An introduction to random walks on groups

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December 2014

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In these notes, we will study a basic question from the theory of random walks on groups: given a random walk

$$e, g_1, g_2g_1, \ldots, g_n \cdots g_1, \ldots$$

on a group G, we will give a criterion for the trajectories to go to infinity with linear speed (for a notion of speed which has to be defined). We will see that this property is related to the notion of amenability of groups: this is a fundamental theorem which was proved by Kesten in the case of discrete groups and extended to the case of continuous groups by Berg and Christensen. We will give examples of this behaviour for random walks on $\mathrm{SL}_2(\mathbb{R})$.

1 Locally compact groups and Haar measure

In order to define random walks on groups, I need to consider probability measures on groups, which will be the distribution of the random walks. When the group is discrete, there is no technical difficulty: a probability measure is a function from the group to $[0, 1]$, the sum of whose values is one. But I also want to consider non discrete groups, such as vector spaces, groups of matrices, groups of automorphisms of trees, etc. Hence, to deal with these examples, I have first to define a proper notion of a continuous group.

Definition 1.1. By a locally compact group, we mean a group G, equipped with a locally compact topology such that the product map $G \times G \to G$ and the inverse map $G \to G$ are continuous.

We will mostly deal with locally compact groups that will be second countable. More precisely, the understanding of the following examples will be our main objective.

Example 1.2. Any group, equipped with the discrete topology, is a locally compact group.

Example 1.3. The usual topology makes the additive groups $\mathbb{R}^d$, $d \geq 1$, locally compact topological groups. The product of matrices on $\mathrm{GL}_d(\mathbb{R})$ and $\mathrm{SL}_d(\mathbb{R})$ and the usual topology induce structures of locally compact topological groups on them. In the same way, the orthogonal group $\mathrm{O}(d)$ and the special orthogonal group $\mathrm{SO}(d)$ have natural topologies of compact groups.

Example 1.4. Let $\Gamma = (V, E)$ be a locally finite graph. In other words, V (vertices) is a set, $E \subset V \times V$ (edges) is a symmetric relation (that is, for any (x, y) in E, one has $(y, x) \in E$) and, for any x in V, the set of y in V such that $(x, y) \in E$ is finite. For x in V and n in $\mathbb{N}$, we let $B_n(x)$ be the ball of center x and radius n in V, that is, the set of y in V such that there exists $0 \leq p \leq n$ and $x = x_0, \ldots, x_p = y$ in V with $(x_{i-1}, x_i) \in E$, $1 \leq i \leq p$.

Let G be the group of automorphisms of Γ, that is, the group of permutations g of V such that, for any (x, y) in E, (gx, gy) also belongs to E. For any g in G, x in V and n in $\mathbb{N}$, define $U(g, x, n)$ as the set of h in G such that $gy = hy$ for any y in $B_n(x)$ (in other words, h is locally equal to g). We equip G with the topology generated by the subsets of the form $U(g, x, n)$. One easily checks that the group operations are continuous and that, due to the fact that the graph is locally finite, the topology is locally compact. It is second countable as soon as V is countable.

For a non empty finite subset W of V, let G_W denote the set of g in G such that $gx = x$, for any x in W. Then the subgroups G_W are compact and open in G and they form a basis of neighborhoods of the identity map.

Thus, the classical groups from geometry all are locally compact groups. In the latter example, the topology is totally discontinuous. One can check that a locally compact group is totally discontinuous if and only if the neutral element e has a basis of neighborhoods consisting of compact open subgroups.

One main feature of locally compact groups is that they come with a natural reference measure.

Theorem 1.5 (Haar). *Let G be a locally compact group. Then there exists a (non zero) Radon measure on G that is invariant under left translations of G, that is such that, for any g in G, for any continuous compactly supported function φ on G, the functions φ and $x \mapsto \varphi(gx)$ have the same integral. This measure is unique up to multiplication by a positive real number.*

By abuse of language (since the measure is not unique), one says that such a measure is the Haar measure of G and one sets $\int_G \varphi(x)\mathrm{d}x$ for the integral of a function φ with respect to Haar measure. Haar measure is finite if and only if G is compact. We denote by $\mathrm{L}^p(G)$, $1 \leq p \leq \infty$, the Lebesgue spaces of Haar measure on G.

In general, in concrete examples, the Haar measure is defined in a concrete way.

Example 1.6. On a discrete group, Haar measure is the counting measure.

Example 1.7. On $\mathbb{R}^d$, $d \geq 1$, Haar measure is Lebesgue measure. On $\mathrm{GL}_d(\mathbb{R})$, it is equal to the restriction of Lebesgue measure of the vector space of square $d \times d$ matrices, multiplied by the function $x \mapsto |\det x|^{-d}$.

Example 1.8. Let G be a locally compact totally discontinous group (for example, the group of automorphisms of a graph). Fix a compact open subgroup H of G (which we will use for chosing the normalization of the measure).

Let φ be a locally constant compactly supported function on G. Then, there exists an open subgroup U of H, such that φ is right-U-invariant. We set

$$\int_G \varphi(x)\mathrm{d}x = \frac{1}{[H:U]} \sum_{y \in G/U} \varphi(y).$$

This formula makes sense since on the one hand, φ being right U-invariant, it may be considered as a finitely supported function on the discrete space G/U and, on the other hand, U being open in H which is compact, it has finite index in H. One easily checks that the definition does not depend on the choice of U.

Remark 1.9. In general, the Haar measure is not invariant under right translations of the groups. This is the case, for example if G is the group of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}$, with $a \neq 0$ and b in $\mathbb{R}$. One easily checks that the measure $|a|^{-1}\mathrm{d}a\mathrm{d}b$ is invariant under left translations, but not under right translations.

2 Amenable groups

We shall now split the locally compact groups into two classes which have a very different behaviour from the measure theoretic point of view.

Let G be a locally compact topological group and X be a locally compact topological space. An action of G on X is said to be continuous if the action map $G \times X \to X$ is continuous when $G \times X$ is equipped with the product topology.

Definition 2.1. We say that a locally compact topological group G is amenable if, for every continuous action of G on a compact space X, there exists a Borel probability measure ν on X that is G-invariant, that is such that, for any continuous function φ on X, for any g in G,

$$\int_X \varphi(gx)\mathrm{d}\nu(x) = \int_X \varphi(x)\mathrm{d}\nu(x).$$

Example 2.2. The groups $\mathrm{GL}_d(\mathbb{R})$ and $\mathrm{SL}_d(\mathbb{R})$, $d \geq 2$, are not amenable. Indeed, they cannot preserve a Borel probability measure on the projective space $\mathbb{P}^{d-1}_{\mathbb{R}}$.

Example 2.3. Let r be a positive integer. We let G be a non-abelian free group of rank r, that is G is spanned by a set $S \subset G$ with r elements and any g in G may be written in a unique way as a product $h_1^{k_1} \cdots h_s^{k_s}$, with $s \geq 0$, $h_1, \dots, h_s$ in S, $k_1, \dots, k_s$ in $\mathbb{Z} \smallsetminus \{0\}$ and $h_i \neq h_{i+1}$ for $1 \leq i \leq s_1$. For $r = 1$, we have $G \simeq \mathbb{Z}$. We claim that G is non amenable as soon as $r \geq 2$. Indeed, since every group spanned by less that r elements is a homomorphic image of G, it suffices to give a group spanned by 2 elements, an action of which does not admit any invariant measure. We set $g = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ and $h = ugu^{-1}$ where $u = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. One checks that no Borel probability measure on $\mathbb{P}^1_{\mathbb{R}}$ is both g and h-invariant.

Here comes a first simple example of an amenable group:

Lemma 2.4. *Abelian groups are amenable.*

Proof. It is a classical fact that every homeomorphism of a compact space preserves a Borel probability measure. In other words, $\mathbb{Z}$ is amenable. The extension of this result to a general abelian group only relies on a formal construction using Zorn's Lemma.

More precisely, le G be an abelian locally compact topological group, acting on a compact space X. Let $\mathcal{S}$ be the set of subgroups of G which preserve a Borel probability measure on X. Equip $\mathcal{S}$ with the inclusion order. We claim that $\mathcal{S}$ is inductive. Indeed, let $\mathcal{T} \subset \mathcal{S}$ be a totally ordered subset. Let $L = \bigcup_{H \in \mathcal{T}} H$, which is a subgroup of G and let us prove that L belongs to $\mathcal{S}$. Let $\mathcal{P}$ be the set of Borel probability measures on X and, for every H in $\mathcal{S}$, let $\mathcal{P}_H \subset \mathcal{P}$ be the non empty subset of H-invariant measures. Equip $\mathcal{P}$ with the weak-$*$ topology, which makes it a compact space. Then, the sets $\mathcal{P}_H$, where H runs in $\mathcal{T}$, form a totally ordered set of compact subsets of $\mathcal{P}$. By compactness, we get

$$\bigcap_{H \in \mathcal{T}} \mathcal{P}_H \neq \emptyset,$$

hence L belongs to $\mathcal{T}$. As L is a majorant of all elements of $\mathcal{T}$, the order is inductive as claimed.

By Zorn's Lemma, $\mathcal{S}$ admits a maximal element H. Let us prove that $H = G$, which finishes the proof. We pick g in G. We will use the classical argument to construct a g-invariant measure in $\mathcal{P}_H$. Let ν be a H-invariant Borel probability measure on X. Then, every limit point as $n \to \infty$ of $\frac{1}{n}\sum_{k=0}^{n-1} g_*^k \nu$ is a g-invariant element. Since g commutes with H, such a limit point is still H-invariant. Hence, as $\mathcal{P}_H$ is weak-$*$ compact, it contains a g-invariant element. In other words, if H' denotes the subgroup of G spanned by g and H, we have $H' \in \mathcal{S}$. As H is maximal, we get $H' = H$, hence $g \in H$, which should be proved. □

Amenable groups can be characterized as groups which admit fixed points in convex invariant subsets for certain actions on Banach spaces.

Given a locally compact group G and a Banach space E, by an isometric continuous action of G on E, we mean a linear action of G by norm isometries on E such that, for any v in E, the orbit map $G \to E, g \mapsto gv$ is continuous.

Example 2.5. The natural action of G by left translations on the spaces $\mathrm{L}^p(G)$, $1 \leq p < \infty$, are isometric and continuous. The one on $\mathrm{L}^\infty(G)$ is not as soon as G is not discrete (there exists bounded measurable functions which are not continuous). If G acts continuously on a compact set X, then the natural action of G on the Banach space $\mathcal{C}^0(X)$ is an isometric continuous action.

When G acts on E in a continuous isometric way, there is a natural action of G on the topological dual space E^* of E (which is isometric but not necessarily continuous).

Lemma 2.6. *Let G be a locally compact topological group. Then G is amenable if and only if, for every isometric continuous action of G on a Banach space E, for every non-empty convex and weak-$*$ compact G-invariant subset $Y \subset E^*$, G admits a fixed point in Y.*

Proof. Clearly, if the fixed point property holds, G is amenable, by the example above. Conversely, let E and Y be as in the statement of the Lemma. Since the action of G on E is continuous, one checks that the ation of G on Y is weak-$*$ continuous, that is, the action map $G \times Y \to Y$ is continuous when Y is equipped with the weak-$*$ topology. As G is amenable, G preserves a Borel probability measure ν on Y. Now, the linear form

$f_\nu : v \mapsto \int_Y \langle f, v\rangle \mathrm{d}\nu(f)$ on E is continuous (note that, by Banach-Steinhaus Theorem, K is bounded in E^*). Since ν is G-invariant, f_ν is G-invariant. Since K is convex, it follows from the geometric form of Hahn-Banach Theorem that f_ν belongs to K. □

Let G be a locally compact group and H be a closed subgroup of G. We shall always equip the space G/H with the quotient topology. We have

Lemma 2.7. *Let G be a locally compact topological group and H be a closed subgroup of G. The space G/H is Hausdorff and the projection map $G \to G/H$ is open. In particular, G/H is a locally compact topological space.*

Proof. Note first that the projection map $\pi : G \to G/H$ is open: indeed, if U is an open subset of G, one has $\pi^{-1}\pi(U) = UH$ which is clearly open in G. In particular, every x in G/H has a compact neighborhood, and it only remains to prove that G/H is Hausdorff.

To do this, consider $x_1 \neq x_2$ in G/H. Write $x_1 = g_1H$ and $x_2 = g_2H$ with $g_1, g_2 \in G$, $g_1^{-1}g_2 \notin H$. Since H is closed, there exists a neighborhood of e in G with $Ug_1^{-1}g_2 \cap H = \emptyset$. Since H is a subgroup, we get

$$g_2H \cap g_1U^{-1}H = \emptyset.$$

Pick a neighborhood V of e with $g_1^{-1}Vg_1 \subset U^{-1}$. We get

$$g_2H \cap Vg_1H = \emptyset.$$

Finally, pick a new neighborhood of e with $W^{-1}W \subset V$. We get

$$Wg_2H \cap Wg_1H = \emptyset,$$

that is $Wx_1 \cap Wx_2 = \emptyset$ and we are done. □

Note that, if G is a locally compact topological group and $H \subset G$ is a closed normal subgroup, then G/H, equipped with the natural group structure, is again a locally compact topological group.

Corollary 2.8. *Let G be a locally compact topological group and $H \subset G$ be a closed normal subgroup. If H and G/H are amenable, so is G.*

Proof. Let G act continuously by isometries on a Banach space E and $Y \subset E^*$ be a convex weak-$*$ compact G-invarant subset. Then H admits fixed points in Y. Let $Z \subset Y$ be the set of those fixed points. Then, as H is normal, Z is G-invariant. Now, to conclude, we must see Z as an invariant subset in the dual space of a Banach space equipped with an action of G/H. Let us construt our candidate for being this Banach space (this is slightly formal). We let $E_H \subset E$ be the closed subspace spanned by the vectors of the form $hv - v$ where v is in E and h is in H and we let F be the quotient Banach space E/E_H. One checks that the dual space of F may be identified with the space of H-invariant elements in E^*. As G/H acts in a natural way on F, we can see Z as a convex and weak-$*$ compact G/H-invariant subset in the dual space of F. Hence G/H admits a fixed point in Z and we are done. □

Corollary 2.9. *Let G be a solvable locally compact topological group. Then G is amenable.*

Proof. We prove this by induction on the length of the derived series of G. If it has length 1, then G is abelian, and we already know the result in this case. If it has length $r \geq 2$, we let H be the closure of the last non zero term in this series. Then H, being the closure of an abelian normal subgroup, is itself an abelian normal subgroup. As G/H is solvable with derived series of length $\leq r - 1$, G/H is amenable. As H is abelian, it is amenable and the result follows. □

Remark 2.10. We shall see in Corollary 3.4 below that, if G is amenable, every closed subgroup of G is amenable. Hence, if G contains a closed non-abelian free group, then G is non-amenable. It was proved by Tits that any subgroup of $\mathrm{GL}_d(\mathbb{C})$, $d \geq 1$, either contains a non-abelian free subgroup or a finite index solvable group, so that for discrete groups that are isomorphic to subgroups of $\mathrm{GL}_d(\mathbb{C})$, amenability amounts to having a finite index solvable group (this is sometimes called virtual solvability). This is untrue in general.

Remark 2.11. We just proved that solvable groups are amenable. In general, the fact that a group is amenable depends on the topology. For example, SO(3) is amenable for its usual compact topology, but, since it contains a non-abelian free group, it is not amenable for the discrete topology.

3 Invariant means

Before starting doing probability theory, we shall need another equivalent definition of amenability. Let (X, μ) be a measure space. We say that a continuous linear functional on $\mathrm{L}^\infty(X, \mu)$ is a mean if it is nonnegative (that is, it takes nonnegative values on nonnegative functions) and $m(1) = 1$. It is a kind of analogue of a measure.

Proposition 3.1. *Let G be a locally compact group. Then G is amenable if and only if $\mathrm{L}^\infty(G)$ admits a left invariant mean.*

Let us first prove this result in cas G is a discrete group.

Proof of Proposition 3.1 when G is discrete. Assume G is amenable. Then the fact that G admits an invariant mean follows from Lemma 2.6 since, on one hand, the action of G on $\ell^\infty(G)$ is isometric (and continuous as G is discrete) and, on the other hand, the set of means is easily seen to be a convex compact set for the weak-$*$ topology in the dual space of $\ell^\infty(G)$.

Conversely assume G admits an invariant mean m. Let X be a compact G-space. Fix x in X and, for any φ in $\mathcal{C}^0(X)$, let $T\varphi$ be the function $g \mapsto \varphi(gx)$ on G. One easily checks that T is a bounded G-equivariant operator $\mathcal{C}^0(X) \to \ell^\infty(G)$. Then, the linear functional T^*m is bounded on $\mathcal{C}^0(X)$ and $T^*m(1) = 1$. Hence, by Riesz's Theorem, T^*m defines a probability measure μ on X. Since m is G-invariant, so is μ. □

Remark 3.2. This proof is not constructive and in general, when a group is amenable but not compact, there is no way of constructing an invariant mean. However, we will now see how to use the existence of this object concretely.

Now, to extend this proof to the general case, we will replace the space $\mathrm{L}^\infty(G)$ with a smaller subspace on which the natural left action of G is continuous.

We say that a continuous functions φ on G is left uniformly continuous if, for any $\varepsilon > 0$, there exists a neighborhood U of e in G such that, for any g in G and u in U,

$$|\varphi(ug) - \varphi(g)| \leq \varepsilon.$$

Compactly supported functions are left uniformly continuous.

Let $\mathrm{BLUC}(G) \subset \mathrm{L}^\infty(G)$ be the space of bounded left uniformly continuous functions on G. Then $\mathrm{BLUC}(G)$ is closed for the uniform topology and invariant by the left action of G. The restriction of this action to $\mathrm{BLUC}(G)$ is continuous. Again, we say that a continuous linear functional on $\mathrm{BLUC}(G)$ is a mean if it is nonnegative and $m(1) = 1$.

Lemma 3.3. *Let G be a locally compact topological group. Then* $\mathrm{L}^\infty(G)$ *admits left-invariant means if and only if* $\mathrm{BLUC}(G)$ *admits left-invariant means.*

Proof. If $\mathrm{L}^\infty(G)$ admits an invariant mean, its restriction to $\mathrm{BLUC}(G)$ is an invariant mean.

Conversely, assume $\mathrm{BLUC}(G)$ admits a left-invariant mean. We want to extend m as an invariant mean on $\mathrm{L}^\infty(G)$: there is a natural way of doing this. Fix ψ in $\mathrm{L}^\infty(G)$. For any continuous compactly supported function φ on G, we let the convolution product $\varphi * \psi$ be the function

$$g \mapsto \int_G \varphi(h)\psi(h^{-1}g)\mathrm{d}h.$$

It is left uniformly continuous and bounded. Set $\langle f_\psi, \varphi\rangle = m(\varphi * \psi)$. If ψ is in $\mathrm{BLUC}(G)$, one has, for any φ,

$$\langle f_\psi, \varphi\rangle = m(\psi)\int_G \varphi(g)\mathrm{d}g.$$

In general, the linear functional f_ψ is left G-invariant. Hence, by uniqueness of the Haar measure, there exists a unique scalar $m(\psi)$ such that, for any φ, one has again

$$\langle f_\psi, \varphi\rangle = m(\psi)\int_G \varphi(g)\mathrm{d}g.$$

One easily checks that m is an invariant mean on $\mathrm{L}^\infty(G)$. □

We can now give the full

Proof of Proposition 3.1. Assume G is amenable. As in the discrete case, since the action of G on $\mathrm{BLUC}(G)$ is continuous, we prove that $\mathrm{BLUC}(G)$ admits a left-invariant mean. By Lemma 3.3, $\mathrm{L}^\infty(G)$ admits a left-invariant mean.

The proof of the converse statement that was given in the discrete case holds in general. □

We now can show that amenability of a group implies amenability of its closed subgroups.

Corollary 3.4. *Let G be a locally compact group and H be a closed subgroup of G. Then H is amenable if and only if $\mathrm{L}^\infty(G)$ admits a left H invariant mean. In particular, if G is amenable, H is amenable.*

To prove this corollary, we will use functions on H to build functions on G. This relies on a process of taking sections of the quotient $G \to G/H$. To build these sections, we will use

Lemma 3.5. *Let G be a locally compact group, H be a closed subgroup of G and U be a neighborhood of e in G. Then there exists a subset $T \subset G/H$ such that $G/H = \bigcup_{t \in T} Ut$ and, for every compact subset of G/H, the set $K \cap T$ is finite.*

Note that this Lemma is empty in case G is discrete.

Proof. We can assume that U is open.

Assume first that G is a countable union of compact subsets, so that G/H also is. We write $G/H = \bigcup_{n \in \mathbb{N}} K_n$, where $K_0 = \emptyset$ and, for any n, K_n is compact and $K_n \subset K_{n+1}$. Now, we chose a neighborhood V of e with $V^{-1}V \subset U$.

We claim that, for any compact subset K of G, there can not exist an infinite sequence (t_k) of elements of K such that, for any $k \neq \ell$, one has $Vt_k \cap Vt_l = \emptyset$. Indeed, if this would be the case, then (t_k) would admit a cluster point t. Then, if W is a neighborhood of e in G with $WW^{-1} \subset V$, for large $k \neq l$, we would have $t_k, t_l \in Wt$, hence in particular, $t_l \in WW^{-1}t_k \subset Vt_k$.

Now, for any n, define recursively a finite subset T_n of G/H as follows. For $n = 0$, we set $T_0 = \emptyset$. If $T_0, \ldots, T_{n-1}$ have already been chosed, we consider the compact set

$$L_n = K_n \smallsetminus (UT_0 \cup \cdots \cup UT_{n-1})$$

and we we chose a finite maximal subset T_n of L_n such that, for any $t \neq t'$ in T_n, one has $Vt \cap Vt' = \emptyset$. The existence of such a finite subset follows from the remark above. By maximality, we have $L_n \subset V^{-1}VT_n \subset UT_n$.

We set $T = \bigcup_n T_n$. By construction, we have $G/H \subset UT$ and, for any $t \neq t'$ in T, $Vt \cap Vt' = \emptyset$, so that, still by the remark above, for any compact subset K of G, $K \cap T$ is finite.

Now, in the general case, let L be the (open) subgroup of G spanned by a compact neighborhood of e, so that L is a countable union of compact sets and L is open in G. We pick a subset $S \subset G/H$ so that $G/H = LS$ and, for any $s \neq s'$ in S, $Ls \cap Ls' = \emptyset$. For any s in S, we let M_s denote its stabilizer in L, so that Ls may be identified with L/M_s. We chose a subset $T_s \subset L/M_s$ which satisfies the conclusion of the Lemma. Then, the set $T = \{ts | s \in S, t \in T_s\}$ works for G/H. □

Now, we can build a nice function on G.

Lemma 3.6. *Let G be a locally compact topological goup and H be a closed subgroup of G. Then there exists a bounded continuous function $\theta : G \to [0, \infty)$ such that*

(i) for any compact subset K of G, the restriction of θ to KH has compact support.

(ii) for any g in G, one has $\int_G \theta(gh)\mathrm{d}h = 1$.

If G is discrete, the choice of θ amounts to the choice of a system of representatives for the quotient map $G \to G/H$.

Proof. Let U be a relatively compact neighborhood of e in G and $T \subset G/H$ be as in Lemma 3.5. For any t in T, chose g_t in G with $g_t H = t$. We pick a nonnegative continuous compactly supported function φ on G which is equal to 1 on U. For any g in G, we set

$$\psi(g) = \sum_{t \in T} \varphi(g g_t^{-1})$$

(the nonzero terms in the sum being only in finite number). Finally, we set, for any g,

$$\theta(g) = \frac{1}{\int_H \psi(gh)\mathrm{d}h} \psi(g).$$

One easily checks that θ satisfies the conclusions of the Lemma. □

Proof of Corollary 3.4. If H is amenable, $\mathrm{L}^\infty(G)$ admits a left H invariant mean: this is proved exactly as in Proposition 3.1 above.

Conversely, assume $\mathrm{L}^\infty(G)$ admits a left H invariant mean m. Fix θ as in Lemma 3.6. For g in G, set $\psi(g) = \theta(g^{-1})$ and, for any φ in $\mathrm{L}^\infty(H)$,

$$\varphi * \psi(g) = \int_H \varphi(h)\psi(h^{-1}g)\mathrm{d}h.$$

Then $\varphi * \psi \in \mathrm{L}^\infty(G)$. One easily checks that $\varphi \mapsto m(\varphi * \psi)$ is a H-invariant mean on $\mathrm{L}^\infty(H)$. □

4 Almost invariant vectors

In this section, we will prove a property of isometric continuous group actions on Banach spaces that will be useful later when we will prove Kesten's theorem that establishes a link between amenability and spectral radius of some convolution operators. It is the way we will use the abstract invariant means that appear in Proposition 3.1.

Proposition 4.1. *Let G be a locally compact topological group and E be a Banach space, equipped with a continuous isometric action of G. Assume that the topological bidual space E^{**} of E admits a nonzero G-invariant element. Then, for every compact subset K of G, for every $\varepsilon > 0$, there exists a unit vector v in E such that $\|gv - v\| \leq \varepsilon$ for any g in K.*

One sometimes says that E admits almost invariant vectors.

Let E and G be as above. If φ is a continuous compactly supported function on G and v is in E, we let $\varphi * v$ denote the vector $\int_G \varphi(g) gv \mathrm{d}g$ (which is well defined as the integral of a continuous compactly supported function with values in E). We have $\|\varphi * v\| \leq \|\varphi\|_1 \|v\|$.

We start with a weaker version of Proposition 4.1.

Lemma 4.2. *Let G be a locally compact topological group and E be a Banach space, equipped with a continuous isometric action of G. Assume that the topological bidual space E^{**} of E admits a nonzero G-invariant element. Then, for any continuous compactly supported functions $\varphi_1, \dots, \varphi_r$ with Haar integral 1 on G, for every $\varepsilon > 0$, there exists a unit vector v in E such that $\|\varphi_i * v - v\| \leq \varepsilon$ for any $1 \leq i \leq r$.*

Note that if G is discrete, Proposition 4.1 directly follows.

Proof. We fix a G-invariant element v in E^{**} with norm 1 and a basis $\mathcal{U}$ of convex neighborhoods of v for the weak topology of E^{**}, seen as the dual space of E^*. As the unit ball of E is dense in the one of E^{**} for this topology, for any U in $\mathcal{U}$, U contains an element v_U of E with norm ≤ 1. As $gv = v$ for any g in G, we have $\varphi_i * v = v$ for $1 \leq i \leq r$, hence $\varphi_i * v_U - v_U \xrightarrow[U \to \{v\}]{} 0$ weakly in E. Let us prove that we can replace this weak limit by a strong one : this is a variation on the fact that convex norm closed subset of E are weakly closed.

We equip the space E^r with the product topology of the norm topology, so that the associated weak topology is the product of the weak topology of E. In particular, for any U in $\mathcal{U}$, let $M_U \subset E^r$ be the set of families $(\varphi_i * v_{U'} - v_{U'})_{1\leq i\leq r}$, where U' runs among the elements of $\mathcal{U}$ that are contained in U. Then, the weak closure of M_U in E^r contains 0. Therefore, the strong closure of the convex hull of M_U in E^G contains 0. By definition of the topology of E^G, and since U is convex, this means that there exists w in U such that $\|\varphi_i * w - w\| \leq \varepsilon$ for $1 \leq i \leq r$.

To conclude, it remains to prove that we can ensure that the norm of w is not close to 0. Indeed, since v has norm 1, there exists f in E^* with $\|f\| \leq 1$ such that $\langle f, v\rangle \geq \frac{2}{3}$. By definition of $\mathcal{U}$, we can ensure that $\langle f, u\rangle \geq \frac{1}{3}$ for any u in U, hence $\|u\| \geq \frac{1}{3}$. We get $\|w\| \geq \frac{1}{3}$ and we are done. □

For φ a continuous compactly supported function on G and g in G, we let $g\varphi$ denote the function $h \mapsto \varphi(g^{-1}h)$. Note that, for any v in E, we have $g(\varphi * v) = (g\varphi) * v$.

We now use the Lemma to establish the

Proof of Proposition 4.1. If G is discrete, we are done, since we can then apply Lemma 4.2 to the continuous characteristic functions of the elements of the finite set K.

In general, we need to deal with vectors v for which we control the modulus of continuity of the map $g \mapsto gv$. More precisely, we fix a nonnegative continuous function φ with compact support on G such that $\int_G \varphi(g)\mathrm{d}g = 1$. Then there exists a neighborhood U of e in G such that $\|u\varphi - \varphi\|_1 \leq \varepsilon$ for any u in U. We fix $g_1, \ldots, g_r$ in K with $K \subset g_1 U \cup \cdots g_r U$. By Lemma 4.1, there exists a unit vector v in E such that $\|\varphi * v - v\| \leq \varepsilon$ and also $\|g_i\varphi * v - v\| \leq \varepsilon$ for $1 \leq i \leq r$. We set $w = \varphi * v$. On one hand, we have $\|w\| \geq 1 - \varepsilon$ which is $\geq \frac{1}{2}$ if $\varepsilon \leq \frac{1}{2}$. On the other hand, if g is in K and i is such that $g \in g_i U$, we have

$$\begin{aligned}\|gw - w\| &\leq \|g\varphi * v - g_i\varphi * v\| + \|g_i\varphi * v - v\| + \|v - \varphi * v\| \\ &\leq \|g\varphi - g_i\varphi\|_1 + 2\varepsilon \leq 3\varepsilon\end{aligned}$$

and we are done. □

5 Random walks and amenable groups

Let G be a locally compact topological group. Given Borel probability measures μ_1 and μ_2 on G, their convolution product $\mu_1 * \mu_2$ is the image measure of $\mu_1 \otimes \mu_2$ under the product map $G \times G \to G$.

Example 5.1. If $G = \mathbb{R}$ and μ_1 (resp. μ_2) is of the form $\varphi_1(x)\mathrm{d}x$ (resp. $\varphi_2(x)\mathrm{d}x$) for some Lebesgue measurable functions φ_1 and φ_2, then $\mu_1 * \mu_2$ is the measure $(\varphi_1 * \varphi_2)(x)\mathrm{d}x$ where $\varphi_1 * \varphi_2$ is the usual convolution product of φ_1 and φ_2.

Given a Borel probability measure μ on X, we define the associated (left) random walk on G as the Markov chain on G whose transition probabilities are the measures $\mu * \delta_x$, x in G. In other words, given a sequence $g_1, \ldots, g_n, \ldots$ of independent random elements of G with law μ, the trajectories of the random walk are of the form $x, g_1 x, \ldots, g_n \cdots g_1 x, \ldots$.

The associated Markov operator P_μ is defined by

$$P_\mu \varphi(x) = \int_G \varphi(gx)\mathrm{d}\mu(g),$$

where φ is a continuous compactly supported function on G and x is in G.

Recall that $\mathrm{L}^p(G)$, $1 \leq p \leq \infty$, denote the Lebesgue spaces of Haar measure on G (and not of μ). Then P_μ extends as a contraction (that is, a bounded operator with norm ≤ 1) in any of these spaces.

Theorem 5.2 (Kesten)**.** *Let G be a locally compact topological group and μ be a Borel probability measure on G.*

If G is amenable, then P_μ has spectral radius 1 in $\mathrm{L}^2(G)$.

Conversely, if the support of μ spans G and P_μ has spectral radius 1 in $\mathrm{L}^2(G)$, then G is amenable.

Remark 5.3. In other words, when the support of μ spans G, the fact that the spectral radius is < 1 or not does not depend on μ.

Remark 5.4. By reasoning as in the proof of Corollary 3.4, one could easily prove that the spectral radius of P_μ in $\mathrm{L}^2(G)$ is < 1 as soon as the closed subgroup spanned by the support of μ is not amenable.

To prove that the property in the Theorem implies that the group is amenable, we will use it to build an invariant mean. This will be done by applying the following Lemma, which is a kind of converse to Proposition 4.1.

Lemma 5.5. *Let G be a locally compact topological group. Assume that there exists a nonnegative sequence (θ_n) in $\mathrm{L}^1(G)$ such that, for any n, $\int_G \theta_n(h)\mathrm{d}h = 1$ and $\|g\theta_n - \theta_n\|_1 \xrightarrow[n\to\infty]{} 0$ for g in a dense subset of G. Then G is amenable.*

Proof. Let, as in Section 3, $\mathrm{BLUC}(G)$ be the space of bounded left uniformly continuous functions on G, eauipped with the supremum norm and the natural left G-action, which is isometric and continuous. For any n, we let m_n be the linear functional $\varphi \mapsto \int_G \theta_n(g)\varphi(g)\mathrm{d}g$ on $\mathrm{BLUC}(G)$, which is bounded with norm 1. By Banach-Alaoglu Theorem, the sequence (m_n) admits a limit point m in the topological dual space of $\mathrm{BLUC}(G)$ for the weak-$*$ topology. Since for any n, θ_n is nonnegative and $\int_G \theta_n(h)\mathrm{d}h = 1$, m is nonnegative and $\langle m, 1\rangle = 1$. Since $\|g\theta_n - \theta_n\|_1 \xrightarrow[n\to\infty]{} 0$ for g in a dense subset of G, we have $gm = m$ for g in the same dense subset, hence, since the action of G on $\mathrm{BLUC}(G)$ is continuous, the same holds for any g in G (which would not be necessarily the case if we would work in $\mathrm{L}^\infty(G)$ instead of $\mathrm{BLUC}(G)$).

In other words, $\mathrm{BLUC}(G)$ admits a left-invariant mean. By Lemma 3.3, so does $\mathrm{L}^\infty(G)$ and we are done by Proposition 3.1. □

Proof of Theorem 5.2. Assume that G is amenable. By Proposition 3.1, it admits an invariant mean. Hence, by Proposition 4.1, for any compact subset K of G, for any $\varepsilon > 0$, there exists φ in $\mathrm{L}^1(G)$ such that $\|\varphi\|_1 = 1$ and $\|g\varphi - \varphi\|_1 \leq \varepsilon$ for any g in K. We claim that the same holds in $\mathrm{L}^2(G)$. Indeed, after replacing φ by $|\varphi|$, we can assume φ is nonnegative. Now, if $\psi = \sqrt{\varphi}$, then ψ belongs to $\mathrm{L}^2(G)$ and, as the square root function is $\frac{1}{2}$-Hölder continuous, for any g in K,

$$\|g\psi - \psi\|_2^2 \leq \|g\varphi - \varphi\|_1 \leq \varepsilon$$

and we are done.

Now, we claim that, for any $\varepsilon > 0$, there exists a unit function φ in $\mathrm{L}^2(G)$ such that $\|P_\mu\varphi - \varphi\|_2 \leq \varepsilon$, which implies that 1 is a spectral value of P_μ. Indeed, fix a compact subset of G with $\mu(K) \geq 1 - \varepsilon$. There exists a unit function φ in $\mathrm{L}^2(G)$ such that $\|g\varphi - \varphi\|_2 \leq \varepsilon$ for any g in K. We have

$$P_\mu\varphi = \int_K g^{-1}\varphi\mathrm{d}\mu(g) + \int_{G\smallsetminus K} g^{-1}\varphi\mathrm{d}\mu(g),$$

hence $\|P_\mu\varphi - \varphi\|_2 \leq 3\varepsilon$, which should be proved.

Conversely, assume that the support of μ spans G and that P_μ has spectral radius 1. Then P_μ has a spectral value λ of modulus 1. This means that $P_\mu - \lambda$ is not injective or that is has non dense image or that it has non closed image.

If $P_\mu - \lambda$ is not injective, we let φ be unit element in $\mathrm{L}^2(G)$ such that $P_\mu \varphi = \lambda \varphi$. Then, we have

$$\lambda = \langle P_\mu \varphi, \varphi \rangle = \int_G \langle \varphi, g\varphi \rangle \mathrm{d}\mu(g).$$

For any g in G, $g\varphi$ has norm 1 and $|\langle \varphi, g\varphi \rangle| \leq 1$. Since λ has modulus one, we get $\langle \varphi, g\varphi \rangle = \lambda$ for μ-almost any g in G, that is, $g\varphi = \overline{\lambda}\varphi$. By replacing φ with $|\varphi|$, we can assume $\lambda = 1$. Then, since $g\varphi = \varphi$ for μ-almost any g in G and the support of μ spans G, $\mathrm{L}^2(G)$ contains a G-invariant non zero vector. One easily checks that this implies that G is compact.

If $P_\mu - \lambda$ has non dense image, then its adjoint operator is non injective. But the adjoint operator P_μ^* of P_μ is given by $P_\mu^* = P_{\mu^\vee}$, where $\mu^\vee$ is the image of μ under the inverse map. Hence, if $P_\mu - \lambda$ has non dense image, $P_{\mu^\vee} - \overline{\lambda}$ is not injective and the group G is compact as above.

It remains to deal with the main case, that is the one where $P_\mu - \lambda$ has non closed image, which means that there exists a sequence (φ_n) of unit elements in $\mathrm{L}^2(G)$ such that $\|P_\mu \varphi_n - \lambda \varphi_n\|_2 \xrightarrow[n\to\infty]{} 0$. This is a refinment of the previous argument, but instead of constructing an invariant function, we will construct an invariant mean by a limit process, which is somehow the converse of Proposition 4.1. Again, we have, for any n,

$$\langle P_\mu \varphi_n, \varphi_n \rangle = \int_G \langle \varphi_n, g\varphi_n \rangle \mathrm{d}\mu(g),$$

hence

$$\int_G \langle \varphi_n, g\varphi_n \rangle \mathrm{d}\mu(g) \xrightarrow[n\to\infty]{} \lambda$$

and in particular

$$\int_G \mathrm{Re}(\langle \overline{\lambda}\varphi_n, g\varphi_n \rangle) \mathrm{d}\mu(g) \xrightarrow[n\to\infty]{} 1.$$

Set, for g in G and n in $\mathbb{N}$,

$$\psi_n(g) = \|g\varphi_n - \overline{\lambda}\varphi_n\|_2^2 = 2 - 2\,\mathrm{Re}(\langle \overline{\lambda}\varphi_n, g\varphi_n \rangle) \geq 0.$$

We have $\int_G \psi_n \mathrm{d}\mu \xrightarrow[n\to\infty]{} 0$ and therefore, $\psi_n \xrightarrow[n\to\infty]{} 0$ in $\mathrm{L}^1(G,\mu)$ (be careful, this is the L^1 space of the measure μ, not of the Haar measure). Now after extracting a subsequence, we can assume that, for μ-almost any g in G, $\psi_n(g) \xrightarrow[n\to\infty]{} 0$, that is

$$\|g\varphi_n - \overline{\lambda}\varphi_n\|_2^2 \xrightarrow[n\to\infty]{} 0.$$

We will use this property to construct an invariant mean on G. We must first bring everything back to $\mathrm{L}^1(G)$. To do this, we note that, if α and β are in $\mathrm{L}^2(G)$, we have

$$||\alpha|^2 - |\beta|^2| \leq ||\alpha| - |\beta|| \times ||\alpha| + |\beta|| \leq |\alpha - \beta| \times ||\alpha| + |\beta||,$$

hence

$$\||\alpha|^2 - |\beta|^2\|_1 \leq \|\alpha - \beta\|_2 \times \||\alpha| + |\beta|\|_2 \leq \|\alpha - \beta\|_2(\|\alpha\|_2 + \|\beta\|_2).$$

Thus, if we set $\theta_n = |\varphi_n|^2$, we get, for μ-almost any g in G,

$$\|g\theta_n - \theta_n\|_1 \xrightarrow[n\to\infty]{} 0.$$

Besides, for any n, $\int_G \theta_n(h)\mathrm{d}h = 1$. By Lemma 5.5, G is amenable. □

6 Growth of groups

We now aim at deducing from Theorem 5.2 properties on the speed of escape of the trajectories of random walks. To do this, we need to define a natural notion of distance in such a group. This will be done in groups where we can write every element as a product of elements from a given compact subset.

Definition 6.1. Let G be a locally compact topological group. We say that G is compactly generated if there exists a compact subset of G which spans G as a group.

Example 6.2. A discrete group is compactly generated if and only if it is finitely generated. The groups $\mathbb{Z}^d$ and $\mathrm{GL}_d(\mathbb{Z})$, $d \geq 1$, are finitely generated. The fundamental group of a compact connected manifold is finitely generated.

If $s \geq 2$ is an integer, the subring $\mathbb{Z}[s^{-1}]$ spanned by s in $\mathbb{Q}$ is not finitely generated as an abelian group.

Example 6.3. A connected locally compact topological group is always compactly generated: indeed, as it is connected, it is generated by any open set. Hence, the groups $\mathbb{R}^d$ or $\mathrm{GL}_d(\mathbb{R})$, $d \geq 1$, are compactly generated (note that $\mathrm{GL}_d(\mathbb{R})$ is not connected, but it has two connected components).

Example 6.4. Say a connected locally finite tree is regular if each of its vertices has the same number of neighboors. Then the group of automorphisms of a connected locally finite regular tree is compactly generated.

Assume that G is compactly generated and fix a compact symmetric subset K of G which spans G, so that $G = \bigcup_{n \in \mathbb{N}} K^n$. For g in G, we define $j_K(g)$ as the minimal n with $g \in K^n$. For any g, h in G, we have $j_K(gh) \leq j_K(g) + j_K(h)$, so that the function $(g,h) \mapsto j_K(g^{-1}h)$ is a distance on G. Note that this distance induces the discrete topology on G! It will be interesting in as much as it gives informations on the large scale geometry of G.

First, the function j_K is roughly independent of the choice of K.

Lemma 6.5. *Let G be a locally compact group and K and L be compact symmetric subsets of G which span G. Then there exists $a > 1$ with $j_L \leq a j_K$.*

Proof. By definition, we have $G = \bigcup_{n \in \mathbb{N}} K^n$. We claim that every g in G lies in the interior of K^n for some n. Indeed, by Baire's Lemma, there exists n such that K^n has non empty interior. In other words, there exists a neighborhood V of e in G and h in G such that $Vh \subset K^n$. Let g be in G and p be such $h^{-1}g \in K^p$. We have

$$Vg = Vh(h^{-1}g) \subset K^{n+p}$$

and we are done.

In particular, since L is compact, there exists m such that $L \subset \bigcup_{n \leq m} K^n$. We get $j_L \leq m j_K$, which should be proved. □

The behaviour of j_K can be read in certain isometric actions of G.

Let (X,d) be a metric space. We say that X is proper if the bounded subsets of X are relatively compact. We say that it is geodesic if, for any x, y in X with $d(x,y) = \delta$, there exists an isometry $\gamma : [0, \delta] \to X$ with $\gamma(0) = x$ and $\gamma(\delta) = y$ (by γ being an isometry, we mean that $d(\gamma(t), \gamma(u)) = u - t$ for any $0 \leq t \leq u \leq \delta$).

Now, if G acts continuously on a locally compact topological space X, we say that the action is proper if, for any compact subset L of X, the set

$\{g \in G | gL \cap L \neq \emptyset\}$ is compact. We say that the action is cocompact if there exists a compact subset L of X with $GL = X$.

Lemma 6.6. *Let G be a locally compact topological group and (X, d) be a proper geodesic metric space, equipped with a continuous isometric G-action. Assume that the action is proper and cocompact. Then G is compactly generated. If K is a compact symmetric subset of G which spans G, for every x in G, there exists $a > 1$ and $b > 0$ such that, for any g in G, one has*

$$\frac{1}{a} j_K(g) - b \leq d(x, gx) \leq a j_K(g) + b. \tag{6.1}$$

One says that the metric spaces (G, j_K) and (X, d) are quasi-isometric.

Remark 6.7. The assumption that the metric space is geodesic is important. For example, if $G = \mathbb{Z}$, $K = \{-1, 1\}$ and $X = \mathbb{R}$, equipped with the action of $\mathbb{Z}$ by translations, but with the distance $(t, u) \mapsto \sqrt{|u - t|}$, the result is false.

Example 6.8. Let G be a group and K be a compact subgroup of G. Then the action of G on G/K is proper.

Assume $G = \mathrm{SL}_2(\mathbb{R})$ and K is $\mathrm{SO}(2)$. The action of G by homographies on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$ identifies $\mathbb{H}$ and G/K. This action preserves a Riemannian complete metric. Hence the associated distance is proper. It can also be shown to be geodesic. The Lemma implies that, if K is a symmetric compact subset of G which spans G, the function j_K is comparable with the function $g \mapsto \log \|g\|$, where $\|.\|$ stands for the operator norm for the action on $\mathbb{R}^2$, equipped with the standard scalar product.

Assume now X is a homogeneous locally finite tree, G is its group of automorphisms and K is the stabilizer of a point x (in particular, K is open in G). The set G/K identifies with the set of vertices of X. The action of G on X preserves the natural tree metric which is proper and geodesic. From the fact that the action of G on the set of vertices is proper, one easily deduces that the action on X is proper.

Proof of Lemma 6.6. By Lemma 6.5, if (6.1) holds for a given compact generating set in G, it holds for any.

Let g be in G and set $y = gx$ and $n = [d(x, y)]$. As X is geodesic, there exists a sequence $x = x_0, \ldots, x_{n+1} = y$ of elements of X with $d(x_{i-1}, x_i) \leq 1$, $1 \leq i \leq n+1$. Let L be a compact subset of X containing x with $X = GL$.

Set $h_0 = e$, $h_{n+1} = g$ and, for any $1 \leq i \leq n$, write $x_i = h_i y_i$ with y_i in L and h_i in G. For $1 \leq i \leq n+1$, set $g_i = h_{i-1}^{-1} h_i$. By construction, one has $g = g_1 \cdots g_{n+1}$. Now, for $1 \leq i \leq n+1$, one has $d(h_{i-1}y_{i-1}, h_{i-1}g_i y_i) \leq 1$, hence g_i belongs to the set

$$K = \{h \in G | \exists y, z \in L \quad d(hy, z) \leq 1\},$$

which is compact since the action is proper and X is a proper metric space. Therefore K spans G and, for any g in G, $j_K(g) \leq d(x, gx) + 1$.

Conversely, let g be in G and $g_1, \ldots, g_n$ be in K with $g_1 \cdots g_n = g$. We have

$$d(x, gx) \leq d(x, g_1 x) + d(g_1 x, g_1 g_2 x) + \cdots + d(g_1 \cdots g_{n-1} x, gx) = n \max_{h \in K} d(x, hx).$$

Thus, we get $d(x, gx) \leq c j_K(g)$ with $c = \max_{h \in K} d(x, hx)$. The Lemma follows. □

7 Speed of escape

Let still G be a locally compact topological group which is spanned by a compact symmetric subset K. We can dominate the growth of the Haar measures of the balls of j_K.

Proposition 7.1. *Let G be a locally compact group, with Haar measure λ, and K be a compact symmetric subset of G with nonempty interior. Then the sequence $(\lambda(K^n)^{\frac{1}{n}})$ converges.*

In other words, the growth of the balls of the metric j_K is at most exponential.

In case the group is discrete, the proof is straightforward, since λ may be chosed to be the counting measure and the sequence $(\lambda(K^n))$ is then subbaditive. In general, the same kind of phenomenon appears due to the following

Lemma 7.2. *Let G be a locally compact group, with Haar measure λ, and A, B, C be compact subsets of G. We have*

$$\lambda(AB)\lambda(C) \leq \lambda(AC)\lambda(C^{-1}B).$$

Proof. The proof relies on the interpretation of the number $\lambda(AC)\lambda(C^{-1}B)$ as the integral of a convolution product. More precisely, let $\varphi = \mathbf{1}_{AC} * \mathbf{1}_{C^{-1}B}$ be the convolution product of the characteristic functions of AC and $C^{-1}B$, that is, for any g in G,

$$\varphi(g) = \int_G \mathbf{1}_{AC}(h) * \mathbf{1}_{C^{-1}B}(h^{-1}g)\mathrm{d}h.$$

By Fubini Theorem, on one hand, we have

$$\int_G \varphi(g)\mathrm{d}g = \lambda(AC)\lambda(C^{-1}B). \tag{7.1}$$

On the other hand, we claim that, for g in AB, we have

$$\varphi(g) \geq \lambda(C). \tag{7.2}$$

This and (7.1) finish the proof.

Let us hence prove (7.2). For g in AB, write $g = ab$ with a in A and b in B. We have

$$\varphi(g) = \int_G \mathbf{1}_{AC}(h)*\mathbf{1}_{C^{-1}B}(h^{-1}g)\mathrm{d}h = \int_G \mathbf{1}_{AC}(ah)*\mathbf{1}_{C^{-1}B}(h^{-1}a^{-1}g)\mathrm{d}h \geq \lambda(C),$$

the inequality being obtained by taking $h \in C$ in the integral. The Lemma follows. □

Proof of Proposition 7.1. For n, m in $\mathbb{N}$, apply Lemma 7.2 with $A = K^n$, $B = K^m$ and $C = C^{-1} = K$. We get

$$\lambda(K^{n+m}) \leq \lambda(K)^{-1}\lambda(K^{n+1})\lambda(K^{m+1}).$$

By extending the usual proof for subadditive sequences, it is not difficult to show that this implies the result. □

Now, Theorem 5.2 and Proposition 7.1 together imply that trajectories of random walks in non amenable groups escape quickly to infinity:

Corollary 7.3. *Let G be a compactly generated locally compact group and μ be a Borel probability measure on G. Assume that the subgroup of G spanned by the support of μ is not amenable. Let K be a compact symmetric subset* *of G which spans G. Then, there exists $\alpha, \varepsilon > 0$ such that, if $g_1, \dots, g_n, \dots$ is a sequence of independent random elements of G with law μ,*

$$\mathbb{P}(j_K(g_n \cdots g_1) \leq \varepsilon n) \ll e^{-\alpha n}.$$

In particular, almost surely, for large n,

$$j_K(g_n \cdots g_1) \geq \varepsilon n.$$

Remark 7.4. Set-theoretically, the conclusion can be rewritten as

$$\mu^{*n}(K^{[\varepsilon n]}) \ll e^{-\alpha n}.$$

Remark 7.5. Let μ be a Borel probability measure on $\mathbb{R}$ with finite first moment and assume that μ has positive drift, that is $\int_{\mathbb{R}} x \mathrm{d}\mu(x) > 0$. Then by the usual law of large numbers, there exists $\varepsilon > 0$ such that, if $x_1, \dots, x_n, \dots$ is a sequence of independent random variables with law μ, we have, almost surely, for large n,

$$x_n + \cdots + x_1 \geq \varepsilon n.$$

Besides, if μ admits exponential moments, by the large deviations principle, we can assume that

$$\mathbb{P}(x_n + \cdots + x_1 \leq \varepsilon n) \ll e^{-\alpha n}$$

for some $\alpha > 0$.

The content of Corollary 7.3 is that, when the ambient group is not amenable, analogue properties hold for any probability measure, with no moment assumption.

Proof of Corollary 7.3. We can assume that K has nonempty interior. Now, let L be any compact subset of G. For any g, x in G, we have

$$\mathbf{1}_L(g)\mathbf{1}_K(x) \leq \mathbf{1}_{LK}(gx)\mathbf{1}_K(x).$$

By integrating, we get, in $\mathrm{L}^2(G)$, for any n,

$$\mu^{*n}(L)\lambda(K) \leq \int_G \left(\int_G \mathbf{1}_{LK}(gx) \mathrm{d}\mu^{*n}(g) \right) \mathbf{1}_K(x) \mathrm{d}x = \langle P_\mu^n \mathbf{1}_{LK}, \mathbf{1}_K \rangle$$

(where λ denotes Haar measure of G and $\langle .,. \rangle$ scalar product in $\mathrm{L}^2(G)$). Fix $\varepsilon > 0$ to be determined later and set $L = K^{[\varepsilon n]}$. Theorem 5.2 and Remark 5.4 and the assumptions imply that there exists $\alpha > 0$ with

$$\langle P_\mu^n \mathbf{1}_{K^{[\varepsilon n]+1}}, \mathbf{1}_K \rangle \ll e^{-\alpha n} \|\mathbf{1}_{K^{[\varepsilon n]+1}}\|_2 \|\mathbf{1}_K\|_2.$$

Proposition 7.1 implies that, by chosing ε small enough, we can assume that the right hand-side is $\ll e^{\frac{\alpha}{2}n}$. We then get

$$\mu^{*n}(K^{[\varepsilon n]}) \ll e^{-\frac{\alpha}{2}n}$$

and the first part of the Corollary is proved. The second follows, by Borel-Cantelli Lemma. □

8 Subgroups of $\mathrm{SL}_2(\mathbb{R})$

From the discussion in Example 6.8, it follows that Corollary 7.3 may be translated in terms of the norm metric for random walks on $\mathrm{SL}_2(\mathbb{R})$. To make this translation complete, we shall describe precisely the closed amenable subgroups of $\mathrm{SL}_2(\mathbb{R})$.

Set $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}(2)$ and

$$A^+ = \left\{ \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \middle| s \in \mathbb{R}, s \geq 1 \right\}.$$

To be able to describe amenable subgroups of G, we will need to describe the way a sequence can go to ∞ in G. This will use

Lemma 8.1 (Cartan decomposition)**.** *One has $G = KA^+K$.*

Proof. Let g be in G. Set $s = {}^t g g$ to be the product of the transpose of the matrix of g with g. Then s is symmetric and positive and may hence be written as r^2, where r is also symmetric and positive. We have ${}^t(gr^{-1})(gr^{-1}) = e$, hence $k = gr^{-1}$ belongs to K. The result follows by diagonalizing r in some direct orthonormal basis. □

Let $\mathbb{P}^1_{\mathbb{R}}$ be the projective line, that is, the set of vector lines of $\mathbb{R}^2$ equipped with the natural topology as a quotient of $\mathbb{R}^2 \smallsetminus \{0\}$. This is a compact set. Lemma 8.1 implies that a continuous Borel probability measure on $\mathbb{P}^1_{\mathbb{R}}$ has a compact stabilizer in G.

Lemma 8.2. *Let ν be a Borel probability measure on $\mathbb{P}^1_{\mathbb{R}}$. Assume the stabilizer of ν in G is unbounded. Then ν is supported on at most two elements of $\mathbb{P}^1_{\mathbb{R}}$.*

Proof. For $s \geq 1$, set $a_s = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$. Note that, if $x = \mathbb{R}(1,0)$ and $y = \mathbb{R}(0,1)$ in $\mathbb{P}^1_{\mathbb{R}}$, for any z in $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{y\}$, one has $a_s z \xrightarrow[s\to\infty]{} x$ and this convergence is uniform on compact subsets of $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{y\}$.

Let now (g_n) be an unbounded sequence of elements of G that preserve ν. Thanks to Lemma 8.1, write, for any n, $g_n = k_n a_{s_n} l_n$, with $k_n, l_n \in K$ and $s_n \geq 1$. After extracting, we can assume $s_n \xrightarrow[n\to\infty]{} \infty$ and there exists k, l in K with $k_n \xrightarrow[n\to\infty]{} k$ and $l_n \xrightarrow[n\to\infty]{} l$. We get $g_n z \xrightarrow[n\to\infty]{} kx$, uniformly for z in a compact subset of $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{l^{-1}y\}$. Since g_n preserves ν, this gives $\nu(\{kx, l^{-1}y\}) = 1$. □

Let us define the subgroups of G which are not too small.

Definition 8.3. We say that a subgroup of G is non-elementary if it is unbounded and does not fix a line in $\mathbb{R}^2$, nor the union of two lines.

Remark 8.4. A subgroup of G is non-elementary if and only if it is neither contained up to conjugacy in SO(2) nor in

$$\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a, b \in \mathbb{R}, a \neq 0 \right\}$$

nor in

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{R}, a \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} \middle| a \in \mathbb{R}, a \neq 0 \right\}.$$

Example 8.5. Let $s, t > 1$ and $\theta \in]0, \frac{\pi}{2}[$ be given. Set $r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. The subgroup spanned by the matrices a_s and $r_\theta a_t r_\theta^{-1}$ is non-elementary.

We get

Proposition 8.6. *Let H be closed subgroup of G. Then H is non-amenable if and only if it is non-elementary.*

Proof. Assume H is amenable and unbounded. Then, it fixes a Borel probability measure ν on $\mathbb{P}^1_{\mathbb{R}}$. Thus, by Lemma 8.2, the support of ν is a singleton or a pair in $\mathbb{P}^1_{\mathbb{R}}$. In any case, since this support is H-invariant, H is elementary.

Conversely, by Remark 8.4, if H is elementary, it is solvable, hence amenable. □

From Example 6.8 and Corollary 7.3, we now have

Theorem 8.7. *Let μ be a Borel probability measure on $G = \mathrm{SL}_2(\mathbb{R})$ such that the closed subgroup of G spanned by the support of μ is non-elementary. Then, there exists $\alpha, \varepsilon > 0$ such that, if $g_1, \dots, g_n, \dots$ is a sequence of independent random elements of G with law μ,*

$$\mathbb{P}(\|g_n \cdots g_1\| \leq e^{\varepsilon n}) \ll e^{-\alpha n}.$$

In particular, almost surely, for large n,

$$\|g_n \cdots g_1\| \geq e^{\varepsilon n}.$$

Example 8.8. Keep the notations of Example 8.5. Then the Theorem applies to the probability measure $\mu = \frac{1}{2}(\delta_{a_s} + \delta_{r_\theta a_t r_\theta^{-1}})$.