Some Families of Polynomial Automorphisms.

Eric EDO,
Département de Mathématiques Pures,
Université Bordeaux I,
351 Cours de la Libération, 33 400 Talence, France,
email : Eric.Edo@math.u-bordeaux.fr

Jean-Philippe FURTER,
Département de Mathématiques,
Université de La Rochelle,
avenue M. Crépeau, 17 000 La Rochelle, France,
email : jpfurter@univ-lr.fr

Abstract. We give a family of polynomial automorphisms of the complex affine plane whose generic length is 3 and degenerating in an automorphism of length 1 with surprisingly high degree.

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1. Introduction.

The group G of polynomial automorphisms of the complex plane can be endowed with the structure of an infinite-dimensional algebraic variety (see [10]). Furthermore, using the structure of amalgamated product of G, one can define the multidegree of an element σ of G (see below). It is a finite sequence $d = (d_1, \ldots, d_l)$ of integers ≥ 2. We then say that l is the length of σ. A natural question is: What are the relations between the (Zariski) topology of G and the multidegree? A first answer is given in [6]. It is shown that the length is a lower semicontinuous function on the group G (a family of automorphisms of length l can only degenerate in an automorphism of length $l' \leq l$).

Let us denote by G_d the set of automorphisms of G whose multidegree is $d = (d_1, \ldots, d_l)$. We obtain a partition of G by the G_d when d describes the set D of finite sequences of integers ≥ 2. It was conjectured in [6] that $\overline{G_d}$

(the Zariski closure of G_d) is satured for the equivalence relation "to have the same multidegree", *i.e.* there should exist a subset $E(d) \subset D$ of multidegrees $e = (e_1, \ldots, e_m)$ such that $\overline{G_d} = \bigcup_{e \in E(d)} G_e$. This is equivalent to saying that if $G_e \cap \overline{G_d} \neq \emptyset$, then $G_e \subset \overline{G_d}$.

In this paper, we show that this conjecture is false. We make explicit a result of [2] to obtain a family of automorphisms belonging generically to $G_{(11,3,3)}$ and degenerating in an automorphism of $G_{(19)}$. Therefore, we have $G_{(19)} \cap \overline{G_{(11,3,3)}} \neq \emptyset$. However, for grounds of dimension, one cannot have $G_{(19)} \subset \overline{G_{(11,3,3)}}$. Indeed, $G_{(d_1,\ldots,d_l)}$ is a constructible subset of G of dimension $d_1 + \cdots + d_l + 6$ (see [4] and [5]).

2. The multidegree of an automorphism.

This notion is introduced by Friedland and Milnor (see [4]).

We denote by G the group of polynomial automorphisms of the complex plane $\mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[X,Y])$. An element σ of G is identified with its sequence (f,g) of coordinate functions, where $f, g \in \mathbb{C}[X,Y]$. We define the degree of σ by $\deg \sigma = \max\{\deg(f), \deg(g)\}$. Let $\sigma = (f,g) \in G$ and $\sigma' = (f',g') \in G$. We denote by $\sigma\sigma'$ the composition, *i.e.* $\sigma\sigma' = (f(f',g'), g(f',g'))$.

Let

$$A := \{(aX + bY + c, a'X + b'Y + c');\ a, b, c, a', b', c' \in \mathbb{C}, ab' - a'b \neq 0\}$$

be the subgroup of affine automorphisms and let

$$B := \{(aX + P(Y), bY + c);\ a, b, c \in \mathbb{C},\ P \in \mathbb{C}[Y],\ ab \neq 0\}$$

be the subgroup of triangular automorphisms (B may be viewed as a Borel subgroup of G).

If $\sigma \in G$, by the Jung-van der Kulk theorem (see [7] and [8]), one can write $\sigma = \alpha_1\beta_1 \ldots \alpha_k\beta_k\alpha_{k+1}$ where the α_j (resp. β_j) belong to A (resp. B). By contracting such an expression, one might as well suppose that it is reduced, *i.e.* $\forall\ j,\ \beta_j \notin A$ and $\forall\ j,\ 2 \leq j \leq k,\ \alpha_j \notin B$. It follows from the amalgamated structure of G that if $\sigma = \alpha'_1\beta'_1 \ldots \alpha'_l\beta'_l\alpha'_{l+1}$ is another reduced expression of f, then $k = l$ and there exist $(\gamma_j)_{1 \leq j \leq k}$, $(\delta_j)_{1 \leq j \leq k}$ in $A \cap B$ such that $\alpha'_1 = \alpha_1\gamma_1^{-1}$, $\alpha'_j = \delta_{j-1}\alpha_j\gamma_j^{-1}$ (for $2 \leq j \leq k$), $\alpha'_{k+1} = \delta_k\alpha_{k+1}$ and $\beta'_j = \gamma_j\beta_j\delta_j^{-1}$ (for $1 \leq j \leq k$). Following [4], we define the multidegree of σ by $d(\sigma) := (\deg \beta_1, \ldots, \deg \beta_k)$ which does not depend on the choice of the reduced expression. Therefore, we have a multidegree function $d: G \to D$, where D denotes the set of finite sequences of integers ≥ 2 (including the empty sequence).

3. The structure of infinite-dimensional variety of G.

This notion is introduced by Shafarevich (see [10]).

When $n \geq 1$ is an integer, we set $G_{\leq n} := \{\sigma \in G,\ \deg \sigma \leq n\}$. The subset $G_{\leq n}$ is naturally endowed with the structure of an algebraic variety (see [BCW]). The equality $G = \bigcup_n G_{\leq n}$ endows G with the structure of an infinite-dimensional algebraic variety.

Let us recall that a set V with a fixed sequence of subsets V_n, each of which has a structure of finite-dimensional algebraic variety, is called an infinite-dimensional algebraic variety if the following conditions are satisfied:

1) $V = \bigcup_n V_n$;

2) V_n is a closed algebraic subvariety of V_{n+1}.

Each of the V_n will be considered with its Zariski topology and we endow V with the topology of the inductive limit, in which a set $W \subset V$ is closed if and only if $W \cap V_n$ is closed in V_n for each n.

4. Families of automorphisms of the affine plane.

4.1. The Nagata automorphism.

The first non trivial example of a family of automorphisms of the plane comes from the Nagata automorphism (see [9]). Let us set $R = \mathbb{C}[Z]$. The Nagata automorphism is the R-automorphism of $\mathbb{A}^2_R = \text{Spec } R[X,Y]$ defined by $N = (X - 2Y(XZ+Y^2) - Z(XZ+Y^2)^2, Y + Z(XZ+Y^2))$. We thus obtain a family of automorphisms of $\mathbb{A}^2_{\mathbb{C}}$ parametrized by $\mathbb{A}^1_{\mathbb{C}} = \text{Spec } \mathbb{C}[Z]$, *i.e.* a morphism from $\mathbb{A}^1_{\mathbb{C}}$ to G.

4.2. The group $G(R)$.

When R is a ring, let us denote by $G(R)$ the group of polynomial R-automorphisms of $\mathbb{A}^2_R = \text{Spec } R[X,Y]$. We still identify an element σ of $G(R)$ with the couple (f,g) of its coordinate functions, where $f, g \in R[X,Y]$.

Let us notice that knowing an element of $G(R)$ is equivalent to knowing an (algebraic) family of automorphisms parametrized by the affine scheme $V = \text{Spec } R$. In particular, when $R = \mathbb{C}[Z]$, an element σ of $G(\mathbb{C}[Z])$ induces a morphism from $\mathbb{A}^1_{\mathbb{C}}$ to G. We denote by $\sigma_{Z\to z}$ the image of $z \in \mathbb{A}^1_{\mathbb{C}} = \mathbb{C}$ by this induced morphism.

For example, if N denotes the Nagata automorphism, we have $N_{Z\to z} = (X - 2Y(zX + Y^2) - z(zX + Y^2)^2, Y + z(zX + Y^2))$ for all $z \in \mathbb{C}$ and in particular $N_{Z\to 0} = (X - 2Y^3, Y)$.

4.3. Degeneration.

We shall say that a family $\sigma \in G(\mathbb{C}[Z])$ degenerates at a point z_0 of $\mathbb{C}$ if the multidegree of $\sigma_{Z\to z_0}$ is different from the generic multidegree of σ, *i.e.* the multidegree of σ as an element of $G(\mathbb{C}(Z))$. For example, one could easily check that the Nagata automorphism splits in $N = (X - Z^{-1}Y^2, Y)(X, Z^2X + Y)(X + Z^{-1}Y^2, Y)$ in $G(\mathbb{C}(Z))$. This last expression shows that the generic multidegree of σ (which is also in this example the multidegree of $N_{Z\to z}$ for all $z \in \mathbb{C}^*$) is $(2, 2)$. However, the multidegree of $N_{Z\to 0} = (X - 2Y^3, Y)$ is (3), hence $G_{(3)} \cap \overline{G_{(2,2)}} \neq \emptyset$.

This family degenerates for $z = 0$ and we observe that the length has strictly decreased. This illustrates the main result of [6]:

Theorem 1. *The length map $l : G \to \mathbb{Z}$ is lower semicontinuous.*

However, even if the length behaves well with respect to the topology, it seems that it is not the case of the multidegree (at least in length 3).

4.4. Constructing the elements of $G(\mathbb{C}[Z])$.

The next result (see Lemma 1.1.8. p. 5 of [3]) is very useful to construct the elements of $G(\mathbb{C}[Z])$.

Proposition. *Let $\sigma = (f, g) \in G(\mathbb{C}(Z))$, then the following assertions are equivalent:*

(i) $\sigma \in G(\mathbb{C}[Z])$;

(ii) $f, g \in \mathbb{C}[Z][X, Y]$ and Jac $\sigma \in \mathbb{C}^*$, where Jac $\sigma := \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X}$.

Remark. We reformulate the assertion $f, g \in \mathbb{C}[Z][X, Y]$ by saying that the coefficients of f and g (or even the ones of σ) belong to $\mathbb{C}[Z]$ (the coefficients are of course the ones of X^pY^q).

Furthermore, since $\mathbb{C}(Z)$ is a field, we know (by the Jung-van der Kulk theorem) that any element of $G(\mathbb{C}(Z))$ can be expressed as a composition of affine and triangular automorphisms.

Therefore, to construct any element of $G(\mathbb{C}[Z])$, it is sufficient to compose affine and/or triangular automorphisms $\sigma_1, \ldots, \sigma_n$ with coefficients in $\mathbb{C}(Z)$, in such a way that:

1) $\sigma = \sigma_1 \cdots \sigma_n$ has its coefficients in $\mathbb{C}[Z]$;

2) $\text{Jac } \sigma = \prod_{j=1}^{n} \text{Jac } \sigma_j \in \mathbb{C}^*$.

For example, using the splitting of the Nagata automorphism in $G(\mathbb{C}(Z))$ (see 4.3.), one can verify assumptions 1) and 2) and the proposition shows that N belongs to $G(\mathbb{C}[Z])$.

5. Some families of automorphisms with generic length 3.

Our main outcome is the following:

Theorem 2. *Let $a, b \geq 2$ and $c \geq 1$ be integers, then*

$$G_{\big(a+c(ab-1)\big)} \cap \overline{G_{\big(a+(c-1)(ab-1),b,a\big)}} \neq \emptyset.$$

Example. Fixing some values for (a, b, c), we obtain:
$G_{(5)} \cap \overline{G_{(2,2,2)}} \neq \emptyset$, $G_{(7)} \cap \overline{G_{(2,3,2)}} \neq \emptyset$ and $G_{(19)} \cap \overline{G_{(11,3,3)}} \neq \emptyset$.

From Theorem 2, we can deduce the

Corollary. *Let $a, b \geq 2$ and $c \geq 1$ be integers with $(a, b) \neq (2, 2)$, then the closure of $G_{\big(a+(c-1)(ab-1),b,a\big)}$ in G is not saturated for the equivalence relation "to have the same multidegree".*

Proof. Let us recall that $G_{(d_1,\ldots,d_l)}$ is a constructible subset of G of dimension $d_1 + \cdots + d_l + 6$. Let us set $d = \big(a + (c-1)(ab-1), b, a\big)$ and $e = \big(a + c(ab-1)\big)$. We argue by contradiction. If $\overline{G_d}$ was saturated for the equivalence relation "to have the same multidegree", then we would have $G_e \subset \overline{G_d} \setminus G_d$ by Theorem 2. Hence the dimension of G_e would be strictly less than that of G_d, whence $a+c(ab-1)+6 < a+(c-1)(ab-1)+b+a+6$, *i.e.* $(a-1)(b-1) < 2$, which is absurd. $\square$

Example. We have thus shown that $\overline{G_{(2,3,2)}}$ and $\overline{G_{(11,3,3)}}$ are not saturated for the equivalence relation "to have the same multidegree".

Theorem 2 is a direct consequence of the following one:

Theorem 3. *Let $a, b \geq 2$ and $c \geq 1$ be integers.*
Let us define triangular automorphisms $\beta, \beta', \beta'' \in G(\mathbb{C}[Z])$ by

$\beta := (X + Y^a, Y)$,

$\beta' := (X + ZY^b, Y)$ *and*

$$\beta'' := \Big(X - Y^a \sum_{n=0}^{c-1} \frac{1}{bn+1} \binom{a(bn+1)}{n} (-ZY^{ab-1})^n, Y\Big)$$

and affine automorphisms $\alpha, \alpha' \in G(\mathbb{C}(Z))$ by

$\alpha := (Z^c X, Y)$ *and*

$\alpha' := (Y, X)$.

If $\sigma \in G(\mathbb{C}(Z))$ is defined by

$\sigma := \alpha^{-1}\beta''\alpha'\beta'\alpha'\beta\alpha$,

then $\sigma \in G(\mathbb{C}[Z])$ and $\sigma_{Z\to 0} = \Big(X + \dfrac{(-1)^c}{bc+1} \dbinom{a(bc+1)}{c} Y^{a+c(ab-1)}, Y\Big)$.

To obtain Theorem 2 it is enough to remark that the multidegree of σ is generically equal to $(\deg \beta'', \deg \beta', \deg \beta) = \big(a + (c-1)(ab-1), b, a\big)$ and that $\sigma_{Z\to 0}$ is a triangular automorphism of degree $a + c(ab-1)$, so that $d(\sigma_{Z\to 0}) = \big(a + c(ab-1)\big)$.

The proof of Theorem 3 is based on the following result:

Lemma 1. *Let $a \geq 1$, $b \geq 0$ be integers. Let $S(T) = \sum_{n\geq 0} s_n T^n \in \mathbb{C}[[T]]$ be the power series defined by the functional equation*

$$S(T) = (1 + TS^b(T))^a,$$

then

(i) $S(T) = \displaystyle\sum_{n\geq 0} \frac{1}{bn+1} \binom{a(bn+1)}{n} T^n$ *and*

(ii) $S\Big(-T(1+T)^{ab-1}\Big) = \dfrac{1}{(1+T)^a}$.

Proof. (i) Let us set $u(T) := TS^b(T)$, then $\frac{u(T)}{(1+u(T))^{ab}} = T$, hence u is a local analytic diffeomorphism around 0 with inverse v, where v is defined by

$v(W) := \dfrac{W}{(1+W)^{ab}}$.

It is clear that $s_0 = 1$.

If $\oint$ denotes integration around a little circle around the origin and if $n > 0$ is an integer, it comes out (by the Lagrange formula):

$$s_n = \frac{1}{2\pi i}\oint \frac{S(T)}{T^{n+1}}dT = \frac{1}{2\pi i}\oint \frac{(1+u(T))^a}{T^{n+1}}dT$$

$$= \frac{1}{2\pi i}\oint \frac{(1+W)^a}{v^{n+1}(W)}v'(W)dW \quad \text{(setting } T = v(W)\text{)}$$

$$= \frac{a}{n}\frac{1}{2\pi i}\oint \frac{(1+W)^{a-1}}{v^n(W)}dW \quad \text{(integrating by parts)}$$

$$= \frac{a}{n}\frac{1}{2\pi i}\oint \frac{(1+W)^{abn+a-1}}{W^n}dW = \frac{a}{n}\binom{abn+a-1}{n-1}$$

$$= \frac{a}{a(bn+1)}\binom{a(bn+1)}{n} = \frac{1}{bn+1}\binom{a(bn+1)}{n}.$$

(ii) Let us set $h(T) := -T(1+T)^{ab-1}$. Then h is a local analytic diffeomorphism around 0. Let us denote by k its inverse. It is sufficient to show that $S(T) = \dfrac{1}{(1+k(T))^a}$.

For this, let us set $\tilde{S}(T) := \dfrac{1}{(1+k(T))^a}$ and let us show that $\tilde{S}$ satisfies the same functional equation than S, that it is to say $\tilde{S}(T) = (1+T\tilde{S}^b(T))^a$.

This comes from the equality $\dfrac{1}{1+k(T)} = 1 + \dfrac{T}{(1+k(T))^{ab}}$, which comes itself from the equality $h(k(T)) = T$. □

Remarks. 1. Let us set $\lambda = ab$. One could show that the radius of convergence of the power series $S(T)$ is equal to $+\infty$ (resp. 1, resp. $\dfrac{(\lambda-1)^{\lambda-1}}{\lambda^\lambda}$), if $\lambda = 0$ (resp. $\lambda = 1$, resp. $\lambda \geq 2$).

2. It is pleasant to explain the Lagrange formula by complex analysis; nevertheless, it would have been possible to use a purely algebraic version of this formula (see Corollary 5.4.3. p. 42 in [11]), which would have allow us to replace the field $\mathbb{C}$ by any field of characteristic 0.

3.Using the functional equation satisfied by $S(T)$, one could show that $S(T) \in \mathbb{Z}[[T]]$. Hence, there exists a version of Theorem 3 where the field $\mathbb{C}$ is replaced by any field (possibly of positive characteristic).

Proof of Theorem 3.

Let us set $P(Y) := Y^a S_{<c}(-ZY^{ab-1}) \in \mathbb{C}[Z][Y]$,

where $S_{<c}(T)$ denotes the series $S(T)$ truncated at the order c,

that is to say $S_{<c}(T) = \sum_{n=0}^{c-1} s_n T^n$.

We have $\beta'' = (X - P(Y), Y)$ by (i) of Lemma 1.

and if we set $\begin{cases} g := Y + Z(Z^c X + Y^a)^b \\ f := X + Z^{-c}(Y^a - P(g)) \quad (1), \end{cases}$

then $\sigma = (f, g)$.

We prove that $\sigma \in G(\mathbb{C}[Z])$ using the proposition of § 4.4. It is clear that $\operatorname{Jac} \sigma = 1$ and that $g \in \mathbb{C}[Z][X,Y]$, hence to show that σ is an automorphism of $\mathbb{C}[Z][X,Y]$, it is sufficient to show that $Y^a - P(g)$ is divisible by Z^c (in the ring $\mathbb{C}[Z][X,Y]$), which can be written using congruences $Y^a - P(g) \equiv 0\ [Z^c]$.

But, to compute $\sigma_{Z \to 0}$, we need to compute $Y^a - P(g)$ modulo Z^{c+1}. Therefore, let us carry out directly this computation.

We have $g \equiv Y(1 + ZY^{ab-1})\ [Z^{c+1}]$, hence

$$\begin{aligned} Y^a - P(g) &= Y^a - g^a S_{<c}(-Zg^{ab-1}) \\ &\equiv Y^a Q(ZY^{ab-1})\ [Z^{c+1}] \quad (2) \end{aligned}$$

where $Q(T) := 1 - (1+T)^a S_{<c}\left(-T(1+T)^{ab-1}\right)$.

But, by (ii) of Lemma 1, we have:

$$(1+T)^a S\left(-T(1+T)^{ab-1}\right) = 1, \quad \text{hence}$$

$$\begin{aligned} Q(T) &= (1+T)^a S_{\geq c}\left(-T(1+T)^{ab-1}\right), \quad \text{with } S_{\geq c}(T) = \sum_{n \geq c} s_n T^n \\ &\equiv (-1)^c s_c T^c\ [T^{c+1}], \\ &\equiv \frac{(-1)^c}{bc+1} \binom{a(bc+1)}{c} T^c\ [T^{c+1}], \qquad \text{by (i) of Lemma 1.} \end{aligned}$$

whence $Y^a - P(g) \equiv \dfrac{(-1)^c}{bc+1}\dbinom{a(bc+1)}{c} Z^c Y^{a+c(ab-1)}\ [Z^{c+1}]$ by (2). By (1) this shows that $f \in \mathbb{C}[Z][X,Y]$ and $\sigma \in G(\mathbb{C}[Z])$ by the proposition of § 4.4. Furthermore we have: $f \equiv X + \dfrac{(-1)^c}{bc+1}\dbinom{a(bc+1)}{c} Y^{a+c(ab-1)}\ [Z]$ by (1) and $g \equiv Y\ [Z]$ which shows that $\sigma_{Z\to 0} = (X + \dfrac{(-1)^c}{bc+1}\dbinom{a(bc+1)}{c} Y^{a+c(ab-1)}, Y)$.
□

6. The collapse power of a multidegree.

Definition. If $d = (d_1, \dots, d_l)$ is a multidegree with $l \geq 1$, let us define the collapse power of d by $\mathrm{cp}(d) := \max\{k \in \mathbb{Z},\ k \geq 2 \text{ and } G_{(k)} \cap \overline{G_d} \neq \emptyset\}$.

Using the well known fact that if the multidegree of σ is $(d_1, \dots, d_l)$, then the multidegree of σ^{-1} is $(d_l, \dots, d_1)$, it is easy to show the next result

Lemma 2. *If $(d_1, \dots, d_l)$ is a multidegree, then*
$$\mathrm{cp}(d_1, \dots, d_l) = \mathrm{cp}(d_l, \dots, d_1).$$

Theorem 2 yields a lower bound for $\mathrm{cp}(d_1, d_2, d_3)$.

Theorem 4. *If $(d_1, d_2, d_3) \in D$ is a multidegree such that $d_1 - d_3$ is a multiple of $d_2 \min(d_1, d_3) - 1$, then $\mathrm{cp}(d_1, d_2, d_3) \geq \max(d_1, d_3) + d_2 \min(d_1, d_3) - 1$.*

Proof. By Lemma 2 and since the assumption in Theorem 4 is invariant by exchanging d_1 and d_3, we can assume that $d_1 \geq d_3$. We set $a = d_3$, $b = d_2$ and $c = 1 + (d_1 - d_3)/(d_2 d_3 - 1)$ then $\max(d_1, d_3) + d_2 \min(d_1, d_3) - 1 = a + c(ab-1)$ and Theorem 4 follows from Theorem 2. □

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