circ \gamma_{l-1} \circ \sigma \circ \gamma_l \circ \beta$.

Proof. Existence : we show it by induction on $\deg(f_2)$. If $\deg(f_2) = 1$, we are done by Lemma 3.

If $\deg(f_2) > 1$, by Lemma 2, f can be writen $f = \gamma \circ \sigma \circ \tilde{f}$ where $(\gamma, \tilde{f}) \in (\mathrm{T}_2 \setminus \mathrm{GL}_2) \times \mathrm{GA}_2$ and $\tilde{f} = (\tilde{f_1}, \tilde{f_2})$ with $\deg(\tilde{f_2}) < \deg(\tilde{f_1}) = \deg(f_2)$. The equality $f(0,0) = (0,0)$ implies the equality $\tilde{f}(0,0) = (0,0)$. We have the result by applying the induction hypothesis to $\tilde{f}$.

Uniqueness : it comes from the uniqueness in Lemmas 2 and 3 by noting that if $l \geq 2$, $(\gamma_i)_{1 \leq i \leq l} \in \left(\mathrm{T}_2 \setminus \mathrm{GL}_2\right)^{l-1} \times \left(\mathrm{UA}_2 \setminus \mathrm{GL}_2\right)$ and $\beta \in \mathrm{GL}_2$, then the automorphism $p = (p_1, p_2) = \sigma \circ \gamma_2 \circ \sigma \circ \cdots \circ \sigma \circ \gamma_{l-1} \circ \sigma \circ \gamma_l \circ \beta$ satisfies $\deg p_1 < \deg p_2$. $\square$

Proof of Theorem 1. Let $f \in \mathrm{GA}_2 \setminus \mathrm{Af}_2$. First of all, whether we are in case i) or ii), it is clear that we must have $(a, b) = f(0,0)$. Conversely, if we set $(a,b) = f(0,0)$, then the automorphism $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) = \tau^{-1}_{(a,b)} \circ f$ satisfies $\tilde{f}(0,0) = (0,0)$. If $\deg(\tilde{f}_1) > \deg(\tilde{f}_2)$, we obtain the result by Lemma 4 applyed to $\tilde{f}$. If $\deg(\tilde{f}_1) \leq \deg(\tilde{f}_2)$, there exists a unique $c \in \mathbb{C}$ such that $\hat{f} = (\hat{f}_1, \hat{f}_2) = \sigma_{-c} \circ \tilde{f}$ satisfies $\deg \hat{f}_1 > \deg \hat{f}_2$. We again apply Lemma 4 to $\hat{f}$ to obtain the result. □

4. On the multidegree of an automorphism of the affine plane.

We introduce here the definition of the multidegree of an automorphism of the affine plane and give a few basic properties of it. The multidegree of an automorphism is an element of D.

Definitions. *D denotes the set of all finite sequences of integers greater than or equal to 2 and D^* denotes the set D with the vacuum sequence omitted.*

We define the following partial order $\leq$ on D :
if $d = (d_1, ..., d_l) \in D$ and if $e = (e_1, ..., e_m) \in D$, then we say that $d \leq e$ if $l \leq m$ and if there exist $1 \leq i_1 < i_2 < \cdots < i_l \leq m$ such that $\forall\ j \in \{1, \ldots, l\}$, $d_j \leq e_{i_j}$. Also, we will denote the concatenation of d and e by $de = (d_1, ..., d_l, e_1, ..., e_m)$.
We set :

$l(d) = l$;
$|d| = d_1 + d_2 + ... + d_l$;
$deg(d) = d_1 \times d_2 \times ... \times d_l$;
$d^{-1} = (d_l, \ldots, d_1)$.

Let us set $\mathrm{B}_2 = \mathrm{Af}_2 \cap \mathrm{BA}_2$. W. van der Kulk proved that GA_2 is the amalgamated product of Af_2 and BA_2 over B_2. The following theorem expresses this result :

Theorem (vdK). *Let $f \in GA_2$, then there exists $l \in \mathbb{N}$, there exist $(\alpha_i)_{1 \leq i \leq l+1} \in (Af_2)^{l+1}$, there exist $(\gamma_i)_{1 \leq i \leq l} \in (BA_2)^l$ such that*

$$\begin{cases} f = \alpha_1 \gamma_1 \alpha_2 \gamma_2 \cdots \alpha_l \gamma_l \alpha_{l+1} \\ \forall\ i \in \{1, \ldots, l\},\ \gamma_i \notin Af_2 \\ \forall\ i \in \{2, \ldots, l\},\ \alpha_i \notin BA_2 \end{cases}$$

Moreover, if $m \in \mathbb{N}$, $(\alpha'_i)_{1\le i\le m+1} \in (Af_2)^{m+1}$ *and* $(\gamma'_i)_{1\le i\le m} \in (BA_2)^m$ *are such that*

$$\begin{cases} f = \alpha'_1\gamma'_1\alpha'_2\gamma'_2\cdots\alpha'_m\gamma'_m\alpha'_{m+1} \\ \forall\ i \in \{1,\dots,m\},\ \gamma'_i \notin Af_2 \\ \forall\ i \in \{2,\dots,m\},\ \alpha'_i \notin BA_2 \end{cases}$$

then $l = m$ *and there exist* $(\beta_i)_{1\le i\le l} \in (B_2)^l$ *and* $(\delta_i)_{1\le i\le l} \in (B_2)^l$ *such that*

$$\begin{cases} \alpha'_1 = \alpha_1\beta_1^{-1} \\ \forall\ i \in \{2,\dots,l\},\ \alpha'_i = \delta_{i-1}\alpha_i\beta_i^{-1} \\ \alpha'_{l+1} = \delta_l\alpha_{l+1} \end{cases}$$

and $\forall\ i \in \{1,\dots,l\},\ \gamma'_i = \beta_i\gamma_i\delta_i^{-1}$.

Remarks.

1. If $(\alpha'_i)_{1\le i\le l+1} \in (\mathrm{Af}_2)^{l+1}$ and $(\gamma'_i)_{1\le i\le l} \in (\mathrm{BA}_2)^l$ satisfy the latter relations, we easily check that

$$\begin{aligned} \alpha'_1\gamma'_1\alpha'_2\gamma'_2\cdots\alpha'_l\gamma'_l\alpha'_{l+1} &= \alpha_1\beta_1^{-1}\beta_1\gamma_1\delta_1^{-1}\delta_1\alpha_2\beta_2^{-1}\beta_2\gamma_2\delta_2^{-1}\delta_2\cdots \\ &\quad \cdots\delta_{l-1}\alpha_l\beta_l^{-1}\beta_l\gamma_l\delta_l^{-1}\delta_l\alpha_{l+1} \\ &= \alpha_1\gamma_1\alpha_2\gamma_2\cdots\alpha_l\gamma_l\alpha_{l+1} \end{aligned}$$

2. If $\gamma \in \mathrm{BA}_2$, then $\gamma \in \mathrm{B}_2$ if and only if $\deg\gamma = 1$. Furthermore, for all β, β' in Af_2, we have $\deg(\beta\gamma\beta') = \deg\gamma$.

Thanks to the Theorem of W. van der Kulk and to the point 2 of the Remarks, we can define the multidegree $d(f)$ of f and the length $l(f)$ of f.

Definition. $d(f) = (\deg\gamma_1,\dots,\deg\gamma_l) \in D$; $l(f) = l(d(f))$.

Remark. The definition of the length $l(f)$ of f that we use in this paper is different from the usual one which is equal to the length of f as an element of the amalgamated product $\mathrm{Af}_2 *_{\mathrm{B}_2} \mathrm{BA}_2$.

Proposition 2. *We have* $\deg f = \deg(d(f))$.

Proof. Let us write $f = \alpha_1\gamma_1\alpha_2\gamma_2\cdots\alpha_l\gamma_l\alpha_{l+1}$ as in the last Theorem.

By exactly the same way as in Proposition 1, we would show by a decreasing induction on i, beginning with $i = l$ and finishing with $i = 1$ that

$$\mathrm{bideg}(\gamma_i\alpha_{i+1}\gamma_{i+1}\cdots\gamma_{l-1}\alpha_l\gamma_l\alpha_{l+1}) = (\prod_{j=i}^{l}\deg\gamma_j,\ \prod_{j=i+1}^{l}\deg\gamma_j).$$

So we have

$$\text{bideg}(\gamma_1\alpha_2\gamma_2\alpha_3\cdots\gamma_{l-1}\alpha_l\gamma_l\alpha_{l+1}) = (\prod_{j=1}^{l} \deg\, \gamma_j, \prod_{j=2}^{l} \deg\, \gamma_j),$$

whence the result. □

The next result is obvious from the definitions.

Proposition 3. $d(f^{-1}) = d(f)^{-1}$. □

Remark. From the last two propositions, we get : $\deg(f^{-1}) = \deg(f)$, which is nothing else than the $n = 2$ case of the following formula of Gabber : any automorphism f of $\mathbb{A}^n_{\mathbb{C}}$ satisfies $\deg\, f^{-1} \le (\deg\, f\,)^{n-1}$.

Proposition 4. *If f and g are elements of GA_2, then $d(fg) \le d(f)d(g)$ and we have the equality if and only if $\deg(fg) = \deg\,(f)\,\deg\,(g)$.*

Proof. Let us prove $d(fg) \le d(f)d(g)$.

If $f \in \mathrm{Af}_2$, we have $d(fg) = d(g)$ and $d(f) = \emptyset$ so that $d(fg) = d(f)d(g)$.

If $f \in \mathrm{BA}_2$, let us write $g = \alpha'_1\gamma'_1\alpha'_2\gamma'_2\cdots\alpha'_m\gamma'_m\alpha'_{m+1}$ as in the last Theorem. If $\alpha'_1 \notin \mathrm{B}_2$, it is clear that $d(fg) = d(f)d(g)$. So, let us suppose $\alpha'_1 \in \mathrm{B}_2$, whence $f\alpha'_1\gamma'_1 \in \mathrm{BA}_2$. We now claim that :

either $f\alpha'_1\gamma'_1 \in \mathrm{B}_2$, in which case $d(fg) = (\deg\, \gamma'_2, \dots, \deg\, \gamma'_m)$ so that $d(fg) \le d(g) \le d(f)d(g)$,

either $f\alpha'_1\gamma'_1 \notin \mathrm{B}_2$, in which case $\deg\, f\alpha'_1\gamma'_1 \le \max\, \{\deg\, f, \deg\, \gamma'_1\}$ so that $d(fg) = (\deg\, f\alpha'_1\gamma'_1, \deg\, \gamma'_2, \dots, \deg\, \gamma'_m) \le d(f)d(g)$.

Now that we have proven $d(fg) \le d(f)d(g)$ when f is either in Af_2 or in BA_2, we can deduce the same inequality for any f by writing $f = \alpha_1\gamma_1\alpha_2\gamma_2\cdots\alpha_l\gamma_l\alpha_{l+1}$ as in the last Theorem.

Indeed

$$\begin{aligned} d(fg) &= d(\alpha_1\gamma_1\alpha_2\gamma_2\cdots\alpha_l\gamma_l\alpha_{l+1}g) \\ &\le d(\alpha_1)d(\gamma_1)d(\alpha_2)d(\gamma_2)\cdots d(\alpha_l)d(\gamma_l)d(\alpha_{l+1})d(g) \\ &\le d(\gamma_1)d(\gamma_2)\cdots d(\gamma_l)d(g) \\ &\le d(f)d(g). \end{aligned}$$

If $d(fg) = d(f)d(g)$, it is clear that $\deg(fg) = \deg\,(f)\,\deg\,(g)$ by Proposition 2. Conversely, if $d(fg) \neq d(f)d(g)$, then $d(fg) < d(f)d(g)$ so that

$\deg d(fg) < \deg \big(d(f)d(g)\big)$ and we obtain

$$\begin{array}{rcl} \deg\ (fg) & = & \deg\big(d(fg)\big) \\ & < & \deg\ \big(d(f)d(g)\big) \quad = \quad \deg f\ \deg g \end{array}$$

□

For any $d \in D$, we define the sets U_d and V_d which will be in the central position of section 6. In the present section, we will show that they have a very simple characterization in terms of the multidegree.

Definition. *We set :*

$$\begin{array}{lccccccc} \phi_d : & Af_2 & \times & \prod_{i=1}^{l} T_{2,d_i} & \times & GL_2 & \to & GA_{2,deg(d)} \\ & (\ \alpha & , & (\gamma_1, ..., \gamma_l) & , & \beta\ ) & \mapsto & \alpha\gamma_1\sigma\gamma_2\cdots\sigma\gamma_l\beta \end{array}$$

$U_d = Im(\phi_d)$;
$\psi_d = \phi_{d|Af_2 \times \prod_{i=1}^{l-1}(T_{2,d_i} \setminus T_{2,d_i-1}) \times (UA_{2,d_l} \setminus UA_{2,d_l-1}) \times GL_2}$ *and finally*
$V_d = Im(\psi_d)$.

Proposition 5. *For any $d \in D$, $V_d = \{f \in GA_2$ such that $d(f) = d\}$.*

Proof. For any $d \in D$, let us set $V'_d = \{f \in \mathrm{GA}_2$ such that $d(f) = d\}$. We have clearly $V_d \subset V'_d$. By the Theorem 1, we have $\mathrm{GA}_2 = \coprod_{d \in D} V_d$. The equality $\mathrm{GA}_2 = \coprod_{d \in D} V'_d$ now shows us that $V_d = V'_d$. □

Before characterizing nicely U_d, we need Proposition 6, which is a decomposition theorem for elements of V_d. When $\alpha = \sigma$, the following lemma is nothing else than the existence result of Lemma 2.

Lemma 5. *Let $f = (f_1, f_2) \in GA_2$ satisfy $\deg\,(f_1) > \deg\,(f_2) > 1$ and let $\alpha \in Af_2 \setminus B_2$, then there exists $(\gamma, p) \in (T_2 \setminus GL_2) \times GA_2$ such that $f = \gamma\alpha p$ with $\deg p_1 > \deg p_2$ and $\deg p_1 = \deg f_2$.*

Proof. Because $\alpha \notin \mathrm{B}_2$, we have $\alpha^{-1} \notin \mathrm{B}_2$ so that $\alpha^{-1} = (aX + bY + c, dX + eY + f)$ where $a,\ b,\ c,\ d,\ e,\ f$ are complex numbers with $d \neq 0$.

On the one hand, we have

$$\alpha^{-1} = (aX + (b - \frac{ae}{d})Y + c, dX + f)(X + \frac{e}{d}Y, Y)$$

which is equivalent to

$$(X - \frac{e}{d}Y, Y) = \alpha(aX + (b - \frac{ae}{d})Y + c, dX + f)$$

and on the other hand, by Lemma 2, we know the existence of $(\gamma', g) \in (\mathrm{T}_2 \setminus \mathrm{GL}_2) \times \mathrm{GA}_2$ such that $\gamma' f = g$ with $\deg g_1 < \deg g_2 = \deg f_2$. So, we have :

$$(X - \frac{e}{d}Y, Y)\gamma' f = \alpha(ag_1 + (b - \frac{ae}{d})g_2 + c, dg_1 + f).$$

By setting

$$\begin{cases} \gamma = \gamma'^{-1}(X + \frac{e}{d}Y, Y) \\ p = (ag_1 + (b - \frac{ae}{d})g_2 + c, dg_1 + f) \end{cases}$$

we have the desired result (we use the fact that $b - \frac{ae}{d} \neq 0$, which is true because $(aX + bY + c, dX + eY + f)$ is an automorphism). □

Proposition 6. *If $f \in GA_2$ satisfies $d(f) = d \in D$ with $l(d) = l \in \mathbb{N}$, then, for any $(\alpha_i)_{1 \le i \le l-1} \in (GL_2 \setminus B_2)^{l-1}$, there exist $(\alpha, \beta) \in Af_2 \times GL_2$ and $(\gamma_i)_{1 \le i \le l} \in (T_2)^l$ such that*

$$f = \alpha\gamma_1\alpha_1 \cdots \gamma_{l-1}\alpha_{l-1}\gamma_l\beta.$$

Proof. If $l = 0$, there is nothing to do. So, let us suppose $l \geq 1$, so that $\deg f \geq 2$. We know the existence of $\alpha \in \mathrm{Af}_2$ such that $g = \alpha^{-1} f$ satisfies $g(0, 0) = (0, 0)$ and $\deg(g_1) > \deg(g_2)$.

Now, the result comes from the fact that if $(\alpha_i)_{i\in\mathbb{N}^*} \in (\mathrm{GL}_2 \setminus \mathrm{B}_2)^{\mathbb{N}^*}$ and if $g \in \mathrm{GA}_2$ satisfies $g(0, 0) = (0, 0)$ and $\deg(g_1) > \deg(g_2)$, then there exist $m \in \mathbb{N}^*$, $(\gamma_i)_{1\le i\le m} \in (T_2 \setminus \mathrm{GL}_2)^m$ and $\beta \in \mathrm{GL}_2$ such that $g = \gamma_1\alpha_1 \cdots \gamma_{m-1}\alpha_{m-1}\gamma_m\beta$. This result is proved by exactly the same way as the Lemma 4 (of course using Lemma 5 instead of Lemma 2). □

Proposition 7. *For any e in D, we have $U_e = \coprod_{d\le e} V_d$.*

Proof. By the Proposition 4, it is clear that $U_e \subset \coprod_{d\le e} V_d$.

Conversely, to prove that $d \le e$ implies $V_d \subset U_e$, it is sufficient to prove that if $e = (e_1, \ldots, e_m)$, $k \in \{1, \ldots, m\}$ and $d = (e_1, \ldots, \hat{e_k}, \ldots, e_m)$ then

$V_d \subset U_e$. Let $f \in V_d$. By definition, there exist $(\alpha, \beta) \in \mathrm{Af}_2 \times \mathrm{GL}_2$ and $(\gamma_i)_{1,\dots,\hat{k},\dots,m} \in (T_2)^{m-1}$ such that

$$f = \alpha\gamma_1\sigma\gamma_2\sigma\cdots\sigma\gamma_{k-1}\sigma\gamma_{k+1}\sigma\cdots\sigma\gamma_m\beta,$$

i.e. $f = \phi_d\big(\alpha, (\gamma_1, \dots, \hat{\gamma_k}, \dots, \gamma_m), \beta\big)$. If $k = 1$, we have

$$f = \phi_e\big(\alpha\sigma, (\mathrm{Id}, \gamma_2, \dots, \gamma_m), \beta\big)$$

and if $k = m$, we have

$$f = \phi_e\big(\alpha, (\gamma_1, \dots, \gamma_{m-1}, \mathrm{Id}), \sigma\beta\big)$$

so that in these two cases we have $f \in U_e$.

Let us suppose that $k \in \{2, \dots, m-1\}$. We set $\theta = (X+Y, Y) \in \mathrm{T}_2$, so that $\sigma\theta\sigma = (X, X+Y) \in \mathrm{GL}_2 \setminus \mathrm{B}_2$. So, if we define $(\alpha_i)_{1 \le i \le m-1} \in (\mathrm{GL}_2 \setminus \mathrm{B}_2)^{m-1}$ by $\alpha_i = \sigma$ if $i \neq k$ and $\alpha_k = \sigma\theta\sigma$, by the Proposition 6, we deduce the existence of $(\alpha', \beta') \in \mathrm{Af}_2 \times \mathrm{GL}_2$ and $(\gamma'_i)_{1,\dots,\hat{k},\dots,m} \in (T_2)^{m-1}$ such that

$$f = \alpha'\gamma'_1\sigma\gamma'_2\sigma\cdots\sigma\gamma'_{k-1}\sigma\theta\sigma\gamma'_{k+1}\sigma\cdots\sigma\gamma'_m\beta',$$

i.e.

$$f = \phi_e\big(\alpha', (\gamma'_1, \dots, \gamma'_{k-1}, \theta, \gamma'_{k+1}, \dots, \gamma'_m), \beta'\big).$$

So $f \in U_e$. □

Corollary 1. $U_d = \{f \in \mathrm{GA}_2,\ \text{such that } d(f) \le d\}$ □

Corollary 2. $d \le e$ is equivalent to $U_d \subset U_e$. □

5. Automorphisms of length less than or equal to one.

The next proposition gives us some simple characterizations for an automorphism to be of length less than or equal to one :

Proposition 8. *Let $f = (f_1, f_2) \in GA_2$. Write $f = \sum_{i=0}^{\infty}(P_i, Q_i)$, where the (P_i, Q_i) are homogeneous $\mathbb{C}$-endomorphisms of $\mathbb{A}^2_{\mathbb{C}}$ of degree i and set $(P, Q) = (\sum_{i=2}^{\infty} P_i, \sum_{i=2}^{\infty} Q_i)$. Then, the six following assertions are equivalent :*

i) P and Q are linearly dependent polynomials.
ii) $l(f) \leq 1$
iii) $\forall\ i,j \geq 2, Jac(P_i, Q_j) = 0$
iv) $Jac(P,Q) = 0$
v) P and Q are algebraically dependent over $\mathbb{C}$.
vi) $\exists\ (u,v) \in \mathbb{C}[t], \exists\ R\ \in \mathbb{C}[X,Y]$ such that $(P,Q) = \big(u(R), v(R)\big)$.

We use the following lemma :

Lemma 6. *Let $(u,v) \in \mathbb{C}[t]^2 \setminus \{(0,0)\}$ and $R \in \mathbb{C}[X,Y]$ such that $u(R)\frac{\partial R}{\partial X} + v(R)\frac{\partial R}{\partial Y} = 0$, then there exist $(a,b) \in \mathbb{C}^2$ and $w \in \mathbb{C}[t]$ such that $R = w(aX + bY)$.*

Proof. We can suppose that u and v are relatively prime, so that $u(0) \neq 0$ or $v(0) \neq 0$. The relation $R \mid u(0)\frac{\partial R}{\partial X} + v(0)\frac{\partial R}{\partial Y}$ shows us that $u(0)\frac{\partial R}{\partial X} + v(0)\frac{\partial R}{\partial Y} = 0$. By making a linear change of coordinates in this last equality, we obtain the existence of $w \in \mathbb{C}[t]$ such that $R = w(v(0)X - u(0)Y)$. □

Proof of Proposition 8. The implications i) ⇒ ii) ⇒ iii) ⇒ iv) are clear. The equivalence of iv), v) and vi) comes from the equivalence of the following three assertions (see [No]), where g_1 and g_2 are any elements of $\mathbb{C}[X,Y]$:
1) $\mathrm{Jac}(g_1, g_2) = 0$
2) g_1 et g_2 are algebraically dependent over $\mathbb{C}$.
3) $\exists\ (u,v) \in \mathbb{C}[t], \exists\ R \in \mathbb{C}[X,Y]$ such that $(g_1, g_2) = \big(u(R), v(R)\big)$.

Let us now prove that vi) ⇒ i). We can suppose that $f = \big(X + u(R), Y + v(R)\big)$ where $u(0) = v(0) = R(0,0) = 0$, $\mathrm{val}\big(u(R)\big) \geq 2$ and $\mathrm{val}\big(v(R)\big) \geq 2$. Since f is an automorphism, we necessarily have Jac(f) = 1, which implies that $u'(R)\frac{\partial R}{\partial X} + v'(R)\frac{\partial R}{\partial Y} = 0$. By Lemma 6, if $(u', v') \neq (0,0)$, there exist $(a,b) \in \mathbb{C}^2$ and $w \in \mathbb{C}[t]$ such that $R = w(aX + bY)$. So, we can suppose that $R = aX + bY$. The condition $u'(R)\frac{\partial R}{\partial X} + v'(R)\frac{\partial R}{\partial Y} = 0$ implies that $u'(R)$ and $v'(R)$ are linearly dependent. It follows easily that $u(R)$ and $v(R)$ are also linearly dependent. □

6. On the irreducible components of $\mathbf{GA}_{2,n}$.

If n is a positive integer, let $\mathrm{E}_{2,n}$ be the vector space of $\mathbb{C}$-endomorphisms f of $\mathbb{A}^2_{\mathbb{C}}$ satisfying $\deg(f) \leq n$. Let us set $\mathrm{GA}_{2,n} = \mathrm{GA}_2 \cap \mathrm{E}_{2,n}$, $\mathrm{T}_{2,n} = \mathrm{T}_2 \cap \mathrm{E}_{2,n}$ and $\mathrm{UA}_{2,n} = \mathrm{UA}_2 \cap \mathrm{E}_{2,n}$.

We set $\mathrm{J}_{2,n} = \{f \in \mathrm{E}_{2,n},\ \text{such that Jac } f = 1\}$ and $\mathrm{G}_{2,n} = \mathrm{GA}_{2,n} \cap \mathrm{J}_{2,n}$.

$\mathrm{E}_{2,n}$ is an affine space and $\mathrm{J}_{2,n}$ is a (Zariski) closed subset of it. H. Bass, E.H. Connell and D. Wright show in the paper [B-C-W] that $G_{2,n}$ is also a closed subset of $\mathrm{E}_{2,n}$. By slightly modifying their proof, we obtain that $\mathrm{GA}_{2,n}$ is a locally closed subset of $\mathrm{E}_{2,n}$. In particular, it is an algebraic variety. We have $\mathrm{GA}_{2,n} \simeq \mathbb{C}^* \times \mathrm{G}_{2,n}$ via $f = (f_1, f_2) \mapsto \big(\mathrm{Jac}(f), (\frac{f_1}{\mathrm{Jac}(f)}, f_2)\big)$, so that the irreducible components of $\mathrm{GA}_{2,n}$ are in one to one correspondence with the irreducible components of $\mathrm{G}_{2,n}$. Our interest in the irreducible components of $\mathrm{G}_{2,n}$ is motivated by the Jacobian Conjecture in dimension 2 and degree n asserting that an element of $\mathrm{E}_{2,n}$ is invertible if and only if its Jacobian is a nonzero constant. This conjecture is equivalent to the equality $\mathrm{J}_{2,n} = \mathrm{G}_{2,n}$.

In Remark (1.7) of the last quoted paper, the authors note that to show this equality, it would suffice to show, if possible, that $\dim \mathrm{J}_{2,n} = \dim \mathrm{G}_{2,n}$ and that the algebraic variety $\mathrm{J}_{2,n}$ is irreducible (because we have of course $\mathrm{J}_{2,n} \subset \mathrm{G}_{2,n}$).

We answer here negatively to this hope. Indeed, on the one hand $\mathrm{J}_{2,n} = \mathrm{G}_{2,n}$ when $n \leq 100$, by T.T. Moh ([Mo]) and on the other hand, the Proposition 11 asserts that $\mathrm{G}_{2,n}$ is reducible when $n \geq 4$. We obtain thus the Corollary of Proposition 11 : the variety $\mathrm{J}_{2,n}$ is reducible when $4 \leq n \leq 100$.

The results of section 3 lead us to believe that we can get some insight into the decomposition in irreducible components of $\mathrm{GA}_{2,n}$ via purely combinatorial means (see the Conjecture formulated in this section). Before stating this conjecture, we will give a few definitions and apply them to obtain various results which will end by the computation of the dimension of $\mathrm{GA}_{2,n}$ (see Proposition 10).

Let us recall that for any $d \in D$, we introduced in section 4 the subsets U_d and V_d of GA_2 which turned out to be

$$\begin{cases} U_d = \{f \in \mathrm{GA}_2 \text{ such that } d(f) \leq d\} \\ V_d = \{f \in \mathrm{GA}_2 \text{ such that } d(f) = d\} \end{cases}$$

Definition. *We note that by the definition of U_d and V_d (see section 4), their Zariski closures in $GA_{2,deg(d)}$ are equal.*

We set $W_d = \overline{U_d} = \overline{V_d}$ (Zariski closure in $GA_{2,deg(d)}$).

If n is an integer greater than or equal to two, the element $(n) \in D$ will play an important role in the following lines. We begin with a crucial result :

Proposition 9. *If n is an integer greater than or equal to two, then $U_{(n)}$ is a closed subvariety of $GA_{2,n}$.*

Proof. $U_{(n)} = \{f \in \mathrm{GA}_{2,n}$ such that $l(f) \leq 1\}$ and the condition iii) (or iv)) of Proposition 8 shows us that this condition is closed. □

Lemma 7. *If $d \in D$, then W_d is an irreducible variety of dimension $|d|+6$.*

Proof. If $d = \emptyset$, the lemma is true. Let us now suppose that $d \in D^*$.

We have :

$$W_d = \overline{\psi_d(\mathrm{Af}_2 \times \prod_{i=1}^{l-1}(\mathrm{T}_{2,d_i} \setminus \mathrm{T}_{2,d_i-1}) \times (\mathrm{UA}_{2,d_l} \setminus \mathrm{UA}_{2,d_l-1}) \times \mathrm{GL}_2)}$$

where $\mathrm{Af}_2 \times \prod_{i=1}^{l-1}(\mathrm{T}_{2,d_i} \setminus \mathrm{T}_{2,d_i-1}) \times (\mathrm{UA}_{2,d_l} \setminus \mathrm{UA}_{2,d_l-1}) \times \mathrm{GL}_2$ is an irreducible variety (as a product of irreducible varieties) so that W_d is an irreducible variety.

Let us set :

$\psi_{1,d} = \psi_{d|\mathrm{A}_1 \times \prod_{i=1}^{l-1}(\mathrm{T}_{2,d_i} \setminus \mathrm{T}_{2,d_i-1}) \times (\mathrm{UA}_{2,d_l} \setminus \mathrm{UA}_{2,d_l-1}) \times \mathrm{GL}_2}$
where A_1 is the group of translations of $\mathbb{A}^2_{\mathbb{C}}$;

$\psi_{2,d} = \psi_{d|\mathrm{A}_2 \times \prod_{i=1}^{l-1}(\mathrm{T}_{2,d_i} \setminus \mathrm{T}_{2,d_i-1}) \times (\mathrm{UA}_{2,d_l} \setminus \mathrm{UA}_{2,d_l-1}) \times \mathrm{GL}_2}$
where A_2 is the set of automorphisms of $\mathbb{A}^2_{\mathbb{C}}$ of the shape $(Y+a, X+cY+b)$ where a, b, c are elements of $\mathbb{C}$;

$V_{1,d} = \mathrm{Im}(\psi_{1,d})$;

$V_{2,d} = \mathrm{Im}(\psi_{2,d})$.

We have $V_d = V_{1,d} \sqcup V_{2,d}$ (by Theorem 1). $\psi_{1,d}$ and $\psi_{2,d}$ being injective morphisms (always by Theorem 1), it comes out : $\dim(\overline{V_{1,d}}) = |d| + 5$ and

$\dim(\overline{V_{2,d}}) = |d| + 6$. Finally,

$$\begin{array}{rcl} \dim W_d & = & \dim \overline{V_d} \\ & = & \dim(\overline{V_{1,d}} \cup \overline{V_{2,d}}) \\ & = & \max\{\dim \overline{V_{1,d}}, \dim \overline{V_{2,d}}\} \\ & = & |d| + 6 \end{array}$$

□

Definition. *For $n \in \mathbb{N}^*$, we set $D_n = \{d \in D, deg(d) \leq n\}$ and C_n denotes the set of maximal elements of D_n.*

For example :
$C_1 = \{\emptyset\}$;
$C_2 = \{(2)\}$;
$C_3 = \{(3)\}$;
$C_4 = \{(4), (2, 2)\}$;
$C_5 = \{(5), (2, 2)\}$;
$C_6 = \{(6), (2, 3), (3, 2)\}$;
$C_7 = \{(7), (2, 3), (3, 2)\}$;
$C_8 = \{(8), (2, 4), (4, 2), (2, 2, 2)\}$;
$C_9 = \{(9), (2, 4), (3, 3), (4, 2), (2, 2, 2)\}$.

The next lemma is obvious :

Lemma 8. *If n is a nonzero positive integer, then we have :*

$$GA_{2,n} = \bigcup_{d \in C_n} U_d = \coprod_{d \in D_n} V_d = \bigcup_{d \in C_n} W_d.$$

□

Proposition 10. *If n is a nonzero positive integer, then $GA_{2,n}$ is an algebraic variety of dimension 6 if $n = 1$ and of dimension $n + 6$ otherwise.*

Proof. By lemma 7 and 8, we have :

$$\dim \text{ GA}_{2,n} = \max_{d \in D_n} |d| + 6$$

If $n = 1$, $D_1 = \{\emptyset\}$ and the proposition is clear.

If $n > 1$, $\forall\ d = (d_1, ..., d_l) \in D_n$, we have :

$$|d| = \sum_{i=1}^{l} d_i \leq \prod_{i=1}^{l} d_i \leq n$$

hence, $\max_{d \in D_n} |d|$ is obtained for $d = (n)$ where it is equal to n. □

Being now familiarized with the definitions, we give the announced conjecture :

Conjecture. *If n is a nonzero positive integer, then $GA_{2,n} = \bigcup_{d \in C_n} W_d$ is the decomposition in irreducible components of $GA_{2,n}$.*

This conjecture is equivalently formulated as :

$$\forall\ (d, e) \in (C_n)^2, d \neq e \Rightarrow W_d \not\subset W_e.$$

We will prove this conjecture in case $n \leq 9$ by using the next lemma :

Lemma 9. *If n is an integer greater than or equal to two and if $d \in C_n$ with $d \neq (n)$, then we have :*

$$W_d \not\subset W_{(n)} \text{ and } W_{(n)} \not\subset W_d.$$

Proof. U_d contains an automorphism f such that $l(f) \geq 2$ and so : $U_d \not\subset U_{(n)}$.

$U_{(n)}$ being closed, this implies $W_d \not\subset W_{(n)}$. We deduce at once from this that $W_{(n)} \not\subset W_d$, because, if we had $W_{(n)} \subset W_d$, the inequality $\dim W_d \leq \dim W_{(n)}$ together with the irreducibility of W_d, would imply $W_{(n)} = W_d$, which is not the case. □

Theorem 2. *If $1 \leq n \leq 9$, then the Conjecture is true.*

Proof. If $n = 1$, then $\mathrm{GA}_{2,n} = W_{(\emptyset)}$ and if $2 \leq n \leq 3$, then it is clear that $\mathrm{GA}_{2,n} = W_{(n)}$ and the conjecture is then checked.

Let us now suppose that $4 \leq n \leq 9$. By Lemma 9, it is sufficient to prove that if d and e are two elements of C_n different from (n), then $W_d \not\subset W_e$. But the condition $n \leq 9$ implies the equality dim $W_d =$ dim W_e. It is then sufficient to show that $W_d \neq W_e$. Now, V_d is a dense constructible subset of W_d and V_e is a dense constructible subset of W_e. Furthermore, $V_d \cap V_e = \emptyset$. This proves that we cannot have $W_d = W_e$. □

Even if we cannot yet compute the number of irreducible components of $\mathrm{GA}_{2,n}$, we can however decide whether this variety is irreducible :

Proposition 11. *If n is a nonzero positive integer, then the variety $GA_{2,n}$ is irreducible if and only if $n \leq 3$.*

Proof. It is sufficient to prove that if $n \geq 4$ is any integer, then $\mathrm{GA}_{2,n}$ is reducible. Now, $W_{(n)}$ is an irreducible component of $\mathrm{GA}_{2,n}$ (because it is an irreducible variety of the same dimension as $\mathrm{GA}_{2,n}$). The element $(X + Y^2, Y + X^2 + 2XY^2 + Y^4)$ is in $W_{(2,2)} \subset \mathrm{GA}_{2,n}$ but not in $W_{(n)}$. □

Corollary of Proposition 11. *If $4 \leq n \leq 100$, then the variety $J_{2,n}$ is reducible.* □

Remark. It is not true that the mapping

$$\begin{array}{rccc} d: & \mathrm{GA}_2 & \to & D \\ & f & \mapsto & d(f) \end{array}$$

is a lower-continuous function, i.e. if $d \in D$, it is not necessarily true that U_d is a closed subset of $\mathrm{GA}_{2,\deg(d)}$.

To point out this fact, we can just consider the family of GA_2 given by the Nagata automorphism :

$$\begin{array}{rccc} \chi: & \mathbb{C} & \to & \mathrm{GA}_{2,4} \\ & Z & \mapsto & \chi(Z), \end{array}$$

where $\chi(Z) = (X - 2Y(XZ + Y^2) - Z(XZ + Y^2)^2, Y + Z(XZ + Y^2))$. Then, if $Z \in \mathbb{C}^*$, we have $\chi(Z) = \gamma_1 \alpha_1 \gamma_2$ where

$$\begin{cases} \gamma_1 = (X - Z^{-1}Y^2, Y) \in \mathrm{BA}_2 \setminus \mathrm{B}_2 \\ \alpha_1 = (X, Y + Z^2 X) \in \mathrm{Af}_2 \setminus \mathrm{B}_2 \\ \gamma_2 = (X + Z^{-1}Y^2, Y) \in \mathrm{BA}_2 \setminus \mathrm{B}_2 \end{cases}$$

so that $d(\chi(Z)) = (2,2)$. However $d(\chi(0)) = (3)$ so that $\chi(0) \in \overline{U_{(2,2)}} \backslash U_{(2,2)}$.

7. More details on the irreducible components of $\mathbf{GA}_{2,4}$.

It results from Theorem 2 that $W_{(4)} \not\subset W_{(2,2)}$. The next proposition gives us an explicit equation satisfied by all elements of $W_{(2,2)}$ but not necessarily by elements of $W_{(4)}$.

Proposition 12. *Let* $f = (f_1, f_2) \in W_{(2,2)}$. *Write* $f = \sum_{i=0}^{4} (P_i, Q_i)$, *where* (P_i, Q_i) *are* $\mathbb{C}$*-endomorphisms of* $\mathbb{A}^2_{\mathbb{C}}$ *which are homogeneous of degree* i. *Then,* $P_2P_3 - 2P_1P_4$ *and* $Q_2Q_3 - 2Q_1Q_4$ *are proportional polynomials.*

This is equivalent to the algebraic condition :

$$Jac(P_2P_3 - 2P_1P_4, Q_2Q_3 - 2Q_1Q_4) = 0.$$

Proof. Let $f = (f_1, f_2) = \sum_{i=0}^{4} (P_i, Q_i) \in \mathrm{Im}(\phi_{(2,2)})$ where (P_i, Q_i) are $\mathbb{C}$-endomorphisms of $\mathbb{A}^2_{\mathbb{C}}$ which are homogeneous of degree i.

We begin by proving that there exists a complex number λ such that

$$(Q_2Q_3 - 2Q_1Q_4)^2 = \lambda (P_1Q_2 - P_2Q_1)^2 Q_4.$$

If $Q_4 = 0$, we see easily that $Q_3 = 0$. In this case, it is sufficient to take $\lambda = 0$.

Otherwise, there exists a complex number α such that $\deg(f_1 - \alpha f_2) = 2$. We set $g = (g_1, g_2) = (f_1 - \alpha f_2, f_2) = (\sum_{i=0}^{2} (P_i - \alpha Q_i), \sum_{i=0}^{4} Q_i) \in \mathrm{GA}_2$. There exists $w \in \mathbb{C}[t]$ such that $\deg(g_2 - w(g_1)) \leq 2$. We necessarily have $\deg w = 2$. Let β be the coefficient of t^2 in w. We obtain the relations :

$$\begin{cases} Q_3 = 2\beta(P_1 - \alpha Q_1)(P_2 - \alpha Q_2) \\ Q_4 = \beta(P_2 - \alpha Q_2)^2 \end{cases}$$

which give us successively :

$$\begin{array}{rcl} Q_2Q_3 - 2Q_1Q_4 & = & 2\beta(P_2 - \alpha Q_2)((P_1 - \alpha Q_1)Q_2 - (P_2 - \alpha Q_2)Q_1) \\ Q_2Q_3 - 2Q_1Q_4 & = & 2\beta(P_2 - \alpha Q_2)(P_1Q_2 - P_2Q_1) \\ (Q_2Q_3 - 2Q_1Q_4)^2 & = & 4\beta^2(P_2 - \alpha Q_2)^2(P_1Q_2 - P_2Q_1)^2 \\ (Q_2Q_3 - 2Q_1Q_4)^2 & = & 4\beta(P_1Q_2 - P_2Q_1)^2Q_4 \end{array}$$

Whence the existence of λ. By the same argument, we show the existence of μ such that $(P_2P_3 - 2P_1P_4)^2 = \mu(P_1Q_2 - P_2Q_1)^2P_4$. Since P_4 and Q_4 are proportional polynomials, $(P_2P_3 - 2P_1P_4)^2$ and $(Q_2Q_3 - 2Q_1Q_4)^2$ also and finally $P_2P_3 - 2P_1P_4$ and $Q_2Q_3 - 2Q_1Q_4$ too.
We have thus proved the relation :

$$\mathrm{Jac}(P_2P_3 - 2P_1P_4, Q_2Q_3 - 2Q_1Q_4) = 0$$

and this relation remains true if we only suppose that $f \in W_{(2,2)} = \overline{\mathrm{Im}(\phi_{(2,2)})}$.
□

Proposition 12 is enough to prove that $g = (X + Y^4, X + Y + Y^4) \in W_{(4)} \setminus W_{(2,2)}$. Indeed, the homogeneous components (P_i, Q_i) of degree i of g are $(P_1, Q_1) = (X, X + Y)$, $(P_2, Q_2) = (0, 0)$, $(P_3, Q_3) = (0, 0)$ and $(P_4, Q_4) = (Y^4, Y^4)$ so that $P_2P_3 - 2P_1P_4$ and $Q_2Q_3 - 2Q_1Q_4$ are equal to $-2XY^4$ and $-2(X + Y)Y^4$ which are not proportional polynomials.

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