LATTICES IN S-ADIC LIE GROUPS

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ABSTRACT. We show that any finite volume quotient of an S-adic Lie group admits a fibration with compact fibers over some finite volume quotient of a product of algebraic semisimple p-adic Lie groups.

We also prove a similar decomposition for lattices in a solvable locally compact group.

1. Introduction

This text can be seen as a short survey of elementary results about lattices Λ in a real Lie group G. However, its main purpose is the extension of some of these results to the context of lattices Λ in more general locally compact groups G. In particular, when G is an S-adic Lie group i.e. a group which is locally the product of real and p-adic Lie groups (see Definition 4.1), we prove in Theorem 6.6 a decomposition theorem of Λ with respect to the adjoint action of G on a suitable semisimple quotient $\mathfrak{s}$ of its Lie algebra $\mathfrak{g}$.

With the same methods we prove also in Proposition 3.4 a similar decomposition for lattices in a solvable locally compact group. The proof relies on a property of certain minimal actions of $\mathbb{R}^d$ that we call *strong minimality*.

Our motivation to prove the decomposition theorem 6.6 comes from our paper [4] : In this paper we prove, when G is a real Lie group, some recurrence properties of random walks on G/Λ which were conjectured in [9]. Our decomposition theorem 6.6 is then the key ingredient which allows us, in the last section of [4], to extend these recurrence properties from the framework of real Lie groups to the one of S-adic Lie groups. These recurrence properties will be used in [5] to extend the results of [3].

Here is the structure of the paper :
– Section 2 : General facts about minimal actions of abelian groups.
– Section 3 : General facts about lattices in locally compact groups and decomposition of lattices in solvable locally compact groups.
– Section 4 : General facts about S-adic Lie groups and Borel density

theorem.
– Section 5 : A cocompactness criterion for lattices in S-adic Lie groups.
– Section 6 : The decomposition theorem for lattices in S-adic Lie groups.

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2. Minimal actions of $\mathbb{R}^d$

In this section we give a criterion for a locally compact space X, equipped with a continuous minimal action of $\mathbb{R}^d$, to be compact.

Let $A = \mathbb{R}^d$ and X be a locally compact A-space, i.e. a space endowed with a continuous action of A. We denote this action by $(a, x) \mapsto a\, x$.

An orbit Ax is said to be *strongly* dense if, for every non-empty open convex cone $C \subset A$, the set Cx is dense.

We recall that the A-space X is minimal if all its A-orbits are dense. The A-space X is said to be *strongly* minimal if all its A-orbits are strongly dense.

Proposition 2.1. *Let X be a minimal locally compact $\mathbb{R}^d$-space.*
a) If the action preserves a Borel probability measure μ on X, then there exists at least one strongly dense orbit in X.
b) If the space X is compact, the action is strongly minimal.
c) Conversely, if the action is strongly minimal, X is compact.

By reading the proof, it is worth keeping in mind the following two examples.

Example 2.2. *There exists a minimal action of $\mathbb{R}^d$ which does not contain any strongly dense orbit.*

Proof. The action by translations of $\mathbb{R}$ on itself, or the product action of $\mathbb{R}^2$ on $\mathbb{R} \times X'$ where X' is a minimal $\mathbb{R}$-space. □

Example 2.3. *There exists a continuous action of $\mathbb{R}$ on a non-compact locally compact space X which is minimal and preserves a Borel probability measure μ on X.*

Proof. Our example is a suspension over an irrational rotation of the circle. Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ be the circle, $\alpha \in \mathbb{T}$ an irrational element, $\mathrm{d}\theta$ the Lebesgue probability on $\mathbb{T}$ and $f : \mathbb{T} \to (0, \infty]$ a continuous function such that $\int_{\mathbb{T}} f(\theta)\, \mathrm{d}\theta = \frac{1}{2}$ and $f^{-1}(\infty) = \{0\}$. We set

$$Y := \{(\theta, t) \in \mathbb{T} \times \mathbb{R} \mid -f(\theta) \leq t \leq f(\theta + \alpha)\}$$

and $X = Y/_\sim$ the quotient space for the identifications

$$(\theta, -f(\theta)) \sim (\theta - \alpha, f(\theta)) \ , \text{ for all } \theta \neq 0 \ .$$

The space X is locally compact but not compact. There exists a continuous flow $s \mapsto \varphi_s$ on X such that $\varphi_s(\theta, t) = (\theta, t+s)$ as soon as both (θ, t) and $(\theta, t+s)$ are in Y. This flow is minimal and it preserves the probability measure $\mu = \mathrm{d}\theta \otimes \mathrm{d}t$. This flow is not strongly minimal since the only limit point of the half orbit $\mathbb{R}_+ x_0$ of the point $x_0 := (-\alpha, 0)$ is the point $x_\infty := (-\alpha, \infty)$. □

Proof of Proposition 2.1. We endow $\mathbb{R}^d$ with the usual euclidean norm $\|.\|$. Let $\mathcal{C}$ be the set of open convex cones $C \subset \mathbb{R}^d$.

For $C \in \mathcal{C}$ and $x \in X$, we define the ω_C-limit set of x to be

$$\omega_C(x) := \bigcap_{a \in A} \overline{(a + C)x}.$$

By definition this set is closed and A-invariant. Since the action is minimal, this set is either empty or equal to X.

a) We will check that

$$\mu(\{x \in X \mid \forall C \in \mathcal{C} \ , \ \omega_C(x) = X\}) = 1. \tag{2.1}$$

For $\varepsilon > 0$, choose a compact set $K \subset X$ with $\mu(K) > 1 - \varepsilon$. By Poincaré recurrence theorem, for μ-almost every x in K, for every rational vector a in $\mathbb{R}^d$, infinitely many translates $(na)x$, $n \in \mathbb{N}$, belong to K. We note that the interior of any $C \in \mathcal{C}$ contains a rational vector a. Hence for such a point x, for every $C \in \mathcal{C}$, one has $\omega_C(x) \cap K \neq \emptyset$ and thus, since the action is minimal, one has $\omega_C(x) = X$. Since ε is arbitrarily small, this proves (2.1). In particular, the set given in (2.1) is non empty. This proves the claim, since every point in this set has a strongly dense orbit.

b) Since X is compact, all the sets $\omega_C(x)$ are non empty. Since the action is minimal, they are equal to X. Hence the action is strongly minimal.

c) We can choose a constant $\varepsilon_0 > 0$ and $d+1$ open convex cones $C_0, \ldots, C_d$ of $\mathbb{R}^d$ such that, for every family of $d+1$ vectors $v_0, \ldots, v_d$ with $\|v_i\| = 1$ and $v_i \in C_i$,

(2.2) *the ball $B(0, \varepsilon_0)$ is contained in the convex hull of $v_0, \ldots, v_d$.*

Let U be a non-empty open subet of X with compact closure K. For $0 \leq i \leq d$, let $T_i : X \to [0, \infty]$ be the "hitting time of U in the direction C_i" :

$$T_i(x) = \inf\{\|c\| \geq 1 \mid c \in C_i \ , \ cx \in U\}. \tag{2.3}$$

Since the action is strongly minimal, the function T_i is finite everywhere. Since U is open, the function T_i is upper semi-continuous. Hence, since K is compact, the constant

$$M_0 := \sup\{T_i(x) \mid 0 \le i \le d\ ,\ x \in K\}$$

is finite. For every $x \in X$, we introduce the set of "hitting times"

$$A_x := \{a \in \mathbb{R}^d \mid ax \in K\}.$$

Since the action is minimal the closed set A_x is non-empty. Let a be an element of A_x with minimal norm. We claim that

$$\|a\| \le \frac{M_0}{2\varepsilon_0}. \tag{2.4}$$

Indeed, for every $0 \le i \le d$, one can find $c_i \in C_i$ with $1 \le \|c_i\| \le M_0$ and $a + c_i \in A_x$. Since the element a has minimal norm in A_x, one has, for all i,

$$\|a + c_i\| \ge \|a\|,$$

that is,

$$\|a\|^2 + 2\langle a, c_i\rangle + \|c_i\|^2 \ge \|a\|^2.$$

Setting, for all i, $v_i := \frac{1}{\|c_i\|} c_i$, we get

$$2\langle a, v_i\rangle \ge -M_0.$$

Therefore, using (2.2), for any v in $B(0, \varepsilon_0)$, one also has

$$2\langle a, v\rangle \ge -M_0.$$

Choosing $v := -\frac{\varepsilon_0}{\|a\|}\, a$, we get the expected bound (2.4).

This bound (2.4) proves that, for all x in X, there exists a in $B(0, \frac{M_0}{2\varepsilon_0})$ and y in K such that $x = a^{-1}y$. In particular, it proves that the space X is compact. □

We conclude this section by noting that these results can easily be adapted to actions of the group $A = \mathbb{Z}^d$.

3. Lattices in locally compact groups

We give elementary properties of lattices and we prove a decomposition result for lattices in a locally compact solvable group.

Let G be a locally compact group and H be a closed subgroup of G. We shall say that H *has finite covolume* in G if the quotient G/H admits a *finite* G-invariant Borel measure. For instance, a *lattice* is by definition a discrete finite covolume subgroup.

Let us state some elementary properties of finite covolume subgroups.

Lemma 3.1. *Let G be a locally compact group and H_1, H_2 be two closed subgroups such that $H_1 \subset H_2$. Then H_1 has finite covolume in G if and only if simultaneously H_1 has finite covolume in H_2 and H_2 has finite covolume in G.*

Proof. This is classical (see [13, Lemma 1.6]). Recall (see [16]) that a quotient G/H admits a G-invariant Radon measure if and only if the modular function of H is the restriction to H of the modular function of G.

If H_1 has finite covolume in H_2 and H_2 has finite covolume in G, the transitivity formula for integration on homogeneous spaces (see [16]) proves that G/H_1 supports a G-invariant measure with total mass 1.

Conversely, if H_1 has finite covolume in G, then the image in G/H_2 of the G-invariant probability measure on G/H_1 is also G-invariant with total mass 1. The same transitivity formula proves that H_1/H_2 supports also a H_1-invariant measure with total mass 1. □

Lemma 3.2. *Let G be a locally compact group, H be a finite covolume closed subgroup of G, G' be an open subgroup of G and $H' := H \cap G'$.*
a) The group H' has finite covolume in G'.
b) If H is cocompact in G then H' is cocompact in G'.
c) Conversely, if H' is cocompact in G' and G' is normal in G then H is cocompact in G.

Proof. a) The restriction of the G-invariant probability on G/H to the G'-orbit G'/H' is a non-zero finite G'-invariant measure.

b) The G'-orbits in the compact space G/H are open hence closed. In particular G'/H' is compact.

c) Since the group $G'H$ is open, the space $G/G'H$ is discrete. This space admits a finite measure which is invariant under the transitive action of the group G/G'. Hence, this space is finite, that is G/H is a finite union of G'-orbits. As each of these orbits is compact, G/H is compact. □

From these results, we at once get the following

Lemma 3.3. *Let N be a nilpotent locally compact group. Then any finite covolume closed subgroup $H \subset N$ is cocompact.*

Proof. Let Z be the center of N and $N' := \overline{HZ}$. By lemma 3.1, N' has finite covolume in N and H has finite covolume in N'.

Now, by an induction argument on the length of the central series of N, the finite covolume subgroup N'/Z of N/Z is cocompact. Hence N' is cocompact in N.

Besides, one has

$$[N', N'] \subset \overline{[HZ, HZ]} \subset \overline{[H, H]} \subset H.$$

Hence H is normal in N' and the quotient N'/H is a group. As its Haar measure is finite, it is compact and H is cocompact in N. □

The proof of the following proposition will be much more delicate.

Proposition 3.4. *Let G be a solvable locally compact group, G_e its connected component, and H a finite covolume closed subgroup of G. Then H is cocompact in the group $\overline{HG_e}$.*

In particular, as proved by Mostow, when G is solvable and connected, any lattice H in G is cocompact in G.

If we set G_d for the totally discontinuous quotient $G_d = G/G_e$ and H_d for the finite covolume closed subgroup $H_d := \overline{HG_e}$, Proposition 3.4 tells us that *the finite volume quotient G/H fibers over the totally discontinuous finite volume homogeneous space G_d/H_d with compact fibers.*

Example 3.5. (Bader, Caprace, Gelander, Mozes) *There exist metabelian locally compact groups containing non-cocompact lattices.*

Proof of Example 3.5. The group G is a semidirect product $G = A \rtimes K$ of a commutative discrete group A by a commutative compact group K, where A is the direct sum of finite fields $\mathbb{F}_p$ for an infinite set S of primes p such that $\sum_{p\in S} p^{-1} < \infty$, and where K is the corresponding product of multiplicative groups $\mathbb{F}_p^*$.

Let H be the subgroup of G generated by the following elements h_p for $p \in S$. The only non-trivial component of h_p is the p^{th} coordinate which is $(1-k_p\,, k_p) \in \mathbb{F}_p \rtimes \mathbb{F}_p^*$ where k_p is a generator of $\mathbb{F}_p^*$. The orbits of H in $A \simeq G/K$ are the sets $A_I := \{a \in A \mid p \in I \Leftrightarrow a_p = 1\}$, for I finite subset of S. The cardinality N_I of the stabilizer of a point $a \in A_I$ is $N_I = \prod_{p\in I}(p-1)$. Since $\sum_I N_I^{-1} < \infty$, H is a lattice in G. □

Proof of Proposition 3.4. We may assume that $G = \overline{HG_e}$. We argue by induction on the length of the derived series of G_e. Let A be the closure of the last non-trivial term of the derived series of G_e and $H' := \overline{HA}$. According to Lemma 3.1, the group H'/A has finite covolume in G/A and the group H has finite covolume in H'.

By the induction hypothesis, the group H'/A is cocompact in G/A hence H' is cocompact in G. By Lemma 3.6 below, H is cocompact in H'. Hence H is cocompact in G. □

Lemma 3.6. *Let G be a locally compact group, $H \subset G$ a finite covolume closed subgroup and $A \subset G$ a normal closed connected abelian subgroup. If $G = \overline{HA}$, then H is cocompact in G.*

Proof. According to [12, §2.21], as A is a connected locally compact abelian group, it contains a largest compact subgroup K_A and the quotient group A/K_A is isomorphic to $\mathbb{R}^d$, for some $d \geq 0$. By uniqueness, K_A is normal in G. Thus, after replacing G by G/K_A and H by HK_A/K_A, we may assume $A = \mathbb{R}^d$.

We will apply Proposition 2.1 to the action of A on the locally compact space $X := G/H$. Since H has finite covolume, this action preserves a Borel probability measure. Since $G = \overline{HA}$, this action is minimal. Hence, by Proposition 2.1.a, there exists at least one orbit Ax_0 in X which is strongly dense. Since G acts transitively on X and normalizes A, all the orbits Ax in X are strongly dense. Hence, by Proposition 2.1.c, the space X is compact. □

More generally, the same argument proves the following

Proposition 3.7. *Let G be a locally compact group, R a closed normal connected amenable subgroup of G, and H a finite covolume closed subgroup of G. Then H is cocompact in the group $\overline{HR}$.*

Proof. We use the following two facts: every connected locally compact group is a compact extension of a connected Lie group (see [12]); every amenable connected Lie group is a compact extension of its solvable radical (see [17]). Hence the group R is a compact extension of a connected solvable Lie group R_1 which is normal in G. We follow then the same proof as for Proposition 3.4 by induction on the length of the derived series of R_1. □

4. S-adic Lie groups

This section contains elementary definitions and facts about S-adic Lie groups. It also contains a version of the Borel density theorem for S-adic Lie groups.

We recall that $\mathbb{Q}_p$ is the field of p-adic numbers and $\mathbb{Q}_\infty = \mathbb{R}$ is the field of real numbers or ∞-adic numbers. Let S be a finite subset of the set of prime numbers including ∞.

Definition 4.1. *An S-adic Lie group G is a locally compact group which contains an open subgroup U isomorphic to a group of the form $(\prod_{p\in S} G_p)/N$ where, for each $p \in S$, G_p is a p-adic Lie group and N is a discrete normal subgroup of this product.*

Let G be an S-adic Lie group. The $\mathbb{Q}$-vector space $\mathfrak{g} := \oplus_{p\in S}\mathfrak{g}_p$ which is the direct sum of the Lie algebras $\mathfrak{g}_p$ of G_p does not depend on the choices and is called the Lie algebra of G. This Lie algebra is an S-adic Lie algebra i.e. a direct sum of p-adic Lie algebras with p in S. The real Lie subalgebra $\mathfrak{g}_\infty$ is called the real factor of $\mathfrak{g}$. We say that $\mathfrak{g}$ is *non-archimedean* if $\mathfrak{g}_\infty = 0$. The Lie subalgebra $\mathfrak{g}_f := \oplus_{p\neq\infty}\mathfrak{g}_p$ is called the non-archimedean factor of $\mathfrak{g}$. We will denote by $\mathrm{Ad}_{\mathfrak{g}_p}$, $\mathrm{Ad}_{\mathfrak{g}}, \ldots$ the adjoint action of G in $\mathfrak{g}_p$, $\mathfrak{g}, \ldots$

Here are the first properties of S-adic Lie groups:
– Real Lie groups and p-adic Lie groups are S-adic Lie groups.
– A product of two S-adic Lie groups is an S-adic Lie group.
– A closed subgroup of an S-adic Lie group is an S-adic Lie group (see [14, Prop. 1.5]).
– The quotient of an S-adic Lie group by a closed normal subgroup is an S-adic Lie group.

An S-adic Lie group can be connected, even if its Lie algebra admits a nontrivial non-archimedean factor, as, for example, the solenoid $(\mathbb{R}\times\mathbb{Q}_p)/\mathbb{Z}[\frac{1}{p}]$, where $\mathbb{Z}[\frac{1}{p}]$ is embedded diagonally in $\mathbb{R}\times\mathbb{Q}_p$, or the group $(\widetilde{\mathrm{SL}(2,\mathbb{R})}\times\mathbb{Z}_p)/\mathbb{Z}$, where $\mathbb{Z}$ is embedded diagonally as a central subgroup in $\widetilde{\mathrm{SL}(2,\mathbb{R})}\times\mathbb{Z}_p$.

The following proposition is a version of the classical Borel density theorem in the framework of S-adic Lie groups (see [13, Chap. 5], [17, Chap. 3] or [11, §2.4]).

Proposition 4.2. *Let G be an S-adic Lie group, $p\in S$, $H\subset G$ a finite covolume closed subgroup, $\pi : G\to \mathrm{GL}(d,\mathbb{Q}_p)$ a continuous morphism.*
a) For any H-invariant line $x_0\in\mathbb{P}(\mathbb{Q}_p^d)$, the G-orbit Gx_0 is compact.
b) If G has no proper cocompact normal subgroups, then any H-invariant line $x_0\in\mathbb{P}(\mathbb{Q}_p^d)$ is G-invariant.
c) If G has no proper cocompact normal subgroups, then the Zariski closures of $\pi(H)$ and $\pi(G)$ are equal.

Example 4.3. *In point a), there does not always exist a cocompact normal subgroup of G stabilizing x_0.*

Proof of 4.3. We give an example with $p=\infty$ and G real connected. We denote by $r_\theta\in\mathrm{SO}(2,\mathbb{R})\subset\mathrm{SO}(3,\mathbb{R})$ the rotation of angle θ and we fix $\alpha\in\mathbb{R}\smallsetminus\mathbb{Q}$. We choose

$$G := \mathrm{SO}(3,\mathbb{R})\times\mathbb{R},$$
$$H := \{(r_\theta,\theta)\mid \theta\in 2\alpha\pi\mathbb{Z}\},$$
$$K := \mathrm{SO}(3,\mathbb{R})\times\mathrm{SO}(2,\mathbb{R})\times\mathrm{SO}(2,\mathbb{R}),\quad\text{and}$$

$$\pi : G \to K; (k,\theta) \mapsto (k, r_\theta, r_{\theta/\alpha}),$$

and we set $M := \overline{\pi(H)} = \{(r_\theta, r_\theta, 1) \mid \theta \in \mathbb{R}\}$. The subgroup H is a cocompact lattice in G but, for no $\beta \in \mathbb{R}$, does the cocompact normal subgroup $G' = \{(1,\theta) \mid \theta \in \beta\mathbb{Z}\}$ fix the base point $x_0 \in K/M$. □

Proof of Proposition 4.2. a) After replacing G by a finite index subgroup, we may assume that $\pi(G)$ is Zariski connected. We may also assume that the orbit Gx_0 spans $\mathbb{Q}_p^d$.

We claim that

(4.1) *the closure* $K := \overline{\pi(G)}$ *is a compact subgroup of* $\mathrm{PGL}(d,\mathbb{Q}_p)$.

Since x_0 is H-invariant, one has a G-equivariant map

$$i : G/H \to \mathbb{P}(\mathbb{Q}_p^d) \text{ given by } i(gH) = gx_0.$$

The probability measure $i_*(\mu)$ on $\mathbb{P}(\mathbb{Q}_p^d)$, which is the image of the G-invariant probability measure μ on G/H, is K-invariant. Lemma 4.4 below shows that the group K is compact, as claimed in (4.1).

Since we did not assume G and π to be algebraic, the group $\pi(G)$ might not be closed as in Example 4.3. This will make the proof a little bit longer. Let $C \subset G$ be an open relatively compact subset so that one has

$$i_*(\mu)(Cx_0) > 0.$$

We can assume π to be injective. The group G is then a p-adic Lie group. Since the continuous morphism π is $\mathbb{Q}_p$-analytic, the set Cx_0 is a $\mathbb{Q}_p$-submanifold of Kx_0. Since $i_*(\mu)$ is the K-invariant probability measure on Kx_0, one has then

$$\dim_{\mathbb{Q}_p} Cx_0 = \dim_{\mathbb{Q}_p} Kx_0$$

and the orbit Gx_0 is open in Kx_0. Since $\pi(G)$ is dense in K, every G-orbit in Kx_0 is dense hence meets the open set Gx_0. This proves the equality $Gx_0 = Kx_0$ and this ends the proof of *a*).

b) We assume now that G does not admit any proper cocompact normal subgroup. In particular, the group $\pi(G)$ is Zariski connected. We may again assume that the orbit Gx_0 spans $\mathbb{Q}_p^d$. We want to prove that this orbit is a singleton or, equivalently, that

(4.2) *the group* $K := \overline{\pi(G)}$ *is trivial.*

We may again assume that π is injective. We note first that the group K is connected : indeed for any open normal subgroup K' in K, the group $G' := \pi^{-1}(K')$ is a finite index normal subgroup of G, hence equals G.

Therefore, we may assume $p = \infty$ and G is a real Lie group. Now, the connected component G_e of G is an open subgroup of G. Thus,

G_eH being an open finite index subgroup of G, it contains an open normal finite index subgroup and we get

$$G = G_eH.$$

Let $S := [K, K]$ and let T be the connected component of the center of K, so that $K = ST$ and $S \cap T$ is finite. We let $\mathfrak{k}$, $\mathfrak{s}$ and $\mathfrak{t}$ denote the Lie algebras of the real compact Lie groups K, S and T.

Let L be the immersed subgroup $\pi(G_e)$ in K and $\mathfrak{l}$ be its Lie algebra. As G_e is normal in G and $\pi(G)$ is dense in K, $\mathfrak{l}$ is an ideal in $\mathfrak{k}$, and one has

$$\mathfrak{l} = (\mathfrak{l} \cap \mathfrak{s}) \oplus (\mathfrak{l} \cap \mathfrak{t}).$$

Set $\mathfrak{l}' = \mathfrak{l} \cap \mathfrak{s}$ and let L' be the closed normal connected subgroup of K with Lie algebra $\mathfrak{l}'$. This group L' is included in L and the group $G' = \pi^{-1}(L')$ is included in G_e. As G_e is a connected Lie group, π induces a homeomorphism from compact subsets of G_e onto their images and G' is a compact connected normal subgroup of G. Now, we have $\mathrm{Ad}_{\mathfrak{l}'}(K) = \mathrm{Ad}_{\mathfrak{l}'}(L')$, hence

$$G = Z_G(G')G',$$

where $Z_G(G')$ denotes the centralizer of G' in G. In particular, $Z_G(G')$ is a normal cocompact subgroup of G. Therefore, $Z_G(G') = G$ and L' is a central subgroup of K, that is

$$L \subset T$$

Let M be the closure of $\pi(H)$. As $G = G_eH$, we have

$$K = TM.$$

As Gx_0 spans $\mathbb{Q}_p^d$, the group K acts faithfully on the orbit $Kx_0 = Gx_0$ and we have

$$\textstyle\bigcap_{k \in K} kMk^{-1} = \{e\} \qquad (4.3)$$

Thus we have $M = \{e\}$, that is $H = \{e\}$. Now, the group G has finite Haar measure, hence is compact. Since G does not admit any proper cocompact normal subgroup, G is trivial and K is trivial too, what we claimed in (4.2).

c) Now, let $\mathbf{G}$ be the Zariski closure of $\pi(G)$ in $\mathrm{GL}(d, \mathbb{Q}_p)$ and $\mathbf{H}$ be the Zariski closure of $\pi(H)$. According to Chevalley Theorem (see [2, 5.1]) there exists an algebraic representation $\rho : \mathbf{G} \to \mathrm{PGL}(m, \mathbb{Q}_p)$ and a line $x_0 \in \mathbb{P}(\mathbb{Q}_p^m)$ whose stabilizer is $\mathbf{H}$. By point *b*) applied to this representation ρ, we get $\pi(G) \subset \mathbf{H}$ hence $\mathbf{G} = \mathbf{H}$. □

Lemma 4.4. *Let ν be a probability measure on $\mathbb{P}(\mathbb{Q}_p^d)$. Suppose that, for any two proper subspaces $E_1, E_2 \subsetneq \mathbb{Q}_p^d$ with $\dim E_1 + \dim E_2 \leq d$, the support of ν is not contained in the union $\mathbb{P}(E_1) \cup \mathbb{P}(E_2)$. Then the stabilizer $S := \{g \in \mathrm{PGL}(\mathbb{Q}_p^d) \mid g_*\nu = \nu\}$ of ν is compact.*

Proof. This lemma due to Furstenberg is proven in [17, §3.2]. □

Corollary 4.5. *Let G be an S-adic Lie group and H be a finite covolume closed subgroup of G, with Lie algebra $\mathfrak{h}$.*
a) Then the normalizer $N_G(\mathfrak{h})$ is cocompact in G.
b) If G has no proper cocompact normal subgroup, G normalizes $\mathfrak{h}$.

Example 4.6. *In point a), there does not always exist a cocompact normal subgroup of G normalizing $\mathfrak{h}$.*

Proof of 4.6. We give an example with $p < \infty$ and G nilpotent with an exact sequence $1 \to G_0 \to G \to \mathbb{Z} \to 1$ where G_0 is an open compact subgroup. Let K be the group of upper triangular unipotent 4×4-matrices u with coefficients $u_{i,j}$ in $\mathbb{Z}_p$, for $1 \leq i < j \leq 4$. The group G is immersed in K as

$$G := \{u \in K \mid u_{1,2} \in \mathbb{Z}\}.$$

The group H is the closed subgroup of G isomorphic to $\mathbb{Z} \times \mathbb{Z}_p$,

$$H := \{u \in G \mid u_{1,3} = u_{2,3} = u_{1,4} = u_{2,4} = 0\}.$$

One computes the normalizer $N_G(\mathfrak{h}) = \{u \in G \mid u_{1,3} = u_{2,3} = 0\}$ and the group $\cap_{g \in G} g N_G(\mathfrak{h}) g^{-1} := \{u \in G \mid u_{1,2} = u_{1,3} = u_{2,3} = 0\}$ is not cocompact in G. □

Proof of Corollary 4.5. a) Applying, for any p in S, Proposition 4.2 to the adjoint representation of G in $\Lambda^{d_p}\mathfrak{g}_p$, where $d_p := \dim_{\mathbb{Q}_p}(\mathfrak{h}_p)$, we get that the G-orbit of the line $x_p := \Lambda^{d_p}\mathfrak{h}_p$ is compact. But the G-orbits in the product $\prod_{p \in S} G x_p$ are open and hence closed. Thus the stabilizer $N_G(\mathfrak{h})$ of the point $x := (x_p)_{p \in S}$ is cocompact in G.

b) Since H normalizes $\mathfrak{h}$, by Proposition 4.2, G normalizes $\mathfrak{h}$ too. □

Corollary 4.7. *For any p in S, let G_p be the group of $\mathbb{Q}_p$-points of a $\mathbb{Q}_p$-algebraic semisimple group with no anisotropic factor and set $G = \prod_{p \in S} G_p$. If H is a finite covolume closed subgroup of G, for any p in S, the image of H in G_p has finite index Zariski closure.*

Proof. For any p in S, let G_p^+ be the subgroup of G_p which is spanned by unipotent one-parameter subgroups in G_p. As G_p does not have anisotropic factors, G_p^+ is open with finite index in G_p and every cocompact normal subgroup of G_p contains G_p^+. The result now follows from Proposition 4.2 applied to the group $G = \prod_{p \in S} G_p^+$. □

Corollary 4.8. *Let G be as above and Λ be a lattice in G, then Λ has finite index in its normalizer $N_G(\Lambda)$.*

Proof. Let $N := N_G(\Lambda)$ be the normalizer of Λ and $\mathfrak{n}$ its Lie algebra. By noetherianity there exists a finitely generated subgroup $\Lambda_0 \subset \Lambda$ whose centralizer in $\mathfrak{g}$ is the same as the one of Λ. Since Λ is discrete, the elements of N which are small enough commute with Λ_0. Hence the group Λ centralizes $\mathfrak{n}$. By corollary 4.7, the centralizer of $\mathfrak{n}$ in G has finite index. Hence $\mathfrak{n}$ is a central ideal of $\mathfrak{g}$. Such an ideal is trivial, hence the group N is discrete. Since $\Lambda \subset N$ and Λ is a lattice in G, this group Λ has finite index in N. □

5. Cocompactness of lattices

We give a sufficient criterion for an S-adic Lie group G to admit only cocompact lattices.

We say that an S-adic Lie algebra $\mathfrak{g}$ is *amenable* if it is the Lie algebra of some amenable S-adic Lie group, that is if $\mathfrak{g}_\infty$ does not admit any noncompact semisimple Lie algebra or, equivalently, if $\mathfrak{g}$ does not contain a copy of $\mathfrak{sl}(2,\mathbb{R})$. In particular, every non-archimedean S-adic Lie algebra is amenable.

Proposition 5.1. *Let G be an S-adic Lie group whose Lie algebra $\mathfrak{g}$ is amenable. Then any finite covolume closed subgroup $H \subset G$ is cocompact.*

We begin the proof of Proposition 5.1 by a special case :

Lemma 5.2. *Let G be a non-archimedean S-adic Lie group. Then any finite covolume closed subgroup $H \subset G$ is cocompact.*

We note that Lemma 5.2 can not be extended to any locally compact totally discontinuous group G. Indeed, for example, if k is the non-archimedean local field with positive characteristic $\mathbb{F}_q((T))$, the group $\mathrm{SL}(2, \mathbb{F}_q[T^{-1}])$ is a non-cocompact lattice in $\mathrm{SL}(2,k)$.

When H is a lattice in G, Lemma 5.2 is [14, Prop. 2]. In this case the proof is very short: Just choose a torsion free compact open subgroup Ω of G, and note successively that the action of Ω on G/H is free, that all the Ω-orbits have same volume, that there are only finitely many Ω orbits and that G/H is compact.

In order to adapt this proof to non discrete groups H, we recall a few facts on standard groups and on invariant measures.

A p-adic Lie group G_p with Lie algebra $\mathfrak{g}_p$ is said to be *standard* if there exists a compact open subgroup O_p of $\mathfrak{g}_p$ which is invariant by the Lie bracket and such that the exponential map $O_p \to G_p$ is well-defined and is a bijection onto G_p (see [7]). A non-archimedean S-adic Lie group is said to be *standard* if it is a product of standard p-adic Lie groups. By [15, Prop. 1.1], if G is standard, every closed subgroup H of G with Lie algebra $\mathfrak{h}$ is contained in $\exp(\mathfrak{h})$. Every non-archimedean S-adic Lie group contains a standard open subgroup.

Let G be an S-adic Lie group. Then the tangent bundle of G identifies G-equivariantly on the right with $G \times \mathfrak{g}$ and, through this identification, the left action of G reads as the map

$$G \times (G \times \mathfrak{g}) \to G \times \mathfrak{g}; (g, h, v) \mapsto (gh, \mathrm{Ad}_{\mathfrak{g}}(g)v).$$

Besides, if λ is a right Haar measure on G, there exists a Haar measure ω on $\mathfrak{g}$ such that λ is the measure associated to the constant field $g \mapsto \omega$ on G. In particular, the modular function of G is the function $g \mapsto |\det(\mathrm{Ad}_{\mathfrak{g}}(g))| := \prod_{p \in S} |\det(\mathrm{Ad}_{\mathfrak{g}_p}(g))|$ and, if H is a closed subgroup of G, the space G/H admits a G-invariant measure if and only if, for any h in H, $|\det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h))| = 1$.

Proof of Lemma 5.2. According to Lemma 3.1, the group H has also finite covolume in the normalizer $N_G(\mathfrak{h})$ of $\mathfrak{h}$ in G. According to Corollary 4.5, $N_G(\mathfrak{h})$ is cocompact in G. Hence, it is enough to show that H is cocompact in $N_G(\mathfrak{h})$. Thus, replacing G by $N_G(\mathfrak{h})$, we may assume that the Lie algebra $\mathfrak{h}$ is $\mathrm{Ad}_{\mathfrak{g}}(G)$-invariant. In this case, the tangent bundle of $X := G/H$ identifies with $X \times \mathfrak{g}/\mathfrak{h}$ and the action of G on this bundle can be read as the map

$$G \times (X \times \mathfrak{g}/\mathfrak{h}) \to G \times \mathfrak{g}/\mathfrak{h}; (g, x, v) \mapsto (gx, \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(g)v).$$

In the same way, if x_0 is the base point of X and λ_X is the G-invariant probability measure, then λ_X comes from the field

$$x = gx_0 \mapsto |\det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(g))|\omega$$

where ω is some fixed Haar measure on $\mathfrak{g}/\mathfrak{h}$ (note that, by the remark above, for any h in H, $|\det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h))| = 1$). Now, we claim that

$$|\det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(g))| = 1 \text{ for all } g \text{ in } G. \tag{5.1}$$

Indeed the character $\chi : G \to \mathbb{R}_+^*; g \mapsto |\det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(g))|$ is trivial on H and the probability measure $\chi_*\lambda_X$ on $\mathbb{R}_+^*$ is $\chi(G)$-invariant, hence $\chi(G) = 1$. Thus, λ_X comes from the constant field $x \mapsto \omega$ on X.

We will then prove that, for any compact open subgroup, Ω one has

$$\inf_{x \in X} \lambda_X(\Omega x) > 0. \tag{5.2}$$

To do this, we may assume that $\Omega = \prod_{p\in S}\Omega_p$ is a *standard* open compact subgroup of G. We write $\Omega = \exp(O)$ where $O = \prod_{p\in S} O_p$ and exp is the componentwise exponential map. Now, since Ω is standard, the set $H_0 := \exp(\mathfrak{h}\cap O)$ is the largest subgroup of Ω with Lie algebra $\mathfrak{h}$. For every $x := gx_0$ in X, the Ω-orbit map at x gives rise to an embedding $\Omega x \simeq \Omega/(\Omega\cap H_x) \hookrightarrow X$, where $H_x := gHg^{-1}$ is the stabilizer of x in G, and λ_X restricts on Ωx as the measure coming from the constant field $y \mapsto \omega$. Since G normalizes $\mathfrak{h}$, the Lie algebra of H_x is equal to $\mathfrak{h}$ and $H_x\cap\Omega$ is contained in H_0 as a subgroup of finite index d_x. Thus, we get

$$\lambda_X(\Omega x) = \tfrac{d_x}{d_{x_0}}\lambda_X(\Omega x_0)$$

and this quantity is bounded below by $\frac{\lambda_X(\Omega x_0)}{d_{x_0}}$, whence Equation (5.2).

Since X has finite volume, this implies that Ω has only finitely many orbits in X and X is compact. □

Proof of Proposition 5.1. We proceed by induction on the dimension of the largest solvable ideal $\mathfrak{r}_\infty$ of $\mathfrak{g}_\infty$.

1^st case : $\mathfrak{r}_\infty = 0$.
In this case, the connected immersed subgroup G_∞ of G with Lie algebra $\mathfrak{g}_\infty$ is compact. According to Lemma 3.1, the group HG_∞/G_∞ has finite covolume in the non-archimedean S-adic Lie group G/G_∞. Hence according to Lemma 5.2, the group H is cocompact in G.

2^nd case : $\mathfrak{r}_\infty \neq 0$.
We argue here exactly as in the proof of Proposition 3.4. Let R_∞ be the connected immersed real Lie group with Lie algebra $\mathfrak{r}_\infty$. Let A be the closure of the last non trivial term of the derived series of R_∞ and $H' := \overline{HA}$. According to Lemma 3.1, the group H'/A has finite covolume in G/A and the group H has finite covolume in H'. By the induction hypothesis, the group H'/A is cocompact in G/A hence H' is cocompact in G. By Lemma 3.6, the group H is cocompact in H'. Hence the group H is cocompact in G. □

6. Projections of lattices

The aim of this section is to prove, for any lattice Λ in any S-adic Lie group G, a decomposition theorem of Λ with respect to the adjoint action on a suitable semisimple quotient $\mathfrak{s}$ of the Lie algebra of G (Theorem 6.6).

Proposition 6.1. *Let G be an S-adic Lie group. Let $\mathfrak{r}_0$ be the smallest ideal of $\mathfrak{g}$ such that the Lie algebra $\mathfrak{s}_0 := \mathfrak{g}/\mathfrak{r}_0$ is semisimple and such* *that, for any non zero G-invariant ideal $\mathfrak{s}_1$ of $\mathfrak{s}_0$, the group* $\mathrm{Ad}_{\mathfrak{s}_1}(G)$ *is an unbounded subgroup of* $\mathrm{Aut}(\mathfrak{s}_1)$. *Let* $R_0 := \mathrm{Ker}(\mathrm{Ad}_{\mathfrak{s}_0})$ *be the kernel of the adjoint map in* $\mathfrak{s}_0$.

Then, for any lattice Λ in G, the Lie algebra $\mathfrak{l}$ of the group $L := \overline{\Lambda R_0}$ *is amenable.*

We postpone the proof of Proposition 6.1 to the end of this section.

If $S = \{\infty\}$, the group ΛR_0 is closed (see lemma 6.4 below). In the general setting, our statement is optimal, as shown by the following two examples.

Example 6.2. *The S-adic Lie group L may contain Lie subgroups isomorphic to* $\mathrm{PSL}(2,\mathbb{Q}_p)$ *with p finite.*

Proof. Let G_0 be a non compact simple p-adic Lie group, for instance $G_0 = \mathrm{PSL}(2,\mathbb{Q}_p)$. We will prove, more generally, that L *may contain Lie subgroups isomorphic to* G_0. Let $Y := G_0/K_0$ where K_0 is a compact open subgroup of G_0 and R_0 be the free group on Y, i.e. the discrete free non-abelian group with infinitely many generators e_y with y in Y. We define G to be the semidirect product $G := G_0 \ltimes R_0$ where the action by conjugation of G_0 on R_0 is given by $g_0 e_y g_0^{-1} = e_{g_0 y}$ for all g_0 in G_0 and all y in Y. Every element g of G can be written in a unique way as $g = r_g s_g$ with r_g in R_0 and s_g in G_0.

Let us now construct a lattice Λ in G. We first construct a discrete subgroup F of G. For y in Y, one chooses an element a_y in G_0 which fixes y. We assume that a_y has infinite order. Let F be the group generated by the elements

$$f_y := e_y a_y = a_y e_y.$$

We claim that

(6.1) the group F is a free discrete subgroup of G and the map $F \to R_0; f \mapsto r_f$ is a bijection.

Indeed, the R_0 component r_w of a word

$$w = f_{y_1}^{n_1} \cdots f_{y_\ell}^{n_\ell} \text{ with } y_i \in Y \text{ and } n_i \in \mathbb{Z}$$

is equal to

$$r_w = e_{z_1}^{n_1} \cdots e_{z_\ell}^{n_\ell} \text{ with } z_j := a_1^{n_1} \cdots a_{j-1}^{n_{j-1}}(y_j).$$

This proves (6.1).

Let Λ_0 be a torsion free cocompact lattice in G_0. Since Λ_0 acts freely on Y, we can choose the a_y, $y \in Y$, in such a way that, for every λ_0 in Λ_0 and y in Y, one has $a_{\lambda_0 y} = \lambda_0 a_y \lambda_0^{-1}$. This ensures that Λ_0

normalizes the group F. We choose Λ to be the group $\Lambda := \Lambda_0 F$. This group is discrete and cocompact in G but its projection on G/R_0 is dense. Hence, in this case, the group L is equal to $G = G_0 \ltimes R_0$. □

Example 6.3. *When Λ is a discrete subgroup of G which is not assumed to be a lattice, the Lie algebra $\mathfrak{l}$ may contain Lie subalgebras isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.*

Proof. Let G_0 be a simple real Lie group, for instance $G_0 = \mathrm{PSL}(2,\mathbb{R})$, and Γ be a dense subgroup of G_0. Let G be the cartesian product $G := G_0 \times R_0$ where R_0 is a copy of Γ endowed with the discrete topology. The discrete subgroup $\Lambda := \{(\gamma,\gamma) \mid \gamma \in \Gamma\}$ is a discrete subgroup of G whose projection in G_0 is dense. Proposition 6.1 tells us that this can not happen if Λ is a lattice in G. □

For real Lie groups G, Proposition 6.1 is a consequence of the following

Lemma 6.4. *Let G be a real Lie group and Λ be a lattice in G. Let $\mathfrak{r}$ be the largest amenable ideal of $\mathfrak{g}$, $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$ and R be the kernel of the adjoint action $\mathrm{Ad}_{\mathfrak{s}} : G \to \mathrm{Aut}(\mathfrak{s})$ in $\mathfrak{s}$. Then, the intersection $\Lambda \cap R$ is a cocompact lattice in R and the image $\mathrm{Ad}_{\mathfrak{s}}(\Lambda)$ is a lattice in $\mathrm{Aut}(\mathfrak{s})$.*

Proof. **1$^{\text{st}}$ case** : G is a connected real Lie group.
Let R_e be the connected component of R. The group R_e is the largest closed connected normal amenable subgroup of G. Since R_e is a compact extension of a solvable group, according to Auslander projection theorem in [13, Chap. 8], the group $L := \overline{\Lambda R_e}$ has an amenable Lie algebra $\mathfrak{l}$. Since $\mathfrak{l}$ is normalized by Λ and $\mathfrak{s}$ has no compact factors, by Borel density theorem ([13, Chap. 5] or Corollary 4.7), $\mathfrak{l}$ is an ideal of $\mathfrak{g}$. Hence $\mathfrak{l} = \mathfrak{r}$, and the group ΛR_e is closed. By Lemma 3.1, $\Lambda \cap R_e$ is a lattice in R_e. Since R_e is amenable, this lattice is cocompact ([13, Chap. 4] or Proposition 5.1). Replacing G by G/R_e, we can assume that G is semisimple connected with no compact factor.

The group R is then discrete and is the center of G. Since the group $L := \overline{\Lambda R}$ has a discrete derived subgroup, its Lie algebra $\mathfrak{l}$ is abelian. Again by Borel density theorem, one gets $\mathfrak{l} = 0$. Hence L is discrete and, by Lemma 3.1, the group $L/\Lambda \simeq R/(\Lambda \cap R)$ is finite. Thus the image $\Lambda_{\mathfrak{s}} = \mathrm{Ad}_{\mathfrak{s}}(\Lambda) \simeq \Lambda/(\Lambda \cap R)$ is a lattice in the adjoint group $\mathrm{Ad}_{\mathfrak{s}}(G)$. Since this adjoint group has finite index in $\mathrm{Aut}(\mathfrak{s})$, $\Lambda_{\mathfrak{s}}$ is also a lattice in $\mathrm{Aut}(\mathfrak{s})$.

2$^{\text{nd}}$ case : G is any real Lie group.
Since the connected component G_e of G is an open subgroup of G, by Lemma 3.2, the intersection $\Lambda_0 := \Lambda \cap G_e$ is a lattice in G_e. According

to the first case, the group $\Lambda_{0,S} := \mathrm{Ad}_{\mathfrak{s}}(\Lambda_0)$ is a lattice in the group $\mathrm{Aut}(\mathfrak{s})$. Since the group $\Lambda_{\mathfrak{s}} := \mathrm{Ad}_{\mathfrak{s}}(\Lambda)$ normalizes $\Lambda_{0,S}$, it is discrete by Corollary 4.8. Hence $\Lambda_{\mathfrak{s}}$ is a lattice. As a consequence, the intersection $\Lambda_R := \Lambda \cap R$ is a lattice in R by Lemma 3.1. According also to the first case, the group $\Lambda_0 \cap R$ is cocompact in $G_e \cap R$, hence, by Lemma 3.2, the lattice Λ_R is also cocompact in R. □

Lemma 6.5. *Let G be a non-archimedean S-adic Lie group, $\mathfrak{r}$ be the largest solvable ideal of $\mathfrak{g}$ and $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$. Assume that the image $\mathrm{Ad}_{\mathfrak{s}}(G)$ of G in $\mathrm{Aut}(\mathfrak{s})$ is a compact group. Then there exists an increasing sequence*

$$G_1 \subset \cdots \subset G_i \subset \cdots \subset G$$

of open subgroups whose union $G' := \cup_i G_i$ is a normal open subgroup of G and such that, for all integers $i \geq 1$, the group $\mathrm{Ad}_{\mathfrak{g}}(G_i)$ is compact.

In order to help the reader to understand the following technical proof, we suggest him to keep in mind the example where $G := (\Gamma \times \mathbb{Q}_p^*) \ltimes \mathbb{Q}_p^2$ where Γ is the group $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$ with the discrete topology and where $\mathbb{Q}_p^*$ acts diagonally on $\mathbb{Q}_p^2$.

Proof of Lemma 6.5. We first note that it is enough to prove this lemma for a finite index subgroup of G.

We will need some notations. Let $\mathfrak{g} = \oplus \mathfrak{g}_p$ be the Lie algebra of G and $\mathfrak{s}_p^0$ be a maximal semisimple Lie subalgebra of $\mathfrak{g}_p$. Let
- $\mathrm{Aut}(\mathfrak{g}_p)$ be the group of automorphisms of the Lie algebra $\mathfrak{g}_p$,
- $\mathbf{A}_p$ the Zariski connected component of $\mathrm{Aut}(\mathfrak{g}_p)$,
- $\mathbf{S}_p$ a maximal semisimple Zariski connected subgroup of $\mathbf{A}_p$,
- $\mathbf{R}_p$ the maximal solvable Zariski connected normal subgroup of $\mathbf{A}_p$,
- $\mathbf{U}_p$ the maximal unipotent normal subgroup of $\mathbf{A}_p$,

and $\mathfrak{a}_p$, $\mathfrak{s}_p$, $\mathfrak{r}_p$ and $\mathfrak{u}_p$ their Lie algebras. We may assume that $\mathfrak{s}_p$ contains the Lie algebra $\mathfrak{s}_p^1 := \mathrm{ad}(\mathfrak{s}_p^0)$. Since the image $\mathrm{ad}(\mathfrak{g}_p)$ is an ideal of $\mathfrak{a}_p$, $\mathfrak{s}_p^1$ is an ideal of $\mathfrak{s}_p$. We denote by $\mathfrak{s}_p^2$ the complementary ideal of $\mathfrak{s}_p^1$ in $\mathfrak{s}_p$. We have the decomposition

$$\mathfrak{s}_p^1 \oplus \mathfrak{s}_p^2 \oplus \mathfrak{r}_p = \mathfrak{a}_p$$

We set $\mathbf{S}_p^1$ and $\mathbf{S}_p^2$ for the Zariski closed and Zariski connected semisimple subgroups of $\mathbf{A}_p$ with Lie algebras respectively $\mathfrak{s}_p^1$ and $\mathfrak{s}_p^2$. The group

$$\mathbf{S}_p^1 \mathbf{S}_p^2 \mathbf{R}_p \subset \mathbf{A}_p$$

is a finite index subgroup.

Let Ω_p be an open compact subgroup of $\mathbf{S}_p^1 \mathbf{R}_p$. By the compactness assumption in Lemma 6.5, and since we are allowed to replace G by a finite index subgroup, we may assume that

$$\mathrm{Ad}_{\mathfrak{g}_p}(g) \in \Omega_p \mathbf{S}_p^2 \mathbf{R}_p \text{ for all } g \in G \text{ and for all } p.$$

On the one hand, since $\mathrm{ad}(\mathfrak{g}_p) \subset \mathfrak{s}_p^1 \oplus \mathfrak{r}_p$, the group

$$G' := \{ g \in G \mid \mathrm{Ad}_{\mathfrak{g}_p}(g) \in \Omega_p \mathbf{U}_p \text{ for all } p \} .$$

is an open subgroup of G.

On the other hand, since $[\mathbf{A}_p, \mathbf{R}_p] \subset \mathbf{U}_p$, the group $\Omega_p \mathbf{U}_p$ is normal in $\Omega_p \mathbf{S}_p^2 \mathbf{R}_p$ and the group G' is normal in G.

To conclude we just apply the following fact with $H := \mathrm{Ad}_{\mathfrak{g}_p}(G')$.

Let $H = \Omega U$ be a non-archimedean linear p-adic Lie group which is generated by a compact subgroup Ω and a normal unipotent subgroup U, then there exists an increasing sequence $(H_i)_{i \geq 1}$ of compact open subgroups of H whose union is equal to H.

We now check this fact. We first notice that this fact is true for the groups U_d of p-adic upper triangular unipotent $d \times d$ matrices. Since any unipotent group U is isomorphic to a closed subgroup of U_d, for some $d \geq 1$, this fact is also true for the group U. Hence, any subgroup of U generated by two compact open subgroups is still an open compact subgroup. Therefore any compact subgroup of U is included in a compact subgroup invariant by conjugation by Ω. Our claim follows. □

Proof of Proposition 6.1. Let $\mathfrak{g} = \mathfrak{g}_f \oplus \mathfrak{g}_\infty$ and $\mathfrak{s}_0 = \mathfrak{s}_{0,f} \oplus \mathfrak{s}_{0,\infty}$ be the decompositions of $\mathfrak{g}$ and $\mathfrak{s}_0$ as a sum of a non-archimedean ideal and a real one and let G_∞ be the connected immersed subgroup of G with Lie algebra $\mathfrak{g}_\infty$. We first begin by a special case.

1$^{\text{st}}$ case : The group $\mathrm{Ad}_{\mathfrak{g}_f}(G)$ is compact.
Let $D := \mathrm{Ker}(\mathrm{Ad}_{\mathfrak{g}})$ be the Kernel of the adjoint action. Since the Lie algebras $\mathfrak{g}_f$ and $\mathfrak{g}_\infty$ commute, there exists a compact subgroup G_f of G with Lie algebra $\mathfrak{g}_f$ which commutes with G_∞. Since $\mathrm{Ad}_{\mathfrak{g}_f}(G)$ is compact, the group $G_\infty G_f D$ is a finite index open subgroup of G. Hence we may assume that

$$G = G_\infty G_f D.$$

For any compactly generated subgroup D_1 of D, the centralizer of D_1 in G_f has finite index in G_f. In particular, D is the union of its G_f-invariant compactly generated subgroups D_1.

We want to prove that the group $L := \overline{\Lambda R_0}$ has an amenable Lie algebra.

We proceed by contradiction. Assume this is not the case. Since every dense subgroup of a real connected Lie group contains a finitely generated dense subgroup, there would exist a finitely generated subgroup Λ_0 of Λ such that the group $\overline{\Lambda_0 R_0}$ has a non-amenable Lie algebra. Let D_1 be an open compactly generated G_f-invariant subgroup of D such that the group $G_1 := G_\infty G_f D_1$ contains Λ_0. By Lemma 3.2, the intersection $\Lambda_1 := \Lambda \cap G_1$ is a lattice in G_1.

It is enough to prove our claim for (G_1, Λ_1). Hence we can assume D to be compactly generated. But then, after replacing G by an open finite index subgroup, we can assume that G_f and D commute. The quotient group $G' := G/G_f$ is then a real Lie group and the image Λ' of Λ in G' is a lattice. Since one has $G_f R_0 = \mathrm{Ker}(\mathrm{Ad}_{\mathfrak{s}_{0,\infty}})$, according to Lemma 6.4 applied to (G', Λ'), the group $\Lambda G_f R_0$ is closed. Hence the group $L := \overline{\Lambda R_0}$ has an amenable Lie algebra, whence a contradiction.

2$^{\mathrm{nd}}$ case : General case.
Let K be a compact open subgroup of $\mathrm{Aut}(\mathfrak{s}_{0,f})$. Since the group

$$G_K := \{g \in G \mid \mathrm{Ad}_{\mathfrak{s}_{0,f}}(g) \in K\} \tag{6.2}$$

is an open subgroup of G, $\Lambda \cap G_K$ is a lattice in G_K by Lemma 3.2. Since G_K contains R_0, the groups $\overline{\Lambda R_0}$ and $\overline{(\Lambda \cap G_K) R_0}$ both have Lie algebra $\mathfrak{l}$. Hence we may assume that $G = G_K$.

According to equality (6.2) and the definition of $\mathfrak{s}_0$, the adjoint group $\mathrm{Ad}_{\mathfrak{s}_f}(G)$ is compact. Hence we can apply Lemma 6.5 to the group $G/\overline{G_\infty}$: there exists an increasing sequence

$$G_1 \subset \cdots \subset G_i \subset \cdots \subset G$$

of open subgroups of G containing G_∞ whose union $G' := \cup_i G_i$ is normal in G and such that, for $i \geq 1$, the group $\mathrm{Ad}_{\mathfrak{g}_f}(G_i)$ is compact.

Again by Lemma 3.2, for all $i \geq 1$, the group $\Lambda_i := \Lambda \cap G_i$ is a lattice in G_i. We denote by

$$\Lambda_{\mathfrak{s}_0, i} := \mathrm{Ad}_{\mathfrak{s}_{0,\infty}}(\Lambda_i)$$

its image in the group $\mathrm{Aut}(\mathfrak{s}_{0,\infty})$ and we set

$$R_{0,\infty} := \mathrm{Ker}(\mathrm{Ad}_{\mathfrak{s}_{0,\infty}}).$$

According to the first case applied to (G_i, Λ_i), the group $\overline{\Lambda_i R_{0,\infty}}/R_{0,\infty}$ has an amenable Lie algebra. Since the group $\mathrm{Aut}(\mathfrak{s}_{0,\infty})$ is semisimple with no compact factor, by the Borel density theorem (Corollary 4.7), this Lie algebra is an ideal of $\mathfrak{s}_{0,\infty}$, hence it is trivial. Therefore, $\Lambda_i R_{0,\infty}$ is closed and, by Lemma 3.1, the group $\Lambda_{\mathfrak{s}_0,i}$ is a lattice in $\mathrm{Aut}(\mathfrak{s}_{0,\infty})$. Since any increasing sequence of lattices in a semisimple real Lie group is stationary (see [10]), there exists $i_0 \geq 1$ such that $\Lambda_{\mathfrak{s}_{0,i}} = \Lambda_{\mathfrak{s}_{0,i_0}}$ for all $i \geq i_0$.

We set $\Lambda_{\mathfrak{s}_0}$ and $\Lambda'_{\mathfrak{s}_0}$ for the images of Λ and $\Lambda' := \Lambda \cap G'$ in $\mathrm{Aut}(\mathfrak{s}_{0,\infty})$. We have just proven that $\Lambda'_{\mathfrak{s}_0} = \Lambda_{\mathfrak{s}_0,i_0}$ is a lattice in $\mathrm{Aut}(\mathfrak{s}_{0,\infty})$. Since $\Lambda_{\mathfrak{s}_0}$ normalizes $\Lambda'_{\mathfrak{s}_0}$, we obtain, by Corollary 4.8, that $\Lambda_{\mathfrak{s}_0}$ is also a lattice in $\mathrm{Aut}(\mathfrak{s}_{0,\infty})$. Hence the group $\Lambda R_{0,\infty}$ is closed in G. This proves that the Lie algebra $\mathfrak{l}$ is amenable. □

Let us now state the main result of this paper.

Theorem 6.6. *Let G be an S-adic Lie group and Λ be a lattice in G. Then, there exists a G-invariant ideal $\mathfrak{r}$ of $\mathfrak{g}$ with the following properties. Let $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$ and R be the kernel of the adjoint map $\mathrm{Ad}_{\mathfrak{s}} : G \to \mathrm{Aut}(\mathfrak{s})$.*
(i) $\mathrm{Aut}(\mathfrak{s})$ is a semisimple S-adic Lie group with no compact factor.
(ii) The group $\mathrm{Ad}_{\mathfrak{s}}(G)$ is a finite index subgroup in $\mathrm{Aut}(\mathfrak{s})$.
(iii) The group $\Lambda_{\mathfrak{s}} := \mathrm{Ad}_{\mathfrak{s}}(\Lambda)$ is a lattice in $\mathrm{Aut}(\mathfrak{s})$.
(iv) The intersection $\Lambda \cap R$ is a cocompact lattice in R.

See [8, Thm 9.5] where a related projection theorem is proven for a general locally compact group G and a normal amenable subgroup R. We note that in our Theorem 6.6, the group R may not be amenable (see Example 6.2).

Proof. Let $L = \overline{\Lambda R_0}$ be as in Proposition 6.1 and let $\mathfrak{r} = \mathfrak{l}$ be the Lie algebra of L. Note that, by [6, 6.14 and 9.10], for any simple real or p-adic Lie algebra $\mathfrak{h}$, any non-compact open subgroup of $\mathrm{Aut}(\mathfrak{h})$ has finite index. Hence, G/R_0 is a finite index subgroup of $\mathrm{Aut}(\mathfrak{s}_0)$. Besides, by Lemma 3.1, L is a finite covolume closed subgroup of G. Therefore, by Corollary 4.7, the Lie algebra $\mathfrak{r}$ is an ideal of $\mathfrak{g}$. In particular, the quotient Lie algebra $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$ is semisimple. We thus get (i) and (ii).

Let R be the kernel of the map $\mathrm{Ad}_{\mathfrak{s}}$ in G and let us prove that $L \cap R$ has finite index in R. As R and $L \cap R$ have Lie algebra $\mathfrak{r}$, through the map $\mathrm{Ad}_{\mathfrak{r}/\mathfrak{r}_0}(G)$, R/R_0 and $L \cap R/R_0$ identify with open subgroups of $\mathrm{Aut}(\mathfrak{r}/\mathfrak{r}_0)$ which are normalized by $\mathrm{Ad}_{\mathfrak{r}/\mathfrak{r}_0}(L)$. Now, by assumption, the group $\mathrm{Ad}_{\mathfrak{r}/\mathfrak{r}_0}(G)$ is a finite index open subgroup of $\mathrm{Aut}(\mathfrak{r}/\mathfrak{r}_0)$ and, since L is a finite covolume subgroup of G and since the Lie algebra of L is $\mathfrak{r}$, the group $\mathrm{Ad}_{\mathfrak{r}/\mathfrak{r}_0}(L)$ is a finite index open subgroup of $\mathrm{Ad}_{\mathfrak{r}/\mathfrak{r}_0}(G)$. Thus, $\mathrm{Ad}_{\mathfrak{r}/\mathfrak{r}_0}(R)$ and $\mathrm{Ad}_{\mathfrak{r}/\mathfrak{r}_0}(L \cap R)$ are also finite index open subgroup of $\mathrm{Aut}(\mathfrak{r}/\mathfrak{r}_0)$ and $L \cap R$ has finite index in R.

In particular, the group LR is closed and has Lie algebra $\mathfrak{r}$. Now, as $R \subset \overline{\Lambda R} \subset LR$, $\overline{\Lambda R}$ also has Lie algebra R, that is ΛR is closed. By Lemma 3.1, ΛR has finite covolume in G/R and $\Lambda \cap R$ has finite covolume in R. By Proposition 6.1, the Lie algebra of R being amenable,

$\Lambda \cap R$ is cocompact in R. Thus, G/R being a finite index open subgroup of $\mathrm{Aut}(\mathfrak{s})$, we get (iii) and (iv). □

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