An overview of Patterson-Sullivan theory

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1 Introduction

1.1 Geometry, groups and measure

Let M be a complete Riemannian manifold with negative sectional curvature. Then the universal cover X of M is diffeomorphic to a Euclidean space and may be geometrically compactified by adding to it a topological sphere ∂X. The fundamental group Γ of M acts by isometries on X and this action extends to the boundary ∂X.

In [13], S.-J. Patterson discovered, in case X is the real hyperbolic plane, how to associate to a point x of X a probability measure ν_x on the boundary ∂X that is Γ-quasi-invariant. In some sense, this measure gives the proportion of elements of the orbit Γx that goes to a given zone in ∂X. The construction of these measures was extended to all hyperbolic spaces by D. Sullivan in [15]. What's more, Sullivan discovered deep connections between the measures ν_x and harmonic analysis and ergodic theory of the geodesic flow of M. Today, we know how to construct Patterson-Sullivan measures when X is any simply connected complete Riemannian manifold with negative curvature.

If X is a symmetric space, that is if X possesses a very large group of isometries (we shall have a precise definition later), as real or complex hyperbolic space, and if Γ is cocompact, that is if M is compact, the Patterson-Sullivan measures ν_x are invariant measures for some subgroups of the group of isometries that act transitively on the boundary and most of the results of the theory are consequences of theorems about harmonic analysis in the group of isometries. So the power of Patterson-Sullivan theory is to allow to draw an analogy between the case where X is symmetric and Γ cocompact and the general case.

In these notes, we shall focus on this analogic point of view. Therefore in section 2 we will briefly recall the definition of classical symmetric spaces and their basic geometric properties. In section 3 we will show how to construct measures on the boundary of symmetric spaces with good geometric properties, only by group-theoretic methods. Finally, in section 4, which is the core of these notes, we will explain how to construct measures in general manifolds that have close properties to the ones appearing in the homogeneous situation.

But first of all, we begin by giving an example of boundary measures coming from classical analysis. It will later appear to be related to our geometric problems.

1.2 Harmonic measures in the disk

Let us recall some well-known facts on the integral representation of harmonic functions.

We equip $\mathbb{R}^2$ with the canonical scalar product, that is, for $x = (x_1, x_2)$ in $\mathbb{R}^2$, we put $\|x\|^2 = x_1^2 + x_2^2$. We denote by $\mathbb{D}$ the open unit disk $\{x \in \mathbb{R}^2 | \, \|x\| < 1\}$ and by $\mathbb{S}^1$ the unit circle $\partial\mathbb{D} = \{x \in \mathbb{R}^2 | \, \|x\| = 1\}$. We let σ be the uniform probability measure on $\mathbb{S}^1$, that is its measure as a Riemannian submanifold of $\mathbb{R}^2$, normalized in such a way that $\sigma(\mathbb{S}^1) = 1$. For x in $\mathbb{D}$ and ξ in $\mathbb{S}^1$, we define $P(x, \xi)$ by

$$P(x,\xi) = \frac{1 - \|x\|^2}{\|\xi - x\|^2}.$$

The function P is known as the *Poisson kernel*. For x in $\mathbb{D}$, we set $\nu_x = P(x, .)\sigma$: it is a probability measure on $\mathbb{S}^1$. We call it the harmonic measure associated to x. Note that $\nu_0 = \sigma$.

A $\mathcal{C}^2$ function $\varphi : \mathbb{D} \to \mathbb{C}$ is said to be *harmonic* if $\Delta\varphi = 0$ where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_1^2}$ denotes the Laplace operator. A direct calculation shows that, for ξ in $\mathbb{S}^1$, the function $P(., \xi)$ is harmonic. Therefore, for each f in $\mathrm{L}^\infty(\mathbb{S}^1)$ its Poisson transform $\mathcal{P}f$, defined by

$$\forall x \in \mathbb{D} \quad \mathcal{P}f(x) = \int_{\mathbb{S}^1} f(\xi)\mathrm{d}\nu_x(\xi) = \int_{\mathbb{S}^1} f(\xi)P(x,\xi)\mathrm{d}\sigma(\xi),$$

is a bounded harmonic function on $\mathbb{D}$.

We have the following

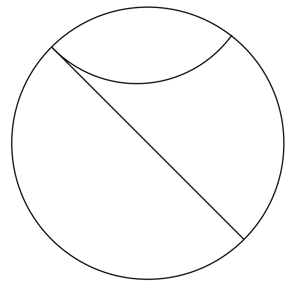

Figure 1: Geodesics in the hyperbolic disk

Theorem 1.1. *The map $f \mapsto \mathcal{P}f$ gives an isomorphism between $\mathrm{L}^\infty(\mathbb{S}^1)$ and the space of bounded harmonic functions on $\mathbb{D}$. It preserves the L^∞-norm.*

Let us give an other interpretation of the harmonic measures and the Poisson kernel in terms of the hyperbolic geometry in the disk. The hyperbolic Riemannian metric of the disk is defined by

$$\forall x \in \mathbb{D} \quad g_x = \frac{4}{\left(1 - \|x\|^2\right)^2} g^e_x$$

where g^e is the Euclidean metric; that is, the hyperbolic length of a $\mathcal{C}^1$ curve $\gamma : [0,1] \to \mathbb{D}$ is

$$\int_0^1 \frac{2}{1 - \|\gamma(t)\|^2} \|\gamma'(t)\| \,\mathrm{d}t.$$

This metric is complete and the complete geodesics are the diameters of $\mathbb{D}$ and the arcs of circles orthogonal to $\mathbb{S}^1$. Any geodesic thus possesses two limit points in $\mathbb{S}^1$.

Let ξ be in $\mathbb{S}^1$. A *horocycle* with center ξ is the intersection of $\mathbb{D}$ with a Euclidean circle (of radius < 1) passing through ξ. The horocycles of $\mathbb{D}$ are the curves which are orthogonal to the geodesics having ξ as a limit point. For x and y in $\mathbb{D}$ and ξ in $\mathbb{S}^1$, define the *Busemann function* $b_\xi(x, y)$ as the hyperbolic distance between x and the point where the geodesic line passing through x and having ξ as a limit point hits the horocycle centered at ξ and passing through y, this distance being counted positively if this hitting point lies between x and ξ and negatively else (see figure 2). The geodesics and horospheres appearing in this definition being orthogonal, if x' (resp. y') lies

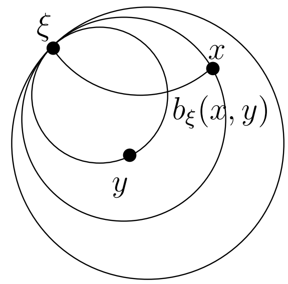

Figure 2: Horocycles and Busemann function

in the same horocycle centered at ξ as x (resp. y), one has $b_\xi(x,y) = b_\xi(x',y')$. In particular, the Busemann function satisfies the cocyle identity:

$$\forall x,y,z \in \mathbb{D} \quad b_\xi(x,z) = b_\xi(x,y) + b_\xi(y,z).$$

From the definition of the metric, the Busemann function may be computed explicitly; we then get:

$$\forall \xi \in \mathbb{S}^1 \quad \forall x,y \in \mathbb{D} \quad b_\xi(x,y) = \log\left(\frac{P(y,\xi)}{P(x,\xi)}\right).$$

In other words, the harmonic measures associated to points of $\mathbb{D}$ satisfy:

$$\forall \xi \in \mathbb{S}^1 \quad \forall x,y \in \mathbb{D} \quad \frac{\mathrm{d}\nu_y}{\mathrm{d}\nu_x}(\xi) = e^{-b_\xi(y,x)}.$$

Note in particular that this relation implies the cocycle identity.

In the sequel, we will be interested in finding measures satisfying analogous properties in the context of manifolds of negative curvature. We begin by studying the ones possessing a large group of isometries.

2 Rank one symmetric spaces of noncompact type

2.1 Symmetric spaces

We recall material from [6] and [10]. Given a connected Riemannian manifold M, consider a point x of M and a symmetric neighbourhood U of M in the tangent space $\mathrm{T}_x M$ possessing the property that the exponential map $\mathrm{Exp}_x : U \to M$ is well-defined and is a diffeomorphism onto its image V. The symmetry $u \mapsto -u$ of U then induces a map s_x on V. We shall call s_x the *local geodesic symmetry* centered at x.

We say that M is a Riemannian *locally symmetric space* if, for any x in M, the local symmetry s_x is a local isometry of M. We say that M is *globally symmetric* if, for any x, this isometry may be extended (necessarily uniquely) to M. A complete simply connected locally symmetric space is globally symmetric. Globally symmetric spaces are complete spaces which possess a very large group of isometries (in particular their group of isometries is transitive).

Let M be a connected Riemannian manifold, with isometry group G. If x is a point of G, consider the map $G \to \mathrm{T}^*{}_x M \otimes \mathrm{T}M, g \mapsto (gx, \mathrm{d}g(x))$: it is injective and has closed image. From this we deduce that the group of isometries of M is separable, locally compact and second countable for the compact-open topology. In case M is globally symmetric, G can be proved to carry a structure of Lie group compatible with this topology. The structure of Riemannian globally symmetric spaces is therefore intrinsically linked with the theory of Lie groups. In particular, globally symmetric spaces have been classified by E. Cartan.

It turns out that every globally symmetric space M is the Cartesian Riemannian product of three globally symmetric spaces M_0, M_+ and M_-, where M_0 is isometric to some $\mathbb{R}^k$, with its canonical Euclidean structure, and M_- (resp. M_+) has nonpositive (resp. nonnegative) curvature, but may not be written as a product of $\mathbb{R}$ with some other Riemannian manifold. Spaces of the form M_- (resp. M_+) are said to be of noncompact (resp. compact) type. In the sequel, we shall be concerned by symmetric spaces of noncompact type.

A (connected) Lie group is said to be semisimple if it has no non-trivial abelian connected normal closed subgroup. In other words, a Lie group is semisimple if its Lie algebra is semisimple. If G is a semisimple Lie group it possesses maximal compact subgroups: these subgroups are all conjugate to each other and equal to their normalizer. Therefore, if K is such a maximal compact subgroup, the manifold G/K may be seen as the set of maximal compact subgroups of G. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ the one of K. As K is compact, its adjoint action preserves a scalar product on the vector space $\mathfrak{g}/\mathfrak{k}$ which is naturally identified to the tangent space at K of G/K. This scalar product then induces a G-invariant metric on G/K. This metric can be proved to make G/K a globally symmetric space of noncompact type and we have the following theorem, due to E. Cartan:

Theorem 2.1. *The Riemannian globally symmetric spaces of noncompact type are the spaces of the form G/K equipped with a G-invariant metric, where G is a connected semisimple Lie group and K a maximal compact subgroup of G.*

As every complete Riemannian manifold with nonpositive curvature, all these spaces are diffeomorphic to some $\mathbb{R}^k$.

Example 2.1. The Lie group $\mathrm{SL}_n(\mathbb{R})$ can be checked to be semisimple (this is in fact the generic example of a semisimple group). As every compact group of linear automorphism of $\mathbb{R}^n$ preserves a scalar product, the group $\mathrm{SO}(n)$ of orthogonal matrices with determinant 1 is a maximal compact subgroup of $\mathrm{SL}_n(\mathbb{R})$ and every maximal compact subgroup of $\mathrm{SL}_n(\mathbb{R})$ is conjugate to it. The tangent space to $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ at $\mathrm{SO}(n)$ may be $\mathrm{SO}(n)$-equivariantly identified with the vector space of symmetric matrices. On this space, the bilinear form $(A, B) \mapsto \mathrm{Tr}(AB)$ is a $\mathrm{SO}(n)$-invariant scalar product. We therefore get a $\mathrm{SL}_n(\mathbb{R})$-invariant Riemannian metric on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. The automorphism $g \mapsto (g^{-1})^t$ (where ^{t} denotes the transpose matrix) of $\mathrm{SL}_n(\mathbb{R})$ fixes $\mathrm{SO}(n)$ and induces on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ an isometry which extends the local geodesic symmetry centered at $\mathrm{SO}(n)$.

Consider the particular case $n = 2$. The group $\mathrm{SL}_2(\mathbb{R})$ acts on the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ by projective automorphisms and preserves the circle $\mathbb{P}^1_{\mathbb{R}}$. As this group is connected, it preserves the upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$, we get $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$. This action is transitive, as upper triangular matrices already act transitively, and the stabilizer of i is $\mathrm{SO}(2)$. Therefore it identifies $\mathrm{SL}_2(\mathbb{R})$-equivariantly $\mathbb{H}$ and $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$. The $\mathrm{SL}_2(\mathbb{R})$-invariant metric g we defined above can be written $g_{x+iy} = \frac{2}{y^2} g^e_{x+iy}$ where g^e is the Euclidean metric: in other word it is (up to a scalar multiple) the upper half plane model for hyperbolic plane (see paragraph 2.2).

A totally geodesic submanifold of a globally symmetric space M is necessarily itself a globally symmetric space. If M is of noncompact type, totally geodesic submanifolds have nonpositive curvature and, thus, don't have compact type factors. In that case, we say that M has rank k if it contains a flat totally geodesic submanifold of dimension k and if every other flat totally geodesic submanifold has rank $\leq k$. Maximal flat subspaces can be shown to be conjugate under the group of isometries. As M contains geodesics, its rank is ≥ 1. A symmetric space has rank one if and only if it has negative curvature, that is its sectional curvature as a function on the Grassmannian bundle $\mathcal{G}^2 M$ of tangent 2-planes of M, is everywhere negative.

Example 2.2. The rank of $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ is $n-1$. More precisely, if A is the group of diagonal matrices with positive entries in $\mathrm{SL}_n(\mathbb{R})$, the set $F = A\mathrm{SO}(n)$ is a flat totally geodesic submanifold of $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. The other maximal flat subspaces are of the form gF for some g in $\mathrm{SL}_n(\mathbb{R})$. In particular, for $n = 2$, the hyperbolic plane has rank one and the geodesics are the curves of the form $t \mapsto g \cdot e^t i$ for some g in $\mathrm{SL}_2(\mathbb{R})$.

2.2 Rank one symmetric spaces

The classification of globally symmetric spaces of noncompact type is the same as the classification of semisimple Lie groups. As often in Lie group theory, the classification contains a finite number of infinite lists (as the one of special linear groups $\mathrm{SL}_n(\mathbb{R})$, $n \geq 2$), the so-called classical groups, and a finite set of "exceptional" examples.

For rank one symmetric spaces, there are three lists of classical spaces: real, complex and quaternionic hyperbolic spaces. There is only one exceptional one, the Cayley hyperbolic plane, which we will not describe here.

2.2.1 Real hyperbolic spaces

Fix an integer $n \geq 1$ and equip $\mathbb{R}^{n+1}$ with the quadratic form $q(x_0, \ldots, x_n) = x_0^2 - x_1^2 - \ldots - x_n^2$ of signature $(1, n)$. Denote by $\mathbb{H}^n_{\mathbb{R}}$ the set $\{x \in \mathbb{R}^{n+1} | q(x) = 1 \text{ and } x_0 > 0\}$: this is one of the two connected components of the set $\{q = 1\}$. For x in $\mathbb{H}^n_{\mathbb{R}}$, the tangent space at x of $\mathbb{H}^n_{\mathbb{R}}$ identifies with its q-orthogonal hyperplane. On that space, by Sylvester's theorem, the restriction of q is negative definite. Denote by g_x its opposite: the field of bilinear forms g is a Riemannian metric on $\mathbb{H}^n_{\mathbb{R}}$. We call this Riemannian manifold real hyperbolic space of dimension n.

The other classical models for hyperbolic space may be recovered from this one, which, as we shall soon see, is the most practical one to describe the group of isometries. First of all, we can identify $\mathbb{H}^n_{\mathbb{R}}$ with the set of vector lines

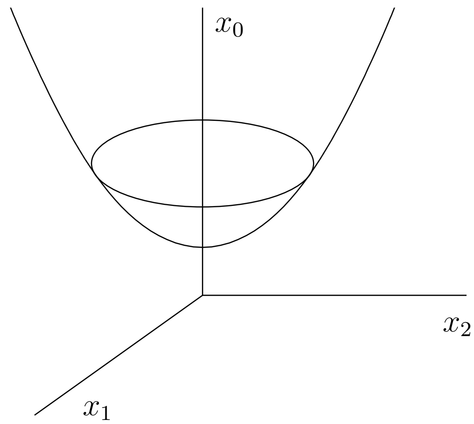

Figure 3: The hyperbolic hypersurface

containig a q-positive vector, as such a line hits the hyperbolic hypersurface at only one point: we then get the Klein model of the hyperbolic space, which is seen as an open subset in $\mathbb{P}^n_{\mathbb{R}}$. We can then project by stereographic projection from the point $(-1, 0, \ldots, 0)$ the set $\mathbb{H}^n_{\mathbb{R}}$ onto the ball $\mathbb{B}^n = \{x \in \mathbb{R}^n | \, \|x\| < 1\}$ (where the norm is the canonical Euclidean norm of $\mathbb{R}^n$) which we view as a subset of $\{0\} \times \mathbb{R}^n$. We get the ball model for real hyperbolic space, that is the metric $x \mapsto \frac{4}{\left(1-\|x\|^2\right)^2} g^e_x$, where, as usual, g^e denotes the Euclidean metric. Finally, we can apply to the ball model the Euclidean inversion of $\mathbb{R}^n$ with center $(-1, 0, \ldots, 0)$ and radius $\sqrt{2}$: we then get the upper half space model, that is the set $\{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 > 0\}$ equipped with the metric $x \mapsto \frac{1}{x_1^2} g^e_x$.

Return now to the original model. We will exhibit strong properties of transitivity of some groups of isometries $\mathbb{H}^n_{\mathbb{R}}$. We shall need the following

Lemma 2.2. *Let V be a vector subspace of $\mathbb{R}^{n+1}$ containing an element x of $\mathbb{H}^n_{\mathbb{R}}$. Then $V \cap \mathbb{H}^n_{\mathbb{R}}$ is a totally geodesic submanifold of $\mathbb{H}^n_{\mathbb{R}}$. It is isometric to real hyperbolic space of dimension* $\dim V - 1$*. Every complete totally geodesic submanifold of $\mathbb{H}^n_{\mathbb{R}}$ is of this form. In particular, complete geodesics of $\mathbb{H}^n_{\mathbb{R}}$ are the nonempty intersections of $\mathbb{H}^n_{\mathbb{R}}$ with planes of $\mathbb{R}^{n+1}$.*

Proof. As x is a q-anisotropic vector we have $\mathbb{R}^{n+1} = \mathbb{R}x \oplus x^{\perp}$ (where $\perp$ refers to orthogonality with respect to q). As V contains x, we get $V = \mathbb{R}x \oplus (x^{\perp} \cap V)$. Since the restriction of q to $x^{\perp}$ is negative definite, the restriction of q

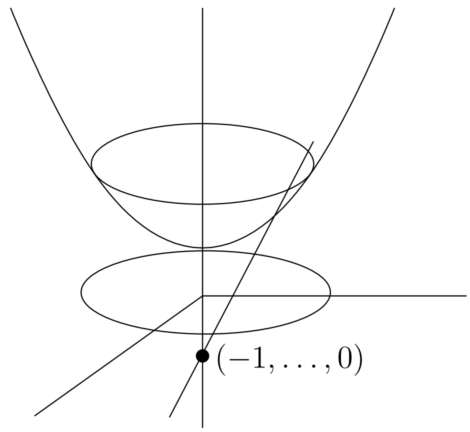

Figure 4: Stereographic projection

to V is nondegenerate and has signature $(1, \dim V - 1)$. Therefore $V \cap \mathbb{H}^n_{\mathbb{R}}$ is isometric to hyperbolic space of dimension $\dim V - 1$. Let us show that it is totally geodesic. As the restriction of q to V is nondegenerate, we have $\mathbb{R}^{n+1} = V \oplus V^{\perp}$. Let s denote the q-orthogonal reflection with respect to V, that is the linear automorphism that is $y \mapsto y$ on V and $y \mapsto -y$ on $V^{\perp}$: s is a q-isometry and, as it fixes x, stabilizes $\mathbb{H}^n_{\mathbb{R}}$, where it therefore induces an isometry. The fixed point set of this isometry is exactly $V \cap \mathbb{H}^n_{\mathbb{R}}$. Thus this set is a totally geodesic submanifold, by the local uniqueness of geodesics. Finally, let $M \subset \mathbb{H}^n_{\mathbb{R}}$ be a complete totally geodesic submanifold and let x be a point of M. If W is the tangent space to M at x and $V = \mathbb{R}x \oplus W$, M and $V \cap \mathbb{H}^n_{\mathbb{R}}$ are complete totally geodesic submanifolds having the same tangent space at x and are therefore equal. $\square$

Denote by $\mathrm{O}(1,n)$ the orthogonal group of q, by $\mathrm{SO}(1,n)$ the special orthogonal group and by $\mathrm{SO}^{\circ}(1,n)$ the connected component of the identity in $\mathrm{SO}(1,n)$: it has index 2 in $\mathrm{SO}(1,n)$ as each element of $\mathrm{O}(1,n)$ either stabilizes $\mathbb{H}^n_{\mathbb{R}}$ or exchanges $\mathbb{H}^n_{\mathbb{R}}$ and $-\mathbb{H}^n_{\mathbb{R}}$. Let $(e_0, e_1, \dots, e_n)$ be the canonical basis of $\mathbb{R}^{n+1}$. We shall write K for the subgroup of $\mathrm{SO}^{\circ}(1,n)$ consisting of isometries of the form $(x_0, x_1, \dots, x_n) \mapsto (x_0, g(x_1, \dots, x_n))$ where g lies in $\mathrm{SO}(n)$: it is the stabilizer of e_0 in $\mathrm{SO}^{\circ}(1,n)$ and e_0 is the unique fixed point of K in $\mathbb{H}^n_{\mathbb{R}}$. Let us denote by $(f_0, f_1, \dots, f_n)$ the basis of $\mathbb{R}^{n+1}$ such that $f_0 = \frac{e_0+e_n}{\sqrt{2}}, f_1 = e_1, \dots, f_{n-1} = e_{n-1}, f_n = \frac{e_0-e_n}{\sqrt{2}}$ and $y_0, \dots, y_n$ the coordinates in this base. We have $q = 2y_0y_n - y_1^2 - \ldots - y_{n-1}^2$. For $t \in \mathbb{R}$, the linear operator a_t whose matrix with respect to $(f_0, f_1, \dots, f_n)$ is

$$\begin{pmatrix} e^t & 0 & 0 \\ 0 & \mathrm{I}_{n-1} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}$$

is a q-isometry; its matrix with respect to $(e_0, e_1, \dots, e_n)$ is

$$\begin{pmatrix} \cosh t & 0 & -\sinh t \\ 0 & \mathrm{I}_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}.$$

In particular the curve $t \mapsto a_t e_0$ is a unit speed geodesic. We write A for the subgroup $\{a_t | t \in \mathbb{R}\}$ of $\mathrm{SO}^\circ(1,n)$.

Let $\partial\mathbb{H}^n_{\mathbb{R}}$ denote the boundary of $\mathbb{H}^n_{\mathbb{R}}$ as a subset of projective space, that is the projective image of the isotropic cone of q. It is diffeomorphic to the sphere $\mathbb{S}^{n-1}$.

Lemma 2.3. *The group* $\mathrm{SO}^\circ(1,n)$ *acts transitively on* $\mathbb{H}^n_{\mathbb{R}}$ *and the group* K *acts transitively on* $\partial\mathbb{H}^n_{\mathbb{R}}$. *The sectional curvature of* $\mathbb{H}^n_{\mathbb{R}}$ *has constant value* -1.

Proof. Let $x = (x_0, \dots, x_n)$ be in $\mathbb{H}^n_{\mathbb{R}}$. By applying an element of K, we can suppose $(x_2, \dots, x_{n-1}) = 0$. Then x lies in Ae_0. The proof is analogous for isotropic vectors.

As $\mathrm{SO}(n)$ acts transitively on 2-planes, the group K acts transitively on 2-planes of $\mathrm{T}_{e_0}\mathbb{H}^n_{\mathbb{R}}$ and $\mathrm{SO}^\circ(1,n)$ acts transitively on the Grassmannian bundle $\mathcal{G}^2\mathbb{H}^n_{\mathbb{R}}$ of 2-planes tangent to $\mathbb{H}^n_{\mathbb{R}}$. Therefore, the sectional curvature of $\mathbb{H}^n_{\mathbb{R}}$ is constant. We postpone the calculus of its value to appendix A. □

Let N be the subgroup of elements of $\mathrm{SO}^\circ(1,n)$ whose matrix with respect to the basis $(f_0, f_1, \dots, f_n)$ is of the form

$$n_u = \begin{pmatrix} 1 & u & -\frac{\|u\|^2}{2} \\ 0 & \mathrm{I}_{n-1} & -u^t \\ 0 & 0 & 1 \end{pmatrix}$$

where u is some line vector in $\mathbb{R}^{n-1}$ and u^t designs the transpose column vector. The map $u \mapsto n_u$ is an isomorphism from $\mathbb{R}^{n-1}$ onto N. The group N is normalized by A and one has $a_t n_u a_{-t} = n_{e^t u}$. Finally let M be the subgroup of elements of $\mathrm{SO}^\circ(1,n)$ whose matrix with respect to the basis $(f_0, f_1, \ldots, f_n)$ has the form

$$m_g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some g in $\mathrm{SO}(n-1)$: it normalizes N and one has $m_g n_u m_{g^{-1}} = n_{gu}$. Let $P = MAN$.

The links between all these subgroups are explained in the following:

Lemma 2.4. *One has* $\mathrm{SO}^\circ(1,n) = KP = KAN$ *and* P *is the stabilizer of* $\xi_0 = \mathbb{R}f_0$ *in* $\mathrm{SO}^\circ(1,n)$. *For every* ξ *in* $\partial\mathbb{H}^n_\mathbb{R}$, *the stabilizer* P_ξ *of* ξ *is conjugate to* P *and one has* $\mathrm{SO}^\circ(1,n) = KP_\xi$.

Proof. Let us show that $\mathrm{SO}^\circ(1,n) = KAN$: it suffices to show that AN acts transitively on $\mathbb{H}^n_\mathbb{R}$ let $(y_0, \ldots, y_n)$ be the coordinates in $(f_0, f_1, \ldots, f_n)$ of an element y of $\mathbb{H}^n_\mathbb{R}$. Then $n_{(y_2,\ldots,y_{n-1})}y$ belongs to Ae_0 and this implies the result.

Let g be an element of $\mathrm{SO}^\circ(1,n)$ stabilizing $\mathbb{R}f_0$. From the preceding, we can suppose that g fixes e_0. Therefore, it stabilizes the vector plane spanned by f_0 and f_n. As the only isotropic lines of this plane are $\mathbb{R}f_0$ and $\mathbb{R}f_n$, it stabilizes both these lines. Since it fixes e_0, the eigenvalue on these lines must be 1. Now g must stabilize the q-orthogonal space of the plane spanned by f_0 and f_n, hence g belongs to M. $\square$

As we shall see in the next section these objects will play a key role in the description of harmonic measures on $\partial\mathbb{H}^n_\mathbb{R}$. We will now generalize their construction to the other classical rank one symmetric spaces.

2.2.2 Complex hyperbolic spaces

We now consider on $\mathbb{C}^{n+1}$ the hermitian quadratic form $q(x_0, x_1, \ldots, x_n) = |x_0|^2 - |x_1|^2 - \ldots - |x_n|^2$ of signature $(1,n)$ corresponding to the hermitian sesquilinear form $\langle x, y\rangle = \overline{x_0}y_0 - \overline{x_1}y_1 - \ldots - \overline{x_n}y_n$. Consider the open set $U = \{q > 0\}$ in $\mathbb{C}^{n+1}$. On U, we define a (complex) subbundle E of the tangent bundle as follows: for each x, we take E_x as being the q-orthogonal space of x. Then, on E_x, by Sylvester's theorem, the restriction of q is negative definite. We define g_x to be the restriction to E_x of $-\frac{1}{q(x)}q$: this hermitian metric on U is invariant by multiplication by complex scalars. Let

$\mathbb{H}^n_{\mathbb{C}}$ be the open subset of $\mathbb{P}^n_{\mathbb{C}}$ which is the image of U and let $\pi : U \to \mathbb{H}^n_{\mathbb{C}}$ be the natural map. Then $\mathbb{H}^n_{\mathbb{C}}$ is diffeomorphic to the ball $\mathbb{B}^{2n}$. The differential $\mathrm{d}\pi : E \to \mathrm{T}\mathbb{H}^n_{\mathbb{C}}$ is surjective on fibers. As the metric g is invariant by multiplication by scalars, it induces a hermitian metric on $\mathbb{H}^n_{\mathbb{C}}$: we call this space complex hyperbolic space of dimension n.

We know the structure of the complex hyperbolic line:

Lemma 2.5. *The space $\mathbb{H}^1_{\mathbb{C}}$ is isometric to the hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$ equipped with the metric which is equal to $\frac{1}{4}$ times the usual one.*

Proof. The map $\mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$, $z \mapsto [1, z]$ induces a (holomorphic) diffeomorphism from the disk $\mathbb{D} = \{z \in \mathbb{C} |\, |z| < 1\}$ onto $\mathbb{H}^1_{\mathbb{C}}$. The pulled back metric may be written $z \mapsto \frac{1}{\left(1-|z|^2\right)^2} g^e_z$ where g^e denotes the canonical Euclidean metric, that is $\frac{1}{4}$ times the hyperbolic metric in the ball model and the lemma follows. □

We say that a $\mathbb{R}$-subspace of $\mathbb{C}^{n+1}$ is *Lagrangian* if it is totally isotropic for the skew-symmetric $\mathbb{R}$-bilinear form $(x, y) \mapsto \mathrm{Im}\langle x, y\rangle$. The following result is an analogue of lemma 2.2:

Lemma 2.6. *Let V be a complex vector subspace of $\mathbb{C}^{n+1}$ containing an element x such that $q(x) > 0$. Then $\mathbb{P}(V) \cap \mathbb{H}^n_{\mathbb{C}}$ is a totally geodesic complex submanifold of $\mathbb{H}^n_{\mathbb{C}}$. It is isometric to complex hyperbolic space of dimension $\dim_{\mathbb{C}} V - 1$. Every complete totally geodesic complex submanifold of $\mathbb{H}^n_{\mathbb{C}}$ is of this form.*

Let V be a Lagrangian real vector subspace of $\mathbb{C}^{n+1}$ containing an element x such that $q(x) > 0$. Then the image of V in $\mathbb{P}^n_{\mathbb{C}}$ intersects $\mathbb{H}^n_{\mathbb{C}}$ on a totally geodesic real submanifold. It is isometric to real hyperbolic space of dimension $\dim_{\mathbb{R}} V - 1$.

Every complete totally geodesic submanifold of $\mathbb{H}^n_{\mathbb{C}}$ is of one of these forms. In particular, complete geodesics of $\mathbb{H}^n_{\mathbb{C}}$ are the image in $\mathbb{H}^n_{\mathbb{C}}$ of Lagrangian $\mathbb{R}$-planes of $\mathbb{C}^{n+1}$.

Proof. The proof that the image of a complex subspace of $\mathbb{C}^{n+1}$ in $\mathbb{H}^n_{\mathbb{C}}$ is totally geodesic and that every totally geodesic complex submanifold is of this form is analogous to the proof of lemma 2.2.

Suppose now V is a Lagrangian $\mathbb{R}$-subspace of $\mathbb{C}^{n+1}$ containing a vector x such that $q(x) > 0$. As V is Lagrangian, $x^{\perp} \cap V = \{y \in V |\, \mathrm{Re}\langle x, y\rangle = 0\}$ is a $\mathbb{R}$-hyperplane of V and the restriction of $\mathrm{Re}\langle ., .\rangle$ to V is a nondegenerate $\mathbb{R}$-bilinear form of signature $(1, \dim_{\mathbb{R}} V - 1)$. Since V is Lagrangian, we have

$V \cap iV = \{0\}$ and the projection map $\{y \in V | q(y) = 1\} \to \mathbb{H}^n_{\mathbb{C}}$ induces an isometry of its image with real hyperbolic space of dimension $\dim_{\mathbb{R}} V - 1$. Finally let us choose a maximal Lagrangian $\mathbb{R}$-subspace $W \supset V$ of $\mathbb{C}^{n+1}$. Then one has $W \oplus iW = \mathbb{C}^{n+1}$ and conjugation with respect to this decomposition (that is $x + iy \mapsto x - iy$) induces an anti-q-isometry of $\mathbb{C}^{n+1}$ and an isometry of $\mathbb{H}^n_{\mathbb{C}}$, with fixed points set the image of W in $\mathbb{H}^n_{\mathbb{C}}$. Therefore this image is totally geodesic and isometric to $\mathbb{H}^n_{\mathbb{R}}$. Hence the image of V, which is contained in the one of W, is totally geodesic by lemma 2.2.

The classification of totally geodesic submanifolds of complex hyperbolic space will be achieved in appendix A. $\square$

Denote by $\mathrm{U}(1, n)$ the unitary group of the form q and by $\mathrm{PU}(1, n)$ its projective image: these are connected groups. As before, we shall denote by $(e_0, e_1, \ldots, e_n)$ the canonical basis of $\mathbb{C}^{n+1}$ and write K for the image in $\mathrm{PU}(1, n)$ of the group of isometries of q of the form $(x_0, x_1, \ldots, x_n) \mapsto (x_0, g(x_1, \ldots, x_n))$ where g lies in $\mathrm{U}(n)$: it is the stabilizer of $\mathbb{C}e_0$ in $\mathrm{PU}(1, n)$ and $\mathbb{C}e_0$ is the unique fixed point of K in $\mathbb{H}^n_{\mathbb{C}}$. Let us still denote by $(f_0, f_1, \ldots, f_n)$ the basis $(\frac{e_0+e_n}{\sqrt{2}}, e_1, \ldots, e_{n-1}, \frac{e_0-e_n}{\sqrt{2}})$ and $y_0, \ldots, y_n$ the coordinates in this base. We have $q = 2\,\mathrm{Re}(\overline{y_0} y_n) - |y_1|^2 - \ldots - |y_{n-1}|^2$. For $t \in \mathbb{R}$, the linear operator a_t whose matrix with respect to $(f_0, f_1, \ldots, f_n)$ is

$$\begin{pmatrix} e^t & 0 & 0 \\ 0 & \mathrm{I}_{n-1} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}$$

is a q-isometry with matrix with respect to $(e_0, e_1, \ldots, e_n)$:

$$\begin{pmatrix} \cosh t & 0 & -\sinh t \\ 0 & \mathrm{I}_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix},$$

and the curve $t \mapsto a_t \mathbb{C}e_0$ is a unit speed geodesic. We write A for the image of the group $\{a_t | t \in \mathbb{R}\}$ in $\mathrm{PU}(1, n)$.

Let $\partial\mathbb{H}^n_{\mathbb{C}}$ denote the boundary of $\mathbb{H}^n_{\mathbb{C}}$ as a subset of projective space. It is the projective image of the isotropic cone of q and is diffeomorphic to the sphere $\mathbb{S}^{2n-1}$.

Lemma 2.7. *The group $\mathrm{PU}(1, n)$ acts transitively on $\mathbb{H}^n_{\mathbb{C}}$ and the group K acts transitively on $\partial\mathbb{H}^n_{\mathbb{C}}$. The sectional curvature of $\mathbb{H}^n_{\mathbb{C}}$ lies everywhere between -4 and -1 and reaches the value -1 exactly on Lagrangian real 2-planes and the value -4 exactly on complex lines, viewed as real 2-planes.*

Proof. The first part is proved the same way as for lemma 2.3. The computation of the curvature will be achieved in appendix A. $\square$

Let now N be the the image in $\mathrm{PU}(1,n)$ of the group of elements of $\mathrm{SU}(1,n)$ whose matrix with respect to the basis $(f_0, f_1, \dots, f_n)$ is of the form

$$n_{(u,s)} = \begin{pmatrix} 1 & u & is - \frac{\|u\|^2}{2} \\ 0 & \mathrm{I}_{n-1} & -\overline{u}^t \\ 0 & 0 & 1 \end{pmatrix}$$

where u is some line vector in $\mathbb{C}^{n-1}$ and s is a real number. The group N is $(2n-1)$-dimensional Heisenberg group, that is its Lie algebra is isomorphic to $\mathbb{R}^{2n-2} \times \mathbb{R}$ equipped with the Lie bracket defined by $[(U,S),(V,R)] = (0, \omega(U,V))$ where ω is some skew-symmetric nondegenate bilinear form on $\mathbb{R}^{2n-2}$. As before, this group is normalized by A and one has $a_t n_{(u,s)} a_{-t} = n_{(e^t u, e^{2t} s)}$. Finally let M be the subgroup of elements of $\mathrm{PU}(1,n)$ which are images of q-isometries with matrix with respect to the basis $(f_0, f_1, \dots, f_n)$ of the form

$$m_g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some g in $\mathrm{PU}(n-1)$: it normalizes N and one has $m_g n_{(u,s)} m_{g^{-1}} = n_{(gu,s)}$. Let $P = MAN$.

We now have an analogue of lemma 2.4:

Lemma 2.8. *One has* $\mathrm{PU}(1,n) = KP = KAN$ *and* P *is the stabilizer of* $\xi_0 = \mathbb{C} f_0$ *in* $\mathrm{PU}(1,n)$. *For every* ξ *in* $\partial \mathbb{H}^n_{\mathbb{C}}$, *the stabilizer* P_ξ *of* ξ *is conjugate to* P *and one has* $\mathrm{PU}(1,n) = KP_\xi$.

2.2.3 Quaternionic hyperbolic space

We let $\mathbb{Q}$ be the set of quaternionic numbers and, as usual, i, j, k be three fixed elements of $\mathbb{Q}$ such that $i^2 = j^2 = k^2 = -1$ and $ij = k$, $jk = i$ and $ki = j$. We write $x \mapsto \overline{x}$ for the quaternionic conjugation, $x \mapsto |x| = \sqrt{\overline{x}x}$ for the quaternionic module and $\mathbb{Q}_0$ for the space of pure quaternions, that is those satisfying $\overline{x} = -x$ (it is the $\mathbb{R}$-vector space spanned by i, j, k).

If E is a right quaternionic vector space, a (quaternionic) hermitian form on E is a $\mathbb{R}$-bilinear map $\varphi : E \times E \to \mathbb{Q}$ such that, for α, β in $\mathbb{Q}$, for x, y in E, one gets $\varphi(x\alpha, y\beta) = \overline{\alpha}\varphi(x,y)\beta$ and $\varphi(y,x) = \overline{\varphi(x,y)}$. For such maps, the analogue of Sylvester's theorem holds: we have classification by signature for nondegenerate forms. The unitary group of such a form is the group of $\mathbb{Q}$-linear automorphisms of E which preserve it. The unitary group of the standard form of signature (p,q) on $\mathbb{Q}^{p+q}$, $\langle x,y\rangle = \overline{x_1}y_1 + \ldots + \overline{x_p}y_p - \overline{x_{p+1}}y_{p+1} - \ldots - \overline{x_q}y_q$, which is the polar form of $q(x) = |x_1|^2 + \ldots + |x_p|^2 - |x_{p+1}|^2 - \ldots - |x_q|^2$, is denoted by $\mathrm{Sp}(p,q)$. It is a real form of the symplectic group $\mathrm{Sp}_{2(p+q)}(\mathbb{C})$.

Let us now fix the standard form $\langle x,y\rangle = \overline{x_0}y_0 - \overline{x_1}y_1 - \ldots - \overline{x_n}y_n$ of signature $(1,n)$ on $\mathbb{Q}^{n+1}$, which we consider as a right vector space, and set $q(x) = |x_0|^2 - |x_1|^2 - \ldots - |x_n|^2$. As before, we let $\mathbb{H}^n_{\mathbb{Q}}$ denote the open set which is the image of $\{q > 0\}$ in the set $\mathbb{P}^n_{\mathbb{Q}}$ of one-dimensional right $\mathbb{Q}$-subspaces of $\mathbb{Q}^{n+1}$; it is diffeomorphic to the ball $\mathbb{B}^{4n}$. For each x with $q(x) > 0$ the form $-\frac{1}{q(x)}q$ is positive definite on the orthogonal of x. This defines a hermitian metric on $\mathbb{H}^n_{\mathbb{Q}}$. We call this space quaternionic hyperbolic space of dimension n.

As the complex hyperbolic line, the quaternonic hyperbolic line has a special structure: the same proof as the one of lemma 2.5 gives the

Lemma 2.9. *The space $\mathbb{H}^1_{\mathbb{Q}}$ is isometric with the hyperbolic hyperspace $\mathbb{H}^4_{\mathbb{R}}$ equipped with the metric which is equal to $\frac{1}{4}$ times the usual one.*

Let $p_{\mathbb{R}}$ be the $\mathbb{R}$-projection $x \mapsto \frac{1}{2}(x - \overline{x})$ of $\mathbb{Q}$ onto $\mathbb{Q}_0$. A real subspace V of $\mathbb{Q}^{n+1}$ is said to be *Lagrangian* if the restriction to V of the skew-symmetric $\mathbb{R}$-bilinear map $(x,y) \mapsto p_{\mathbb{R}}(\langle x,y\rangle)$ is trivial. In the same way, if $\mathbb{K}$ is a subfield of $\mathbb{Q}$ which is isomorphic to $\mathbb{C}$, there exists a $\mathbb{R}$-subspace W of $\mathbb{Q}$, supplementary to $\mathbb{K}$, which is invariant by both left and right multiplication by elements of $\mathbb{K}$: this is the set of y in $\mathbb{Q}$ such that, for any x in $\mathbb{K}$, one has $xy = y\overline{x}$ (for $\mathbb{K} = \mathbb{R}[i]$, one has $W = \mathbb{R}j \oplus \mathbb{R}k$). It is contained in $\mathbb{Q}_0$. We let $p_{\mathbb{K}}$ denote the $\mathbb{R}$-projection onto W with kernel $\mathbb{K}$: it is left and right $\mathbb{K}$-linear. In particular, if E is a right $\mathbb{Q}$-vector space and φ a hermitian form on E, then $p_{\mathbb{K}} \circ \varphi$ is a skew-symmetric $\mathbb{K}$-bilinear map on E. A right $\mathbb{K}$-subspace of $\mathbb{Q}^{n+1}$ is said to be *Lagrangian* if it is totally isotropic for the skew-symmetric $\mathbb{K}$-bilinear map $(x,y) \mapsto p_{\mathbb{K}}(\langle x,y\rangle)$. We still have an analogous result to lemmas 2.2 and 2.6:

Lemma 2.10. *Let V be a quaternionic vector subspace of $\mathbb{Q}^{n+1}$ containing an element x such that $q(x) > 0$. Then $\mathbb{P}(V) \cap \mathbb{H}^n_{\mathbb{Q}}$ is a totally geodesic quaternionic submanifold of $\mathbb{H}^n_{\mathbb{C}}$. It is isometric to quaternionic hyperbolic* *space of dimension* $\dim_{\mathbb{C}} V - 1$*. Every complete totally geodesic quaternionic submanifold of* $\mathbb{H}^n_{\mathbb{C}}$ *is of this form.*

Let $\mathbb{K}$ *be a subfield of* $\mathbb{Q}$ *which is isomorphic to* $\mathbb{C}$ *and let* V *be a Lagrangian* $\mathbb{K}$*-vector subspace of* $\mathbb{Q}^{n+1}$ *containing an element* x *such that* $q(x) > 0$*. Then the image of* V *in* $\mathbb{P}^n_{\mathbb{Q}}$ *intersects* $\mathbb{H}^n_{\mathbb{Q}}$ *on a totally geodesic submanifold. It is isometric to complex hyperbolic space of dimension* $\dim_{\mathbb{K}} V - 1$*.*

Every complete totally geodesic submanifold of $\mathbb{H}^n_{\mathbb{Q}}$ *either is of the first form, or is contained in a manifold of the first form and of* $\mathbb{Q}$*-dimension* 1*, or is contained in a submanifold of the second form. In particular, complete geodesics of* $\mathbb{H}^n_{\mathbb{Q}}$ *are the image in* $\mathbb{H}^n_{\mathbb{Q}}$ *of Lagrangian* $\mathbb{R}$*-planes of* $\mathbb{Q}^{n+1}$*.*

Denote by $\mathrm{PSp}(1,n)$ the projective image of $\mathrm{Sp}(1,n)$, that is its quotient by the group of diagonal real matrices. Let still $(e_0, e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{Q}^{n+1}$ and $(f_0, f_1, \ldots, f_n)$ the basis $(\frac{e_0+e_n}{\sqrt{2}}, e_1, \ldots, e_{n-1}, \frac{e_0-e_n}{\sqrt{2}})$ (with coordinates $y_0, \ldots, y_n$ in such a way that $q = 2\,\mathrm{Re}(\overline{y_0}y_n) - |y_1|^2 - \ldots - |y_{n-1}|^2$). Write K for the image in $\mathrm{PSp}(1,n)$ of the group of isometries of q of the form $(x_0, x_1, \ldots, x_n) \mapsto (\alpha x_0, g(x_1, \ldots, x_n))$ where α is a unit modulus quaternion and g lies in $\mathrm{Sp}(n)$: it is the stabilizer of $e_0\mathbb{Q}$ in $\mathrm{PSp}(1,n)$ and $e_0\mathbb{Q}$ is the unique fixed point of K in $\mathbb{H}^n_{\mathbb{Q}}$. As before, for $t \in \mathbb{R}$, the linear operator a_t whose matrix with respect to $(f_0, f_1, \ldots, f_n)$ is

$$\begin{pmatrix} e^t & 0 & 0 \\ 0 & \mathrm{I}_{n-1} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}$$

is a q-isometry and the curve $t \mapsto a_t e_0\mathbb{Q}$ is a unit speed geodesic. We write A for the image of the group $\{a_t | t \in \mathbb{R}\}$ in $\mathrm{PSp}(1,n)$.

Let $\partial\mathbb{H}^n_{\mathbb{Q}}$ denote the boundary of $\mathbb{H}^n_{\mathbb{Q}}$ as a subset of projective space. It is the projective image of the isotropic cone of q and is diffeomorphic to the sphere $\mathbb{S}^{4n-1}$. We have a quaternionic version of lemmas 2.3 and 2.7:

Lemma 2.11. *The group* $\mathrm{PSp}(1,n)$ *acts transitively on* $\mathbb{H}^n_{\mathbb{Q}}$ *and the group* K *acts transitively on* $\partial\mathbb{H}^n_{\mathbb{Q}}$*. The sectional curvature of* $\mathbb{H}^n_{\mathbb{Q}}$ *lies everywhere between* -4 *and* -1 *and reaches the value* -1 *exactly on Lagrangian real* 2*-planes and the value* -4 *exactly on real* 2*-planes that are* $\mathbb{K}$*-lines for some maximal commutative subfield* $\mathbb{K}$ *of* $\mathbb{Q}$*.*

Let now N be the the image in $\mathrm{PSp}(1,n)$ of the group of elements of $\mathrm{PSp}(1,n)$ whose matrix with respect to the basis $(f_0, f_1, \ldots, f_n)$ is of the form

$$n_{(u,s)} = \begin{pmatrix} 1 & u & s - \frac{\|u\|^2}{2} \\ 0 & \mathrm{I}_{n-1} & -\overline{u}^t \\ 0 & 0 & 1 \end{pmatrix}$$

where u is some line vector in $\mathbb{Q}^{n-1}$ and s lies in $\mathbb{Q}_0$. The group N has Lie algebra isomorphic to $\mathbb{Q}^{n-1} \times \mathbb{Q}_0$ equiped with the Lie bracket defined by $[(U,S),(V,R)] = (0, p_{\mathbb{R}}(\langle U,V \rangle))$, where $\langle .,. \rangle$ denotes the standard $\mathbb{Q}$-hermitian scalar product on $\mathbb{Q}^{n-1}$. This group is still normalized by A and one has $a_t n_{(u,s)} a_{-t} = n_{(e^t u, e^{2t} s)}$. Finally let M be the subgroup of elements of $\mathrm{PSp}(1,n)$ who are images of q-isometries with matrix with respect to the basis $(f_0, f_1, \dots, f_n)$ of the form

$$m_{(\alpha,g)} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

for some unit quaternion α and some g in $\mathrm{PSp}(n-1)$: it normalizes N and one has $m_{(\alpha,g)} n_{(u,s)} m_{(\alpha^{-1},g^{-1})} = n_{(gu, \alpha s \alpha^{-1})}$. Let $P = MAN$.

We still have an analogue of lemma 2.4 and 2.8:

Lemma 2.12. *One has* $\mathrm{PSp}(1,n) = KP = KAN$ *and* P *is the stabilizer of* $\xi_0 = f_0\mathbb{Q}$ *in* $\mathrm{PSp}(1,n)$*. For every* ξ *in* $\partial\mathbb{H}^n_{\mathbb{Q}}$*, the stabilizer* P_ξ *of* ξ *is conjugate to* P *and one has* $\mathrm{PSp}(1,n) = KP_\xi$*.*

3 Homogeneous harmonic measures for rank one classical symmetric spaces

Here we shall develop the formalism of harmonic measures for the spaces introduced above. We begin by a fact from general group theory.

3.1 A Haar measure computation

Let us recall some general notions from Haar measure theory. Let G be a separable, locally compact and second countable topological group. Then up to homothety G possesses a unique Radon measure which is invariant by left (resp. right) translations by elements of G: such a measure is called a left (resp. right) *Haar measure* for G. Let λ be a left Haar measure. Then, for each g in G, the image of λ by right translation by g^{-1} is still a left Haar measure for G. It is therefore of the form $\Delta_G(g)\lambda$ for some $\Delta_G(g)$ in $\mathbb{R}_+^*$, which does not depend on the choice of λ. The function Δ_G is a continuous homomorphism from G into the multiplicative group $\mathbb{R}_+^*$; it is called the *modular function* of G. The measure $\frac{1}{\Delta_G}\lambda$ is a right Haar measure for G and, for every continuous function φ with compact support on G, one gets $\int_G \varphi(g^{-1})\mathrm{d}g = \int_G \Delta_G(g)^{-1}\varphi(g)\mathrm{d}g$.

The group G is said to be *unimodular* if $\Delta_G = 1$, that is if it possesses a bi-invariant Radon measure. Compact groups are unimodular: they don't possess any non-trivial homomorphism into $\mathbb{R}_+^*$ (as this latter group doesn't have any non-trivial compact subgroup). Abelian groups are unimodular as for them both left and right multiplication coincide and, by an induction argument, this implies that nilpotent groups are unimodular. Discrete groups are unimodular as their Haar measure is the counting measure which is invariant under any bijection. The groups P_ξ from the preceding section are not unimodular (we shall soon compute their modular function).

Lemma 3.1. *The groups* $\mathrm{SO}^\circ(1,n)$*,* $\mathrm{PU}(1,n)$ *and* $\mathrm{PSp}(1,n)$ *are unimodular.*

Proof. Choose one of these groups, denote it by G and let X be the associated hyperbolic space. Let us keep the notations of the previous section and let o be the point of X associated to e_0. Then, as K acts transitively on the unit sphere of T_oX, every point x of X belongs to Ka_ro where $r = d(o,x)$. Therefore, we have $G = K\{a_t | t \geq 0\}K$, and, as K is compact, the modular function Δ_G is determined by its restriction to A. But, for t in $\mathbb{R}$, a_t belongs to $Ka_{-t}K$ and, therefore, $\Delta_G(a_t) = 0$. The conclusion follows. $\square$

Let H be a closed subgroup of G. Then the homogeneous space G/H possesses a G-invariant measure if and only if the modular functions of G and H are equal on H. Such a measure is then unique. If H is compact, there exists an invariant measure on G/H: it is the projection of some Haar measure of G onto G/H. In particular, if G is compact, this measure is finite and can therefore be uniquely normalized to have total mass 1.

Let us focus on the situation provided by lemmas 2.4, 2.8 and 2.12. We therefore fix a unimodular group G, a compact subgroup K of G and a closed subgroup P, this last one with modular function Δ. We fix some left Haar measures $\mathrm{d}g$, $\mathrm{d}k$ and $\mathrm{d}p$ on G, K and P.

Lemma 3.2. *Suppose we have $G = KP$. Then the Haar measures can be normalized in such a way that, for any continuous function φ with compact support on G one gets:*

$$\int_G \varphi(g)\mathrm{d}g = \int_{K\times P} \Delta(p)^{-1}\varphi(kp)\mathrm{d}p\mathrm{d}k.$$

Proof. Consider the topological group $K \times P$ and let it act on G in such a way that $(k,p)\cdot g = kgp^{-1}$. By the hypothesis, this action is transitive. It induces an homeomorphism from G onto $(K\times P)/H$ where H is the compact subgroup $\{(h,h)|h \in K\cap P\}$. As the Haar measure of G is right P-invariant and left K-invariant, it induces a $K \times P$ invariant measure on $(K \times P)/H$, which is the projection of some Haar measure of $K \times P$. By normalizing suitably, we therefore get, for a continuous function φ with compact support on G:

$$\int_G \varphi(g)\mathrm{d}g = \int_{K\times P} \varphi(kp^{-1})\mathrm{d}p\mathrm{d}k = \int_{K\times P} \Delta(p)^{-1}\varphi(kp)\mathrm{d}p\mathrm{d}k.$$

□

Suppose $G = KP$. Then K acts transitively on G/P and thus preserves a unique probability measure on G/P. The action of G on this set preserves the measure class of this measure and we will give an explicit description of the associated Radon-Nikodym cocycle.

Choose once for all a Borel section $s : G/P \equiv K/(K\cap P) \to K$, that is a Borel map such that, for every g in G, g belongs to $s(g)P$ (such a map always exists for quotients of separable, locally compact and second countable topological groups). For every g in G and ξ in G/P, there exists a unique $\sigma(g,\xi)$ in P such that $gs(\xi) = s(g\xi)\sigma(g,\xi)$. The Borel function $\sigma : G \times (G/P) \to P$ clearly satisfies te cocycle identity:

$$\forall g,h \in G \quad \forall \xi \in G/P \quad \sigma(gh,\xi) = \sigma(g,h\xi)\sigma(h,\xi).$$

As the restriction of Δ to $K \cap P$ is trivial, the $\mathbb{R}_+^*$-valued cocycle $\theta = \Delta\circ\sigma$ doesn't depend on the choice of the section s and is continuous.

We now can prove the general formula we will later use in the context of hyperbolic spaces:

Proposition 3.3. *Suppose $G = KP$ and let ν be the unique K-invariant measure on G/P. Then, for every g in G, one gets $g_*\nu = \theta(g^{-1},.)\nu$.*

Proof. From the definition of s, the action of G on itself is Borel equivalent to its action on $(G/P) \times P$ defined by $g \cdot (\xi, p) = (g\xi, \sigma(g, \xi)p)$. From lemma 3.2, we know that, under this equivalence, the Haar measure of G may be written is the measure λ on $(G/P) \times P$ defined by

$$\int_{(G/P)\times P} \varphi(\xi, p)\mathrm{d}\lambda(\xi, p) = \int_{(G/P)\times P} \Delta(p)^{-1}\varphi(\xi, p)\mathrm{d}p\mathrm{d}\nu(\xi),$$

for every continuous function φ with compact support on $G \times (G/P)$. As this measure is G-invariant, for such a φ, we get, for every g in G:

$$\begin{aligned}\int_{G\times(G/P)} \Delta(p)^{-1}\varphi(\xi, p)\mathrm{d}p\mathrm{d}\nu(\xi) &= \int_{(G/P)\times P} \Delta(p)^{-1}\varphi(g\xi, \sigma(g, \xi)p)\mathrm{d}p\mathrm{d}\nu(\xi) \\ &= \int_{(G/P)} \theta(g, \xi) \int_P \Delta(p)^{-1}\varphi(g\xi, p)\mathrm{d}p\mathrm{d}\nu(\xi).\end{aligned}$$

Therefore, for every continuous function φ on G/P and every g in G we get:

$$\int_{G/P} \varphi\mathrm{d}\nu = \int_{G/P} \theta(g, \xi)\varphi(g\xi)\mathrm{d}\nu(\xi)$$

and the conclusion follows. □

3.2 Harmonic densities

We return to the study of hyperbolic spaces. We let X be $\mathbb{H}^n_{\mathbb{R}}$, $\mathbb{H}^n_{\mathbb{C}}$ or $\mathbb{H}^n_{\mathbb{Q}}$ and we denote by ∂X the boundary introduced in paragraph 2.2. We set $G = \mathrm{SO}^\circ(1, n)$, $\mathrm{PU}(1, n)$ or $\mathrm{PSp}(1, n)$, following the nature of X and we conserve the notations A, $t \mapsto a_t$, N, M, P and ξ_0 introduced above. In order to apply proposition 3.3, we need to compute the modular function of P:

Lemma 3.4. *Let Δ be the modular function of P. Then, for t in $\mathbb{R}$, n in N and m in M, we get*

(i) if $X = \mathbb{H}^n_{\mathbb{R}}$, $\Delta(ma_tn) = e^{-(n-1)t}$.

(ii) if $X = \mathbb{H}^n_{\mathbb{C}}$, $\Delta(ma_tn) = e^{-2nt}$.

(iii) if $X = \mathbb{H}^n_{\mathbb{Q}}$, $\Delta(ma_tn) = e^{-2(2n+1)t}$.

Proof. For a Lie group H the modular function is $h \mapsto \det(\mathrm{Ad}_h)^{-1}$ where Ad denotes the adjoint action of H on its Lie algebra. The result easily follows. $\square$

Let us now introduce the Busemann function. Denote by o the fixed point of K in X. In projective space, we have $a_t o \xrightarrow[t\to\infty]{} \xi_0$. Let us describe the links between the boundary ∂X and the geodesics of X:

Lemma 3.5. *Let $\sigma :]-\infty, \infty[\to X$ be a geodesic. Then σ has two distinct limit points $\sigma(+\infty)$ and $\sigma(-\infty)$ in ∂X. Conversely, if $\xi \neq \eta$ are two points of ∂X, there exists up to parameter translation a unique geodesic σ such that $\sigma(+\infty) = \xi$ and $\sigma(-\infty) = \eta$. If x is a point of X, there exists a unique geodesic σ such that $\sigma(+\infty) = \xi$ and $\sigma(0) = x$.*

The proof requires an intermediate lemma.

Let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$, following the nature of X and let n be the $\mathbb{K}$-dimension of X. Recall that we wrote $\langle ., . \rangle$ for the hermitian product of signature $(1, n)$ on $\mathbb{K}^{n+1}$ that allowed us to define the metric of X and q for the form $x \mapsto \langle x, x \rangle$. We write $\perp$ for orthogonality with respect to q.

Lemma 3.6. *Let ξ be in ∂X, that is ξ is a q-isotropic line. Then the form induced by q on the quotient vector space $\xi^\perp/\xi$ is negative definite.*

Proof. Let v be a non-zero element of ξ and let w be a vector such that $\langle v, w \rangle \neq 0$. As v is isotropic, if V is the plane spanned by v and w, the restriction of q to V is nondegenerate and has signature $(1, 1)$. Therefore, the restriction of q to $V^\perp$ is negative definite. The result follows, since we have $\xi^\perp = \xi \oplus V^\perp$. $\square$

Proof of lemma 3.5. The first point is true for the geodesic $t \mapsto a_t o$ and therefore for a general geodesic as G acts transitively on the set of geodesics, since it acts transitively on points of X and K acts transitively on the unit sphere of $\mathrm{T}_o X$. The third point is true for $x = o$ as K acts transitively on ∂X and therefore for any x, as, for every ξ in ∂X, its stabilizer P_ξ in G acts transitively on X by lemmas 2.4, 2.8 and 2.12.

For the second point, let ξ and η be distinct points of the boundary and choose non-zero vectors v and w in the $\mathbb{K}$-lines ξ and η. There exists α in $\mathbb{K}$ such that $p_{\mathbb{R}}(\langle v, w\alpha \rangle) = 0$. Let P be the Lagrangian $\mathbb{R}$-plane spanned by v and $w\alpha$. Then the restriction of q to P is a real quadratic form which has isotropic vectors and is nondegenerate by lemma 3.6. Therefore, it contains positive vectors and, by lemmas 2.2, 2.6 and 2.10, its image in X is a geodesic with limit points ξ and η. It is clearly unique. $\square$

We shall need the following:

Lemma 3.7. *For every n in N, one gets $d(no, a_t o) - t \xrightarrow[t\to\infty]{} 0$.*

Proof. For any t, we have

$$\begin{aligned} |d(no, a_t o) - t| &= |d(no, a_t o) - d(o, a_t o)| \\ &= \left|d(o, n^{-1} a_t o) - d(o, a_t o)\right| \\ &\leq d(n^{-1} a_t o, a_t o) \\ &= d(o, a_{-t} n a_t o) \xrightarrow[t\to\infty]{} 0, \end{aligned}$$

as $a_{-t} n a_t \xrightarrow[t\to\infty]{} e$ in G. $\square$

Let x, y be in X and ξ be in ∂X the *Busemann function* $b_\xi(x, y)$ is the limit $\lim_{t\to\infty} t - d(y, r(t))$ where $r : [0, \infty[\to X$ is the unique geodesic ray sutch that $r(0) = x$ and $r(t) \xrightarrow[t\to\infty]{} \xi$. Such a limit always exists, as we shall see later, by general arguments on negative curved spaces. However, we can proove its existence and describe its value by a group-theoretic method:

Lemma 3.8. *Let g in G, s in $\mathbb{R}$ and n in N be such that $\xi = g\xi_0$, $x = go$ and $y = ga_s no$. Then $b_\xi(x, y) = s$.*

Note that such g, s and n always exist by both transitivities of the action of K on $\partial X \equiv G/P$ and of the action on AN on $X \equiv G/K$. As in paragraph 3.1, this formula implies b to satisfy the cocycle identity:

$$\forall x, y, z \in X \quad \forall \xi \in \partial X \quad b_\xi(x, z) = b_\xi(x, y) + b_\xi(y, z).$$

Proof. As the Busemann function is invariant under the natural action of G, it suffices to prove the lemma for $g = e$. Then, the geodesic ray going from o to ξ_0 is $t \mapsto a_t o$ and we get $t - d(a_s no, a_t 0) = t - d(no, a_{t-s} o) \xrightarrow[t\to\infty]{} s$, by lemma 3.7, what should be proved. $\square$

We can now come to the extension to hyperbolic spaces of some of the objects appearing in paragraph 1.2:

Proposition 3.9. *For each x in X, let ν_x be the unique probability measure on ∂X which is invariant under the stabilizer of x in G. Then, for x, y in X, ν_x and ν_y are equivalent and one has*

$$\forall \xi \in \partial X \quad \frac{\mathrm{d}\nu_y}{\mathrm{d}\nu_x}(\xi) = e^{-\delta_X b_\xi(y,x)}.$$

where $\delta_X = n-1$ if X is $\mathbb{H}^n_{\mathbb{R}}$, $\delta_X = 2n$ if X is $\mathbb{H}^n_{\mathbb{C}}$ and $\delta_X = 2(2n+1)$ if X is $\mathbb{H}^n_{\mathbb{Q}}$.

Proof. The measures are clearly equivalent as they all belong to the Lebesgue class of the manifold ∂X. By the equivariance of the formula, it suffices to show that, for every g in G and ξ in ∂X, $\frac{\mathrm{d}\nu_{go}}{\mathrm{d}\nu_o}(\xi) = \frac{\mathrm{d}g_*\nu_o}{\mathrm{d}\nu_o}(\xi) = e^{-\delta_X b_\xi(go,o)}$. Let us write $\xi = k\xi_0$ and $k^{-1}g \in a_s nK$, for some k in K, s in $\mathbb{R}$ and n in N. Then, by lemma 3.8, we get $b_\xi(o, go) = s$. On the other hand, we have $g^{-1}k \in Kn^{-1}a_{-s}$ and, thus, with the notations of paragraph 3.1, $\theta(g^{-1}, \xi) = \Delta(n^{-1}a_{-s}) = e^{\delta_X s}$, the value of the modular function being given by lemma 3.4. The result now follows from proposition 3.3. □

3.3 The space of geodesics

We will now describe the connection between the measures $(\nu_x)_{x\in X}$ and the homogeneous invariant measures for geodesic flows of compact manifolds with universal cover isometric to X.

Consider the homogeneous space G/M. The group M is the stabilizer in K of the unit vector tangent at o to the geodesic $t \mapsto a_t o$. As K acts transitively on the set of unit vectors in $\mathrm{T}_o X$, the unit tangent bundle of X identifies G-equivariantly with G/M and the geodesic flows reads as the action of A by right translations on G/M.

Set $\partial^2 X = \partial X \times \partial X - \{(\xi, \xi) | \xi \in \partial X\}$. By lemma 3.5, the map which assigns to a geodesic its limit points in $+\infty$ and in $-\infty$ induces a G-equivariant surjection onto $\partial^2 X$. In particular, G acts transitively on $\partial^2 X$ (of course, you can show this last point directly by proving that P acts transitively on $\partial X - \{\xi_0\}$). Let $\xi_0^\vee$ be the limit in $X \cup \partial X$ of $a_t o$ as t goes to $-\infty$ (that is $f_n\mathbb{K}$). Then MA both fixes ξ_0 and $\xi_0^\vee$ and, thus, the surjection $G/M \to \partial^2 X$ identifies with the natural map $G/M \to G/MA$. In other terms, G/M is the set of pointed oriented complete geodesics of X and $G/MA \equiv \partial^2 X$ is the set of oriented geodesics up to parameter translation.

As MA is an unimodular group, G preserves a measure on G/MA; this measure has Lebesgue class. For x in X, the measure $\nu_x \otimes \nu_x$ has Lebesgue class on $\partial^2 X$. Therefore it is equivalent to the invariant measure. In other terms, there exists a Borel function $f_x : \partial^2 X \to \mathbb{R}_+^*$ such that for (ξ, η) in $\partial^2 X$ and g in G, one gets $f_x(\xi, \eta) = e^{-\delta_X(b_\xi(gx,x)+b_\eta(gx,x))} f_x(g^{-1}\xi, g^{-1}\eta)$. We shall make such a function explicit; this can be done by a geometric way as we shall see later. In this paragraph, we will use an algebraic method.

Let x be in X, a point which we view as a vector line in $\mathbb{K}^{n+1}$. We denote by $\langle ., . \rangle_x$ the hermitian scalar product for which x and $x^\perp$ are orthogonal and such that $\langle ., . \rangle_x = \langle ., . \rangle$ on x and $\langle ., . \rangle_x = -\langle ., . \rangle$ on $x^\perp$ and we let $\|.\|_x$ be the hermitian norm associated to $\langle ., . \rangle_x$. For g in G and v in $\mathbb{K}^{n+1}$, we have $\|v\|_{gx} = \|g^{-1}v\|_x$. In particular, $\|.\|_x$ is invariant by the stabilizer of x in G. Let us give a way of computing the Busemann function:

Lemma 3.10. *Let x, y be in X and ξ be in ∂X. Then, if v is a non-zero element of the vector line ξ, we get*

$$b_\xi(x, y) = \log \frac{\|v\|_x}{\|v\|_y}.$$

Proof. As usual, it suffices to check this formula for $\xi = \xi_0$, $x = o$, $y = a_s n o$, with s in $\mathbb{R}$ and n in N. Then we have $\|f_0\|_x = 1$, $\|f_0\|_y = e^s$ and, by lemma 3.8, $b_\xi(x, y) = s$. □

Let ξ and η be two different points of ∂X and let v and w be non-zero vectors in the vector lines ξ and η. By lemma 3.6, if $\xi \neq \eta$, the q-isotropic vector w cannot belong to $\xi^\perp$, that is we have $\langle v, w \rangle \neq 0$. If x is a point of X, we set

$$d_x(\xi, \eta) = \sqrt{\frac{|\langle v, w \rangle|}{\|v\|_x \|w\|_x}}.$$

Although we shall not use it, we prove incidentally the following

Lemma 3.11. *For any x in X, the function $d_x : \partial X \times \partial X \to \mathbb{R}_+$ is a distance.*

Proof. Symmetry is evident and, as pointed above, separation follows from lemma 3.6. Let us prove the triangle inequality. For this take ξ, η, ζ in ∂X and u, v, w non-zero vectors in ξ, η, ζ. Choose a vector a in x such that $q(a) = \|a\|_x^2 = 1$. Then we can write, for some α, β, γ in $\mathbb{K}$ and some u_0, v_0, w_0

in $x^\perp$, $u = a\alpha + u_0$, $v = a\beta + v_0$ and $w = a\gamma + w_0$. As u, v, w are q-isotropic vectors, we have $|\alpha| = \|u_0\|_x$, $|\beta| = \|v_0\|_x$,$|\gamma| = \|w_0\|_x$. By multiplying u, v, w by suitable elements of $\mathbb{K}$, we can suppose that $\|u_0\|_x = \|v_0\|_x = \|w_0\|_x = 1$, that $\langle u_0, w_0\rangle_x \in \mathbb{R}$ and that $\alpha = \beta = \gamma$. Then, we have:

$$\begin{aligned} d_x(\xi,\zeta) &= \sqrt{|\langle u, w\rangle|} = \sqrt{|1 - \langle u_0, w_0\rangle|} \\ &= \sqrt{\left|1 - \frac{1}{2}\left(\|u_0 - w_0\|_x^2 - \|u_0\|_x^2 - \|w_0\|_x^2\right)\right|} \\ &= \frac{1}{\sqrt{2}}\|u_0 - w_0\|_x \leq \frac{1}{\sqrt{2}}\|u_0 - v_0\|_x + \frac{1}{\sqrt{2}}\|v_0 - w_0\|_x . \end{aligned}$$

But

$$\|u_0 - v_0\|_x^2 = 2(1 - \mathrm{Re}(\langle u_0, w_0\rangle_x)) \leq 2\,|1 - \langle u_0, w_0\rangle_x| = 2\,|\langle u_0, v_0\rangle_x|$$

(where, for t in $\mathbb{K}$, $\mathrm{Re}(t)$ stands for $\frac{1}{2}(t + \bar{t})$). Therefore we have

$$\frac{1}{\sqrt{2}}\|u_0 - v_0\|_x \leq d_x(\xi,\eta) \text{ and, similarly, } \frac{1}{\sqrt{2}}\|v_0 - w_0\|_x \leq d_x(\eta,\zeta)$$

and the result follows. □

By lemma 3.10, if y is another point of X, we get

$$d_y(\xi,\eta) = e^{\frac{1}{2}(b_\xi(x,y) + b_\eta(x,y))} d_x(\xi,\eta).$$

Summarizing the discussion above, we have the

Proposition 3.12. *For x in X, the measure $d_x^{-2\delta_X}\nu_x \otimes \nu_x$ on $\partial^2 X$ doesn't depend on x. It's up to homothety the unique G-invariant measure on $\partial^2 X$.*

3.4 Geodesic flows

Suppose Γ is a discrete subgroup of G. If Γ doesn't contain any element of finite order, then $\Gamma\backslash X$ is a manifold. With its quotient metric, it is a locally symmetric space. The unit tangent bundle of this manifold is $\Gamma\backslash G/M$ and the geodesic flow on this space reads as the action of A by right translation.

Let μ be a Radon measure on G/MA. For every continuous function φ with compact support on G/M, set

$$\int_{G/M} \varphi \mathrm{d}\tilde{\mu} = \int_{G/MA}\int_{\mathbb{R}} \varphi(ga_t)\mathrm{d}t\mathrm{d}\mu(gMA).$$

The correspondence $\mu \mapsto \tilde{\mu}$ establishes a G-equivariant bijection between the set of Radon measures on G/MA and the set of right A-invariant Radon measures on G/M. In the same way, given a measure λ on $\Gamma\backslash G/M$, define a measure $\tilde{\lambda}$ on G/M by setting, for any continuous function φ with compact support on G/M,

$$\int_{G/M} \varphi \mathrm{d}\tilde{\lambda} = \int_{\Gamma\backslash G/M} \sum_{\gamma\in\Gamma} \varphi(g\gamma)\mathrm{d}\lambda(g\Gamma).$$

This defines a G-equivariant bijection between the set of Radon measures on $\Gamma\backslash G/M$ and the set of left Γ-invariant Radon measures on G/M. Thus, there is a natural one-to-one correspondence between the set of left Γ-invariant Radon measures on G/M and the set of invariant measures for the geodesic flow on $\Gamma\backslash G/M$.

We shall now concentrate on the case where all measures we consider are in fact G-invariant. A discrete subgroup of G is said to be a lattice if the G-invariant measure on $\Gamma\backslash G$ is finite. A cocompact subgroup is a lattice, but there exists both cocompact and non-cocompact lattices. We have the following first step in the ergodic theory of geodesic flows of finite volume locally symmetric spaces:

Theorem 3.13. *Let Γ be a lattice in G and normalize the Haar measure of G in such a way that the associated measure μ on $\Gamma\backslash G/M$ has total mass one. Then the action of A on this measure is mixing, that is*

$$\forall\varphi,\psi \in \mathrm{L}^2(\mu) \quad \int_{\Gamma\backslash G/M} \varphi(x)\psi(xa_t)\mathrm{d}\mu(x) \xrightarrow[|t|\to\infty]{} \int_{\Gamma\backslash G/M} \varphi\mathrm{d}\mu \int_{\Gamma\backslash G/M} \psi\mathrm{d}\mu.$$

Proof. This is a consequence of the Howe-Moore theorem about unitary representations of G, for which we refer, for example, to [2]. □

4 Conformal densities

We now come to the core of our subject, that is the construction of conformal densities on general manifolds with negative curvature. We begin by recalling general notions about these spaces.

4.1 Manifolds with negative curvature

The reader may find a more detailed exposition in [5] and [8].

Let X be a complete simply connected Riemannian manifold with nonpositive sectional curvature, that is, as a function on the Grassmannian bundle $\mathcal{G}^2X$ of tangent 2-planes of X, the sectional curvature is everywhere nonpositive. Then for every point x of X, the exponential map $\mathrm{Exp}_x : \mathrm{T}_xX \to X$ is a diffeomorphism. The space X is thus diffeomorphic to an open ball. There exists a compactification $\bar{X} = X \cup \partial X$ which extends this diffeomorphism to an homeomorphism with the closed ball. To describe it, consider the set of geodesic rays $r : [0, \infty[\to X$. Two rays r_1 and r_2 are said to be *asymptotic* if and only if the function $t \mapsto d(r_1(t), r_2(t))$ is bounded. The boundary ∂X is the set of equivalence classes of geodesic rays. If r is a geodesic ray, we denote by $r(\infty)$ its equivalence class. If ξ lies in ∂X and x in X there exists exactly one geodesic ray with origin x such that $r(\infty) = \xi$. We therefore put on $X \cup \partial X$ the topology inherited from the sphere compactification of T_xX: it doesn't depend on the base point x. If $\sigma : \mathbb{R} \to X$ is a complete geodesic, it has to limit points in ∂X which we denote by $\sigma(+\infty)$ and $\sigma(-\infty)$.

For x, y, z in X, set $b_z(x, y) = d(x, z) - d(y, z)$. Then, for any ξ in ∂X $b_z(x, y)$ has a limit as z goes to ξ. We still denote this limit by $b_\xi(x, y)$. The function $b : \bar{X} \times X \times X \to \mathbb{R}$ is continuous. We call it the *Busemann function* of X. For x, y, z in X and ξ in ∂X, we have $b_\xi(x, z) = b_\xi(x, y) + b_\xi(y, z)$. If σ is a complete unit speed geodesic such that $\sigma(+\infty) = \xi$, for any s, t in $\mathbb{R}$, one has $b_\xi(\sigma(s), \sigma(t)) = t - s$. For x in X and ξ in ∂X, the *horosphere* with center ξ based at x is the set of y in X such that $b_\xi(x, y) = 0$.

These results are based on the fact that the triangles of X are finer than the ones of Euclidean space: let x, y, z be points of X and let x_0, y_0, z_0 be points of $\mathbb{R}^2$, equipped with its canonical Euclidean structure, such that $d(x, y) = d(x_0, y_0)$, $d(x, z) = d(x_0, z_0)$ and $d(y, z) = d(y_0, z_0)$. For s, t in $[0, 1]$, let u (resp. v) be the point of the unique geodesic joining x to y (resp. z) such that $d(x, u) = sd(x, y)$ (resp. $d(x, v) = td(x, z)$) and let $u_0 = (1 - s)x_0 + sy_0$ and $v_0 = (1 - t)x_0 + tz_0$ (see figure 5). Then we have $d(u, v) \leq d(u_0, v_0)$.

Example 4.1. If X is $\mathbb{R}^n$ equipped with the Euclidean structure associated with the canonical scalar product $\langle ., . \rangle$, the boundary ∂X naturally identifies with the unit sphere $\mathbb{S}^{n-1}$. If u is a unit vector and x and y are two points of $\mathbb{R}^n$, we have $b_u(x, y) = \langle u, y - x \rangle$. If u and v are unit vectors, there exists a complete geodesic with limit points u and v if and only if $v = -u$.

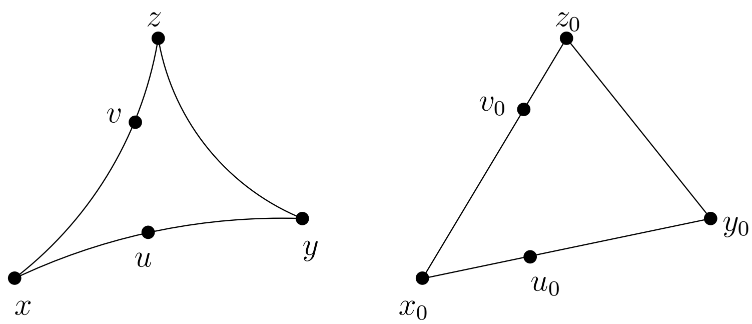

Figure 5: Comparison of distances

Suppose now the sectional curvature of X is bounded above by some negative constant $-c$ for some $c > 0$. By normalizing the metric, we can suppose $c = 1$. The triangles of X are now finer than those of real hyperbolic plane. For any two points $\xi \neq \eta$, there exists an unique complete geodesic with limit points ξ and η.

Let $\xi \neq \eta$ be in the boundary and let x be in X. The Gromov product $(\xi|\eta)_x$ of ξ and η viewed from x is the quantity $\frac{1}{2}(b_\xi(x,y)+b_\eta(x,y))$ where y is any point of the geodesic with limit points ξ and η; it does not depend on y. The map $d_x = e^{-(\cdot|\cdot)_x}$ is a distance on ∂X (with the convention $d_x(\xi,\xi) = 0$, of course). It clearly satisfies

$$\forall x, y \in X \quad \forall \xi, \eta \in \partial X \quad d_y(\xi,\eta) = e^{\frac{1}{2}(b_\xi(x,y)+b_\eta(x,y))} d_x(\xi,\eta)$$

(compare with paragraph 3.3). In particular, if g is an isometry of X, its action extends to the boundary and, for x in X and ξ, η in ∂X, one has $d_x(g\xi, g\eta) = d_{g^{-1}x}(\xi,\eta)$ so that $d_x(g\xi, g\eta) \leq e^{d(x,gx)} d_x(\xi,\eta)$.

The compact-open topology makes the group of isometries of X a separable, locally compact and second countable topological group (see paragraph 2.1). Let us recall the usual classification of its elements. An isometry is said to be elliptic if it fixes a point in X. It is then contained in a compact group of isometries. An isometry is said to be parabolic if it fixes exactly one point in ∂X. It then stabilizes every horosphere centered at its fixed point. Finally, a non-elliptic isometry is said to be hyperbolic if it fixes two points in ∂X.

Remark 4.1. All the objects introduced above come from CAT(-1)-

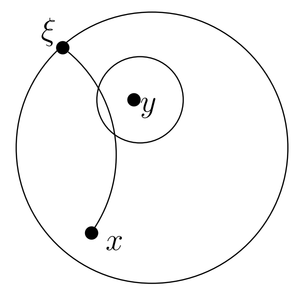

Figure 6: Shadow

geometry and the theory developed below can in fact be extended to the study of isometric actions of discrete groups on CAT(-1)-spaces (and especially on trees). The reader may refer to [4] and [14].

4.2 Shadows and isometries

Let always X be complete simply connected with curvature ≤ -1. If x and y are points of X and r is a positive real number, we define the *shadow* $\mathcal{O}_r(x,y)$ to be the set of ξ in ∂X such that the geodesic ray issued from x with limit point ξ hits the closed ball of center y with radius r.

We shall use the following

Lemma 4.1. *Let x, y be in X and $r > 0$. For ξ in $\mathcal{O}_r(x,y)$, one has*

$$d(x,y) - 2r \leq b_\xi(x,y) \leq d(x,y).$$

Proof. Let $t \mapsto x_t$ be the geodesic ray such that $x_0 = x$ and $x_\infty = \xi$ and let z be a point of that geodesic ray being at distance $\leq r$ to y. For $t \geq 0$, we have

$$\begin{aligned} d(y,x_t) \leq d(y,z) + d(z,x_t) &= d(y,z) + d(x,x_t) - d(x,z) \\ &\leq 2d(y,z) + d(x,x_t) - d(x,y) \leq 2r + d(x,x_t) - d(x,y) \end{aligned}$$

and thus $b_\xi(x,y) = \lim_{t\to\infty} d(x,x_t) - d(y,x_t) \geq d(x,y) - 2r$. The lemma follows, the other inequality being obvious. $\square$

Note that shadows may in some sense be very large:

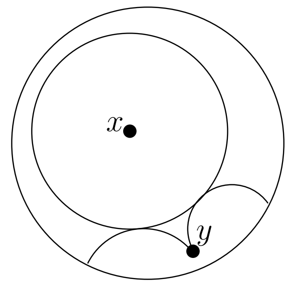

Figure 7: Large shadow

Lemma 4.2. *Let x be in X. For every $y \neq x$ let η_y be the limit point of the geodesic going from x to y Then as r goes to ∞, one has*

$$\sup_{\substack{y\neq x \\ \xi\notin\mathcal{O}_r(y,x)}} d_x(\xi,\eta_y) \xrightarrow[r\to\infty]{} 0.$$

Proof. Suppose to the contrary there are $\varepsilon > 0$, $r_n \to \infty$, y_n in X and ξ_n in $X - \mathcal{O}_{r_n}(y_n, x)$ such that $d_x(\xi, \eta_{y_n}) \geq \varepsilon$ for each n. Then after eventually choosing a subsequence, we can suppose that, for some ξ and η in ∂X, one has $\xi_n \to \xi$ and $y_n \to \eta$. We then have $\eta_{y_n} \to \eta$ and $d_x(\xi, \eta) \geq \varepsilon$. Therefore there exists a ball with center x that hits the geodesic line from η to ξ, what contradicts the fact that $r_n \to \infty$. □

Shadows will allows us to find pieces of the boundary where isometries contract the distances:

Lemma 4.3. *Let g be an isometry of X, x a point and $r > 0$. Then for ξ, η in $\mathcal{O}_r(g^{-1}x, x)$, one has*

$$d_x(g\xi, g\eta) \leq e^{2r-d(x,gx)} d_x(\xi, \eta).$$

Proof. We have

$$d_x(g\xi, g\eta) = d_{g^{-1}x}(\xi, \eta) = e^{\frac{1}{2}(b_\xi(x,g^{-1}x)+b_\eta(x,g^{-1}x))} d_x(\xi, \eta).$$

By lemma 4.1, we have

$$b_\xi(g^{-1}x, x) \geq d(x, gx) - 2r \text{ and } b_\eta(g^{-1}x, x) \geq d(x, gx) - 2r,$$

what implies the result. □

Let us use this lemma to give more details on hyperbolic isometries:

Proposition 4.4. *Let g be a hyperbolic isometry of X, $\xi \neq \eta$ two fixed points of g in ∂X and x a point of the geodesic D joining ξ to η. Suppose that gx lies between x and ξ. Then $l(g) = d(x, gx) > 0$ doesn't depend on x on D and for any y in X, one has $l(g) = b_\xi(y, gy) \leq d(y, gy)$. For any $\zeta \neq \eta$ in ∂X, one has $d_x(g^n\zeta, \xi) = O(e^{-nl(g)})$, uniformly for ζ bounded away from η. In particular, ξ and η are the unique fixed points of g in ∂X.*

The quantity $l(g)$ is called the *translation length* of g. In the sequel, we shall denote by g^+ and g^- the attractive and repulsive fixed points of g.

Proof. As g fixes ξ and η, it stabilizes D and induces a translation on it. Since we assumed g were not elliptic, this translation is not trivial and $l(g)$ doesn't depend on x and is not trivial. Let y be in X and z be the unique point of D such that $b_\xi(y, z) = 0$. Then one has $b_\xi(y, gy) = b_\xi(y, z) + b_\xi(z, gz) + b_\xi(gz, gy)$. As g fixes ξ, $b_\xi(gz, gy) = b_\xi(z, y)$ and thus $b_\xi(y, gy) = b_\xi(z, gz) = d(z, gz) = l(g)$. Finally let r be a positive number. Then, by lemma 4.3, for every ζ in $\mathcal{O}_r(g^{-1}x, x)$, one has $d_x(g\zeta, \xi) \leq e^{2r-l(g)} d_x(\zeta, \xi)$. The result follows now by lemma 4.2. $\square$

4.3 Groups of isometries

Let Γ be a discrete group of isometries of X. We will say that Γ is *non-elementary* if it does not stabilize any finite subset of $X \cup \partial X$. If Γ is such a subgroup, its limit set Λ_Γ is the set of limit points of Γx in ∂X where x is any point of X (by definition of the boundary, this set doesn't depend on x). The exponent of growth of Γ is the exponent of convergence of the Dirichlet series

$$\sum_{\gamma \in \Gamma} e^{-sd(x, \gamma x)} \quad (s \in \mathbb{R}),$$

that is the quantity

$$\limsup_{r \to \infty} \frac{1}{r} \log(\mathrm{card}\{\gamma \in \Gamma | d(x, \gamma x) \leq r\}).$$

We will assume this quantity to be finite. The next lemma shows that it is the case in most interesting examples. Let m denote the Riemannian volume of X and δ_X its volume entropy

$$\delta_X = \limsup_{r \to \infty} \frac{1}{r} \log(m(B(x, r))) < \infty.$$

Lemma 4.5. *Assume $\delta_X < \infty$. Let Γ be a discrete group of isometries of X. Then one has $\delta_\Gamma \leq \delta_X < \infty$. If Γ is cocompact, then $\delta_\Gamma = \delta_X$ and $\Lambda_\Gamma = \partial X$.*

For symmetric spaces, δ_X is finite and is the number appearing in section 3. More generally, δ_X is finite, if the sectional curvature of X is negatively pinched, that is if it lies between two negative constants. It is the case when X possesses a cocompact group of isometries.

Proof. As Γ is discrete, there exists a real number $s > 0$ such that, for every γ in Γ, $B(\gamma x, s) \cap B(x, s) \neq \emptyset \Rightarrow \gamma x = x$. Let n be the number of elements of Γ that fix x. For $r > 0$, we have

$$\operatorname{card}\{\gamma \in \Gamma | d(x, \gamma x) \leq r\} m(B(x, s)) \leq n m(B(x, r + s))$$

and therefore $\delta_\Gamma \leq \delta_X$. Suppose now Γ is cocompact. There exists $s > 0$ such that $X = \bigcup_{\gamma \in \Gamma} B(\gamma x, s)$. Thus, for $r \geq 0$, we have

$$B(x, r) \subset \bigcup_{\substack{\gamma \in \Gamma \\ d(x, \gamma x) \leq r + s}} B(\gamma x, s)$$

and

$$m(B(x, r)) \leq \operatorname{card}\{\gamma \in \Gamma | d(x, \gamma x) \leq r + s\} m(B(x, s))$$

so that $\delta_X \leq \delta_\Gamma$. Finally, let ξ be a point of ∂X and (x_n) a sequence of points of X converging to ξ. There exists (γ_n) in Γ such that, for each n, $d(x_n, \gamma_n x) \leq s$. Therefore $\gamma_n x \to \xi$ and ξ belongs to Λ_X. $\square$

Let us see how to construct non-elementary discrete groups of isometries. This is the classical example of Schottky groups.

Lemma 4.6. *Let g and h be two hyperbolic isometries of X with no common fixed point and let x in X. Then, after eventually having replaced g and h by powers of themselves, the group Γ of isometries generated by g and h is free and discrete and there exists $0 < k < \inf(d(x, gx), d(x, hx))$ such that, for γ in Γ, if $\gamma = g_1 \dots g_n$ is the decomposition of γ as a reduced word in g and h, one has*

$$d(x, \gamma x) \geq d(x, g_1 x) + \ldots + d(x, g_n x) - nk.$$

In particular $0 < \delta_\Gamma < \infty$.

Proof. Fix $\varepsilon > 0$ such that the d_x-balls of radius ε centered at the fixed points of g and h are all disjoint and their union does not cover ∂X. By proposition 4.4, $g^{-n}x \xrightarrow[n\to\infty]{} g^-$ so that, after having replaced g by a power, by lemma 4.2, we can find an $r > 0$ such that $\mathcal{O}_r(g^{-1}x, x)$ contains all points of $\partial X - B_x(g^-, \varepsilon)$. Then, by lemma 4.3, for ξ and η in this shadow, we have $d_x(g\xi, g\eta) \leq e^{2r-d(x,gx)} d_x(\xi, \eta) \leq e^{2r-l(g)} d_x(\xi, \eta)$. After again having replaced g by some power, we can suppose $l(g) > 2r$ and $g(\partial X - B_x(g^-, \varepsilon)) \subset B_x(g^+, \varepsilon)$. Doing the same job for g^{-1}, h and h^{-1}, we finally get on one hand $\partial X - B_x(g^+, \varepsilon) \subset \mathcal{O}_r(gx, x)$, $\partial X - B_x(h^-, \varepsilon) \subset \mathcal{O}_r(h^{-1}x, x)$ and $\partial X - B_x(g^+, \varepsilon) \subset \mathcal{O}_r(gx, x)$ and on the other hand $g^{-1}(\partial X - B_x(g^+, \varepsilon)) \subset B_x(g^-, \varepsilon)$, $h(\partial X - B_x(h^-, \varepsilon)) \subset B_x(h^+, \varepsilon)$ and $h^{-1}(\partial X - B_x(h^+, \varepsilon)) \subset B_x(h^-, \varepsilon)$.

Let now $\gamma = g_1 \dots g_n$ be a reduced word in g and h, that is each g_i is g, h, g^{-1} or h^{-1} and $g_{i+1} \neq g_i^{-1}$. We have to show that γ, as an isometry of X, is far away from the identity. Take ξ in ∂X which does not belong to the union of the four balls constructed above. Then $g_n\xi$ belongs to the ball centered at the attractive fixed point of g_n and, by induction, $\gamma\xi$ belongs to the ball centered at the attractive fixed point of g_1, what implies the result.

In the preceding construction, for each $1 \leq i \leq n$, we have $g_{i+1} \dots g_n\xi \in \mathcal{O}_r(g_i^{-1}x, x)$ so that, by lemma 4.1, $b_{g_{i+1}\cdots g_n\xi}(g_i^{-1}x, x) \geq d(g_i^{-1}x, x) - 2r = d(x, g_ix) - 2r$. Therefore we have

$$\begin{aligned} d(x, \gamma x) = d(\gamma^{-1}x, x) \geq b_\xi(\gamma^{-1}x, x) &= \sum_{i=1}^{n} b_\xi(g_n^{-1} \dots g_i^{-1}x, g_n^{-1} \dots g_{i+1}^{-1}x) \\ &= \sum_{i=1}^{n} b_{g_{i+1}\cdots g_n\xi}(g_i^{-1}x, x) \\ &\geq \sum_{i=1}^{n} d(x, g_ix) - 2nr, \end{aligned}$$

what should be proved. The estimates on δ_Γ now follow from exponential growth of the free group. $\square$

We can now say something about general discrete subgroups:

Proposition 4.7. *Let Γ be a non-elementary discrete group of isometries of X. Then Γ contains hyperbolic isometries. The set Λ_Γ is the closure of the set of fixed points of hyperbolic elements of Γ and is the smallest nonempty closed Γ-invariant subset of ∂X. The exponent δ_Γ is positive.*

Proof. Let x be a point of X, ξ a point of ∂X and (γ_n) be a sequence of elements of Γ such that $\gamma_n x \to \xi$. For each n, let ξ_n be the limit point of the geodesic ray joining x to $\gamma_n x$ and η_n the limit point of the ray joining x to $\gamma_n^{-1}x$. Then, $\xi_n \to \xi$ and after extracting a subsequence, we can suppose that $\eta_n \to \eta$ for some η. As Γ is non-elementary, there exists f in Γ such that $d_x(\xi, f\eta) > 0$ and after replacing (γ_n) by $(\gamma_n f^{-1})$, we can suppose that $\eta \neq \xi$. Choose $0 < \varepsilon \leq \frac{1}{5} d_x(\xi, \eta)$. By lemma 4.2, there exists an $r > 0$ such that, for any $y \neq x$ in X, $\partial X - \mathcal{O}_r(y, x)$ is contained in the d_x-ball of radius ε with center the limit point of the geodesic ray joining x to y. Then, for sufficiently large n, we have $\partial X - \mathcal{O}_r(\gamma_n^{-1}x, x) \subset B_x(\eta, 2\varepsilon)$ and $\partial X - \mathcal{O}_r(\gamma_n x, x) \subset B_x(\xi, 2\varepsilon)$. What's more, by lemma 4.3, γ_n is $e^{2r-d(x,\gamma_n x)}$-Lipschitz on $\mathcal{O}_r(\gamma_n^{-1}x, x)$. As $\gamma_n \mathcal{O}_r(\gamma_n^{-1}x, x) = \mathcal{O}_r(x, \gamma_n x) \ni \xi_n$, for n sufficiently large, γ_n possesses an attractive fixed point in $B_x(\xi_n, \varepsilon) \subset B_x(\xi, 2\varepsilon)$. In the same way, γ_n^{-1} possesses an attractive fixed point in $B_x(\eta, 2\varepsilon)$. Thus γ_n is hyperbolic and $d_x(\gamma_n^+, \xi) \leq \varepsilon$, what should be proved.

Let now F be a closed Γ-invariant nonempty set in ∂X and let γ be a hyperbolic element of Γ. Let ξ be an element of F. Then there exists f in Γ such that $f\xi \neq \gamma^-$. We then have $\gamma^n f\xi \to \gamma^+$, thus γ^+ belongs to F. As this is true for any hyperbolic isometry γ in Γ, we have $\Lambda_\Gamma \subset F$.

Finally, let γ be an hyperbolic element of Γ. There exists an element f of Γ such that $f\gamma^+ \neq \gamma^+$, $f\gamma^- \neq \gamma^+$, $f\gamma^+ \neq \gamma^-$ and $f\gamma^- \neq \gamma^-$. In other words, the isometries $g = f\gamma f^{-1}$ and $h = \gamma$ satisfy the hypothesis of lemma 4.6. Therefore, by this lemma, Γ contains a subgroup Γ_0 with $\delta_{\Gamma_0} > 0$. Hence we have $\delta_\Gamma > 0$. $\square$

4.4 Patterson construction

Let always Γ be a (non-elementary) discrete group of isometries of X and let β be a real number. A Γ-*conformal density* of dimension β is a map $x \mapsto \nu_x$ from X to the set of Radon measures on ∂X which is Γ-equivariant, that is $\gamma_* \nu_x = \nu_{\gamma x}$ for γ in Γ and x in X, and such that, for each x, y in X, ν_y and ν_x are equivalent and we have

$$\forall \xi \in \partial X \quad \frac{\mathrm{d}\nu_y}{\mathrm{d}\nu_x}(\xi) = e^{-\beta b_\xi(y,x)}.$$

Fixing a base point x in X, the data of a conformal density of dimension β is equivalent to the one of a Radon measure ν_x such that, for each γ in Γ, one has $\gamma_* \nu_x = e^{-\beta b_\cdot(\gamma x, x)}$.

Example 4.2. From proposition 3.9, we know that if X is a hyperbolic space, there exists a conformal density of dimension δ_X which is equivariant under the full group of isometries.

Conformal densities have been originally constructed by Patterson for isometries of the real hyperbolic plane but this construction extends to our general situation:

Theorem 4.8. *Let Γ be a non-elementary discrete group of isometries of X. Then there exists a Γ-conformal density of dimension δ_Γ with support Λ_Γ.*

Proof. Fix x in X and suppose first that $\sum_{\gamma\in\Gamma} e^{-\delta_\Gamma d(x,\gamma x)} = \infty$. Set, for $s > \delta_\Gamma$, $\Phi(s) = \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)}$ and

$$\nu_s = \frac{1}{\Phi(s)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)} D_{\gamma x}$$

where, for y in X, D_y is the Dirac measure at y. Then, for $s > \delta_\Gamma$, ν_s may be seen as a probability measure on the compact space $X \cup \partial X$. Therefore, there exists a sequence $s_n \to \delta_\Gamma$ such that ν_{s_n} converges weakly to some probability measure ν. As $\Phi(s_n) \to \infty$, for each $r \geq 0$, $\nu_{s_n}(B(x,r)) \to 0$ and ν is concentrated on ∂X. As the support of each ν_s is $\Gamma x \cup \Lambda_\Gamma$, ν has support $\subset \Lambda_\Gamma$. Let now θ be an element of Γ. For $s > \delta_\Gamma$, we have

$$\begin{aligned}
\theta_*\nu_s &= \frac{1}{\Phi(s)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)} D_{\theta\gamma x} \\
&= \frac{1}{\Phi(s)} \sum_{\gamma\in\Gamma} e^{-sd(x,\theta^{-1}\gamma x)} D_{\gamma x} \\
&= \frac{1}{\Phi(s)} \sum_{\gamma\in\Gamma} e^{-s(d(\theta x,\gamma x)-d(x,\gamma x))} e^{-sd(x,\gamma x)} D_{\gamma x}.
\end{aligned}$$

Consider the function $\varphi : X \cup \partial X \to \mathbb{R}$ such that $\varphi(y) = d(\theta x, y) - d(x, y)$ for y in X and $\varphi(\xi) = b_\xi(\theta x, x)$ for ξ in ∂X. This function is continuous and for each $s > \delta_\Gamma$, we have $\theta_*\nu_s = e^{-s\varphi}\nu_s$. Therefore $\theta_*\nu = e^{-\delta_\Gamma b_{.}(\theta x,x)}$ and we have a Γ-conformal density of dimension δ_Γ, by the remark above. Finally, as ν is Γ-quasi-invariant its support is Γ-invariant. As this support is contained in Λ_Γ, it is Λ_Γ by proposition 4.7.

Let now $\sum_{\gamma\in\Gamma} e^{-\delta_\Gamma d(x,\gamma x)}$ be $< \infty$. Then we can make the same construction by setting $\Phi(s) = \sum_{\gamma\in\Gamma} h(d(x,\gamma x))e^{-sd(x,\gamma x)}$ and

$$\nu_s = \frac{1}{\Phi(s)} \sum_{\gamma\in\Gamma} h(d(x,\gamma x))e^{-sd(x,\gamma x)} D_{\gamma x},$$

where h is the function provided by the following lemma applied to the measure $\lambda = \sum_{\gamma\in\Gamma} D_{d(x,\gamma x)}$ on $\mathbb{R}_+$. ☐

Lemma 4.9. *Let λ be a Radon measure on $\mathbb{R}_+$, such that the Laplace transform of λ*

$$\int_{\mathbb{R}_+} e^{-st}\mathrm{d}\lambda(t) \quad (s \in \mathbb{R})$$

has critical exponent $\delta \in \mathbb{R}$. Then there exists a nondecreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+^*$ with the following properties:

(i) one has

$$\int_{\mathbb{R}_+} h(t)e^{-\delta t}\mathrm{d}\lambda(t) = \infty.$$

(ii) for every $\varepsilon > 0$, there exists $t_0 \geq 0$ such that, for any $u \geq 0$ and $t \geq t_0$, one has

$$h(u+t) \leq e^{\varepsilon u}h(t).$$

In particular, the Laplace transform of $h\lambda$ has critical exponent δ.

Proof. Choose a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers, going to 0. By induction, we will construct an increasing sequence of real numbers $(t_n)_{n\in\mathbb{N}}$ with $t_0 = 0$ and a function $h : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that, for n in $\mathbb{N}$, the logarithm of h will be affine with slope ε_n on $[t_n, t_{n+1}]$ and that

$$\int_{[t_n,t_{n+1}[} h(t)e^{-\delta t}\mathrm{d}\lambda(t) \geq 1.$$

Such a function will clearly satisfy the conclusions of the lemma.

Let therefore n be a nonnegative integer and suppose $t_0 < \ldots < t_n$ and $h : [t_0, t_n] \to \mathbb{R}_+^*$ as above are constructed. Let $h_n : [t_n, \infty[\to \mathbb{R}_+^*$ the function which is logarithmically affine with slope ε_n and such that $h_n(t_n) = h(t_n)$. As

$$\int_{\mathbb{R}_+} e^{-(\delta-\varepsilon_n)t}\mathrm{d}\lambda(t) = \infty,$$

we have

$$\int_{[t_n,\infty[} h_n(t)e^{-\delta t}\mathrm{d}\lambda(t) = \infty$$

and, thus, there exists a real number $t_{n+1} > t_n$ such that

$$\int_{[t_n,t_{n+1}[} h_n(t)e^{-\delta t}\mathrm{d}\lambda(t) \geq 1.$$

We then set, for t in $]t_n, t_{n+1}]$, $h(t) = h_n(t)$ and this construction may be pursued by induction. $\square$

4.5 Sullivan's shadow lemma

We shall now emphasize the connection between growth of groups and densities. The key point there is the following lemma due to Sullivan, which allows to estimate the measure of certain subsets of the boundary with respect to conformal densities:

Lemma 4.10. *Let Γ be a non-elementary group of isometries of X, ν a Γ-conformal density of dimension β and x a point of X. Then there exists $r_0 > 0$ such that, for every $r \geq r_0$, there exists $C > 0$ such that, for any γ in Γ, one has*

$$\frac{1}{C}e^{-\beta d(x,\gamma x)} \leq \nu_x(\mathcal{O}_r(x,\gamma x)) \leq Ce^{-\beta d(x,\gamma x)}.$$

Proof. As Γ is non-elementary, the support of ν_x is not a point and, for every ξ in ∂X, one has $\nu_x(\{\xi\}) < \nu_x(\partial X)$. Therefore, by compacity of the boundary, there exists $\epsilon > 0$ such that, for any ξ in ∂X, $\nu_x(\partial X - B_x(\xi,\varepsilon)) \geq \varepsilon$. By lemma 4.2, there exists $r_0 > 0$ such that, for $r \geq r_0$, for y in X, one has $\partial X - B_x(\xi,\varepsilon) \subset \mathcal{O}_r(y,x)$. Let γ be in Γ. We have

$$\begin{aligned}\nu_x(\mathcal{O}_r(x,\gamma x)) = \nu_x(\gamma\mathcal{O}_r(\gamma^{-1}x,x)) &= \nu_{\gamma^{-1}x}(\mathcal{O}_r(\gamma^{-1}x,x)) \\ &= \int_{\mathcal{O}_r(\gamma^{-1}x,x)} e^{-\beta b_\xi(\gamma^{-1}x,x)}\mathrm{d}\nu_x(\xi).\end{aligned}$$

Thus, by lemma 4.1, if $\beta \geq 0$,

$$\varepsilon e^{-\beta d(x,\gamma x)} \leq \nu_x(\mathcal{O}_r(x,\gamma x)) \leq \nu_x(\partial X)e^{2\beta r}e^{-\beta d(x,\gamma x)},$$

and the lemma follows, the case $\beta < 0$ being handled similarly (and being empty, as we shall see below !) $\square$

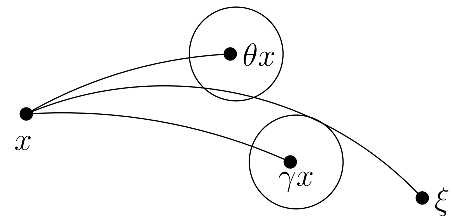

Figure 8: Shadow lemma and covering

From this lemma we deduce the

Theorem 4.11. *Let Γ be a discrete group of isometries of X. If there exists a Γ-conformal density of dimension β, then one has $\beta \geq \delta_\Gamma$.*

Proof. Let x be a point of X and $r > 0$ and $C > 0$ as in lemma 4.10. For n in $\mathbb{N}$, let Γ_n be the set of elements γ in Γ such that $n \leq d(x, \gamma x) < n+1$ and a_n the cardinal of Γ_n. Then we have $\delta_\Gamma = \limsup_{n\to\infty} \frac{1}{n} \log a_n$. Let γ and θ be in Γ_n and suppose $\mathcal{O}_r(x, \gamma x) \cap \mathcal{O}_r(x, \theta x) \neq \emptyset$. Let ξ be in this set. Then the geodesic ray going from x to ξ contains a point y such that $d(y, \gamma x) \leq r$. As $n \leq d(x, \gamma x) \leq n + 1$, one has $n - r \leq d(x, y) \leq n + 1 + r$. In the same way, choose a point z on the geodesic ray from x to ξ such that $d(z, \theta x) \leq r$, and hence $n-r \leq d(x, z) \leq n+1+r$ (see figure 8). As y and z ly in the same geodesic ray, we have $d(y, z) \leq 1 + 2r$ and, therefore, $d(\gamma x, \theta x) \leq 1 + 4r$. In other words, if p is be the number of γ in Γ such that $d(x, \gamma x) \leq 1 + 4r$, for every ξ in ∂X, one has $\mathrm{card}\{\gamma \in \Gamma_n | \xi \in \mathcal{O}_r(x, \gamma x)\} \leq p$. Hence, if $\beta \geq 0$, for any n, we have

$$\begin{aligned}
\nu_x(\partial X) &\geq \nu_x\left(\bigcup_{\gamma\in\Gamma_n} \mathcal{O}_r(x, \gamma_n x)\right) \\
&\geq \frac{1}{p} \sum_{\gamma\in\Gamma_n} \nu_x(\mathcal{O}_r(x, \gamma_n x)) \\
&\geq \frac{1}{pC} \sum_{\gamma\in\Gamma_n} e^{-\beta d(x,\gamma_n x)} \geq \frac{e^{-\beta(n+1)}}{pC} a_n,
\end{aligned}$$

that is $a_n \leq pC\nu_x(\partial X)e^{\beta(n+1)}$. Therefore $\beta \geq \delta_\Gamma$. The case $\beta < 0$ is handled similarly and reveals to be empty. ∎

4.6 Geodesic flows

Conformal densities have two major applications. The first one is the investigation of the geodesic flow of the quotient orbifold $\Gamma\backslash X$, in analogy with what has been done for lattices in classical symmetric spaces at paragraphs 3.3 and 3.4.

As every geodesic in X has two limit points in ∂X, the set of oriented geodesics of X naturally identifies with $\partial^2 X = \partial X \times \partial X - \{(\xi, \xi) | \xi \in \partial X\}$. Let $\mathrm{T}^1 X$ be the unit tangent bundle of X. Then the action of the geodesic flow (g^t) in $\mathrm{T}^1 X$ is proper and the quotient of $\mathrm{T}^1 X$ by this action is the set of oriented geodesics $\partial^2 X$. As a consequence, there is a natural bijection between the set of invariant Radon measures for the geodesic flow in $\mathrm{T}^1 X$ and the set of Radon measure on $\partial^2 X$. As the action of the group of isometries of X on $\mathrm{T}^1 X$ commutes with the geodesic flow, the bijection above is equivariant under this action.

Let Γ be a non-elementary discrete group of isometries. If no non-trivial element of γ has a fixed point in X, that is if Γ doesn't contain any element of finite order, then $\Gamma\backslash X$ has a natural structure of Riemannian manifold. Its geodesic flow is the quotient action of (g^t) on $\Gamma\backslash \mathrm{T}^1 X$. In any case, this action exists. We still denote it by (g^t). The discussion above shows that there exist a natural bijection between the set of (g^t)-invariant Radon measures on $\Gamma\backslash \mathrm{T}^1 X$ and the set of Γ-invariant Radon measures on $\partial^2 X$.

In the same way, there is a bijection between the set of closed (g^t)-invariant subsets of $\Gamma\backslash \mathrm{T}^1 X$ and the set of closed Γ-invariant subsets of $\partial^2 X$. In particular, let $F_\Gamma = (\Lambda_\Gamma \times \Lambda_\Gamma) \cap \partial^2 X$ and let E_Γ be the corresponding (g^t)-invariant subset of $\Gamma\backslash \mathrm{T}^1 X$: E_Γ is the image of the set of unit vectors whose tangent geodesic has both its limit points in Λ_Γ. Let γ be a hyperbolic element of Γ. Then (γ^+, γ^-) belongs to F_Γ. As the geodesic going from γ^- to γ^+ is stable by Γ, the image of any of its unit tangent vectors in $\Gamma\backslash \mathrm{T}^1 X$ has closed orbit under the flow (g^t). Conversely, every closed orbit is obtained this way.

Lemma 4.12. *The set E_Γ is the closure of the union of closed (g^t)-orbits in $\Gamma\backslash \mathrm{T}^1 X$. Equivalently, the points of the form (γ^+, γ^-), where γ is a hyperbolic element of Γ, is dense in F_Γ.*

Proof. Let $\xi \neq \eta$ be elements of Λ_Γ. By proposition 4.7, there exists sequences of hyperbolic elements (γ_n) and (η_n) in Γ such that $\gamma_n^+ \to \xi$ and $\theta_n^- \to \eta$. After extracting subsequences, we can suppose that $\gamma_n^- \to \zeta$ and that $\theta_n^+ \to \varsigma$ for some ζ and ς in Λ_Γ. Let f be an element of Γ such that $f\varsigma \neq \zeta$. Let V and W be open neighborhoods of ξ and η that do not intersect. Then, for large n, the isometry $\rho_n = \gamma_n f \theta_n$ acts on $\partial X - W$ as a contraction with image $\subset V$, and ρ_n^{-1} acts on $\partial X - V$ as a contraction with image $\subset W$, that is ρ_n is hyperbolic and $\rho_n^+ \to \xi$ and $\rho_n^- \to \eta$, what should be proved. $\square$

Consider now a Γ-conformal density ν of dimension β and let x be a point of X. As Γ is non-elementary, ν_x is not a Dirac measure and therefore the restriction to $\partial^2 X$ of the measure $\nu_x \otimes \nu_x$ is non-trivial. On ∂X, the product of this measure by the function $d_x^{-2\beta}$ is Γ-invariant and does not depend on x. To this measure, we associate a (g^t)-invariant measure μ on $\Gamma \backslash \mathrm{T}^1 X$: we say that μ is the *Bowen-Margulis measure* associated to ν. Note that μ is not necessarily finite (compare with paragraph 3.3).

Let G be a locally compact topological group and let (E, λ) be a σ-finite measure space with a measure preserving Borel action of G. Choose a right Haar measure on G. A measurable subset A of E is said to be wandering if, for λ-allmost every x in A, the set $\{g \in G | gx \in A\}$ has finite Haar measure. There exists up to sets of measure 0 a unique partition $E = E_C \cup E_D$ into G-invariant measurable subsets such that E_D is a countable union of wandering sets and every wandering set is contained in E_D. The set E_D is called the dissipative part of E and E_C its conservative part. The system is said to be conservative (resp. dissipative) if $\lambda(E_D) = 0$ (resp. $\lambda(E_C) = 0$).

The following theorem has a long story. One of its ingredients is the original proof of the ergodicity of the geodesic flow of surfaces by Hopf. This method has been extended to the study of conformal densities by Sullivan. The general form we give here is due to Roblin in [14], where the reader may find its proof.

Theorem 4.13. *Let Γ be a non-elementary discrete group of isometries of X and let ν be a Γ-conformal density of dimension β with associated Bowen-Margulis measure μ. Fix an arbitrary point x of X. Then either one has simultaneously*

(i) $\sum_{\gamma \in \Gamma} e^{-\beta d(x, \gamma x)} < \infty$.

(ii) the action of Γ on $(\partial^2 X, \nu_x \otimes \nu_x)$ is dissipative.

(iii) the action of (g^t) on $(\Gamma \backslash \mathrm{T}^1 X, \mu)$ is dissipative.

or

(i) $\sum_{\gamma\in\Gamma} e^{-\beta d(x,\gamma x)} = \infty$.

(ii) the action of Γ *on* $(\partial^2 X, \nu_x \otimes \nu_x)$ *is conservative and ergodic.*

(iii) the action of (g^t) *on* $(\Gamma\backslash \mathrm{T}^1 X, \mu)$ *is conservative and ergodic.*

Note that if the second condition is true one has necessarily $\beta = \delta_\Gamma$ thanks to theorem 4.11.

A group Γ such that $\sum_{\gamma\in\Gamma} e^{-\delta_\Gamma d(x,\gamma x)} = \infty$ is said to be of divergence type. The groups constructed in lemma 4.6 can be showed to be of divergence type.

Remark 4.2. Let E be a countable set and $\left(\sum_{y\in E} p(x,y) D_y\right)_{x\in E}$ a family of transition probabilities on E (where D stands for Dirac measure). Then, if x is a point of E, the associated Markov chain starting at x is recurrent if and only if $\sum_{n=0}^{\infty} p^n(x,x) = \infty$, with

$$p^n(x,x) = \sum_{x_1,\ldots,x_{n-1}\in E} p(x,x_1)p(x_1,x_2),\ldots,p(x_{n-1},x)$$

for $n \geq 0$. This fact relies on the Borel-Cantelli lemma. A version of the Borel-Cantelli lemma appears in the proof of theorem 4.13 too.

Corollary 4.14. *Let* Γ *be a non-elementary discrete group of isometries of* X*. If* Γ *is of divergence type, there exists up to homothety a unique* Γ*-conformal density of dimension* δ_Γ*. It is concentrated on* Λ_Γ*.*

Proof. The existence is provided by theorem 4.8. For the uniqueness, consider a density ν of dimension δ_Γ.

Let us show that ν has no atoms. By theorem 4.13, the associated Bowen-Margulis measure μ is (g^t) conservative and therefore, if x is a point of X, and r and C are as in lemma 4.10, for ν_x-almost ξ in Λ_Γ, if $t \mapsto x_t$ is the geodesic ray joining x to ξ, there exists a sequence $t_n \to \infty$ and a sequence (γ_n) of elements of Γ such that $d(\gamma_n x, x_{t_n}) \leq r$. One then has $d(x, \gamma_n x) \to \infty$ and, for any n, $\xi \in \mathcal{O}_r(x, \gamma_n x)$. Therefore, for any n, $\nu_x(\{\xi\}) \leq C e^{-\delta_\Gamma d(x,\gamma_n x)}$, what implies $\nu_x(\{\xi\}) = 0$, what should be proved.

As ν_x has no atoms, the set $\{(\xi,\xi) | \xi \in \partial X\}$ has $\nu_x \otimes \nu_x$-measure 0 and the projection on the first component from $\partial^2 X$ onto ∂X maps $\nu_x \otimes \nu_x$ to ν_x.

In particular the action of Γ on $(\partial X, \nu_x)$ is ergodic. As this is true for any density of dimension δ_Γ and as a convex combination of two densities of the same dimension is again a density, there exists only one density of dimension δ_Γ. □

For cocompact groups, the measure μ is finite and therefore conservative by Poincaré recurrence theorem. We therefore have the

Corollary 4.15. *Suppose X possesses a discrete cocompact group of isometries. Then there exists up to homothety a unique map $x \mapsto \nu_x$ from X to the set of Radon measures on ∂X such that*

(i) for any x, y in X, ν_x and ν_y are equivalent and

$$\forall \xi \in \partial X \quad \frac{\mathrm{d}\nu_y}{\mathrm{d}\nu_x}(\xi) = e^{-\delta_X b_\xi(y,x)}.$$

(ii) for any isometry g of X one has $g_*\nu_x = \nu_{gx}$.

Proof. Let Γ be a discrete cocompact group of isometries. Uniqueness comes from the fact that such a map is a Γ-conformal density of dimension δ_X and that Γ is of divergence type. For the existence, consider a Γ-conformal density ν of dimension δ_X and fix a point x in X. Let G be the group of isometries of X. Then Γ is a discrete cocompact subgroup of G and, as G possesses such a subgroup, it is unimodular. Fix a Haar measure $\mathrm{d}g$ on G and still denote by $\mathrm{d}g$ the induced finite measure on G/Γ.

Let φ be a continuous function on ∂X. Consider the function $\psi(\varphi) : g \mapsto \int_{\partial X} e^{\delta_X b_{g\xi}(gx,x)} \varphi(gx) \mathrm{d}\nu_x(\xi)$. For γ in Γ, one has

$$\begin{aligned}
\psi(\varphi)(g\gamma) &= \int_{\partial X} e^{\delta_X b_{g\xi}(g\gamma x, x)} \varphi(g\xi) \mathrm{d}(\gamma_*\nu_x)(\xi) \\
&= \int_{\partial X} e^{\delta_X (b_{g\xi}(g\gamma x,x) - b_\xi(\gamma x, x))} \varphi(g\xi) \mathrm{d}\nu_x(\xi) \\
&= \int_{\partial X} e^{\delta_X (b_{g\xi}(g\gamma x,x) - b_{g\xi}(g\gamma x, gx))} \varphi(g\xi) \mathrm{d}\nu_x(\xi) = \psi(\varphi)(g).
\end{aligned}$$

We define a measure $\tilde{\nu}_x$ on ∂X by setting, for any continuous function φ on ∂X, $\int_{\partial X} \varphi \mathrm{d}\tilde{\nu}_x = \int_{G/\Gamma} \psi(\varphi)(g) \mathrm{d}g$. Let h be in G and let us compute $h_*\tilde{\nu}_x$.

For φ a continuous function on ∂X, for g in G, we have

$$\begin{aligned}\psi(\varphi \circ h)(g) &= \int_{\partial X} e^{\delta_X b_{g\xi}(gx,x)} \varphi(hgx) \mathrm{d}\nu_x(\xi) \\ &= \int_{\partial X} e^{\delta_X (b_{hg\xi}(hgx,x) - b_{hg\xi}(hx,x))} \varphi(hgx) \mathrm{d}\nu_x(\xi) \\ &= \psi\left(e^{-\delta_X b_\cdot(hx,x)}\varphi\right)(hg)\end{aligned}$$

and therefore

$$\begin{aligned}\int_{\partial X} \varphi \mathrm{d}(h_* \tilde{\nu}_x) &= \int_{G/\Gamma} \psi(\varphi \circ h)(g) \mathrm{d}g = \int_{G/\Gamma} \psi(e^{-\delta_X b_\cdot(hx,x)}\varphi)(hg)\mathrm{d}g \\ &= \int_{G/\Gamma} \psi(e^{-\delta_X b_\cdot(hx,x)}\varphi)(g)\mathrm{d}g = \int_{\partial X} e^{-\delta_X b_\xi(hx,x)} \varphi(\xi) \mathrm{d}\tilde{\nu}_x(\xi),\end{aligned}$$

what is to say $h_* \tilde{\nu}_x = e^{-\delta_X b_\cdot(hx,x)} \tilde{\nu}_x$. By setting, for y in X, $\tilde{\nu}_y = e^{-\delta_X b_\cdot(y,x)} \tilde{\nu}_x$, we get a map satisfying the properties required in the corollary. Note that, by uniqueness, we have in fact $\tilde{\nu} = \nu$! □

We finally may ask if the analogy with the homogeneous theory may be pursued and if we have an analogous result to theorem 3.13. If Γ is cocompact and has no finite order element, the geodesic flow of $\Gamma \backslash \mathrm{T}^1 X$ is an Anosov contact flow and the Bowen-Margulis measure associated to its unique conformal density of dimension δ_X is the measure of maximal entropy of this flow (this fact explains the terminology). It is therefore mixing. The reader may refer to [9] for the general notions involved in the discussion above.

To extend theorem 3.13, we however need a supplementary hypothesis on Γ: we say that Γ has *non-arithmetic spectrum* if the subgroup of $\mathbb{R}$ generated by the numbers $l(\gamma)$, where γ is an hyperbolic element of Γ, is dense. This may be shown to always hold if Γ is cocompact or if X is a symmetric space. Note that we don't have any example where we can show this to be false.

Suppose that for some Γ-conformal density the associated Bowen-Margulis measure has finite mass. By theorem 4.13 and Poincaré recurrence theorem, the dimension of the density is δ_Γ and, by corollary 4.14, it is the unique conformal density of dimension δ_Γ. We then say that the associated Bowen-Margulis measure is the Bowen-Margulis measure of Γ and that Γ has finite Bowen-Margulis measure.

We have the following theorem, due to Roblin ([14]) in this general form:

Theorem 4.16. *Let Γ be a non-elementary discrete group of isometries of X with non-arithmetic spectrum and finite Bowen-Margulis measure. Then this measure is mixing under the action of the geodesic flow.*

4.7 Conformal densities and potential theory

Conformal densities appear to be a powerful tool in some rigidity problems connected to the barycenter method. As an example, of such a connection, we give here a characterization of symmetric spaces due to Ledrappier [12] and Besson, Courtois and Gallot [3]. We suppose here the curvature of X is pinched.

Let Δ be the Laplace-Beltrami operator on X. A $\mathcal{C}^2$ function $\varphi : X \to \mathbb{C}$ is said to be harmonic if $\Delta\varphi = 0$. There exists an extension of theorem 1.1, which is due to Anderson and Schoen [1]:

Theorem 4.17. *There exists a map $x \mapsto \mu_x$ from X to the set of probability measures on ∂X with the following properties:*

(i) The measures μ_x, for x in X, are all equivalent and, for x, y in X, $\frac{\mathrm{d}\mu_y}{\mathrm{d}\mu_x}$ is a continuous function on ∂X.

(ii) Set, for any essentially bounded function f on ∂X with respect to the measure class of the $(\mu_x)_{x\in X}$ and for any x in X, $\mathcal{P}f(x) = \int_{\partial X} f \mathrm{d}\mu_x$. Then the correspondence $f \mapsto \mathcal{P}f$ establishes a L^∞-norm preserving isomorphism between $\mathrm{L}^\infty(\partial X, (\mu_x)_{x\in X})$ and the space of bounded harmonic functions on X.

The map $\mu : x \mapsto \mu_x$ is called the *harmonic density* of X.

Suppose that X possesses a discrete cocompact group of isometries. On the boundary we therefore have the harmonic density μ of X and the conformal density ν of dimension δ_X provided by corollary 4.15. If X is a symmetric space we have $\mu = \nu$ (see appendix B). This is the only case where it can happen:

Theorem 4.18. *Suppose there exists x in X such that ν_x and μ_x are proportional. Then X is a symmetric space.*

A Curvature and submanifolds of symmetric spaces

We finish here the proofs of lemmas 2.3, 2.6, 2.7, 2.10 and 2.11 by determining the sectional curvature of hyperbolic spaces and the totally geodesic submanifolds of complex and quaternionic hyperbolic spaces.

We go back to notations of paragraph 2.2.

A.1 Curvature of real hyperbolic space

We refer to [7] for general notions of Riemannian geometry.

Denote by D the covariant derivative for $\mathbb{H}^n_{\mathbb{R}}$. Let x be a point of $\mathbb{H}^n_{\mathbb{R}}$, X and Y two vector fields defined in an $\mathbb{R}^{n+1}$-neighbourhood of x. Then $(D_X Y)_x$ is simply the q-orthogonal projection on $x^\perp = \mathrm{T}_x\mathbb{H}^n_{\mathbb{R}}$ of the Euclidean $\mathbb{R}^{n+1}$-derivative at x of Y in the direction X_x at x: indeed this defines a connection on $\mathbb{H}^n_{\mathbb{R}}$, which clearly satisfies the Levi-Civita axioms.

Let R be the curvature tensor of $\mathbb{H}^n_{\mathbb{R}}$. We have the following:

Proposition A.1. *Let v and w be unit orthogonal tangent vectors of $\mathbb{H}^n_{\mathbb{R}}$ at some point x. Then we have $R(v,w)v = -w$. In particular, $\mathbb{H}^n_{\mathbb{R}}$ has constant sectional curvature -1.*

Proof. By transitivity of the group of isometries and by lemma 2.2, it suffices to prove the result for $n = 2$, $x = (1,0,0)$, $v = (0,0,1)$ and $w = (0,1,0)$. For this, we shall construct a Jacobi vector field on the unit speed geodesic $c : t \mapsto (\cosh t, 0, \sinh t)$. As for each s in $\mathbb{R}$, the matrix $k_s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos s & \sin s \\ 0 & -\sin s & \cos s \end{pmatrix}$ belongs to $\mathrm{SO}^\circ(1,n)$, the map

$$H : (s,t) \mapsto (\cosh t, \sin s \sinh t, \cos s \cosh t) = k_s c(t)$$

is a geodesic variation with $H(0,.) = c$. Therefore, the vector field

$$X(t) = \left.\frac{\partial H(s,t)}{\partial s}\right|_{s=0} = (0, \sinh t, 0)$$

is a Jacobi vector field on c, that is is satisfies $D^2_{c'}X = -R(c',X)c'$ where R is the curvature tensor. But, by the remark above, the vector field $t \mapsto (0,1,0)$ is parallel along c and, therefore, we have $D^2_{c'}X = X$ so that $R(c',X)c' = -X$ and the sectional curvature has constant value -1. $\square$

A.2 Curvature and submanifolds of complex hyperbolic space

Let still R denote the curvature tensor. Note that, by lemma 2.5 and proposition A.1, the complex hyperbolic line $\mathbb{H}^1_{\mathbb{C}}$ has constant sectional curvature -4. We can use the computation of the curvature of real hyperbolic space and complex hyperbolic line to prove the following

Proposition A.2. *Let v and w be unit tangent vectors of $\mathbb{H}^n_{\mathbb{C}}$ at some point x such that $\mathrm{Re}\langle v, w\rangle_x = 0$. If $\langle v, w\rangle_x = 0$, we have $R(v,w)v = -w$. If $w = iv$, we have $R(v,w)v = -4w$. In particular, $\mathbb{H}^n_{\mathbb{C}}$ has sectional curvature lying everywhere between -4 and -1 and the value -1 is reached exactly on Lagrangian tangent real 2-planes and the value -4 on tangent complex lines, viewed as real 2-planes.*

Proof. Let v and w be as above and let u be a unit vector such that $\langle v, u\rangle_x = 0$ and $w = siv + tu$ for some s, t in $\mathbb{R}$ with $s^2 + t^2 = 1$. Then, u and v span a Lagrangian plane and hence, by lemma 2.2, they are tangent to a totally geodesic submanifold which is isometric to real hyperbolic plane. Therefore, by proposition A.1, we have $R(v,u)v = -u$. In the same way, v and iv are tangent to a totally geodesic submanifold which is isometric to complex hyperbolic line and, therefore, by lemma 2.5, we have $R(v, iv) = -4iv$. Thus, we have $R(v,w)v = -4siv - tu$ and the lemma follows. □

Now we can use these informations to determine the totally geodesic submanifolds. To do this, we must first complete the information provided by proposition A.2:

Lemma A.3. *Let v and w be unit tangent vectors of $\mathbb{H}^n_{\mathbb{C}}$ at some point x such that $\langle v, w\rangle_x = 0$. Then we have $R(v, iv)w = -2iw$ and $R(v,w)(iv) = -iw$.*

Proof. The sectional curvature of $\mathbb{H}^n_{\mathbb{C}}$ being known, we just have to use classical Riemannian geometry machinery. Note that, by proposition A.2, for t in $\mathbb{R}$, one has $R(v+tw, i(v+tw))(v+tw) = -4i(1+t^2)(v+tw)$. Identifying the coefficient of t in this equation, we get $R(w, iv)v + R(v, iw)v + R(v, iv)w = -4iw$. By proposition A.2, $R(v, iw)v = -iw$, thus

$$R(v, iv)w - R(iv, w)v = -3iw.$$

Exchanging the roles of v and iv, we have

$$R(v, iv)w + R(v, w)(iv) = -3iw.$$

Finally, the Bianchi identity for the three vectors v, iv and w reads

$$R(v,iv)w + R(iv,w)v - R(v,w)(iv) = 0.$$

Solving this system of equations, we get the result. $\square$

We now have the following proposition, which achieves the proof of lemma 2.6:

Proposition A.4. *Let M be a totally geodesic real submanifold of $\mathbb{H}^n_{\mathbb{C}}$ and let x be a point of M. Then the real tangent space of M at x is either a real Lagrangian subspace or a complex subspace.*

The key point is the

Lemma A.5. *Let v, w be non-zero tangent vectors of M at x that are $\mathbb{R}$-independent. Then we are in one of the following mutually excluding situations:*

(i) we have $\mathrm{Im}\langle v,w\rangle_x = 0$ and the intersection of T_xM with the $\mathbb{C}$-plane spanned by v and w is the $\mathbb{R}$-plane spanned by v and w.

(ii) the $\mathbb{C}$-subspace spanned by v and w is contained in T_xM.

Proof. We can suppose that v and w are unitary and that $\mathrm{Re}\langle v,w\rangle_x = 0$. Let V be the tangent space to M at x. We have $R(v,w)w \in V$. Let u be a vector of $v^\perp$ such that $w = siv + tu$, for some s,t in $\mathbb{R}$. Then, by lemma A.3, we have $R(v,w)v = -(4isv, tu)$.

If s and t are both non-zero, then iv and u both belong to V and, therefore, as by lemma A.3 $iu = -\frac{1}{2}R(v,iv)u$, iu belongs to V and we are in the second case.

If $s = 0$ (and $t = 1$), then $\mathbb{R}v \oplus \mathbb{R}u \subset W$. But, by lemma A.3, $R(v,u)$ has no stable $\mathbb{R}$-line in $\mathbb{R}iv \oplus \mathbb{R}iu$ and, therefore, $V = \mathbb{C}v \oplus \mathbb{C}u$, that is we are in the second case, or $W = \mathbb{R}v \oplus \mathbb{R}u$ and we are in the first case.

If $t = 0$ (and $s = 1$), then $\mathbb{C}v \subset V$ and as, by lemma A.3, $R(v,iv)$ has no stable $\mathbb{R}$-line in $\mathbb{C}u$, we have $V = \mathbb{C}v \oplus \mathbb{C}u$ or $W = \mathbb{C}v$: we are in the second case. $\square$

Proof of proposition A.4. Suppose first there exists two tangent vectors v and w of M at x that are $\mathbb{R}$-independent and such that the $\mathbb{C}$-subspace V spanned by v and w is contained in T_xM. Then, if u is a vector of T_xM that does not belong to V, as $\mathbb{C}v$ is contained in T_xM, by lemma A.5, we have $\mathbb{C}u \subset \mathrm{T}_xM$ and T_xM is a complex subspace. Suppose now that for every pair v, w of $\mathbb{R}$-independent vectors in T_xM, the $\mathbb{C}$-subspace they span is not included in T_xM. Then, by lemma A.5, the space T_xM is Lagrangian, what should be proved. □

A.3 Curvature and submanifolds of quaternionic hyperbolic space

From lemma 2.9, we know that the quaternionic hyperbolic line $\mathbb{H}^1_{\mathbb{Q}}$ has constant sectional curvature -4.

Let x be in $\mathbb{Q}$. Then x is contained in a maximal commutative subfield of $\mathbb{Q}$. Therefore, if V is a right quaternionic vector space and v and w are vectors in V, there exists a maximal commutative subfield $\mathbb{K}$ of $\mathbb{Q}$ and a $\mathbb{K}$-subspace of V that contains v and w. From this and lemma 2.10, we deduce the

Proposition A.6. *Let v and w be unit tangent vectors of $\mathbb{H}^n_{\mathbb{Q}}$ at some point x such that $\mathrm{Re}\langle v, w\rangle_x = 0$. If $\langle v, w\rangle_x = 0$, we have $R(v, w)v = -w$. If $w \in v\mathbb{Q}_0$, we have $R(v, w)v = -4w$. In particular, $\mathbb{H}^n_{\mathbb{Q}}$ has sectional curvature lying everywhere between -4 and -1 and the value -1 is reached exactly on Lagrangian tangent real 2-planes and the value -4 on tangent planes that are $\mathbb{K}$-lines for some maximal commutative subfield $\mathbb{K}$ of $\mathbb{Q}$.*

Knowing the curvature tensor, we can complete the proof of lemma 2.10 by the

Proposition A.7. *Let M be a totally geodesic real submanifold of $\mathbb{H}^n_{\mathbb{Q}}$ and let x be a point of M. Then the real tangent space of M at x is either contained in a quaternionic line or in a Lagrangian $\mathbb{K}$-subspace, for some maximal commutative subfield $\mathbb{K}$, or is a quaternionic subspace.*

As in the complex case, this relies on a technical lemma:

Lemma A.8. *Let v, w be non-zero tangent vectors of M at x that are $\mathbb{R}$-independent. Then we are in one of the following mutually excluding situations:*

(i) v and w are contained in the same $\mathbb{Q}$-line.

(ii) v and w are $\mathbb{Q}$-independent, $\langle v, w\rangle_x \in \mathbb{R}$ and the intersection of $\mathrm{T}_x M$ with the $\mathbb{Q}$-plane spanned by v and w is the $\mathbb{R}$-plane spanned by v and w.

(iii) v and w are $\mathbb{Q}$-independent, and there exists a maximal commutative subfield $\mathbb{K} \ni \langle v, w\rangle_x$ such that the intersection of $\mathrm{T}_x M$ with the $\mathbb{Q}$-plane spanned by v and w is the $\mathbb{K}$-plane spanned by v and w.

(iv) v and w are $\mathbb{Q}$-independent and $\mathrm{T}_x M$ contains the $\mathbb{Q}$-plane spanned by v and w.

Proof. As usual, we suppose v and w are unit vectors and $\langle v, w\rangle_x \in \mathbb{Q}_0$. Choose a unit vector u with $\langle v, u\rangle_x = 0$, s in $\mathbb{Q}_0$ and t in $\mathbb{R}$ such that $w = vs + ut$. Then, by proposition A.6, we have $R(v, w)v = -v4s - ut$ and, therefore, vs and ut belong to V. Suppose now v and w are $\mathbb{Q}$-independent. Then $t \neq 0$ and u belongs to V. If $V \cap v\mathbb{Q} \oplus w\mathbb{Q} = v\mathbb{R} \oplus w\mathbb{R}$, then $s = 0$, and we are in the second case. Else, suppose V contains an element of the form $vq + ur$ for some q and r in $\mathbb{Q}_0$ that are not both zero. By lemma A.6, we have $R(vq + ur, v)v = v4q + ur$ and, therefore, vq and ur belong to V. By lemma A.3, we have $R(ur, u)v = v2r$ and $R(ur, u)vq = v2qr$ and, thus v, vq, vr and vqr belong to V. In the same way, u, uq, ur and uqr belong to V. Hence, either V contains an element of the form $uq+vr$ for some q and r who does not belong to the same real line of $\mathbb{Q}_0$ and, then, $u\mathbb{Q} \oplus v\mathbb{Q} \subset V$, or there exists a maximal commutative subfield $\mathbb{K}$ such that $u\mathbb{Q} \oplus v\mathbb{Q} \cap V = u\mathbb{K} \oplus v\mathbb{K}$. In particular, we then have $\langle v, w\rangle_x \in \mathbb{K}$ and the claim follows. □

Proof of proposition A.7. Let the tangent space to M at x not be contained in a $\mathbb{Q}$-line. Let v be a non-zero vector in $\mathrm{T}_x M$. Then, by lemma A.8, there exists a subfield $\mathbb{K}$ of $\mathbb{Q}$ such that $v\mathbb{Q} \cap \mathrm{T}_x M = v\mathbb{K}$ and that, for every w in $\mathrm{T}_x M - v\mathbb{Q}$, one has $\langle v, w\rangle_x \in \mathbb{K}$. As this is true for w in $v\mathbb{Q} \cap \mathrm{T}_x M$ too, the result follows. □

B Busemann functions of symmetric spaces

Let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$ and $X = \mathbb{H}^n_{\mathbb{K}}$. We shall prove here that the functions of the form $y \mapsto e^{-\delta_X b_\xi(y,x)}$ on X are harmonic. Let us freely use the notations of paragraphs 3.2 and 3.3.

Observe that the definition of ∂X given in paragraph 3.2 coincides we the one of paragraph 4.1, by lemma 3.3. Therefore Busemann functions defined in paragraph 3.2 coincide with the ones of paragraph 4.1. What's more, note that, as the distances d_x, for x in X, defined in both contexts satisfy the same homogeneity properties, it suffices to check that they coincide for a particular choice of x and of two points in the boundary. For this, take x to be the image of e_0 in X, and ξ and η to be the images of the vectors f_0 and f_n. Then, on one hand, the Gromov product of ξ and η viewed from x is 0 as x ly in the geodesic from ξ to η, and, on the other hand, we have $\langle f_0, f_n\rangle = \|f_0\|_x = \|f_n\|_x = 1$: the distances are the same.

We now can prove the

Proposition B.1. *For every x in X and ξ in ∂X, the function $X \to \mathbb{R}, y \mapsto e^{-\delta_X b_\xi(y,x)}$ is harmonic.*

Proof. Note that a function f is harmonic if and only if, for every x in X and $r > 0$, if $S(x,r)$ denotes the sphere with center x and radius r and $\sigma_{x,r}$ the normalized Riemannian measure on this sphere, one has $f(x) = \int_{S(x,r)} f \mathrm{d}\sigma_{x,r}$. Let o be our usual base point in X, that is the image of e_0 in X. Then, as K acts transitively on the sphere of the tangent space at o, for any $r > 0$, one has $S(o,r) = Ka_r o$ (where $t \mapsto a_t$ is as in paragraph 3.2) and the measure $\sigma_{o,r}$ is the unique K-invariant measure on that subset. What's more, for every x in X, there exists g in G such that $go = x$ and, thus, $S(x,r) = gKa_r o$. Therefore, a function f on X is harmonic if and only if, for any g, h in G, we have $\int_K f(gkho)\mathrm{d}k = f(go)$.

Let us check this equation for the functions we are studying. Let so g, h be in G and consider the measure μ on ∂X such that, for any continuous function φ on ∂X, we have

$$\int_{\partial X} \varphi \mathrm{d}\mu = \int_K \int_{\partial X} \varphi(ghk\xi)\mathrm{d}\nu_o(\xi)\mathrm{d}k.$$

Then this measure is gKg^{-1}-invariant and therefore it is equal to ν_{go}. In other words, for any continuous function φ on ∂X, we have

$$\begin{aligned}\int_{\partial X} e^{-b_\xi(go,o)}\varphi(\xi)\mathrm{d}\nu_0(\xi) = \int_{\partial X} \varphi \mathrm{d}\nu_{go} &= \int_K \int_{\partial X} \varphi(ghk\xi)\mathrm{d}\nu_o(\xi)\mathrm{d}k \\ &= \int_{\partial X} \left(\int_K e^{-b_\xi(gkho,o)}\mathrm{d}k \right) \varphi(\xi)\mathrm{d}\nu_o(\xi),\end{aligned}$$

that is $e^{-b_\xi(go,o)} = \int_K e^{-b_\xi(gkho,o)}\mathrm{d}k$ for ν_o-almost ξ and, hence, for every ξ as the functions are continuous. The claim follows. ☐

References

[1] M. Anderson, R. Schoen, Positive harmonic functions on complete manifolds of negative curvature, *Annals of mathematics* **121** (1985), 429-461.

[2] B. Bekka, M. Mayer, *Ergodic theory and topological dynamics of group actions on homogeneous spaces*, London Mathematical Society Lecture Notes Series 269, Cambridge University Press, Cambridge, 2000.

[3] G. Besson, G. Courtois, S. Gallot, Entropies et rigidités des espaces localement symétriques de courbure strictement négative, Geometric and functional analysis **5** (1995), 731-799.

[4] M. Burger, S. Mozes, CAT(-1)-spaces, divergence groups and their commensurators, *Journal of the American Mathematical Society* **9** (1996), 57-93.

[5] M. Bourdon, Structure conforme au bord et flot géodésique d'un CAT(-1)-espace, *L'enseignement mathématique* **41** (1995), 63-102.

[6] P. Eberlein, *Geometry of nonpositively curved manifolds*, Chicago lectures in mathematics, Chicago, 1996.

[7] S. Gallot, D. Hulin, J. Lafontaine, *Riemannian geometry*, Universitext, Springer Verlag, Berlin, 1993.

[8] E. Ghys, P. de la Harpe (eds), *Sur les groupes hyperboliques, d'après Mikhael Gromov*, Progress in mathematics 83, Birkhäuser, Boston, 1990

[9] B. Hasselblatt, A. Katok, *Introduction to the modern theory of dynamical systems*, Encyclopedia of mathematics and its applications 54, Cambridge university press, Cambridge, 1995.

[10] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Pure and applied mathematics 80, Academic press, San Diego, 1978.

[11] S. Helgason, *Groups and geometric analysis*, Pure and applied mathematics 113, Academic press, San Diego, 1984.

[12] F. Ledrappier, Harmonic measures and Bowen-Margulis measures, *Israel Journal of Mathematics* **71** (1990), 275-287.

[13] S.-J. Patterson, The limit set of a fuchsian group, *Acta mathematica* **136** (1976), 241-273.

[14] Th. Roblin, Ergodicité et équidistribution en courbure négative, *Mémoires de la SMF* **95** (2003).

[15] D. Sullivan, The density at infinity of a discrete group of hyperbolic motions, *Publications mathématiques de l'IHES* **50** (1979), 171-202.

[16] R. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, Boston, 1984.