RANDOM WALKS ON PROJECTIVE SPACES

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Abstract. Let $G$ be a connected real semisimple Lie group, $V$ be a finite dimensional representation of $G$, and $\mu$ be a probability measure on $G$ whose support spans a Zariski dense subgroup. We prove that the set of ergodic $\mu$-stationary probability measures on the projective space $\mathbb{P}(V)$ is in one-to-one correspondence with the set of compact $G$-orbits in $\mathbb{P}(V)$. When $V$ is strongly irreducible, we prove the existence of limits for the empirical measures.

We prove related results over local fields as the finiteness of the set of ergodic $\mu$-stationary measures on the flag variety of $G$.

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1. Introduction

1.1. Random walks on $\mathbb{P}(V)$. Let $\mathbb{K}$ be a local field of characteristic 0, i.e. $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or a finite extension of $\mathbb{Q}_p$. Let $V$ be the $\mathbb{K}$-vector space $V = \mathbb{K}^d$, $X$ be the projective space $X = \mathbb{P}(V)$ and $\mu$ be a probability measure on the linear group $GL(V)$. In this text, “probability measure” will stand for “Borel probability measure”. We set $\Gamma_\mu$ for the smallest closed subsemigroup of $GL(V)$ such that $\mu(\Gamma_\mu) = 1$ and $G_\mu$ for the Zariski closure of $\Gamma_\mu$ in $GL(V)$.

We assume that the action of $\Gamma_\mu$ on $V$ is semisimple i.e. every $\Gamma_\mu$-invariant vector subspace of $V$ admits a $\Gamma_\mu$-invariant complementary subspace. Equivalently, the algebraic group $G_\mu$ is reductive.

A Borel probability measure $\nu$ on $X$ is said to be $\mu$-stationary if $\mu^* \nu = \nu$. It is said to be $\mu$-ergodic if it is extremal among the $\mu$-stationary probability measures. We denote by $F_\nu = \text{supp}(\nu)$ the support of $\nu$.

A closed subset $F \subset X$ is said to be $\Gamma_\mu$-invariant if $gF \subset F$ for all $g$ in $\Gamma_\mu$. It is said to be $\Gamma_\mu$-minimal if it is minimal for the inclusion among the non-empty $\Gamma_\mu$-invariant closed subsets. If $\nu$ is a $\mu$-stationary Borel probability measure on $X$, its support $F_\nu$ is a $\Gamma_\mu$-invariant closed subset.

The aim of this text is to describe the asymptotic properties of the random walk on $\mathbb{P}(V)$ associated to $\mu$. We will also describe the $\mu$-ergodic $\mu$-stationary Borel probability measures on $\mathbb{P}(V)$ and check that they are in one-to-one correspondence with the $\Gamma_\mu$-minimal subsets of $\mathbb{P}(V)$.

This paper extends previous works of Furstenberg, Guivarc’h and Raugi.

1.2. Empirical measures on $\mathbb{P}(V)$. Let $V = \mathbb{K}^d$, $X = \mathbb{P}(V)$ and $x \in X$. Our first result describes the asymptotic behavior in law at time $n$ of the random walk induced by $\mu$ on $\mathbb{P}(V)$ starting from $x$. This behavior is given by the probability measure $\mu^m * \delta_x$. We want to prove the existence of a limit for this sequence in the set of probability measures on $X$ endowed with the $*$-weak topology. We will assume that $\Gamma_\mu$ is strongly irreducible i.e. that the only $\Gamma_\mu$-invariant finite union of vector subspaces of $V$ is $\{0\}$ or $V$. 

Theorem 1.1. (Asymptotic law) Let $\mathbb{K}$ be a local field of characteristic 0, $X := \mathbb{P}(\mathbb{K}^d)$, $\mu$ be a probability measure on $\text{GL}(\mathbb{K}^d)$ such that the action of $\Gamma_\mu$ on $\mathbb{K}^d$ is strongly irreducible. 
(i) Then for every $x$ in $X$, the limit probability measure
\begin{equation}
\nu_x := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mu^{*k} * \delta_x
\end{equation}
exists, is $\mu$-stationary and depends continuously on $x$.
(ii) When $\mathbb{K} = \mathbb{R}$ and the Zariski closure of $\Gamma_\mu$ is semisimple, one has
\begin{equation}
\nu_x = \lim_{n \to \infty} \mu^{*n} * \delta_x.
\end{equation}

Remarks 1.2. 1. Theorem 1.1 is due to Guivarc’h and Raugi when $X$ is an “isometric extension” of a flag variety of $G$ and $\mathbb{K} = \mathbb{R}$ (see [16] and [14]).

2. Theorem 1.1.ii can not be extended to any local field $\mathbb{K}$. For instance when $\mathbb{K} = \mathbb{Q}_p$, and when the support $\text{Supp}(\mu)$ is included in the compact open group $K = \text{SL}(d, \mathbb{Z}_p)$ and is equal to a translate of a small open normal subgroup of $K$, Equation (1.2) may not be satisfied.

3. When $\mathbb{K} = \mathbb{R}$, semisimplicity of the Zariski closure of $\Gamma_\mu$ is necessary for Theorem 1.1.ii to be true. For instance when $\mu$ is a Dirac mass supported by an irrational rotation of $\mathbb{R}^2$, Equation (1.2) is not satisfied.

Our second result describes, when $V$ is strongly irreducible, the asymptotic behavior of the trajectories of the random walk induced by $\mu$ on $\mathbb{P}(V)$ starting from $x$. We denote $\mathbb{N}^* = \{1, 2, \ldots\}$. This behavior is given by the empirical measures $\frac{1}{n} \sum_{k=1}^{n} \delta_{b_k \cdots b_1 x}$ for a sequence $(b_n)_{n \geq 1}$ of elements of $\text{GL}(\mathbb{K}^d)$ chosen independently with law $\mu$ i.e. for $\beta$-almost all such sequences where $\beta = \mu^\otimes \mathbb{N}^*$.

Theorem 1.3. (Empirical measures) Let $\mathbb{K}$ be a local field of characteristic 0, $X := \mathbb{P}(\mathbb{K}^d)$, $\mu$ be a probability measure on $\text{GL}(\mathbb{K}^d)$ such that the action of $\Gamma_\mu$ on $\mathbb{K}^d$ is strongly irreducible. Then, for every $x$ in $X$, for $\beta$-almost all sequences $b = (b_n)_{n \geq 1}$, the limit of the empirical probability measures
\begin{equation}
\nu_{x,b} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{b_k \cdots b_1 x}
\end{equation}
exists and is a $\mu$-ergodic $\mu$-stationary probability measure on $X$. Moreover one has
\begin{equation}
\nu_x = \int \nu_{x,b} \, d\beta(b).
\end{equation}

Remarks 1.4. 1. We note that, the assumption “$V$ is strongly irreducible” is crucial for Theorems 1.1 and 1.3 to be true (see Example 3.3).
2. Even when $\mathbb{K} = \mathbb{R}$, the limit measures $\nu_x$ might be non $\mu$-ergodic and hence the limit measures $\nu_{x,b}$ might not be equal to $\nu_x$. See Remark 1.9 and Example 2.11 (for a Markov chain which does not come from a group action).

1.3. **Stationary measures on $\mathbb{P}(V)$**. In our third result, we do not assume that the representation $V$ is irreducible, and we describe the $\mu$-stationary probability measures on $\mathbb{P}(V)$.

**Theorem 1.5. (Stationary measures)** Let $\mathbb{K}$ be a local field of characteristic 0, $X := \mathbb{P}(\mathbb{K}^d)$, $\mu$ be a probability measure on $\text{GL}(\mathbb{K}^d)$ such that the action of $\Gamma_\mu$ on $\mathbb{K}^d$ is semisimple. Then the map $\nu \mapsto \text{supp}(\nu)$ is a bijection between the sets

$$\{\text{\mu-ergodic probability on } X\} \leftrightarrow \{\Gamma_\mu\text{-minimal subset of } X\}.$$  

**Remarks 1.6.**

1. Theorem 1.5 is due to Furstenberg when the action of $\Gamma_\mu$ on $\mathbb{K}^d$ is strongly irreducible, and “proximal” i.e. when there exists a sequence $g_n$ in $\Gamma_\mu$ such that the sequence $\frac{g_n}{\|g_n\|}$ converges to a rank-one endomorphism $\pi$ in $\text{End}(V)$. In this case, $\mathbb{P}(\mathbb{K}^d)$ supports a unique $\mu$-stationary probability measure called “Furstenberg measure” (see the book [7]).

2. When $\mathbb{K} = \mathbb{R}$ and the action of $\Gamma_\mu$ on $\mathbb{K}^d$ is strongly irreducible, Theorem 1.5 can also be seen as a corollary of the main result of Y. Guivarc’h and A. Raugi in [16] where a bijection like (1.4) is obtained for “isometric extensions” $X$ of flag varieties.

3. We note also that even for a deterministic topological dynamical system on a compact space $X$, the support of an ergodic probability measure is not always minimal. For instance the Lebesgue probability measure on the circle $T = \mathbb{R}/\mathbb{Z}$ is ergodic for the map $t \mapsto 2t$. It might also happen that $X$ is minimal without being uniquely ergodic (see [10, p. 585]).

4. We note that, when the action of $\Gamma_\mu$ is not supposed to be semisimple, the support of a $\mu$-ergodic probability measure is not always $\Gamma_\mu$-minimal. Here is an example with $V = \mathbb{R}^2$ and $\mu$ the finitely supported measure

$$\mu = \frac{1}{2}(\delta_{a_0} + \delta_{a_1})$$

where $a_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $a_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

In this case one has $\mathbb{P}(V) = \mathbb{R} \cup \{\infty\}$ and $\{\infty\}$ is the only minimal $\Gamma_\mu$-invariant subset of $\mathbb{P}(V)$ while there exists a $\mu$-ergodic $\mu$-stationary probability on $\mathbb{P}(V)$ whose support is $[0, \infty]$.

When $\mathbb{K} = \mathbb{R}$ and $X = \mathbb{P}(\mathbb{R}^d)$, applying the following Theorem 1.7 with $\Gamma = \Gamma_\mu$, one can describe more precisely the $\mu$-ergodic probability measures on $X$ (see Proposition 5.5).
Theorem 1.7. (Minimal subsets) Let $\Gamma \subset \text{GL}(\mathbb{R}^d)$ be a subsemigroup whose action on $\mathbb{R}^d$ is semisimple and let $G$ be the Zariski closure of $\Gamma$. Every minimal $\Gamma$-invariant subset $F$ of $X := \mathbb{P}(\mathbb{R}^d)$ is supported by a compact $G$-orbit $O_F$, and the map $F \mapsto O_F$ is a bijection between the sets

$$\{ \Gamma\text{-minimal subset of }X\} \longleftrightarrow \{ \text{compact } G\text{-orbit in }X\}.$$  

Remark 1.8. It is easy to describe the set of compact orbits of this real reductive group $G$. Indeed, let $MAN$ be a minimal parabolic subgroup of $G$, $AN$ its maximal $\mathbb{R}$-split solvable subgroup and $X^{AN}$ the set of fixed point of $AN$ in $X$. Then, the map $O \mapsto O \cap X^{AN}$ is a bijection between the sets

$$\{ \text{compact } G\text{-orbit in }X\} \longleftrightarrow \{ M\text{-orbit in }X^{AN}\}$$

(see Lemma 4.15). In particular, one recovers the well-known fact due to Furstenberg, Guivarc’h, Raugi, Goldsheid and Margulis ([11], [13], [15]): there exists a unique $\mu$-stationary probability measure $\nu_P$ on the flag variety $\mathcal{P}$ of $G$. This measure is called Furstenberg measure.

Remark 1.9. Even when $V$ is strongly irreducible the sets (1.6) may be uncountable. For instance for $G := \text{SO}(n,1)$ acting on $V := \Lambda^3 \mathbb{R}^{n+1}$ with $n \geq 5$. In this case the compact group $M$ is isomorphic to $O(n-1)$, the set $X^{AN}$ is $\mathbb{P}(W)$ where $W = \Lambda^2 \mathbb{R}^{n-1}$ and $M$ has uncountably many orbits in $X^{AN}$.

1.4. Stationary measures on the flag variety. Let $p$ be a prime number. When $\mathbb{K} = \mathbb{Q}_p$ there may exist more than one $\mu$-stationary probability measure on the flag variety $\mathcal{P}$ of $G$ (see Section 4.1 for the definition of $\mathcal{P}$). However one has the following finiteness result. We recall that the expression $\mathbb{K}$-group is a shortcut for algebraic group defined over $\mathbb{K}$.

Theorem 1.10. (Finiteness) Let $G$ be the group of $\mathbb{Q}_p$-points of a reductive $\mathbb{Q}_p$-group, $\mu$ be a probability measure on $G$ such that $\Gamma_\mu$ is Zariski dense in $G$. Then there exist only finitely many $\mu$-ergodic $\mu$-stationary probability measures on the flag variety $\mathcal{P}$ of $G$.

1.5. Strategy of proofs. In order to prove Theorems 1.1 and 1.3, we will introduce the averaging operator

$$P_\mu : C^0(X) \to C^0(X) ; \varphi \mapsto P_\mu(\varphi) = \int_{\Gamma_\mu} \varphi(gx) \, d\mu(g),$$

and prove in Proposition 3.1 that, as soon as $\Gamma_\mu$ acts strongly irreducibly on $V$, this Markov-Feller operator is equicontinuous (see [23, 22] and section 2 below for definitions ; this strategy is inspired by the
work of Guivarc’h and Raugi [16]). When $\mathbb{K} = \mathbb{R}$ and $\Gamma_\mu$ has semisimple Zariski closure, the only eigenvalue of modulus 1 of this operator $P_\mu$ is 1 (Lemma 5.6). Then Theorems 1.1 and 1.3 will occur as special cases of statements about equicontinuous Markov-Feller operators.

For Theorem 1.1, we will use well-known decomposition theorems for operators in Banach spaces spanning a compact semigroup (Propositions 2.2 and 2.3), that we will recall in section 2.1, and that we will apply to equicontinuous Markov Feller operators (Proposition 2.9).

For Theorem 1.3, we will use a general fact due to Raugi [22] about equicontinuous Markov-Feller operators $P$ : for such operators the empirical measures converge almost surely toward a $P$-ergodic probability measure (Proposition 2.9.e).

In the setting of Theorem 1.5 and 1.7, the Markov-Feller operator $P_\mu$ might be non-equicontinuous. Hence we have to develop new tools (Lemmas 5.3 and 5.4) to be able to describe the algebraic homogeneous $G$-spaces which support a $\mu$-stationary probability measure. When $\mathbb{K} = \mathbb{R}$, those homogeneous spaces are exactly the compact ones, and each of them supports a unique $\mu$-stationary probability measure (Proposition 5.5). When $\mathbb{K}$ is any local field, those homogeneous space are exactly those containing a $\Gamma_\mu$-invariant compact subset (Proposition 5.1). The description of these homogeneous spaces (Proposition 4.2) occupies most of Chapter 4. An important tool that we have to introduce is a compact group $M_\Gamma$ that we associate to any Zariski dense subsemigroup $\Gamma$ and that we call the limit group of $\Gamma$ (Propositions 4.5 and 4.9).

In the setting of Theorem 1.10, the Markov-Feller operator $P_\mu$ is again equicontinuous. We can use directly Proposition 2.9 and we only have to check that there exist only finitely many $\Gamma_\mu$-minimal subsets in the flag variety (Proposition 4.17) using again the limit group $M_\Gamma$.

2. EQUICONTOINUOUS OPERATORS

The aim of this chapter is to recall decomposition theorems for bounded operators on Banach spaces spanning a compact semigroup (Propositions 2.2 and 2.3), to recall Breiman law of large numbers for Markov-Feller operators on a compact space (Proposition 2.4) and the results of Raugi about equicontinuous Markov-Feller operators (Propositions 2.7 and 2.9).

2.1. Decomposition theorems. We begin by recalling the JLG decomposition theorem for bounded operators spanning a compact semigroup.
Let $(E, \|\cdot\|)$ be a Banach space. We endow the space $\mathcal{L}(E)$ of bounded linear operators with the strong topology: a sequence $P_n$ in $\mathcal{L}(E)$ converges strongly towards $P$ in $\mathcal{L}(E)$ if and only if, for any $f$ in $E$, one has $\lim_{n \to \infty} \|P_n f - Pf\| = 0$.

**Definition 2.1.** We say that an operator $P$ in $\mathcal{L}(E)$ spans a strongly compact semigroup if $P$ belongs to a semigroup of $\mathcal{L}(E)$ which is compact for the strong topology. Equivalently, the operators $P_n$ have uniformly bounded norms: $\sup_{n \geq 1} \|P^n\| < \infty$, and for every $f$ in $E$, the orbit $(P^n f)_{n \geq 1}$ is strongly relatively compact in $E$.

We endow the dual Banach space $E^*$ with the $*$-weak topology: a sequence $\nu_n$ in $E^*$ converges $*$-weakly towards $\nu$ in $E^*$ if and only if, for any $f$ in $E$, one has $\lim_{n \to \infty} \nu_n(f) = \nu(f)$. For any operator $P$ in $\mathcal{L}(E)$, we will write $\nu \mapsto \nu P$ for the adjoint operator of $P$ in $E^*$, $E^P$ for the set of $P$-invariant vectors and $(E^*)^P$ for the set of $P$-invariant linear forms:

\[(2.1) \quad E^P := \{ f \in E \mid Pf = f \} \quad \text{and} \quad (E^*)^P := \{ \nu \in E^* \mid \nu P = \nu \}.\]

We also introduce the Banach subspaces

\[(2.2) \quad E_P := \{ f \in E \mid \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P^k f = 0 \, \text{strongly} \} \]

\[(2.3) \quad (E^*)_P := \{ \nu \in E^* \mid \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \nu P^k = 0 \, \text{$*$-weakly} \}.\]

The following proposition is known as “von Neumann functional ergodic theorem”.

**Proposition 2.2.** Let $E$ be a Banach space and $P \in \mathcal{L}(E)$ be an operator spanning a strongly compact semigroup. Then

a) The restriction map $\nu \mapsto \nu|_{E^P}$ is an isomorphism $(E^*)^P \simeq (E^P)^*$.

b) One has the decomposition $E = E^P \oplus E_P$.

c) One also has the decomposition $E^* = (E^*)^P \oplus (E^*)_P$.

**Sketch of proof.**

a) To prove injectivity, we start with a linear form $\nu$ on $E$ which is zero on $E^P$. Since, by [24, Th. 3.20.c)], the convex hull of a compact subset of $E$ is relatively compact, for any $f$ in $E$, we can choose a cluster point $y_\infty$ of the sequence $\frac{1}{n} \sum_{k=1}^{n} P^k f$. This point is $P$-invariant and one has $\nu(f) = \nu(y_\infty) = 0$. Hence, one has $\nu = 0$.

To prove surjectivity, we start with a linear form on $E^P$ extend it by Hahn-Banach Theorem to a linear form $\nu$ on $E$ and notice that any cluster point of the weakly relatively compact sequence $\frac{1}{n} \sum_{k=1}^{n} \nu P^k$ is $P$-invariant and has same restriction to $E^P$ as $\nu$. 


b) Again, for \( f \) in \( E \), the sequence \( \frac{1}{n} \sum_{k=1}^{n} P^k f \) is relatively compact. If \( y_\infty \) is a cluster point of it, for any \( P \)-invariant linear form \( \nu \), one has \( \nu(y_\infty) = \nu(f) \). Hence the sequence \( \frac{1}{n} \sum_{k=1}^{n} P^k f \) admits a unique cluster point, that is it converges to some \( \pi_P f \in E^P \). The map \( \pi_P : E \to E \) is then a \( P \)-invariant projector whose image is \( E^P \) and whose kernel is \( E_P \).

\[ E^r := \{ f \in E \mid \lim_{n \to \infty} P^nf = 0 \ \text{strongly} \} \]

The following JLG decomposition is a strong improvement of Proposition 2.2. We will only use it to prove Theorem 1.1.ii.

For any complex number \( \chi \) we consider the eigenspace

\[ E_\chi := \{ f \in E \mid Pf = \chi f \} \]

set \( E_r \) for the linear closure of \( \oplus_{|\chi|=1} E_\chi \) and \( E_s \) for the space

\[ E_s := \{ f \in E \mid \lim_{n \to \infty} P^nf = 0 \ \text{strongly} \} \]

Proposition 2.3. (Jacobs, de Leeuw, Glicksberg) Let \( E \) be a Banach space and \( P \in \mathcal{L}(E) \) be an operator spanning a strongly compact semigroup. Then

a) One has the decomposition \( E = E_r \oplus E_s \).

b) In particular, if 1 is the only eigenvalue of \( P \) with modulus 1, then the following limits exist:

\[
\begin{align*}
(i) & \quad \text{for every } f \text{ in } E, \quad \lim_{n \to \infty} P^n f = \pi_P f \quad \text{strongly,} \\
(ii) & \quad \text{for every } \nu \text{ in } E^*, \quad \lim_{n \to \infty} \nu P^n = \nu \pi_P \quad \text{*-weakly.}
\end{align*}
\]

Sketch of proof. Let \( S \) be the closure of the semigroup spanned by \( P \) in \( \mathcal{L}(E) \) for the strong topology. Then one easily checks that \( S \) is compact and that the composition map \( S \times S \to S \) is continuous. We let \( T \) be a non-empty minimal closed subset of \( S \) such that \( ST \subset T \).

One checks that one has \( T = Su \) where \( u \) is an idempotent element and that the composition map induces a group structure on \( T \) with identity element \( u \). We have \( E = \ker u \oplus \im u \) and since \( S \) is abelian, both these subspaces are \( S \)-invariant. Since the image of \( S \) in \( \mathcal{L}(\im u) \) is a strongly compact abelian group, we have \( \im u \subset E_r \) and it only remains to prove one has \( \ker u \subset E_s \). Indeed, if \( f \) belongs to \( \ker u \), as \( u \) belongs to the strong closure of the sequence \( P^n \), there exists a sequence \( n_k \) of integers with \( \|P^{n_k} f\| \to 0 \). Now, by Banach-Steinhaus Theorem, the sequence \( \|P^n\| \) is bounded and hence \( \|P^n f\| \to 0 \), what should be proved.

For a detailed proof, see [9, Chap. 12].
2.2. Empirical measures for Markov-Feller operators. For further quotation, we recall in this section Breiman law of large numbers.

Let $X$ be a compact metrizable space, $E = C^0(X)$ be the Banach space of continuous functions on $X$ endowed with the supremum norm. Its dual space $E^*$ is the space $\mathcal{M}(X)$ of complex measures on $X$. We denote by $X$ the compact set $X = X^\mathbb{N}$ of infinite sequences $\underline{x} = (x_0, x_1, x_2, \ldots)$.

Let $P : C^0(X) \to C^0(X)$ be a Markov-Feller operator i.e. a bounded operator such that $\|P\| \leq 1$, $P1 = 1$ and such that $Pf \geq 0$ for all function $f \geq 0$. Such a Markov-Feller operator can be seen alternatively as a continuous map $x \mapsto P_x$ from $X$ to the set of probability measures on $X$, where $P_x$ is defined by $P_x(f) = (Pf)(x)$ for all $f$ in $C^0(X)$. We denote by $\mathbb{P}_x$ the Markov probability measure on $X$ which gives the law of the trajectories of the Markov chain starting from $x$ associated to $P$.

For any trajectory $\underline{x} \in X$, and $n \geq 1$ the probability measures $\nu_{\underline{x},n} := \frac{1}{n} \sum_{k=1}^{n} \delta_{x_k}$ are called empirical measures. Heuristically this sequence of measures tells us where the trajectories spend a positive proportion of their time. We want to understand the behavior of this sequence of measures. The first result in that direction is Breiman’s law of large numbers:

Proposition 2.4. (Breiman) Let $X$ be a compact metrizable space and $P$ be a Markov-Feller operator on $X$. Then, for every point $x$ in $X$, for $\mathbb{P}_x$-almost every trajectory $\underline{x} \in X$, every $\ast$-weak cluster point $\nu_\infty$ of the sequence of empirical measures $\frac{1}{n} \sum_{k=1}^{n} \delta_{x_k}$ is $P$-invariant.

In particular, if $P$ is uniquely ergodic i.e. admits a unique $P$-invariant probability measure $\nu$ on $X$, then, for every point $x$ in $X$, for $\mathbb{P}_x$-almost every trajectory $\underline{x} \in X$, one has

\begin{equation}
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{x_k} = \nu.
\end{equation}

For a proof, see [8] or [4].

Example 2.5. When $P$ is not uniquely ergodic, the limit (2.5) does not always exists. This is already the case for deterministic operators: for example, if $P$ is the Markov-Feller operator on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ such that $P_x = \delta_{2x}$, $x \in \mathbb{T}$.

2.3. Equicontinuous Markov-Feller operators. We now recall the description of $P$-invariant measures of an equicontinuous Markov-Feller operator $P$. 


Definition 2.6. We say that the Markov-Feller operator $P$ is equicontinuous if, for every $f$ in $C^0(X)$, the family of functions $(P^nf)_{n\geq 1}$ is equicontinuous.

Equivalently, by Ascoli Theorem, this means that $P$ spans a strongly compact semigroup in $L(C^0(X))$.

Let $P$ be a Markov-Feller operator on $X$. A closed subset $F \subset X$ is said to be $P$-invariant if, for all $x$ in $X$, one has $P_x(F) = 1$. A $P$-invariant subset $F$ is said to be $P$-minimal if it is minimal among the non-empty closed $P$-invariant subsets of $X$.

We shall now describe the structure of the $P$-minimal subsets of $X$. We recall from [24, Th. 11.12] that if $A$ is a commutative $C^*$-algebra, there exists a unique compact space $Z$ such that $A$ is isomorphic to the algebra $C^0(Z)$. The space $Z$ is called the spectrum of $A$. If $\varphi : A_1 \rightarrow A_2$ is a morphism of commutative $C^*$-algebras and if $Z_1$ and $Z_2$ are the spectra of $A_1$ and $A_2$, then there exists a unique continuous map $\theta : Z_2 \rightarrow Z_1$ such that, for any $f$ in $A_1$, one has $\varphi(f) = f \circ \theta$.

Proposition 2.7. Let $X$ be a compact metrizable space and $P$ be an equicontinuous Markov-Feller operator on $X$. Let $Y$ be the closure of the union of $P$-minimal subsets of $X$. Then

a) The restriction map

$$C^0(X)^P \rightarrow C^0(Y)^P$$

is an isometry of Banach spaces.

b) More generally, when $|\chi| = 1$, the restriction map between the eigenspaces

$$C^0(X)_\chi \rightarrow C^0(Y)_\chi$$

is an isometry of Banach spaces.

c) Each $P$-invariant function $f \in C^0(X)^P$ is constant on the $P$-minimal subsets and hence $C^0(Y)^P$ is a Banach sub-$C^*$-algebra of $C^0(Y)$.

Let $Z$ be the spectrum of $C^0(X)^P$ and $\pi : Y \rightarrow Z$ be the surjective continuous map associated with the inclusion $C^0(Y)^P \rightarrow C^0(Y)$.

d) For any $z$ in $Z$, the set $\pi^{-1}(z)$ is $P$-invariant and contains a unique $P$-minimal subset $F_z$.

This result is essentially due to Raugi [22, Th. 2.6].

Example 2.8. It might happen that, for some $z$ in $Z$, the preimage $\pi^{-1}(z)$ is not minimal. This is the case when $X = \mathbb{N}^* \cup \{0, 1\}$, where $\mathbb{N}^* = \mathbb{N}^* \cup \{\infty\}$ is the one-point compactification of $\mathbb{N}^*$, and $P$ is the
Markov operator such that, for any \( n \) in \( \mathbb{N}^* \), one has
\[
P_{(n,0)} = P_{(n,1)} = \frac{1}{n} \delta_{n,0} + (1 - \frac{1}{n}) \delta_{n,1}
\]
and \( P_{(\infty,0)} = P_{(\infty,1)} = \delta_{\infty,1} \). Then one has \( Y = X, \ Z = \overline{\mathbb{N}}^* \) and \( \pi \) is the map \((n,u) \mapsto n\). In particular \( \pi^{-1}(\infty) = \{(\infty,0), (\infty,1)\} \) and this set is not \( P \)-minimal.

**Proof of Proposition 2.7. a)&b)** We first prove that this restriction map is injective. Let \( f \) be a continuous function on \( X \) such that \( Pf = \chi f \) with \(|\chi| = 1\). Assume that the restriction of \( f \) to \( Y \) is zero. We want to prove that \( f = 0 \). The function \( g := |f| \) satisfies \( Pg \geq g \). Let \( M := \sup_{x \in X} g(x) \). Then \( g^{-1}(M) \) is then a closed \( P \)-invariant subset of \( X \) and hence contains a \( P \)-minimal subset. This proves that \( M = 0 \) as required.

We now prove that this restriction map is a surjective isometry. Let \( g \) be a continuous function on \( Y \) such that \( Pg = \chi g \). This function can be extended as a continuous function \( h \) on \( X \) with \(|h|_{C^0(X)} = |g|_{C^0(Y)} \). Since \( g \) is an eigenfunction of \( P \), \( g \) is also the restriction of the functions \( h_n := \frac{1}{n} \sum_{k=1}^{n} \chi^{-k} P^k h \), for \( n \geq 1 \). This sequence is equicontinuous and admits a cluster value \( f \) in \( C^0(X) \). By construction, this function \( f \) belongs to the eigenspace \( C^0(X)_{\chi} \), the restriction of \( f \) to \( Y \) is equal to \( g \) and one has \(|f|_{C^0(X)} = |g|_{C^0(Y)} \) as required.

\( c) \) Let \( f \) be a \( P \)-invariant continuous function with real values, \( F \subset X \) be a closed \( P \)-minimal subset and \( M = \sup_{F} f \). Then the set \( f^{-1}(M) \cap F \) is closed and \( P \)-invariant, hence \( f = M \) on \( F \), what should be proved.

\( d) \) Equip \( Z \) with a distance which defines its topology, fix \( z \) in \( Z \) and, for \( y \) in \( Y \), set \( f(y) = d(\pi(y), z) \). By definition of \( \pi \), \( f \) is \( P \)-invariant, so that, if \( f(x) = 0 \), one has \( f(y) = 0 \) for \( P_x \)-almost any \( y \), that is the set \( \pi^{-1}(z) \) is \( P \)-invariant.

Let \( F_1 \neq F_2 \) be closed \( P \)-minimal subsets. Then, as \( F_1 \cap F_2 \) is closed and \( P \)-minimal, one has \( F_1 \cap F_2 = \emptyset \). Let \( f \) be in \( C^0(Y) \) with \( f = 0 \) on \( F_1 \) and \( f = 1 \) on \( F_2 \) and set \( g = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P^k f \). Then \( g \) belongs to \( C^0(X)_{P} \) and \( g \) does not take the same value on \( F_1 \) and \( F_2 \), so that \( \pi(F_1) \neq \pi(F_2) \), what should be proved. \( \square \)

Recall a probability measure \( \nu \) on \( X \) is said to be \( P \)-invariant\(^1\) if \( \nu P = \nu \). It is then said to be \( P \)-ergodic if it is an extremal point of the compact convex set of \( P \)-invariant probability measures on \( X \).

\(^1\)Many synonyms for the word “invariant” have been used in the litterature like “stationary”, “harmonic” or even “regular” in [25].
For an equicontinuous Markov-Feller operator $P$, one can describe the $P$-invariant probability measures. For $x$ in $X$, we denote by $\delta_x$ the Dirac mass at $x$ and $\delta_x P^k$ its image by the transpose of $P^k$.

We have the following results by Raugi [22, Prop. 3.2 and 3.3].

From Propositions 2.2 and 2.7, we get

**Proposition 2.9.** Let $X$ be a compact metrizable space, $P$ be an equicontinuous Markov-Feller operator on $X$, $Y$ be the closure of the union of the $P$-minimal subsets of $X$ and $Z$ be the spectrum of the Banach algebra $C^0(Y)^P$.

a) Any $P$-ergodic $P$-invariant probability measure on $X$ has $P$-minimal compact support and any $P$-minimal closed subset $F$ of $X$ carries a unique $P$-invariant probability measure $\nu_F$. The set of $P$-ergodic $P$-invariant probability measures on $X$ is compact for the weak-$*$ topology.

b) The map

$$\mathcal{M}(Z) \to \mathcal{M}(X)^P; \alpha \mapsto \int_Z \nu_F(z) d\alpha(z)$$

is an isomorphism.

c) For every $x$ in $X$, the limit probability measure

$$\nu_x := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \delta_x P^k$$

exists, is $P$-invariant and depends continuously on $x$.

d) Seeing these $\nu_x$ as measures on $Z$, the map

$$C^0(Z) \to C^0(X)^P; \varphi \mapsto (x \mapsto \int_Z \varphi(z) d\nu_x(z))$$

is a Banach spaces isomorphism.

e) For every $x$ in $X$, for $\mathbb{P}_x$-almost every trajectory $\bar{x} \in X$, the limit

$$\nu_{\bar{x}} := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \delta_{x_k}$$

exists, is $P$-invariant and $P$-ergodic, and one has the equality

$$\nu_x = \int_X \nu_{\bar{x}} d\mathbb{P}_x(\bar{x}).$$

**Remarks 2.10.**

1. In particular, any limit of a sequence of $P$-ergodic probability measures on $X$ is also $P$-ergodic.

2. Formula (2.10) is a kind of Poisson formula expressing harmonic functions thanks to continuous functions on a “boundary”.

**Proof of Proposition 2.9.** a), b), c) and d) directly follow from Propositions 2.2 and 2.7. For a), note that, if $\pi : Y \to Z$ is the natural map, necessarily, for any $z$ in $Z$, the set $\pi^{-1}(z)$ carries a unique $P$-invariant probability measure $\nu$. Since, by definition, $F_z \subset \pi^{-1}(z)$ and $F_z$ also carries a $P$-invariant probability measure, one has $\nu(F_z) = 1$, and a) follows.
Let us prove $e$). We fix $x$ in $X$. By Breiman’s proposition 2.4, we already know that the cluster points of the sequence of empirical measures $\nu_{x,n}$ are $P$-invariant probability measures. Hence, by Proposition 2.2.a and since $X$ is metrizable, to prove convergence in (2.11), we only have to check that for every $P$-invariant function $f$ on $X$, for $\mathbb{P}_x$-almost all trajectories $x$ in $X$, the sequence $\nu_{x,n}(f)$ converges. For that, we note that, since $f$ is $P$-invariant, the sequence of functions $\Phi_n : x \mapsto f(x_n)$ is a bounded martingale on $X$, with respect to the natural filtration. Hence, by Doob’s martingale theorem, for $\mathbb{P}_x$-almost all $x$ in $X$, the sequence $f(x_n)$ converges. Therefore the Cesaro average $\nu_{x,n}(f)$ converges too.

It remains to check that, for $\mathbb{P}_x$-almost all trajectories $x$ in $X$, the limit $\nu_x$ is $P$-ergodic. Indeed, for any $P$-invariant continuous function $f$ on $X$, for $\mathbb{P}_x$-almost all trajectories $x$ in $X$, the sequence $f(x_n)$ converges to $\ell = \nu_x(f)$. Hence, all the cluster points in $X$ of the trajectory $x$ belong to the level set $f^{-1}(\ell)$ and the support of $\nu_x$ is contained in this level set. In particular the set $\pi(\text{supp} \nu_x)$ is a singleton $z$ and ergodicity follows from (a). Formula (2.12) is obvious.

Example 2.11. Here is an example where the limits of empirical measures $\nu_x$ given in (2.11) are not equal to $\nu_x$. Choose $X := \mathbb{Z} \cup \{-\infty, \infty\}$ to be the two points compactification of $\mathbb{Z}$ and $P$ to be the Markov Feller operator on $X$ such that

$$P_{-\infty} = \delta_{-\infty} \text{ and } P_n = a_n \delta_{n-1} + (1 - a_n) \delta_{n+1} \quad (n \in \mathbb{Z})$$

with $a_n = \frac{1}{3}$ and $a_{-n} = \frac{2}{3}$ for $n > 0$, and $a_0 = \frac{1}{2}$. This operator $P$ is equicontinuous and $P$ has two ergodic measures $\delta_{-\infty}$ and $\delta_{\infty}$. One computes using (2.10) that for $x$ in $\mathbb{Z}$ the limit probability measure $\nu_x$ in (2.12) is given by

$$\nu_x = (1 - 2^{x-1}) \delta_{-\infty} + 2^{x-1} \delta_{\infty} \quad \text{for } x \leq 0,$$

$$\nu_x = 2^{x-1} \delta_{-\infty} + (1 - 2^{-x-1}) \delta_{\infty} \quad \text{for } x \geq 0,$$

and hence $\nu_x$ is not $P$-ergodic.

A very similar example is obtained by choosing $P = P_\mu$ to be the averaging operator of a Zariski dense probability measure $\mu$ on the group $SO(2,1)$ acting on the projective sphere $X = \mathbb{S}^2$ of $\mathbb{R}^3$. In this case, $P$ is equicontinuous and there exists exactly two $P$-ergodic measures on $X$, $\nu_+$ and $\nu_-$ and two extremal dual $P$-invariant continuous functions $\varphi_+$ and $\varphi_-$ on $X$.

Another very similar example (in the setting of Theorem 1.3) can be obtained by choosing $P = P_\mu$ to be the averaging operator of a probability measure $\mu$ on the group $G = SO(5,1)$ with $\Gamma_\mu = G$ acting
on the projective space $X = \mathbb{P}(V)$ for the irreducible representation $V = \Lambda^3 \mathbb{R}^6$ of $G$ introduced in Remark 1.9, and by choosing a point $x = \mathbb{R}v$ in $\mathbb{P}(V)$ for which the orbit closure $\overline{Gx}$ contains uncountably many compact $G$-orbits. For instance $v = v_1 + wv_2$ where $v_1$ and $v_2$ are non zero $N$-invariant vectors in $V$ belonging to distinct $MA$-orbits and where $w$ is the non trivial element of the Weyl group.

Example 2.12. When $P$ has a unique $P$-ergodic probability measure $\nu$, Equation (2.5) gives us an information on the statistical behavior of a typical trajectory starting from $x$. In particular this trajectory spends most of the time near the support of $\nu$. However, even when $P$ is equicontinuous, the limit set of $(x_k)_{k \geq 1}$ may be strictly larger than $\text{Supp}(\nu)$. Here is an example: choose $X := \mathbb{Z} \cup \{\infty\}$ to be the one point compactification of $\mathbb{Z}$ and $P$ to be the Markov Feller operator on $X$ for which $P_x = \mu * \delta_x$ where $\mu$ is the probability measure $\mu := \frac{1}{2}(\delta_{-1} + \delta_1)$, $x \neq \infty$, and $P_\infty = \delta_\infty$. The operator $P$ is equicontinuous and is uniquely ergodic with invariant measure $\delta_\infty$, but, for all $x$ in $\mathbb{Z}$, $\mathbb{P}_x$-almost all trajectories visit infinitely often every point in $\mathbb{Z}$.

3. Linear random walks

In this chapter, we use the results of Chapter 2 in order to prove Theorems 1.1.1 and 1.3.

3.1. Equicontinuity on the projective spaces. The main step will be to understand when the Markov-Feller operator $P_\mu$ in (1.7) is equi-continuous (see Proposition 3.1).

Let $K$ be a local field of characteristic $0$, $V = K^d$, $X = \mathbb{P}(V)$ and $\mu$ be a probability measure on the linear group $\text{GL}(V)$. We set $\Gamma_\mu$ for the smallest closed subsemigroup of $\text{GL}(V)$ such that $\mu(\Gamma_\mu) = 1$.

We recall the averaging operator that we introduced in (1.7): this operator is the Markov-Feller operator $P = P_\mu : C^0(X) \to C^0(X)$ whose transition probabilities are given by $P_x = \mu * \delta_x$ for all $x$ in $X$.

We set $(B, \mathcal{B}, \beta, T)$ to be the one-sided Bernoulli shift with alphabet $(\Gamma_\mu, \mu)$. This means that $B$ is the set of sequences $b = (b_1, \ldots, b_n, \ldots)$ with $b_n$ in $\text{GL}(V)$, $\mathcal{B}$ is its Borel $\sigma$-algebra, $\beta$ is the product probability measure $\beta = \mu^\otimes \mathbb{N}$ and $T$ is the shift: $Tb := (b_2, b_3, \ldots)$.

For every $x$ in $X$, the Markov measure $\mathbb{P}_x$ is the image of $\beta$ by the map

$$B \to X ; \quad b \mapsto (x, b_1x, b_2b_1x, b_3b_2b_1x, \ldots).$$

Proposition 3.1. Let $K$ be a local field of characteristic $0$, $V = K^d$, $X := \mathbb{P}(V)$, $\mu$ be a probability measure on $\text{GL}(V)$ such that the action
of $\Gamma_\mu$ on $V$ is strongly irreducible. Then the Markov-Feller operator $P_\mu$ on $X$ is equicontinuous.

We will need the following lemma.

We introduce a distance on $\mathbb{P}(V)$. We fix a norm $\|\cdot\|$ on $V$: we choose it to be euclidean when $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, and to be ultrametric when $\mathbb{K}$ is non-archimedean. We endow $\Lambda^2V$ with a compatible norm also denoted $\|\cdot\|$. The formula

$$d(x, y) = \frac{\|v \wedge w\|}{\|v\|\|w\|}, \quad \text{for } x = \mathbb{K}v \text{ and } y = \mathbb{K}w \in \mathbb{P}(V),$$

defines a distance on $\mathbb{P}(V)$ which induces the usual compact topology.

**Lemma 3.2.** Let $V = \mathbb{K}^d$ and $\mu$ be a probability measure on $\text{GL}(V)$ such that the action of $\Gamma_\mu$ on $V$ is strongly irreducible. For all $\varepsilon > 0$, a) there exists $c_\varepsilon > 0$ such that, for all $v \in V \setminus \{0\}$, one has

$$\beta(\{b \in B \mid \inf_{n \geq 1} \frac{\|b_n \cdots b_1 v\|}{\|b_n \cdots b_1\| \| v\|} \geq c_\varepsilon\}) \geq 1 - \varepsilon,$$

b) there exists $M_\varepsilon > 0$ such that, for all $x, y \in \mathbb{P}(V)$, one has

$$\beta(\{b \in B \mid \sup_{n \geq 1} d(b_n \cdots b_1 x, b_n \cdots b_1 y) \leq M_\varepsilon d(x, y)\}) \geq 1 - \varepsilon.$$

We recall that the **proximal dimension** of a subsemigroup $\Gamma \subset \text{GL}(V)$ is the smallest integer $r \geq 1$ for which there exists an endomorphism $\pi$ in $\text{End}(V)$ of rank $r$ such that $\pi = \lim_{n \to \infty} \lambda_n g_n$ with $\lambda_n$ in $\mathbb{K}$ and $g_n$ in $\text{G}$. The semigroup $\Gamma$ is **proximal** if and only if $r = 1$.

**Proof of Lemma 3.2.** a) By [7, Th. 3.1], we know that there exists a Borel map $b \mapsto W_b$ from $B$ to the Grassmannian variety $\text{Gr}_{d-r}(V)$, where $r$ is the proximal dimension of $\Gamma_\mu$ in $V$, such that, for $\beta$-almost all $b \in B$, $W_b$ is the kernel of all the matrices $\pi \in \text{End}(V)$ which are cluster points of the sequence $\frac{b_n \cdots b_1}{\|b_n \cdots b_1\|}$. By [7, Pr. 2.3], we also know that, for all $x$ in $\mathbb{P}(V)$, one has

$$\beta(\{b \in B \mid x \in \mathbb{P}(W_b)\}) = 0.$$

Hence, for all $\varepsilon > 0$, there exists $\alpha_\varepsilon > 0$ such that, for all $x$ in $\mathbb{P}(V)$,

$$\beta(\{b \in B \mid d(x, \mathbb{P}(W_b)) \geq \alpha_\varepsilon\}) \geq 1 - \varepsilon/2.$$

By definition of $W_b$, for all $\alpha > 0$, for $\beta$-almost all $b \in B$, there exists $c_{\alpha, b} > 0$ such that, for all non-zero vector $v$ in $V$ with $d(\mathbb{K}v, \mathbb{P}(W_b)) \geq \alpha$, one has

$$\inf_{n \geq 1} \frac{\|b_n \cdots b_1 v\|}{\|b_n \cdots b_1\| \| v\|} \geq c_{\alpha, b}.$$
We choose then the constant $c_\varepsilon > 0$ such that

\begin{equation}
\beta(\{b \in B \mid c_{\alpha,b} \geq \varepsilon\}) \geq 1 - \varepsilon/2.
\end{equation}

Then (3.1) follows from (3.3), (3.4) and (3.5).

b) For $p_n = b_n \cdots b_1$, $v$ in $x$ and $w$ in $y$, we have

\[ \frac{d(p_n,b_n \cdots b_1)}{d(x,y)} \leq \frac{\|p_n v \wedge p_n w\|}{\|v \wedge w\|} \leq \frac{\|p_n\|\|v\|\|w\|}{\|p_n v\|\|p_n w\|}, \]

hence (3.2) follows from (3.1) with $M_\varepsilon = (c_\varepsilon/2)^{-2}$.

**Proof of Proposition 3.1.** Let $\varphi$ be a continuous function on $X$. We want to prove that the family of functions $(P^n \varphi)_{n \geq 1}$ is equicontinuous. We can assume $\|\varphi\|_\infty \leq 1$. We fix $\varepsilon > 0$. By uniform continuity of $\varphi$, there exists $\eta_\varepsilon > 0$ such that, for all $x', y'$ in $\mathcal{P}(V)$,

\[ d(x', y') \leq \eta_\varepsilon \Rightarrow |\varphi(x') - \varphi(y')| \leq \varepsilon. \]

Let $x$, $y$ be in $\mathcal{P}(V)$ such that $d(x, y) \leq \eta_\varepsilon / M_\varepsilon$ where $M_\varepsilon$ is as in Lemma 3.2. We know from this lemma that the set

\[ B_{\varepsilon,x,y} := \{ b \in B \mid \sup_{n \geq 1} d(b_n \cdots b_1 x, b_n \cdots b_1 y) \leq M_\varepsilon d(x, y) \} \]

satisfies $\beta(B_{\varepsilon,x,y}^c) \leq \varepsilon$. We compute then by decomposing the following integral into two pieces,

\[ 2 \beta(B_{\varepsilon,x,y}^c) + 2 \beta(B_{\varepsilon,x,y}) \leq 3 \varepsilon. \]

Since this upperbound does not depend on $n$, this computation proves that the family $(P^n \varphi)_{n \geq 1}$ is equicontinuous. \hfill \square

**Example 3.3.** Lemma 3.2 and Proposition 3.1 are not always true when $V$ is a semisimple representation of $\Gamma_\mu$ which is not strongly irreducible. For instance, when $V = W \oplus K$ is a direct sum of an irreducible proximal representation of $\Gamma_\mu$ and the trivial representation, then the operator $P_\mu$ on $\mathcal{P}(V)$ is not equicontinuous. Indeed, in this case there are only two $P_\mu$-ergodic probability measures on $\mathcal{P}(V)$: $\nu$ which is supported by $\mathcal{P}(W)$ and the Dirac mass $\delta_{x_0}$ where $x_0$ is the $\Gamma_\mu$-invariant point in $\mathcal{P}(V)$. For every $x \neq x_0$, one has $\nu_x = \lim_{n \to \infty} \mu^n \ast \delta_x = \nu$ while $\nu_{x_0} = \delta_{x_0}$. Hence the map $x \mapsto \nu_x$ is not continuous and, according to Proposition 2.9.c, the operator $P_\mu$ is not equicontinuous.

However Proposition 3.1 is also true under a slightly more general assumption than strong irreducibility. This fact will be useful in the proof of Proposition 5.1.
Corollary 3.4. Let \( K \) be a local field of characteristic 0, \( V = K^d \), \( X := \mathbb{P}(V) \), \( \mu \) be a probability measure on \( \text{GL}(V) \). We assume that \( V \) is a direct sum of strongly irreducible representations \( V_i \) of \( \Gamma_\mu \) such that
\[
\sup_{g \in \Gamma_\mu} \frac{\|g|v_i\|}{\|g|v_j\|} < \infty, \quad \text{for all } i, j.
\]
Then the Markov-Feller operator \( P_\mu \) is equicontinuous.

Remark 3.5. One can prove that the converse is also true: when \( P_\mu \) is equicontinuous, condition (3.6) is satisfied.

Proof of Corollary 3.4. This is a corollary of the proofs of Proposition 3.1, Lemma 3.2 which are true with the same proof under this assumption (3.6). \( \square \)

3.2. Limit law on projective spaces. We can now prove part of the first two theorems of the introduction:

Proof of Theorem 1.1.i. By Proposition 3.1, \( P_\mu \) is equicontinuous. Our statement follows then from Proposition 2.9. \( \square \)

Proof of Theorem 1.3. Just apply Propositions 2.9.e and 3.1. \( \square \)

Remark 3.6. When \( V \) is not irreducible, the limit (1.1) in Theorem 1.1 does not always exist. Indeed, an example can be constructed with \( V = \mathbb{R}^2 \) and \( \mu \) a probability measure (with infinite moments) on the group of diagonal matrices \( \Gamma := \{ \text{diag}(e^t, e^{-t}) \mid t \in \mathbb{R} \} \).

4. Compact minimal subsets in homogeneous spaces

In this chapter \( G \) will be the group of \( K \)-points of a reductive \( K \)-group and \( \Gamma \) a Zariski-dense subsemigroup of \( G \). Our main goal is to describe the compact \( \Gamma \)-minimal subsets on an algebraic homogeneous space \( G/H \) (Proposition 4.2) and, in particular when \( K = \mathbb{R} \), to prove Theorem 1.7.

Studying the compact \( \Gamma \)-minimal subsets on algebraic homogeneous spaces is equivalent to studying the \( \Gamma \)-minimal subsets on projective spaces. Indeed, by Chevalley theorem, every algebraic homogeneous space \( G/H \) can be realized as an orbit in the projective space \( \mathbb{P}(V) \) of an algebraic representation \( V \) of \( G \). Conversely, since the \( G \)-orbits in the projective space \( \mathbb{P}(V) \) of an algebraic representation of \( G \) are locally closed, any compact \( \Gamma \)-minimal subset on \( \mathbb{P}(V) \) is supported by a \( G \)-orbit i.e. by an algebraic homogeneous space \( G/H \).
4.1. Zariski-dense subsemigroups. In this section we recall well-known definitions and properties of reductive groups and their Zariski dense subsemigroups.

Let $K$ be a local field of characteristic 0, $G$ be the group of $K$-points of a connected reductive $K$-group $G$, and $g$ be the Lie algebra of $G$. Let $A$ be a maximal $K$-split torus of $G$, $Z$ be the centralizer of $A$ in $G$, and $Z \to \mathfrak{a}; z \mapsto z^\omega$ the universal morphism of $Z$ in a real vector space. Since $A$ is central and cocompact in $Z$, any continuous morphism $A \to \mathbb{R}$ extends in a unique way as a continuous morphism $Z \to \mathbb{R}$ and hence defines a linear form on $\mathfrak{a}$. Thus, for any algebraic character $\chi$ of $A$, we let $\chi^\omega$ be the unique linear form on $\mathfrak{a}$, such that, for any $z$ in $A$, $|\chi(z)| = e^{\chi^\omega(z^\omega)}$. Let $\Sigma$ be the set of restricted roots of $A$ in $Z$. The set $\Sigma^\omega$ is a root system in the real vector space $\mathfrak{a}^*$. Let $\mathfrak{a}^+ \subset \mathfrak{a}$ be a closed Weyl chamber, $Z^+ := \{z \in Z \mid z^\omega \in \mathfrak{a}^+\}$, $\Pi$ be the corresponding set of simple restricted roots, $N$ be the corresponding maximal unipotent subgroup of $G$, $P := ZN$ be the corresponding minimal parabolic subgroup, $\mathcal{P} \simeq G/P$ be the full flag variety and $x_\Pi \in \mathcal{P}$ be the base point whose stabilizer is $P$.

Let $K$ be a “good” maximal compact subgroup of $G$ with respect to $\mathfrak{a}$, so that one has the Cartan decomposition $G = KZ^+K$ and the Iwasawa decomposition $G = KZN$. Every element $g$ of $G$ can be written as

\begin{equation}
(4.1) \quad g = k_{g,1}z^+_g k_{g,2} \text{ with } k_{g,1} \in K, \ z^+_g \in Z^+, \ k_{g,2} \in K.
\end{equation}

The element $\kappa(g) := (z^+_g)^\omega \in \mathfrak{a}^+$ is uniquely defined and called the Cartan projection of $g$.

For every $g$ in $G$ and $x = kx_\Pi$ in $\mathcal{P}$ with $k$ in $K$, there exists an element $z_{gk}$ in $Z$ such that

$$
gk \in Kz_{gk}N.
$$

The element $\sigma(g, x) := (z_{gk})^\omega \in \mathfrak{a}$ is uniquely defined and this map $\sigma : G \times \mathcal{P} \to \mathfrak{a}$ is a cocycle which is called the Iwasawa cocycle.

For any set $\Theta \subset \Pi$ of simple restricted roots, we let $A_\Theta$ be the centralizer in $A$ of the sum of the root spaces associated to the elements of $\Theta^c$, we let $Z_\Theta$ be the centralizer in $G$ of $A_\Theta$, we let $N_\Theta$ be the smallest unipotent normal subgroup of $N$ whose Lie algebra contains the root spaces associated to the elements of $\Theta$, we let $P_\Theta = Z_\Theta N_\Theta$ be the normalizer in $G$ of $N_\Theta$, we let $\mathcal{P}_\Theta = G/P_\Theta$ be the associated partial flag variety and $x_\Theta \in \mathcal{P}_\Theta$ be the base point whose stabilizer is $P_\Theta$. In particular when $\Theta = \Pi$, one has

$$
A_\Pi = A, \ Z_\Pi = Z, \ N_\Pi = N, \ P_\Pi = P, \ \mathcal{P}_\Pi = \mathcal{P}.
$$
Let $\Gamma$ be a Zariski-dense semigroup in $G$. Let $\Theta = \Theta_\Gamma \subset \Pi$ be the set of simple restricted roots $\alpha$ for which the set $\alpha(\kappa(\Gamma)) \subset \mathbb{R}$ is unbounded. Since the action of $\Gamma$ on $\mathcal{P}_\Theta$ is proximal, there exists a unique $\Gamma$-minimal subset $\Lambda_\Gamma \subset \mathcal{P}_\Theta$: it is called the limit set of $\Gamma$ in $\mathcal{P}_\Theta$ (see [2, 3.6]). For a suitable choice of torus $A$ and Weyl chamber $a^+$, we may assume that

(4.2) the base point $x_\Theta$ belongs to the limit set $\Lambda_\Gamma$.

Let $A_\Gamma$ be the smallest subtorus $A'$ of $A$ such that

(4.3) $\kappa(\Gamma)$ stays at bounded distance from $\omega(A')$.

and let $H_\Gamma$ be the following solvable subgroup of $G$

(4.4) $H_\Gamma := A_\Gamma N_\Theta$.

Let $Z_\Gamma$ be the group

(4.5) $Z_\Gamma := Z_\Theta/A_\Gamma$.

The following $G$-equivariant fibration

(4.6) $Y_\Gamma = G/H_\Gamma \rightarrow \mathcal{P}_\Theta = G/P_\Theta$

is a principal $Z_\Gamma$-bundle. This homogeneous space $Y_\Gamma$ will play a crucial role in our analysis. We will denote by $y_\Gamma$ the base point of $Y_\Gamma$.

\textit{Remark 4.1.} When $\mathbb{K} = \mathbb{R}$, according to [13], one has $\Theta_\Gamma = \Pi$ and according to [2], one has $A_\Gamma = A$, and hence $H_\Gamma = A N$ is a maximal $\mathbb{R}$-split solvable algebraic subgroup of $G$ and the principal bundle (4.6) is $Y_\Gamma = G/AN \rightarrow \mathcal{P}_\Pi = G/P$.

4.2. \textbf{Minimal subsets in homogeneous spaces}. The following Proposition 4.2, describes exactly which algebraic homogeneous spaces support a compact $\Gamma$-invariant subset.

\textbf{Proposition 4.2.} Let $\mathbb{K}$ be a local field of characteristic 0, $G$ be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, $\Gamma$ be a Zariski-dense subsemigroup of $G$, $H$ be an algebraic subgroup of $G$ and $X = G/H$. Then the following two assertions are equivalent:

(i) There exists a compact $\Gamma$-invariant subset in $X$,

(ii) $H$ contains a conjugate of the group $H_\Gamma := A_\Gamma N_{\Theta_\Gamma}$.

We will need the following Lemma which does not involve Zariski dense subsemigroups and which describes the cluster points of a $G$-orbit in a projective space.
Lemma 4.3. Let $\mathbb{K}$ be a local field of characteristic 0, $G$ be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, $(V, \rho)$ be an algebraic representation of $G$ and $\Theta \subset \Pi$ a subset of restricted simple roots. Let $g_k$ be a sequence in $G$ such that

$$\text{(4.7)} \quad \text{for all } \alpha \text{ in } \Theta, \text{ one has } \alpha^\omega(\kappa(g_k)) \xrightarrow[k \to \infty]{} \infty.$$  

and $\pi$ be a non zero limit point in $\text{End}(V)$ of a sequence $\lambda_k \rho(g_k)$ with $\lambda_k$ in $\mathbb{K}$.

a) For all $x$ in $\mathbb{P}(V) \setminus \mathbb{P}(\ker \pi)$, the limit $\lim_{k \to \infty} g_k x$ exists and belongs to the projective space $\mathbb{P}(\text{im } \pi)$.
b) This space $\mathbb{P}(\text{im } \pi)$ is included in the set of fixed points of a conjugate of the unipotent group $N_\Theta$.
c) More precisely, let $A' \subset A$ be the smallest subtorus of $A$ such that $\sup_k d(\kappa(g_k), A') < \infty$. This space $\mathbb{P}(\text{im } \pi)$ is included in the set of fixed points of a conjugate of the solvable group $A'N_\Theta$.

Proof of Lemma 4.3. a) The endomorphism $\pi$ induces a well-defined map from $\mathbb{P}(V) \setminus \mathbb{P}(\ker \pi)$ to $\mathbb{P}(V)$ and the sequence $g_k$ converges toward $\pi$ uniformly on compact subsets of $\mathbb{P}(V) \setminus \mathbb{P}(\ker \pi)$.

b) and c) Using the Cartan decomposition $G = KZ^+K$ and using the compactness of the quotient $Z/A$, we may assume that the sequence $g_k$ is in $A^+$. We may also assume that, for any pair of weights $\chi_1$, $\chi_2$ of $A$ in $V$, the sequence $\chi_1^\omega(\kappa(g_k)) - \chi_2^\omega(\kappa(g_k))$ converges to a limit $\ell_{\chi_1, \chi_2} \in \mathbb{R} \cup \{\pm \infty\}$. Let $S$ be the non-empty set of weights of $A$ in $V$ such that, for all $\chi_1$ in $S$, when $\chi_2$ is also in $S$, the limit $\ell_{\chi_1, \chi_2}$ is finite and, when $\chi_2$ is not in $S$, the limit $\ell_{\chi_1, \chi_2}$ is $+\infty$. The image of $\pi$ is then the direct sum $\text{im } \pi = \bigoplus_{\chi \in S} V_\chi$ of the weight spaces $V_\chi$ of $A$ in $V$ such that $\chi$ is in $S$. By definition of $\Theta$, if $\chi$ belongs to $S$ and $\alpha \in \Sigma^+$ is a positive root whose decomposition into simple roots contains elements of $\Theta$, the character $\chi + \alpha$ is not a weight of $V$. This proves that $\text{im } \pi$ is included in the space $V^{N_\Theta}$ of fixed points of $N_\Theta$. Moreover, by definition, all the characters of $S$ coincide on $A'$, hence this subtorus acts by a character on $\text{im } \pi$. \hfill \Box

Proof of Proposition 4.2. We first want to prove $(i) \Rightarrow (ii)$. As the limit cone $\ell_\Gamma$ of $\kappa(\Gamma)$ in $\mathfrak{a}$ is convex (see [2, § 4]), there exists a sequence $g_k$ in $\Gamma$ such that, for any weight $\chi$ of $A$ that is non trivial on $A^\Gamma$, one has $|\chi^\omega(\kappa(g_k))| \xrightarrow[k \to \infty]{} \infty$. Now, by Chevalley Theorem [6, 5.1], there exists an algebraic representation $(V, \rho)$ of $G$ and a point $y_0$ in $\mathbb{P}(V)$ such that the stabilizer of $y_0$ in $G$ is equal to $H$. We may assume that the $G$-orbit $Gy_0$ spans the $\mathbb{K}$-vector space $V$. After extraction, we may
assume that, for some $\lambda_k$ in $K$, the sequence $\lambda_k \rho(g_k)$ has a non-zero limit $\pi$ in $\text{End}(V)$.

By assumption there exists a point $y$ on the $G$-orbit $Gy_0$ such that the orbit closure $\overline{\Gamma y}$ is a compact subset of $Gy_0$. As $Gy_0$ spans $V$ and $\Gamma$ is Zariski dense in $G$, we can assume $y \notin \mathbb{P}(\ker \pi)$. According to Lemma 4.3, the limit $hy = \lim_{k \to \infty} g_k y$ exists and is invariant by a conjugate of $A_{\Gamma}N_\Theta$. Since this point $\pi y$ is still on the $G$-orbit $Gy_0$, this proves that the group $A_{\Gamma}N_\Theta$ is contained in a conjugate of $H$.

Implication $(ii) \Rightarrow (i)$ follows from the following more precise Proposition 4.5.

□

Remark 4.4. The reader who is only interested in real Lie groups may avoid the next three Sections 4.3, 4.4 and 4.6 and go directly to Section 4.5. Indeed, when $K = \mathbb{R}$, one has $\Theta = \Theta_{\Gamma}$ and $A_{\Gamma} = A$, so that the whole space $Y_{\Gamma} = G/AN$ is compact and implication $(ii) \Rightarrow (i)$ is trivial.

4.3. Minimal subsets in $Y_{\Gamma}$. We will now describe the set of compact $\Gamma$-minimal subsets of the homogeneous space $Y_{\Gamma} = G/H_{\Gamma}$. The main point will be to prove that this set is non empty.

We recall that $y_\Gamma$ is the base point of $Y_{\Gamma}$, that $Y_{\Gamma}$ is endowed with a left-action of $G$ and a commuting free right-action of $Z_{\Gamma}$, and that the set of $N_\Theta$-fixed points $Y_{\Gamma}^{N_\Theta}$ is equal to the fiber $\pi^{-1}(x_\Theta) = Z_{\Theta}y_\Gamma = y_\Gamma Z_{\Gamma}$ of the principal $Z_{\Gamma}$-bundle $Y_{\Gamma} \to \mathbb{P}_{\Theta}$.

Proposition 4.5. Let $\mathbb{K}$ be a local field of characteristic 0, $G$ be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, $\Gamma$ be a Zariski-dense subsemigroup of $G$ and $\Theta = \Theta_{\Gamma}$. Let $H_{\Gamma} = A_{\Gamma}N_\Theta$, $Y_{\Gamma} := G/H_{\Gamma}$ and $y$ be a point of $Y_{\Gamma}$ whose image $\pi(y)$ in $\mathbb{P}_{\Theta}$ is in the limit set $\Lambda_{\Gamma}$.

a) The orbit closure $\overline{\Gamma y}$ is compact and $\Gamma$-minimal.

b) The set $M_y := \{ z \in Z_{\Gamma} \mid yz \in \overline{\Gamma y} \}$ is a compact subgroup of $Z_{\Gamma}$.

c) For any $y'$ in $\overline{\Gamma y}$, one has $M_{y'} = M_y$.

d) For every $z$ in $Z_{\Gamma}$, one has $M_{yz} = z^{-1}M_y z$. When $y = y_\Gamma$, the group $M_{\Gamma} := M_{y_\Gamma}$ is called the limit group of $\Gamma$.

e) The map $F \mapsto \{ z \in Z_{\Gamma} \mid y_\Gamma z \in F \}$ is a bijection between the sets

\{compact $\Gamma$-minimal subset in $Y_{\Gamma}$\} $\longleftrightarrow M_{\Gamma} \backslash Z_{\Gamma}$

In case $\mathbb{K} = \mathbb{R}$, the limit group was introduced by Benoist [3] and Proposition 4.5 was proved by Guivarch and Raugi [16].

We will need a few lemmas. First, to exhibit compact orbits on non compact homogeneous spaces, we will use Lemma 4.6 below, which, in a given linear representation, produces subspaces where $\Gamma$ almost acts by similarities:
Lemma 4.6. Let $\mathbb{K}$ be a local field of characteristic 0, $V = \mathbb{K}^d$, $\Gamma$ be a subsemigroup of $\text{GL}(V)$ and $r$ be its proximal dimension. There exists $C > 1$ such that, for every $\gamma$ in $\Gamma$, $\pi$ in $\mathbb{K}\Gamma$ with rank $r$ and $v, v' \neq 0$ in $W = \text{im} \pi$, one has

$$\|\gamma v'\| \leq C \|\gamma v\|.$$

Proof. First, note that, for any $\varepsilon > 0$, there exists $\alpha > 0$ such that, for any $x \in \mathbb{P}(V)$ and $\pi$ in $\mathbb{K}\Gamma$ with rank $r$, if $d(x, \mathbb{P}(\ker \pi)) \geq \varepsilon$, one has $\|\pi w\| \geq \alpha \|\pi\| \|w\|$. Indeed, if this were not the case, one could find a sequence of elements of $\mathbb{K}\Gamma$ with rank $r$ but with a non-zero cluster point of rank $< r$.

Using the compactness of the Grassmann varieties, we pick $\varepsilon > 0$ such that, for any $U$ in $\mathbb{G}_{n-r}(V)$ and $U'$ in $\mathbb{G}_{n-r+1}(V)$, there exists $x$ in $\mathbb{P}(U')$ with $d(x, \mathbb{P}(U)) \geq \varepsilon$, and we let $\alpha$ be as above. For $\gamma$ in $\Gamma$, $W = \text{im} \pi$ in $\Lambda^r \Gamma$ and $v \neq 0$ in $W$, we can find $w$ in $V$ such that $\pi w = v$ and $d(\mathbb{K}w, \mathbb{P}(\ker \pi)) \geq \varepsilon$. We get

$$\alpha \|\pi\| \|w\| \leq \|v\| \leq \|\pi\| \|w\|$$

$$\alpha \|\gamma \pi\| \|w\| \leq \|\gamma v\| \leq \|\gamma \pi\| \|w\|$$

hence

$$\alpha \|\gamma \pi\| \leq \frac{\|\gamma v\|}{\|v\|} \leq \frac{1}{\alpha} \|\gamma \pi\|.$$

(4.8) follows immediately. \qed

Now, the following Lemma constructs a representation that is adapted to the setting of Proposition 4.5.

Lemma 4.7. Let $\mathbb{K}$ be a local field of characteristic 0, $G$ be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, $\Gamma$ be a Zariski-dense subsemigroup of $G$ and $H$ be an algebraic subgroup containing the group $H_{\Gamma} := A_{\Gamma}N_{\Theta}$.

a) Then there exists an algebraic representation $V$ of $G$ and a point $x$ in $\mathbb{P}(V)$ whose stabilizer in $G$ is equal to $H$ and whose orbit spans $V$.

b) For such a representation $V$, the group $A_{\Gamma}$ acts by a character on the space $V^{N_{\Theta}}$.

c) There exists $C > 1$ such that, for every $\gamma$ in $\Gamma$, and $v, v'$ non-zero in $V^{N_{\Theta}}$, one has

$$\|\gamma v'\| \leq C \|\gamma v\|.$$

Proof of Lemma 4.7. a) This is a special case of Chevalley Theorem [6, 5.1].

b) We write $x = \mathbb{K}v$ and $V = \bigoplus_i V_i$, where each $V_i$ is an irreducible subrepresentation with highest weight $\chi_i$. We have $V^{N_{\Theta}} = \bigoplus_i V_i^{N_{\Theta}}$.
and, for any $i$, $V_{i,\Theta}$ is the sum of the weight spaces $V_{i,\chi'}$ of $V_i$ associated to characters $\chi'$ of $A$ such that $\chi_i - \chi'$ is a sum of elements of $\Theta^c$. In particular, since $A_\Gamma \subset A_\Theta$, $A_\Gamma$ acts by a character on $V_{i,\Theta}$. Now, write $v = \sum_i v_i$. Since $Gv$ spans $V$, for any $i$, we have $v_i \neq 0$. As $A_\Gamma$ fixes $Kv$, $A_\Gamma$ acts by a character on this line, hence all the characters $\chi_i$ have the same restriction to $A_\Gamma$, what should be proved.

c) Let us prove that the proximal dimension of $\rho(\Gamma)$ is the dimension of $V_{i,\Theta}$ and that, due to (4.2), $V_{i,\Theta}$ is the image of an element of $K\rho(\Gamma)$: this and Lemma 4.6 will imply the result.

Indeed, let $g_k$ be a sequence in $\Gamma$ and assume, for some $\lambda_k \in K$, the sequence $\lambda_k \rho(g_k)$ converges towards a non zero endomorphism $\pi$ of $V$. For any $k$, let $g_k = h_kz_k\ell_k$ be a Cartan decomposition of $g_k$ with $h_k$, $\ell_k$ in $K$ and $z_k$ in $Z^+$. After extracting a subsequence, we may assume $\lambda_k \rho(z_k)$ converges towards a non zero endomorphism $\varpi$ of $V$ and $\pi$ and $\varpi$ have the same rank. Since $\varpi$ is not zero, we must have

$$\sup_k \left| \log \| \rho(z_k) \| - \log |\lambda_k| \right| < \infty.$$  

Now, since $A$ is cocompact in $Z$ and acts by characters on the weight spaces of $V$, we have

$$\sup_{z \in Z^+} \left| \log \| \rho(z) \| - \max_i \chi_i^\omega(z^\omega) \right| < \infty.$$  

As, for any $k$, $z_k^\omega = \kappa(g_k)$ and all the characters $\chi_i$ have the same restriction to $A_\Gamma$, we get

$$\sup_{i,k} |\chi_i^\omega(z_k^\omega) - \log |\lambda_k|| < \infty.$$  

Finally, for any $i$, we let $X_i$ be the set of characters of $A$ such that $\chi_i - \chi'$ is a sum of elements of $\Theta^c$. By definition of $\Theta$, and still since $z_k^\omega = \kappa(g_k)$, we get

$$\sup_{i,k} \sup_{\chi' \in X_i} |(\chi')^\omega(z_k^\omega) - \log |\lambda_k|| < \infty.$$  

Hence, we have $\bigoplus_{i,\chi' \in X_i} V_{i,\chi'} = V_{i,\Theta} \subset \im \varpi$ and $\varpi$ has rank $\geq \dim V_{N,\Theta}$. Conversely, since the limit cone $\ell_\Gamma$ of $\kappa(\Gamma)$ in $a$ is convex (see [2, §4]), we can chose $g_k$ in such a way that, for any $\alpha \in \Theta$, one has $\alpha^\omega(\kappa(g_k)) \to \infty$. Since by (4.2) $x_\Pi$ belongs to the inverse image of $\Lambda_\Gamma$ in $P$, we can assume $g_kV_{N,\Theta} \to V_{N,\Theta}$. Then, we get $\im \pi = \im \varpi = V_{N,\Theta}$ and we are done.

Proof of Proposition 4.5. We will first prove that the orbit closure $\overline{\Gamma y_\Gamma}$ in $Y_\Gamma$ is compact. We pick a representation $V$ of $G$ as in Lemma 4.7
with $H = H_\Gamma$ and we let $d = \dim V^{N_\Theta}$. We set

$$
\mathcal{R} = \{(x_1, \ldots, x_{d+1}) \in \mathbb{P}(V)^{d+1} \text{ invariant by a conjugate of } N_\Theta \\
\text{and } d \text{ by } d \text{ linearly independant}\}.
$$

We claim that the $G$-orbit of any element $x = (x_1, \ldots, x_{d+1}) \in \mathcal{R}$ is closed in $\mathcal{R}$. Indeed, we will check that the stabilizer $G_\mathcal{R}$ of such an element $x$ is conjugate to $H_\Gamma$. We can assume $x$ to be $N_\Theta$-invariant.

By Lemma 4.7.c, if $(x_1, \ldots, x_{d+1})$ is in $\mathcal{R}$ and $x_1, \ldots, x_{d+1}$ belong to $V^{N_\Theta}$, the $\Gamma$-orbit of $(x_1, \ldots, x_{d+1})$ in $\mathcal{R}$ has compact closure. Hence the orbit closure $\overline{\Gamma y_0}$ in $Y_\Gamma = G/H_\Gamma$ is compact.

The remaining statements follow from the following Lemma 4.8, applied to the principal $Z_\Gamma$-bundle $\pi^{-1}(\Lambda_\Gamma) \xrightarrow{\pi} \Lambda_\Gamma$. □

**Lemma 4.8.** Let $\Gamma$ and $Z$ be locally compact topological groups. Let $Y$ be a locally compact topological space, equipped with a continuous left-action of $\Gamma$ and a continuous right-action of $Z$ that commute to each other, such that the action of $Z$ is proper and cocompact and the action of $\Gamma$ on $X = Y/Z$ is minimal. Assume that there exists a point $y_0$ in $Y$ such that the orbit closure $\overline{\Gamma y_0}$ is compact. Then, for all $y$ in $Y$,

a) The orbit closure $\overline{\Gamma y}$ is also compact and is $\Gamma$-minimal.

b) The set $M_y := \{z \in Z \mid yz \in \overline{\Gamma y}\}$ is a compact subgroup of $Z$.

c) For any $y'$ in $\overline{\Gamma y}$, one has $M_{y'} = M_y$.

d) For every $z$ in $Z$, one has $M_{yz} = z^{-1}M_yz$.

e) The map $F \mapsto \{z \in Z \mid y_0z \in F\}$ is a bijection between the sets

$$
\{\text{compact } \Gamma \text{-minimal subset in } Y\} \leftrightarrow M_{y_0}\setminus Z.
$$

**Proof of Lemma 4.8.** a) Since $F_0 = \overline{\Gamma y_0}$ contains a $\Gamma$-minimal closed subset, we may assume it is $\Gamma$-minimal. Since $X$ is $\Gamma$-minimal, one has $\pi(F_0) = X$. Hence for every $y$ in $Y$, there exists $z$ in $Z$ such that $y$ belongs to $F_0z$. Since the actions of $\Gamma$ and $Z$ commute the set $F_0z$ is $\Gamma$-invariant and $\Gamma$-minimal and the orbit closure $\overline{\Gamma y}$ is equal to $F_0z$.

b) Since $\overline{\Gamma y}$ is $\Gamma$-minimal, the set $M_y$ can also be defined as

$$
M_y = \{z \in Z \mid \overline{\Gamma y}z = \overline{\Gamma y}\}.
$$

Hence $M_y$ is a compact subgroup of $Z$.

c), d) and e) follow also from (4.10). □
4.4. Limit group of a Zariski dense semigroup. In this section we give another definition of the limit group $M_{\Gamma}$ of a Zariski dense subgroup that will be useful for the proof of Theorem 1.10. This definition is similar to the one which has been introduced for real Lie groups in the appendix of [3, Th. 8.2].

Let $\mathbb{K}$ be a local field of characteristic 0, $G$ be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group and $\Gamma$ be a Zariski dense subsemigroup of $G$. We keep the notations

$$\Theta = \Theta_{\Gamma}, Z_{\Theta}, N_{\Theta}, P_{\Theta}, x_{\Theta}, Z_{\Gamma}, Y_{\Gamma}, y_{\Gamma}, \ldots$$

from Section 4.1. Let $C_{\Gamma}$ be the center of $Z_{\Gamma}$. By construction this group $C_{\Gamma}$ is compact modulo $A_{\Theta}/A_{\Gamma}$.

Let $N_{\Theta}^{-}$ be the $A$-invariant unipotent subgroup of $G$ opposite to $P_{\Theta}$. According to the Bruhat decomposition [6, 21.15], the set

$$U_{\Theta} = N_{\Theta}^{-} Z_{\Theta} N_{\Theta}$$

is a Zariski open subset of $G$ and every element $g$ of $U_{\Theta}$ can be written in a unique way as a product $g = n_{\Theta}^{-} z_{\Theta} n_{\Theta}$ with $n_{\Theta}^{-}$, $z_{\Theta}$ and $n_{\Theta}$ in $N_{\Theta}^{-}$, $Z_{\Theta}$ and $N_{\Theta}$ respectively. We introduce the Bruhat projection $m$ as the map

$$m : U_{\Theta} \to Z_{\Gamma} ; \; g \mapsto m(g) := z_{\Theta} A_{\Gamma} = \text{image of } z_{\Theta} \text{ in } Z_{\Gamma}.$$  

By definition of $\Theta = \Theta_{\Gamma}$, we can find a semisimple element $\gamma_{0}$ of $\Gamma$ whose action on $P_{\Theta}$ is proximal (see [2, 3.6]). Hence, for a suitable choice of a torus $A$ and Weyl chamber $a^{+}$ we may assume a stronger condition than (4.2), namely that

$$\gamma_{0} \in Z_{\Theta} \cap \Gamma \text{ with } x_{\Theta} \text{ as attractive fixed point.}$$

Here are the alternative definition and the main properties of the limit group $M_{\Gamma}$.

Proposition 4.9. Let $\mathbb{K}$ be a local field of characteristic 0, $G$ be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, $\Gamma$ be a Zariski dense subsemigroup of $G$. We choose $A$ and $a^{+}$ satisfying (4.13).

a) The limit group $M_{\Gamma}$ is equal to the closure $M_{\Gamma} = \overline{m(\Gamma \cap U_{\Theta})}$.

b) This group $M_{\Gamma}$ is a Zariski dense and compact subgroup of $Z_{\Gamma}$.

c) Moreover, if $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_{p}$, the group $C_{\Gamma} M_{\Gamma}$ is open in $Z_{\Gamma}$.

Remark 4.10. By reasoning as in the proof of [20, 1.3], one could also prove that if $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_{p}$ the group $M_{\Gamma}$ is open in $Z_{\Gamma}$.

We need the following Lemma
Lemma 4.11. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$, $G$ be the group of $\mathbb{K}$-points of a connected semisimple $\mathbb{K}$-group and $H$ be a compact Zariski-dense subgroup of $G$. Then $H$ is open in $G$.

An example of such a group is $H = \text{SL}(d, \mathbb{Z}_p)$ in $G = \text{SL}(d, \mathbb{Q}_p)$.

Proof of Lemma 4.11. Since $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$, the Lie algebra $\mathfrak{h}$ of $H$ is a $\mathbb{K}$-subspace of the Lie algebra $\mathfrak{g}$ of $G$. Since $H$ is Zariski dense in $G$, $\mathfrak{h}$ is $\text{Ad} G$-invariant and hence $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Let $H'$ be the Kernel of the adjoint action in $\mathfrak{g}/\mathfrak{h}$. This group $H'$ is an algebraic subgroup of $G$ with Lie algebra $\mathfrak{h}$. Since $H$ is compact, and since $H \cap H'$ is open in $H$, the group $H \cap H'$ has finite index in $H$. Since $H$ is Zariski dense in $G$, $H \cap H'$ and also $H'$ are Zariski dense in $G$. Hence one has $\mathfrak{h} = \mathfrak{g}$.

Proof of Proposition 4.9. a) We set $M_\Gamma' = m(\Gamma \cap U_\Theta)$. We want to prove that $M_\Gamma = M_\Gamma'$. We only have to check

\begin{equation}
\prod_{\gamma \in \Gamma} \gamma_{\Gamma} \cap \prod_{\gamma \in \Gamma} \gamma_{\Gamma} = y_{\Gamma} M_{\Gamma}'.
\end{equation}

We first prove the inclusion $\subseteq$ in (4.14). Let $g_k$ be a sequence in $\Gamma$ such that the limit $y_{\infty} = \lim_{k \to \infty} g_k y_{\Gamma}$ exists and belongs to the fiber $y_{\Gamma} Z_{\Gamma}$. We want to prove that $y_{\infty}$ belongs to the set $y_{\Gamma} M_{\Gamma}$. We first notice that, for $k$ large, $g_k$ belongs to $U_\Theta$ and we write as in (4.11) $g_k = n_{g_k} z_{g_k} n_{g_k}$. Since $y_{\infty}$ belongs to the fiber $y_{\Gamma} Z_{\Gamma}$, we must have $\lim_{k \to \infty} n_{g_k} = e$ and the sequence $m(g_k)$ must converge to some $m_{\infty} \in M_{\Gamma}$. But then, one has the equality

\begin{equation}
y_{\infty} = \lim_{k \to \infty} z_{g_k} y_{\Gamma} = \lim_{k \to \infty} y_{\Gamma} m(g_k) = y_{\Gamma} m_{\infty}
\end{equation}

and $y_{\infty}$ belongs to $y_{\Gamma} M_{\Gamma}'$.

Finally, we prove the inverse inclusion $\supseteq$ in (4.14). By construction the image $m(\gamma_0)$ of $\gamma_0$ in $Z_{\Gamma}$ is an elliptic element. In particular, there exists a sequence $k_i \to \infty$ such that

\begin{equation}
\lim_{i \to \infty} m(\gamma_0)^{k_i} = e.
\end{equation}

Because of (4.13), the Bruhat decomposition (4.11) is related to the element $\gamma_0$ by the formulas

\begin{equation}
N_{\Theta}^{-} := \{ g \in G \mid \lim_{k \to \infty} \gamma_0^k g \gamma_0^{-k} = e \},
\end{equation}

\begin{equation}
Z_{\Theta} := \{ g \in G \mid \lim_{i \to \infty} \gamma_0^{k_i} g \gamma_0^{-k_i} = \lim_{i \to \infty} \gamma_0^{-k_i} g \gamma_0^{k_i} = g \},
\end{equation}

\begin{equation}
N_{\Theta} := \{ g \in G \mid \lim_{k \to \infty} \gamma_0^k g \gamma_0^{-k} = e \}.
\end{equation}
In particular, for \( g \) in \( \Gamma \cap U_{\Theta} \), one has \( y_{\Gamma} m(g) = z_{\Gamma} y_{\Gamma} = \lim_{i \to \infty} \gamma_{\Theta}^{k_i} g y_{\Gamma} \), hence \( y_{\Gamma} M_{\Gamma} \subset \Gamma y_{\Gamma} \).

b) By Proposition 4.5, \( M_{\Gamma} \) is a compact subgroup of \( Z_{\Gamma} \). Since \( \Gamma \) is Zariski dense in \( G \) and \( m \) is a rational map, it follows from a), that \( M_{\Gamma} \) is Zariski dense in \( Z_{\Gamma} \).

c) Since the quotient group \( Z_{\Gamma}/C_{\Gamma} \) is a finite index subgroup in the group of \( \mathbb{K} \)-points of a semisimple \( \mathbb{K} \)-group and since the image of \( M_{\Gamma} \) in this quotient is compact and Zariski dense, our claim follows from Lemma 4.11.

This ends the proof of Proposition 4.2.

4.5. **Minimal subsets and compact orbits for real groups.** In this section one has \( \mathbb{K} = \mathbb{R} \) and we prove Theorem 1.7.

**Proof of Theorem 1.7.** Since the \( G \)-orbits in \( \mathbb{P}(V) \) are locally closed, any \( \Gamma \)-minimal closed subset of \( \mathbb{P}(V) \) is contained in a \( G \)-orbit and Theorem 1.7 follows from Proposition 4.12 below.

Proposition 4.12 strengthens Proposition 4.2 when \( \mathbb{K} = \mathbb{R} \).

**Proposition 4.12.** Let \( G \) be the group of real points of a connected reductive \( \mathbb{R} \)-group, \( \Gamma \) be a Zariski-dense subsemigroup of \( G \), \( H \) be an algebraic subgroup of \( G \) and \( X = G/H \).

a) \( X \) contains a compact \( \Gamma \)-minimal subset if and only if \( X \) is compact.

b) In this case, there exists a unique \( \Gamma \)-minimal subset in \( X \).

**Proof.** a) If \( X \) is compact, it contains a \( \Gamma \)-minimal subset. Conversely, if \( X \) contains a compact \( \Gamma \)-minimal subset, by Proposition 4.2, we can assume \( A_{\Gamma} N_{\Theta_{\Gamma}} \subset H \). Since \( \mathbb{K} = \mathbb{R} \), one has \( \Theta_{\Gamma} = \Pi \), \( A_{\Gamma} = A \) and \( N_{\Theta} = N \). As \( P = ZN \) is cocompact in \( G \) and \( A \) is cocompact in \( Z \), \( AN \) is cocompact in \( G \) and \( X \) is compact.

b) If the homogeneous space \( X = G/H \) is compact, by Proposition 4.2 applied to \( \Gamma = G \), the algebraic group \( H \) contains a conjugate of \( AN \). The last statement then follows from Lemma 4.13 below.

**Lemma 4.13.** Let \( G \) be the group of real points of a connected reductive \( \mathbb{R} \)-group, \( H = AN \) be a maximal \( \mathbb{R} \)-split solvable algebraic subgroup of \( G \) and \( \Gamma \) be a Zariski-dense subsemigroup of \( G \). Then there exists a unique \( \Gamma \)-minimal subset \( F \) in \( G/AN \).

This Lemma is a special case of a result of Guivarc’h and Raugi in [16, Th. 2] relying on the appendix of [3].

**Remark 4.14.** Since \( A \) is an \( \mathbb{R} \)-split torus, the number of connected components of \( A \) is \( 2^{\dim A} \). There may exist more than one \( \Gamma \)-minimal
subset in $G/A_eN$ where $A_e$ is the connected component of $A$. For instance, when $G = \text{SL}(3, \mathbb{R})$ and $\Gamma$ preserves a properly convex subset $\Omega \subset \mathbb{P}(\mathbb{R}^3)$, there are exactly four $\Gamma$-minimal subsets in $G/A_eN$. See [16] for more details.

Proof of Lemma 4.13. By Proposition 4.5, this amounts to proving that $M_\Gamma = Z_\Gamma = Z/A$. Now, by definition, $M_\Gamma$ is a compact subgroup of $Z_\Gamma$, so that, by Godement Theorem, it is Zariski closed. The result follows since, by Proposition 4.9, it is also Zariski dense. □

To conclude this section, we will establish bijection (1.6). This will follow from Proposition 4.2 applied to $\Gamma = G$ and the following

Lemma 4.15. Let $G$ be the group of real points of a connected reductive $\mathbb{R}$-group, $P = MAN$ a minimal parabolic subgroup, $H$ an algebraic subgroup containing $AN$ and $X = G/H$. Then, the set $X^{AN}$ of fixed points of $AN$ in $X$ is an $M$-orbit.

This will be a consequence of the following classical

Lemma 4.16. Let $K$ be a field, $G$ be the group of $K$-points of a connected reductive $K$-group and $P$ be the group of $K$-points of a minimal parabolic $K$-subgroup. Then, for any $g$ in $G$, $g$ belongs to the subgroup of $G$ spanned by $P$ and $gPg^{-1}$.

Proof. We let $A$ be the group of $K$-points of a maximal $K$-split torus contained in $P$, $\Sigma$ be the set of restricted roots of $A$ in the Lie algebra of $G$, $\Sigma^+$ be the set of positive roots associated to the choice of $P$, $\Pi$ be the basis of $\Sigma^+$ and $W = N_G(A)/Z_G(A)$ be the Weyl group of $A$. For $w$ in $W$, let us prove by induction on $\ell_w = \sharp(\Sigma^+ \cap w(-\Sigma^+))$ that $w$ may be written as a product of reflections $s_\alpha$ associated to elements $\alpha$ of $\Pi \cap w(-\Sigma^+)$. Indeed, if $\ell_w = 0$, there is nothing to prove. If $\ell_w > 0$, we have necessarily $\Pi \cap w(-\Sigma^+) \neq \emptyset$. We pick $\alpha \in \Pi \cap w(-\Sigma^+)$. For any $\beta \in \Sigma^+ \setminus \mathbb{R}\alpha$, since $s_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha$ may be written as linear combination of elements of $\Pi$ in which either all coefficients are $\geq 0$ or all coefficients are $\leq 0$, we have $s_\alpha(\beta) \in \Sigma^+$. Thus $s_\alpha$ permutes the elements of $\Sigma^+ \setminus \mathbb{R}\alpha$ and, if $w' = s_\alpha w$, we have $\Sigma^+ \cap w'(-\Sigma^+) = \Sigma^+ \cap w(-\Sigma^+) \setminus \mathbb{R}\alpha$.

The result follows by induction.

Now, let $g$ be in $G$ and let us prove $g$ belongs to the subgroup $Q$ spanned by $P$ and $gPg^{-1}$. By Bruhat decomposition, we can assume $g$ normalizes $A$. Set $w = gZ_G(A) \in W$. By construction, for any $\alpha$ in $\Sigma^+ \cap w(-\Sigma^+)$, $N_Q(A)/Z_G(A)$ contains the reflection $s_\alpha$ associated to
α. Since we have proved that \( w \) may be written as the product of such reflections, we get \( w \in N_Q(A)/Z_G(A) \), hence \( g \in Q \).

**Proof of Lemma 4.15.** Let \( x \) and \( x' = gx \) be two points of \( X^{AN} \). Still by Bruhat decomposition, we can assume \( g \) normalizes \( A \) and hence \( M \). We get \( P = MAN \) and \( g^{-1}Pg = MA(g^{-1}Ng) \). As \( x' = gx \) is \( N \)-invariant, \( x = g^{-1}Ng \)-invariant and \( Mx = Px = g^{-1}Pg \). Since, by Lemma 4.16, \( g \) belongs to the subgroup spanned by \( P \) and \( g^{-1}Pg \), we get \( gMx = Mx \), hence \( x' \in Mx \). □

4.6. **Minimal subsets on the flag variety.** In this section \( K = \mathbb{R} \) or \( \mathbb{Q}_p \) for a prime number \( p \). We prove that the flag variety \( P = G/P \) supports only finitely many \( \Gamma \)-minimal subsets. This result is easier to prove when \( K = \mathbb{R} \) since in this case there exists only one \( \Gamma \)-minimal subset on the flag variety.

**Proposition 4.17 (Finiteness).** Let \( K = \mathbb{R} \) or \( \mathbb{Q}_p \), \( G \) be the group of \( K \)-points of a reductive \( K \)-group, \( \Gamma \) be a Zariski dense subsemigroup in \( G \) and \( P \) be a minimal parabolic subgroup of \( G \). Then there exists only finitely many \( \Gamma \)-minimal subsets in the flag variety \( P = G/P \).

**Remarks 4.18.**

1. When the field \( K \) is \( \mathbb{R} \), or more generally when the set \( \Theta_{\Gamma} \) is the whole set \( \Pi \) of simple restricted roots, the action on the full flag variety is proximal and there exists only one \( \Gamma \)-minimal subset in \( P \).

2. When the field \( K \) is \( \mathbb{C} \), there exists also only one \( \Gamma \)-minimal subset in \( P \). Indeed the Zariski closure \( H \) of \( \Gamma \) in \( G \) for the real Zariski topology is a reductive group which contains a real form of \( G \). Such a group \( H \) has only one compact orbit in the flag variety \( P \) and this orbit is a partial flag variety \( H/Q \) of \( H \). Hence our claim follows from the first remark combined with Proposition 4.12.

3. When the field \( K \) is \( \mathbb{Q}_p \), there may exist more than one \( \Gamma \)-minimal subset in \( P \). This is the case when \( \Gamma \) is a small open compact subgroup of \( G \).

4. When the field \( K \) is an extension of \( \mathbb{Q}_p \), there may exist uncountably many \( \Gamma \)-minimal subsets in \( P \). This is the case, when \( G = SL(2, \mathbb{K}) \) and \( \Gamma = SL(2, \mathbb{Z}_p) \) as soon as \( K \) is an extension of \( \mathbb{Q}_p \) of degree \( d \geq 4 \), because, in this example, \( \dim_{\mathbb{Q}_p} P = d > \dim_{\mathbb{Q}_p} \Gamma = 3 \).

**Proof of Proposition 4.17.** We set \( \Theta = \Theta_{\Gamma} \) and we use freely the notations from the previous sections. We consider the fibrations

\[
Y_{\Gamma} = G/A_{\Gamma}N_\Theta \xrightarrow{\pi} P = G/P \xrightarrow{\pi} P_\Theta = G/P_\Theta.
\]

Let \( x \) be in \( P = G/P \) be such \( \overline{\Gamma x} \) is minimal. Then by uniqueness of the \( \Gamma \)-minimal subset in \( G/P_\Theta \), we get \( \varpi(\overline{\Gamma x}) = \Lambda_{\Gamma} \) and we can
assume \( \varpi(x) = x_{\Theta} \). Note that the left action of \( Z_\Theta \) on the fibers \( \varpi^{-1}(x) \) and \( (\pi \varpi)^{-1}(x) \) factors as an action of \( Z_\Gamma \). Pick \( y \) in \( Y_\Gamma \) such that \( \pi(y) = x \). By Proposition 4.5.e, we have \( \Gamma y \cap Z_\Gamma y = M_\Gamma y \), hence \( \Gamma x \cap Z_\Gamma x \) contains \( M_\Gamma x \). Now, by Proposition 4.9.c, the group \( M_\Gamma \) has open orbits in \( \varpi^{-1}(x_{\Theta}) \) which is a compact set. The result follows. \( \square \)

5. Finite stationary measures on homogeneous spaces

In this chapter we describe the stationary probability measures on projective spaces and prove Theorems 1.1.\( ii \), 1.5 and 1.10. More precisely we describe exactly which algebraic homogeneous spaces support a stationary probability measure. Those are the ones that support a compact minimal subset and that were described in Chapter 4.

We keep the notations of Chapter 4. Let \( \mu \) be a Zariski-dense probability measure on \( G \) i.e. a probability measure such that the semigroup \( \Gamma = \Gamma_\mu \) is Zariski-dense in \( G \). We will shorten the notations, writing

\[ \Theta_\mu = \Theta_{\Gamma_\mu} \].

5.1. Stationary measures on homogeneous spaces. Studying \( \mu \)-ergodic probability measures on projective spaces is equivalent to studying \( \mu \)-ergodic probability measures on homogeneous algebraic spaces. Indeed, by Chevalley Theorem [6, 5.1], every algebraic homogeneous space \( G/H \) can be realized as an orbit in the projective space \( \mathbb{P}(V) \) of an algebraic representation \( V \) of \( G \). Conversely, since the \( G \)-orbits in the projective space \( \mathbb{P}(V) \) of an algebraic representation of \( G \) are locally closed, any \( \mu \)-ergodic probability measure on \( \mathbb{P}(V) \) is supported by a \( G \)-orbit i.e. by an algebraic homogeneous space \( G/H \).

Proof of Theorem 1.5. According to the previous discussion, Theorem 1.5 follows from Proposition 5.1 below. \( \square \)

Proposition 5.1. Let \( \mathbb{K} \) be a local field of characteristic 0, \( G \) be the group of \( \mathbb{K} \)-points of a connected reductive \( \mathbb{K} \)-group, \( \mu \) be a Zariski-dense probability measure on \( G \), \( H \) be an algebraic subgroup of \( G \) and \( X = G/H \).

a) The following three assertions are equivalent :
(i) There exists a \( \mu \)-stationary probability measure on \( X \),
(ii) There exists a compact \( \Gamma_\mu \)-invariant subset in \( X \),
(iii) \( H \) contains a conjugate of the group \( H_{\Gamma_\mu} = A_{\Gamma_\mu}N_{\Theta_\mu} \).

b) Every \( \mu \)-ergodic probability measure on \( G/H \) has compact support.

c) The map \( \nu \mapsto \text{supp}(\nu) \) is a bijection between the sets

\[ \{ \text{\( \mu \)-ergodic probability on \( X \)} \} \longleftrightarrow \{ \text{\( \Gamma_\mu \)-minimal compact subset of \( X \)} \} \].
Remark 5.2. When $\mathbb{K} = \mathbb{R}$, one can improve the statement of Proposition 5.1: see Proposition 5.5.

The proof of Proposition 5.1 will occupy the next three sections.

5.2. $N_{\Theta_\mu}$ is in the stabilizer. The aim of this section is to prove part of the implication $(i) \Rightarrow (iii)$ in Proposition 5.1.a. More precisely, we will check that a conjugate of $N_{\Theta_\mu}$ is included in $H$ or equivalently we will prove the following

**Lemma 5.3.** Let $\mathbb{K}$ be a local field of characteristic 0, $G$ be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, $\mu$ be a Zariski-dense probability measure on $G$, $V = \mathbb{K}^d$ be an algebraic representation of $G$ and $\nu$ be a $\mu$-stationary probability measure on $\mathbb{P}(V)$. Let $Y$ be the set of points of $\mathbb{P}(V)$ which are invariant by a conjugate of $N_{\Theta_\mu}$. Then we have $\nu(Y) = 1$.

**Proof.** We can assume $\nu$ to be $\mu$-ergodic and, by induction on the dimension of $V$, for any proper subspace $W$ of $V$, one has $\nu(\mathbb{P}(W)) < 1$. Let us prove this implies, for any such $W$, one has $\nu(\mathbb{P}(W)) = 0$. This is a variation on a classical argument due to Furstenberg.

Indeed, let $r$ be the smallest integer $> 0$ such that there exists an $r$-dimensional subspace $W$ of $V$ with $\nu(\mathbb{P}(W)) > 0$. For any $W \neq W'$ in $G_r(V)$, one has $\nu(\mathbb{P}(W) \cap \mathbb{P}(W')) = 0$, hence, if $W_i$ is a finite or countable family of distinct elements of $G_r(V)$, one has

$$\nu\left(\bigcup_i \mathbb{P}(W_i)\right) = \sum_i \nu(\mathbb{P}(W_i)).$$

Thus, if, for any subset $E$ of $G_r(V)$, we set $\nu'(E) = \sum_{W \in E} \nu(\mathbb{P}(W))$, the function $\nu'$ is a finite measure defined on all the subsets of $G_r(V)$. Moreover, the measure $\nu'$ is atomic and $\mu$-stationary. Hence, it may be written as a countable sum of invariant measures carried by finite orbits of $\Gamma_\mu$ in $G_r(V)$. (see for example [7, Prop 2.3] or [5]). Since $\nu$ is ergodic, $\nu'$ is ergodic, hence it is supported on a unique finite $\Gamma_\mu$-orbit $W \subset G_r(V)$. Now, as $\Gamma_\mu$ is Zariski dense in $G$, $W$ is also $G$-invariant, and, as $G$ is Zariski connected, $W$ is a singleton $\{W\}$. In other terms, there exists a $G$-invariant subspace $W \in G_r(V)$ with $\nu(\mathbb{P}(W)) > 0$. By ergodicity of $\nu$, we get $\nu(\mathbb{P}(W)) = 1$, hence by assumption, $W = V$, that is $r = d$ and we are done.

Let $B = G^{N_r}$ and $\beta = \mu^{N_r}$. According to a result of Furstenberg and Guivarc'h-Raugi, for $\beta$-almost any $b$ in $B$, for any $\alpha$ in $\Theta_\mu$, one has $\alpha(\kappa(b_1 \ldots b_n)) \xrightarrow{n \to \infty} \infty$ (see [7, Prop. 3.2] or [5]). Thus, by Lemma 4.3.b, for $\beta$-almost all $b$ in $B$, the image $\mathbb{P}(\text{im} \pi)$ of any non-zero limit point $\pi$ in $\text{End}(V)$ of a sequence $\lambda_k b_1 \ldots b_{n_k}$ with $\lambda_k$ in $\mathbb{K}$ is contained
in $Y$. Now, according to another result of Furstenberg and Guivarc’h-Raugi, for $\beta$-almost any $b$ in $B$, the measure $(b_1 \cdots b_n)\nu$ converges towards a probability measure $\nu_b$ on $\mathbb{P}(V)$ and $\nu = \int_B \nu_b \, d\beta(b)$ (see [7, Lem. 2.1]). If $\pi$ is as above, since $\nu(\ker\pi) = 0$, we get $\nu_b(\im\pi) = 1$, hence $\nu_b(Y) = 1$. Thus $\nu(Y) = 1$ and we are done. \hfill $\square$

5.3. $A_{\Gamma_\mu}$ is in the stabilizer. The aim of this section is to prove the second half of the implication $(i) \Rightarrow (iii)$ in Proposition 5.1.a, namely, that a conjugate of $A_{\Gamma_\mu}$ is contained in $H$.

Proof of Proposition 5.1.a. The equivalence $(ii) \iff (iii)$ follows from Proposition 4.2. The implication $(ii) \Rightarrow (i)$ is clear since any compact $\Gamma_\mu$-invariant set supports a $\mu$-stationary probability measure.

It only remains to prove the implication $(i) \Rightarrow (iii)$. By Lemma 5.3, we can assume that $H$ contains $N_{\Theta_\mu}$. Since every algebraic subgroup $H$ of $G$ contains a cocompact algebraic subgroup which is $\mathbb{K}$-split solvable, we can assume that $H$ is $\mathbb{K}$-split solvable. Since $AN$ is a maximal $\mathbb{K}$-split solvable subgroup of $G$, after conjugation, we may assume that $H = A'N'$ with $N'$ a unipotent subgroup such that $N_{\Theta_\mu} \subset N' \subset N$ and $A'$ a subtorus of $A$ normalizing $N'$. Enlarging $H$, we may assume that $N \subset H$.

Now, according to Lemma 5.4 below, the torus $A'$ contains $A_{\Gamma_\mu}$ and we are done. \hfill $\square$

In this proof, we used the following

Lemma 5.4. Let $\mathbb{K}$ be a local field of characteristic 0, $G$ be the group of $\mathbb{K}$-points of a connected reductive $\mathbb{K}$-group, $\mu$ be a Zariski-dense probability measure on $G$. Let $A$ be a maximal $\mathbb{K}$-split torus of $G$, $N$ be a maximal unipotent subgroup normalized by $A$, $A'$ be a subtorus of $A$ and $H = A'N$. If $G/H$ supports a $\mu$-ergodic $\mu$-stationary probability measure $\nu$ then $\nu$ has compact support and the torus $A'$ contains $A_{\Gamma_\mu}$.

Proof of Lemma 5.4. We let $a' = \omega(A')$ and $Z' = \omega^{-1}(a')$, so that $A'$ is a cocompact subgroup of $Z'$. We consider the action of $G$ on $\mathcal{P} \times a/a'$ such that, for any $g$ in $G$, $x$ in $\mathcal{P}$ and $t$ in $a/a'$, one has
\[
g(x,t) = (gx, t + \sigma(g,x)),
\]
where $\sigma : G \times \mathcal{P} \to a$ is the Iwasawa cocycle and $\sigma$ denotes its composition with the natural map $a \to a/a'$.

We claim the stabilizer of $(x_{\Pi},0)$ for this action is $Z'N$ and the orbit map $G/Z'N \to \mathcal{P} \times a/a'$ is proper. Indeed, if, for some $g$ in $G$, one has $g(x_{\Pi},0) = (x_{\Pi},0)$, then $g = zn$ belongs to $P = ZN$ and $\omega(z) = \sigma(g,x_{\Pi}) \in a'$. Now, if $g_n$ is a sequence in $G$ such that $g_nZ'N$ leaves every compact subset of $G/Z'N$, since $G/P$ is compact, we can
assume \( g_n \) belongs to \( ZN \). Since \( N \) is normal in \( Z \), we can assume \( g_n = z_n \) belongs to \( Z \) and \( z_n \) leaves every compact subset of \( Z \). Now, since \( \omega \) is a proper morphism \( Z \to a \), the image of \( \omega(z_n) \) in \( a/a' \) leaves every compact subset and we are done.

We let \( \nu' \) be the image of \( \nu \) under the maps
\[
G/A'N \to G/Z'N \to \mathcal{P} \times a/a',
\]
so that \( \nu' \) is a \( \mu \)-ergodic \( \mu \)-stationary probability measure on \( \mathcal{P} \times a/a' \) and we will prove \( \nu' \) has compact support. Since \( A' \) is cocompact in \( Z' \) and the orbit map \( G/Z'N \to \mathcal{P} \times a/a' \) is proper, this will imply \( \nu \) has compact support too.

The dynamical system
\[
B \times (\mathcal{P} \times a/a') \to B \times (\mathcal{P} \times a/a')
\]
\[
(b, (x, t)) \mapsto (Tb, b_1(x, t)) = (Tb, (b_1x, t + \sigma(b_1, x)))
\]
preserves the probability measure \( \beta \otimes \nu' \) and is ergodic. Hence by Birkhoff ergodic theorem, for all \( M > 0 \), for \( \nu' \)-almost all \( (x, t) \) in \( \mathcal{P} \times a/a' \), for \( \beta \)-almost all \( b \) in \( B \), one has
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} 1\{\|t + \sigma(b_k \cdots b_1, x)\| \leq M\} = \nu(\mathcal{P} \times B(0, M)),
\]
where \( B(0, M) \) is the ball of radius \( M \) and center 0 in \( a/a' \).

We will need the following fact which is an intrinsic reformulation of (3.1), and relates the Iwasawa cocycle and the Cartan projection for a random trajectory (see also [5]): for all \( \varepsilon > 0 \), there exists \( M_\varepsilon > 0 \), such that, for all \( x \) in \( \mathcal{P} \),
\[
(5.1) \quad \beta(\{b \in B \mid \sup_{n \geq 1} \|\sigma(b_n \cdots b_1, x) - \kappa(b_n \cdots b_1)\| \leq M_\varepsilon\}) \geq 1 - \varepsilon.
\]

Fix \( \varepsilon > 0 \). One can find \( M_1 > 0 \) such that, for \( \nu' \)-almost all \( (x, t) \) in \( \mathcal{P} \times a/a' \),
\[
(5.2) \quad \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \beta(\{b \in B \mid \|t + \sigma(b_k \cdots b_1, x)\| \leq M_1\}) \geq 1 - \varepsilon.
\]
Then, using (5.1), one can find \( M_2 > 0 \) such that,
\[
(5.3) \quad \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \beta(\{b \in B \mid \|\pi(b_k \cdots b_1)\| \leq M_2\}) \geq 1 - 2\varepsilon
\]
(where \( \pi \) denotes the image of the Cartan projection in \( a/a' \)). Using again (5.1), one can find \( M_3 > 0 \) such that, for all \( x \) in \( \mathcal{P} \),
\[
(5.4) \quad \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \beta(\{b \in B \mid \|\pi(b_k \cdots b_1, x)\| \leq M_3\}) \geq 1 - 3\varepsilon.
\]
If \( \text{supp}(\nu') \) were not compact, the number \( t \) in (5.2) could be chosen to be arbitrarily large. This would contradict (5.4) when \( \varepsilon < \frac{1}{3} \) and
t > M_1 + M_3. Hence \( \nu' \) has compact support and so does \( \nu \) as remarked above.

In particular, \( \Gamma_\mu \)-preserves a compact subset in \( G/A'N \), so that, by Proposition 4.2, \( A_{\Gamma_\mu}N_{\Theta_\mu} \) fixes a point in \( G/A'N \). Since, by Bruhat decomposition, the set of fixed points of \( N_{\Theta_\mu} \) in \( \mathcal{P} = G/ZN \) is \( Z_{\Theta_\mu}x_\Pi \), we get \( A_{\Gamma_\mu} \subset A' \), what should be proved. \( \square \)

5.4. Equicontinuity on homogeneous spaces. In this section we finish the proof of the classification of \( \mu \)-stationary probability measures on a homogeneous space \( G/H \) by using a compactification of \( G/H \) for which the Markov-Feller operator \( P_\mu \) is equicontinuous.

Proof of Proposition 5.1. By point a), one can assume that \( H \) contains \( H_{\Gamma_\mu} \). By Lemma 4.7, the homogeneous space \( G/H \) occurs as a \( G \)-orbit in a projective space \( \mathbb{P}(V) \) where \( (\rho, V) \) is a representation of \( G \) which is the direct sum of strongly irreducible representations \( (\rho_i, V_i) \) with highest weight \( \chi_i \), such that all the \( \chi_i \) have the same restriction to \( A_{\Gamma_\mu} \). By Cartan decomposition, for any \( i \), there exists \( C_i > 0 \) such that, for any \( g \) in \( G \), one has

\[
\frac{1}{C_i} \| \rho_i(g) \| \leq \exp(\chi_i^\omega(\kappa(g))) \leq C_i \| \rho_i(g) \| .
\]

Thus, the assumptions of Corollary 3.4 are satisfied and hence the Markov-Feller operator \( P_\mu \) on \( \mathbb{P}(V) \) is equicontinuous. Our statement then follows from Proposition 2.9.a. \( \square \)

We end this chapter by discussing a few properties of stationary measures which are different over the real numbers and over the non-archimedean local fields: we conclude the proof of Theorem 1.1.ii, Theorem 1.7 and Theorem 1.10.

5.5. Stationary measures for real groups. We strengthen here Proposition 5.1 when \( \mathbb{K} = \mathbb{R} \).

**Proposition 5.5.** Let \( G \) be the group of real points of a connected reductive \( \mathbb{R} \)-group, \( \mu \) be a Zariski-dense probability measure on \( G \), \( H \) be an algebraic subgroup of \( G \) and \( X = G/H \).

a) There exists a \( \mu \)-stationary probability measure on \( X \) if and only if \( X \) is compact.

b) In this case, (i) the Markov-Feller operator \( P_\mu \) on \( X \) is equicontinuous, (ii) there exists a unique \( \mu \)-stationary probability measure on \( X \).

**Proof of Proposition 5.5.** a) Since \( \mathbb{K} = \mathbb{R} \), one knows that \( \Theta_\mu = \Pi \), \( A_{\Gamma_\mu} = A \) and \( N_{\Theta_\mu} = N \) and our claims follow from Proposition 5.1 and the compactness of \( G/AN \).
b) When the homogeneous space \( X = G/H \) is compact, the algebraic group \( H \) contains a conjugate of \( AN \). By Lemma 4.7 and Corollary 3.4, the Markov-Feller operator \( P_\mu \) on \( X \) is equicontinuous. The last statement then follows from Proposition 2.9.a and Lemma 4.13. □

5.6. Eigenvalues of \( P_\mu \). In this section one has \( \mathbb{K} = \mathbb{R} \) and we end the proof of Theorem 1.1.

Proof of Theorem 1.1.ii. Our statement will follow from Proposition 2.3.b and the following Lemma 5.6. □

Lemma 5.6. Let \( X = \mathbb{P}(\mathbb{R}^d) \) and \( \mu \) be a probability measure on \( GL(\mathbb{R}^d) \) such that the action of \( \Gamma_\mu \) on \( \mathbb{R}^d \) is strongly irreducible and the Zariski closure of \( \Gamma_\mu \) is semisimple. Then, the only eigenvalue of modulus 1 of the averaging operator \( P_\mu \) in \( C^0(X) \) is 1.

Proof of Lemma 5.6. Let \( \varphi \) be a non-zero continuous function on \( X \) such that \( P_\mu \varphi = \chi \varphi \) with \( \chi \in S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \} \). We want to prove that \( \chi = 1 \). According to Proposition 2.7.b, there exists a \( \Gamma_\mu \)-minimal subset of \( X \) on which \( \varphi \) is non-zero. By Theorem 1.7, this minimal subset is supported by a compact orbit \( G/H \) of the Zariski closure \( G \) of \( \Gamma_\mu \). By Lemma 4.15, \( H \) contains a conjugate of the maximal \( \mathbb{R} \)-split solvable subgroup \( AN \) of \( G \).

We construct this way a non-zero continuous function \( \psi \) on \( Y = G/AN \) such that \( P_\mu \psi = \chi \psi \) with \( \chi \in S^1 \). We want to prove that \( \chi = 1 \). This space \( Y \) is then an isometric extension of the flag variety \( \mathcal{P} \) and this statement is due to Guivarc’h and Raugi in [16, Th. 3]. Here is a short proof of it.

We assume first that \( \chi \) is a \( n \)-th-root of unity. We note that \( P_\mu^{*n} \psi = \psi \) and, since \( G \) is semisimple, that the probability measure \( \mu^{*n} \) is still Zariski dense in \( G \). Hence, by Propositions 2.7.c and 4.12, the \( P_\mu^{*n} \)-invariant function \( \psi \) is constant and \( \chi = 1 \).

We assume now that \( \chi \) is not a root of unity. We introduce the probability measure

\[
\mu' := \mu \otimes \delta_\chi \quad \text{on} \quad G' := G \times S^1.
\]

Since \( G \) is semisimple and since \( \chi \) is not a root of unity, the probability measure \( \mu' \) is Zariski dense in the real algebraic reductive group \( G' \). We also introduce the continuous function \( \psi' \) on \( Y' := G'/AN \cong Y \times S^1 \) given by

\[
\psi'(y, z) = z^{-1} \psi(y) \quad \text{for all} \ y \in Y, \ z \ in \ S^1.
\]
This function $\psi'$ is $P_\mu'$-invariant since one has
\[
P_\mu' \psi'(y, z) = \int_G \psi'(gy, \chi z) \, d\mu(g) \\
= z^{-1} \chi^{-1} P_\mu \psi(y) = z^{-1} \psi(y) = \psi'(y, z).
\]
Hence, by Propositions 2.7.c and 4.12, the $P_\mu'$-invariant function $\psi'$ is constant. Contradiction. \qed

5.7. Stationary measures on the flag variety. In this section $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$. We prove Theorem 1.10 which says that the flag variety $\mathcal{P} = G/P$ supports only finitely many $\mu$-stationary measures. This statement is interesting only when $\mathbb{K}$ is non-archimedean since, when $\mathbb{K} = \mathbb{R}$, one knows that there exists only one $\mu$-stationary measure on the flag variety (see for instance Proposition 5.5).

Proof of Theorem 1.10. By Proposition 5.1, the set of $\mu$-ergodic probability measures on $X$ is in bijection with the set of $\Gamma_\mu$-minimal subsets of $X$. According to Proposition 4.17 this set is finite. \qed

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