Plane Polynomial Automorphisms of Fixed Multidegree

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Abstract

Let $\mathcal{G}$ be the group of polynomial automorphisms of the complex affine plane. On one hand, $\mathcal{G}$ can be endowed with the structure of an infinite dimensional algebraic group (see [26]) and on the other hand there is a partition of $\mathcal{G}$ according to the multidegree (see [6]). Let $\mathcal{G}_d$ denote the set of automorphisms whose multidegree is equal to d. We prove that $\mathcal{G}_d$ is a smooth, locally closed subset of $\mathcal{G}$ and show some related results. We give some applications to the study of the varieties $\mathcal{G}_{=m}$ (resp. $\mathcal{G}_{\leq m}$) of automorphisms whose degree is equal to m (resp. is less than or equal to m).

Keywords

Affine space, Polynomial automorphisms.

Introduction

The study of the infinite dimensional algebraic variety of polynomial automorphisms of the affine space has been initiated by Shafarevich in [25]. However, this paper contains some inaccuracies and this theory remains mysterious (see [26, 27, 13, 14]) . In the present paper, we carry on with the work begun in [7, 8]. We try to relate the algebraic and the amalgamated structures of the group of complex plane polynomial automorphisms.

The complex affine N-space is denoted by $\mathbb{A}^N$. A polynomial endomorphism of $\mathbb{A}^2$ is identified with its sequence $f = (f_1, f_2)$ of coordinate functions $f_j \in \mathbb{C}[X, Y]$. We define its degree by $\deg f = \max\{\deg f_1, \deg f_2\}$.

A subset of some topological space is called locally closed when it is the intersection of an open and a closed subset. If Z is such a subset and $\overline{Z}$ its closure, this amounts to saying that $\overline{Z} \setminus Z$ is closed.

The space $\mathcal{E} := \mathbb{C}[X, Y]^2$ of polynomial endomorphisms of $\mathbb{A}^2$ is naturally an infinite dimensional algebraic variety (see [25, 26] for the definition). This roughly means that $\mathcal{E}_{\leq m} := \{f \in \mathcal{E}, \deg f \leq m\}$ is a (finite dimensional) algebraic variety for any $m \geq 1$, which comes out from the fact that it is an affine space. If $Z \subseteq \mathcal{E}$, we set $Z_{\leq m} := Z \cap \mathcal{E}_{\leq m}$. The space $\mathcal{E}$ is endowed with the topology of the inductive limit, in which Z is closed (resp. open, resp. locally closed) if and only if $Z_{\leq m}$ is closed (resp. open, resp. locally closed) in $\mathcal{E}_{\leq m}$ for any m. In the same way, the space $\mathcal{P} := \mathbb{C}[X,Y]$ is naturally an infinite dimensional algebraic variety. Let $\mathcal{G}$ be the group of polynomial automorphisms of $\mathbb{A}^2$. Since $\mathcal{G}$ is locally closed in $\mathcal{E}$ (see [2, 25, 26]), it is naturally an infinite dimensional algebraic variety.

Using the amalgamated structure of $\mathcal{G}$ (see [12, 16, 21]), one can define the multidegree (see [6, 7, 4]) and length (see [7]) of any of its elements. Let $\mathcal{A}$ be the group of affine automorphisms of $\mathbb{A}^2$ and let $\mathcal{B} := \{(aX + p(Y), bY + c),\ a, b, c \in \mathbb{C}, p \in \mathbb{C}[Y],\ ab \neq 0\}$ be the group of triangular automorphisms ($\mathcal{B}$ may be viewed as a Borel subgroup of $\mathcal{G}$). Any automorphism admits a *reduced* expression

$$f = \alpha_1 \circ \beta_1 \circ \cdots \circ \alpha_k \circ \beta_k \circ \alpha_{k+1}$$

where the α_j's (resp. β_j's) belong to $\mathcal{A}$ (resp. $\mathcal{B}$) and where the β_j's do not belong to $\mathcal{A}$ and the α_j's (for $2 \leq j \leq k$) do not belong to $\mathcal{B}$. The multidegree and length are then defined by

$$\operatorname{mdeg} f := (\deg\, \beta_1, \ldots, \deg\, \beta_k) \quad \text{and} \quad l(f) := k.$$

This definition does not depend on the chosen reduced expression, but only on f. We recall that degree and multidegree are related by the formula:

$$\deg f = \deg\, \beta_1 \times \cdots \times \deg\, \beta_k.$$

The set of multidegrees, i.e. of finite sequences of integers ≥ 2 (including the empty sequence) is denoted by $\mathcal{D}$. If $d \in \mathcal{D}$, let us set $\mathcal{G}_d = \{f \in \mathcal{G},\ \operatorname{mdeg} f = d\}$.

By an algebraic family of automorphisms, we mean a morphism from a complex algebraic variety to $\mathcal{G}$. If the variety is connected, we say that the family is connected. What can be said on a family of automorphisms with respect to the multidegree? A source of inspiration is given by the Nagata automorphism (see [21]):

$$f := (X - 2Y(XZ + Y^2) - Z(XZ + Y^2)^2,\ \ Y + Z(XZ + Y^2),\ \ Z).$$

This automorphism of $\mathbb{A}^3$ can be seen as an automorphism of $\mathbb{A}^2_{\mathbb{C}[Z]}$ inducing as well the family of automorphisms $\mathbb{A}^1 \to \mathcal{G}$, $z \mapsto f_z$. If $z \neq 0$, the factorization

$$f_z = (X - z^{-1}Y^2, Y) \circ (X, Y + z^2X) \circ (X + z^{-1}Y^2, Y)$$

shows that f_z has multidegree $(2,2)$. If $z = 0$, $f_0 = (X - 2Y^3, Y)$ so that f_0 has multidegree (3). We make two simple observations:

1) the length has decreased at $z = 0$;

2) the change of length has occured together with a change of degree.

The first observation led us to prove the following generalization in [8]: locally, the length of a family of automorphisms can only decrease. In other words, the length is a lower semicontinuous map on the variety of automorphisms.

The second observation also suggests some generalization. Let $\mathcal{G}_{=m}$ denote the set of automorphisms whose degree is equal to m and recall that $\mathcal{G}_{\leq m}$ is the set of automorphisms whose degree is $\leq m$. Since $\mathcal{G}_{\leq m}$ is closed in $\mathcal{G}$, it is clear that $\mathcal{G}_{=m}$ is locally closed so that it is naturally an algebraic variety. In the present paper, we show the following result which has been suggested to us by David Wright:

Theorem A. If $d = (d_1, \ldots, d_l)$ and $m = d_1 \ldots d_l$, then $\mathcal{G}_d$ is closed in $\mathcal{G}_{=m}$.

Corollary 1. $\mathcal{G}_d$ is locally closed in $\mathcal{E}$.

Corollary 2. The irreducible components of $\mathcal{G}_{=m}$ are the $\mathcal{G}_d$'s, where d runs through the multidegrees $(d_1, \ldots, d_l)$ satisfying $d_1 \ldots d_l = m$.

Corollary 3. For any connected family of automorphisms, the multidegree is constant if and only if the degree is constant.

We have a partition of $\mathcal{G}$ by the $\mathcal{G}_d$'s where d runs through $\mathcal{D}$. If $d = (d_1, \ldots, d_l)$, it is easy to show that $\mathcal{G}_d$ is an irreducible constructible subset of dimension $d_1 + \cdots + d_l + 6$ (see [6, 7]). In [6], Friedland and Milnor show that $\mathcal{G}_d$ forms a smooth analytic manifold (see their lemma 2.4). Roughly speaking, they construct a bijective morphism from a smooth algebraic variety to $\mathcal{G}_d$. In our paper, we slightly refine their proof. By corollary 1 above, $\mathcal{G}_d$ is naturally an algebraic variety. By showing that their morphism is an isomorphism, we prove the following result:

Theorem B. Each $\mathcal{G}_d$ is a smooth, locally closed subset of $\mathcal{G}$.

Theorems A and B directly imply the following result:

Corollary 4. $\mathcal{G}_{=m}$ is a smooth variety.

If an algebraic group acts morphically on a variety, each orbit is a smooth, locally closed subset. Moreover, its boundary is a union of orbits of strictly lower dimension (see e.g. [11], prop. 8.3). Let $\overline{\mathcal{G}}_d$ denote the closure of $\mathcal{G}_d$ in $\mathcal{G}$. Unfortunately, it is not true that $\overline{\mathcal{G}}_d$ is a union of $\mathcal{G}_e$'s (see [3]). Actually, it is proved there that $\mathcal{G}_{(19)} \cap \overline{\mathcal{G}}_{(11,3,3)} \neq \emptyset$ and by dimension count we cannot have $\mathcal{G}_{(19)} \subseteq \overline{\mathcal{G}}_{(11,3,3)}$. However, we define a natural partial order $\sqsubseteq$ on $\mathcal{D}$ (see 7.1) by $d \sqsubseteq e \iff \overline{\mathcal{G}}_d \subseteq \overline{\mathcal{G}}_e$. For general multidegrees d, e, we are not yet able to decide whether $d \sqsubseteq e$ or not. However, if d, e have the same length, the situation gets lucid due to the following theorem.

Theorem C. If $d = (d_1, \ldots, d_l)$, $e = (e_1, \ldots, e_l)$ have the same length, the following assertions are equivalent:

(i) $\mathcal{G}_d \subseteq \overline{\mathcal{G}}_e$; (ii) $\mathcal{G}_d \cap \overline{\mathcal{G}}_e \neq \emptyset$; (iii) $d_i \leq e_i$ $(\forall\, i)$.

This paper is divided into seven sections. Section 1 is devoted to preliminary results. The proofs of theorems A and B are given in sections 2 and 3 respectively. Section 4 is devoted to semicontinuity results to be used in section 5 where we prove theorem C. In section 6, we prove an analogous of theorem B for variables (see 1.1 for the definition of a variable). Finally, in section 7, we discuss the order $\sqsubseteq$ and the variety $\mathcal{G}_{\leq m}$. In particular, we give the irreducible components of $\mathcal{G}_{\leq m}$ when $m \leq 27$.

1. Preliminary results

1.1. Variables

An element v of $\mathbb{C}[X,Y]$ is called a variable if it is the component of a plane polynomial automorphism. Let $\mathcal{V}$ denote the set of variables. Since in dimension 2, automorphisms and variables are intimately connected, one can also define the multidegree of a variable (see [8]). If $v,w \in \mathcal{V}$, we say that w is a predecessor of v if $(v,w) \in \mathcal{G}$ and $\deg w < \deg v$. The following result is classical (see e.g. [8], lemma 2):

Lemma 1.1. If $v \in \mathcal{V}$ has degree ≥ 2, then v admits a predecessor w and any other predecessor is of the form $w' = aw + b$ where $a,b \in \mathbb{C}$ with $a \neq 0$.

Definition 1.1. If v is a variable, we define its multidegree by $\operatorname{mdeg} v = \emptyset$ if $\deg v = 1$ and by $\operatorname{mdeg} v = \operatorname{mdeg}(v,w)$ if $\deg v \geq 2$ and w is any predecessor of v.

If $d = (d_1, \ldots, d_k) \in \mathcal{D}$ with $k \geq 1$, let us set $d' := (d_2, \ldots, d_k)$. If some variable has multidegree d, it is clear that any of its predecessors has multidegree d'. By the way, one also defines the length of a variable v of multidegree $(d_1, \ldots, d_k)$ by setting $l(v) = k$.

If K is any field, the multidegree of an automorphism of $\mathbb{A}^2_K$ or of a variable of $K[X,Y]$ would be defined in exactly the same way.

The following easy result is useful. If $f \in \mathcal{E}$, its Jacobian determinant is denoted by $\operatorname{Jac} f$.

Lemma 1.2. Let $v \in \mathcal{V}$ be a variable.

1. If $p \in \mathbb{C}[T]$ is non-constant and $u := p(v)$, the kernel of the derivation $q \mapsto \operatorname{Jac}(u,q)$ is equal to $\mathbb{C}[v]$.

2. If $w \in \mathbb{C}[v]$, the three following assertions are equivalent:
(i) $w \in \mathcal{V}$; (ii) w is irreducible; (iii) $w = av + b$ for some $a,b \in \mathbb{C}$ with $a \neq 0$.

Proof. We have $\operatorname{Jac}(u,q) = p'(v)\operatorname{Jac}(v,q)$ so that the kernel of the derivations $q \mapsto \operatorname{Jac}(u,q)$ and $q \mapsto \operatorname{Jac}(v,q)$ are equal. However, for any $a,b \in \mathbb{C}[X,Y]$, it is well known that $\operatorname{Jac}(a,b) = 0$ if and only if a,b are algebraically dependent (over $\mathbb{C}$). Therefore, the first part of the lemma is proved. Finally (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) is obvious. □

1.2. Valuative criterion

We will often use the valuative criterion that we state below. We are indebted to Michel Brion for his useful advice on this subject. Even if such a criterion sounds familiar (see e.g. [19], chap. 2, § 1, pp 52-54 or [10], § 7), we give a brief proof of it for the sake of completeness.

Let $\mathbb{C}[[T]]$ be the algebra of complex formal power series and let $\mathbb{C}((T))$ be its quotient field. If V is a complex algebraic variety and A an algebra over $\mathbb{C}$, $V(A)$ will denote the points of V with values in A, i.e. the set of morphisms $\operatorname{Spec} A \to V$. If v is a closed point of V and $\varphi \in V\Big(\mathbb{C}((T))\Big)$, we will write $v = \lim\limits_{T \to 0} \varphi(T)$ when:

(i) the point $\varphi : \operatorname{Spec} \mathbb{C}((T)) \to V$ is a composition $\operatorname{Spec} \mathbb{C}((T)) \to \operatorname{Spec} \mathbb{C}[[T]] \to V$;

(ii) v is the point $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C}[[T]] \to V$.

For example, if $V = \mathbb{A}^1$ and $\varphi \in V\Big(\mathbb{C}((T))\Big) = \mathbb{C}((T))$, we will write $v = \lim\limits_{T \to 0} \varphi(T)$ when $\varphi \in \mathbb{C}[[T]]$ and $v = \varphi(0)$.

Valuative criterion. Let $f : V \to W$ be a morphism of complex algebraic varieties and let w be a closed point of W. The two following assertions are equivalent:

(i) $w \in \overline{f(V)}$;

(ii) $w = \lim\limits_{T \to 0} f(\varphi(T))$ for some $\varphi \in V\Big(\mathbb{C}((T))\Big)$.

Proof. (i) $\Longrightarrow$ (ii). If $w \in \overline{f(V)} \setminus f(V)$, there exists an irreducible curve $\mathcal{C}$ of V such that $z \in \overline{f(\mathcal{C})}$ (see [15], p. 262, cor.). Therefore, we may assume that V is an irreducible curve. By normalizing V and by Nagata's theorem (see [20]), we may suppose that V is smooth and that W is complete. Let C be "the completion" of V, i.e. a smooth projective curve containing V as an open subset. Since W is complete, f can be (uniquely) extended in a morphism $f : C \to W$. We have $\overline{f(V)} = f(C)$, so that it is enough to show that for any point $x \in C$, there exists $\varphi \in V\Big(\mathbb{C}((T))\Big)$ such that $x = \lim\limits_{T \to 0} \varphi(T)$. We can assume that $x \notin V$ because otherwise there is nothing to do. Finally, taking a well chosen affine neighborhood of x in C, we can suppose that C is affine and that $V = C \setminus \{x\}$. Let $\mathcal{O}(C)$ be the algebra of regular functions on C, let $\mathcal{O}_{C,x}$ be the local ring of x on C and let $\widehat{\mathcal{O}_{C,x}}$ be its completion. We have natural injections $\mathcal{O}(C) \hookrightarrow \mathcal{O}_{C,x} \hookrightarrow \widehat{\mathcal{O}_{C,x}}$ and it is well known that $\widehat{\mathcal{O}_{C,x}} \simeq \mathbb{C}[[T]]$. Let $\mathbb{C}(C) \hookrightarrow \mathbb{C}((T))$ be the extension to fields of fractions of the map $\mathcal{O}(C) \hookrightarrow \mathbb{C}[[T]]$. We have the commutative diagram:

$$\begin{tikzcd}
\mathcal{O}(C) \arrow[r, hook] \arrow[d, hook] & \mathcal{O}_{C,x} \arrow[r, hook] & \widehat{\mathcal{O}_{C,x}} \arrow[r, "\sim"] & \mathbb{C}[[T]] \arrow[dd, hook] \\
\mathcal{O}(V) \arrow[d, hook] \arrow[drrr, hook, "\varphi^*"] & & & \\
\mathbb{C}(C) \arrow[rrr, hook] & & & \mathbb{C}((T))
\end{tikzcd}$$

where $\varphi^* : \mathcal{O}(V) \to \mathbb{C}((T))$ is the algebra morphism corresponding to the point $\varphi : \operatorname{Spec} \mathbb{C}((T)) \to V$ which we were looking for.

(ii) $\Longrightarrow$ (i). This is well known. $\square$

Remark. Note the analogy with the metric case where $w \in \overline{f(V)}$ if and only if there exists a sequence $(v_n)_{n \geq 1}$ of V such that $w = \lim\limits_{n \to +\infty} f(v_n)$.

Let $\mathcal{G}_d\Big(\mathbb{C}((T))\Big)$ be the set of automorphisms of $\mathbb{A}^2_{\mathbb{C}((T))}$ of multidegree d and let $\mathcal{G}_d\Big(\mathbb{C}[[T]]\Big)$ be the subset of elements which are also endomorphisms of $\mathbb{A}^2_{\mathbb{C}[[T]]}$, i.e. which admit a limit when T goes to zero. Later on, we will show that $\mathcal{G}_d$ is locally closed in $\mathcal{G}$, so that it is an algebraic variety. It will then be clear that $\mathcal{G}_d\Big(\mathbb{C}((T))\Big)$, resp. $\mathcal{G}_d\Big(\mathbb{C}[[T]]\Big)$, is actually the set of points of $\mathcal{G}_d$ with values in $\mathbb{C}((T))$, resp. $\mathbb{C}[[T]]$. Therefore, there will be no clash of notations.

Corollary 1.1. If $d \in \mathcal{D}$ and $f \in \mathcal{G}$, the following assertions are equivalent:
(i) $f \in \overline{\mathcal{G}}_d$;
(ii) $f = \lim\limits_{T\to 0} g_T$ for some $g \in \mathcal{G}_d\Big(\mathbb{C}((T))\Big)$.

Proof. It is enough to express $\mathcal{G}_d$ as the image of a morphism of algebraic varieties as follows. If $d = (d_1, \dots, d_l)$, set $m := d_1 \dots d_l$.

Let us set $\mathcal{A}' := \mathcal{A} \setminus \mathcal{B}$. For $1 \le k \le l$, let $\mathcal{B}_k$ denote the set of triangular automorphisms whose degree is equal to d_k. Note that $\mathcal{A}'$ and $\mathcal{B}_k$ are algebraic varieties. It is clear that $\mathcal{G}_d$ is equal to the image of the morphism $\mathcal{A} \times \mathcal{B}_1 \times \mathcal{A}' \times \cdots \times \mathcal{A}' \times \mathcal{B}_l \times \mathcal{A} \to \mathcal{G}_{\le m}$ sending $(\alpha_1, \beta_1, \dots, \beta_l, \alpha_{l+1})$ to $\alpha_1 \circ \beta_1 \circ \cdots \circ \beta_l \circ \alpha_{l+1}$.

Therefore, $f \in \overline{\mathcal{G}}_d$ if and only if there exists an automorphism g of $\mathbb{A}^2_{\mathbb{C}((T))}$ of multidegree d such that $f = \lim\limits_{T\to 0} g_T$. □

Let us set $\mathcal{V}_d := \{v \in \mathcal{V},\, \operatorname{mdeg} v = d\}$ and let $\overline{\mathcal{V}}_d$ be the closure of $\mathcal{V}_d$ in $\mathcal{P} = \mathbb{C}[X,Y]$. In the same way, we define $\mathcal{V}_d\Big(\mathbb{C}((T))\Big)$ as the set of variables of $\mathbb{C}((T))[X,Y]$ of multidegree d. Let $\mathcal{V}_d\Big(\mathbb{C}[[T]]\Big)$ be the subset of elements which also belong to $\mathbb{C}[[T]][X,Y]$, i.e. which admit a limit when T goes to zero. We will later on show that $\mathcal{V}_d$ is locally closed in $\mathcal{P}$. Therefore, $\mathcal{V}_d\Big(\mathbb{C}((T))\Big)$, resp. $\mathcal{V}_d\Big(\mathbb{C}[[T]]\Big)$, will actually be the set of points of $\mathcal{V}_d$ with values in $\mathbb{C}((T))$, resp. $\mathbb{C}[[T]]$. We omit the proof of the following result.

Corollary 1.2. If $d \in \mathcal{D}$ and $p \in \mathcal{P} = \mathbb{C}[X,Y]$, the following assertions are equivalent:
(i) $p \in \overline{\mathcal{V}}_d$;
(ii) $p = \lim\limits_{T\to 0} v_T$ for some $v \in \mathcal{V}_d\Big(\mathbb{C}((T))\Big)$.

2. Proof of theorem A

The leading term of a polynomial will denote its homogeneous component of highest degree. The following fundamental fact is taken from [16]:

Lemma 2.1. Let K be any field and let $f = (f_1, f_2)$ be a polynomial automorphism of $\mathbb{A}^2_K$ which is not affine.

(i) There exists a linear form $\varphi = aX + bY$, where $a, b \in K$, such that the leading term of f_i is proportional to $\varphi^{\deg f_i}$ for $i = 1, 2$;

(ii) $\deg f_1$ divides $\deg f_2$ or $\deg f_2$ divides $\deg f_1$.

Our proof of theorem A relies on the following analogous result dealing with variables instead of automorphisms.

Lemma 2.2. Let $d = (d_1, \ldots, d_l)$ be a multidegree. If $v \in \overline{\mathcal{V}}_d$ is a variable of degree $d_1 \ldots d_l$, then $v \in \mathcal{V}_d$.

Proof. By induction on l. The case $l = 0$ being clear, let us assume that $l \geq 1$. Let us set $m = d_1 \ldots d_l$ and $n = d_2 \ldots d_l$.

First step. Preliminary reduction.

The leading term of v is of the form $(\alpha X + \beta Y)^m$, where α, β are complex numbers. Therefore, up to some linear change of coordinates, we may assume that this leading term is Y^m.

Let $v_T \in \mathcal{V}_d\Big(\mathbb{C}((T))\Big)$ be such that $v = \lim\limits_{T \to 0} v_T$.

The leading term of v_T is of the form $\lambda_T(\alpha_T X + \beta_T Y)^m$, where $\lambda_T, \alpha_T, \beta_T$ belong to $\mathbb{C}((T))$. Up to replacing T by T^m, we may assume that $\lambda_T = (\mu_T)^m$ for some $\mu_T \in \mathbb{C}((T))$. Therefore, up to replacing (α_T, β_T) by $(\mu_T\, \alpha_T, \mu_T\, \beta_T)$, we may assume that $\lambda_T = 1$. Looking at the coefficient of X^m, we get $\lim\limits_{T \to 0} (\alpha_T)^m = 0$, so that $\lim\limits_{T \to 0} \alpha_T = 0$. Looking at the coefficient of Y^m, we get $\lim\limits_{T \to 0} (\beta_T)^m = 1$, so that $\beta_T \in \mathbb{C}[[T]]$ and $\lim\limits_{T \to 0} \beta_T$ is equal to some m-th root of unity ω. Up to replacing (α_T, β_T) by $(\alpha_T/\omega, \beta_T/\omega)$, we may assume that $\lim\limits_{T \to 0} \beta_T = 1$.

Up to replacing v_T by $v_T \circ (X, \alpha_T X + \beta_T Y)^{-1}$, we may assume that the leading term of v_T is Y^m so that v_T is of the form:

$$v_T = Y^m + a_{m-1} Y^{m-1} + \cdots + a_0, \text{ where the } a_k\text{'s belong to } \mathbb{C}[[T]][X].$$

Let $w_T \in \mathcal{V}_{d'}\Big(\mathbb{C}((T))\Big)$ be a predecessor of v_T.

By lemma 2.1 and up to multiplying w_T by some element of $\mathbb{C}((T))$, we may assume that the leading term of w_T is Y^n so that w_T is of the form

$$w_T = Y^n + b_{n-1} Y^{n-1} + \cdots + b_0, \text{ where the } b_k\text{'s belong to } \mathbb{C}((T))[X].$$

We may also assume that $w_T(0,0) = 0$, i.e. that b_0 is of the form $b_0 = c_p X^p + \cdots + c_1 X$ where the c_i's belong to $\mathbb{C}((T))$.

Second step. Let us show that $\lim\limits_{T \to 0} w_T$ exists.

a) Let us begin by showing that $b_{n-1}, \ldots, b_1$ belong to $\mathbb{C}[[T]][X]$.

Since $(v_T - (w_T)^{d_1}, w_T)$ is an automorphism, we get $\deg(v_T - (w_T)^{d_1}) \leq n(d_1 - 1) = m - n$. As a consequence, for $1 \leq i \leq n-1$, the Y^{m-i}-coefficients (as polynomials in the indeterminate Y) of v_T and $(w_T)^{d_1}$ coincide. However, the Y^{m-i}-coefficient of $(w_T)^{d_1} = (Y^n + b_{n-1}Y^{n-1} + \cdots + b_0)^{d_1}$ is equal to $d_1 b_{n-i} + p_i(b_{n-1}, \ldots, b_{n-i+1})$ for some polynomial $p_i(A_1, \ldots, A_{i-1}) \in \mathbb{Z}[A_1, \ldots, A_{i-1}]$.

Therefore, $b_{n-i} = 1/d_1\,[a_{m-i} - p_i(b_{n-1}, \ldots, b_{n-i+1})]$ so that we get $b_{n-i} \in \mathbb{C}[[T]]$ by an immediate induction.

b) Let us now show by contradiction that b_0 also belongs to $\mathbb{C}[[T]][X]$. Otherwise, there would exist $k > 0$ and a non-constant polynomial $u \in \mathbb{C}[X]$ such that $\lim_{T\to 0} T^k b_0 = u$. We would also have $\lim_{T\to 0} T^k w_T = u$. Since $(v, u) = \lim_{T\to 0}(v_T, T^k w_T)$, we get $(v, u) \in \overline{\mathcal{G}}$ so that $\mathrm{Jac}(v, u) \in \mathbb{C}$. Let q be the degree of u. Looking at the leading terms of u and v, we get $\mathrm{Jac}(Y^m, X^q) = 0$ which is false.

Therefore, $b_{n-1}, \ldots, b_0$ belong to $\mathbb{C}[[T]][X]$ which means that $w = \lim_{T\to 0} w_T$ exists.

Third step. The actual induction.

It is clear that $\mathrm{Jac}(v, w) \in \mathbb{C}$. If $\mathrm{Jac}(v, w) = 0$, then w should be a polynomial in v which is impossible for grounds of degrees. Consequently, $\mathrm{Jac}(v, w) \in \mathbb{C}^*$ showing that (v, w) is an automorphism and w a variable. Since $w \in \overline{\mathcal{V}}_{d'}$ is a variable of degree n, we get $w \in \mathcal{V}_{d'}$ by the induction hypothesis. It is now clear that $v \in \mathcal{V}_d$. □

Remark. Our proof of lemma 2.2 strongly relies on the fact that we are working in characteristic zero. Let us note in particular that we do a division by d_1.

Proof of theorem A. If $f = (f_1, f_2) \in \overline{\mathcal{G}}_d \cap \mathcal{G}_{=m}$, let us show that $f \in \mathcal{G}_d$. Since $\mathcal{A} \circ f \subseteq \overline{\mathcal{G}}_d \cap \mathcal{G}_{=m}$, we may assume that $\deg f_1 = m$ and $\deg f_2 < m$. However, since $f_1 \in \overline{\mathcal{V}}_d$, we get $f_1 \in \mathcal{V}_d$ by the previous lemma, so that $f \in \mathcal{G}_d$. □

3. Proof of theorem B

Let us assume that $d = (d_1, \ldots, d_l)$ with $l \geq 1$.

It is enough to show that $G_d := \{f \in \mathcal{G}_d,\ f(0,0) = (0,0)\}$ is smooth.

There are two steps:

1) We recall the construction given in [6] of the locally trivial fibration $\pi : G_d \to \mathbb{P}^1 \times \mathbb{P}^1$ over the product $\mathbb{P}^1 \times \mathbb{P}^1$ of two projective lines. At this point, it is sufficient to show that the fiber F_d is smooth.

2) We show that the bijective morphism given in [6] from a smooth variety to F_d is an isomorphism.

First step. The locally trivial fibration $\pi : G_d \to \mathbb{P}^1 \times \mathbb{P}^1$.

Let G be the subgroup of automorphisms of $\mathcal{G}$ fixing the origin. Let GL be the linear group and E be the group of elementary (i.e. triangular) automorphisms fixing the origin. Note that G is the amalgamated product of GL and E over their intersection B, which turns out to be a Borel subgroup of GL. We identify the projective line $\mathbb{P}^1$ with the coset space GL/B. Any element of G_d can be written as a reduced word of the form

$$f = a_0 \circ e_1 \circ a_1 \circ \cdots \circ e_l \circ a_l$$

where the a_i (resp. e_i) belong to GL (resp. E). Due to the amalgamated structure, the cosets a_0B and Ba_l do not depend on the reduced word. Hence, the projection π is well defined by the formula $\pi(f) := (a_0B, a_l^{-1}B)$. It is straightforward that π is a locally trivial fibration whose fiber is $F_d := \pi^{-1}(B, B)$.

Second step. Let us prove that the fiber F_d is smooth.

The fiber F_d consists of all group elements which can be written as reduced words of the form $f = e_1 \circ a_1 \circ \cdots \circ e_{l-1} \circ a_{l-1} \circ e_l$ with elementary transformation at both ends and with $\deg e_i = d_i$. Let $\sigma := (Y, X) \in \mathcal{G}$, let $\mathbb{T} := \{(aX, bY),\, a, b \in \mathbb{C}^*\}$ be a maximal torus of GL and for $1 \leq i \leq l$, let us set $E_i := \{(X + p(Y), Y),\, p \in \mathbb{C}[Y],\, p(0) = 0,\, \deg p = d_i\}$. One can easily show that the following morphism is bijective (see [6]):

$$\prod_{1 \leq i \leq l} E_i \times \mathbb{T} \to F_d, \qquad (e_1, \ldots, e_l, t) \mapsto e_1 \circ \sigma \circ \cdots \circ \sigma \circ e_l \circ t.$$

Since $\mathbb{T}$ and the E_i's are smooth (affine) varieties, it is sufficient to show that it is an isomorphism. Using induction on l, it is sufficient to show that the following map is regular:

$$\alpha : F_d \to E_1, \qquad f = e_1 \circ \sigma \circ \cdots \circ \sigma \circ e_l \circ t \ \mapsto \ e_1.$$

The case $l = 1$ being clear, let us assume that $l \geq 2$. But $\alpha(f)$ is the unique element $(X + p(Y), Y)$ of E_1 such that

$$\deg\Big(f_1 - p(f_2)\Big) < \deg f_2.$$

Writing $p = \sum_{1 \leq i \leq d_1} p_i Y^i$, we want to show that the p_i's : $F_d \to \mathbb{C}$ are regular.

Let us set $m := d_1 \ldots d_l$ and $n := d_2 \ldots d_l$.

If $q = \sum_{i,j \geq 0} q_{i,j}\, X^i Y^j \in \mathbb{C}[X, Y]$, we denote its X^iY^j-coefficient by $c(X^iY^j, q) := q_{i,j}$.

If $f \in F_d$, it is easy to check that $c(Y^n, f_2) \neq 0$. Furthermore, the p_i's may be computed by a decreasing induction, using the following recurrence relation:

$$p_i = c(Y^n, f_2)^{-i}\, c(Y^{ni}, f_1 - \sum_{i < j \leq d_1} p_j\, f_2^j\,) \qquad \text{for } i = d_1, \ldots, 1.$$

This proves that the p_i's are regular. □

4. The lower semicontinuity of the length of a variable revisited

4.1. The closure of the set of variables

We begin by noting that the set $\mathcal{V}$ of variables is not locally closed in the infinite dimensional variety $\mathcal{P}$ of polynomials. Let ε and ζ be non-zero complex numbers. On the one hand the polynomial $X + \varepsilon X^2$ belongs to $\overline{\mathcal{V}}$ since it is the limit of the variable $\zeta Y + X + \varepsilon X^2$ when ζ goes to zero and on the other hand it does not belong to $\mathcal{V}$ since it is reducible. The polynomial $X + \varepsilon X^2$ belongs to $\overline{\mathcal{V}} \setminus \mathcal{V}$ while its limit when ε goes to zero does not. This proves that $\overline{\mathcal{V}} \setminus \mathcal{V}$ is not closed, i.e. $\mathcal{V}$ is not locally closed. As we prefer working with subvarieties, we are naturally led to introduce the variety $\overline{\mathcal{V}}$. By [8], th. 3, we have $\overline{\mathcal{V}} = \mathcal{U}$, where $\mathcal{U} := \{p(v),\, p \in \mathbb{C}[T],\, v \in \mathcal{V}\}$. It turns out that this set appears in the literature. We now recall a geometric and an algebraic characterization of it.

The following geometric characterization is known as the parallel lines lemma. It is proved in [24], cor. 1 or [23], lemma 1.2.1. As usual, a line denotes any variety isomorphic to $\mathbb{A}^1$. Furthermore, two lines of $\mathbb{A}^2$ are called parallel if they are either equal or disjoint.

Lemma 4.1 (parallel lines lemma). Let $u : \mathbb{A}^2 \to \mathbb{A}^1$ be a non-constant morphism. The following assertions are equivalent:

(i) $u \in \mathcal{U}$;

(ii) any fiber of u is a union of parallel lines;

(iii) some fiber of u is a union of parallel lines.

Remarks. 1. The conditions (i-iii) are still equivalent to saying that $\mathbb{C}[X, Y]/(u)$ is isomorphic to some $B[T]$ where B is a $\mathbb{C}$-algebra and T an indeterminate (see [23]).

2. Lemma 4.1 is both a consequence and a generalization of the Abhyankar-Moh-Suzuki theorem (see [1, 28]) asserting that for any morphism $v : \mathbb{A}^2 \to \mathbb{A}^1$ the following assertions are equivalent:

(i) $v \in \mathcal{V}$; (ii) any fiber of v is a line; (iii) some fiber of v is a line.

The following algebraic characterization of $\mathcal{U}$ (explicitely stated in [17], §3 or [5], cor. 4.7) is an easy consequence of the famous result of Rentschler (see [22]) asserting that any locally nilpotent derivation of $\mathbb{C}[X, Y]$ is conjugate (by an automorphism of $\mathbb{C}[X, Y]$) to a triangular derivation $p(X)\, \partial_Y$ (see also [17, 4]).

Lemma 4.2. Let u be an element of $\mathbb{C}[X, Y]$. The following assertions are equivalent:

(i) $u \in \mathcal{U}$;

(ii) The Jacobian derivation $q \mapsto \mathrm{Jac}(u, q)$ of $\mathbb{C}[X, Y]$ is locally nilpotent.

Moreover, any locally nilpotent derivation of $\mathbb{C}[X, Y]$ is of the form $q \mapsto \mathrm{Jac}(u, q)$.

This last approach allows us to recover the fact that $\mathcal{U}$ is closed in $\mathcal{P} = \mathbb{C}[X, Y]$. Let $\mathfrak{Der} := \{a\, \partial_X + b\, \partial_Y,\, a, b \in \mathcal{P}\} \simeq \mathcal{P}^2$ be the infinite dimensional variety of derivations of $\mathbb{C}[X, Y]$ and let $\mathfrak{LND}$ be the subset of locally nilpotent derivations.

Lemma 4.3. $\mathfrak{LND}$ is closed in $\mathfrak{Der}$.

Proof. Let $D = a\, \partial_X + b\, \partial_Y$ be a derivation and let $m := \max\{\deg a, \deg b\}$. According to

[4], th. 1.3.52 or [5], prop. 8.4, D is locally nilpotent if and only if $D^{m+2}X = D^{m+2}Y = 0$. $\square$

Here is a direct proof of the closed nature of $\mathcal{U}$:

Proposition 4.1. $\mathcal{U}$ is closed in $\mathcal{P} = \mathbb{C}[X, Y]$.

Proof. Let $\varphi : \mathcal{P} \to \mathfrak{Der}$ be the morphism sending $p \in \mathcal{P}$ to the derivation $q \mapsto \mathrm{Jac}(p, q)$. We have $\mathcal{U} = \varphi^{-1}(\mathfrak{LND})$ by lemma 4.2 and we conclude by lemma 4.3. $\square$

4.2. Semicontinuity results

We recall the main result of [8]:

Theorem 4.1. The length maps $\mathcal{G} \to \mathbb{Z}$, $f \mapsto l(f)$ and $\mathcal{V} \to \mathbb{Z}$, $v \mapsto l(v)$ are lower semicontinuous.

If $f = (f_1, f_2) \in \mathcal{G}$, we have $l(f) = \max\{l(f_1), l(f_2)\}$. Therefore, the first semicontinuity is a consequence of the second one.

Let $\mathcal{H} := \{aT + b,\ a, b \in \mathbb{C},\ a \neq 0\}$ be the group of automorphisms of $\mathbb{A}^1$. If a *non-constant* element of $\mathcal{U}$ is written as above $u = p \circ v$, let us note that the cosets $p \circ \mathcal{H}$ and $\mathcal{H} \circ v$ are uniquely determined. Indeed, if $p \circ v = q \circ w$, we get $\mathrm{Jac}(v, w) = 0$ so that there exist $a, b \in \mathbb{C}$ with $a \neq 0$ such that $v = aw + b$. If $u = p \circ v$ is *any* element of $\mathcal{U}$, the coset $p \circ \mathcal{H}$ is still uniquely determined, but no longer the coset $\mathcal{H} \circ v$. As a consequence, $\deg p$ is uniquely determined (by convention, we set $\deg 0 = -\infty$). However, one could check that the induced map $\mathcal{U} \to \mathbb{Z} \cup \{-\infty\}$, $p \circ v \mapsto \deg p$ is neither lower or upper semicontinuous. Conversely, we will see that the map sending $u \in \mathcal{U}$ to the smallest integer $k \geq 0$ such that u belong to $\mathbb{C}[v]$ for some variable v of length k, has nicer properties. First, it extends the length map $l : \mathcal{V} \to \mathbb{Z}$. Secondly, it is still lower semicontinuous. However, for technical grounds (see th. 4.2 below), if u is constant, we will set $l(u) = -1$ rather than $l(u) = 0$ (see definition 4.1). For any $k \geq 0$, let $\mathcal{V}^{\leq k}$ be the set of variables of length $\leq k$. We know that $\mathcal{V}^{\leq k}$ is closed in $\mathcal{V}$. More precisely, if we set $\mathcal{U}^{\leq k} := \{p \circ v,\ p \in \mathbb{C}[T],\ v \in \mathcal{V}^{\leq k}\}$ and $\mathcal{U}^{\leq -1} := \mathbb{C}$, according to [8], th. 4 we have:

Theorem 4.2. $\overline{\mathcal{V}^{\leq k}} = \mathcal{V}^{\leq k} \cup \mathcal{U}^{\leq k-1}$.

As a consequence:

Corollary 4.1. $\overline{\mathcal{V}^{\leq k}} \subseteq \mathcal{U}^{\leq k}$.

The length map $l : \mathcal{V} \to \mathbb{Z}$ is naturally extended to a map $l : \mathcal{U} \to \mathbb{Z}$:

Definition 4.1. If $u \in \mathcal{U}$, we set $l(u) := \min\{k \in \mathbb{Z},\, u \in \mathcal{U}^{\leq k}\}$.

We have already said that the lower semicontinuity of the map $l : \mathcal{G} \to \mathbb{Z}$ is a consequence of the lower semicontinuity of the map $l : \mathcal{V} \to \mathbb{Z}$. In fact, this latter semicontinuity is itself a consequence of the following one:

Theorem 4.3. The map $l : \mathcal{U} \to \mathbb{Z}$ is lower semicontinous.

Proof. We take up the proof of [8]. We want to show that $\mathcal{U}^{\leq k}$ is closed in $\mathcal{U}$. For $k = -1$, it is obvious. So, let us assume that $k \geq 0$.

First step. Preliminary reduction.

Let us set $P := \{p \in \mathcal{P},\, p(0,0) = 0\}$ and $P_{\leq n} := \{p \in P,\, \deg p \leq n\}$. Since $\mathcal{U}^{\leq k}$ is invariant by any translation $u \mapsto u + c$ where $c \in \mathbb{C}$, it is sufficient to show that $U^{\leq k} := \mathcal{U}^{\leq k} \cap P$ is closed in P. We will also need the set $V^{\leq k} := \mathcal{V}^{\leq k} \cap P$. A subset Z is closed in P if $Z \cap P_{\leq n}$ is closed in $P_{\leq n}$ for any $n \geq 1$.

Second step. Reduction to a projective problem.

We denote by $\mathbb{P}$ (resp. $\mathbb{P}_{\leq n}$) the set of vectorial lines of P (resp. $P_{\leq n}$). The equality $\mathbb{P} = \bigcup_n \mathbb{P}_{\leq n}$ endows $\mathbb{P}$ with the structure of an infinite dimensional algebraic variety. We recall that there exists a natural correspondence between the cones of P and the subsets of $\mathbb{P}$. Furthermore, the cone is closed if and only if the subset of $\mathbb{P}$ is closed. Let D_k be the subset corresponding to the cone $U^{\leq k}$ of P. We want to show that D_k is closed in $\mathbb{P}$. Let F_k be the closed subset of $\mathbb{P}$ corresponding to the closed cone $\overline{V^{\leq k}}$ of P.

Third step. The Jacobian variety.

The map $P \times P \to \mathbb{C}[X, Y]$ sending (p, q) to $\mathrm{Jac}(p, q)$ is bilinear. As a result, the equality $\mathrm{Jac}(p, q) = 0$ defines a closed subset $J_0 \subseteq \mathbb{P} \times \mathbb{P}$ which we call the Jacobian variety. Note the difference with another Jacobian variety $J \subseteq \mathbb{P} \times \mathbb{P}$ introduced in [8],3.c which was defined by the relation $\mathrm{Jac}(p, q) \in \mathbb{C}$.

We will denote by p_1 (resp. p_2): $\mathbb{P} \times \mathbb{P} \to \mathbb{P}$ the first (resp. second) projection.

It is clear that $Z_k := J_0 \cap p_2^{-1}(F_k)$ is a closed subset of $\mathbb{P} \times \mathbb{P}$. The main idea is to establish that $D_k = p_1(Z_k)$. In fact, we will need the stronger equality:

$$D_k \cap \mathbb{P}_{\leq n} = p_1\Big(Z_k \cap (\mathbb{P}_{\leq n} \times \mathbb{P}_{\leq n})\Big) \text{ for } n \geq 1. \qquad \text{(E)}$$

Indeed, the map $p_1 : \mathbb{P}_{\leq n} \times \mathbb{P}_{\leq n} \to \mathbb{P}_{\leq n}$ is closed by the fundamental theorem of elimination theory (see [18], I, §9, th. 1). Hence $D_k \cap \mathbb{P}_{\leq n}$ is closed in $\mathbb{P}_{\leq n}$ for any $n \geq 1$ showing that D_k is closed in $\mathbb{P}$.

Let us finish the proof by establishing (E). We begin with the inclusion $p_1(Z_k) \subseteq D_k$. This amounts to proving that if $\mathrm{Jac}(p, q) = 0$ where $p \in P$ and q is a non-zero element of $\overline{V^{\leq k}}$, then $p \in U^{\leq k}$. But $\overline{V^{\leq k}} \subseteq U^{\leq k}$, by corollary 4.1, so that there exist a non-constant polynomial $r \in \mathbb{C}[T]$ and $v \in V^{\leq k}$ such that $q = r(v)$. The equality $\mathrm{Jac}(r(v), p) = 0$ gives us $p \in \mathbb{C}[v]$ so that $p \in U^{\leq k}$.

Now, we must show that $D_k \cap \mathbb{P}_{\leq n} \subseteq p_1\Big(Z_k \cap (\mathbb{P}_{\leq n} \times \mathbb{P}_{\leq n})\Big)$ for $n \geq 1$. Equivalently, we must prove that if p is a non-zero element of $U^{\leq k}$, then there exists a non-zero element q of $\overline{V^{\leq k}}$ satisfying $\operatorname{Jac}(p,q)=0$ and $\deg q \leq \deg p$. But, by definition of $U^{\leq k}$, we can write $p = r \circ v$ where $r \in \mathbb{C}[T]$ and $v \in \mathcal{V}^{\leq k}$. Using a translation, there is no restriction to assume that $v \in V^{\leq k}$. It is clear that $\deg v \leq \deg p$, so we can take $q = v$. $\square$

5. Proof of theorem C.

Since (iii) $\Longrightarrow$ (i) $\Longrightarrow$ (ii) is clear, let us show (ii) $\Longrightarrow$ (iii).

We use induction on l.

If $l = 0$, then $d = e = \emptyset$ and there is nothing to show.

If $l = 1$, let us note that $\forall\, f \in \mathcal{G}_{(e_1)}$, $\deg f = e_1$, so that $\forall\, f \in \overline{\mathcal{G}}_{(e_1)}$, $\deg f \leq e_1$ and it is done.

If $l \geq 2$, let us take $f \in \mathcal{G}_d \cap \overline{\mathcal{G}}_e$. Since $\mathcal{A} \circ f \subseteq \mathcal{G}_d \cap \overline{\mathcal{G}}_e$, we can assume that $f(0,0) = (0,0)$ and $\deg f_1 > \deg f_2$. It follows that the length of the variable f_1 (resp. f_2) is equal to l (resp. $l-1$).

Since $f \in \overline{\mathcal{G}}_e$, there exists $g \in \mathcal{G}_e\Big(\mathbb{C}((T))\Big)$ such that $f = \lim\limits_{T\to 0} g(T)$. We can of course assume that $g(0,0) = (0,0)$.

<u>First step.</u> We will come down to the case where $\deg g_1 > \deg g_2$.

First of all, we prove by contradiction that $\deg g_1 \geq \deg g_2$. Otherwise, we would have $l(g_1) \leq l-1$, where g_1 is seen as a variable of $\mathbb{C}((T))[X,Y]$. Therefore, by the semicontinuity of the length of a variable, we would get $l(f_1) \leq l-1$. A contradiction.

Let then λ be the unique element of $\mathbb{C}((T))$ such that $\deg(g_2 - \lambda g_1) < \deg g_1$. As above, we prove by contradiction that $\lambda \in \mathbb{C}[[T]]$. Otherwise, $\dfrac{1}{\lambda} \in T\mathbb{C}[[T]]$ and $f = \lim\limits_{T\to 0} \widetilde{g}(T)$, where $\widetilde{g} := (g_1 - \dfrac{1}{\lambda} g_2, g_2)$. Yet, the length of the variable $\widetilde{g_1} = g_1 - \dfrac{1}{\lambda} g_2$ is equal to $l-1$ and we have previously seen that this led to a contradiction.

From $(f_1, f_2 - \lambda(0) f_1) = \lim\limits_{T\to 0} \widehat{g}$, where $\widehat{g} := (g_1, g_2 - \lambda g_1)$, we still deduce by contradiction that $\lambda(0) = 0$. Otherwise, the variable $f_2 - \lambda(0) f_1$ would be of length l while being the limit of the variable $\widehat{g}_2 = g_2 - \lambda g_1$ which is of length $l-1$.

Replacing g by $\widehat{g}$, we can actually assume that $\deg g_1 > \deg g_2$.

Since $\deg g_1 > \deg g_2$, the automorphism $g \in \mathcal{G}_e\Big(\mathbb{C}((T))\Big)$ can uniquely be expressed as the composition $g = t \circ \sigma \circ h$, where $t = \left(X + \sum\limits_{1 \leq i \leq e_1} a_i Y^i, Y\right)$ is a triangular automorphism, $\sigma = (Y,X) \in \mathcal{G}$ and $h = (h_1,h_2) \in \mathcal{G}_{e'}\Big(\mathbb{C}((T))\Big)$ satisfies $\deg h_1 > \deg h_2$. The a_i's are of course assumed to belong to $\mathbb{C}((T))$. Let us also note that $h(0,0) = (0,0)$.

<u>Second step.</u> Let us show that $\lim\limits_{T\to 0} h(T)$ exists.

We have $h_1 = g_2$, hence $\lim\limits_{T\to 0} h_1 = f_2$. Let us show by contradiction that $\lim\limits_{T\to 0} h_2$

exists. Otherwise, let $k \geq 1$ be the least integer such that $\lim\limits_{T\to 0} T^k h_2$ exists. We set $(p,q) := \lim\limits_{T\to 0}(h_1, T^k h_2)$. Since $\mathrm{Jac}(h_1, T^k h_2) = T^k \mathrm{Jac}\, h$ with $\mathrm{Jac}\, h = -\mathrm{Jac}\, g$, we get $\mathrm{Jac}(p,q) = 0$. The relation $h_2(0,0) = 0$ implies that q is non-constant. As $T^k h_2$ is a variable of length $l-2$, we get $q \in \overline{\mathcal{V}^{\leq l-2}} \subseteq \mathcal{U}^{\leq l-2}$ (by corollary 4.1), so that we get the existence of a non-constant $r \in \mathbb{C}[T]$ and $v \in \mathcal{V}^{\leq l-2}$ such that $q = r(v)$. Since $\mathrm{Jac}(r(v), p) = 0$ with p irreducible, there exist $a, b \in \mathbb{C}$ with $a \neq 0$ such that $p = av + b$ (by lemma 1.2). A contradiction, because $p = f_2$ is of length $l-1$ while v is of length $\leq l-2$.

We have proved the existence of an endomorphism $\overline{h} = (\overline{h}_1, \overline{h}_2)$ such that $\overline{h} = \lim\limits_{T\to 0} h$. Since $\mathrm{Jac}\, \overline{h} = -\mathrm{Jac}\, f \in \mathbb{C}^*$ and $\overline{h}_1 = f_2$ is a variable, it is clear that $\overline{h} \in \mathcal{G}$.

Third step. The actual induction.

Since $\overline{h} = \lim\limits_{T\to 0} h$ and $g = t.\sigma.h$, there exist $\overline{a}_i$'s $\in \mathbb{C}$ and a triangular automorphism $\overline{t} := \left(X + \sum\limits_{1 \leq i \leq e_1} \overline{a}_i Y^i, Y \right)$ such that $\overline{t} = \lim\limits_{T\to 0} t$. Then, we have $f = \overline{t} \circ \sigma \circ \overline{h}$, so that $l(\overline{h}) \geq l-1$. But $\overline{h} = \lim\limits_{T\to 0} h$, where $l(h) = l-1$, so that $l(\overline{h}) \leq l-1$ by the semicontinuity of the length of an automorphism. Finally $l(\overline{h}) = l-1$ and the multidegree of f is obtained by the concatenation of the ones of $\overline{t}$ and $\overline{h}$. We get $d_1 = \deg \overline{t}$ and $\overline{h} \in \mathcal{G}_{d'}$. It is now clear that $d_1 \leq e_1$. Since $\overline{h} \in \mathcal{G}_{d'} \cap \overline{\mathcal{G}}_{e'}$, we get $d_i \leq e_i$ for $i \geq 2$ by the induction hypothesis. $\square$

Here is the analogous result for variables:

Corollary 5.1. If $d = (d_1, \dots, d_l)$ and $e = (e_1, \dots, e_l) \in \mathcal{D}$ are multidegrees with the same length such that $\mathcal{V}_d \cap \overline{\mathcal{V}}_e \neq \emptyset$, then $d_i \leq e_i$ for any i.

Proof. For $l = 0$ and 1, it is clear. If $l \geq 2$ and $u \in \mathcal{V}_d \cap \overline{\mathcal{V}}_e$, there exists $p \in \mathcal{V}_e(\mathbb{C}((T)))$ such that $u = \lim\limits_{T\to 0} p(T)$. Let $q \in \mathcal{V}_{e'}(\mathbb{C}((T)))$ be such that $(p,q) \in \mathcal{G}_e((\mathbb{C}((T)))$. We may assume that $q(0,0) = 0$. Possibly replacing q by $T^k q$ where $k \in \mathbb{Z}$, we may assume that $\lim\limits_{T\to 0} q$ exists and is non-constant. Let us set $v = \lim\limits_{T\to 0} q$. It is clear that $\mathrm{Jac}(u,v) \in \mathbb{C}$. If $\mathrm{Jac}(u,v) = 0$, we would get $v \in \mathbb{C}[u]$ where u has length l and v has length $\leq l-1$. A contradiction. Therefore $\mathrm{Jac}(u,v) \in \mathbb{C}^*$, so that (u,v) is an automorphism of multidegree d. Finally $(u,v) \in \mathcal{G}_d \cap \overline{\mathcal{G}}_e$ and we conclude by theorem C. $\square$

6. Variables of fixed multidegree

Even if $\mathcal{V}$ is not locally closed in $\mathcal{P}$ (see section 4), we show:

Lemma 6.1. $\mathcal{V}_d$ is locally closed in $\mathcal{P}$.

Proof. We may assume that $d = (d_1, \ldots, d_l)$ with $l \geq 1$. We set $A_d := \{e = (e_1, \ldots, e_l) \in \mathcal{D},\, e_i \leq d_i, \forall\, i$ and $e \neq d\}$. If $k \geq 0$, we recall that $\mathcal{V}^{\leq k} = \{v \in \mathcal{V},\, l(v) \leq k\}$ and that $\mathcal{U}^{\leq k} = \{p \circ v,\, p \in \mathbb{C}[T],\, v \in \mathcal{V}^{\leq k}\}$. Using theorem 4.2 and corollary 5.1, we get:

$$\overline{\mathcal{V}}_d \setminus \mathcal{V}_d = \left(\overline{\mathcal{V}}_d \cap \mathcal{U}^{\leq l-1}\right) \quad \cup \bigcup_{e \in A_d} \overline{\mathcal{V}}_e$$

so that $\overline{\mathcal{V}}_d \setminus \mathcal{V}_d$ is closed by theorem 4.3. $\square$

Remark. We could show in the same way that $\mathcal{G}_d$ is locally closed in $\mathcal{G}$. If $k \geq 0$, we set $\mathcal{G}^{\leq k} := \{f \in \mathcal{G},\, l(f) \leq k\}$. Using theorem 4.1 and theorem C, we get:

$$\overline{\mathcal{G}}_d \setminus \mathcal{G}_d = \left(\overline{\mathcal{G}}_d \cap \mathcal{G}^{\leq l-1}\right) \quad \cup \bigcup_{e \in A_d} \overline{\mathcal{G}}_e$$

so that $\overline{\mathcal{G}}_d \setminus \mathcal{G}_d$ is closed.

Here is the analogous of theorem B for variables:

Proposition 6.1. Each $\mathcal{V}_d$ is a smooth, locally closed subset of $\mathcal{P}$.

Proof. It is enough to show that $V_d := \{v \in \mathcal{V}_d,\, v(0,0) = 0\}$ is smooth.

There will be two steps:

1) If H_d is the subset of $\mathcal{G}_d$ composed of the automorphisms satisfying the three conditions $f(0,0) = (0,0)$, $\deg f_1 > \deg f_2$ and $\operatorname{Jac} f = 1$, we show that H_d is a smooth, locally closed subset of $\mathcal{E}$.

2) We show that the first projection $p_1 : H_d \to V_d$, $(f_1, f_2) \mapsto f_1$ is an isomorphism.

First step. Let us show that H_d is a smooth, locally closed subset of $\mathcal{E}$.

We take back the notations used in the proof of theorem B.

a) The locally trivial fibration $\pi : G_d \to \mathbb{P}^1 \times \mathbb{P}^1$ with fiber F_d induces the locally trivial fibration $\widetilde{\pi} : \widetilde{G}_d \to \mathbb{P}^1 \times \mathbb{P}^1$ with fiber $\widetilde{F}_d$, where we have set

$$\widetilde{G}_d := \{f \in G_d,\ \operatorname{Jac} f = 1\} \qquad \text{and} \qquad \widetilde{F}_d := \{f \in F_d,\ \operatorname{Jac} f = 1\}.$$

b) It is clear that $\widetilde{F}_d$ is locally closed in $\mathcal{E}$. Let us check that it is smooth.

Let us set $\widetilde{\mathbb{T}} := \{f \in \mathbb{T}, \operatorname{Jac} f = (-1)^{l-1}\} \subseteq \mathbb{T}$. It is sufficient to note that the isomorphism

$$\prod_{1 \leq i \leq l} E_i \times \mathbb{T} \to F_d, \qquad (e_1, \ldots, e_l, t) \mapsto e_1 \circ \sigma \circ \cdots \circ \sigma \circ e_l \circ t$$

(given in the proof of theorem B) induces the isomorphism

$$\prod_{1 \leq i \leq l} E_i \times \widetilde{\mathbb{T}} \to \widetilde{F}_d, \qquad (e_1, \ldots, e_l, t) \mapsto e_1 \circ \sigma \circ \cdots \circ \sigma \circ e_l \circ t.$$

c) Since $\widetilde{\pi} : \widetilde{G}_d \to \mathbb{P}^1 \times \mathbb{P}^1$ is a locally trivial fibration with smooth fiber and since

$\{B\} \times \mathbb{P}^1$ is a smooth closed subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$, $H_d = \widetilde{\pi}^{-1}(\{B\} \times \mathbb{P}^1)$ is a smooth closed subvariety of $\widetilde{G}_d$. Indeed $\widetilde{\pi}$ induces a locally trivial fibration $H_d \to \{B\} \times \mathbb{P}^1 \simeq \mathbb{P}^1$ with fiber $\widetilde{F}_d$.

Second step. Let us show that $p_1 : H_d \to V_d$, $(f_1, f_2) \mapsto f_1$ is an isomorphism.

Let us set $m = d_1 \dots d_l$ and let A be the vector space of polynomials $p \in \mathbb{C}[X,Y]$ satisfying $p(0,0) = 0$ and $\deg p < m$. Since p_1 is a bijective morphism, it is sufficient to show that the map $\alpha : V_d \to A$ sending f_1 to the unique f_2 such that $(f_1, f_2) \in H_d$ is regular. But $\alpha(f_1)$ is implicitely defined by the equality $\mathrm{Jac}(f_1, \alpha(f_1)) = 1$.

Let B be the vector space of polynomials $q \in \mathbb{C}[X,Y]$ satisfying $\deg p \leq 2m$.

We conclude by applying the following implicit function lemma to the morphism $\varphi : V_d \times A \to B$, $(f_1, f_2) \mapsto \mathrm{Jac}(f_1, f_2)$ and by setting $b = 1 \in B$. Indeed:

(i) the map $f_2 \mapsto \mathrm{Jac}(f_1, f_2)$ is linear;

(ii) if $\mathrm{Jac}(f_1, f_2) = 0$, where $(f_1, f_2) \in V_d \times A$, then $f_2 \in \mathbb{C}[f_1]$ and $\deg f_2 < \deg f_1$, so that $f_2 = 0$.

(iii) for any $f_1 \in V_d$, there exists a unique $f_2 \in A$ such that $\mathrm{Jac}(f_1, f_2) = 1$. $\square$

Lemma 6.2. Let $\varphi : W \times A \to B$ be a morphism, where W is a variety and A, B are finite dimensional vector spaces. Let b be a given vector of B. If for any $w \in W$, the map $\varphi_w : A \to B$, $a \mapsto \varphi(w, a)$ is such that:

(i) φ_w is linear; (ii) φ_w is injective; (iii) b belongs to the image of φ_w;

then the map $\alpha : W \to A$ implicitely defined by $\varphi(w, \alpha(w)) = b$ is regular.

Proof. If $w_0 \in W$, there exists an open neighborhood U of w_0 and a linear map $p : B \to A$ such that $\forall\, w \in U$, $p \circ \varphi_w \in GL(A)$. Therefore, we may assume that $B = A$ and that $\varphi_w \in GL(A)$. The equality $\alpha(w) = (\varphi_w)^{-1}(b)$ shows that α is regular. $\square$

Of course, if W is smooth, there exists a stronger statement. Let $\varphi : W \times A \to B$ be a morphism, where W, A, B are varieties, W being smooth. Let b be a given point of B. If for any $w \in W$, there exists a unique $a \in A$ such that $\varphi(w, a) = b$, then the map $\alpha : W \to A$ implicitely defined by $\varphi(w, \alpha(w)) = b$ is regular. Indeed, let Γ be the closed subset of $W \times A$ defined by $\Gamma := \{(w, a),\ \varphi(w, a) = b\}$ and let $p_1 : W \times A \to W$ (resp. $p_2 : W \times A \to A$) be the first (resp. second) projection. The map $p_{1|\Gamma} : \Gamma \to W$ being a bijective morphism, it is an isomorphism by Zariski's main theorem. We conclude by the equality $\alpha = p_2 \circ (p_{1|\Gamma})^{-1}$.

7. Three partial orders on multidegrees

7.1. The natural partial order

Let $\sqsubseteq$ be the relation on multidegrees defined by $d \sqsubseteq e \iff \overline{\mathcal{G}}_d \subseteq \overline{\mathcal{G}}_e$.

We begin with the following result:

Lemma 7.1. The binary relation $\sqsubseteq$ is a partial order.

Proof. It is clear that $\sqsubseteq$ is reflexive and transitive. Let us show that it is antisymmetric. If $d \sqsubseteq e$ and $e \sqsubseteq d$, then $\mathcal{G}_d$ and $\mathcal{G}_e$ are both dense open subsets of the (irreducible) variety $\overline{\mathcal{G}}_d = \overline{\mathcal{G}}_e$. Therefore, $\mathcal{G}_d \cap \mathcal{G}_e \neq \emptyset$ showing that $d = e$. $\square$

Remarks. 1. In the last proof, theorem B is useless. Indeed, it is enough to note that $\mathcal{G}_d$ is constructible.

2. Let $m \geq 1$ be an integer. If $\mathcal{D}_{\leq m}$ is the set of multidegrees $(d_1, \ldots, d_l)$ satisfying $d_1 \ldots d_l \leq m$, then the irreducible components of $\mathcal{G}_{\leq m}$ are the $\overline{\mathcal{G}}_d$'s, where d runs through the maximal elements of $\mathcal{D}_{\leq m}$ for the order $\sqsubseteq$.

We will show that the partial order $\sqsubseteq$ may also have been defined by $d \sqsubseteq e \iff \overline{\mathcal{V}}_d \subseteq \overline{\mathcal{V}}_e$. The proof is quite technical and uses the two following lemmas:

Lemma 7.2. If $p(v) \in \overline{\mathcal{V}}_d$, where $p \in \mathbb{C}[T]$ is non-constant and $v \in \mathcal{V}$, then $v \in \overline{\mathcal{V}}_d$.

Proof. By induction on the length of d. If this length is 0, it is obvious, so let us assume that $d = (d_1, \ldots, d_l)$ with $l \geq 1$. We can also suppose that $\deg p \geq 2$, because otherwise there is nothing to show.

There exists an automorphism $f = (f_1, f_2) \in \mathcal{G}_d\Big(\mathbb{C}((T))\Big)$ such that $p(v) = \lim\limits_{T \to 0} f_1(T)$. Furthermore, we may assume that $f_2 \in \mathcal{V}_{d'}\Big(\mathbb{C}((T))\Big)$ and that $\lim\limits_{T \to 0} f_2(T)$ exists and is a non-constant polynomial r. We have $\operatorname{Jac}(p(v), r) = p'(v) \operatorname{Jac}(v, r) \in \mathbb{C}$, so that $\operatorname{Jac}(v, r) = 0$ showing that $r = q(v)$ for some non-constant $q \in \mathbb{C}[T]$. We get $q(v) \in \overline{\mathcal{V}}_{d'}$, so that $v \in \overline{\mathcal{V}}_{d'}$ by the induction hypothesis. We conclude by noting that $\overline{\mathcal{V}}_{d'} \subseteq \overline{\mathcal{V}}_d$. $\square$

Lemma 7.3. If a variable belongs to $\overline{\mathcal{V}}_d$, then any of its predecessors does too.

Proof. We show by induction on $l(d) - l(v)$ that if a variable v belongs to $\overline{\mathcal{V}}_d$, then any of its predecessors does too.

If $l(d) - l(v) = 0$, then by corollary 5.1 the multidegree of v is of the form $(e_1, \ldots, e_l)$ where $e_k \leq d_k$ for each k. Therefore, any predecessor of v has multidegree $(e_2, \ldots, e_l)$ and it is clear that it belongs to $\overline{\mathcal{V}}_d$.

Let us now assume that $l(d) - l(v) > 0$. If $d = (d_1, \ldots, d_l)$, let k be the biggest integer such that v belongs to $\overline{\mathcal{V}}_{(d_k, \ldots, d_l)}$. Up to replacing d by $(d_k, \ldots, d_l)$, we may assume that v belongs to $\overline{\mathcal{V}}_d$, but not to $\overline{\mathcal{V}}_{d'}$. Let $f = (f_1, f_2) \in \mathcal{G}_d\Big(\mathbb{C}((T))\Big)$ be such that $(v, r) = \lim\limits_{T \to 0} f(T)$, where r is non-constant and $f_2 \in \mathcal{V}_{d'}\Big(\mathbb{C}((T))\Big)$.

We have $\operatorname{Jac}(v, r) \in \mathbb{C}$, but we cannot have $\operatorname{Jac}(v, r) = 0$, because otherwise $r = p(v)$ for some non-constant polynomial p and since $p(v) \in \overline{\mathcal{V}}_{d'}$, lemma 7.2 gives us $v \in \overline{\mathcal{V}}_{d'}$. A contradiction. Therefore $\operatorname{Jac}(v, r) \in \mathbb{C}^*$, so that (v, r) is an automorphism.

We cannot have $\deg v < \deg r$, because otherwise v would be a predecessor of r and since $r \in \overline{\mathcal{V}}_{d'}$, with $l(d') - l(r) = l(d) - l(v) - 2$, the induction hypothesis would give us $v \in \overline{\mathcal{V}}_{d'}$. A contradiction. Therefore, $\deg r \leq \deg v$.

Let $\alpha \in \mathbb{C}$ be such that $w := r - \alpha v$ is a predecessor of v (i.e. $\deg w < \deg v$). We have $w = \lim\limits_{T \to 0} (f_2 - \alpha f_1)$. Furthermore, $f_2 - \alpha f_1$ belongs to $\mathcal{V}_d \Big(\mathbb{C}((T))\Big)$ (if $\alpha \neq 0$) or to $\mathcal{V}_{d'} \Big(\mathbb{C}((T))\Big)$ (if $\alpha = 0$). Since $\overline{\mathcal{V}}_{d'} \subseteq \overline{\mathcal{V}}_d$, we get $w \in \overline{\mathcal{V}}_d$ in both cases. Some predecessor of v belonging to $\overline{\mathcal{V}}_d$, it is clear that any predecessor does too. $\square$

Proposition 7.1. $\mathcal{V}_d \subseteq \overline{\mathcal{V}}_e \iff \mathcal{G}_d \subseteq \overline{\mathcal{G}}_e$.

Proof. Let us write $d = (d_1, \ldots, d_l)$ and $e = (e_1, \ldots, e_m)$. Using the lower semicontinuity of the length, we may assume that $1 \leq l \leq m$.

($\Longrightarrow$) We suppose that $\mathcal{V}_d \subseteq \overline{\mathcal{V}}_e$.

If $f \in \mathcal{G}_d$, we want to show that $f \in \overline{\mathcal{G}}_e$. Since $\mathcal{G}_d$ and $\overline{\mathcal{G}}_e$ are stable by the left action of $\mathcal{A}$, we may assume that $f_1 \in \mathcal{V}_d$ and $f_2 \in \mathcal{V}_{d'}$.

Let k be the biggest integer such that f_1 belongs to $\overline{\mathcal{V}}_{(e_k, \ldots, e_m)}$. There exists $g_1 \in \mathcal{V}_{(e_k, \ldots, e_m)} \Big(\mathbb{C}((T))\Big)$ such that $f_1 = \lim\limits_{T \to 0} g_1(T)$. Let $g_2 \in \mathcal{V}_{(e_{k+1}, \ldots, e_m)} \Big(\mathbb{C}((T))\Big)$ be such that $g = (g_1, g_2) \in \mathcal{G}_{(e_k, \ldots, e_m)} \Big(\mathbb{C}((T))\Big)$. Up to replacing g_2 by $T^s g_2 + c$, where $s \in \mathbb{Z}$ and $c \in \mathbb{C}((T))$, we may assume that $\lim\limits_{T \to 0} g_2(T)$ exists and is non-constant. Let us set $h = \lim\limits_{T \to 0} g(T)$.

We have $\operatorname{Jac} h \in \mathbb{C}$, but we cannot have $\operatorname{Jac} h = 0$, because otherwise we would have $h_2 = p(h_1)$ for some non-constant p and $h_2 \in \overline{\mathcal{V}}_{(e_{k+1}, \ldots, e_m)}$, so that $h_1 \in \overline{\mathcal{V}}_{(e_{k+1}, \ldots, e_m)}$ by lemma 7.2, contradicting the definition of k.

Therefore, $\operatorname{Jac} h \in \mathbb{C}^*$, so that h is an automorphism and $h \in \overline{\mathcal{G}}_{(e_k, \ldots, e_m)}$.

We cannot have $\deg h_1 < \deg h_2$, because otherwise h_1 would be a predecessor of $h_2 \in \overline{\mathcal{V}}_{(e_{k+1}, \ldots, e_m)}$, so that $h_1 \in \overline{\mathcal{V}}_{(e_{k+1}, \ldots, e_m)}$ by lemma 7.3. A contradiction.

Hence $\deg h_2 \leq \deg h_1$. Let $\alpha \in \mathbb{C}$ be such that $\deg(h_2 - \alpha h_1) < \deg h_1$.

We have $(h_1, h_2 - \alpha h_1) = \lim\limits_{T \to 0} (g_1, g_2 - \alpha g_1)$, so that $(h_1, h_2 - \alpha h_1) \in \overline{\mathcal{G}}_{(e_k, \ldots, e_m)} \subseteq \overline{\mathcal{G}}_e$.

Since $(h_1, h_2 - \alpha h_1)$ and f have the same first component, it is clear that one can pass from one to the other by composing on the left by an affine automorphism. As a conclusion, we get $f \in \overline{\mathcal{G}}_e$.

($\Longleftarrow$) We suppose that $\mathcal{G}_d \subseteq \overline{\mathcal{G}}_e$ and we want to show that $\mathcal{V}_d \subseteq \overline{\mathcal{V}}_e$.

If $v \in \mathcal{V}_d$, let $w \in \mathcal{V}_{d'}$ be such that $f := (v, w)$ belongs to $\mathcal{G}_d$. There exists $g \in \mathcal{G}_e \Big(\mathbb{C}((T))\Big)$ such that $f = \lim\limits_{T \to 0} g(T)$. Up to replacing g_1 by $g_1 + T g_2$, we may assume that the multidegree of g_1 is e. Since $v = \lim\limits_{T \to 0} g_1(T)$, we have shown that $v \in \overline{\mathcal{V}}_e$. $\square$

Question. Is it true that $\overline{\mathcal{G}}_d = \mathcal{G} \cap (\overline{\mathcal{V}}_d \times \overline{\mathcal{V}}_d)$?

7.2. Three partial orders on multidegrees

In this subsection, we consider three partial orders $\sqsubseteq$, $\leq$ and $\preceq$ on $\mathcal{D}$ and we try to relate them.

1) $\sqsubseteq$ is the natural partial order which has been previously introduced. Recall that $d \sqsubseteq e \Longleftrightarrow \overline{\mathcal{G}}_d \subseteq \overline{\mathcal{G}}_e \Longleftrightarrow \overline{\mathcal{V}}_d \subseteq \overline{\mathcal{V}}_e$ and that for general d, e, we are not yet able to decide whether $d \sqsubseteq e$ or not.

2) $\preceq$ is introduced in [8],1. It is the concrete partial order induced by the following three relations:
(i) $\emptyset \preceq (d_1, \ldots, d_k)$;
(ii) $(d_1, \ldots, d_k) \preceq (e_1, \ldots, e_k)$ if $d_j \leq e_j$ for any j;
(iii) $(d_1, \ldots, d_{j-1}, d_j + d_{j+1} - 1, d_{j+2}, \ldots, d_k) \preceq (d_1, \ldots, d_k)$ if $1 \leq j \leq k-1$.

Here is our most ambitious conjecture (see [8]):

Conjecture 7.1. The partial orders $\sqsubseteq$ and $\preceq$ coincide, i.e. $d \sqsubseteq e \Longleftrightarrow d \preceq e$.

According to [9], if the conjecture 7.2 below holds, we get:
(i) $d \preceq e \Longrightarrow d \sqsubseteq e$;
(ii) if d and e have lengths ≤ 2, we even have $d \preceq e \Longleftrightarrow d \sqsubseteq e$.

Conjecture 7.2. For any $m, n \geq 1$, the following assertion is fulfilled.

R(m,n). Let $a = X(1+a_1X + \cdots + a_mX^m)$ and $b = X(1+b_1X + \cdots + b_nX^n)$ belong to $\mathbb{C}[X]$, where the a_i's and b_j's belong to $\mathbb{C}$. Let us write $a \circ b = X(1+c_1X + \cdots + c_NX^N)$, where $N = (m+1)(n+1) - 1$ and the c_k's belong to $\mathbb{C}$. If $c_1 = \cdots = c_{m+n} = 0$, then $a = b = X$.

3) $\leq$ is introduced in [7],4. If $d = (d_1, \ldots, d_k)$, $e = (e_1, \ldots, e_l)$, we say that $d \leq e$ if $k \leq l$ and if there exists a finite sequence $1 \leq i_1 < i_2 < \ldots < i_k \leq l$ such that $d_j \leq e_{i_j}$ for $1 \leq j \leq k$.

If $d \leq e$, it is easy to show that $d \sqsubseteq e$ and $d \preceq e$. Furthermore:

Lemma 7.4. The maximal elements of $\mathcal{D}_{\leq m}$ for $\leq$ and $\preceq$ coincide.

Proof. Since $d \leq e \Longrightarrow d \preceq e$, any maximal element for $\preceq$ is maximal for $\leq$.

Let us show the converse by contradiction. Otherwise, there would exist an element e of $\mathcal{D}_{\leq m}$ which is maximal for $\leq$ but not for $\preceq$. Therefore, there exist $d = (d_1, \ldots, d_k) \in \mathcal{D}_{\leq m}$ and $1 \leq j \leq k-1$ such that $e = (d_1, \ldots, d_{j-1}, d_j + d_{j+1} - 1, d_{j+2}, \ldots, d_k)$. Since $d_j + d_{j+1} \leq d_j d_{j+1}$, the multidegree $e' := (d_1, \ldots, d_{j-1}, d_j + d_{j+1}, d_{j+2}, \ldots, d_k)$ belongs to $\mathcal{D}_{\leq m}$. However, $e < e'$. A contradiction. $\square$

Let C_m be the set of maximal elements of $\mathcal{D}_{\leq m}$ for $\leq$ (or $\preceq$). It is clear that the irreducible components of $\mathcal{G}_{\leq m}$ are among the $\overline{\mathcal{G}}_d$'s, where d runs through C_m. The following conjecture (made in [7],6) asserts that there is no superfluous term. Note that due to lemma 7.4, this is a consequence of conjecture 7.1.

Conjecture 7.3. The irreducible components of $\mathcal{G}_{\leq m}$ are exactly the $\overline{\mathcal{G}}_d$'s, where d runs through C_m.

7.3. Proof of conjecture 7.3 for $m \leq 27$

If $d = (d_1, \dots, d_k)$ is a multidegree, we set $l(d) = k$ and $|d| = d_1 + \cdots + d_k$.

Lemma 7.5. If $\overline{\mathcal{G}}_d$ is strictly included into $\overline{\mathcal{G}}_e$, then $|d| < |e|$. If we assume furthermore that $d, e \in C_m$ for some m, we also have $l(d) < l(e)$.

Proof. The first part is clear since $\overline{\mathcal{G}}_d$ is irreducible of dimension $|d| + 6$. Let us show the second part by contradiction. Otherwise, we would get $l(d) = l(e)$ (by theorem 4.1), so that $d \leq e$ (by theorem C). A contradiction. □

In [7], we prove conjecture 7.3 for $m \leq 9$. We give the following improvement:

Proposition 7.2. Conjecture 7.3 holds for $m \leq 27$.

Proof. If $m \leq 27$, it is enough to check by hand that for any $d, e \in C_m$, we have $|e| \leq |d|$ or $l(e) \leq l(d)$. This amounts to checking that $l(d) < l(e) \Longrightarrow |d| \geq |e|$. For example, if $m = 27$, C_{27} is composed of any permutation of any of the following finite sequences: (27), $(2, 13)$, $(3, 9)$, $(4, 6)$, $(5, 5)$, $(2, 2, 6)$, $(2, 3, 4)$, $(3, 3, 3)$, $(2, 2, 2, 3)$ and the check is straightforward. □

Remark. Let us take $m = 28$. The multidegrees $(5, 5)$ and $(2, 2, 7)$ belong to C_{28}. However, we do not know whether $\mathcal{G}_{(5,5)} \subseteq \overline{\mathcal{G}}_{(2,2,7)}$ or not.

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