ADDITIVE REPRESENTATIONS OF TREE LATTICES 3. SPECTRAL THEORY OF EUCLIDEAN FIELDS

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Abstract. Building on [6] and [7], we continue to study unitary representations of tree lattices. We prove a Plancherel formula for representations obtained by orthogonal extension from a Euclidean field. This allows us to compute the spectrum of natural operators associated to these representations.

Contents

1. Introduction

1.1. **The spectral problem for unitary representaions of tree lattices.** We fix an integer $q \geq 2$ and a homogeneous tree X of degree $q+1$: in other words, every vertex of X has $q+1$ neighbours. We let Γ be a cofinite lattice of X, that is, Γ is a discrete group of automorphisms of X and the quotient $\Gamma\backslash X$ is finite. The purpose of this article is to develop a spectral theory for certain unitary representations of Γ.

Let us precise what kind of spectral theory we have in mind. Given a vector space V equipped with an action of Γ, we let $\mathcal{F}(X,V)^\Gamma$ denote the space of all maps $X \to V$ which are Γ-equivariant. This space comes with a natural linear operator Q associated to the geometry of X. For f in $\mathcal{F}(X,V)^\Gamma$ and x in X, we write

$$Qf(x) = \frac{1}{q+1} \sum_{y \sim x} f(y).$$

Assume V is equipped with a Hilbert space structure and the action of Γ on V is unitary. Then, we can equip $\mathcal{F}(X,V)^\Gamma$ with the natural scalar product defined by, for f, g in $\mathcal{F}(X,V)^\Gamma$,

$$\langle f, g \rangle = \sum_{x \in \Gamma\backslash X} \frac{1}{|\Gamma_x|} \langle f(x), g(x) \rangle$$

(where, for x in X, Γ_x is the stabilizer of x in Γ). One easily checks that the operator Q has norm ≤ 1 and that it is self-adjoint with respect to this scalar product. We are interested in understanding the spectral invariants of this self-adjoint operator.

Example 1.1. Assume Γ is the free group generated by a finite set S with $|S| = r \geq 2$ and that X is the natural tree associated with this data (so that $q = 2r - 1$). Let x be the vertex of X whose neighbours are the sx and $s^{-1}x$ for s in S. Then, the map $f \mapsto f(x)$ is an isometry from $\mathcal{F}(X, V)^\Gamma$ onto V which conjugates the operator Q and the operator $v \mapsto \frac{1}{2r} \sum_{s \in S}(sv + s^{-1}v)$ on V.

Example 1.2. Let V be $\ell^2(\Gamma)$, equipped with the left regular representation. In this case, elements of $\mathcal{F}(X, V)^\Gamma$ can be thought of as Γ-invariant functions f on $\Gamma \times X$. The map $f \mapsto f(e, x)$ is an isometry from $\mathcal{F}(X, V)^\Gamma$ onto $\ell^2(X)$. Then, we can see Q as the natural Markov operator on X defined by

$$Qg(x) = \frac{1}{q+1} \sum_{y \sim x} g(y), \quad g \in \ell^2(X), \quad x \in X.$$

The spectrum of this operator was computed by Kesten [5]: this is the interval $[-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}]$. We give a proof of this fact in Corollary 4.10.

According to our knowledge, very few examples are known, where the spectral invariants of Q can be explicitely computed. The purpose of this article is to provide a wide family of unitary representations where such computations can be achieved.

1.2. **Euclidean fields.** In [6], we have introduced the notion of a k-quadratic field, where $k \geq 2$ is an integer. The vector space $\mathcal{F}_k$ of all Γ-invariant k-quadratic fields has finite dimension. It can be thought of as an analogue of a space of sections of a vector bundle over the quotient space $\Gamma \backslash X_1$, where X_1 is the set of oriented edges of X. There is a natural surjective map

$$\mathcal{F}_{k+1} \to \mathcal{F}_k, p \mapsto p^- \tag{1.1}$$

and the projective limit of the system $(\mathcal{F}_k)_{k \geq 2}$ may be identified with the space of Γ-invariant symmetric bilinear forms on the space $\overline{\mathcal{D}}(\partial X)$, where ∂X is the boundary of X, $\mathcal{D}(\partial X)$ is the space of locally constant functions on ∂X and $\overline{\mathcal{D}}(\partial X)$ is the quotient of the latter by constant functions.

In $\mathcal{F}_k$, there is a non empty convex open cone $\mathcal{P}_k$ whose elements are called Γ-invariant k-Euclidean fields. This set comes with a natural non linear map

$$\mathcal{P}_k \to \mathcal{P}_{k+1}, p \mapsto p^+,$$

which is called the orthogonal extension, and which plays the role of a section of the map in (1.1). In other words, we have $(p^+)^- = p$ for p in $\mathcal{P}_k$.

Starting from p in $\mathcal{P}_k$, by iterating orthogonal extension, one ends up with an element of the projective limit of the system $(\mathcal{F}_j)_{j\geq k}$, which turns out to be a Γ-invariant scalar product p^∞ on the space $\overline{\mathcal{D}}(\partial X)$. The completion of this space with respect to p^∞ is denoted by H^p. Thus, H^p is a unitary representations of Γ.

It is our hope that a very precise understanding of the spectral questions of Subsection 1.1 for these particular representations would lead to a better knowledge of the general case, and in particular of the case of certain representations appearing in probability theory and dynamical systems.

In this article, we build a complete spectral theory that allows a detailed description of the spectral invariants of Q in $\mathcal{F}(X, H^p)^\Gamma$. This is achieved in Theorem 13.1 where we establish a Plancherel formula. The elements of this formula will be built along the paper. It yields

Corollary 1.3. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and H^p be the associated unitary representation of Γ. The spectrum of the operator Q in $\mathcal{F}(X, H^p)^\Gamma$ is $\Sigma_p \cup [-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}]$. The associated spectral measures are absolutely continuous on the interval $[-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}]$.*

The set Σ_p is a finite subset of $(-1, -\frac{2\sqrt{q}}{q+1}) \cup (\frac{2\sqrt{q}}{q+1}, 1)$ which can be empty in certain cases. It is defined precisely in (8.1) by means of the spectral values of the simple transfer operator S_p, which is a linear endomorphism of a finite-dimensional vector space. The simple transfer operator is an analogue of the quadratic transfer operator T_p that appeared in [6]. The quadratic transfer operator T_p acted on the space $\mathcal{L}_{k-1}$ of Γ-invariant $(k-1)$-pseudokernels, which is a space of functions with two variables. The simple transfer operator will act on the space $\mathcal{H}_{k-1}$ of Γ-invariant $(k-1)$-pseudofunctions, which is a space of functions with one variable. This space will play a key role in all our constructions.

1.3. **The Ihara trace formula.** Part of these constructions rely on analogies with the case where the representation V of Γ is the trivial representation. Then, the space $\mathcal{F}(X, V)^\Gamma$ is simply the space $\mathcal{H}_0$ of Γ-invariant functions on X. In that case, the spectrum of the operator Q is related to the spectrum of an other operator, acting on the space $\mathcal{H}_1$ of Γ-invariant functions on X_1: in our study the role of the latter operator will be played by the simple transfer operator mentioned above. The relations between those two operators on $\mathcal{H}_0$ and $\mathcal{H}_1$ are at the core of the Ihara trace formula [4]. To motivate the reader, we will now briefly recall the proof of this formula.

We equip the space $\mathcal{H}_1$ with the operator T defined by

$$Tu(x,y) = \sum_{\substack{z\sim y\\ z\neq x}} u(y,z), \quad u \in \mathcal{H}_1, \quad x \sim y \in X.$$

The original reason for studying this operator was that, for $n \geq 1$, the number $\frac{1}{n}\operatorname{tr}(T^n)$ can be interpretated as a number of closed loops of length n in the quotient graph $\Gamma\backslash X$ when this makes sense.

To relate the spectrum of Q, acting on $\mathcal{H}_0$, to the one of T, acting on $\mathcal{H}_1$, we start by equipping these spaces with the natural scalar products defined by, for a, b in $\mathcal{H}_0$ and u, v in $\mathcal{H}_1$,

$$\begin{aligned} \langle a, b\rangle &= \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|} a(x)b(x) \\ \text{and } \langle u, v\rangle &= \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} u(x,y)v(x,y). \end{aligned}$$

There are two natural embeddings $L : \mathcal{H}_0 \to \mathcal{H}_1$ and $R : \mathcal{H}_0 \to \mathcal{H}_1$ which may be defined by

$$La(x,y) = a(x) \text{ and } Ra(x,y) = a(y), \quad a \in \mathcal{H}_0, \quad x \sim y \in X.$$

With respect to the natural Euclidean structures, the adjoint operators $L^\dagger$ and $R^\dagger$ may be defined by

$$L^\dagger u(x) = \sum_{y\sim x} u(x,y) \text{ and } R^\dagger u(x) = \sum_{y\sim x} u(y,x), \quad u \in \mathcal{H}_1, \quad x \in X.$$

Let $\mathcal{E}$ be the orthogonal complement in $\mathcal{H}_1$ of the space $L\mathcal{H}_0 + R\mathcal{H}_0$. It will turn out that both $\mathcal{E}$ and $\mathcal{E}^\perp$ are T-invariant.

Indeed, the space $\mathcal{E}$ is the space of all functions u on X_1 wich satisfy, for any x in X,

$$\sum_{y\sim x} u(x,y) = \sum_{y\sim x} u(y,x) = 0.$$

In particular, for such a function, we have $Tu(x,y) = -u(y,x)$ for any $x \sim y$ in X. Thus, $\mathcal{E}$ is stable under T and the spectrum of T in $\mathcal{E}$ is (at most) $\{-1, 1\}$. The dimensions of the associated eigenspaces are combinatorial invariants of the action of Γ on X which may be related to the cardinalities of the sets $\Gamma\backslash X$ and $\Gamma\backslash X_1$ when Γ is torsion free.

Now, we analyse the action of T on $\mathcal{E}^\perp = L\mathcal{H}_0 + R\mathcal{H}_0$. For a, b in $\mathcal{H}_0$, set $B\begin{pmatrix} a \\ b \end{pmatrix} = La + Rb$. A direct computation shows the relation

$$TB\begin{pmatrix} a \\ b \end{pmatrix} = B\begin{pmatrix} 0 & -1 \\ q & (q+1)Q \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Note that, for t in $\mathbb{R}$ with $(q+1)^2t^2 \neq 4q$, the eigenvalues of the matrix $\begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix}$ are the two roots of the equation $q + u^2 = (q+1)tu$. For $t = \frac{2\sqrt{q}}{q+1}$ (resp. $t = -\frac{2\sqrt{q}}{q+1}$), the matrix is not diagonalizable and its eigenvalue is $\sqrt{q}$ (resp. $-\sqrt{q}$). Thus, the spectrum of the operator $\begin{pmatrix} 0 & -1 \\ q & (q+1)Q \end{pmatrix}$, acting on $\mathcal{H}_0^2$, is the set of all roots of the equation $q + u^2 = (q+1)tu$, where t runs among the eigenvalues of Q, acting on $\mathcal{H}_0$.

To conclude, we need to analyse the null space of the linear map B. Say that a function u on X is constant on neighbours if, for every x, y, z in X with $x \sim y$ and $x \sim z$, we have $u(y) = u(z)$. If u is such a function, let Iu be the function on X whose value on x in X is the value of u on the neighbours of x: again, Iu is constant on neighbours. Write $\mathcal{H}_{-1}$ for the space of Γ-invariant functions that are constant on neighbours. Then, the null space of B is the space

$$\left\{ \begin{pmatrix} u \\ -Iu \end{pmatrix} \middle| u \in \mathcal{H}_{-1} \right\} \subset \mathcal{H}_0^2.$$

Let us describe $\mathcal{H}_{-1}$ more precisely. There are two cases, depending on whether the action of Γ on X is bipartite or not: we say that the action of Γ on X is bipartite if, for some (equivalently any) x in X, all the integral numbers $d(x, \gamma x)$, $\gamma \in \Gamma$, are even. If the action of Γ on X is not bipartite, the space $\mathcal{H}_{-1}$ is reduced to constant functions. If the action of Γ on X is bipartite, the space $\mathcal{H}_{-1}$ has dimension 2. Depending on the case, the spectrum of Q in $\mathcal{H}_{-1}$ is either $\{1\}$ or $\{-1, 1\}$. Besides, an application of the maximum principle shows that all the eigenvalues of Q in $\mathcal{H}_0/\mathcal{H}_{-1}$ are contained in $(-1, 1)$.

Summarizing this discussion, we can relate the spectra of Q and T as follows.

Theorem 1.4 (Ihara [4])**.** *The eigenspace of T associated to 1 (resp. -1) is the space of all skew symmetric (resp. symmetric) functions u on X_1 with $\sum_{y \sim x} u(x, y) = 0$ for any x in X. The operator T admits q as an eigenvalue with multiplicity 1. The associated eigenline is the space of constant functions. If the action of Γ on X is bipartite, the* *operator T admits $-q$ as an eigenvalue with multiplicity 1. The other eigenvalues of T are the roots of the equation $q+u^2=(q+1)tu$ where t runs among the eigenvalues of P in $(-1,1)$. For each such t and u, the dimension of the eigenspace of T associated to u is equal to the dimension of the eigenspace of P associated to t. If $(q+1)^2t^2\neq 4q$, then u is a simple eigenvalue of T. If $(q+1)^2t^2=4q$, the characteristic space associated to u has dimension twice the dimension of the eigenspace.*

In a finite-dimensional vector space, an eigenvalue of a linear operator is said to be simple if the associated eigenspace is equal to the associated characteristic subspace; in other words, the dimension of the eigenspace is equal to the multiplicity of the eigenvalue as a root of the characteristic polynomial.

The spaces $\mathcal{H}_{-1}$, $\mathcal{H}_0$ and $\mathcal{H}_1$ will play a role in this article. More generally, we will handle a sequence of finite-dimensional spaces $(\mathcal{H}_k)_{k\geq -1}$. For $k\geq -1$, the elements of $\mathcal{H}_k$ will be called Γ-invariant k-pseudofunctions. The space $\mathcal{H}_k$ will be equipped with an automorphism $H\mapsto H^\vee$ and an embedding $H\mapsto H^>$ into $\mathcal{H}_{k+1}$. The embeddings L and R will be respectively written as $H\mapsto H^>$ and $H\mapsto H^{\vee>\vee}$.

The construction of the objects that appear in the Plancherel formula of Theorem 13.1 will use the action of the matrix $\begin{pmatrix}0 & -1\\ q & (q+1)t\end{pmatrix}$ on the space $\mathcal{H}_k^2$, for t in $[-\frac{2\sqrt{q}}{q+1},\frac{2\sqrt{q}}{q+1}]\cup\Sigma_p$.

1.4. **Structure of the article.** References to [6] and to [7] are indicated with I and II.

In Section 2, we introduce the language of pseudofunctions. The spectral problem of Subsection 1.1 was set by means of the space $\mathcal{F}(X,H^p)^\Gamma$ of Γ-equivariant maps $X\to H^p$, where, as in Subsection 1.2, H^p is the completion of $\overline{\mathcal{D}}(\partial X)$ with respect to the scalar product associated to a Euclidean field p. It will actually be more convenient to use instead the space $\mathcal{F}(X_1,H^p)^\Gamma$ of Γ-equivariant maps $X_1\to H^p$. Pseudofunctions are truncated maps $X_1\to\overline{\mathcal{D}}(\partial X)$ in the same way as the pseudokernels of [6] and [7] could be thought of as truncated maps $X_1\to\mathcal{D}(\partial^2X)$. The space of all Γ-equivariant maps $X_1\to\overline{\mathcal{D}}(\partial X)$ will be denoted by $\mathcal{H}_\infty$. For any $k\geq -1$, we introduce the notion of a k-pseudofunction. The boundary cases $k\in\{-1,0,1\}$ are related to the objects appearing in the proof of Theorem 1.4. The space of all Γ-invariant k-pseudofunctions is denoted by $\mathcal{H}_k$: this is a finite-dimensional vector space with a natural embedding in the space $\mathcal{H}_\infty$. We define operations on pseudofunctions in analogy with the operations on pseudokernels appearing in Section II.2. We use these operations to define the polyextension map $E_k : \mathcal{H}_k^{(\mathbb{N})} \to \mathcal{H}_\infty$, where $\mathcal{H}_k^{(\mathbb{N})}$ is the space of finitely supported sequences of elements of $\mathcal{H}_k$. The range of the polyextension map is the span in $\mathcal{H}_\infty$ of the image of $\mathcal{H}_k$ by the natural operators associated to the geometry of the tree. The polyextension map allows to encode the action of these operators by operators acting on sequences of real numbers, which we call the model operators. Model operators appear implicitely in the study of the harmonic analysis on X and X_1 in [3]. The polyextension map is not injective but its null space may be determined in an explicit manner.

In the next three Sections, we study the model operators.

In Section 3, we collect independent facts from algebra and harmonic analysis that will be used throughout the article in the construction of spectral transforms and spectral formulas.

In the parallel Sections 4 and 5, we study the model operators. Indeed, as for pseudokernels in Section I.8, the definition of a k-pseudofunction depends on the parity of the integer $k \geq -1$. So do the definitions of the model operators. Section 4 is devoted to the study of the even model operators whereas Section 5 is devoted to the study of the odd model operators. In both cases, we define spectral transforms which are isomorphisms from $\mathbb{R}^{(\mathbb{N})}$ onto the space $\mathbb{R}^2[t]$ of polynomial functions with values in $\mathbb{R}^2$. The spectral transforms conjugate the model operators with actions on $\mathbb{R}^2[t]$ of 2 by 2 matrices with coefficients in $\mathbb{R}[t]$. By computing the resolvent of certain operators, we determine their spectral measures in order to state a Plancherel formula for the spectral transforms. These computations will serve as a model for the statement of the Plancherel formula for Euclidean fields.

In Section 6, we use the study of the model operators on scalar sequences to define the spectral transform of sequences of pseudofunctions. For $k \geq 0$, the spectral transform is a linear isomorphism $\mathcal{H}_k^{(\mathbb{N})} \xrightarrow{\sim} \mathcal{H}_k^2[t]$, from the set of finitely supported sequences of elements of $\mathcal{H}_k$ onto the space of polynomial functions with values in $\mathcal{H}_k^2$. Its definition relies on the constructions of the previous sections. We describe the image under the spectral transform of the null space of the polyextension map E_k in $\mathcal{H}_k^{(\mathbb{N})}$ by means of the action of the matrix $\begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix}$ of Subsection 1.3 on the space $\mathcal{H}_{k-1}^2[t]$. The computations of this Section rely on the use of the dual space $\mathcal{S}_k$ of the space $\mathcal{H}_k$, which is called the space of k-simple pseudofields by analogy with the space of k-quadratic pseudofields of Section I.10.

Now that we have built a spectral transform, we aim at proving the Plancherel formula in Theorem 13.1.

In Section 7, we introduce the simple transfer operator S_p, by analogy with the quadratic transfer operator of Section I.10. For $k \geq 2$, if p is a Γ-invariant k-Euclidean field, the quadratic transfer operator T_p was an endomorphism of the space $\mathcal{L}_{k-1}$ of $(k-1)$-pseudokernels. Now, the simple transfer operator S_p is an endomorphism of the space $\mathcal{H}_{k-1}$ of $(k-1)$-pseudofunctions. We give informations on the spectral structure of S_p.

In Section 8, we give a formula for the resolvent of a certain self-adjoint operator on $\mathcal{F}(X_1, H^p)^\Gamma$. This formula relates the resolvent with the meromorphic function $u \mapsto (u - S_p)^{-1}$ with values in the space of endomorphisms of the finite-dimensional vector space $\mathcal{H}_{k-1}$.

We aim at using this resolvent formula to describe the spectral measures of this operator. To better understand the formula, we will proceed to new algebraic constructions.

In Section 9, we study u-radical simple pseudofields. For $u \in \mathbb{C}^*$, the notion of a u-radical simple pseudofields is defined by analogy with the notion of a radical quadratic pseudofield in Section II.3 and that of a q-radical quadratic pseudofield in Section II.6. For $k \geq 0$, the space $\mathcal{S}^u_{k,\mathbb{C}}$ of Γ-invariant u-radical k-simple pseudofields is a subspace of the space $\mathcal{S}_{k,\mathbb{C}}$ of complex Γ-invariant k-simple pseudofields, which is the complexification of $\mathcal{S}_k$. Given $k \geq 2$ and a Γ-invariant k-Euclidean field p, we associate to p a natural complex symmetric bilinear form p^*_u on $\mathcal{S}^u_{k,\mathbb{C}}$.

In Section 10, we study t-radical pairs of simple pseudofields. For t in $\mathbb{C}$, the notion of a t-radical pair of simple pseudofields is defined as above, but the complex number u is replaced by the matrix $\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix}$. For $k \geq 0$, the space $\mathcal{S}^{2,t}_{k,\mathbb{C}}$ of Γ-invariant t-radical pairs of k-simple pseudofields is a subspace of $\mathcal{S}^2_{k,\mathbb{C}}$. For $(q+1)^2 t^2 \neq 4q$, by diagonalizing the matrix, one gets an identification of $\mathcal{S}^{2,t}_{k,\mathbb{C}}$ with $\mathcal{S}^u_{k,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$ where u is a root of the equation $q + u^2 = (q+1)tu$. When $k \geq 2$ and p is a Γ-invariant k-Euclidean field, we use this structure and the previous construction of p^*_u to define a complex symmetric bilinear form $p^{2,*}_t$ on $\mathcal{S}^{2,t}_{k,\mathbb{C}}$. When t belongs to the critical interval $(-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1})$, the bilinear form $p^{2,*}_t$ is real and positive definite on $\mathcal{S}^{2,t}_k$. The dual object is a non-negative symmetric bilinear form p^2_t on $\mathcal{H}^2_k$ whose null space is exactly the image of the null space of the polyextension map E_k of Section 2 by the evaluation at t of the spectral transform of Section 6. We call p^2_t the spectral bilinear form associated to t. It will allow to write the Plancherel formula on the interval $(-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1})$.

In Section 11, we introduce the objects that will allow to write the Plancherel formula at the points of the set Σ_p of Subsection 1.2, which we call the exceptional spectrum. Exceptional spectral values are real numbers t for which one of the roots of the equation

$$q + u^2 = (q+1)tu$$

belongs to $(-q, -\sqrt{q}) \cup (\sqrt{q}, q)$ and is an eigenvalue of the simple transfer operator S_p. The study of the spectrum of S_p in Section 7 and that of the notion of a t-radical pair of simple pseudofields in Section 10 allow to associate to t in Σ_p a non-negative symmetric bilinear form $p_t^{2,\mathrm{ex}}$ on $\mathcal{H}_k^2$. We call $p_t^{2,\mathrm{ex}}$ the exceptional spectral bilinear form associated to t.

In Section 12, we handle the remaining two points of the spectrum, which are -1 and 1. Those two points do not appear in Corollary 1.3 which only deals with harmonic analysis in the space of Γ-equivariant functions $X \to H^p$. When studying instead Γ-equivariant functions $X_1 \to H^p$, these two spectral values appear in the computations. The same phenomenon happens in the spectral decompositions of the action of the full group of automorphisms G of X on the spaces $\ell^2(X)$ and $\ell^2(X_1)$ in [3]. It is then related to the construction of the special unitary representations of G. It also takes place in the Ihara trace formula, Theorem 1.4, through the study of the space that was denote by $\mathcal{E}$ in Subsection 1.3. In our case, in analogy with the notion of a skew quadratic field in II.6, we define the notions of a skew field and of a reverse skew field. The space $\mathcal{G}_k^1$ of Γ-invariant k-skew fields is a subspace of $\mathcal{S}_k^1$; the space $\mathcal{G}_k^{(-1)}$ of Γ-invariant reverse k-skew fields is a subspace of $\mathcal{S}_k^{(-1)}$. If $k \geq 2$ and p is a Γ-invariant k-Euclidean field, we can associate to p scalar products $p_1^{\mathrm{sp},*}$ and $p_{(-1)}^{\mathrm{sp},*}$ on $\mathcal{G}_k^1$ and $\mathcal{G}_k^{(-1)}$. This construction is a direct translation for k-skew fields and reverse k-skew fields of the construction of the skew weight metric on k-skew quadratic fields in Section II.7. From $p_1^{\mathrm{sp},*}$ and $p_{(-1)}^{\mathrm{sp},*}$, we build non-negative symmetric bilinear forms $p_1^{2,\mathrm{sp}}$ and $p_{(-1)}^{2,\mathrm{sp}}$ on $\mathcal{H}_k^2$, which we call the special spectral bilinear forms associated to 1 and -1.

In the final Section 13, we gather the previous constructions and results to state the Plancherel formula. For $k \geq 2$ and p a Γ-invariant

k-Euclidean field, we write this formula as

$$p^\infty(E_kH,E_kJ)=\frac{q+1}{2\pi(q-1)}\int_{-\frac{2\sqrt{q}}{q+1}}^{\frac{2\sqrt{q}}{q+1}}p_t^2(\widehat{H}(t),\widehat{J}(t))\sqrt{4q-(q+1)^2t^2}\mathrm{d}t$$
$$+(q-1)\sum_{t\in\Sigma_p}p_t^{2,\mathrm{ex}}(\widehat{H}(t),\widehat{J}(t))+\frac{q-1}{2(q+1)}p_1^{2,\mathrm{sp}}(\widehat{H}(1),\widehat{J}(1))$$
$$+\frac{q-1}{2(q+1)}p_{(-1)}^{2,\mathrm{sp}}(\widehat{H}(-1),\widehat{J}(-1)),$$

where H and J are in $\mathcal{H}_k^{(\mathbb{N})}$. We prove this formula by comparing the boundary values of the resolvent function constructed in Section 8 with the formulas established in Section 10 for the spectral bilinear forms, in Section 11 for the exceptional spectral bilinear forms and in Section 12 for the special spectral bilinear forms.

1.5. **Notation.** We use the general notation introduced in Subsection I.1.8 and Subsection I.2.1. In particular, we have $d(x)=q+1$ for any x in X.

If E is a set, we let $E^{\mathbb{N}}$ be the set of sequences of elements of E.

If V is a vector space, we write $V^{(\mathbb{N})}\subset V^{\mathbb{N}}$ for the set of finitely supported sequences, that is, sequences $(x_i)_{i\geq 0}$ such that $x_i=0$ for all large enough i. For v in V and $i\geq 0$, we let $v\mathbf{1}_i$ be the sequence in $V^{(\mathbb{N})}$ all of whose coefficients are 0, except the i-th one which is equal to v.

We will often write elements u of V^2 as column matrices $u=\begin{pmatrix}v\\w\end{pmatrix}$, where v and w are in V. In this case, we write $u^*=\begin{pmatrix}v & w\end{pmatrix}$ for the associate line matrix.

Let V be a real vector space. We write $V_{\mathbb{C}}$ for the complexification of V, that is, $V_{\mathbb{C}}=\mathbb{C}\otimes V$. One has a natural embedding $V\subset V_{\mathbb{C}}$ and $V_{\mathbb{C}}$ may be written as $V_{\mathbb{C}}=V\oplus iV$. We then write $v\mapsto\overline{v}$ for the natural complex conjugation in $V_{\mathbb{C}}$, so that, for v in V, one has $\overline{v}=v$ and $\overline{iv}=-iv$

Any object defined in terms of real linear algebra on real vector spaces defines an analoguous complexified object on complexified spaces: we usually denote the two objects by the same letters. For example, if $T:V\to V$ is a real linear map, we still write $T:V_{\mathbb{C}}\to V_{\mathbb{C}}$ for the complex linear map defined by $T(v+iw)=Tv+iTw$ for v,w in V. In particular, we adopt the following convention: if p is a symmetric bilinear form on V, then p also stands for the complex bilinear form on $V_{\mathbb{C}}$ whose restriction to V is p. Thus, the natural Hermitian form on $V_{\mathbb{C}}$ associated to p is defined as $\tilde{p} : (v, w) \mapsto p(\overline{v}, w), V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$. Then, p is positive definite as a real bilinear form on V if and only if $\tilde{p}$ is positive definite as a Hermitian form on $V_{\mathbb{C}}$.

The space of polynomial functions $\mathbb{R} \to V$ is denoted by $V[t]$.

2. Pseudofunctions

We introduce a family $(\mathcal{H}_k)_{k \geq -1}$ of vector spaces which will serve to define the spectral invariants associated to a Euclidean field. In case $k = -1, 0, 1$, these spaces are essentially the same as the ones used in the proof of the Ihara trace formula, Theorem 1.4.

2.1. Pseudofunctions and the special cases $k = -1, 0, 1$.

We start by defining k-pseudofunctions for $k \geq 1$. Recall that, for x in X and $\ell \geq 0$, we write $S^\ell(x)$ for the sphere with radius ℓ and center x and $\overline{V}^\ell(x)$ for the quotient of the space of real valued functions on $S^\ell(x)$ by the constant functions. In the same way, for $x \sim y$ in X, we set

$$S^\ell(xy) = \{z \in X \,|\, \min(d(x, z), d(y, z)) = \ell\}$$

and we let $\overline{V}^\ell(xy)$ be the quotient of the space of real valued functions on $S^\ell(xy)$ by the constant functions.

Definition 2.1. Let $k \geq 1$. If k is even, $k = 2\ell$, $\ell \geq 1$, a k-pseudofunction is a family $(H_{xy})_{(x,y) \in X_1}$ where, for any $x \sim y$ in X, H_{xy} is an element of $\overline{V}^\ell(x)$. If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, a k-pseudofunction is a family $(H_{xy})_{(x,y) \in X_1}$ where, for any $x \sim y$ in X, H_{xy} is an element of $\overline{V}^\ell(xy)$.

The finite dimensional vector space of Γ-invariant k-pseudofunctions is denoted by $\mathcal{H}_k$.

In case $k = 1$, for $x \sim y$ in X, the space $\overline{V}^0(xy)$ has dimension 1. Therefore, if H is a 1-pseudofunction, we can write $H_{xy} = u(xy)\mathbf{1}_y$, $x \sim y \in X$, for some uniquely defined function u on X_1. In the sequel, we shall use this convention to identify 1-pseudofunctions with functions on X_1. In particular, as in Subsection 1.3, the space $\mathcal{H}_1$ will also be seen as the space of Γ-invariant functions on X_1.

Following this identification, let us define 0-pseudofunctions and (-1)-pseudofunctions.

By convention, we say that a 0-pseudofunction is a pair $(0, u)$, where u is a function on X and we write $\mathcal{H}_0$ for the space of Γ-invariant 0-pseudofunctions.

As in Subsection 1.3, we will say that a function u on X is constant on neighbours if, for any x in X and for any neighbours y, z of X, one has $u(y) = u(z)$. The space of functions that are constant on neighbours has dimension 2. The space of Γ-invariant functions that are constant on neighbours has dimension 1 or 2. The action of Γ on X (or if this makes sense, the associated quotient graph) is called bipartite in case this dimension is 2. There is a natural involution on the space of functions on X that are constant on neighbours: if u is such a function, we define the opposite of u as the function whose value on x in X is the value of u on neighbours of x.

Again by convention, a (-1)-pseudofunction is a pair $(-1, u)$, where u is a function on X that is constant on neighbours and we write $\mathcal{H}_{-1}$ for the space of Γ-invariant (-1)-pseudofunctions.

Remark 2.2. We have slightly changed the definitions of $\mathcal{H}_0$ and $\mathcal{H}_{-1}$ with respect to the ones used in Subsection 1.3. The reason for these formal modifications will be clear in the next Subsection.

2.2. **Operations on pseudofunctions.** We define natural algebraic operations on pseudofunctions which are analogues of the ones defined on pseudokernels in Section II.2.1.

Definition 2.3. Let $k \geq 1$ and H be a k-pseudofunction.

If k is even, for any $x \sim y$ in X, we set $H^\vee_{xy} = \sum_{\substack{z \sim x \\ z \neq y}} H_{xz}$. We call $H^\vee$ the reversal of H. The map $H \mapsto H^\vee$ is a linear automorphism of the space of k-pseudofunctions.

If k is odd, for any $x \sim y$ in X, we set $H^\vee_{xy} = H_{yx}$. We call $H^\vee$ the inversion of H. The map $H \mapsto H^\vee$ map is an involution of the space of k-pseudofunctions.

Note that, if $k = 1$, and H is the 1-pseudofunction associated to the function u on X_1, then $H^\vee$ is the 1-pseudofunction associated to the function $(x, y) \mapsto -u(yx)$ on X_1. By convention, if $k = 0$ and H is the 0-pseudofunction associated to the function u on X, we set $H^\vee$ to be the 0-pseudofunction associated to the function $x \mapsto -u(x)$ on X. If $k = -1$ and H is the (-1)-pseudofunction associated to the function u on X that is constant on neighbours, we set $H^\vee$ to be the (-1)-pseudofunction associated to the function $-v$ on X, where v is the opposite of u. These choices will be justified by Lemma 2.6 below.

If $k \geq -1$ is odd, the inversion is an involution of $\mathcal{H}_k$ and we will use the notation

$$\mathcal{H}_{k,+} = \{H \in \mathcal{H}_k | H^\vee = H\} \text{ and } \mathcal{H}_{k,-} = \{H \in \mathcal{H}_k | H^\vee = -H\}.$$

In particular, we have $\mathcal{H}_k = \mathcal{H}_{k,+} \oplus \mathcal{H}_{k,-}$. If k is even (and ≥ 2), the reversal is not an involution, but a direct computation gives (see Lemma II.6.15):

Lemma 2.4. *Let $k \geq 0$ be even and H be a k-pseudofunction. We have $H^{\vee\vee} = qH + (q-1)H^{\vee}$.*

In other words, if $k \geq 0$ is even, the endomorphism $H \mapsto H^{\vee}$ of $\mathcal{H}_k$ is diagonalizable, with eigenvalues q and -1. We then set

$$\mathcal{H}_{k,+} = \{H \in \mathcal{H}_k | H^{\vee} = qH\} \text{ and } \mathcal{H}_{k,-} = \{H \in \mathcal{H}_k | H^{\vee} = -H\}.$$

We still have $\mathcal{H}_k = \mathcal{H}_{k,+} \oplus \mathcal{H}_{k,-}$.

Let us now define extensions of pseudofunctions. For $x \sim y$ in X, we use the notation of Subsection I.4.2 for the natural injections:

$$I^{\ell}_{xy} : \overline{V}^{\ell}(xy) \to \overline{V}^{\ell+1}(x), \quad \ell \geq 0,$$
$$J^{\ell}_{xy} : \overline{V}^{\ell}(x) \to \overline{V}^{\ell}(xy), \quad \ell \geq 1.$$

Definition 2.5. Let $k \geq 1$ and H be a k-pseudofunction. We define the direct extension $H^{>}$ of H, which is a $(k+1)$-pseudofunction, as follows.

If k is odd, $k = 2\ell+1$, $\ell \geq 0$, for any $x \sim y$ in X, we set $H^{>}_{xy} = I^{\ell}_{xy} H_{xy}$.

If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we set $H^{>}_{xy} = J^{\ell}_{xy} H_{xy}$.

Direct extension is a linear embedding from the space of k-pseudofunctions into the space of $(k+1)$-pseudofunctions.

By convention, if $k = 0$ and H is the 0-pseudofunction associated to the function u on X, we set $H^{>}$ to be the 1-pseudofunction associated to the function $(x, y) \mapsto u(x)$ on X_1. If $k = -1$ and H is the (-1)-pseudofunction associated to the function u on X, we set $H^{>}$ to be the 0-pseudofunction associated to the function u on X.

By analogy with the case of pseudokernels (see Lemma II.2.3), if H is a pseudofunction, we will sometimes write H^{+} instead of $H^{>\vee}$ and call it the orthogonal extension of H.

As for pseudokernels, we have a commutation property.

Lemma 2.6. *Let $k \geq -1$ and H be a k-pseudofunction. We have $L^{\vee>>} = L^{>>\vee}$.*

Proof. In case $k \geq 1$, this is proved as in Lemma II.2.4. Let us prove the cases where $k = 0$ and $k = -1$. This will justify our conventions.

If $k = 0$, assume that H is the 0-pseudofunction associated with the function u on X. Then, we need to prove that $H^{>>\vee} = -H^{>>}$. Indeed, for every $x \sim y$ in X, we have $H^{>}_{xy} = u(x)\mathbf{1}_y$ (in $\overline{V}^0(xy)$), hence $H^{>>}_{xy} = u(x)\mathbf{1}_y$ (in $\overline{V}^1(x)$). We get

$$H^{>>\vee}_{xy} = u(x) \sum_{\substack{z \sim x \\ z \neq y}} \mathbf{1}_z = -u(x)\mathbf{1}_y = -H^{>>}_{xy},$$

which should be proved.

If $k = -1$, assume that H is the (-1)-pseudofunction associated with the function u on X, where u is constant on neighbours. Let v be the opposite of u. Then, on one hand, $H^{>>}$ is the 1-pseudofunction associated with the function $(x, y) \mapsto u(x)$ on X_1, hence $H^{>>\vee}$ is associated with the function $(x, y) \mapsto -u(y)$. On the other hand, $H^{\vee}$ is the (-1)-pseudofunction associated with the function $-v$ on X, hence $H^{\vee>>}$ is associated with the function $(x, y) \mapsto -v(x)$. The conclusion follows since by definition, for (x, y) in X_1, one has $u(y) = v(x)$. □

We will later need to know that the eigenspaces of the $\vee$ operator are not reduced to $\{0\}$.

Corollary 2.7. *For $k \geq -1$, one has $\mathcal{H}_{k,-} \neq \{0\}$. If k is large enough, one has $\mathcal{H}_{k,+} \neq \{0\}$.*

Proof. Let H_{-1} and $H_0 = H_{-1}^{>}$ be respectively the (-1)-pseudofunction and the 0-pseudofunction associated with the constant function with value 1 on X. By construction, one has $H_{-1}^{\vee} = -H_{-1}$ and $H_0^{\vee} = -H_0$. For $k \geq 1$, define by induction a k-pseudofunction H_k by setting $H_k = H_{k-1}^{>}$. By Lemma 2.6, we get $H_k^{\vee} = -H_k$ for any $k \geq -1$, hence $\mathcal{H}_{k,-} \neq \{0\}$.

Let us now show that we have $\mathcal{H}_{k,+} \neq \{0\}$ if k is large enough.

Assume that k is even, $k = 2\ell$, $\ell \geq 1$. Let $S \subset X$ be a system of representatives for the action of Γ: thus, we have $\Gamma S = X$ and $\Gamma x \cap S = \{x\}$ for x in S. Then, an element H in $\mathcal{H}_{k,+}$ may be seen as a family $(H_x)_{x \in S}$, where, for x in S, H_x is a Γ_x-invariant element of $\overline{V}^{\ell}(x)$. Fix x in S. If ℓ is large enough, the finite group Γ_x has more than one orbit in the sphere $S^{\ell}(x)$. Therefore, the space $\overline{V}^{\ell}(x)$ contains a non zero Γ_x-invariant element. We get $\mathcal{H}_{k,+} \neq \{0\}$.

Assume that k is odd, $k = 2\ell + 1$, $\ell \geq 0$. Let Y be the set of non ordered pairs $\{x, y\}$ with x, y in X and $x \sim y$. The set Y may be seen as the quotient of X_1 by the natural involution $(x, y) \mapsto (y, x)$. For $\{x, y\}$ in Y, we let $\Gamma_{\{x,y\}}$ be the stabilizer of $\{x, y\}$ in Γ. Let now $S \subset Y$ be a system of representatives for the action of Γ. An element H in $\mathcal{H}_{k,+}$ may be seen as a family $(H_{\{x,y\}})_{\{x,y\} \in S}$, where, for $\{x, y\}$ in S, $H_{\{x,y\}}$ is a $\Gamma_{\{x,y\}}$-invariant element of $\overline{V}^{\ell}(xy)$. Fix $\{x, y\}$ in S. As above, if ℓ is large enough, the finite group $\Gamma_{\{x,y\}}$ has more than one orbit in the sphere $S^{\ell}(xy)$. Again, this implies $\mathcal{H}_{k,+} \neq \{0\}$. □

Let us now state an analogue of Lemma II.2.5.

Lemma 2.8. *Let $k \geq -1$ and G, H be k-pseudofunctions. Assume that we have $G^{>} = H^{>\vee}$.*

If $k \geq 0$, there exists a $(k-1)$-pseudofunction F with $G = F^{\vee>}$ and $H = F^{>}$.

If $k = -1$, we have $G + H = 0$.

Proof. The cases $k = 0, -1$ directly follow from the definitions.

Assume $k = 1$. Let u and v be the functions on X_1 associated with G and H. For $x \sim y$ in X, we have, in $\overline{V}^1(x)$,

$$u(xy)\mathbf{1}_y = I^0_{xy} G_{xy} = \sum_{\substack{z \sim x \\ z \neq y}} I^0_{xz} H_{xz} = \sum_{\substack{z \sim x \\ z \neq y}} v(xz)\mathbf{1}_z.$$

As $\mathbf{1}_y = -\sum_{\substack{z \sim x \\ z \neq y}} \mathbf{1}_z$, this gives

$$\sum_{\substack{z \sim x \\ z \neq y}} (v(xz) + u(xy))\mathbf{1}_z = 0.$$

Thus, for any $y, z \sim x$, $y \neq z$, we have $v(xz) + u(xy) = 0$. As $q \geq 2$, there exists a function w on X such that, for any $x \sim y$ in X, one has $v(xy) = w(x) = -u(xy)$. The conclusion follows by our conventions on 0-pseudofunctions.

Assume k is even, $k = 2\ell$, $\ell \geq 1$. For $x \sim y$ in X, we have, in $\overline{V}^\ell(xy)$,

$$J^\ell_{xy} G_{xy} = J^\ell_{yx} H_{yx}.$$

In particular, for every z in $S^{\ell-1}(x)$ with $y \notin [xz]$, the function $G_{xy} \in \overline{V}^\ell(x)$ is constant on the set

$$\{w \in S^\ell(x) | z \in [xw]\},$$

that is, we may write $G_{xy} = I^{\ell-1}_{xy} F_{yx}$ where F_{yx} is in $\overline{V}^{\ell-1}(xy)$. In other words, we have $G = F^{\vee>}$. As $G^{>} = H^{>\vee}$, we get, by using Lemma 2.6, $H^{>\vee} = F^{\vee>>} = F^{>>\vee}$, hence $H = F^{>}$.

The proof in the odd case is analogous. □

2.3. **The polyextension map.** The spaces $\mathcal{H}_k$, $k \geq -1$, are embedded into each other through the direct extension map. We will now study their direct limit and encode some elements of this direct limit by sequences of pseudofunctions.

We shall say that a ∞-pseudofunction is any map $X_1 \to \overline{\mathcal{D}}(\partial X)$. The space of Γ-invariant ∞-pseudofunctions is denoted by $\mathcal{H}_\infty$. Recall from Subsections I.4.4 and I.7.3 that we have defined natural operators

$$N^\ell_x : \overline{V}^\ell(x) \hookrightarrow \overline{\mathcal{D}}(\partial X) \qquad x \in X, \ell \geq 1$$

$$\text{and } N^\ell_{xy} : \overline{V}^\ell(xy) \hookrightarrow \overline{\mathcal{D}}(\partial X) \qquad x \sim y \in X, \ell \geq 0.$$

For $k \geq 1$, if H is a k-pseudofunction, we set $H^{>^\infty}$ to be the ∞-pseudofunction defined by, for $x \sim y$ in X,

$$\begin{aligned} H_{xy}^{>^\infty} &= N_x^\ell H_{xy} && k = 2\ell, \ell \geq 1 \\ H_{xy}^{>^\infty} &= N_{xy}^\ell H_{xy} && k = 2\ell+1, \ell \geq 0. \end{aligned}$$

One easily checks that one has

$$(H^{>})^{>^\infty} = H^{>^\infty}. \tag{2.1}$$

Thus, one can extend the construction in case $k = 0$ or $k = -1$ by using (2.1) as a definition.

Let $i \geq 0$, $k \geq -1$ and H be a k-pseudofunction. We denote by $H^{>^i}$ and H^{+^i} the $(k+i)$-pseudofunctions obtained from H respectively by i sucessive direct extensions and by i successive orthogonal extensions. Note that, by Lemma 2.6, we have

$$H^{+^i>>} = H^{>>+^i}. \tag{2.2}$$

This formalism allows us to compare the different notions of extension through the following generalization of Lemma 2.8.

Proposition 2.9. *Let $k \geq 0$ and $(H_i)_{i\geq 0}$ be a sequence of k-pseudofunctions with $H_i = 0$ for i large enough. Set*

$$H = \sum_{i\geq 0} H_i^{+^i>^\infty}$$

Then $H = 0$ if and only if there exists a sequence $(G_i)_{i\geq 0}$ of $(k-1)$-pseudofunctions with $G_i = 0$ for i large enough such that

$$\begin{aligned} H_0 &= G_1^{\vee>\vee} + G_0^{\vee>} \\ H_i &= G_{i+1}^{\vee>\vee} - G_{i-1}^{>} && i \geq 1. \end{aligned} \tag{2.3}$$

The proof uses the following consequence of Lemma 2.8.

Lemma 2.10. *Let $i \geq 1$, $k \geq 0$ and H be a k-pseudofunction. Assume that there exists a $(k+i-1)$-pseudofunction G with $H^{+^i} = G^{>}$. Then, there exists a $(k-1)$-pseudofunction F with $H = F^{>}$ and $G = F^{\vee+^{i-1}>}$.*

Proof. We show the result by induction on $i \geq 1$. If $i = 1$, this is Lemma 2.8. Assume $i \geq 2$ and the result holds for $i-1$. By Lemma 2.8, there exists a $(k+i-2)$-pseudofunction G_1 with $H^{+^{i-1}} = G_1^{>}$ and $G = G_1^{\vee>}$. The induction assumption says that there exists a $(k-1)$-pseudofunction F with $H = F^{>}$ and $G_1 = F^{\vee+^{i-2}>}$. We get $G = F^{\vee+^{i-2}>\vee>} = F^{\vee+^{i-1}>}$ and the conclusion follows. □

Proof of Proposition 2.9. Assume that (2.3) holds. Then, a direct computation gives

$$H = G_1^{\vee+>\infty} + G_0^{\vee>\infty} + \sum_{i\geq 1} G_{i+1}^{\vee+^{i+1}>\infty} - G_{i-1}^{>+^{i}>\infty}$$
$$= \sum_{i\geq 0} G_i^{\vee+^{i}>\infty} - G_i^{>+^{i+1}>\infty}.$$

For $i \geq 0$, Lemma 2.6, (2.1) and (2.2) give

$$G_i^{\vee+^{i}>\infty} - G_i^{>+^{i+1}>\infty} = (G_i^{\vee+^{i}>>} - G_i^{>+^{i+1}})^{>\infty} = (G_i^{>>\vee+^{i}} - G_i^{>+^{i+1}})^{>\infty} = 0.$$

We get $H = 0$ as required.

Conversely, we may assume that there exists i with $H_i \neq 0$. Then, we set $j = \max\{i \geq 0 | H_i \neq 0\}$ and we prove the statement by induction on j. If $j = 0$, there is nothing to prove. If $j = 1$, this is Lemma 2.8. Thus, we assume $j \geq 2$ and the statement holds for $j - 1$. By assumption, we have the following equality of $(k + j)$-pseudofunctions:

$$-H_j^{+^{j}} = \sum_{i=0}^{j-1} H_i^{+^{i}>^{j-i}} = \left(\sum_{i=0}^{j-1} H_i^{+^{i}>^{j-i-1}} \right)^{>}.$$

Lemma 2.10 tells us that we can find a $(k - 1)$-pseudofunction A with $H_j = A^{>}$ and

$$\sum_{i=0}^{j-1} H^{+^{i}>^{j-i-1}} + A^{\vee+^{j-1}>} = 0. \tag{2.4}$$

We define a new sequence $(J_i)_{i\geq 0}$ of k-pseudofunctions by setting

$$\begin{aligned} J_i &= H_i & & 0 \leq i \leq j-3 \\ J_{j-2} &= H_{j-2} + A^{\vee>\vee} & & \\ J_{j-1} &= H_{j-1} & & \\ J_i &= 0 & & i \geq j. \end{aligned}$$

We want to apply the induction assumption to this new sequence. Indeed, we compute

$$\sum_{i=0}^{j-1} J_i^{+^{i}>^{j-1-i}} = \sum_{i=0}^{j-1} H_i^{+^{i}>^{j-1-i}} + A^{\vee>\vee+^{j-2}>} = 0,$$

where the last equality follows from (2.4). Thus, the induction assumption tells us that there exists a sequence $(F_i)_{i\geq 0}$ of $(k-1)$-pseudofunctions, with finitely many non zero terms, such that

$$\begin{aligned} J_0 &= F_1^{\vee>\vee} + F_0^{\vee>} \\ \text{and } J_i &= F_{i+1}^{\vee>\vee} - F_{i-1}^{>}, \qquad i \geq 1. \end{aligned} \tag{2.5}$$

We set $G_i = F_i$ for $i \neq j-1$ and $G_{j-1} = F_{j-1} - A$. One easily checks that (2.3) holds. Indeed, for $i \notin \{j-2, j\}$, it is equivalent to (2.5) since $H_i = J_i$. For $i = j-2$, it also follows from (2.5) since $H_{j-2} - J_{j-2} = -A^{\vee>\vee} = (G_{j-1} - F_{j-1})^{\vee>\vee}$. Finally, for $i = j$, (2.5) gives $F_{j+1}^{\vee>\vee} - F_{j-1}^{>} = 0$, hence $G_{j+1}^{\vee>\vee} - G_{j-1}^{>} = A^{>} = H_j$ as required. □

This result leads us to define two natural linear maps.

Definition 2.11. Let $k \geq 0$. For $(H_i)_{i\geq 0}$ in $\mathcal{H}_k^{(\mathbb{N})}$, we set

$$E_k(H_i)_{i\geq 0} = \sum_{i\geq 0} H_i^{+^i>^\infty} \in \mathcal{H}_\infty.$$

We call the linear map $E_k : \mathcal{H}_k^{(\mathbb{N})} \to \mathcal{H}_\infty$ the k-polyextension map.

Definition 2.12. Let $k \geq 0$. For $(G_i)_{i\geq 0}$ in $\mathcal{H}_{k-1}^{(\mathbb{N})}$, we set $D_k(G_i)_{i\geq 0} \in \mathcal{H}_k^{(\mathbb{N})}$ to be the sequence $(H_i)_{i\geq 0}$ given by (2.3). We call the linear map $D_k : \mathcal{H}_{k-1}^{(\mathbb{N})} \to \mathcal{H}_k^{(\mathbb{N})}$ the k-default map.

2.4. **Injectivity of the default map.** To complete the preceding picture, we will show that the default map D_k is injective. More generally, we have

Proposition 2.13. *Let $k \geq -1$ and $(G_i)_{i\geq 0}$ be a sequence of k-pseudofunctions. Assume that we have*

$$G_1^{\vee>\vee} = -G_0^{\vee>} \tag{2.6}$$

$$G_{i+1}^{\vee>\vee} = G_{i-1}^{>} \qquad i \geq 1. \tag{2.7}$$

Then, if k is odd, there exists a (-1)-pseudofunction J such that, for any $i \geq 0$, one has

$$G_{2i} = G_{2i+1} = (-1)^i J^{\vee^i>^{k+1}}. \tag{2.8}$$

If k is even, there exists a (-1)-pseudofunction J such that, for any $i \geq 0$, one has

$$G_{2i} = (-1)^i J^{\vee^{i+1}>^{k+1}} \text{ and } G_{2i+1} = (-1)^{i+1} J^{\vee^i>^{k+1}}. \tag{2.9}$$

Conversely, in both cases, if $(G_i)_{i\geq 0}$ is of the form in (2.8) or (2.9), then (2.6) and (2.7) hold.

Proof. First we check that the condition is sufficient. Let J be a (-1)-pseudofunction.

Assume k is odd and (2.8) holds. Then, we have $G_0 = G_1 = J^{>^{k+1}}$. Hence, by Lemma 2.6, $G_0^{\vee>} = G_1^{\vee>} = J^{\vee>^{k+2}}$ and, as the $\vee$-operator is (-1) on 0-pseudofunctions,

$$G_1^{\vee>\vee} = J^{\vee>^{k+2}\vee} = J^{\vee>\vee>^{k+1}} = -J^{\vee>^{k+2}} = -G_0^{\vee>}.$$

By the same arguments, for $i \geq 1$,

$$\begin{aligned} G_{2i}^{\vee>\vee} - G_{2i-2}^{>} = G_{2i+1}^{\vee>\vee} - G_{2i-1}^{>} &= (-1)^i J^{\vee^i>^{k+1}\vee>\vee} - (-1)^{i-1} J^{\vee^{i-1}>^{k+2}} \\ &= (-1)^{i+1} J^{\vee^{i+1}>^{k+2}} - (-1)^{i-1} J^{\vee^{i-1}>^{k+2}} = 0, \end{aligned}$$

which should be proved.

Assume k is even and (2.9) holds. On one hand, we have

$$G_1^{\vee>\vee} + G_0^{\vee>} = -J^{>^{k+1}\vee>\vee} + J^{\vee>^{k+1}\vee>} = J^{\vee>^{k+2}} - J^{\vee>^{k+2}} = 0.$$

On the other hand, for $i \geq 1$,

$$\begin{aligned} G_{2i+1}^{\vee>\vee} - G_{2i-1}^{>} &= (-1)^{i+1} J^{\vee^i>^{k+1}\vee>\vee} - (-1)^i J^{\vee^{i-1}>^{k+2}} \\ &= (-1)^{i+2} J^{\vee^{i+1}>^{k+2}} - (-1)^i J^{\vee^{i-1}>^{k+2}} = 0 \end{aligned}$$

and, for $i \geq 0$,

$$\begin{aligned} G_{2i+2}^{\vee>\vee} - G_{2i}^{>} &= (-1)^{i+1} J^{\vee^{i+2}>^{k+1}\vee>\vee} - (-1)^i J^{\vee^{i+1}>^{k+2}} \\ &= (-1)^{i+2} J^{\vee^{i+3}>^{k+2}} - (-1)^i J^{\vee^{i+1}>^{k+2}} = 0. \end{aligned}$$

Conversely, whether k is odd or even, let $(G_i)_{i\geq 0}$ be a sequence of k-pseudofunctions such that (2.6) and (2.7) hold. We will prove by induction on $k \geq -1$ that $(G_i)_{i\geq 0}$ is of the form in (2.8) or (2.9), depending on the parity of k.

If $k = -1$, the equations read as $G_0 = G_1$ and $G_{i-1} + G_{i+1}^{\vee} = 0$ for $i \geq 1$. The conclusion directly follows by taking $J = G_0$.

Assume $k \geq 0$ and the result is true for $k-1$. By Lemma 2.8, as (2.6) holds, there exists a $(k-1)$-pseudofunction H_0 with

$$G_1^{\vee} = H_0^{>} \text{ and } G_0^{\vee} = -H_0^{\vee>}. \tag{2.10}$$

Still by Lemma 2.8, as (2.7) holds, for $i \geq 1$, there exists a $(k-1)$-pseudofunction H_i with

$$G_{i+1}^{\vee} = H_i^{>} \text{ and } G_{i-1} = H_i^{\vee>}. \tag{2.11}$$

From (2.10) and (2.11) in case $i = 1$, we get $G_0^{\vee} = -H_0^{\vee>}$ and $G_0 = H_1^{\vee>}$, hence

$$H_1^{\vee>\vee} + H_0^{\vee>} = 0. \tag{2.12}$$

Besides, by (2.10) and (2.11), we have, for all $i \geq 1$, $G_i^\vee = H_{i-1}^>$, whereas, by (2.11), $G_i = H_{i+1}^{\vee>}$. Thus, we get

$$H_{i+1}^{\vee>\vee} = H_{i-1}^>. \tag{2.13}$$

By (2.12) and (2.13), we can apply the induction assumption to the sequence $(H_i)_{i\geq 0}$.

If k is even, this tells us that there exists a (-1)-pseudofunction J with, for $i \geq 0$, $H_{2i} = H_{2i+1} = (-1)^i J^{\vee^i >^k}$. By applying (2.11), we get, thanks to Lemma 2.6,

$$G_{2i} = H_{2i+1}^{\vee>} = (-1)^i J^{\vee^i >^k \vee >} = (-1)^i J^{\vee^{i+1} >^{k+1}}$$
$$G_{2i+1} = H_{2i+2}^{\vee>} = (-1)^{i+1} J^{\vee^{i+1} >^k \vee >} = (-1)^{i+1} J^{\vee^i >^{k+1}}.$$

If k is odd, there exists a (-1)-pseudofunction J with, for $i \geq 0$, $H_{2i} = (-1)^i J^{\vee^{i+1} >^k}$ and $H_{2i+1} = (-1)^{i+1} J^{\vee^i >^k}$. We get, as the $\vee$ operator is -1 on 0-pseudofunctions,

$$G_{2i} = H_{2i+1}^{\vee>} = (-1)^{i+1} J^{\vee^i >^k \vee >} = (-1)^i J^{\vee^i >^{k+1}}$$
$$G_{2i+1} = H_{2i+2}^{\vee>} = (-1)^{i+1} J^{\vee^i >^k \vee >} = (-1)^i J^{\vee^i >^{k+1}}.$$

The Proposition follows by induction. □

From Proposition 2.9 and Proposition 2.13, we get

Corollary 2.14. *Let $k \geq 0$. Then, the k-default map D_k is injective and the null space of the k-polyextension map E_k is the range of D_k. In other words, we have an exact sequence*

$$0 \longrightarrow \mathcal{H}_{k-1}^{(\mathbb{N})} \xrightarrow{D_k} \mathcal{H}_k^{(\mathbb{N})} \xrightarrow{E_k} \mathcal{H}_\infty.$$

Proof. Let $(G_i)_{i\geq 0}$ be a sequence of $(k-1)$-pseudofunctions such that $G_i = 0$ for large enough i. Assume that (2.6) and (2.7) hold and let J be the (-1)-pseudofunction as in Proposition 2.13. As $G_i = 0$ for large enough i, we have $J = 0$, hence $G = 0$. In particular, this tells us that the default map D_k is injective.

Besides, let H be in $\mathcal{H}_k^{(\mathbb{N})}$ and assume that $E_k H = 0$. The above tells us that the sequence $(G_i)_{i\geq 0}$ of $(k-1)$-pseudofunctions obtained by applying Proposition 2.9 to H is uniquely determined by H. In particular, it is Γ-invariant, that is, H belongs to the range of the default map D_k, as should be proved. □

2.5. **Operations on ∞-pseudofunctions.** We will now define natural operations on $\mathcal{H}_\infty$ and show that they can be transfered to operations on $\mathcal{H}_k^{(\mathbb{N})}$ thanks to the polyextension map.

Let H be an ∞-pseudofunction. By analogy with Definition 2.3, we define the reversal RH and the inversion SH of H by

$$RH_{xy} = \sum_{\substack{z\sim x\\ z\neq y}} H_{xz} \text{ and } SH_{xy} = H_{yx}, \quad x \sim y \in X.$$

The maps R and S define linear operators of $\mathcal{H}_\infty$ and we have $S^2 = 1$ and, as in Lemma 2.4, $R^2 = q + (q-1)R$. Frome these definitions, we directly get

Lemma 2.15. *Let $k \geq -1$ and H be a k-pseudofunction. If k is even, we have $R(H^{>\infty}) = H^{\vee>\infty}$. If k is odd, we have $S(H^{>\infty}) = H^{\vee>\infty}$.*

We will now define analogues of these maps on $\mathcal{H}_k^{(\mathbb{N})}$.

Definition 2.16. (k even) Let $k \geq 0$ be an even integer, $k = 2\ell$, $\ell \geq 0$, and $H = (H_i)_{i\geq 0}$ be a sequence of k-pseudofunctions.

The reversal RH of H is the sequence of k-pseudofunctions defined by

$$\begin{aligned} (RH)_0 &= H_0^\vee & & \\ (RH)_i &= H_{i-1} + (q-1)H_i & & i \text{ even}, i \geq 2 \\ (RH)_i &= qH_{i+1} & & i \text{ odd.} \end{aligned}$$

The inversion SH of H is the sequence of k-pseudofunctions defined by

$$\begin{aligned} (SH)_i &= H_{i+1} & & i \text{ even} \\ (SH)_i &= H_{i-1} & & i \text{ odd.} \end{aligned}$$

Definition 2.17. (k odd) Let $k \geq -1$ be an odd integer, $k = 2\ell + 1$, $\ell \geq -1$, and $H = (H_i)_{i\geq 0}$ be a sequence of k-pseudofunctions.

The reversal RH of H is the sequence of k-pseudofunctions defined by

$$\begin{aligned} (RH)_i &= qH_{i+1} & & i \text{ even} \\ (RH)_i &= H_{i-1} + (q-1)H_i & & i \text{ odd.} \end{aligned}$$

The inversion SH of H is the sequence of k-pseudofunctions defined by

$$\begin{aligned} (SH)_0 &= H_0^\vee & & \\ (SH)_i &= H_{i-1} & & i \text{ even}, i \geq 2 \\ (SH)_i &= H_{i+1} & & i \text{ odd.} \end{aligned}$$

One directly checks that the relations $R^2 = q + (q-1)R$ and $S^2 = 1$ still hold. Besides, these maps are compatible with the polyextension map.

Lemma 2.18. *Let $k \geq 0$ and $H = (H_i)_{i\geq 0}$ be in $\mathcal{H}_k^{(\mathbb{N})}$. We have*

$$E_k(RH) = RE_k(H) \text{ and } E_k(SH) = SE_k(H).$$

Proof. Assume k is even. By definition, we get

$$E_k(H) = \sum_{i\geq 0} H^{+^i > \infty} = H_0^{>\infty} + \sum_{i\geq 1} (H_{2i-1}^{+^{2i-1}>} + H_{2i}^{+^{2i}})^{>\infty}.$$

Now, H_0 belongs to $\mathcal{H}_k$ and, for $i \geq 1$, $H_{2i-1}^{+^{2i-1}>} + H_{2i}^{+^{2i}}$ belongs to $\mathcal{H}_{k+2i}$, so that Lemma 2.15 gives $RH_0^{>\infty} = H_0^{\vee>\infty}$ and

$$R(H_{2i-1}^{+^{2i-1}>} + H_{2i}^{+^{2i}})^{>\infty} = (H_{2i-1}^{+^{2i-1}>\vee} + H_{2i}^{+^{2i}\vee})^{>\infty}.$$

Using Lemma 2.4, we get

$$\begin{aligned} H_{2i-1}^{+^{2i-1}>\vee} + H_{2i}^{+^{2i}\vee} &= H_{2i-1}^{+^{2i}} + H_{2i}^{+^{2i-1}>\vee\vee} \\ &= qH_{2i}^{+^{2i-1}>} + (H_{2i-1} + (q-1)H_{2i})^{+^{2i}}, \end{aligned}$$

which gives $RE_k(H) = Ek(RH)$.

In the same way, we have

$$E_k(H) = \sum_{i\geq 0} (H_{2i}^{+^{2i}>} + H_{2i+1}^{+^{2i+1}})^{>\infty}.$$

As, for $i \geq 0$, $H_{2i}^{+^{2i}>} + H_{2i+1}^{+^{2i+1}}$ belongs to $\mathcal{H}_{k+2i+1}$, Lemma 2.15 gives

$$SE_k(H) = \sum_{i\geq 0} (H_{2i+1}^{+^{2i}>} + H_{2i}^{+^{2i+1}})^{>\infty},$$

which should be proved.

The proof in case k is odd is analogous. □

Finally, to describe the behaviour of these operators R and S in the null space of the polyextension map, we define a new pair of operators R' and S' as follows.

Definition 2.19. (k even) Let $k \geq 0$ be an even integer, $k \geq 2\ell$, $\ell \geq 0$, and $G = (G_i)_{\geq 0}$ be a sequence of k-pseudofunctions. The antireversal of G is the sequence $R'G$ of k-pseudofunctions defined by

$$\begin{aligned} (R'G)_0 &= (q-1)G_0 - G_0^{\vee} & & \\ (R'G)_i &= G_{i-1} + (q-1)G_i & & i \text{ even}, i \geq 2 \\ (R'G)_i &= qG_{i+1} & & i \text{ odd}. \end{aligned}$$

Definition 2.20. (k odd) Let $k \geq -1$ be an odd integer, $k \geq 2\ell + 1$, $\ell \geq -1$, and $G = (G_i)_{\geq 0}$ be a sequence of k-pseudofunctions. The antiinversion of G is the sequence $S'G$ of k-pseudofunctions defined by

$$\begin{aligned} (S'G)_0 &= -G_0^{\vee} && \\ (S'G)_i &= G_{i-1} && i \text{ even}, i \geq 2 \\ (S'G)_i &= G_{i+1} && i \text{ odd.} \end{aligned}$$

Note that, in the even case, the maps R and R' only differ by their action on the first component. So do the maps S and S' in the odd case.

These maps allow to describe the action of the reversal and the inversion on the range of the default map.

Lemma 2.21. *Let $k \geq 0$ and $G = (G_i)_{i\geq 0}$ be in $\mathcal{H}_{k-1}^{(\mathbb{N})}$. If k is even, we have*

$$D_k(RG) = RD_k(G) \text{ and } D_k(S'G) = SD_k(G).$$

If k is odd, we have

$$D_k(R'G) = RD_k(G) \text{ and } D_k(SG) = SD_k(G).$$

Proof. These are straightforward computations. For example, we deal with the reversion action in case k is even. We set $H = D_kG$. For $i = 0$, we have $H_0 = G_1^{\vee>\vee} + G_0^{\vee>}$, hence

$$\begin{aligned} (RH)_0 = G_1^{\vee>\vee\vee} + G_0^{\vee>\vee} &= (G_0 + (q-1)G_1)^{\vee>\vee} + qG_1^{\vee>} \\ &= (RG)_1^{\vee>\vee} + (RG)_0^{\vee>} \end{aligned}$$

as required. For $i \geq 1$, we have $H_i = G_{i+1}^{\vee>\vee} - G_{i-1}^{>}$, hence, if i is odd,

$$(RH)_i = qH_{i+1} = qG_{i+2}^{\vee>\vee} - qG_i^{>} = (RG)_{i+1}^{\vee>\vee} - (RG)_{i-1}^{\vee>\vee};$$

and if i is even,

$$\begin{aligned} (RH)_i = H_{i-1} + (q-1)H_i &= G_i^{\vee>\vee} - G_{i-2}^{>} + (q-1)G_{i+1}^{\vee>\vee} - (q-1)G_{i-1}^{>} \\ &= (RG)_{i+1}^{\vee>\vee} - (RG)_{i-1}^{>}. \end{aligned}$$

The other three computations are analogous. □

2.6. **Pseudofunctions in the bipartite case.** In this Subsection, we assume that Γ is bipartite. In that case, the spaces of pseudofunctions may be equipped with an additional natural operation. We introduce this operation and explain how it is related to the previous constructions.

We fix a function $\chi : X \to \{1, -1\}$ such that, for every $x \sim y$ in X, one has $\chi(y) = -\chi(x)$. This function is uniquely defined up to a sign change and saying that Γ is bipartite amounts to saying that χ is Γ-invariant.

For $k \geq 1$ and H a k-pseudofunction, we define the twist $H^{\wr}$ of H the k-pseudofunction defined by

$$H^{\wr}_{xy} = \chi(x) H_{xy} \quad x \sim y \in X.$$

If $k = 0$ (resp. $k = -1$) and H is the k-pseudofunction associated to the function u on X (resp. to the function u on X that is constant on neighbours), we let $H^{\wr}$ be the k-pseudofunction associated with the function χu.

We directly get

Lemma 2.22. *Let $k \geq -1$ and H be a k-pseudofunction. We have $H^{\wr >} = H^{>\wr}$. If k is even, we have $H^{\wr \vee} = H^{\vee \wr}$. If k is odd, we have $H^{\wr \vee} = -H^{\vee \wr}$.*

In the same way, if H is an ∞-pseudofunction, we set UH to be the ∞-pseudofunction defined by

$$(UH)_{xy} = \chi(x) H_{xy} \quad x \sim y \in X.$$

As in Lemma 2.22, we have the relations

$$UR = RU \text{ and } US = -SU.$$

If $k \geq -1$ and H is a k-pseudofunction, we have $U(H^{>\infty}) = H^{\wr >\infty}$.

To check the compatibility properties of this operation with the polyextension map, we introduce an operator on sequences. For $k \geq -1$ and $(H_i)_{i \geq 0}$ a sequence of k-pseudofunctions, we set UH to be the sequence defined by

$$\begin{aligned} (UH)_i &= (-1)^{\frac{i}{2}} H_i^{\wr} && i \text{ even} \\ (UH)_i &= (-1)^{\frac{i+1}{2}} H_i^{\wr} && i \text{ odd}, k \text{ even} \\ (UH)_i &= (-1)^{\frac{i-1}{2}} H_i^{\wr} && i \text{ odd}, k \text{ odd}. \end{aligned}$$

A direct computation using Definition 2.11 and Definition 2.12 yields

Lemma 2.23. *Let $k \geq -1$. For H in $\mathcal{H}_k^{(\mathbb{N})}$, we have*

$$E_k U H = U E_k H.$$

For G in $\mathcal{H}_{k-1}^{(\mathbb{N})}$, we have

$$\begin{aligned} D_k U G &= -U D_k G && k \text{ even} \\ D_k U G &= U D_k G && k \text{ odd.} \end{aligned}$$

3. Preliminaries for the study of the model operators

Lemma 2.18 above suggests that, in order to study the natural operations on $\mathcal{H}_\infty$, we need to analyse operators defined on sequences of real numbers by formulae as in Definition 2.16 and 2.17. We call these operators the model operators. The purpose of the next three Sections is to develop a precise spectral analysis of the model operators.

Our general strategy for doing this relies on considerations of abstract harmonic analysis. Indeed, the involutive algebra generated by two self-adjoint elements R and S with the relations $R^2 = qR + (q-1)R$ and $S^2 = 1$ is the algebra of the infinite diedral group. In particular, any $*$-representation of this algebra on a Hilbert space may be decomposed (maybe continuously) into irreducible components and these irreducible components have dimension at most 2, as the infinite diedral group has a normal abelian subgroup of index 2.

It turns out that, for the model operators, this decomposition maybe constructed in a very explicit way. We shall build this decomposition. We will need to split these construction according to the parity of k and to the eigenvalues of the $\vee$ operator in the formulae from Definition 2.16 and 2.17. Thus, we need to consider four different cases which will be carried out in the Subsections 4.1, 4.3, 5.1 and 5.2.

In the present preliminary Section, we introduce notation and facts that we will use in the construction of the spectral theory of model operators. Some of them will also be later required in the spectral analysis of Euclidean fields.

3.1. **Algebraic preliminaries.** We introduce a general algebraic framework for the actions of operators R and S which satisfy the properties of the R and S operators on ∞-pseudofunctions.

Let R and S be two symbols and $\mathcal{A}$ be the real algebra spanned by R and S with the relations $S^2 = 1$ and $R^2 = q + (q-1)R$. In other words, $\mathcal{A}$ is the quotient of the tensor algebra of the vector space $\mathbb{R}R \oplus \mathbb{R}S$ by the ideal spanned by $S^2 - 1$ and $R^2 - q - (q-1)R$. We equip $\mathcal{A}$ with the unique involutive anti-automorphism $A \mapsto A^*$ such that $S^* = S$ and $R^* = R$.

Set $S' = \frac{1}{q+1}(2R - (q-1))$. A direct computation shows that $(S')^2 = 1$. Therefore, $\mathcal{A}$ may be seen as the group algebra of the group G generated by S and S', which is isomorphic to the infinite diedral group. In particular, the element $T = SS'$ is unitary and the element P defined by

$$P = \frac{1}{2}(T + T^{-1}) = \frac{1}{q+1}(RS + SR - (q-1)S)$$

is self-adjoint. Set $\mathcal{B}$ to be the subalgebra spanned by T. By construction, $\mathcal{B}$ is the algebra of Laurent polynmials $\mathbb{R}[T, T^{-1}]$.

Lemma 3.1. *We have $\mathcal{A} = \mathcal{B} \oplus S\mathcal{B}$. The center $\mathcal{C}$ of $\mathcal{A}$ is the subalgebra spanned by P, which is isomorphic to the polynomial algebra $\mathbb{R}[P]$.*

The explicit form of the center will play a role for determining spectral measures in Section 4 and 5.

Proof. The first property is a direct translation of the fact that $\mathcal{A}$ is the group algebra of G and that G may be written as $G = T^{\mathbb{Z}} \sqcup ST^{\mathbb{Z}}$. Now, let C be in $\mathcal{C}$ and write

$$C = \sum_n a_n T^n + S \sum_n b_n T^n$$

for some sequences (a_n) and (b_n) in $\mathbb{R}^{(\mathbb{Z})}$. We have

$$C = SCS = \sum_n a_n T^{-n} + S \sum_n b_n T^{-n},$$

hence $a_n = a_{-n}$ and $b_n = b_{-n}$ for n in $\mathbb{Z}$. Besides, as

$$S'ST^nS' = (SS)(S'S)(S'S')T^nS' = S(SS')(SS')(S'T^nS') = ST^{2-n},$$

we get

$$C = S'CS' = \sum_n a_n T^{-n} + S \sum_n b_n T^{2-n},$$

hence, for n in $\mathbb{Z}$, $b_n = b_{2-n} = b_{n-2}$. As b_n is zero for $|n|$ large, we get $b_n = 0$ for any n. The result easily follows. □

We shall also need the existence of nice complementary subspaces for some natural left ideals in $\mathcal{A}$.

Corollary 3.2. *One has the following decompositions:*

$$\begin{aligned} \mathcal{A} &= \mathcal{C} \oplus S\mathcal{C} \oplus \mathcal{A}(R - q) \\ &= \mathcal{C} \oplus S\mathcal{C} \oplus \mathcal{A}(R + 1) \\ &= \mathcal{C} \oplus R\mathcal{C} \oplus \mathcal{A}(S - 1) \\ &= \mathcal{C} \oplus R\mathcal{C} \oplus \mathcal{A}(S + 1) \end{aligned}$$

Proof. Let us show first the decomposition

$$\mathcal{A} = \mathcal{C} \oplus S\mathcal{C} \oplus \mathcal{A}(S' - 1). \tag{3.1}$$

Indeed, Lemma 3.1 gives (by exchanging the roles of S and S')

$$\mathcal{A} = \mathcal{B} \oplus \mathcal{B}S'.$$

By using the identities, for B_0, B_1 in $\mathcal{B}$,

$$B_0 + B_1S' = (B_0 + B_1) + B_1(S' - 1) = (B_0 - B_1) + B_1(S' + 1),$$

we get

$$\mathcal{A} = \mathcal{B} \oplus \mathcal{B}(S' - 1) = \mathcal{B} \oplus \mathcal{B}(S' + 1). \tag{3.2}$$

Still by Lemma 3.1, as $\mathcal{B}$ is the algebra of Laurent polynomials $\mathcal{B} = \mathbb{R}[T, T^{-1}]$ and $\mathcal{C}$ is the polynomial algebra $\mathcal{C} = \mathbb{R}[T + T^{-1}]$, we get

$$\mathcal{B} = \mathcal{C} \oplus \mathcal{C}(T - T^{-1}).$$

As we have $T - T^{-1} = -(T + T^{-1}) + 2T = -2P + 2T$, we get

$$\mathcal{B} = \mathcal{C} \oplus \mathcal{C}T,$$

hence, from the first first equality in (3.2),

$$\mathcal{A} = \mathcal{C} \oplus \mathcal{C}T \oplus \mathcal{B}(S' - 1). \tag{3.3}$$

In particular, since $T = S + S(S' - 1)$, this gives

$$\mathcal{A} = \mathcal{C} + \mathcal{C}S + \mathcal{A}(S' - 1). \tag{3.4}$$

To conclude the proof of (3.1), we need to show that this sum is a direct sum. Thus, we take A in $\mathcal{A}$ and C_0, C_1 in $\mathcal{C}$ such that we have

$$C_0 + C_1 S + A(S' - 1) = 0$$

and we will show that $C_0 = C_1 = A(S' - 1) = 0$. Indeed, still as $T = S + S(S' - 1)$, we have

$$C_0 + C_1 T + (A - C_1 S)(S' - 1) = 0.$$

Use the second equality in (3.2) to find B_0, B_1 in $\mathcal{B}$ with $A - C_1 S = B_0 + B_1(S' + 1)$, so that

$$(A - C_1 S)(S' - 1) = B_0(S' - 1). \tag{3.5}$$

We get

$$C_0 + C_1 T + B_0(S' - 1) = 0,$$

so that (3.3) gives $C_0 = C_1 = 0$ and $B_0(S' - 1) = 0$. By (3.5), we get $A(S' - 1) = 0$ and (3.1) follows.

By changing the roles and signs of S and S', we also get

$$\begin{aligned} \mathcal{A} &= \mathcal{C} \oplus S\mathcal{C} \oplus \mathcal{A}(S' + 1) \\ &= \mathcal{C} \oplus S'\mathcal{C} \oplus \mathcal{A}(S - 1) \\ &= \mathcal{C} \oplus S'\mathcal{C} \oplus \mathcal{A}(S + 1). \end{aligned}$$

The result follows by using the relations $(q + 1)(S' - 1) = 2(R - q)$, $(q + 1)(S' + 1) = 2(R + 1)$ and $\mathcal{C} \oplus S'\mathcal{C} = \mathcal{C} \oplus R\mathcal{C}$. $\square$

As T spans a normal subgroup of index 2 in G, any self-adjoint representation of $\mathcal{A}$ in a Hilbert space H may be decomposed into a direct integral of irreducible representations of $\mathcal{A}$ and these irreducible representations have dimension 1 or 2.

There are four representations in dimension 1 which are determined by letting R (resp. S) be the scalars q or -1 (resp. 1 or -1). Let us now give explicit generators for the irreducible representations in dimension 2. The four different versions of the generators will be adapted to the four different sets of model operators that we will consider in Section 4 and Section 5.

The $++$ and $+-$ matrices. Set $s_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and, for t in $\mathbb{C}$,

$$r_{++}(t) = \begin{pmatrix} q & (q+1)t \\ 0 & -1 \end{pmatrix} \qquad a_{++}(t) = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$$
$$r_{+-}(t) = \begin{pmatrix} -1 & (q+1)t \\ 0 & q \end{pmatrix} \qquad a_{+-}(t) = \begin{pmatrix} 1 & -t \\ -t & 1 \end{pmatrix}.$$

The $-+$ and $--$ matrices. Set $r_- = \begin{pmatrix} q-1 & q \\ 1 & 0 \end{pmatrix}$ and, for t in $\mathbb{C}$,

$$s_{-+}(t) = \begin{pmatrix} 1 & (q+1)t-(q-1) \\ 0 & -1 \end{pmatrix}$$
$$a_{-+}(t) = \begin{pmatrix} 2 & (q+1)t-(q-1) \\ (q+1)t-(q-1) & q^2+1-(q^2-1)t \end{pmatrix}$$
$$s_{--}(t) = \begin{pmatrix} -1 & (q+1)t+(q-1) \\ 0 & 1 \end{pmatrix}$$
$$a_{--}(t) = \begin{pmatrix} 2 & -(q+1)t-(q-1) \\ -(q+1)t-(q-1) & q^2+1+(q^2-1)t \end{pmatrix}.$$

The proof of the following Lemma follows from straightforward computations.

Lemma 3.3. *Let t be in $\mathbb{C}$. The action of $\mathcal{A}$ on $\mathbb{C}^2$ defined by letting R act as $r_{++}(t)$ (resp. $r_{+-}(t)$, resp. r_-, resp. r_-) and S act as s_+ (resp. s_+, resp. $s_{-+}(t)$, resp. $s_{--}(t)$) has a scalar commutant. In all four cases, the central element P of $\mathcal{A}$ acts as the scalar matrix $\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$.*

This representation is self-adjoint with respect to the symmetric bilinear form defined by $a_{++}(t)$ (resp. $a_{+-}(t)$, resp. $a_{-+}(t)$, resp. $a_{--}(t)$) on $\mathbb{C}^2$ and this symmetric bilinear form is the unique one with this property, up to a scalar multiple. If t is real, this symmetric bilinear *form is positive definite on* $\mathbb{R}^2$ *if and only if* $|t| < 1$. *If* $|t| > 1$, *it has signature* $(1,1)$.

For $t^2 \neq 1$, *these actions are all irreducible and isomorphic to each other.*

3.2. **An abstract spectral theorem.** In the sequel, we will encounter self-adjoint representations of the algebra $\mathcal{A}$. To study their spectral theory, we will need the following abstract form of the spectral theorem [8, Theorem 12.23].

Lemma 3.4. *Let* V *be a finite dimensional real vector space. Let* p *be a non-negative symmetric bilinear form on the space* $V[t]$ *of polynomial functions* $\mathbb{R} \to V$ *and assume that the operator* $f(t) \mapsto tf(t)$ *on* $V[t]$ *is bounded and symmetric with respect to* p. *Then, there exists a compactly supported Radon measure* μ *on* $\mathbb{R}$ *and a* μ*-integrable function* $\pi : \mathbb{R} \to \mathcal{Q}_+(V)$ *such that, for any* f, g *in* $V[t]$, *one has*

$$p(f,g) = \int_{\mathbb{R}} \pi(t)(f(t), g(t)) \mathrm{d}\mu(t).$$

We wrote $\mathcal{Q}_+(V)$ for the space of non-negative symmetric bilinear forms on V. When applied to the concrete examples of representations of the algebra $\mathcal{A}$ from Subsection 3.1, this gives

Corollary 3.5. *Let* p *be non-negative symmetric bilinear form on* $\mathbb{R}^2[t]$ *such that the operators* $f(t) \mapsto r_{++}(t)f(t)$ *(resp.* $f(t) \mapsto r_{-+}(t)f(t)$, *resp.* $f(t) \mapsto r_- f(t)$, *resp.* $f(t) \mapsto r_- f(t)$*) as well as* $f(t) \mapsto s_+ f(t)$ *(resp.* $f(t) \mapsto s_+ f(t)$, *resp.* $f(t) \mapsto s_{-+}(t)f(t)$, *resp.* $f(t) \mapsto s_{--}(t)f(t)$*) are bounded and self-adjoint with respect to* p. *Then, there exists a Radon measure* μ *on* $[-1,1]$ *such that, for any* f, g *in* $\mathbb{R}^2[t]$, *one has*

$$p(f,g) = \int_{[-1,1]} \pi(t)(f(t), g(t)) \mathrm{d}\mu(t),$$

where, for $-1 \leq t \leq 1$, $\pi(t)$ *is the symmetric bilinear form with matrix* $a_{++}(t)$ *(resp.* $a_{+-}(t)$, *resp.* $a_{-+}(t)$, *resp* $a_{--}(t)$*) on* $\mathbb{R}^2$.

Proof. We show the $++$ case, the other ones being analogous.

By Lemma 3.3, for f in $\mathbb{R}^2[t]$, and t in $\mathbb{R}$, we have

$$\frac{1}{q+1}(r_{++}(t)s_+ + s_+ r_{++}(t) - (q-1)s_+)f(t) = tf(t).$$

Therefore, by Lemma 3.4, there exists a compactly supported Radon measure μ on $\mathbb{R}$ and a μ-integrable function $\pi : \mathbb{R} \to \mathcal{Q}_+(\mathbb{R}^2)$ such that, for any f, g in $\mathbb{R}^2$, one has

$$\pi(f,g) = \int_{\mathbb{R}} \pi(t)(f(t), g(t)) \mathrm{d}\mu(t).$$

By assumption, for any polynomial functions $f, g : \mathbb{R} \to \mathbb{R}^2$, we have

$$\int_{\mathbb{R}} \pi(t)(r_{++}(t)f(t), g(t))\mathrm{d}\mu(t) = \int_{\mathbb{R}} \pi(t)(f(t), r_{++}(t)g(t))\mathrm{d}\mu(t)$$
$$\int_{\mathbb{R}} \pi(t)(s_+ f(t), g(t))\mathrm{d}\mu(t) = \int_{\mathbb{R}} \pi(t)(f(t), s_+ g(t))\mathrm{d}\mu(t).$$

Therefore, for μ-almost any t in $\mathbb{R}$, the operators $r_{++}(t)$ and s_+ are self-adjoint with respect to the symmetric bilinear form $\pi(t)$. By Lemma 3.3, there exists a μ-integrable function $\varphi : \mathbb{R} \to \mathbb{R}_+$ with support in $[-1, 1]$ such that, for t in $\mathbb{R}$,

$$\pi(t)(v, w) = \varphi(t)v^* a_{++}(t)w, \quad v, w \in \mathbb{R}^2$$

(where v^* is the transpose of the column vector v in $\mathbb{R}^2$). The conclusion follows by replacing μ with the finite measure $\varphi\mu$. □

3.3. **Spectral parametrization.** As when studying spectral theory on X (see [3, Chapter II]), the formulae that will allow to decompose the model operators rely on the use of a certain rational function on $\mathbb{C}$. Set as usual

$$\mathbb{H} = \{t \in \mathbb{C} | \Im t > 0\} \text{ as well as } \mathbb{H}_q = \{u \in \mathbb{C} | \Im u > 0, |u| > \sqrt{q}\}.$$

We will write $\overline{\mathbb{H}}$ and $\overline{\mathbb{H}}_q$ for the closures of these open subsets of $\mathbb{C}$, that is,

$$\overline{\mathbb{H}} = \{t \in \mathbb{C} | \Im t \geq 0\} \text{ and } \overline{\mathbb{H}}_q = \{u \in \mathbb{C} | \Im u \geq 0, |u| \geq \sqrt{q}\}.$$

We will always denote by $\mathcal{I}_q$ the critical interval

$$\mathcal{I}_q = \left[-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1} \right].$$

By a result of Kesten [5] (see also [3, Proposition II.6.3] whose statement is unfortunately mistakenly written), the critical interval is the spectrum of the natural Markov operator on the space $\ell^2(X)$ of square-integrable functions of X. We retrieve this fact in Subsection 4.2 below.

We shall repeatedly use

Lemma 3.6. *The function*

$$u \mapsto t = \frac{1}{q+1}\left(u + \frac{q}{u}\right)$$

maps $\mathbb{C}^*$ onto $\mathbb{C}$. It induces a biholomorphism from $\{u \in \mathbb{C} | \, |u| > \sqrt{q}\}$ onto $\mathbb{C} \smallsetminus \mathcal{I}_q$ which sends $\mathbb{H}_q$ onto $\mathbb{H}$. It also induces a homeomorphism from $\overline{\mathbb{H}}_q$ onto $\overline{\mathbb{H}}$. Finally, it maps the circle $\{u \in \mathbb{C} | |u| = \sqrt{q}\}$ onto $\mathcal{I}_q$.

The proof is straightforward.

From now on, we will stick to the notational convention that t and u are complex numbers that are related through the equation

$$u^2 - (q+1)tu + q = 0.$$

Note that the other root of the equation is $\frac{q}{u}$.

To determine spectral measures of the model operators, we will use a standard technique from spectral analysis (see for example [2, Theorem X.6.1]), which relies on the

Lemma 3.7. *Let μ be a Borel measure on $\mathbb{R}$ with $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$. Then the Poisson transform of μ may be defined on $\mathbb{H}$ by the formula*

$$\mathcal{P}\mu(z) = \frac{1}{\pi} \int_{\mathbb{R}} \Im\left(\frac{1}{t-z}\right) \mathrm{d}\mu(t) \quad z \in \mathbb{H}.$$

Proof. Indeed, for z in $\mathbb{H}$ and t in $\mathbb{R}$, if $z = x + iy$, $x, y \in \mathbb{R}$, we have

$$\frac{1}{\pi}\Im\left(\frac{1}{t-z}\right) = \frac{1}{\pi}\frac{y}{(t-x)^2 + y^2},$$

which is the standard formula for the Poisson kernel of the upper half plane (see [1, Chapter 7]). □

This Lemma will be applied to a particular harmonic function which plays a central role in the spectral theory of the tree X. We denote by μ_q the absolutely continuous measure with density $\frac{q+1}{2\pi}\frac{\sqrt{4q-(q+1)^2t^2}}{1-t^2}$ on the interval $\mathcal{I}_q = \left[-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}\right]$. We will show later that μ_q is a probability measure by getting it as the spectral measure associated to a unit vector and a self-adjoint operator on a Hilbert space. This construction will rely on

Lemma 3.8. *The Poisson transform of μ_q on $\mathbb{H}$ may be defined as follows. For any t in $\mathbb{H}$, let u be the unique element of $\mathbb{H}_q$ with $u^2 - (q+1)tu + q = 0$. Then, we have*

$$\mathcal{P}\mu_q(t) = \frac{q+1}{\pi}\Im\left(\frac{u}{1-u^2}\right).$$

Proof. For t in $\overline{\mathbb{H}}$, we set $F(t) = \frac{q+1}{\pi}\Im\left(\frac{u}{1-u^2}\right)$, where u is the unique element of $\overline{\mathbb{H}}_q$ with $q + u^2 = (q+1)t$. As the function $G : u \mapsto \Im\left(\frac{u}{1-u^2}\right)$ is harmonic on $\mathbb{H}_q$ and continuous on $\overline{\mathbb{H}}_q$, by Lemma 3.6, the function F is harmonic on $\mathbb{H}$ and continuous on $\overline{\mathbb{H}}$. As $G(u) \xrightarrow[u\to\infty]{} 0$, by standard properties of harmonic functions (see for example [1, Theorem 7.5]), we may write F as the Poisson transform of the measure $\nu = F(t)\mathrm{d}t$ on $\mathbb{R}$. Let us show that $\nu = \mu_q$.

As G vanishes on the set $\{u \in \mathbb{R} | \, |u| \geq \sqrt{q}\}$, F vanishes on the set $\{t \in \mathbb{R} | (q+1)\,|t| \geq 2\sqrt{q}\}$. Therefore, it suffices to determine the value of G on the set

$$\partial \mathbb{H}_q \smallsetminus \mathbb{R} = \{u \in \mathbb{C} | \Im u > 0, |u| = \sqrt{q}\}.$$

Thus, let u in $\mathbb{C}$ with $|u| = \sqrt{q}$ and set $t = \frac{1}{q+1}(u + q/u) = \frac{2}{q+1}\Re u$. We get

$$\begin{aligned} 2i\Im\left(\frac{u}{1-u^2}\right) &= \frac{u}{1-u^2} - \frac{q/u}{1-q^2/u^2} = \frac{u - q^2/u - q/u + qu}{(1-u^2)(1-q^2/u^2)} \\ &= (q+1)\frac{u - q/u}{q^2 + 1 - u^2 - q^2/u^2} = \frac{1}{q+1}\frac{u-q/u}{1-t^2}, \end{aligned}$$

which gives

$$\Im\left(\frac{u}{1-u^2}\right) = \frac{1}{q+1}\frac{\Im u}{1-t^2}.$$

The conclusion follows as, if $\Im u \geq 0$, we have

$$\Im u = \sqrt{q - (\Re u)^2} = \frac{1}{2}\sqrt{4q - (q+1)^2 t^2}.$$

□

3.4. **Hilbert spaces of sequences.** The construction of the spectral decomposition of the model operators will rely on the application of spectral constructions in certain Hilbert spaces of sequences. We now define precisely these spaces.

For x in $\mathbb{R}^{(\mathbb{N})}$ and y in $\mathbb{R}^{\mathbb{N}}$, we set

$$\begin{aligned} \langle x, y\rangle_+ &= \sum_{i\geq 0} q^i (x_{2i}y_{2i} + x_{2i+1}y_{2i+1}) \\ \langle x, y\rangle_- &= x_0 y_0 + \sum_{i\geq 1} q^i (x_{2i-1}y_{2i-1} + x_{2i}y_{2i}) \end{aligned}$$

We denote the associated real Hilbert spaces of sequences by H_+ and H_-, that is, H_+ (resp. H_-) is equal to the space

$$\left\{ x \in \mathbb{R}^{\mathbb{N}} \middle| \sum_{i\geq 0} q^{\frac{i}{2}} x_i^2 < \infty \right\} < \infty,$$

which we equip with the scalar product $\langle ., .\rangle_+$ (resp. $\langle ., .\rangle_-$). We stress out that the Hilbert spaces H_+ and H_- have the same underlying vector space, but not the same scalar product.

Spectral analysis will require us to use the complexifications $H_{+,\mathbb{C}}$ and $H_{-,\mathbb{C}}$ of H_+ and H_-. In this framework, we recall our non standard convention: for any x in $\mathbb{C}^{(\mathbb{N})}$ and y in $\mathbb{C}^{\mathbb{N}}$, we still set

$$\langle x, y\rangle_+ = \sum_{i\geq 0} q^i(x_{2i}y_{2i} + x_{2i+1}y_{2i+1})$$

$$\langle x, y\rangle_- = x_0y_0 + \sum_{i\geq 1} q^i(x_{2i-1}y_{2i-1} + x_{2i}y_{2i}).$$

So that we can define $H_{+,\mathbb{C}}$ (resp. $H_{-,\mathbb{C}}$) as the space of complex sequences

$$\left\{ x \in \mathbb{C}^{\mathbb{N}} \,\middle|\, \sum_{i\geq 0} q^{\frac{i}{2}}|x_i|^2 < \infty \right\} < \infty,$$

equipped with the Hermitian scalar product $(x, y) \mapsto \langle \overline{x}, y\rangle_+$ (resp. $(x, y) \mapsto \langle \overline{x}, y\rangle_-$), where $\overline{x}$ is obtained by taking the complex conjugates of the coordinates of x.

4. The even model operators

In this Section, we study operators on scalar sequences defined in analogy with the operators of Definition 2.16. We actually split the definition of the model operators according to the eigenvalue of the $\vee$ operator, which in the even case, can be q or -1. For these model operators, we build a complete spectral theory.

4.1. **The ++ model operators.** In this Subsection, following Definition 2.16, we consider the operators R_{++} and S_{++} defined on sequences $x = (x_i)_{i\geq 0}$ of real numbers by

$$\begin{aligned}
&(R_{++}x)_0 = qx_0 && \\
&(R_{++}x)_i = x_{i-1} + (q-1)x_i && i \text{ even}, i \geq 2\\
&(R_{++}x)_i = qx_{i+1} && i \text{ odd}.
\end{aligned}$$

and

$$\begin{aligned}
&(S_{++}x)_i = x_{i+1} && i \text{ even}\\
&(S_{++}x)_i = x_{i-1} && i \text{ odd}.
\end{aligned}$$

These operators preserve the space of sequences with only finitely many non zero entries. They satisfy the relations $R_{++}^2 = q + (q-1)R_{++}$ and $S_{++}^2 = 1$. In other words, they define a representation of the algebra $\mathcal{A}$ of Subsection 3.1 in the space of sequences of real numbers. In particular, we set $P_{++} = \frac{1}{q+1}(R_{++}S_{++} + S_{++}R_{++} - (q-1)S_{++})$.

The scalar product of H_+ is adapted to the study of this situation.

Lemma 4.1. *The operators R_{++} and S_{++} are bounded and self-adjoint in H_+.*

Thus, these operators define a self-adjoint representation of the algebra $\mathcal{A}$ of Subsection 3.1 in H_+.

Proof. The fact that the operators are bounded is easy. From the formulas, it is clear that S_{++} is self-adjoint. Let us check that R_{++} also is. Indeed, for x, y in H_+, we have

$$\langle x, y\rangle_+ = x_0 y_0 + \sum_{i\geq 1} q^{i-1}(x_{2i-1}y_{2i-1} + q x_{2i} y_{2i}).$$

Therefore,

$$\begin{aligned}\langle R_{++}x, y\rangle_+ &= q x_0 y_0 + \sum_{i\geq 1} q^{i-1}(q x_{2i} y_{2i-1} + q(x_{2i-1} + (q-1)x_{2i})y_{2i}) \\ &= q x_0 y_0 + \sum_{i\geq 1} q^{i}(x_{2i} y_{2i-1} + x_{2i-1} y_{2i} + (q-1)x_{2i} y_{2i}).\end{aligned}$$

The conclusion follows as the latter formula is symmetric in x and y. □

Recall that $\mathbf{1}_0$ stands for the sequence (x_i) with $x_0 = 1$ and $x_i = 0$ for $i \geq 1$. In this Subsection, we prove the following result which defines a spectral analysis for the operators R_{++} and S_{++}.

Proposition 4.2. *There exists a unique linear map $x \mapsto \widehat{x}(t)$ from $\mathbb{R}^{(\mathbb{N})}$ to the space $\mathbb{R}^2[t]$ of polynomial functions $\mathbb{R} \to \mathbb{R}^2$ such that $\widehat{\mathbf{1}_0}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and that, for any x in $\mathbb{R}^{(\mathbb{N})}$, one has*

$$\widehat{R_{++}x}(t) = r_{++}(t)\widehat{x}(t) \text{ and } \widehat{S_{++}x}(t) = s_+\widehat{x}(t) \quad t \in \mathbb{R}.$$

This map is a linear isomorphism from $\mathbb{R}^{(\mathbb{N})}$ onto $\mathbb{R}^2[t]$.

The spectrum of the operator P_{++} in H_+ is the interval $\mathcal{I}_q$ and, for any x, y in $\mathbb{R}^{(\mathbb{N})}$, we have

$$\langle x, y\rangle_+ = \int_{\mathcal{I}_q} \widehat{x}(t)^* a_{++}(t)\widehat{y}(t)\mathrm{d}\mu_q(t). \tag{4.1}$$

We have used the notation introduced in Subsection 3.1 for the ++ matrices. Recall that v^* means the transpose of a column vector v in $\mathbb{R}^2$ and that the measure μ_q is the absolutely continuous measure with density function $t \mapsto \frac{q+1}{2\pi}\frac{\sqrt{4q-(q+1)^2t^2}}{1-t^2}$ on $\mathcal{I}_q$ (see Subsection 3.3). Note that the statement implies that μ_q is a probability measure.

An other way of stating the integral formula would be to say that the transform $x \mapsto \widehat{x}$ induces an isometry from H_+ onto the space of classes of measurable functions $v : \mathcal{I}_q \to \mathbb{R}^2$ with $\int_{\mathcal{I}_q} v(t)^* a_{++}(t) v(t) \mathrm{d}\mu_q(t) < \infty$.

We begin the proof of Proposition 4.2 by showing that the images of $\mathbf{1}_0$ under R_{++} and S_{++} span the set of sequences with finitely many non zero entries.

Lemma 4.3. *We have* $\mathcal{A}\mathbf{1}_0 = \mathbb{R}^{(\mathbb{N})}$.

Proof. This is a standard triagularization argument. Indeed, the formulae defining the operators R_{++} and S_{++} imply by a straightforward induction that, for $i \geq 0$,

$$((R_{++}S_{++})^i \mathbf{1}_0)_{2i} = 1 \text{ and } ((R_{++}S_{++})^i \mathbf{1}_0)_j = 0, \quad j > 2i;$$
$$(S_{++}(R_{++}S_{++})^i \mathbf{1}_0)_{2i+1} = 1 \text{ and } (S_{++}(R_{++}S_{++})^i \mathbf{1}_0)_j = 0, \quad j > 2i+1.$$

The result follows. □

Now, let us construct the joint spectral theory of R_{++} and S_{++}. For u in $\mathbb{C}^*$, we let $a_{++}(u)$ and $b_{++}(u)$ be the sequences of complex numbers defined by, for $i \geq 0$,

$$a_{++}(u)_{2i} = b_{++}(u)_{2i+1} = -qu^{-i} \text{ and } a_{++}(u)_{2i+1} = b_{++}(u)_{2i} = u^{1-i}.$$

These sequences are built in order to satisfy the following relations.

Lemma 4.4. *For u in $\mathbb{C}^*$, we have $S_{++}a_{++}(u) = b_{++}(u)$ and, for any $i \geq 1$,*

$$\begin{aligned} (R_{++}a_{++}(u))_i &= \left(\frac{q}{u} + u\right) b_{++}(u)_i + q a_{++}(u)_i \\ \text{and } (R_{++}b_{++}(u))_i &= -b_{++}(u)_i. \end{aligned}$$

The above relations do not work for $i = 0$. To correct this, we introduce new sequences as follows. For t in $\mathbb{C}$, we choose as in Lemma 3.6, some u in $\mathbb{C}^*$ with $u^2 - (q+1)tu + q = 0$. We assume that we have $(q+1)^2t^2 \neq 4q$, so that $u^2 \neq q$, and we set

$$\begin{aligned} \alpha_{++}(t) &= \frac{1}{u^2 - q} a_{++}(u) + \frac{u^2}{q(q - u^2)} a_{++}\left(\frac{q}{u}\right) \\ \beta_{++}(t) &= S\alpha_{++}(t). \end{aligned}$$

The notation is justified by the fact that, since the right hand-side is invariant by the involution $u \mapsto \frac{q}{u}$, it only depends on t. By construction, one has

$$\alpha_{++}(t)_0 = 1 \text{ and } \beta_{++}(t)_0 = 0. \tag{4.2}$$

Besides, note that if t is real, both $\alpha_{++}(t)$ and $\beta_{++}(t)$ are real sequences.

We can now get the missing case in Lemma 4.4. Indeed, we have the following relation between these sequences and the operators introduced in Subsection 3.1.

Lemma 4.5. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we have the matrix relations:*

$$\begin{pmatrix} S_{++}\alpha_{++}(t) \\ S_{++}\beta_{++}(t) \end{pmatrix} = s_+ \begin{pmatrix} \alpha_{++}(t) \\ \beta_{++}(t) \end{pmatrix}$$
$$\text{and } \begin{pmatrix} R_{++}\alpha_{++}(t) \\ R_{++}\beta_{++}(t) \end{pmatrix} = r_{++}(t) \begin{pmatrix} \alpha_{++}(t) \\ \beta_{++}(t) \end{pmatrix}.$$

For x in $\mathbb{C}^{(\mathbb{N})}$ and t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we set

$$\widehat{x}(t) = \begin{pmatrix} \langle x, \alpha_{++}(t)\rangle_+ \\ \langle x, \beta_{++}(t)\rangle_+ \end{pmatrix}.$$

Note again, that if t is real and x has real coefficients, the vector $\widehat{x}(t)$ has real coordinates. Besides, (4.2) reads as

$$\widehat{\mathbf{1}_0}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.3}$$

From Lemma 4.1 and Lemma 4.5, we directly get

Lemma 4.6. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$ and x in $\mathbb{R}^{(\mathbb{N})}$, we have*

$$\widehat{R_{++}x}(t) = r_{++}(t)\widehat{x}(t) \text{ and } \widehat{S_{++}x}(t) = s_+\widehat{x}(t).$$

Let us describe in more details the map $x \mapsto \widehat{x}$.

Lemma 4.7. *For x in $\mathbb{R}^{(\mathbb{N})}$, the function $t \mapsto \widehat{x}(t)$ is polynomial. The map $x \mapsto \widehat{x}$ induces a linear isomorphism from the space $\mathbb{R}^{(\mathbb{N})}$ onto the space $\mathbb{R}^2[t]$ of polynomial functions $\mathbb{R} \to \mathbb{R}^2$.*

Proof. Indeed, (4.3) implies that $\widehat{1_0}$ is polynomial, since it is constant. Besides, by Lemma 4.6, the space of x in $\mathbb{R}^{(\mathbb{N})}$ such that $\widehat{x}$ is polynomial is stable under the action of the algebra $\mathcal{A}$. Thus, by Lemma 4.3, this space is equal to $\mathbb{R}^{(\mathbb{N})}$.

Now, by Lemma 3.3, Equation (4.3) and Lemma 4.6, for $i \geq 0$, we have

$$\widehat{P_{++}^i\mathbf{1}_0}(t) = \begin{pmatrix} t^i \\ 0 \end{pmatrix} \text{ and } \widehat{S_{++}P_{++}^i}\mathbf{1}_0(t) = \begin{pmatrix} 0 \\ t^i \end{pmatrix}. \tag{4.4}$$

Thus, the map $x \mapsto \widehat{x}$ maps $\mathbb{R}^{(\mathbb{N})}$ onto polynomial functions $\mathbb{R} \to \mathbb{R}^2$.

It remains to prove that this map is injective. By Lemma 4.3, we have $\mathbb{R}^{(\mathbb{N})} = \mathcal{A}\mathbf{1}_0$. Besides, recall from Corollary 3.2 the decomposition $\mathcal{A} = \mathbb{R}[P] \oplus S\mathbb{R}[P] \oplus \mathcal{A}(R-q)$. As $R_{++}\mathbf{1}_0 = q\mathbf{1}_0$, we get

$$\mathbb{R}^{(\mathbb{N})} = (\mathbb{R}[P_{++}] \oplus S_{++}\mathbb{R}[P_{++}])\mathbf{1}_0.$$

Therefore, if x is in $\mathbb{R}^{(\mathbb{N})}$, we can write $x = f(P_{++})\mathbf{1}_0 + S_{++}g(P_{++})\mathbf{1}_0$ where f and g are polynomial functions on $\mathbb{R}$. By (4.4), for t in $\mathbb{R}$, we have

$$\widehat{x}(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

and injectivity follows. □

The sequences $\alpha_{++}(t)$ and $\beta_{++}(t)$ are uniquely defined by the relations in Lemma 4.5.

Corollary 4.8. *Let t_0 be in $\mathbb{R}$ and γ, δ be in $\mathbb{R}^{\mathbb{N}}$ with*

$$\begin{pmatrix} S_{++}\gamma \\ S_{++}\delta \end{pmatrix} = s_+ \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \text{ and } \begin{pmatrix} R_{++}\gamma \\ R_{++}\delta \end{pmatrix} = r_{++}(t_0) \begin{pmatrix} \gamma \\ \delta \end{pmatrix}.$$

Then we have $\gamma = \gamma_0 \alpha_{++}(t_0)$ and $\delta = \gamma_0 \beta_{++}(t_0)$.

Proof. Define a linear map $\varphi : \mathbb{R}^2[t] \to \mathbb{R}^2$ as follows. For $\begin{pmatrix} f \\ g \end{pmatrix}$ in $\mathbb{R}^2[t]$, by Lemma 4.7, there exists a unique x in $\mathbb{R}^{(\mathbb{N})}$ with $\widehat{x} = \begin{pmatrix} f \\ g \end{pmatrix}$. Then, we set $\varphi \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \langle \gamma, x \rangle_+ \\ \langle \delta, x \rangle_+ \end{pmatrix}$. From Lemma 4.6 and the assumption, we have

$$\varphi \left(r_{++}(t) \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \right) = \begin{pmatrix} \langle \gamma, R_{++}x \rangle_+ \\ \langle \delta, R_{++}x \rangle_+ \end{pmatrix} = \begin{pmatrix} \langle R_{++}\gamma, x \rangle_+ \\ \langle R_{++}\delta, x \rangle_+ \end{pmatrix} = r_{++}(t_0) \varphi \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \tag{4.5}$$

and in the same way,

$$\varphi \left(s_+ \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \right) = \begin{pmatrix} \langle \gamma, S_{++}x \rangle_+ \\ \langle \delta, S_{++}x \rangle_+ \end{pmatrix} = s_+ \varphi \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}. \tag{4.6}$$

By Lemma 3.3, we get

$$\varphi \begin{pmatrix} tf(t) \\ tg(t) \end{pmatrix} = \begin{pmatrix} \langle \gamma, P_{++}x \rangle_+ \\ \langle \delta, P_{++}x \rangle_+ \end{pmatrix} = t_0 \varphi \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

and hence, if $f(t_0) = g(t_0) = 0$, $\varphi \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = 0$. Therefore, there exists a linear map $m : \mathbb{R}^2 \to \mathbb{R}^2$ such that, for any f, g in $\mathbb{R}[t]$, $\varphi \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = m \begin{pmatrix} f(t_0) \\ g(t_0) \end{pmatrix}$. Now, (4.5) and (4.6) imply that m commutes with the matrices $r_{++}(t_0)$ and s_+. Therefore, by Lemma 3.3, m is a scalar matrix, that is, there exists a real number λ with, for any f, g in $\mathbb{R}[t]$, $\varphi\begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = \lambda \begin{pmatrix} f(t_0) \\ g(t_0) \end{pmatrix}$. By Lemma 4.7, we get, for any x in $\mathbb{R}^{(\mathbb{N})}$,

$$\langle \gamma, x \rangle_+ = \lambda \langle \alpha_{++}(t), x \rangle_+ \text{ and } \langle \delta, x \rangle_+ = \lambda \langle \beta_{++}(t), x \rangle_+.$$

The conclusion follows. □

So far, we have constructed the spectral transform $x \mapsto \widehat{x}$ in Proposition 4.2. It remains to establish the Plancherel formula (4.1). To this aim, we will use Corollary 3.5. Indeed, Lemma 4.7 tells us that we can identify $\mathbb{R}^{(\mathbb{N})}$ with the space $\mathbb{R}^2[t]$, whereas Lemma 4.6 tells us that the operators R_{++} and S_{++} then act as the matrices $r_{++}(t)$ and s_+ of Subsection 3.1. Therefore, to get (4.1), it suffices to compute the measure μ provided by Corollary 3.5. This computation will use the notation of Subsection 3.3 and the standard method following from Lemma 3.7, which relies on the computation of the resolvent function $\langle \mathbf{1}_0, (P_{++} - t)^{-1} \mathbf{1}_0 \rangle_+$ for t in $\mathbb{H}$. To achieve this computation, we introduce a last family of sequences in $\mathbb{C}^{(\mathbb{N})}$. For u in $\mathbb{C}$, $u \notin \{-1, 0, 1\}$, we set

$$c_{++}(u)_{2i} = \frac{(q+1)u^{1-i}}{1-u^2} \text{ and } c_{++}(u)_{2i+1} = \frac{(q+1)u^{-i}}{1-u^2} \quad i \geq 0.$$

Note that $c_{++}(u)$ belongs to $H_{+,\mathbb{C}}$ if and only if $|u| > \sqrt{q}$. A direct computation gives

Lemma 4.9. *Let u be in $\mathbb{C}$ with $u \notin \{-1, 0, 1\}$. Set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Then we have*

$$P_{++} c_{++}(u) = t c_{++}(u) + \mathbf{1}_0.$$

We gather the arguments of this Subsection.

Proof of Proposition 4.2. The uniqueness of the transform $x \mapsto \widehat{x}$ follows from Lemma 4.3. The fact that it satisfies the required properties was obtained in (4.3) and Lemma 4.6. We know that it induces a linear isomorphism from $\mathbb{R}^{(\mathbb{N})}$ onto $\mathbb{R}^2[t]$ thanks to Lemma 4.7.

It remains to show the Plancherel formula (4.1). Let μ be the measure from Corollary 3.5 applied to the scalar product on $\mathbb{R}^2[t]$ which is obtained by pulling back the scalar product $\langle ., . \rangle_+$ under the inverse of the transform $x \mapsto \widehat{x}$. Let t be in $\mathbb{H}$. By Lemma 3.3 and Lemma 4.6, as for s in $\mathbb{R}$, the matrix $a_{++}(s)$ has upper left coefficient 1, we have,

$$\langle \mathbf{1}_0, (P_{++} - t)^{-1} \mathbf{1}_0 \rangle_+ = \int_{[-1,1]} \frac{1}{s-t} \mathrm{d}\mu(s).$$

By Lemma 4.9, this gives, for u in $\mathbb{H}_q$ with $u^2-(q+1)tu+q=0$,

$$\int_{[-1,1]} \frac{1}{s-t}\mathrm{d}\mu(s) = \langle \mathbf{1}_0, c_{++}(u)\rangle_+ = (q+1)\frac{u}{1-u^2},$$

hence, by Lemma 3.7,

$$\mathcal{P}\mu(t) = \frac{q+1}{\pi}\Im\left(\frac{u}{1-u^2}\right).$$

The conclusion now follows from Lemma 3.8. □

4.2. **Spectral measures in $\ell^2(X)$.** To motivate the reader, and in the hope that this will make our strategy more understandable, we explain how Proposition 4.2 can be used in order to recover the computation of spectral measures in $\ell^2(X)$ for the natural Markov operator. This result is due to Kesten [5]. It plays a key role in the spectral theory developed in [3].

We temporarily come back to the language of trees. We equip the space $\ell^2(X)$ with the bounded self-adjoint operator Q defined by, for f in $\ell^2(X)$,

$$Qf(x) = \frac{1}{q+1}\sum_{y\sim x} f(y), \quad x \in X.$$

Corollary 4.10 (Kesten [5])**.** *For any a in X, the spectral measure of $\mathbf{1}_a$ with respect to Q is the measure μ_q.*

Proof. In this proof, we use the letters R and S for the operators on functions on X_1 defined by, for any $g : X_1 \to \mathbb{R}$,

$$Rg(x,y) = \sum_{\substack{z\sim x\\ z\neq y}} g(x,z) \text{ and } Sg(x,y) = g(y,x), \quad x\sim y \in X.$$

As usual, we set $P = \frac{1}{q+1}(RS+SR-(q-1)S)$. For any function f on X define the function Lf on X_1 by

$$Lf(x,y) = f(x), \quad x\sim y\in X.$$

When f is in $\ell^2(X)$, we get

$$\|Lf\|_2^2 = \sum_{(x,y)\in X_1} Lf(x,y)^2 = (q+1)\sum_{x\in X} f(x)^2 = (q+1)\|f\|_2^2.$$

Besides, for (x,y) in X_1, we have

$$\begin{aligned}(q+1)PLf(x,y) &= (RS+SR-(q-1)S)Lf(x,y)\\ &= \sum_{\substack{z\sim x\\ z\neq y}} Lf(z,x) + \sum_{\substack{z\sim y\\ z\neq x}} Lf(y,z) - (q-1)Lf(y,x)\\ &= \sum_{\substack{z\sim x\\ z\neq y}} f(z) + qf(y) - (q-1)f(y) = \sum_{z\sim x} f(z).\end{aligned}$$

We get $PL = LQ$. Thus, determining spectral measures of the elements of $\ell^2(X)$ with respect to Q is equivalent to determine the spectral measures of their images by L with respect to P.

Now, we fix a in X and we define an orthogonal sequence $(g_i)_{i\geq 0}$ of elements of $\ell^2(X_1)$ as follows. For any $i \geq 0$ and (x,y) in X_1, we set

$$g_{2i}(x,y) = \mathbf{1}_{\substack{d(x,a)=i\\ d(y,a)=i+1}} \text{ and } g_{2i+1}(x,y) = \mathbf{1}_{\substack{d(x,a)=i+1\\ d(y,a)=i}}.$$

We have $\|g_{2i}\|_2^2 = \|g_{2i+1}\|_2^2 = (q+1)q^i$. Besides, the definition immediately gives $Sg_{2i} = g_{2i+1}$, whereas a direct computation yields

$$\begin{aligned} Rg_0 &= qg_0 && \\ Rg_i &= qg_{i-1} + (q-1)g_i && i \text{ even}, i\geq 2 \\ Rg_i &= g_{i+1} && i \text{ odd}.\end{aligned} \tag{4.7}$$

For a sequence $x = (x_i)_{i\geq 0}$ of real numbers, we set Cx to be the function $\sum_{i\geq 0} x_i g_i$ on X_1. With the notation of Subsection 4.1, the relations (4.7) give $RC = CR_{++}$ and $SC = CS_{++}$. Moreover, the computation of the norm of the g_i, $i \geq 0$, implies that, for x in H_+, we have $\|Cx\|_2^2 = (q+1)\|x\|_+^2$. As $C\mathbf{1}_0 = g_0 = L\mathbf{1}_a$, the conclusion follows from Proposition 4.2. □

4.3. **The $+-$ model operators.** We come back to the study of the model operators. We now consider the case of the other eigenvalue of the $\vee$ operator in Definition 2.16, so that we define the operators R_{+-} and S_{+-} acting on sequences $x = (x_i)_{i\geq 0}$ of real numbers by

$$\begin{aligned} (R_{+-}x)_0 &= -x_0 && \\ (R_{+-}x)_i &= x_{i-1} + (q-1)x_i && i \text{ even}, i\geq 2 \\ (R_{+-}x)_i &= qx_{i+1} && i \text{ odd}.\end{aligned}$$

and

$$\begin{aligned} (S_{+-}x)_i &= x_{i+1} && i \text{ even} \\ (S_{+-}x)_i &= x_{i-1} && i \text{ odd}.\end{aligned}$$

Again, these operators preserve the space of sequences with only finitely many non zero entries. They satisfy the relations $R_{+-}^2 = q+(q-1)R_{++}$ and $S_{+-}^2 = 1$. Thus we still have a representation of the algebra $\mathcal{A}$ of Subsection 3.1 in the space of sequences of real numbers. We now set $P_{+-} = \frac{1}{q+1}(R_{+-}S_{+-} + S_{+-}R_{+-} - (q-1)S_{+-})$.

The same proof as above gives

Lemma 4.11. *The operators R_{+-} and S_{+-} are bounded and self-adjoint in H_+.*

The spectral analysis for this new self adjoint representation of the algebra $\mathcal{A}$ is now defined as follows.

Proposition 4.12. *There exists a unique linear map $x \mapsto \widehat{x}(t)$ from $\mathbb{R}^{(\mathbb{N})}$ to the space $\mathbb{R}^2[t]$ of polynomial functions $\mathbb{R} \to \mathbb{R}^2$ such that $\widehat{\mathbf{1}}_0(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and that, for any x in $\mathbb{R}^{(\mathbb{N})}$, one has*

$$\widehat{R_{+-}x}(t) = r_{+-}(t)\widehat{x}(t) \text{ and } \widehat{S_{+-}x}(t) = s_+\widehat{x}(t) \quad t \in \mathbb{R}.$$

This map is a linear isomorphism from $\mathbb{R}^{(\mathbb{N})}$ onto $\mathbb{R}^2[t]$.

The spectrum of the operator P_{+-} in H_+ is the set $\mathcal{I}_q \cup \{-1, 1\}$ and, for any x, y in $\mathbb{R}^{(\mathbb{N})}$, we have

$$\begin{aligned} \langle x, y\rangle_+ = {} & \frac{1}{q}\int_{\mathcal{I}_q} \widehat{x}(t)^* a_{+-}(t)\widehat{y}(t)\mathrm{d}\mu_q(t) \\ & + \frac{q-1}{2q}(\widehat{x}(-1)^* a_{+-}(-1)\widehat{y}(-1) + \widehat{x}^*(1)a_{+-}(1)\widehat{y}(1)) \end{aligned} \tag{4.8}$$

The existence of a discrete component in the spectral measure correspond to the existence of joint eigenvectors for the operators R_{+-} and S_{+-} in H_+.

We start proving Proposition 4.12. As for the ++-model, we can show

Lemma 4.13. *We have $\mathcal{A}\mathbf{1}_0 = \mathbb{R}^{(\mathbb{N})}$.*

We now adapt the construction of the spectral transform by setting, for u in $\mathbb{C}^*$ and $i \geq 0$,

$$a_{+-}(u)_{2i} = b_{+-}(u)_{2i+1} = u^{-i} \text{ and } a_{+-}(u)_{2i+1} = b_{+-}(u)_{2i} = u^{1-i}.$$

A direct computation gives

Lemma 4.14. *For u in $\mathbb{C}^*$, we have $S_{+-}a_{+-}(u) = b_{+-}(u)$ and, for any $i \geq 1$,*

$$(R_{+-}a_{+-}(u))_i = \left(\frac{q}{u} + u\right) b_{+-}(u)_i - a_{+-}(u)_i$$
$$\text{and } (R_{+-}b_{+-}(u))_i = qb_{+-}(u)_i.$$

Again, in order to have the relations above working for every $i \geq 0$, we introduce new sequences. For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we chose u in $\mathbb{C}^*$ with $u^2 - (q+1)tu + q = 0$ and we set

$$\alpha_{+-}(t) = \frac{q}{q-u^2}a_{+-}(u) + \frac{u^2}{u^2-q}a_{+-}\left(\frac{q}{u}\right)$$
$$\beta_{+-}(t) = S\alpha_{+-}(t).$$

As before, these are functions of t, since they are invariant under the involution $u \mapsto \frac{q}{u}$. We still have

$$\alpha_{+-}(t)_0 = 1 \text{ and } \beta(t)_{+-}0 = 0. \tag{4.9}$$

We now get

Lemma 4.15. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we have the matrix relations:*

$$\begin{pmatrix} S_{+-}\alpha_{+-}(t) \\ S_{+-}\beta_{+-}(t) \end{pmatrix} = s_+ \begin{pmatrix} \alpha_{+-}(t) \\ \beta_{+-}(t) \end{pmatrix}$$
$$\text{and } \begin{pmatrix} R_{+-}\alpha_{+-}(t) \\ R_{+-}\beta_{+-}(t) \end{pmatrix} = r_{+-}(t) \begin{pmatrix} \alpha_{+-}(t) \\ \beta_{+-}(t) \end{pmatrix}.$$

For x in $\mathbb{C}^{(\mathbb{N})}$ and t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we set

$$\widehat{x}(t) = \begin{pmatrix} \langle x, \alpha_{+-}(t)\rangle_+ \\ \langle x, \beta_{+-}(t)\rangle_+ \end{pmatrix}.$$

Note again, that if t is real and x has real coefficients, the vector $\widehat{x}(t)$ has real coordinates and that (4.9) gives

$$\widehat{\mathbf{1}_0}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.10}$$

From Lemma 4.11 and Lemma 4.15, we directly get

Lemma 4.16. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$ and x in $\mathbb{R}^{(\mathbb{N})}$, we have*

$$\widehat{R_{+-}x}(t) = r_{+-}(t)\widehat{x}(t) \text{ and } \widehat{S_{+-}x}(t) = s_+\widehat{x}(t).$$

As in the $++$ case, we get

Lemma 4.17. *For x in $\mathbb{R}^{(\mathbb{N})}$, the function $t \mapsto \widehat{x}(t)$ is polynomial. The map $x \mapsto \widehat{x}$ induces a linear isomorphism between the space $\mathbb{R}^{(\mathbb{N})}$ and the space of polynomial functions $\mathbb{R} \to \mathbb{R}^2$.*

The proof is analogous to the one of Lemma 4.7. Note that it uses the relation $\mathcal{A} = \mathcal{C} \oplus S\mathcal{C} \oplus \mathcal{A}(R+1)$ of Corollary 3.2. In the same way, we also get

Corollary 4.18. *Let t_0 be in $\mathbb{R}$ and γ, δ be in $\mathbb{R}^{\mathbb{N}}$ with*

$$\begin{pmatrix} S_{+-}\gamma \\ S_{+-}\delta \end{pmatrix} = s_+ \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} R_{+-}\gamma \\ R_{+-}\delta \end{pmatrix} = r_{+-}(t_0) \begin{pmatrix} \gamma \\ \delta \end{pmatrix}.$$

Then we have $\gamma = \gamma_0 \alpha_{+-}(t_0)$ and $\delta = \gamma_0 \beta_{+-}(t_0)$.

We now start establishing the Plancherel formula (4.8). To this aim, we will compute the measure μ associated to the scalar product $\langle ., . \rangle_+$ on $\mathbb{R}^{(\mathbb{N})}$ through Corollary 3.5 and the new spectral transform. The novelty will be the presence of an atomic part. For u in $\mathbb{C}$, $u \notin \{-q, 0, q\}$, we set

$$c_{+-}(u)_{2i} = \frac{(q+1)u^{1-i}}{q^2 - u^2} \text{ and } c_{+-}(u)_{2i+1} = \frac{q(q+1)u^{-i}}{u^2 - q^2} \quad i \geq 0.$$

A direct computation gives

Lemma 4.19. *Let u be in $\mathbb{C}$ with $u \notin \{-q, 0, q\}$. Set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Then, we have*

$$P_{+-} c_{+-}(u) = t c_{+-}(u) + \mathbf{1}_0.$$

We may now conclude.

Proof of Proposition 4.12. The existence and uniqueness of the spectral transform $x \mapsto \widehat{x}$ and its algebraic properties follow from Lemma 4.13, (4.10), Lemma 4.16 and Lemma 4.17.

It remains to show the Plancherel formula (4.8). Let μ be the measure from Corollary 3.5 applied to the scalar product $\langle ., . \rangle_+$. By Lemma 3.3 and Lemma 4.16, we have, for t in $\mathbb{H}$,

$$\langle \mathbf{1}_0, (P_{+-} - t)^{-1} \mathbf{1}_0 \rangle_+ = \int_{[-1,1]} \frac{1}{s-t} \mathrm{d}\mu(s).$$

By Lemma 4.19, this gives, for u in $\mathbb{H}_q$ with $u^2 - (q+1)tu + q = 0$,

$$\int_{[-1,1]} \frac{1}{s-t} \mathrm{d}\mu(s) = \langle \mathbf{1}_0, c_{+-}(u) \rangle_+ = (q+1) \frac{u}{q^2 - u^2}.$$

Here comes the main difference with the ++ case: the holomorphic function $u \mapsto (q+1)\frac{u}{q^2-u^2}$ on $\mathbb{H}_q$ has no continuous extension to $\overline{\mathbb{H}}_q$. To correct this, we will remove the singularities as $u = -q$ and $u = q$. Indeed, a direct computation shows that

$$(q+1) \frac{u}{q^2 - u^2} = \frac{q+1}{q} \frac{u}{1-u^2} + \frac{q-1}{2q} \left(\frac{1}{1-t} - \frac{1}{1+t} \right),$$

hence, by Lemma 3.7 and Lemma 3.8,

$$\mathcal{P}\mu(t) = \frac{q+1}{q\pi}\Im\left(\frac{u}{1-u^2}\right) + \frac{q-1}{2q\pi}\Im\left(\frac{1}{1-t} - \frac{1}{1+t}\right)$$
$$= \frac{1}{q}\mathcal{P}\mu_q(t) + \frac{q-1}{2q}(\mathcal{P}\delta_1(t) + \mathcal{P}\delta_{-1}(t)),$$

where δ stands for Dirac measures. The conclusion follows. □

4.4. **The + twist.** We complete the description of the even model operators by describing the additional structure related to the twist operator of bipartite graphs introduced in Subsection 2.6.

Let $(x_i)_{i\geq 0}$ be a sequence of real numbers. We set

$$(U_+x)_i = (-1)^{\frac{i}{2}}x_i \qquad i \text{ even}$$
$$(U_+x)_i = (-1)^{\frac{i+1}{2}}x_i \qquad i \text{ odd.}$$

This defines a unitary operator of H_+. Besides, a direct computation yields

$$R_{++}U_+ = U_+R_{++} \qquad R_{+-}U_+ = U_+R_{+-}$$
$$S_{++}U_+ = -U_+S_{++} \qquad S_{+-}U_+ = -U_+S_{+-}.$$

By using the definitions, we get

Lemma 4.20. *Let t be in $\mathbb{C}$. We have*

$$U_+\alpha_{++}(t) = \alpha_{++}(-t) \qquad U_+\alpha_{+-}(t) = \alpha_{+-}(-t)$$
$$U_+\beta_{++}(t) = -\beta_{++}(-t) \qquad U_+\beta_{+-}(t) = -\beta_{+-}(-t).$$

5. The odd model operators

We now study operators on sequences defined in analogy with the operators from Definition 2.17. This study will be lead as in Subsections 4.1 and 4.3. Again, we split the definition of the model operators according to the eigenvalue of the $\vee$ operator, which in the odd case, can be 1 or -1.

5.1. **The $-+$ model operators.** We mimic Definition 2.17, and we consider the operators R_{-+} and S_{-+} defined on sequences $x = (x_i)_{i\geq 0}$ of real numbers by

$$(R_{-+}x)_i = qx_{i+1} \qquad i \text{ even}$$
$$(R_{-+}x)_i = x_{i-1} + (q-1)x_i \qquad i \text{ odd.}$$

and

$$\begin{aligned}
(S_{-+}x)_0 &= x_0 \\
(S_{-+}x)_i &= x_{i-1} && i \text{ even}, i \geq 2 \\
(S_{-+}x)_i &= x_{i+1} && i \text{ odd.}
\end{aligned}$$

We still have the relations $R_{-+}^2 = q + (q-1)R_{-+}$ and $S_{-+}^2 = 1$. Thus, we get a representation of the algebra $\mathcal{A}$ of Subsection 3.1 in the space of sequences of real numbers. We set $P_{-+} = \frac{1}{q+1}(R_{-+}S_{-+} + S_{-+}R_{-+} - (q-1)S_{-+})$.

Now, we change the Hilbert space and use H_- instead.

Lemma 5.1. *The operators R_{-+} and S_{-+} are bounded and self-adjoint in H_-.*

We hence have a self-adjoint representation of the algebra $\mathcal{A}$ in H_-.

Proof. One easily checks that the operators are bounded. Recall that, for any x, y in H_-, we have

$$\langle x, y\rangle_- = x_0y_0 + \sum_{i\geq 1} q^i(x_{2i-1}y_{2i-1} + x_{2i}y_{2i}) = \sum_{i\geq 0} q^i(x_{2i}y_{2i} + qx_{2i+1}y_{2i+1}).$$

We get, from the first formula,

$$\langle S_{-+}x, y\rangle_- = x_0y_0 + \sum_{i\geq 1} q^i(x_{2i}y_{2i-1} + x_{2i-1}y_{2i})$$

and, from the second formula,

$$\begin{aligned}
\langle R_{-+}x, y\rangle_- &= \sum_{i\geq 0} q^i(qx_{2i+1}y_{2i} + q(x_{2i} + (q-1)x_{2i+1})y_{2i+1}) \\
&= \sum_{i\geq 0} q^{i+1}(x_{2i+1}y_{2i} + x_{2i}y_{2i+1} + (q-1)x_{2i+1}y_{2i+1}).
\end{aligned}$$

The conclusion follows. □

As above, we define a spectral analysis for the operators R_{-+} and S_{-+}.

Proposition 5.2. *There exists a unique map $x \mapsto \widehat{x}(t)$ from $\mathbb{R}^{(\mathbb{N})}$ to the space $\mathbb{R}^2[t]$ of polynomial functions $\mathbb{R} \to \mathbb{R}^2$ such that $\widehat{\mathbf{1}_0}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and that, for any x in $\mathbb{R}^{(\mathbb{N})}$, one has*

$$\widehat{R_{-+}x}(t) = r_-\widehat{x}(t) \text{ and } \widehat{S_{-+}x}(t) = s_{-+}(t)\widehat{x}(t) \quad t \in \mathbb{R}.$$

This map is a linear isomorphism from $\mathbb{R}^{(\mathbb{N})}$ onto $\mathbb{R}^2[t]$.

The spectrum of the operator P_{-+} in H_- is the set $\mathcal{I}_q \cup \{-1\}$ and, for any x, y in $\mathbb{R}^{(\mathbb{N})}$, we have

(5.1)
$$\langle x, y\rangle_- = \frac{2}{q+1}\int_{\mathcal{I}_q} \widehat{x}(t)^* a_{-+}(t)\widehat{y}(t)\mathrm{d}\mu_q(t) + \frac{q-1}{q+1}\widehat{x}(-1)^* a_{-+}(-1)\widehat{y}(-1).$$

We start the construction of these objects.

Lemma 5.3. *We have $\mathcal{A}\mathbf{1}_0 = \mathbb{R}^{(\mathbb{N})}$.*

Proof. This now follows from the following property which is obtained by a straightforward induction: for $i \geq 0$,

$$\begin{aligned} ((S_{-+}R_{-+})^i\mathbf{1}_0)_{2i} &= 1 \\ \text{and } ((S_{-+}R_{-+})^i\mathbf{1}_0)_j &= 0, \qquad & j > 2i; \\ (R_{-+}(S_{-+}R_{-+})^i\mathbf{1}_0)_{2i+1} &= 1 \\ \text{and } (R_{-+}(S_{-+}R_{-+})^i\mathbf{1}_0)_j &= 0, \qquad & j > 2i+1. \end{aligned}$$

□

We define sequences in analogy with the previous cases. For u in $\mathbb{C}^*$, we let $a_{-+}(u)$ and $b_{-+}(u)$ be the sequences of complex numbers defined by, for $i \geq 0$,

$$\begin{aligned} a_{-+}(u)_{2i} &= -qu^{-i} \qquad & a_{-+}(u)_{2i+1} &= (u-q+1)u^{-i} \\ b_{-+}(u)_{2i} &= u^{1-i} \qquad & b_{-+}(u)_{2i+1} &= -u^{-i}. \end{aligned}$$

By construction, one has

Lemma 5.4. *For u in $\mathbb{C}^*$, we have*

$$R_{-+}b_{-+}(u) = a_{-+}(u) \text{ and } R_{-+}a_{-+}(u) = qb_{-+}(u) + (q-1)a_{-+}(u).$$

and, for any $i \geq 1$,

$$\begin{aligned} (S_{-+}b_{-+}(u))_i &= -b_{-+}(u)_i \\ \text{and } (S_{-+}a_{-+}(u))_i &= a_{-+}(u)_i + \left(u + \frac{q}{u} - q + 1\right) b_{-+}(u)_i. \end{aligned}$$

As in the even cases, to get the formulae above working for every $i \geq 0$, we introduce new sequences, defined as follows. For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we take u in $\mathbb{C}^*$ with $u^2 - (q+1)tu + q = 0$ and we set

$$\begin{aligned} \alpha_{-+}(t) &= \frac{1}{u^2-q}a_{-+}(u) + \frac{u^2}{q(q-u^2)}a_{-+}\left(\frac{q}{u}\right) \\ \beta_{-+}(t) &= \frac{1}{u^2-q}b_{-+}(u) + \frac{u^2}{q(q-u^2)}b_{-+}\left(\frac{q}{u}\right). \end{aligned}$$

As usual, one has

$$\alpha_{-+}(t)_0 = 1 \text{ and } \beta_{-+}(t)_0 = 0. \tag{5.2}$$

Besides, note that if t is real, both $\alpha_{-+}(t)$ and $\beta_{-+}(t)$ are real sequences.

We can now get the missing case in Lemma 5.4. Indeed, we have the following relation between these sequences and the operators introduced in Subsection 3.1.

Lemma 5.5. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we have the matrix relations:*

$$\begin{pmatrix} S_{-+}\alpha_{-+}(t) \\ S_{-+}\beta_{-+}(t) \end{pmatrix} = s_{-+}(t) \begin{pmatrix} \alpha_{-+}(t) \\ \beta_{-+}(t) \end{pmatrix}$$
$$\text{and } \begin{pmatrix} R_{-+}\alpha_{-+}(t) \\ R_{-+}\beta_{-+}(t) \end{pmatrix} = r_- \begin{pmatrix} \alpha_{-+}(t) \\ \beta_{-+}(t) \end{pmatrix}.$$

For x in $\mathbb{C}^{(\mathbb{N})}$ and t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we set

$$\widehat{x}(t) = \begin{pmatrix} \langle x, \alpha_{-+}(t)\rangle_- \\ \langle x, \beta_{-+}(t)\rangle_- \end{pmatrix}.$$

By (5.2) we get

$$\widehat{\mathbf{1}_0}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5.3}$$

From Lemma 5.1 and Lemma 5.5, we directly get

Lemma 5.6. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$ and x in $\mathbb{R}^{(\mathbb{N})}$, we have*

$$\widehat{R_{-+}x}(t) = r_-\widehat{x}(t) \text{ and } \widehat{S_{-+}x}(t) = s_{-+}(t)\widehat{x}(t).$$

We have again defined an isomorphism onto polynomial functions with values in $\mathbb{R}^2$.

Lemma 5.7. *For x in $\mathbb{R}^{(\mathbb{N})}$, the function $t \mapsto \widehat{x}(t)$ is polynomial. The map $x \mapsto \widehat{x}$ induces a linear isomorphism between the space $\mathbb{R}^{(\mathbb{N})}$ and the space of polynomial functions $\mathbb{R} \to \mathbb{R}^2$.*

Proof. This is proved as Lemma 4.7. Indeed, by Lemma 5.6, for any $i \geq 0$, we have

$$\widehat{P^i_{-+}\mathbf{1}_0}(t) = \begin{pmatrix} t^i \\ 0 \end{pmatrix} \text{ and } \widehat{R_{-+}P^i_{-+}\mathbf{1}_0}(t) = \begin{pmatrix} (q-1)t^i \\ t^i \end{pmatrix},$$

which shows that the spectral transform maps $\mathbb{R}^{(\mathbb{N})}$ into polynomial functions and that it is surjective. To prove injectivity, one uses the decomposition $\mathcal{A} = \mathbb{R}[P] \oplus R\mathbb{R}[P] \oplus \mathcal{A}(S-1)$ from Corollary 3.2. □

We still get a uniqueness result:

Corollary 5.8. *Let t_0 be in $\mathbb{R}$ and γ, δ be in $\mathbb{R}^{\mathbb{N}}$ with*

$$\begin{pmatrix} S_{-+}\gamma \\ S_{-+}\delta \end{pmatrix} = s_{-+}(t_0) \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \text{ and } \begin{pmatrix} R_{-+}\gamma \\ R_{-+}\delta \end{pmatrix} = r_- \begin{pmatrix} \gamma \\ \delta \end{pmatrix}.$$

Then we have $\gamma = \gamma_0 \alpha_{-+}(t_0)$ and $\delta = \gamma_0 \beta_{-+}(t_0)$.

We now focus on the Plancherel formula (5.1). As above, to compute the resolvent function $t \mapsto \langle \mathbf{1}_0, (P_{-+} - t)^{-1} \mathbf{1}_0 \rangle$, we introduce a last family of sequences in $\mathbb{C}^{(\mathbb{N})}$. For u in $\mathbb{C}$, $u \notin \{-q, 0, 1\}$, we set

$$c_{-+}(u)_{2i} = \frac{(q+1)u^{1-i}}{(q+u)(1-u)} \text{ and } c_{-+}(u)_{2i+1} = \frac{(q+1)u^{-i}}{(q+u)(1-u)} \quad i \geq 0.$$

We still have

Lemma 5.9. *Let u be in $\mathbb{C}$ with $u \notin \{-q, 0, 1\}$. Set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Then, we have*

$$P_{-+} c_{-+}(u) = t c_{-+}(u) + \mathbf{1}_0.$$

We are now ready to give the

Proof of Proposition 5.2. The existence and uniqueness of the spectral transform $x \mapsto \widehat{x}$ and its algebraic properties follow from Lemma 5.3, (5.3), Lemma 5.6 and Lemma 5.7.

To show the Plancherel formula (5.1), we let μ be the measure from Corollary 3.5 applied to the scalar product $\langle ., . \rangle_-$. By Lemma 3.3, Lemma 5.6 and Lemma 5.9, we have, for t in $\mathbb{H}$ and u in $\mathbb{H}_q$ with $u^2 - (q+1)tu + q = 0$,

$$\int_{[-1,1]} \frac{1}{s-t} \mathrm{d}\mu(s) = \frac{(q+1)u}{(q+u)(1-u)}.$$

We remove the singularity of this function at $u = -q$. A direct computation shows that

$$\frac{(q+1)u}{(q+u)(1-u)} = \frac{2u}{1-u^2} - \frac{q-1}{q+1} \frac{1}{1+t}.$$

Hence, by Lemma 3.7 and Lemma 3.8,

$$\mathcal{P}\mu(t) = \frac{2}{q+1} \mathcal{P}\mu_q(t) + \frac{q-1}{q+1} \mathcal{P}\delta_{-1}(t).$$

The conclusion follows. $\square$

5.2. **The $--$ model operators.** We describe the final set of model operators. Following Definition 2.17, we define operators R_{--} and S_{--} on sequences $x = (x_i)_{i\geq 0}$ of real numbers by

$$(R_{--}x)_i = qx_{i+1} \qquad i \text{ even}$$
$$(R_{--}x)_i = x_{i-1} + (q-1)x_i \qquad i \text{ odd.}$$

and

$$(S_{--}x)_0 = -x_0$$
$$(S_{--}x)_i = x_{i-1} \qquad i \text{ even}, i \geq 2$$
$$(S_{--}x)_i = x_{i+1} \qquad i \text{ odd.}$$

They still define a representation of the algebra $\mathcal{A}$ of Subsection 3.1 in the space of sequences of real numbers and we set $P_{--} = \frac{1}{q+1}(R_{--}S_{--} + S_{--}R_{--} - (q-1)S_{--})$. As for Lemma 5.1, we show

Lemma 5.10. *The operators R_{--} and S_{--} are bounded and self-adjoint in H_-.*

We hence have a self-adjoint representation of the algebra $\mathcal{A}$ in H_-. The spectral analysis of the operators R_{--} and S_{--} states as

Proposition 5.11. *There exists a unique map $x \mapsto \widehat{x}(t)$ from $\mathbb{R}^{(\mathbb{N})}$ to the space $\mathbb{R}^2[t]$ of polynomial functions $\mathbb{R} \to \mathbb{R}^2$ such that $\widehat{\mathbf{1}}_0(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and that, for any x in $\mathbb{R}^{(\mathbb{N})}$, one has*

$$\widehat{R_{--}x}(t) = r_-\widehat{x}(t) \text{ and } \widehat{S_{--}x}(t) = s_{--}(t)\widehat{x}(t) \quad t \in \mathbb{R}.$$

This map is a linear isomorphism from $\mathbb{R}^{(\mathbb{N})}$ onto $\mathbb{R}^2[t]$.

The spectrum of the operator P_{--} in H_- is the set $\mathcal{I}_q \cup \{1\}$ and, for any x, y in $\mathbb{R}^{(\mathbb{N})}$, we have

(5.4)
$$\langle x, y\rangle_- = \frac{2}{q+1}\int_{\mathcal{I}_q} \widehat{x}(t)^* a_{--}(t)\widehat{y}(t)\mathrm{d}\mu_q(t) + \frac{q-1}{q+1}\widehat{x}(1)^* a_{--}(1)\widehat{y}(1).$$

As for Lemma 5.3, we show

Lemma 5.12. *We have $\mathcal{A}\mathbf{1}_0 = \mathbb{R}^{(\mathbb{N})}$.*

We now set, for u in $\mathbb{C}^*$ and $i \geq 0$,

$$a_{--}(u)_{2i} = qu^{-i} \qquad a_{--}(u)_{2i+1} = (u+q-1)u^{-i}$$
$$b_{--}(u)_{2i} = u^{1-i} \qquad b_{--}(u)_{2i+1} = u^{-i}.$$

By construction, one has

Lemma 5.13. *For u in $\mathbb{C}^*$, we have*

$$R_{--}b_{--}(u) = a_{--}(u) \text{ and } R_{--}a_{--}(u) = qb_{--}(u) + (q-1)a_{--}(u).$$

and, for any $i \geq 1$,

$$(S_{--}b_{--}(u))_i = b_{--}(u)_i$$
$$\text{and } (S_{--}a_{--}(u))_i = -a_{--}(u)_i + \left(u + \frac{q}{u} + q - 1\right) b_{--}(u)_i.$$

To get the relations above working for every $i \geq 0$, for t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we take u in $\mathbb{C}^*$ with $u^2 - (q+1)tu + q = 0$ and we set

$$\alpha_{--}(t) = \frac{1}{q-u^2}a_{--}(u) + \frac{u^2}{q(u^2-q)}a_{--}\left(\frac{q}{u}\right)$$
$$\beta_{--}(t) = \frac{1}{q-u^2}b_{--}(u) + \frac{u^2}{q(u^2-q)}b_{--}\left(\frac{q}{u}\right).$$

One still has

$$\alpha_{--}(t)_0 = 1 \text{ and } \beta_{--}(t)_0 = 0. \tag{5.5}$$

We get the missing case in Lemma 5.13.

Lemma 5.14. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we have the matrix relations:*

$$\begin{pmatrix} S_{--}\alpha_{--}(t) \\ S_{--}\beta_{--}(t) \end{pmatrix} = s_{--}(t) \begin{pmatrix} \alpha_{--}(t) \\ \beta_{--}(t) \end{pmatrix}$$
$$\text{and } \begin{pmatrix} R_{--}\alpha_{-+}(t) \\ R_{--}\beta_{--}(t) \end{pmatrix} = r_{-} \begin{pmatrix} \alpha_{--}(t) \\ \beta_{--}(t) \end{pmatrix}.$$

For x in $\mathbb{C}^{(\mathbb{N})}$ and t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$, we set

$$\widehat{x}(t) = \begin{pmatrix} \langle x, \alpha_{--}(t)\rangle_{-} \\ \langle x, \beta_{--}(t)\rangle_{-} \end{pmatrix}.$$

By (5.5) we have

$$\widehat{\mathbf{1}_0}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5.6}$$

By Lemma 5.10 and Lemma 5.14, we get

Lemma 5.15. *For t in $\mathbb{C}$ with $(q+1)^2t^2 \neq 4q$ and x in $\mathbb{R}^{(\mathbb{N})}$, we have*

$$\widehat{R_{--}x}(t) = r_{-}\widehat{x}(t) \text{ and } \widehat{S_{--}x}(t) = s_{--}(t)\widehat{x}(t).$$

We still have

Lemma 5.16. *For x in $\mathbb{R}^{(\mathbb{N})}$, the function $t \mapsto \widehat{x}(t)$ is polynomial. The map $x \mapsto \widehat{x}$ induces a linear isomorphism between the space $\mathbb{R}^{(\mathbb{N})}$ and the space of polynomial functions $\mathbb{R} \to \mathbb{R}^2$.*

Corollary 5.17. *Let t_0 be in $\mathbb{R}$ and γ, δ be in $\mathbb{R}^{\mathbb{N}}$ with*

$$\begin{pmatrix} S_{--}\gamma \\ S_{--}\delta \end{pmatrix} = s_{--}(t_0) \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \text{ and } \begin{pmatrix} R_{--}\gamma \\ R_{--}\delta \end{pmatrix} = r_{-} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}.$$

Then we have $\gamma = \gamma_0 \alpha_{--}(t_0)$ and $\delta = \delta_0 \beta_{--}(t_0)$.

We now prove the Plancherel formula (5.4). To compute the resolvent function $t \mapsto \langle \mathbf{1}_0, (P_{--} - t)^{-1} \mathbf{1}_0 \rangle$, we set, for u in $\mathbb{C}$, $u \notin \{-1, 0, q\}$,

$$c_{--}(u)_{2i} = \frac{(q+1)u^{1-i}}{(q-u)(1+u)} \text{ and } c_{--}(u)_{2i+1} = \frac{(q+1)u^{-i}}{(u-q)(1+u)} \quad i \geq 0.$$

Lemma 5.18. *Let u be in $\mathbb{C}$ with $u \notin \{-1, 0, q\}$. Set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Then we have*

$$P_{--} c_{--}(u) = t c_{--}(u) + \mathbf{1}_0.$$

We conclude by the

Proof of Proposition 5.11. This is proved as the previous analogous results by using the relation, for $u \in \mathbb{C}$, $u \notin \{-1, 0, 1, q\}$ and $t = \frac{1}{q+1}(u + \frac{q}{u})$,

$$\frac{(q+1)u}{(q-u)(1+u)} = \frac{2u}{1-u^2} + \frac{q-1}{q+1} \frac{1}{1-t}.$$

□

5.3. **The $-$ twist.** As in Subsection 4.4, we describe for the odd model operators the additional structure related to the twist operator of bipartite graphs from Subsection 2.6.

Let $(x_i)_{i \geq 0}$ be a sequence of real numbers. We set

$$\begin{aligned} (U_{-}x)_i &= (-1)^{\frac{i}{2}} x_i && i \text{ even} \\ (U_{-}x)_i &= (-1)^{\frac{i-1}{2}} x_i && i \text{ odd.} \end{aligned}$$

This defines a unitary operator of H_{-}. We now get

$$R_{-+}U_{-} = U_{-}R_{-+} \quad R_{--}U_{-} = U_{-}R_{--} \quad S_{-+}U_{-} = -U_{-}S_{--}.$$

By a direct computation, we have

Lemma 5.19. *Let t be in $\mathbb{C}$. We have*

$$U_{-}\alpha_{-+}(t) = -\alpha_{--}(-t) \text{ and } U_{-}\beta_{-+}(t) = -\beta_{--}(-t).$$

6. Spectral transforms and the default map

We go back to the framework of Subsection 2.3 and Subsection 2.5. There, we defined an action of the algebra $\mathcal{A}$ on the space of ∞-pseudofunctions as well as two actions of the algebra $\mathcal{A}$ on spaces of sequences of pseudofunctions.

We will now transport the results on the model operators to these two actions and see how they are compatible with the default map.

6.1. Spectral transforms.

6.1. Spectral transforms. We set the definition of the model operators in order to mimick the one of the actions of $\mathcal{A}$ on spaces of sequences given in Definitions 2.16, 2.17, 2.19 and 2.20. Therefore, we can transport back the definition of the spectral transforms associated to the different model operators to get spectral transforms of sequences of pseudofunctions. We begin with the actions defined in Definitions 2.16 and 2.17.

Let $k \geq 0$, t be in $\mathbb{R}$ and $H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$ be in $\mathcal{H}^2_k$. If k is even, we set

$$\mathfrak{S}_t H = \begin{pmatrix} q^{-1} H_1^\vee \\ H_0^\vee - (q-1)H_0 \end{pmatrix} \tag{6.1}$$

$$\mathfrak{R}_t H = \begin{pmatrix} H_0^\vee + q^{-1}(q+1)tH_1^\vee \\ (q-1)H_1 - H_1^\vee \end{pmatrix}.$$

If k is odd, we set

$$\mathfrak{R}_t H = \begin{pmatrix} H_1^\vee \\ (q-1)H_1 + qH_0^\vee \end{pmatrix} \tag{6.2}$$

$$\mathfrak{S}_t H = \begin{pmatrix} -H_0^\vee \\ H_1^\vee + (q+1)tH_0^\vee - (q-1)H_0 \end{pmatrix}.$$

Note that, even when the operators do not depend on t, we mention t in the notation. This will avoid splitting certain statements according to the parity of k.

In both cases, a direct computation gives

Lemma 6.1. *Let $k \geq 0$, t be in $\mathbb{R}$ and H be in $\mathcal{H}^2_k$. We have*

$$\mathfrak{R}_t \mathfrak{S}_t H + \mathfrak{S}_t \mathfrak{R}_t H - (q-1)\mathfrak{S}_t H = (q+1)tH.$$

Remark 6.2. The reader may wonder why our definition of the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ is not consistent with our choices for the matrices in Subsection 3.1. Indeed, for example, when k is even, the action of $\mathfrak{R}_t$ on $\mathcal{H}^2_{k,-}$ is not the one given by the matrix $r_{+-}(t)$. The reason for this, and for the choice of normalization in the definition of the operators

$\mathfrak{R}'_t$ and $\mathfrak{S}'_t$ in (6.3) and (6.4) below, is that it will allow for a very simple formulation of Proposition 6.5 which describes the image under the spectral transform of the range of the default map of Definition 2.12.

Proposition 6.3. *Let $k \geq 0$. There exists a unique linear map $H \mapsto \widehat{H}(t)$ from $\mathcal{H}_k^{(\mathbb{N})}$ to the space $\mathcal{H}_k^2[t]$ of $\mathcal{H}_k^2$ valued polynomial functions with the following properties:*
(i) *For any H in $\mathcal{H}_k$, one has*

$$\begin{aligned}\widehat{H\mathbf{1}_0}(t) &= \begin{pmatrix} q^{-1}H^\vee \\ 0 \end{pmatrix} && \text{if } k \text{ is even} \\ &= \begin{pmatrix} 0 \\ H \end{pmatrix} && \text{if } k \text{ is odd.}\end{aligned}$$

(ii) *For any H in $\mathcal{H}_k^{(\mathbb{N})}$, one has*

$$\widehat{RH}(t) = \mathfrak{R}_t \widehat{H}(t) \text{ and } \widehat{SH}(t) = \mathfrak{S}_t \widehat{H}(t).$$

This map is a linear isomorphism from $\mathcal{H}_k^{(\mathbb{N})}$ onto $\mathcal{H}_k^2[t]$.

This result is a more or less direct consequence of Propositions 4.2, 4.12, 5.2 and 5.11, as we will soon check. Now, we would like to describe the image under the spectral transform $H \mapsto \widehat{H}$ of the range of the default map of Definition 2.12. To this aim, we first consider the actions introduced in Definitions 2.19 and 2.20.

Let $k \geq 0$, t be in $\mathbb{R}$ and $G = \begin{pmatrix} G_0 \\ G_1 \end{pmatrix}$ in $\mathcal{H}_k^2$. If k is even, we set

$$\text{(6.3)} \quad \begin{aligned}\mathfrak{S}'_t G &= \begin{pmatrix} q^{-1}(q-1)G_1 - q^{-1}G_1^\vee \\ -G_0^\vee \end{pmatrix} \\ \mathfrak{R}'_t G &= \begin{pmatrix} (q-1)G_0 - G_0^\vee + q^{-1}(q^2-1)tG_1 - q^{-1}(q+1)tG_1^\vee \\ G_1^\vee \end{pmatrix}.\end{aligned}$$

If k is odd, we set

$$\text{(6.4)} \quad \begin{aligned}\mathfrak{R}'_t G &= \begin{pmatrix} -G_1^\vee \\ (q-1)G_1 - qG_0^\vee \end{pmatrix} \\ \mathfrak{S}'_t G &= \begin{pmatrix} G_0^\vee \\ -G_1^\vee - (q+1)tG_0^\vee - (q-1)G_0 \end{pmatrix}.\end{aligned}$$

Proposition 6.4. *Let $k \geq 0$. There exists a unique linear map $G \mapsto \breve{G}(t)$ from $\mathcal{H}_k^{(\mathbb{N})}$ to the space $\mathcal{H}_k^2[t]$ of $\mathcal{H}_k^2$ valued polynomial functions with the following properties:*

(i) *For any G in $\mathcal{H}_k$, one has*

$$\widetilde{G\mathbf{1}_0}(t) = \begin{pmatrix} q^{-1}G^\vee - q^{-1}(q-1)G \\ 0 \end{pmatrix} \qquad \text{if } k \text{ is even}$$
$$= \begin{pmatrix} 0 \\ -G \end{pmatrix} \qquad \text{if } k \text{ is odd.}$$

(ii) *For any G in $\mathcal{H}_k^{(\mathbb{N})}$, one has*

$$\widetilde{R'G}(t) = \mathfrak{R}'_t\check{G}(t) \text{ and } \widetilde{SG}(t) = \mathfrak{S}'_t\check{G}(t).$$

This map is a linear isomorphism from $\mathcal{H}_k^{(\mathbb{N})}$ onto $\mathcal{H}_k^2[t]$.

Again, we will see that this result is essentially a translation of Propositions 4.2, 4.12, 5.2 and 5.11.

Let us now state a less evident result, which will be the principal objective of this Section. Its purpose is to explicitely transport the default map of Definition 2.12 by the spectral transforms. Its simple formulation justifies our choices of normalization in Propositions 6.3 and 6.4.

Proposition 6.5. *Let $k \geq 1$, G be in $\mathcal{H}_{k-1}^{(\mathbb{N})}$ and set $H = D_kG$. Fix t in $\mathbb{R}$. If k is even, we have*

$$\widehat{H}(t) = \begin{pmatrix} \check{G}_0(t)^> \\ \check{G}_1(t)^> \end{pmatrix} - \begin{pmatrix} q^{-1}(q+1)t & q^{-1} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \check{G}_0(t)^{\vee>\vee} \\ \check{G}_1(t)^{\vee>\vee} \end{pmatrix}.$$

If k is odd, we have

$$\widehat{H}(t) = \begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix} \begin{pmatrix} \check{G}_0(t)^> \\ \check{G}_1(t)^> \end{pmatrix} - \begin{pmatrix} \check{G}_0(t)^{\vee>\vee} \\ \check{G}_1(t)^{\vee>\vee} \end{pmatrix}.$$

As in Definition 2.12, we write D_k for the default map.

Remark 6.6. Note that the matrix $\begin{pmatrix} q^{-1}(q+1)t & q^{-1} \\ -1 & 0 \end{pmatrix}$ is the inverse of the matrix $\begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix}$. The latter matrix already appeared in the proof of the Ihara trace formula, Theorem 1.4.

6.2. **Simple pseudofields and duality.** The proof of Proposition 6.5 will rely on a duality argument. Thus, for $k \geq 0$, we will need to introduce the dual space of the space $\mathcal{H}_k$. As usual, we construct this space as a space of families of objects parametrized by X_1.

Let $k \geq 1$. If k is even, $k = 2\ell$, $\ell \geq 1$, as in Subsection I.5.1, we write $V_0^\ell(x)$, $x \in X$, for the space of real valued functions on the sphere $S^\ell(x)$, the sum of whose values is 0. Then, a k-simple pseudofield is a family $(s_{xy})_{(x,y)\in X_1}$ where, for any $x \sim y$ in X, s_{xy} is an element of $V_0^\ell(x)$.

If k is odd, $k = 2\ell+1$, $\ell \geq 0$, still as in Subsection I.5.1, we write $V_0^\ell(xy)$, $x \sim y \in X$, for the space of real valued functions on the sphere $S^\ell(xy)$, the sum of whose values is 0. Then, a k-simple pseudofield is a family $(s_{xy})_{(x,y)\in X_1}$ where, for any $x \sim y$ in X, s_{xy} is an element of $V_0^\ell(xy)$.

As for 1-pseudofunctions, we have a natural identification of 1-simple pseudofields with functions on X_1. Indeed, if u is a function on X_1, we can associate to u the 1-simple pseudofield s such that

$$s_{xy} = u(xy)(\mathbf{1}_y - \mathbf{1}_x), \quad x \sim y \in X.$$

We define a 0-simple pseudofield as a function on X. We do not introduce a notion of a (-1)-simple pseudofield.

The finite-dimensional vector space of Γ-invariant k-simple pseudofields is denoted by $\mathcal{S}_k$.

Recall from Subsection I.5.1 that, for $\ell \geq 1$ and x in X, we have identified the dual space of $\overline{V}^\ell(x)$ with $V_0^\ell(x)$. In the same way, for $\ell \geq 0$ and $x \sim y$ in X, we have identified the dual space of $\overline{V}^\ell(xy)$ with $V_0^\ell(xy)$. We use this convention and our usual construction to identify $\mathcal{S}_k$ with the dual space of $\mathcal{H}_k$. If $k \geq 1$, s is in $\mathcal{S}_k$ and H is in $\mathcal{H}_k$, we set

$$\langle s, H\rangle = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|}\langle s_{xy}, H_{xy}\rangle.$$

If $k = 0$, and if s in $\mathcal{S}_0$ is associated with the function u on X and H in $\mathcal{H}_0$ is associated with the function v on X, we set

$$\langle s, H\rangle = \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|}u(x)v(x).$$

6.3. **Operations on simple pseudofields.** As usual, we define natural operations on pseudofields.

Let $k \geq 0$ and s be a k-simple pseudofield. If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we set $s^\vee_{xy} = \sum_{\substack{z\sim x\\ z\neq y}} s_{xz}$. If k is odd, $k = 2\ell+1$, $\ell \geq 0$, for any $x \sim y$ in X, we set $s^\vee_{xy} = s_{yx}$. If $k = 0$, we set $s^\vee = -s$.

We get, by using Lemma I.9.11,

Lemma 6.7. *Let $k \geq 0$, s be in $\mathcal{S}_k$ and H be in $\mathcal{H}_k$. We have $\langle s^\vee, H\rangle = \langle s, H^\vee\rangle$.*

If $k \geq 0$ is even, we set

$$\mathcal{S}_{k,+} = \{s \in \mathcal{S}_k | s^\vee = qs\} \text{ and } \mathcal{S}_{k,-} = \{s \in \mathcal{S}_k | s^\vee = -s\}.$$

If k is odd, we set

$$\mathcal{S}_{k,+} = \{s \in \mathcal{S}_k | s^\vee = s\} \text{ and } \mathcal{S}_{k,-} = \{s \in \mathcal{S}_k | s^\vee = -s\}.$$

In both cases, we have $\mathcal{S}_k = \mathcal{S}_{k,+} \oplus \mathcal{S}_{k,-}$.

Finally, if $k \geq 1$, and s is a k-simple pseudofield, we define the direct restriction $s^<$ of s, which is a $(k-1)$-simple pseudofield, as follows.

If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we set

$$s^<_{xy} = I^{\ell-1,*}_{xy} s_{xy},$$

where, as in Subsection I.5.3, $I^{\ell-1,*}_{xy} : V^\ell_0(x) \to V^{\ell-1}_0(xy)$ is the adjoint of the natural operator $I^{\ell-1}_{xy} : \overline{V}^{\ell-1}(xy) \to \overline{V}^\ell(x)$.

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for any $x \sim y$ in X, we set

$$s^<_{xy} = J^{\ell,*}_{xy} s_{xy},$$

where, as in Subsection I.5.3, $J^{\ell,*}_{xy} : V^\ell_0(xy) \to V^\ell_0(x)$ is the adjoint of the natural operator $J^\ell_{xy} : \overline{V}^\ell(x) \to \overline{V}^\ell(xy)$.

Lastly, if $k = 1$ and if v is the function on X_1 such that, for any $x \sim y$ in X, $s_{xy} = v(xy)(\mathbf{1}_y - \mathbf{1}_x)$, we let $s^<$ be the 0-simple pseudofield associated with the function u defined by

$$u(x) = \sum_{y \sim x} v(xy), \quad x \in X. \tag{6.5}$$

Again, we have

Lemma 6.8. *Let $k \geq 1$, s be in $\mathcal{S}_k$ and H be in $\mathcal{H}_{k-1}$. We have $\langle s^<, H\rangle = \langle s, H^>\rangle$.*

As usual, the double commutation property holds.

Lemma 6.9. *Let $k \geq 2$ and s be a k-simple pseudofield. We have $s^{<<\vee} = s^{\vee<<}$.*

Proof. If $k \geq 3$, this directly follows from Lemma I.4.4. Let us study the case $k = 2$.

In case $k = 2$, let u and v be the functions on X which are associated with the 0-simple pseudofields $s^{<<\vee}$ and $s^{\vee<<}$. We must show that $u = v$.

For $x \sim y$ in X, s_{xy} is a function in $V^1_0(x)$ and, by the definition of the objects in Subsection I.5.3, we have $s^<_{xy}(y) = s_{xy}(y)$, hence $s^<_{xy} = s_{xy}(y)(\mathbf{1}_y - \mathbf{1}_x)$. Therefore, by (6.5), for x in X,

$$u(x) = -\sum_{y \sim x} s_{xy}(y).$$

Besides, for $x \sim y$ in X, we have, $s^{\vee}_{xy}(y) = \sum_{\substack{z\sim x\\ z\neq y}} s_{xz}(y)$. Again by (6.5), this gives, for x in X,

$$v(x) = \sum_{y\sim x}\sum_{\substack{z\sim x\\ z\neq y}} s_{xz}(y) = \sum_{z\sim x}\sum_{\substack{y\sim x\\ y\neq z}} s_{xz}(y) = -\sum_{z\sim x} s_{xz}(z),$$

where the latter follows from the fact that s_{xz} belongs to $V^1_0(x)$ for $z \sim x$. We get $u = v$ as required. □

Contrarily to what happens in the quadratic case, direct restriction may be injective on eigenspaces of the $\vee$ operator.

Lemma 6.10. *Let $k \geq 2$ and s be a k-simple pseudofield with $s^< = 0$. If k is even and $s^{\vee} = qs$, then $s = 0$. If k is odd and $s^{\vee} = s$ or $s^{\vee} = -s$, then $s = 0$.*

Proof. Assume k is even, $k = 2\ell$, $\ell \geq 1$. Let s be a k-simple pseudofield with $s^{\vee} = qs$. For $y, z \sim x$, we have $s_{xy} = s_{xz}$. We write s_x for this element of $V^{\ell}_0(x)$ that only depends on x. If $s^< = 0$, for any $y \sim x$, when seen as a linear functional on $\overline{V}^{\ell}(x)$, s_x is 0 on the space $I^{\ell-1}_{xy}\overline{V}^{\ell-1}(xy)$. By Proposition I.4.5, as y runs among the neighbours of x, these spaces span $\overline{V}^{\ell}(x)$, hence $s_x = 0$ as required.

If k is odd, we proceed in the same way by using Proposition I.4.6 □

6.4. **Duality on spaces of sequences.** Now that, for $k \geq 0$, we have introduced the dual space of the space of Γ-invariant k-pseudofunctions, we can define a duality at the level of spaces of sequences. We adopt the same convention as the one used for defining Hilbert spaces of sequences of real numbers adapted to the study of the model operators in Subsection 3.4.

Thus, for $k \geq 0$, if $s = (s_i)_{i\geq 0}$ is in $\mathcal{S}^{\mathbb{N}}_k$ and $H = (H_i)_{i\geq 0}$ is in $\mathcal{H}^{(\mathbb{N})}_k$, we set

$$\begin{aligned}\langle s, H\rangle &= \sum_{i\geq 0} q^i(\langle s_{2i}, H_{2i}\rangle + \langle s_{2i+1}, H_{2i+1}\rangle) && \text{if } k \text{ is even}\\ &= \sum_{i\geq 0} q^i(\langle s_{2i}, H_{2i}\rangle + q\langle s_{2i+1}, H_{2i+1}\rangle) && \text{if } k \text{ is odd.}\end{aligned}$$

Besides, we use the same symbols as in Definitions 2.16, 2.17, 2.19 and 2.20 for the analogous operations on sequences of simple pseudofields.

Thus, for $k \geq 0$ and s in $\mathcal{S}_k$, if k is even, we set

$$\begin{aligned}
(Rs)_0 &= s_0^\vee \\
(R's)_0 &= (q-1)s_0 - s_0^\vee \\
(Rs)_i = (R's)_i &= s_{i-1} + (q-1)s_i && i \text{ even}, i \geq 2 \\
(Rs)_i = (R's)_i &= qs_{i+1} && i \text{ odd.} \\
(Ss)_i &= s_{i+1} && i \text{ even} \\
(Ss)_i &= s_{i-1} && i \text{ odd.}
\end{aligned}$$

If k is odd, we set

$$\begin{aligned}
(Rs)_i &= qs_{i+1} && i \text{ even} \\
(Rs)_i &= s_{i-1} + (q-1)s_i && i \text{ odd.} \\
(Ss)_0 &= s_0^\vee \\
(S's)_0 &= -s_0^\vee \\
(Ss)_i = (S's)_i &= s_{i-1} && i \text{ even}, i \geq 2 \\
(Ss)_i = (S's)_i &= s_{i+1} && i \text{ odd.}
\end{aligned}$$

As in Lemmas 4.1, 4.11, 5.1 and 5.10, we get

Lemma 6.11. *Let $k \geq 0$, s be in $\mathcal{S}_k^{\mathbb{N}}$ and H be in $\mathcal{H}_k^{(\mathbb{N})}$. We have*

$$\langle Rs, H \rangle = \langle s, RH \rangle \text{ and } \langle Ss, H \rangle = \langle s, SH \rangle.$$

If k is even, we have $\langle R's, H \rangle = \langle s, R'H \rangle$ and if k is odd, we have $\langle S's, H \rangle = \langle s, S'H \rangle$.

If $s = (s_i)_{i \geq 0}$ is a sequence of simple pseudofields, we will write $s^\vee$ for the sequence $(s_i^\vee)_{i \geq 0}$. All the operators that we have defined on spaces of sequences commute with this operation.

Lemma 6.12. *Let $k \geq 0$ and $s = (s_i)_{i \geq 0}$ be a sequence of k-simple pseudofields. We have $R(s^\vee) = (Rs)^\vee$ and $S(s^\vee) = (Ss)^\vee$. If k is even, we have $R'(s^\vee) = (R's)^\vee$ and if k is odd, we have $S'(s^\vee) = (S's)^\vee$.*

6.5. **Construction of the spectral transforms: the even case.** We will build the spectral transform of sequences of pseudofunctions by analogy with the case of sequences of real numbers. We will use freely the notation of Subsections 4.1, 4.3, 5.1 and 5.2.

Fix t in $\mathbb{R}$. Let $k \geq 0$ be even. As in Subsection 4.1, if s is in $\mathcal{S}_{k,+}$, we set $\alpha_s(t)$ and $\beta_s(t)$ to be the sequences in $\mathcal{S}_k^{\mathbb{N}}$ defined by, for $i \geq 0$,

$$\alpha_s(t)_i = \alpha_{++}(t)_i s \text{ and } \beta_s(t)_i = \beta_{++}(t)_i s.$$

In the same way, as in Subsection 4.3, if s is in $\mathcal{S}_{k,-}$, we set $\alpha_s(t)$ and $\beta_s(t)$ to be the sequences in $\mathcal{S}_k^{\mathbb{N}}$ defined by, for $i \geq 0$,

$$\alpha_s(t)_i = \alpha_{+-}(t)_i s \text{ and } \beta_s(t)_i = \beta_{+-}(t)_i s.$$

Finally, if s is in $\mathcal{S}_k$ and $s = s_+ + s_-$ with s_+ in $\mathcal{S}_{k,+}$ and s_- in $\mathcal{S}_{k,-}$, we set $\alpha_s(t) = \alpha_{s_+}(t) + \alpha_{s_-}(t)$ and $\beta_s(t) = \beta_{s_+}(t) + \beta_{s_-}(t)$. Note that, by Lemma 4.7 and Lemma 4.17, the coordinates of these sequences are polynomial functions of t.

We get

Lemma 6.13. *Let $k \geq 0$ be even and s be in $\mathcal{S}_k$. For any t in $\mathbb{R}$, we have the matrix relations*

(6.6)

$$\begin{pmatrix} S\alpha_s(t) \\ S\beta_s(t) \end{pmatrix} = \begin{pmatrix} \beta_s(t) \\ \alpha_s(t) \end{pmatrix} \text{ and } \begin{pmatrix} R\alpha_s(t) \\ R\beta_s(t) \end{pmatrix} = \begin{pmatrix} \alpha_{s^\vee}(t) + (q+1)t\beta_s(t) \\ \beta_{(q-1)s - s^\vee}(t) \end{pmatrix}.$$

Converserly, assume γ and δ are in $\mathcal{S}_k^{\mathbb{N}}$ and we have

$$\begin{pmatrix} S\gamma \\ S\delta \end{pmatrix} = \begin{pmatrix} \delta \\ \gamma \end{pmatrix} \text{ and } \begin{pmatrix} R\gamma \\ R\delta \end{pmatrix} = \begin{pmatrix} \gamma^\vee + (q+1)t\delta \\ (q-1)\delta - \delta^\vee \end{pmatrix}. \tag{6.7}$$

Then, we have $\gamma = \alpha_s(t)$ and $\delta = \beta_s(t)$ where $s = \gamma_0$.

Proof. This is a direct translation of the analogous results for the model operators. First, we show (6.6). By construction of the objects, it suffices to prove the result when s is an eigenvector of the $\vee$ operator. Then, if $s^\vee = qs$, we have $(q-1)s - s^\vee = -s$, hence

$$\begin{pmatrix} \alpha_{s^\vee}(t) + (q+1)t\beta_s(t) \\ \beta_{(q-1)s - s^\vee}(t) \end{pmatrix} = \begin{pmatrix} q\alpha_s(t) + (q+1)t\beta_s(t) \\ -\beta_s(t) \end{pmatrix} = r_{++}(t) \begin{pmatrix} \alpha_s(t) \\ \beta_s(t) \end{pmatrix}$$

and (6.6) follows from Lemma 4.6. In the same way, if $s^\vee = -s$, we have $(q-1)s - s^\vee = qs$, hence

$$\begin{pmatrix} \alpha_{s^\vee}(t) + (q+1)t\beta_s(t) \\ \beta_{(q-1)s - s^\vee}(t) \end{pmatrix} = r_{+-}(t) \begin{pmatrix} \alpha_s(t) \\ \beta_s(t) \end{pmatrix}$$

and (6.6) follows from Lemma 4.16.

Now, we take γ and δ in $\mathcal{S}_k^{\mathbb{N}}$ such (6.7) holds. We write $\gamma = \gamma_+ + \gamma_-$ and $\delta = \delta_+ + \delta_-$ where $\gamma_+^\vee = q\gamma_+$, $\gamma_-^\vee = -\gamma_-$, $\delta_+^\vee = q\delta_+$ and $\delta_-^\vee = -\delta_-$. By using Lemma 6.12, we get

$$\begin{pmatrix} S\gamma_+ \\ S\delta_+ \end{pmatrix} = s_+ \begin{pmatrix} \gamma_+ \\ \delta_+ \end{pmatrix} \text{ and } \begin{pmatrix} R\gamma_+ \\ R\delta_+ \end{pmatrix} = r_{++}(t) \begin{pmatrix} \gamma_+ \\ \delta_+ \end{pmatrix}.$$

From Lemma 4.8, we get $\gamma_+ = \alpha_{\gamma_{+,0}}(t)$ and $\delta_+ = \beta_{\gamma_{+,0}}(t)$. In the same way, Lemma 4.18 ensures that $\gamma_- = \alpha_{\gamma_{-,0}}(t)$ and $\delta_- = \beta_{\gamma_{-,0}}(t)$. The conclusion follows. □

Still by analogy with the case of the model operators, but with a change of normalization that will be justified later, for H in $\mathcal{H}_k^{(\mathbb{N})}$ and t in $\mathbb{R}$, we define $\widehat{H}(t) = \begin{pmatrix} \widehat{H}_0(t) \\ \widehat{H}_1(t) \end{pmatrix}$ to be the unique element of $\mathcal{H}_k^2$ such that, for any s in $\mathcal{S}_k$, one has

$$\begin{pmatrix} \langle s, \widehat{H}_0(t)^\vee - (q-1)\widehat{H}_0(t)\rangle \\ \langle s, \widehat{H}_1(t)\rangle \end{pmatrix} = \begin{pmatrix} \langle \alpha_s(t), H\rangle \\ \langle \beta_s(t), H\rangle \end{pmatrix}. \tag{6.8}$$

Proof of Proposition 6.3 in case k is even. Take H in $\mathcal{H}_k$. By the constructions in Subsections 4.1 and 4.3, for s in $\mathcal{S}_k$, we have $\alpha_s(t)_0 = s$ and $\beta_s(t)_0 = 0$. Thus, for t in $\mathbb{R}$, we get, from (6.8),

$$\begin{pmatrix} \langle s, \widehat{(H\mathbf{1}_0)}_0(t)^\vee - (q-1)\widehat{(H\mathbf{1}_0)}_0(t)\rangle \\ \langle s, \widehat{(H\mathbf{1}_0)}_1(t)\rangle \end{pmatrix} = \begin{pmatrix} \langle s, H\rangle \\ 0 \end{pmatrix},$$

which gives $\widehat{H\mathbf{1}_0}(t) = \begin{pmatrix} q^{-1}H^\vee \\ 0 \end{pmatrix}$ as required.

Let us now show that the equivariance properties actually hold. Fix t in $\mathbb{R}$ and H in $\mathcal{H}_k^{(\mathbb{N})}$. Let $J = \begin{pmatrix} J_0 \\ J_1 \end{pmatrix}$ and $K = \begin{pmatrix} K_0 \\ K_1 \end{pmatrix}$ be in $\mathcal{H}_k^2$ such that, for s in $\mathcal{S}_k$,

$$\begin{pmatrix} \langle s, J_0\rangle \\ \langle s, J_1\rangle \end{pmatrix} = \begin{pmatrix} \langle \alpha_s(t), H\rangle \\ \langle \beta_s(t), H\rangle \end{pmatrix} \text{ and } \begin{pmatrix} \langle s, K_0\rangle \\ \langle s, K_1\rangle \end{pmatrix} = \begin{pmatrix} \langle \alpha_s(t), RH\rangle \\ \langle \beta_s(t), RH\rangle \end{pmatrix},$$

so that $\widehat{H}(t) = \begin{pmatrix} q^{-1}J_0^\vee \\ J_1 \end{pmatrix}$ and $\widehat{RH}(t) = \begin{pmatrix} q^{-1}K_0^\vee \\ K_1 \end{pmatrix}$. From Lemma 6.11 and Lemma 6.13,

$$\begin{aligned} \begin{pmatrix} \langle s, K_0\rangle \\ \langle s, K_1\rangle \end{pmatrix} &= \begin{pmatrix} \langle R\alpha_s(t), H\rangle \\ \langle R\beta_s(t), H\rangle \end{pmatrix} \\ &= \begin{pmatrix} 0 & (q+1)t \\ 0 & (q-1) \end{pmatrix} \begin{pmatrix} \langle \alpha_s(t), H\rangle \\ \langle \beta_s(t), H\rangle \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \langle \alpha_{s^\vee}(t), H\rangle \\ \langle \beta_{s^\vee}(t), H\rangle \end{pmatrix} \\ &= \begin{pmatrix} 0 & (q+1)t \\ 0 & (q-1) \end{pmatrix} \begin{pmatrix} \langle s, J_0\rangle \\ \langle s, J_1\rangle \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \langle s^\vee, J_0\rangle \\ \langle s^\vee, J_1\rangle \end{pmatrix}. \end{aligned}$$

We get $K_0 = J_0^\vee + (q+1)tJ_1$ and $K_1 = (q-1)J_1 - J_1^\vee$, hence

$$\widehat{RH}(t) = \begin{pmatrix} q^{-1}K_0^\vee \\ K_1 \end{pmatrix} = \begin{pmatrix} J_0 + q^{-1}(q-1)J_0^\vee + q^{-1}(q+1)tJ_1^\vee \\ (q-1)J_1 - J_1^\vee \end{pmatrix}.$$

As $J_1 = \widehat{H}_1(t)$, we get $\widehat{RH}_1(t) = (q-1)\widehat{H}_1(t) - \widehat{H}_1(t)^\vee$. As

$$J_0 = \widehat{H}_0(t)^\vee - (q-1)\widehat{H}_0(t),$$

a straightforward computation gives

$$\widehat{RH}_0(t) = \widehat{H}_0(t)^\vee + q^{-1}(q+1)t\widehat{H}_1(t)$$

and the equivariance property for the operator R holds.

In the same way, we let $L = \begin{pmatrix} L_0 \\ L_1 \end{pmatrix}$ be in $\mathcal{H}_k^2$ such that, for s in $\mathcal{S}_k$, $\begin{pmatrix} \langle s, L_0 \rangle \\ \langle s, L_1 \rangle \end{pmatrix} = \begin{pmatrix} \langle \alpha_s(t), SH \rangle \\ \langle \beta_s(t), SH \rangle \end{pmatrix}$, so that $\widehat{SH}(t) = \begin{pmatrix} q^{-1}L_0^\vee \\ L_1 \end{pmatrix}$. Lemma 6.11 and Lemma 6.13 now give,

$$\begin{pmatrix} \langle s, L_0 \rangle \\ \langle s, L_1 \rangle \end{pmatrix} = \begin{pmatrix} \langle S\alpha_s(t), H \rangle \\ \langle S\beta_s(t), H \rangle \end{pmatrix} = \begin{pmatrix} \langle \beta_s(t), H \rangle \\ \langle \alpha_s(t), H \rangle \end{pmatrix} = \begin{pmatrix} \langle s, J_1 \rangle \\ \langle s, J_0 \rangle \end{pmatrix}.$$

Reasoning as above, we get $\widehat{SH}_1(t) = J_0 = \widehat{H}_0(t)^\vee - (q-1)\widehat{H}_0(t)$ and $\widehat{SH}_0(t) = q^{-1}J_1^\vee = q^{-1}\widehat{H}_1(t)^\vee$.

The uniqueness statement follows from the analogous ones in Proposition 4.2 and Proposition 4.12. □

To construct the $\check{\cdot}$ transform, we do the same constructions, but the roles of $\mathcal{S}_{k,+}$ and $\mathcal{S}_{k,-}$ are exchanged. Thus, if s is in $\mathcal{S}_{k,+}$, we set, for $i \geq 0$,

$$\alpha'_s(t)_i = \alpha_{+-}(t)_i s \text{ and } \beta'_s(t)_i = \beta_{+-}(t)_i s.$$

If s is in $\mathcal{S}_{k,-}$, we set, for $i \geq 0$,

$$\alpha'_s(t)_i = \alpha_{++}(t)_i s \text{ and } \beta'_s(t)_i = \beta_{++}(t)_i s.$$

As above, if s is in $\mathcal{S}_k$ and $s = s_+ + s_-$ with s_+ in $\mathcal{S}_{k,+}$ and s_- in $\mathcal{S}_{k,-}$, we set $\alpha'_s(t) = \alpha'_{s_+}(t) + \alpha'_{s_-}(t)$ and $\beta'_s(t) = \beta_{s_+}(t) + \beta_{s_-}(t)$. By Lemma 4.7 and Lemma 4.17, the coordinates of these sequences are polynomial functions of t.

We now get

Lemma 6.14. *Let $k \geq 0$ be even and s be in $\mathcal{S}_k$. For any t in $\mathbb{R}$, we have the matrix relations*

$$\begin{pmatrix} S\alpha'_s(t) \\ S\beta'_s(t) \end{pmatrix} = \begin{pmatrix} \beta'_s(t) \\ \alpha'_s(t) \end{pmatrix}$$

and

$$\begin{pmatrix} R'\alpha'_s(t) \\ R'\beta'_s(t) \end{pmatrix} = \begin{pmatrix} \alpha'_{(q-1)s-s^\vee}(t) + (q+1)t\beta'_s(t) \\ \beta'_{s^\vee}(t) \end{pmatrix}.$$

Converserly, assume γ and δ are in $\mathcal{S}_k^{\mathbb{N}}$ and we have

$$\begin{pmatrix} S\gamma \\ S\delta \end{pmatrix} = \begin{pmatrix} \delta \\ \gamma \end{pmatrix} \text{ and } \begin{pmatrix} R'\gamma \\ R'\delta \end{pmatrix} = \begin{pmatrix} (q-1)\gamma - \gamma^\vee + (q+1)t\delta \\ \delta^\vee \end{pmatrix}.$$

Then, we have $\gamma = \alpha'_s(t)$ and $\delta = \beta'_s(t)$ where $s = \gamma_0$.

Now, for G in $\mathcal{H}_k^{(\mathbb{N})}$ and t in $\mathbb{R}$, we define $\breve{G}(t) = \begin{pmatrix} \breve{G}_0(t) \\ \breve{G}_1(t) \end{pmatrix}$ to be the unique element of $\mathcal{H}_k^2$ such that, for any s in $\mathcal{S}_k$, one has

$$\begin{pmatrix} \langle s, \breve{G}_0(t)^\vee \rangle \\ \langle s, -\breve{G}_1(t) \rangle \end{pmatrix} = \begin{pmatrix} \langle \alpha'_s(t), G \rangle \\ \langle \beta'_s(t), G \rangle \end{pmatrix}. \tag{6.9}$$

Proof of Proposition 6.4 in case k is even. This can be obtained from Lemma 6.14 and (6.9) as in the proof of Proposition 6.3. □

6.6. **Construction of the spectral transforms: the odd case.** We proceed to analogous constructions in the odd case. Let $k \geq 0$ be odd.

For t in $\mathbb{R}$, as in Subsection 5.1, if s is in $\mathcal{S}_{k,+}$, we set, for $i \geq 0$,

$$\alpha_s(t)_i = \alpha_{-+}(t)_i s \text{ and } \beta_s(t)_i = \beta_{-+}(t)_i s.$$

In the same way, as in Subsection 5.2, if s is in $\mathcal{S}_{k,-}$, we set, for $i \geq 0$,

$$\alpha_s(t)_i = \alpha_{--}(t)_i s \text{ and } \beta_s(t)_i = \beta_{--}(t)_i s.$$

And, for s in $\mathcal{S}_k$, $s = s_+ + s_-$ with s_+ in $\mathcal{S}_{k,+}$ and s_- in $\mathcal{S}_{k,-}$, we set $\alpha_s(t) = \alpha_{s_+}(t) + \alpha_{s_-}(t)$ and $\beta_s(t) = \beta_{s_+}(t) + \beta_{s_-}(t)$.

From Lemma 5.6, Lemma 5.8, Lemma 5.15 and Lemma 5.17, we get

Lemma 6.15. *Let $k \geq 0$ be odd and s be in $\mathcal{S}_k$. For any t in $\mathbb{R}$, we have the matrix relations*

$$\begin{pmatrix} S\alpha_s(t) \\ S\beta_s(t) \end{pmatrix} = \begin{pmatrix} \alpha_{s^\vee}(t) + \beta_{(q+1)ts-(q-1)s^\vee}(t) \\ -\beta_{s^\vee}(t) \end{pmatrix}$$
$$\text{and } \begin{pmatrix} R\alpha_s(t) \\ R\beta_s(t) \end{pmatrix} = \begin{pmatrix} (q-1)\alpha_s(t) + q\beta_s(t) \\ \alpha_s(t) \end{pmatrix}.$$

Converserly, assume γ and δ are in $\mathcal{S}_k^{\mathbb{N}}$ and we have

$$\begin{pmatrix} S\gamma \\ S\delta \end{pmatrix} = \begin{pmatrix} \gamma^\vee + (q+1)t\delta - (q-1)\delta^\vee \\ -\delta^\vee \end{pmatrix}$$
$$\text{and } \begin{pmatrix} R\gamma \\ R\delta \end{pmatrix} = \begin{pmatrix} (q-1)\gamma + q\delta \\ \gamma \end{pmatrix}.$$

Then, we have $\gamma = \alpha_s(t)$ and $\delta = \beta_s(t)$ where $s = \gamma_0$.

For H in $\mathcal{H}_k^{(\mathbb{N})}$ and t in $\mathbb{R}$, we define $\widehat{H}(t) = \begin{pmatrix} \widehat{H}_0(t) \\ \widehat{H}_1(t) \end{pmatrix}$ to be the unique element of $\mathcal{H}_k^2$ such that, for any s in $\mathcal{S}_k$, one has

$$\begin{pmatrix} \langle s, \widehat{H}_1(t) \rangle \\ \langle s, \widehat{H}_0(t)^\vee \rangle \end{pmatrix} = \begin{pmatrix} \langle \alpha_s(t), H \rangle \\ \langle \beta_s(t), H \rangle \end{pmatrix}. \tag{6.10}$$

Proof of Proposition 6.3 in case k is odd. This follows from Lemma 6.15 and (6.10). □

Again, to construct the $\check{}$ transform, we exchange the roles of $\mathcal{S}_{k,+}$ and $\mathcal{S}_{k,-}$. If s is in $\mathcal{S}_{k,+}$, we set, for $i \geq 0$,

$$\alpha'_s(t)_i = \alpha_{--}(t)_i s \text{ and } \beta'_s(t)_i = \beta_{--}(t)_i s.$$

If s is in $\mathcal{S}_{k,-}$, we set, for $i \geq 0$,

$$\alpha'_s(t)_i = \alpha_{-+}(t)_i s \text{ and } \beta'_s(t)_i = \beta_{-+}(t)_i s.$$

If s is in $\mathcal{S}_k$ and $s = s_+ + s_-$ with s_+ in $\mathcal{S}_{k,+}$ and s_- in $\mathcal{S}_{k,-}$, we still set $\alpha'_s(t) = \alpha'_{s_+}(t) + \alpha'_{s_-}(t)$ and $\beta'_s(t) = \beta_{s_+}(t) + \beta_{s_-}(t)$.

We get again

Lemma 6.16. *Let $k \geq 0$ be odd and s be in $\mathcal{S}_k$. For any t in $\mathbb{R}$, we have the matrix relations*

$$\begin{pmatrix} S'\alpha'_s(t) \\ S'\beta'_s(t) \end{pmatrix} = \begin{pmatrix} -\alpha'_{s^\vee}(t) + \beta'_{(q+1)ts+(q-1)s^\vee}(t) \\ \beta'_{s^\vee}(t) \end{pmatrix}$$

$$\begin{pmatrix} R\alpha'_s(t) \\ R\beta'_s(t) \end{pmatrix} = \begin{pmatrix} (q-1)\alpha'_s(t) + q\beta'_s(t) \\ \alpha'_s(t) \end{pmatrix}.$$

Converserly, assume γ and δ are in $\mathcal{S}_k^{\mathbb{N}}$ and we have

$$\begin{pmatrix} S'\gamma \\ S'\delta \end{pmatrix} = \begin{pmatrix} -\gamma^\vee + (q+1)t\delta + (q-1)\delta^\vee \\ \delta^\vee \end{pmatrix}$$

$$\text{and } \begin{pmatrix} R\gamma \\ R\delta \end{pmatrix} = \begin{pmatrix} (q-1)\gamma + q\delta \\ \gamma \end{pmatrix}.$$

Then, we have $\gamma = \alpha'_s(t)$ and $\delta = \beta'_s(t)$ where $s = \gamma_0$.

Finally, for G in $\mathcal{H}_k^{(\mathbb{N})}$ and t in $\mathbb{R}$, we define $\check{G}(t) = \begin{pmatrix} \check{G}_0(t) \\ \check{G}_1(t) \end{pmatrix}$ to be the unique element of $\mathcal{H}_k^2$ such that, for any s in $\mathcal{S}_k$, one has

$$\begin{pmatrix} \langle s, -\check{G}_1(t) \rangle \\ \langle s, \check{G}_0(t)^\vee \rangle \end{pmatrix} = \begin{pmatrix} \langle \alpha'_s(t), G \rangle \\ \langle \beta'_s(t), G \rangle \end{pmatrix}. \tag{6.11}$$

Proof of Proposition 6.4 in case k is odd. This is now a consequence of Lemma 6.16 and (6.11). □

6.7. **Twist and spectral transform.** In this Subsection, we assume Γ to be bipartite. Then, we can use the language of Subsection 2.6. We will describe the action of the twist operator on the spectral transform.

For $k \geq 0$, we let $s \mapsto s^{\wr}$ be the adjoint operations to the twist operator on k-simple pseudofields. More concretely, if $k \geq 1$ and s is a k-simple pseudofield, for $x \sim y$ in X, we set

$$s^{\wr}_{xy} = \chi(x) s_{xy}.$$

If $k = 0$ and s is the 0-simple pseudofield associated to the function u on X, we set $s^{\wr}$ to be 0-simple pseudofield associated to the function χu.

Still by duality with Subsection 2.6, we set, for $k \geq 0$ and s in $\mathcal{S}_k^{\mathbb{N}}$,

$$\begin{aligned} (Us)_i &= (-1)^{\frac{i}{2}} s^{\wr}_i && i \text{ even} \\ (Us)_i &= (-1)^{\frac{i+1}{2}} s^{\wr}_i && i \text{ odd}, k \text{ even} \\ (Us)_i &= (-1)^{\frac{i-1}{2}} s^{\wr}_i && i \text{ odd}, k \text{ odd}. \end{aligned}$$

By using Lemma 4.20 and Lemma 5.19, we get

Lemma 6.17. *Let $k \geq 0$ and s be in $\mathcal{S}_k$. For t in $\mathbb{C}$, we have*

$$\begin{aligned} U\beta_s(t) &= -\beta_{s^{\wr}}(-t) && \\ U\alpha_s(t) &= \alpha_{s^{\wr}}(-t) && \text{if } k \text{ is even} \\ U\alpha_s(t) &= -\alpha_{s^{\wr}}(-t) && \text{if } k \text{ is odd.} \end{aligned}$$

Thanks to Lemma 2.22 and (6.8) and (6.10), this yields

Lemma 6.18. *Let $k \geq 0$ and H be in $\mathcal{H}_k^{(\mathbb{N})}$. For t in $\mathbb{C}$, we have*

$$\widehat{UH}(t) = \begin{pmatrix} \widehat{H}_0(t)^{\wr} \\ -\widehat{H}_1(t)^{\wr} \end{pmatrix}.$$

6.8. **The adjoint of the default map.** Recall that, for $k \geq 0$, the default map $D_k : \mathcal{H}_{k-1}^{(\mathbb{N})} \to \mathcal{H}_k^{(\mathbb{N})}$ was defined in Definition 2.12. By Corollary 2.14, the range of the default map is the null space of the polyextension map of Definition 2.11. In Subsection 6.4 above, we have identified the dual space of $\mathcal{H}_k^{(\mathbb{N})}$ with $\mathcal{S}_k^{\mathbb{N}}$. Therefore, the default map gives rise to an adjoint linear map $D_k^* : \mathcal{S}_k^{\mathbb{N}} \to \mathcal{S}_{k-1}^{\mathbb{N}}$. We can give a direct description of D_k^*.

Lemma 6.19. *Let $k \geq 1$ and s be in $\mathcal{S}_k^{\mathbb{N}}$. If k is even, we have*

$$\begin{aligned} (D_k^* s)_0 &= s_0^{<\vee} - s_1^{<} && \\ (D_k^* s)_i &= \frac{1}{q} s_{i-1}^{\vee<\vee} - s_{i+1}^{<}, && i \geq 1. \end{aligned}$$

If k is odd, we have

$$(D_k^* s)_0 = s_0^{<\vee} - q s_1^{<}$$
$$(D_k^* s)_i = s_{i-1}^{\vee<\vee} - q s_{i+1}^{<}, \qquad i \geq 1.$$

Proof. This is a direct computation. For example, we explicit the even case. If k is even, for G in $\mathcal{H}_{k-1}^{(\mathbb{N})}$, we have by definition

$$\langle D_k^* s, H\rangle = \langle s, D_k H\rangle = \langle s_0, G_1^{\vee>\vee} + G_0^{\vee>}\rangle + \sum_{i=1}^{\infty} q^{i-1}(\langle s_{2i-1}, G_{2i}^{\vee>\vee} - G_{2i-2}^{>}\rangle + q\langle s_{2i}, G_{2i+1}^{\vee>\vee} - G_{2i-1}^{>}\rangle).$$

By using Lemma 6.7 and Lemma 6.8, we get

$$\langle D_k^* s, H\rangle = \langle s_0^{<\vee} - s_1^{<}, G_0\rangle + \sum_{i=1}^{\infty}\langle q^{i-1} s_{2i-2}^{\vee>\vee} - q^i s_{2i}^{<}, G_{2i-1}\rangle + \langle q^{i-1} s_{2i-1}^{\vee>\vee} - q^i s_{2i+1}^{<}, G_{2i}\rangle.$$

The conclusion follows. The odd case is analogous. □

6.9. **The spectral default.** We now prove Proposition 6.5 by using the precise definition of the spectral transforms given in Subsection 6.5 and Subsection 6.6. The main point of the proof is the following delicate computation.

Lemma 6.20. *Let $k \geq 1$ be even, s be in $\mathcal{S}_k^{\mathbb{N}}$ and t be in $\mathbb{R}$. We have*

$$D_k^* \beta_s(t) = -\alpha'_{s^<}(t) + \beta'_{s^{\vee<}}(t). \tag{6.12}$$

If k is even, we have

$$D_k^* \alpha_s(t) = \alpha'_{s^{<\vee}}(t) + \beta'_{s^{\vee<\vee} - (q-1)s^{<\vee} - (q+1)ts^<}(t). \tag{6.13}$$

If k is odd, we have

$$D_k^* \alpha_s(t) = \alpha'_{s^{<\vee} - (q-1)s^<}(t) + \beta'_{s^{\vee<\vee} - (q+1)ts^<}(t). \tag{6.14}$$

One way to prove this is to use the explicit definitions of the objects given in Subsections 4.1, 4.3, 5.1, 5.2, 6.5 and 6.6 and the formulas for the operator D_k^* given in Lemma 6.19. We will instead use an argument based on the uniqueness property in Lemmas 6.14 and 6.16.

We will split the proof according to the parity of k.

Proof of Lemma 6.20 in case k is even. Looking at (6.12) and (6.13) suggests to introduce $\delta = D_k^* \beta_s(t) + (D_k^* \alpha_s(t))^\vee$ and $\gamma = R\delta$. As $R^2 = q + (q-1)R$, we get the second relation in the uniqueness part of

Lemma 6.16. We will conclude by proving that the first relation also holds. By Lemma 2.21, Lemma 6.12 and Lemma 6.13, we have

$$S'\delta = D_k^* S\beta_s(t) + (D_k^* S\alpha_s(t))^\vee = D_k^*\alpha_s(t) + (D_k^*\beta_s(t))^\vee = \delta^\vee.$$

It remains to compute $S'\gamma$. Again by Lemma 2.21, Lemma 6.12 and Lemma 6.13, we have

$$\begin{aligned}\gamma = R\delta &= D_k^* R\beta_s(t) + (D_k^* R\alpha_s(t))^\vee \\ &= D_k^*\beta_{(q-1)s-s^\vee}(t) + (D_k^*\alpha_{s^\vee}(t))^\vee + (q+1)t(D_k^*\beta_s(t))^\vee, \qquad (6.15)\end{aligned}$$

hence

$$S'\gamma = D_k^*\alpha_{(q-1)s-s^\vee}(t) + (D_k^*\beta_{s^\vee}(t))^\vee + (q+1)t(D_k^*\alpha_s(t))^\vee.$$

Applying the $\vee$ operator to (6.15) and summing with the last equation gives

$$\begin{aligned}S'\gamma + \gamma^\vee &= (q-1)D_k^*\alpha_s(t) + (q-1)(D_k^*\beta_s(t))^\vee \\ &\quad + (q+1)t(D_k^*\alpha_s(t))^\vee + (q+1)tD_k^*\beta_s(t) = (q-1)\delta^\vee + (q+1)t\delta,\end{aligned}$$

that is, we have $S'\gamma = -\gamma^\vee + (q+1)t\delta + (q-1)\delta^\vee$. Therefore, by Lemma 6.16, we have $\gamma = \alpha'_{\gamma_0}(t)$ and $\delta = \beta'_{\gamma_0}(t)$. Let us determine γ_0. By construction in Subsections 4.1, 4.3 and 6.5, we have

$$\alpha_s(t)_0 = \beta_s(t)_1 = s \text{ and } \alpha_s(t)_1 = \beta_s(t)_0 = 0.$$

By using Lemma 6.19 and (6.15), we get

$$\begin{aligned}\gamma_0 = -((q-1)s - s^\vee)^< + s^{\vee<} + (q+1)t(-s^<)^\vee \\ = 2s^{\vee<} - (q-1)s^< - (q+1)ts^{<\vee}.\end{aligned}$$

Thus, we have shown that

$$D_k^*\beta_s(t) + (D_k^*\alpha_s(t))^\vee = \delta = \beta'_{2s^{\vee<}-(q-1)s^<-(q+1)ts^{<\vee}}(t).$$

Applying this identity to $s^\vee$ yields

$$(D_k^*\alpha_{s^\vee}(t))^\vee = \beta'_{2qs^<+(q-1)s^{\vee<}-(q+1)ts^{\vee<\vee}}(t) - D_k^*\beta_{s^\vee}(t)$$

and therefore, by using the value for γ given by (6.15),

(6.16)

$$\begin{aligned}D_k^*\beta_{(q-1)s-2s^\vee}(t) + \beta'_{2qs^<+(q-1)s^{\vee<}-(q+1)ts^{\vee<\vee}}(t) + (q+1)t(D_k^*\beta_s(t))^\vee \\ = \gamma = \alpha'_{2s^{\vee<}-(q-1)s^<-(q+1)ts^{<\vee}}(t).\end{aligned}$$

To conclude, we will temporarily split the proof according to the eigenvalues of the $\vee$ operator in $\mathcal{S}_k$.

If $s^\vee = qs$, (6.16) says

$$-D_k^*\beta_s(t) + t(D_k^*\beta_s(t))^\vee = \alpha'_{s^<-ts^{<\vee}}(t) - q\beta'_{s^<-ts^{<\vee}}(t).$$

As the linear map $r \mapsto r - tr^\vee$ is injective on $\mathcal{S}_{k-1}$ for all $t \notin \{-1, 1\}$ and as the functions are polynomial in t, we get

$$D_k^*\beta_s(t) = -\alpha'_{s^<}(t) + q\beta'_{s^<}(t) = -\alpha'_{s^<}(t) + \beta'_{s^{\vee<}}(t). \tag{6.17}$$

If $s^\vee = -s$, (6.16) says

$$D_k^*\beta_s(t) + t(D_k^*\beta_s(t))^\vee = -\alpha'_{s^< + ts^{<\vee}}(t) - \beta'_{s^< + ts^{<\vee}}(t).$$

As above, we get

$$D_k^*\beta_s(t) = -\alpha'_{s^<}(t) - \beta'_{s^<}(t) = -\alpha'_{s^<}(t) + \beta'_{s^{\vee<}}(t). \tag{6.18}$$

Joining the two cases, we get from (6.17) and (6.18) that (6.12) holds for any s in $\mathcal{S}_k$. We now apply the operator S' to this identity. By Lemma 2.21, Lemma 6.12, Lemma 6.13 and Lemma 6.16, this gives

$$D_k^*\alpha_s(t) = \alpha'_{s^{<\vee}}(t) - \beta'_{(q+1)ts^< + (q-1)s^{<\vee}}(t) + \beta'_{s^{\vee<\vee}}(t)$$

and (6.13) follows. □

Proof of Lemma 6.20 in case k is odd. We now consider (6.12) and (6.14) and we set $\delta = qD_k^*\beta_s(t) + (D_k^*\alpha_s(t))^\vee$ and $\gamma = S\delta$, so that the first relation in the uniqueness part of Lemma 6.14 holds. As above, we will conclude by proving that the second relation also holds. By Lemma 2.21, Lemma 6.12 and Lemma 6.15, we have

$$\begin{aligned} R'\delta &= qD_k^*R\beta_s(t) + (D_k^*R\alpha_s(t))^\vee \\ &= qD_k^*\alpha_s(t) + (D_k^*(q\beta_s(t) + (q-1)\alpha_s(t)))^\vee \\ &= qD_k^*\alpha_s(t) + (q-1)(D_k^*\alpha_s(t))^\vee + q(D_k^*(\beta_s(t)))^\vee = \delta^\vee. \end{aligned}$$

We now compute $R'\gamma$. Still by Lemma 2.21, Lemma 6.12 and Lemma 6.15, we have

$$\begin{aligned} \gamma = S\delta &= qD_k^*S\beta_s(t) + (D_k^*S\alpha_s(t))^\vee \\ &= -qD_k^*\beta_{s^\vee}(t) + (D_k^*\alpha_{s^\vee}(t))^\vee + (D_k^*\beta_{(q+1)ts-(q-1)s^\vee}(t))^\vee, \end{aligned} \tag{6.19}$$

hence

$$\gamma^\vee - (q-1)\gamma = -q(D_k^*\beta_{s^\vee}(t))^\vee + qD_k^*\alpha_{s^\vee}(t) + qD_k^*\beta_{(q+1)ts}(t).$$

and

$$\begin{aligned} R'\gamma = &-qD_k^*\alpha_{s^\vee}(t) + q(D_k^*\beta_{s^\vee}(t))^\vee \\ &+ (q-1)(D_k^*\alpha_{s^\vee}(t))^\vee + (D_k^*\alpha_{(q+1)ts-(q-1)s^\vee}(t))^\vee \\ &= -qD_k^*\alpha_{s^\vee}(t) + q(D_k^*\beta_{s^\vee}(t))^\vee + (D_k^*\alpha_{(q+1)ts}(t))^\vee \end{aligned}$$

Summing up the last two relations gives

$$R'\gamma + \gamma^\vee - (q-1)\gamma = (D_k^*\alpha_{(q+1)ts}(t))^\vee + qD_k^*\beta_{(q+1)ts}(t) = (q+1)t\delta.$$

That is, we have $R'\gamma = -\gamma^\vee + (q-1)\gamma + (q+1)t\delta$. Therefore, by Lemma 6.14, we have $\gamma = \alpha'_{\gamma_0}(t)$ and $\delta = \beta'_{\gamma_0}(t)$. We compute γ_0. By construction in Subsections 5.1, 5.2 and 6.6, we have

$$\alpha_s(t)_0 = s \qquad \alpha_s(t)_1 = q^{-1}(q-1)s$$
$$\beta_s(t)_0 = 0 \qquad \beta_s(t)_1 = q^{-1}s.$$

By using Lemma 6.19 and (6.19), we get

$$\gamma_0 = q^{\vee<} + (s^{\vee<\vee} - (q-1)s^{\vee<})^\vee + ((q-1)s^{\vee<} - (q+1)ts^{<})^\vee$$
$$= 2qs^{\vee<} + (q-1)s^{\vee<\vee} - (q+1)ts^{<\vee}.$$

We have shown that

$$qD_k^*\beta_s(t) + (D_k^*\alpha_s(t))^\vee = \delta = \beta'_{2qs^{\vee<}+(q-1)s^{\vee<\vee}-(q+1)ts^{<\vee}}(t).$$

Applying this identity to $s^\vee$, we get

$$(D_k^*\alpha_{s^\vee}(t))^\vee = \beta'_{2qs^{<}+(q-1)s^{<\vee}-(q+1)ts^{\vee<\vee}}(t) - qD_k^*\beta_{s^\vee}(t),$$

whence, by using the value for γ given by (6.19),

(6.20)
$$-2qD_k^*\beta_{s^\vee}(t) + \beta'_{2qs^{<}+(q-1)s^{<\vee}-(q+1)ts^{\vee<\vee}}(t) + (D_k^*\beta_{(q+1)ts-(q-1)s^\vee}(t))^\vee$$
$$= \gamma = \alpha'_{2qs^{\vee<}+(q-1)s^{\vee<\vee}-(q+1)ts^{<\vee}}(t).$$

To conclude, we will again temporarily split the proof according to the eigenvalues of the $\vee$ operator in $\mathcal{S}_k$.

If $s^\vee = s$, (6.20) says

$$-2qD_k^*\beta_s(t) + ((q+1)t - (q-1))(D_k^*\beta_s(t))^\vee$$
$$= \alpha'_{2qs^{<}-((q+1)t-(q-1))s^{<\vee}}(t) - \beta'_{2qs^{<}-((q+1)t-(q-1))s^{<\vee}}(t).$$

As the eigenvalues of the $\vee$ operator on $\mathcal{S}_{k-1}$ are q and -1, the linear map $r \mapsto 2qr - ((q+1)t - (q-1))r^\vee$ is injective on $\mathcal{S}_{k-1}$ for all $t \notin \{-1, 1\}$. As the functions are polynomial in t, we get

$$D_k^*\beta_s(t) = -\alpha'_{s^<}(t) + \beta'_{s^<}(t) = -\alpha'_{s^<}(t) + \beta'_{s^{\vee<}}(t). \tag{6.21}$$

If $s^\vee = -s$, (6.20) says

$$2qD_k^*\beta_s(t) + ((q+1)t + (q-1))(D_k^*\beta_s(t))^\vee$$
$$= -\alpha'_{2qs^{<}+((q+1)t+(q-1))s^{<\vee}}(t) - \beta'_{2qs^{<}+((q+1)t+(q-1))s^{<\vee}}(t).$$

As above, we get

$$D_k^*\beta_s(t) = -\alpha'_{s^<}(t) - \beta'_{s^<}(t) = -\alpha'_{s^<}(t) + \beta'_{s^{\vee<}}(t). \tag{6.22}$$

Joining the two cases, we get from (6.21) and (6.22) that (6.12) holds for any s in $\mathcal{S}_k$. We now apply the operator R' to this identity. By Lemma 2.21, Lemma 6.12, Lemma 6.15 and Lemma 6.14, this gives

$$D_k^* \alpha_s(t) = \alpha'_{s^{<\vee} - (q-1)s^<}(t) - \beta'_{(q+1)ts^<}(t) + \beta'_{s^{\vee<\vee}}(t)$$

and (6.14) follows. □

We conclude the Section by the proof of Proposition 6.5, which is a direct consequence of Lemma 6.20.

Proof of Proposition 6.5. Let G be in $\mathcal{H}_{k-1}^{(\mathbb{N})}$ and set $H = D_k G$. We write as usual $\check{G} = \begin{pmatrix} \check{G}_0 \\ \check{G}_1 \end{pmatrix}$ and $\widehat{H} = \begin{pmatrix} \widehat{H}_0 \\ \widehat{H}_1 \end{pmatrix}$.

First assume that k is even. By the definition of the spectral transform in (6.8), we have, for t in $\mathbb{R}$ and s in $\mathcal{S}_k$,

$$\begin{aligned} \langle s, \widehat{H}_0(t)\rangle &= \langle q^{-1}s^\vee, \widehat{H}_0(t)^\vee - (q-1)\widehat{H}_0(t)\rangle = q^{-1}\langle \alpha_{s^\vee}(t), H\rangle \\ &= q^{-1}\langle \alpha_{s^\vee}(t), D_k G\rangle = q^{-1}\langle D_k^*\alpha_{s^\vee}(t), G\rangle. \end{aligned}$$

Lemma 6.20 gives

$$\begin{aligned} D_k^*\alpha_{s^\vee}(t) &= \alpha'_{s^{\vee<\vee}}(t) + \beta'_{s^{\vee\vee<\vee} - (q-1)s^{\vee<\vee} - (q+1)ts^{\vee<}}(t) \\ &= \alpha'_{s^{\vee<\vee}}(t) + \beta'_{qs^{<\vee} - (q+1)ts^{\vee<}}(t), \end{aligned}$$

hence, by the definition of the spectral transform in (6.11),

$$\begin{aligned} \langle s, \widehat{H}_0(t)\rangle &= q^{-1}\langle \alpha'_{s^{\vee<\vee}}(t) + \beta'_{qs^{<\vee} - (q+1)ts^{\vee<}}(t), G\rangle \\ &= -q^{-1}\langle s^{\vee<\vee}, \check{G}_1(t)\rangle + \langle s^< - q^{-1}(q+1)ts^{\vee<\vee}, \check{G}_0(t)\rangle. \end{aligned}$$

We get $\widehat{H}_0(t) = -q^{-1}\check{G}_1(t)^{\vee<\vee} + \check{G}_0(t)^> - q^{-1}(q+1)t\check{G}_0(t)^{\vee>\vee}$ as required. In the same way, the definitions (6.8) and (6.11) and Lemma 6.20 give

$$\begin{aligned} \langle s, \widehat{H}_1(t)\rangle &= \langle \beta_s(t), H\rangle = \langle D_k^*\beta_s(t), G\rangle \\ &= \langle -\alpha'_{s^<}(t) + \beta'_{s^{\vee<}}(t), G\rangle = \langle s^<, \check{G}_1(t)\rangle + \langle s^{\vee<\vee}, \check{G}_0(t)\rangle, \end{aligned}$$

hence $\widehat{H}_1(t) = \check{G}_1(t)^> + \check{G}_0(t)^{\vee>\vee}$.

Assume now that k is odd. We use (6.9) and (6.10) and Lemma 6.20 to get

$$\begin{aligned} \langle s, \widehat{H}_0(t)\rangle &= \langle \beta_{s^\vee}(t), H\rangle = \langle D_k^*\beta_{s^\vee}(t), G\rangle \\ &= \langle -\alpha'_{s^{\vee<}}(t) + \beta'_{s^<}(t), G\rangle = -\langle s^{\vee<\vee}, \check{G}_0(t)\rangle - \langle s^<, \check{G}_1(t)\rangle, \end{aligned}$$

hence $\widehat{H}_0(t) = -\check{G}_0(t)^{\vee>\vee} - \check{G}_1(t)^{>}$. Finally, the same arguments give

$$\begin{aligned}\langle s, \widehat{H}_1(t)\rangle &= \langle \alpha_s(t), H\rangle = \langle D_k^* \alpha_s(t), G\rangle \\ &= \langle \alpha'_{s^{<\vee}-(q-1)s^{<}}(t) + \beta'_{s^{\vee<\vee}-(q+1)ts^{<}}(t), G\rangle \\ &= \langle s^{<\vee\vee} - (q-1)s^{<\vee}, \check{G}_0(t)\rangle - \langle s^{\vee<\vee} - (q+1)ts^{<}, \check{G}_1(t)\rangle \\ &= q\langle s^{<}, \check{G}_0(t)\rangle + \langle (q+1)ts^{<} - s^{\vee<\vee}, \check{G}_1(t)\rangle,\end{aligned}$$

that is, $\widehat{H}_1(t) = q\check{G}_0(t)^{>} + (q+1)t\check{G}_1(t)^{>} - \check{G}_1(t)^{\vee>\vee}$ and the Proposition follows. □

7. The simple transfer operator

Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Then, the successive orthogonal extensions of p define a scalar product p^∞ on $\overline{\mathcal{D}}(\partial X)$ (see Subsection I.4.5). Recall from Subsection 2.3 that we write $\mathcal{H}_\infty$ for the space of Γ-invariant ∞-pseudofunctions, that is, the space of Γ-invariant maps $X_1 \to \overline{\mathcal{D}}(\partial X)$. By abuse of notation, we will still denote by p^∞ the scalar product associated with p on $\mathcal{H}_\infty$, which is defined by, for any H, J in $\mathcal{H}_\infty$,

$$p^\infty(H, J) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} p^\infty(H_{xy}, J_{xy}). \tag{7.1}$$

Our aim until the end of the article is to establish a Plancherel formula for p in the spirit of (4.1), (4.8), (5.1) and (5.4). Following the strategy of the proofs of this formulae, we need to establish a formula for the resolvent of a certain self-adjoint operator acting on (the completion of) $\mathcal{H}_\infty$. To state this resolvent formula, we will use a linear operator that is an analogue of the quadratic transfer operator of Subsection I.10.3. The purpose of the present Section is to define this operator and give some informations on its spectrum.

7.1. **Adjoint operations and simple transfer operator.**

Our definition will be formulated in the spirit of Subsection II.6.2. In particular, it requires for us to introduce the adjoint operation of direct extension.

Let $k \geq 2$ and p be a k-Euclidean field. Recall from Subsection I.10.3 that, if k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y \in X$, we write $I_{xy}^{\ell-1,\dagger p} : \overline{V}^{\ell}(x) \to \overline{V}^{\ell-1}(xy)$ for the surjective linear map that is the adjoint with respect to p_x of the natural injective map $I_{xy}^{\ell-1} : \overline{V}^{\ell-1}(xy) \to \overline{V}^{\ell}(x)$. Then, if H is a k-pseudofunction, we let $H^{<_p}$ be the $(k-1)$-pseudofunction with $H^{<_p}_{xy} = I_{xy}^{\ell-1,\dagger p} H_{xy}$, $x \sim y \in X$. We equip $\mathcal{H}_k$ with the scalar product defined by

$$p(H,J) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} p_x(H_{xy}, J_{xy}), \quad H, J \in \mathcal{H}_k.$$

In the same way, if k is odd, $k = 2\ell+1$, $\ell \geq 1$, for any $x \sim y \in X$, we write $J_{xy}^{\ell,\dagger p} : \overline{V}^\ell(xy) \to \overline{V}^\ell(x)$ for the surjective linear map that is the adjoint with respect to p_{xy} of the natural injective map $J_{xy}^\ell : \overline{V}^\ell(x) \to \overline{V}^\ell(xy)$. Then, if H is a k-pseudofunction, we let $H^{<_p}$ be the $(k-1)$-pseudofunction with $H_{xy}^{<_p} = J_{xy}^{\ell,\dagger p} H_{xy}$, $x \sim y \in X$. We equip $\mathcal{H}_k$ with the scalar product defined by

$$p(H,J) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} p_{xy}(H_{xy}, J_{xy}), \quad H, J \in \mathcal{H}_k.$$

By construction, we always have

$$H^{><_p} = H.$$

By using Lemma I.9.11, we directly get the adjointness property.

Lemma 7.1. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For any H in $\mathcal{H}_k$ and J in $\mathcal{H}_{k-1}$, we have*

$$p(H, J^>) = p^-(H^{<_p}, J).$$

Note that Lemma I.10.8 may be translated into

Lemma 7.2. *Let $k \geq 2$, p be a k-Euclidean field with orthogonal extension p^+ and H be a k-pseudofunction. We have*

$$H^{>\vee<_{p^+}} = H^{<_p\vee>}.$$

As in Subsection I.4.5, we write p^+ for the orthogonal extension of p. Now, by analogy with I.10.4 and I.10.5, we set

Definition 7.3. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. The simple transfer operator S_p of p is the linear endomorphism of $\mathcal{H}_{k-1}$ defined by

$$S_p H = H^{>\vee<_p\vee} \quad H \in \mathcal{H}_{k-1}.$$

By abuse of notation, we will usually also write $S_p : \mathcal{H}_k \to \mathcal{H}_k$ for the transfer operator of the orthogonal extension p^+ of p. By Lemma 7.2, for H is in $\mathcal{H}_k$, we have

$$S_p H = H^{<_p\vee>\vee}. \tag{7.2}$$

Thus, we get

Lemma 7.4. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For H in $\mathcal{H}_{k-1}$, we have*

$$(S_p H)^{>\vee} = S_p(H^{>\vee}).$$

Note that the $\vee$ operator is self-adjoint with respect to p, as follows directly from the definitions (and from Lemma I.9.11). We write $S_p^\dagger$ for the adjoint of S_p with respect to p. We get

Lemma 7.5. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For H in $\mathcal{H}_{k-1}$, we have*

$$S_p^\dagger H = H^{\vee>\vee<_p}.$$

In particular, the operators S_p and $S_p^\dagger$ are conjugated by the $\vee$ operator, that is, we have $S_p(H^\vee) = (S_p^\dagger H)^\vee$.

Recall from Subsection 2.6 that, when Γ is bipartite, we have introduced the twist operator of pseudofunctions. This operator is then self-adjoint with respect to p. By Lemma 2.22, we obtain

Lemma 7.6. *Assume Γ is bipartite. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For H in $\mathcal{H}_{k-1}$, we have*

$$S_p(H^\wr) = -(S_p H)^\wr \text{ and } S_p^\dagger(H^\wr) = -(S_p^\dagger H)^\wr.$$

7.2. **Spectrum of the simple transfer operator.** The resolvent formula for Euclidean fields will be written by using rational functions of the simple transfer operator. For this to make sense, we will need some information on the spectrum of S_p.

We start by exhibiting a subspace of $\mathcal{H}_{k-1}$ where the simple transfer operator has a very simple behaviour. Recall from Subsection 2.2 that we have introduced the notion of a (-1)-pseudofunction and that the space $\mathcal{H}_{-1}$ has dimension 1 or 2.

Lemma 7.7. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Let H be in $\mathcal{H}_{-1}$. Then, we have*

$$S_p\left(H^{>^k}\right) = -H^{\vee>^k} = S_p^\dagger\left(H^{>^k}\right).$$

In other words, the operators S_p and $S_p^\dagger$ preserve the space of (-1)-pseudofunctions and may be seen on that space as the operator that maps a function on X that is constant on neighbours to the opposite function. Note that this implies in particular that, contrarily to the quadratic transfer operator, the simple transfer operator always admits 1 as an eigenvalue.

Proof. We prove the statement for S_p, the other proof being analogous by using Lemma 7.5. Assume that k is even. Then, by Lemma 2.6, we have

$$S_p\left(H^{>^k}\right) = H^{>^{k+1}\vee <_p \vee} = H^{>\vee>^k <_p\vee} = H^{>\vee>^{k-1}\vee}.$$

As $H^{>}$ is a 0-pseudofunction, by convention, $H^{>\vee} = -H^{>}$. Hence again by Lemma 2.6,

$$S_p\left(H^{>^k}\right) = -H^{>^k\vee} = -H^{\vee>^k}$$

as required. Now, if k is odd, we have

$$S_p\left(H^{>^k}\right) = H^{>^{k+1}\vee<_p\vee} = H^{\vee>^{k+1}<_p\vee} = H^{\vee>^k\vee} = H^{\vee>\vee>^{k-1}}$$

and, again, as $H^{\vee>}$ is a 0-pseudofunction, $H^{\vee>\vee} = -H^{\vee>}$. □

We notice that the spectrum of the simple transfer operator essentially does not depend on the space.

Lemma 7.8. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Then, the spectrum of S_p in $\mathcal{H}_k$ is the union of $\{0\}$ and the spectrum of S_p in $\mathcal{H}_{k-1}$.*

See Corollary I.10.7 for the analogous result for quadratic transfer operators.

Proof. By Lemma 7.4, for J in $\mathcal{H}_{k-1}$, we have $(S_pJ)^{>\vee} = S_p(J^{>\vee})$. By (7.2), we have $S_p\mathcal{H}_k \subset \mathcal{H}_{k-1}^{>\vee}$. The result follows. □

Now we prove a universal bound on the spectral radius.

Proposition 7.9. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Then, S_p has spectral radius $< q$ in $\mathcal{H}_{k-1}$.*

The proof of the Proposition essentially relies on the following norm estimate for the adjoint simple transfer operator.

Lemma 7.10. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and H be in $\mathcal{H}_{k-1}$. Then, we have $\left\|S_p^\dagger H\right\|_p \leq q \left\|H\right\|_p$. Besides, if k is even (resp. odd), then equality holds, that is, $\left\|S_p^\dagger H\right\|_p = q\left\|H^p\right\|$ if and only if there exists J in $\mathcal{H}_{k-2}$ (resp. $\mathcal{H}_{k-3}$) with $J^\vee = qJ$ and $H = J^{>\vee}$ (resp. $H = J^{>>}$).*

We have denoted by $\|.\|_p$ the natural norm induced by p on $\mathcal{H}_{k-1}$, that is, $\|H\|_p = \sqrt{p^-(H,H)}$ for H in $\mathcal{H}_{k-1}$.

Proof. Recall that, if $j \geq 1$ is odd, the $\vee$ operator is an isometry of $\mathcal{H}_j$ with respect to p, whereas if j is even, it is a self-adjoint operator with spectrum $\{-1, q\}$. In particular, for any H in $\mathcal{H}_{k-1}$, we have $\|H^{\vee>\vee}\|_p \leq q\,\|H\|_p$. As the $<_p$ operator is the adjoint of the isometric embedding given by direct extension $\mathcal{H}_{k-1} \hookrightarrow \mathcal{H}_k$, we get

$$\left\|S_p^\dagger H\right\|_p = \|H^{\vee>\vee<_p}\| \leq \|H^{\vee>\vee}\|_p \leq q\,\|H\|_p\,. \tag{7.3}$$

If $H^\vee = J^>$ for some J in $\mathcal{H}_{k-2}$, by Lemma 2.6, we have

$$S_p^\dagger H = H^{\vee>\vee<_p} = J^{>>\vee<_p} = J^{\vee>><_p} = J^{\vee>}. \tag{7.4}$$

Thus, if k is even and $J^\vee = qJ$, we get $S_p^\dagger H = qJ^>$ and

$$\left\|S_p^\dagger H\right\|_p = q\,\|J^>\|_p = q\,\|J^{>\vee}\|_p = q\,\|H\|_p\,.$$

If k is odd and $H = K^{>>}$ for some K in $\mathcal{H}_{k-3}$ with $K^\vee = qK$, by Lemma 2.6, we have

$$S_p^\dagger H = K^{>>\vee>\vee<_p} = qK^{>>>\vee<_p} = qK^{>\vee>><_p} = qK^{>\vee>}$$

and again

$$\left\|S_p^\dagger H\right\|_p = q\,\|K^{>\vee>}\|_p = q\,\|K^{>\vee}\|_p = q\,\|K^>\|_p = q\,\|K^{>>}\|_p = q\,\|H\|_p\,.$$

Conversely, assume $\left\|S_p^\dagger H\right\|_p = q\,\|H\|_p$, so that the two inequalities in (7.3) must be equalities. We get $\|H^{\vee>\vee<_p}\| = \|H^{\vee>\vee}\|_p$, so that $H^{\vee>\vee}$ belongs to $\mathcal{H}_{k-1}^>$. By Lemma 2.8, there exists J in $\mathcal{H}_{k-2}$ with $H^\vee = J^>$.

Assume that k is even. Then, by using (7.4), we get

$$\|J^\vee\|_p = \|J^{\vee>}\|_p = \left\|S_p^\dagger H\right\|_p = q\,\|H\|_p = q\,\|H^\vee\|_p = q\,\|J^>\|_p = q\,\|J\|_p\,.$$

Hence $J^\vee = qJ$ as required.

Assume that k is odd. Then, again by (7.4), we have

$$\|H^\vee\|_p = \|J^>\|_p = \|J\|_p = \|J^\vee\|_p = \|J^{\vee>}\|_p = \left\|S_p^\dagger H\right\|_p = q\,\|H\|_p\,.$$

Therefore, we obtain $H^\vee = qH$. As $H^\vee = J^>$, we have $J^{>\vee} = qJ^>$ which, by Lemma 2.8, implies that we may write $J = K^>$ for some K in $\mathcal{H}_{k-3}$ with $K^\vee = qK$. We have $H = q^{-1}J^> = q^{-1}K^{>>}$ and the conclusion follows. □

Proposition 7.9 will follow from Lemma 7.10 by an induction argument which relies on

Lemma 7.11. *Let $i \geq 0$ and $\ell \geq 1$ be integer. Let H be a 2ℓ-pseudofunction and G be a $2(i+\ell)$-pseudofunction with $G^\vee = qG$. Assume that we have $H^{>^{2i+1}} = G^{>\vee}$. Then, there exists a $2(\ell-1)$-pseudofunction J with $J^\vee = qJ$ and $H = J^{>\vee>}$ and $G = J^{>^{2(i+1)}}$.*

Proof. We fix $\ell \geq 1$ and we prove this statement by induction on $i \geq 0$.

For $i = 0$, we have $H^{>} = G^{>\vee}$. Thus, Lemma 2.8 gives a $(2\ell-1)$-pseudofunction K with $G = K^{>}$ and $H = K^{\vee>}$. As $G^{\vee} = qG$, we have $K^{>\vee} = qK^{>}$. Still by Lemma 2.8, there exists a $2(\ell-1)$-pseudofunction J with $J^{\vee} = qJ$ and $K = J^{>}$. We get $H = K^{\vee>} = J^{>\vee>}$ and $G = J^{>>}$ and we are done.

Now, assume that $i \geq 1$ and the result is true for $i-1$. As above, by Lemma 2.8, there exists a $(2i+2\ell-1)$-pseudofunction K with $G = K^{>}$ and $H^{>^{2i}} = K^{\vee>}$. As $G^{\vee} = qG$, there exists a $2(i+\ell-1)$-pseudofunction G_1 with $G_1^{\vee} = qG_1$ and $K = G_1^{>}$. We get $H^{>^{2i}} = K^{\vee>} = G_1^{>\vee>}$, hence $H^{>^{2i-1}} = G_1^{>\vee}$. Therefore, the induction assumption says that there exists a $2(\ell-1)$-pseudofunction J with $J^{\vee} = qJ$ and $H = J^{>\vee>}$ and $G_1 = J^{>^{2i}}$. As $G = G_1^{>>}$, we get $G = J^{>^{2(i+1)}}$ and the conclusion follows by induction. □

Proof of Proposition 7.9. Note that Lemma 7.10 directly implies the Proposition when $k = 2, 3$. Indeed from the definitions in Subsection 2.1, the $\vee$ operator on $\mathcal{H}_0$ is -1. Thus, if $k = 2, 3$, then $S_p^{\dagger}$ has norm $< q$ and the conclusion follows. The proof for arbitrary $k \geq 2$ will follows from the same reason by an induction argument.

If k is even, $k = 2\ell$, $\ell \geq 1$, we claim that, for any $1 \leq i \leq \ell$, if H is in $\mathcal{H}_{k-1}$ and $\left\| \left(S_p^{\dagger}\right)^i H \right\|_p = q^i \left\| H \right\|_p$, then $H = J^{>^{2i-1}\vee}$ for some J in $\mathcal{H}_{k-2i}$ with $J^{\vee} = qJ$. If $i = 1$, this is Lemma 7.10. If $i \leq \ell - 1$ and the claim is true for i, let us show that it is also true for $i+1$. Therefore, we assume that we have

$$\left\| \left(S_p^{\dagger}\right)^{i+1} H \right\|_p = q^{i+1} \left\| H \right\|_p .$$

We set $G = S_p^{\dagger} H$. By the inequality in Lemma 7.10, we have $\left\| \left(S_p^{\dagger}\right)^i G \right\|_p = q^i \left\| G \right\|_p$ and $\left\| G \right\|_p = q \left\| H \right\|_p$. Hence, on one hand, the induction assumption says that we may write $G = K^{>^{2i-1}\vee}$ for some K in $\mathcal{H}_{2(\ell-i)}$ with $K^{\vee} = qK$, whereas Lemma 7.10 says that we may write $H = L^{>\vee}$ for some L in $\mathcal{H}_{2(\ell-1)}$ with $L^{\vee} = qL$. From (7.4), we get $G = S_p^{\dagger} H = L^{\vee>} = qL^{>}$, hence $K^{>^{2i-1}\vee} = qL^{>}$, or equivalently, $K^{>^{2i-1}} = qL^{>\vee}$. As $L^{\vee} = qL$, we can apply Lemma 7.11. Then, we know that there exists J in $\mathcal{H}_{2(\ell-i-1)}$ with $J^{\vee} = qJ$ and $K = qJ^{>\vee>}$ and $L = J^{>^{2i}}$. As $H = L^{>\vee}$, this gives $H = J^{>^{2i+1}\vee}$, which should be proved.

In particular, applying this claim in case $i = \ell$, as the $\vee$ operator is -1 on $\mathcal{H}_0$, we get that $\left(S_p^\dagger\right)^\ell$ has norm $< q^\ell$ on $\mathcal{H}_{k-1}$, which implies the Proposition in the even case.

Assume that k is odd, $k = 2\ell + 1$, $\ell \geq 1$. We now claim that for any $1 \leq i \leq \ell$, if H is in $\mathcal{H}_{k-1}$ and $\left\|\left(S_p^\dagger\right)^i H\right\|_p = q^i \|H\|_p$, then $H = J^{>>^{2i}}$ for some J in $\mathcal{H}_{k-2i-1}$ with $J^\vee = qJ$. If $i = 1$, this is Lemma 7.10. If $i \leq \ell - 1$ and the claim is true for i, let us show that it is also true for $i + 1$. Again, we assume that we have

$$\left\|\left(S_p^\dagger\right)^{i+1} H\right\|_p = q^{i+1} \|H\|_p$$

and we set $G = S_p^\dagger H$, so that, by Lemma 7.10, we have $\left\|\left(S_p^\dagger\right)^i G\right\|_p = q^i \|G\|_p$ and $\|G\|_p = q\|H\|_p$. By the induction assumption, there exists K in $\mathcal{H}_{k-2i-1}$ with $K^\vee = qK$ and $K^{>^{2i}} = G$. By Lemma 7.10, there exists L in $\mathcal{H}_{k-3}$ with $L^\vee = qL$ and $L^{>>} = H$. We have, by Lemma 2.6,

$$K^{>^{2i}} = G = S_p^\dagger H = L^{>>\vee>\vee<_p} = qL^{>>>\vee<_p} = qL^{>\vee>><_p} = qL^{>\vee>},$$

hence $K^{>^{2i-1}} = qL^{>\vee}$. By Lemma 7.11, there exists J in $\mathcal{H}_{k-2i-3}$ with $J^\vee = qJ$ and $K = qJ^{>\vee>}$ and $L = J^{>^{2i}}$. We get $H = J^{>^{2(i+1)}}$ as required.

As above, the case $i = \ell$ implies that $\left(S_p^\dagger\right)^\ell$ has norm $< q^\ell$ on $\mathcal{H}_{k-1}$. The Proposition follows in the odd case. □

7.3. **Spectrum in the admissible case.** In this Subsection, we relate the simple transfer operator with the quadratic transfer operator of Section I.10. We use this relation to show that, when p is admissible, the spectral radius of S_p is not too large. This fact will not be used in the rest of the present article.

Proposition 7.12. *Let $k \geq 2$ and p be an admissible Γ-invariant k-Euclidean field. Then S_p has spectral radius $< \sqrt{q}$ in $\mathcal{H}_{k-1}$.*

See Section I.10 for the definition and equivalent characterizations of admissible Euclidean fields.

The proof of Proposition 7.12 relies on constructions from quadratic algebra which also appear in Appendix I.C. Recall that, if V is a finite-dimensional real vector space, we have a natural map $v \mapsto v^2, V \to \mathcal{Q}(V^*)$ from V to the space of symmetric bilinear forms on V^*. It is defined by setting $v^2(\varphi, \psi) = \varphi(v)\psi(v)$, for v in V and φ, ψ in V^*. This map satisfies Cauchy-Schwarz inequality:

Lemma 7.13. *Let V be a finite-dimensional real vector space. Pick $v_1, \dots, v_r$ in V and $t_1, \dots, t_r$ in $\mathbb{R}$. We have*

$$\left(\sum_{i=1}^{r} t_i v_i\right)^2 \leq \left(\sum_{i=1}^{r} t_i^2\right)\left(\sum_{i=1}^{r} v_i^2\right),$$

meaning that the difference is a non-negative symmetric bilinear form on V^*.

Proof. Indeed, by the standard Cauchy-Schwartz inequality, for any φ in V^*, one has

$$\left(\sum_{i=1}^{r} t_i \varphi(v_i)\right)^2 \leq \left(\sum_{i=1}^{r} t_i^2\right)\left(\sum_{i=1}^{r} \varphi(v_i)^2\right).$$

□

Now, recall the notion of a pseudokernel from Subsection I.8.2. In Subsection II.2.1, we have defined natural operations on pseudokernels that are analogous to the natural operations on pseudofunctions of Subsection 2.2. For $k \geq 1$, if H is a k-pseudofunction we define a k-pseudokernel $L = H^2$ as follows: if k is odd (resp. even), $k = 2\ell + 1$ (resp. $k = 2\ell$), $\ell \geq 0$ (resp. $\ell \geq 1$), for any $x \sim y$ in X, the symmetric bilinear form on $V_0^\ell(xy)$ (resp. $V_0^\ell(x)$) associated with L_{xy} is H_{xy}^2. In other words, for a, b in $S^\ell(x)$ (resp. $S^\ell(xy)$), we have

$$L_{xy}(a, b) = (H_{xy}(a) - H_{xy}(b))^2.$$

Besides, note that we have

$$(H^>)^2 = (H^2)^>. \tag{7.5}$$

This definition and Lemma 7.13 directly give

Corollary 7.14. *Let $k \geq 1$ and H be a k-pseudofunction. If k is odd, we have $(H^\vee)^2 = (H^2)^\vee$. If k is even, we have $(H^\vee)^2 \leq q(H^2)^\vee$, meaning that the difference is a non-negative pseudokernel.*

In the same way, if $k \geq 2$, and p is a k-Euclidean field, let L be a k-pseudokernel. We define a $(k-1)$-pseudokernel $L^{<_p}$ as follows: if k is even (resp. odd), $k = 2\ell$ (resp. $k = 2\ell + 1$), $\ell \geq 1$, and if, for $x \sim y$ in X, the symmetric bilinear form on $V_0^\ell(x)$ (resp. $V_0^\ell(xy)$) associated with L_{xy} is r_{xy}, then the symmetric bilinear form on $V_0^{\ell-1}(xy)$ (resp. $V_0^\ell(x)$) associated with $L_{xy}^{<_p}$ is $(I_{xy}^{\ell-1,*\dagger p})^\star r_{xy}$ (resp. $(J_{xy}^{\ell,*\dagger p})^\star r_{xy}$). The notation is the one used in Subsection I.10.6. Note that, if H is a k-pseudofunction, we have

$$(H^{<_p})^2 = (H^2)^{<_p}. \tag{7.6}$$

Finally, let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. We change the convention of Subsection I.10.3 and we write T_p for the operator that was denoted by T_p^* there. We still call T_p the quadratic transfer operator. This is an endomorphism of $\mathcal{L}_{k-1}$. As in Corollary II.6.11, we can rewrite Lemma I.10.15 as

Lemma 7.15. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For L in $\mathcal{L}_{k-1}$, we have*

$$T_p L = L^{>\vee <_p \vee}.$$

We can now give the

Proof of Proposition 7.12. By Lemma 7.15, for L in $\mathcal{L}_{k-1}$, we have $T_p L = L^{>\vee <_p \vee}$. By Definition 7.3 and Corollary 7.14, we get, for H in $\mathcal{H}_{k-1}$,

$$(S_p H)^2 = (H^{>\vee <_p \vee})^2 \leq q (H^2)^{>\vee <_p \vee} = q T_p(H^2),$$

where we also have used (7.5) and (7.6). Now, iterating this inequality, we get, for any $n \geq 0$, $(q^{-n/2} S_p^n H)^2 \leq T_p^n(H^2)$. As p is admissible, by Theorem I.10.17, T_p has spectral radius < 1, hence $T_p^n(H^2) \xrightarrow[n\to\infty]{} 0$. We get $q^{-n/2} S_p^n H \xrightarrow[n\to\infty]{} 0$ and the result follows. □

7.4. **Exceptional eigenvalues.** When p is not admissible the spectrum of the simple transfer operator may exit the open disk with radius $\sqrt{q}$ in $\mathbb{C}$, but in a controlled way.

Proposition 7.16. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Let $u \in \mathbb{C}$ be a spectral value of S_p in $\mathcal{H}_{k-1}$. If $|u| \geq \sqrt{q}$, then u is real and simple.*

Recall that saying that u is simple amounts to saying that $\mathcal{H}_{k-1} = \ker(S_p - u) \oplus (S_p - u)\mathcal{H}_{k-1}$ or equivalently $\ker(S_p - u)^2 = \ker(S_p - u)$.

The proof of Proposition 7.16 relies on studying the restriction of S_p to a certain subspace of $\mathcal{H}_{k-1}$. The p-orthogonal complement of this subspace will be described thanks to the following Lemmas.

Lemma 7.17. *Let $k \geq -1$ be odd and H be a k-pseudofunction. The following are equivalent.*
(i) *There exists a (-1)-pseudofunction J with $H = J^{>^{k+1}}$.*
(ii) *One has $H^{>\vee} = -H^{>}$ and $H^{\vee>\vee} = -H^{\vee>}$.*

Lemma 7.18. *Let $k \geq 0$ be even, H be a k-pseudofunction and ε be in $\{-1, 1\}$. The following are equivalent.*
(i) *There exists a (-1)-pseudofunction J with $H = J^{>^{k+1}}$ and $J^\vee = \varepsilon J$.*
(ii) *One has $H^{>\vee} = \varepsilon H^{>}$ and $H^{\vee>\vee} = \varepsilon H^{\vee>}$.*

Proof. We prove the two results simultaneously.

(i)⇒(ii) If $k=-1$ or $k=0$, this directly follows from the definitions. Now, Lemma 2.6 implies that if this true for k, this is also true for $k+2$. The conclusion follows.

(ii)⇒(i) If $k=-1$, there is nothing to prove. If $k=0$ and H is a 0-pseudofunction such that $H^{>}=\varepsilon H^{>\vee}$, $\varepsilon\in\{-1,1\}$, by Lemma 2.8, there exists a (-1)-pseudofunction J with $H=J^{>}$. Then, we get

$$J^{>>}=H^{>}=\varepsilon H^{>\vee}=\varepsilon J^{>>\vee}=\varepsilon J^{\vee>>},$$

hence $J=\varepsilon J^{\vee}$ as required.

Let us conclude by a two steps induction argument. Assume $k\geq 1$ and the result holds for $k-2$. Pick H in $\mathcal{H}_k$ and ε in $\{-1,1\}$ and assume that we have $H^{>\vee}=\varepsilon H^{>}$ and $H^{\vee>\vee}=\varepsilon H^{\vee>}$. By Lemma 2.8, as $H^{>\vee}=\varepsilon H^{>}$, we can find a $(k-1)$-pseudofunction K with $K^{\vee}=\varepsilon K$ and $H=K^{>}$. In the same way, as $H^{\vee>\vee}=\varepsilon H^{\vee>}$, we can find a $(k-1)$-pseudofunction L with $L^{\vee}=\varepsilon L$ and $H^{\vee}=L^{>}$. Thus, we have $L^{>}=K^{>\vee}$ and, again by Lemma 2.8, we can find a $(k-2)$-pseudofunction M with $M^{>}=K$ and $M^{\vee>}=L$. The relations $K^{\vee}=\varepsilon K$ and $L^{\vee}=\varepsilon L$ give $M^{>\vee}=\varepsilon M^{>}$ and $M^{\vee>\vee}=\varepsilon M^{\vee>}$ and the conclusion follows by induction. □

Note that, by Lemma 7.7, the spaces of pseudofunctions that appear in Lemma 7.17 and Lemma 7.18 are stable under the action of simple transfer operators and their adjoint operators. We will prove Proposition 7.16 by describing the action of transfer operators on the orthogonal complements of these subspaces.

To this aim, we introduce some notation. If $k\geq 2$ and p is a Γ-invariant k-Euclidean field, for H in $\mathcal{H}_{k,+}$, we set

$$A_pH=H^{<_p>}+H^{<_p>\vee}\text{ and }B_pH=H^{<_p\vee>}+H^{<_p\vee>\vee}.$$

By construction, A_p and B_p map $\mathcal{H}_{k,+}$ in $\mathcal{H}_{k,+}$ We summarize the information that we will need about these operators.

Lemma 7.19. *Let $k\geq 2$ and p be a Γ-invariant k-Euclidean field. The operators $A_p:\mathcal{H}_{k,+}\to\mathcal{H}_{k,+}$ and $B_p:\mathcal{H}_{k,+}\to\mathcal{H}_{k,+}$ are self-adjoint. For H in $\mathcal{H}_{k,+}$, we have*

$$\begin{aligned}p(H,A_pH)&\leq(q+1)p(H,H) && \text{if } k \text{ is even,}\\ &\leq 2p(H,H) && \text{if } k \text{ is odd.}\end{aligned}$$

Equality holds if and only if there exists G in $\mathcal{H}_{k-2,+}$ with $H=G^{>>}$.

Proof. Assume k is even. For H, J in $\mathcal{H}_{k,+}$, we have

$$p(A_pH, J) = p(H^{<_p>}, J) + p(H^{<_p>\vee}, J) = p(H^{<_p}, J^{<_p}) + p(H^{<_p>}, J^{\vee}) \\ = (q+1)p(H^{<_p}, J^{<_p}),$$

which is symmetric in H and J. As the $<_p$ operator is the adjoint of the isometric injection $>$, this gives $p(H, A_pH) \leq (q+1)p(H, H)$ and equality holds if and only if $H = F^{>}$ for some F in $\mathcal{H}_{k-1}$. As $H^{\vee} = qH$, we then get $F^{>\vee} = qF^{>}$, hence, by Lemma 2.8, there exists G in $\mathcal{H}_{k-2}$ with $F = G^{>}$ and $G^{\vee} = qG$, so that $H = G^{>>}$. Besides, we have

$$p(B_pH, J) = (q+1)p(H^{<_p\vee}, J^{<_p}),$$

which is also symmetric in H and J.

The proof in the odd case is analogous. □

The operators A_p and B_p may be used to describe the action of $S_p^{\dagger}$ in a large subspace of $\mathcal{H}_{k-1}$.

Lemma 7.20. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and J, K be in $\mathcal{H}_{k,+}$. If k is even, we have*

$$S_p^{\dagger}(J^{<_p} + K^{<_p\vee}) = (B_pJ + (A_p - 1)K)^{<_p} - J^{<_p\vee}.$$

If k is odd, we have

$$S_p^{\dagger}(J^{<_p} + K^{<_p\vee}) \\ = (B_pJ + (q(A_p - 1) + (q-1)B_p)K)^{<_p} - (J + (q-1)K)^{<_p\vee}.$$

This Lemma is directly inspired by the proof of the Ihara trace formula [4] (see Subsection 1.3).

Proof. This is a direct computation. If k is even, we have, by Lemma 7.5,

$$\begin{aligned} S_p^{\dagger}(J^{<_p} + K^{<_p\vee}) &= J^{<_p\vee>\vee<_p} + K^{<_p\vee\vee>\vee<_p} \\ &= J^{<_p\vee>\vee<_p} + K^{<_p>\vee<_p} \\ &= (B_pJ)^{<_p} - J^{<_p\vee><_p} + (A_pK)^{<_p} - K^{<_p><_p} \\ &= (B_pJ)^{<_p} - J^{<_p\vee} + (A_pK)^{<_p} - K^{<_p}, \end{aligned}$$

as required. If k is odd, in the same way,

$$\begin{aligned}
S_p^\dagger(J^{<_p} + K^{<_p\vee}) &= J^{<_p\vee>\vee<_p} + K^{<_p\vee\vee>\vee<_p} \\
&= J^{<_p\vee>\vee<_p} + qK^{<_p>\vee<_p} + (q-1)K^{<_p\vee>\vee<_p} \\
&= (B_pJ)^{<_p} - J^{<_p\vee><_p} + q(A_pK)^{<_p} - qK^{<_p><_p} \\
&\quad + (q-1)(B_pK)^{<_p} - (q-1)K^{<_p\vee><_p} \\
&= (B_pJ)^{<_p} - J^{<_p\vee} + q(A_pK)^{<_p} - qK^{<_p} \\
&\quad + (q-1)(B_pK)^{<_p} - (q-1)K^{<_p\vee}.
\end{aligned}$$

□

The formulae in Lemma 7.20 will give a control of eigenvalues of S_p due to the following phenomenon.

Lemma 7.21. *Let V be a Euclidean space with scalar product $\langle .,.\rangle$. Let A and B be self-adjoint endomorphisms of V and $r, s \geq 1$ be real numbers with $\langle Av, v\rangle \leq r\|v\|^2$ for any v in V. Let C be the endomorphism of V^2 given in matrix form by*

$$C = \begin{pmatrix} B & s(A-1)+(s-1)B \\ -1 & -(s-1) \end{pmatrix}.$$

Let u be a complex eigenvalue of C. Then, if u is not real or u is not simple, we have $|u|^2 \leq s(r-1)$.

Proof. As in Subsection 1.5, let $V_{\mathbb{C}}$ be the complexification of V and denote by $v \mapsto \overline{v}$ the complex conjugation in $V_{\mathbb{C}}$. We write $\langle .,.\rangle$ for the complex symmetric bilinear form of $V_{\mathbb{C}}$ whose restriction to V is the scalar product of V. Thus, the Hermitian form $(v,w) \mapsto \langle \overline{v}, w\rangle$ is a Hermitian scalar product on $V_{\mathbb{C}}$. For v in $V_{\mathbb{C}}$, we still write $\|v\| = \sqrt{\langle \overline{v}, v\rangle}$.

Let u be a complex non real eigenvalue of C. Then, there exists v, w in $V_{\mathbb{C}}$ which are not both 0 such that

$$Bv + s(A-1)w + (s-1)Bw = uv \tag{7.7}$$

$$-v - (s-1)w = uw. \tag{7.8}$$

The second equation gives $v = -(u+s-1)w$. In particular, $w \neq 0$ and we can assume $\|w\| = 1$. Besides, we can eliminate v from (7.7) to get

$$-uBw + s(A-1)w = -u(u+s-1)w,$$

that is,

$$u^2w + u(s-1-B)w + s(A-1)w = 0. \tag{7.9}$$

By taking the product with $\overline{w}$, as $\|w\| = 1$, we get the scalar equation

$$u^2 + u\langle \overline{w}, (s-1-B)w\rangle + s\langle \overline{w}, (A-1)w\rangle = 0. \tag{7.10}$$

We claim that the coefficients of this equation are real numbers. Indeed, for example, as the endomorphism A is real and self-adjoint, we have,

$$\overline{\langle \overline{w}, Aw\rangle} = \langle w, \overline{Aw}\rangle = \langle w, A\overline{w}\rangle = \langle Aw, \overline{w}\rangle = \langle \overline{w}, Aw\rangle,$$

hence $\langle \overline{w}, Aw\rangle$ and also $\langle \overline{w}, Bw\rangle$ are real numbers. Since by assumption, u is not a real number, the roots of (7.10) are u and $\overline{u}$ and we have

$$|u|^2 = s\langle \overline{w}, (A-1)w\rangle \leq s(r-1)$$

as required.

Now, let u be a real eigenvalue of C that is not simple. Then, we may find v, w, v', w' in V with v, w not both 0 and

$$C\begin{pmatrix} v \\ w \end{pmatrix} = u\begin{pmatrix} v \\ w \end{pmatrix} \text{ and } C\begin{pmatrix} v' \\ w' \end{pmatrix} = u\begin{pmatrix} v' \\ w' \end{pmatrix} + \begin{pmatrix} v \\ w \end{pmatrix}.$$

As above, (7.7) and (7.8) hold and we can assume that w is a unit vector. Besides, we now get

$$\begin{aligned} Bv' + s(A-1)w' + (s-1)Bw' &= uv' + v \\ -v' - (s-1)w' &= uw' + w. \end{aligned} \tag{7.11}$$

Thus, we have

$$v = -(u+s-1)w \text{ and } v' = -(u+s-1)w' - w,$$

so that (7.11) gives

$$-uBw' - Bw + s(A-1)w' = -u((u+s-1)w' + w) - (u+s-1)w,$$

hence

$$u^2w' + u(s-1-B)w' + s(A-1)w' + 2uw + (s-1-B)w = 0.$$

Let w'' be the orthogonal projection of w' on the orthogonal complement of $\mathbb{R}w$ in V. As (7.9) holds, we still have

$$u^2w'' + u(s-1-B)w'' + s(A-1)w'' + 2uw + (s-1-B)w = 0.$$

Since $\langle w, w''\rangle = 0$, by taking the scalar product with w in this equation, we obtain

$$\langle w, (-uB+sA)w''\rangle + 2u + \langle w, (s-1-B)w\rangle = 0. \tag{7.12}$$

As A and B are self-adjoint, by using (7.9), we get

$$\begin{aligned} \langle w, (-uB+sA)w''\rangle &= \langle (-uB+sA)w, w''\rangle \\ &= \langle (s-(s-1)u-u^2)w, w''\rangle = 0. \end{aligned}$$

Therefore, (7.12) yields

$$2u + \langle w, (s - 1 - B)w\rangle = 0,$$

that is, u is a double root of (7.10). We get

$$u^2 = s\langle w, (A - 1)w\rangle \le s(r - 1)$$

as required. □

To deal with certain subtle equality cases in the proof of Proposition 7.16, we will need an enhanced version of Lemma 2.8.

Lemma 7.22. *Let a, b in $\mathbb{R}$, $k \ge -1$ and H be in $\mathcal{H}_{k,+}$. Assume that we have*

$$aH^{>>} + bH^{>\vee>} = H^{>\vee>\vee}. \tag{7.13}$$

Then, if k is even, we have $H = 0$; if k is odd, we have $H = 0$ or $a - b + 1 = 0$.

Proof. Assume $k = -1$. Then, as the $\vee$ operator is -1 on 0-pseudofunctions, the equation reads as

$$(a - b)H^{>>} = -H^{>>\vee} = -H^{\vee>>} = -H^{>>},$$

the latter holding by the assumption that H is in $\mathcal{H}_{-1,+}$. In other words we have $(a - b + 1)H = 0$ and the conclusion follows.

If $k = 0$, the conclusion holds trivially since $\mathcal{H}_{0,+} = \{0\}$.

We will prove the general case by a two steps induction. Thus, we assume that $k \ge 1$ and that the result is true for $k - 2$. Let H be in $\mathcal{H}_{k,+}$ such that (7.13) holds. Thus, we have

$$(aH^{>} + bH^{>\vee})^{>} = H^{>\vee>\vee}.$$

By Lemma 2.8, there exists J in $\mathcal{H}_k$ with $H^{>\vee} = J^{>}$. Again applying Lemma 2.8, we find K in $\mathcal{H}_{k-1}$ with $H = K^{>}$. As H is in $\mathcal{H}_{k,+}$, a final application of Lemma 2.8 allows to find an element L in $\mathcal{H}_{k-2,+}$ with $H = L^{>>}$. Now, as H satisfies (7.13), Lemma 2.6 gives

$$aL^{>>} + bL^{>\vee>} = L^{>\vee>\vee}$$

and the conclusion follows from the induction assumption. □

We can now use these tools to give the

Proof of Proposition 7.16. We will actually prove the analogous result for the adjoint operator $S_p^\dagger$.

Consider the linear map

$$\Delta_p : \mathcal{H}_{k,+}^2 \to \mathcal{H}_{k-1}, (J, K) \mapsto J^{<_p} + K^{<_p\vee}.$$

By elementary duality arguments, the p-orthogonal complement in $\mathcal{H}_{k-1}$ of the range of Δ_p is the space of H in $\mathcal{H}_{k-1}$ such that $H^{>}+H^{>\vee}=0$ and $H^{\vee>}+H^{\vee>\vee}=0$. By Lemma 7.7, Lemma 7.17 and Lemma 7.18, the latter space is stable under $S_p^\dagger$ and the spectrum of $S_p^\dagger$ in this space is contained in $\{-1,1\}$. Now, Lemma 7.20 says that the range of Δ_p is also stable under $S_p^\dagger$. We will study the spectrum of $S_p^\dagger$ in $\Delta_p\mathcal{H}_{k,+}^2$ by means of Lemma 7.21.

First, assume that k is even. Then, for J,K in $\mathcal{H}_{k,+}$, Lemma 7.20 may be written as

$$S_p^\dagger\Delta_p\begin{pmatrix}J\\K\end{pmatrix}=\Delta_p\begin{pmatrix}B_p & A_p-1\\-1 & 0\end{pmatrix}\begin{pmatrix}J\\K\end{pmatrix}. \tag{7.14}$$

Thanks to Lemma 7.19, we can apply Lemma 7.21 to the space $\mathcal{H}_{k,+}$ and the operator $C_p=\begin{pmatrix}B_p & A_p-1\\-1 & 0\end{pmatrix}$ by taking $r=q+1$ and $s=1$. This tells us that all complex eigenvalues of C_p in $\mathcal{H}_{k,+}^2$ which have modulus $>\sqrt{q}$ are real and simple. A fortiori, by (7.14), the same holds for the eigenvalues of $S_p^\dagger$ in $\Delta_p\mathcal{H}_{k,+}^2$. To conclude, we must investigate the delicate case of eigenvalues with modulus exactly $\sqrt{q}$. This will require us to have a closer look at the proof of Lemma 7.21.

Thus, let u in $\mathbb{C}$ with $|u|=\sqrt{q}$ and assume by contradiction that u is an eigenvalue of C_p in $\mathcal{H}_{k,+}^2$ and that, if u is real (that is, if $u\in\{-\sqrt{q},\sqrt{q}\}$), this eigenvalue is not simple. By the proof of Lemma 7.21 (see in particular (7.9) and (7.10)), there exists $K\neq 0$ in the complexification $\mathcal{H}_{k,+,\mathbb{C}}$ of $\mathcal{H}_{k,+}$ with

$$u^2K-uB_pK+(A_p-1)K=0 \text{ and } p((A_p-1)K,\overline{K})=qp(K,\overline{K}). \tag{7.15}$$

By Lemma 7.19, the second equation implies that there exists L in $\mathcal{H}_{k-2,+,\mathbb{C}}$ with $K=L^{>>}$. We get by definition,

$$A_pK=L^{>><_p>}+L^{>><_p>\vee}=L^{>>}+L^{>>\vee}=(q+1)L^{>>}$$

and

$$B_pK=L^{>><_p\vee>}+L^{>><_p\vee>\vee}=L^{>\vee>}+L^{>\vee>\vee},$$

so that, by (7.15), we have

$$\left(u+\frac{q}{u}\right)L^{>>}-L^{>\vee>}=L^{>\vee>\vee}.$$

By Lemma 7.22, we get $L=0$, which contradicts the assumption that $K\neq 0$. The Proposition follows in the even case.

We now assume that k is odd. For J, K in $\mathcal{H}_{k,+}$, Lemma 7.20 now gives

$$S_p^\dagger \Delta_p \begin{pmatrix} J \\ K \end{pmatrix} = \Delta_p \begin{pmatrix} B_p & q(A_p - 1) + (q-1)B_p \\ -1 & -(q-1) \end{pmatrix} \begin{pmatrix} J \\ K \end{pmatrix}. \tag{7.16}$$

By Lemma 7.19, we can apply Lemma 7.21 to the space $\mathcal{H}_{k,+}$ and the operator $C_p = \begin{pmatrix} B_p & q(A_p - 1) + (q-1)B_p \\ -1 & -(q-1) \end{pmatrix}$ by taking $r = 2$ and $s = q$. This again tells us that all complex eigenvalues of C_p in $\mathcal{H}_{k,+}^2$ which have modulus $> \sqrt{q}$ are real and simple. By (7.16), the same holds for the eigenvalues of $S_p^\dagger$ in $\Delta_p \mathcal{H}_{k,+}^2$. As above, we will use the proof of Lemma 7.21 to study the case of eigenvalues with modulus exactly $\sqrt{q}$.

Let u in $\mathbb{C}$ with $|u| = \sqrt{q}$ and assume by contradiction that u is an eigenvalue of C_p in $\mathcal{H}_{k,+}^2$ and that, if u is real, this eigenvalue is not simple. Still by the proof of Lemma 7.21, there exists $K \neq 0$ in $\mathcal{H}_{k,+,\mathbb{C}}$ with

$$u^2 K + u(q-1-B_p)K + q(A_p-1)K = 0 \text{ and } p((A_p-1)K, \overline{K}) = p(K, \overline{K}). \tag{7.17}$$

By Lemma 7.19, the second equation implies that there exists L in $\mathcal{H}_{k-2,+,\mathbb{C}}$ with $K = L^{>>}$. This gives

$$A_p K = L^{>><_p>} + L^{>><_p>\vee} = L^{>>} + L^{>>\vee} = 2L^{>>}$$

and, as above,

$$B_p K = L^{>><_p\vee>} + L^{>><_p\vee>\vee} = L^{>\vee>} + L^{>\vee>\vee}.$$

Hence, (7.17) yields

$$\left(u + \frac{q}{u} + (q-1)\right) L^{>>} - L^{>\vee>} = L^{>\vee>\vee}.$$

As $K \neq 0$, we have $L \neq 0$, so that, by Lemma 7.22, we get

$$u + \frac{q}{u} + q + 1 = 0.$$

The two roots of this equation are $-q$ and -1. This contradicts the assumption that u has modulus $\sqrt{q}$ and the Proposition follows in the odd case. □

7.5. **Exceptional quadratic forms.** The proof of Proposition 7.16 shows a particular structure of the eigenspaces of $S_p^\dagger$ associated to eigenvalues with modulus $> \sqrt{q}$. This will allow us to prove a positivity result on those spaces, which will later play a role in the statement of the Plancherel formula for Euclidean fields.

Proposition 7.23. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Let u in $\mathbb{R}$ be an eigenvalue of $S_p^\dagger$ with $|u| > \sqrt{q}$. Then, the symmetric bilinear form $(H, J) \mapsto p(H, J^\vee)$ is anisotropic on the eigenspace*

$$\{H \in \mathcal{H}_{k-1} | S_p^\dagger H = uH\}.$$

More precisely, if k is even, it is positive definite if $u > \sqrt{q}$ and negative definite if $u < \sqrt{q}$. If k is odd, it is always positive definite.

Recall that a symmetric bilinear form p on a real vector space V is said to be anisotropic, or equivalently definite, if, for any $v \neq 0$ in V, one has $p(v, v) \neq 0$. In that case, p is either positive definite or negative definite.

Proof. Let H be in $\mathcal{H}_{k-1}$ with $S_p^\dagger H = uH$. Assume k is even. Then, by the proof of Proposition 7.16 and by Lemma 7.20, we can find J, K in $\mathcal{H}_{k,+}$ with

$$\begin{aligned} J^{<_p} + K^{<_p \vee} &= H \\ B_p J + (A_p - 1)K &= uJ \\ -J &= uK. \end{aligned}$$

We get

$$\begin{aligned} p(H, H^\vee) &= p(J^{<_p}, J^{<_p \vee}) + 2p(J^{<_p}, K^{<_p}) + p(K^{<_p}, K^{<_p \vee}) \\ &= (u^2 + 1)p(K^{<_p}, K^{<_p \vee}) - 2up(K^{<_p}, K^{<_p}) \\ &= \frac{u^2 + 1}{q + 1} p(K, B_p K) - \frac{2u}{q + 1} p(K, A_p K). \end{aligned}$$

Note that we have $B_p J = (1 - u^2)K - A_p K$, hence $B_p K = (u - u^{-1})K + u^{-1} A_p K$. We get

$$(q + 1)p(H, H^\vee) = (u - u^{-1})((u^2 + 1)p(K, K) - p(K, A_p K)).$$

By assumption, we have $u^2 > q$ and by Lemma 7.19, we have $p(K, A_p K) \leq (q + 1)p(K, K)$. The conclusion follows.

Assume now that k is odd. Again by Proposition 7.16 and Lemma 7.20, we get J, K in $\mathcal{H}_{k,+}$ with

$$J^{<_p} + K^{<_p \vee} = H$$

$$B_p J + q(A_p - 1)K + (q - 1)B_p K = uJ \tag{7.18}$$

$$-J - (q - 1)K = uK \tag{7.19}$$

We have $J = -(u + q - 1)K$, hence

$$H = -(u + q - 1)K^{<_p} + K^{<_p \vee} \text{ and } H^\vee = qK^{<_p} - uK^{<_p \vee}.$$

We get

$$p(H,H^\vee) = -q(u+q-1)p(K^{<p},K^{<p}) + (u(u+q-1)+q)p(K^{<p},K^{<p\vee}) - up(K^{<p},K^{<p\vee\vee}).$$

As $K^{<p\vee\vee} = qK^{<p} + (q-1)K^{<p\vee}$, this gives

$$\begin{aligned} p(H,H^\vee) &= -q(2u+q-1)p(K^{<p},K^{<p}) + (u^2+q)p(K^{<p},K^{<p\vee}) \\ &= -\frac{q(2u+q-1)}{2}p(K,A_pK) + \frac{u^2+q}{2}p(K,B_pK). \end{aligned}$$

Since (7.18) and (7.19) yield

$$B_pK = (u+q-1-qu^{-1})K + qu^{-1}A_pK,$$

we finally get

$$\begin{aligned} &2p(H,H^\vee) = \\ &(u^2+q)(u+q-1-qu^{-1})p(K,K) + ((u^2+q)qu^{-1} - q(2u+q-1))p(K,A_pK) \\ &\qquad = (u+q-1-qu^{-1})((u^2+q)p(K,K) - qp(K,A_pK)). \end{aligned}$$

By Proposition 7.9, as u is an eigenvalue of $S_p^\dagger$, we must have $-q < u < -\sqrt{q}$ or $\sqrt{q} < u < q$. Since $u+q-1-qu^{-1} = u^{-1}(u-1)(u+q)$ this number is positive for all those values of u. Now, by Lemma 7.19, we have $p(K,A_pK) \leq 2p(K,K)$ and the conclusion follows. □

8. A resolvent formula for Euclidean fields

Our objective is still to give a Plancherel formula for $\mathcal{H}_\infty$, in the spirit of (4.1), (4.8), (5.1) and (5.4). To this aim, we will start by establishing a resolvent formula. Let us be more precise.

Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. We have associated to p a scalar product on $\mathcal{H}_\infty$ by the natural formula (7.1). Let R and S be the operators acting on $\mathcal{H}_\infty$ which are defined in Subsection 2.5. One easily shows, by using Lemma I.9.11, that the operators R and S are bounded and self-adjoint with respect to p. As usual, we set $P = \frac{1}{q+1}(RS+SR-(q-1)S)$. We still denote by p^∞ the scalar product of the completion $\mathcal{H}_\infty^p$ of $\mathcal{H}_\infty$ with respect to p^∞. We also denote by p^∞ the natural complex symmetric bilinear form on the complexification $\mathcal{H}_{\infty,\mathbb{C}}^p$ of this completion. Then, for H in $\mathcal{H}_\infty$ and t in the upper half-plane $\mathbb{H}$, the complex number $p^\infty(H,(P-t)^{-1}H)$ is well-defined. In this Section, we establish the following formulas for this number, which only involve operations in finite-dimensional spaces.

Proposition 8.1. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field, u be in $\mathbb{H}_q$ and set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Fix H in $\mathcal{H}_{k,+}$ and J in $\mathcal{H}_{k,-}$.*

If k is even, we have

$$\begin{aligned}
p^\infty(H^{>\infty}, (P-t)^{-1}H^{>\infty}) &= \frac{q+1}{q}\frac{u}{1-u^2}p(H, (qu+S_p)(u-S_p)^{-1}H)\\
p^\infty(J^{>\infty}, (P-t)^{-1}J^{>\infty}) &= (q+1)\frac{u}{q^2-u^2}p(J, (u+qS_p)(u-S_p)^{-1}J)\\
p^\infty(H^{+>\infty}, (P-t)^{-1}J^{>\infty}) &= -\frac{q+1}{q}p(H, S_p(u-S_p)^{-1}J)\\
p^\infty(J^{+>\infty}, (P-t)^{-1}H^{>\infty}) &= (q+1)p(J, S_p(u-S_p)^{-1}H).
\end{aligned}$$

If k is odd, we have

$$\begin{aligned}
p^\infty(H^{>\infty}, (P-t)^{-1}H^{>\infty}) &= \frac{(q+1)u}{(q+u)(1-u)}p(H, (u+S_p)(u-S_p)^{-1}H)\\
p^\infty(J^{>\infty}, (P-t)^{-1}J^{>\infty}) &= \frac{(q+1)u}{(q-u)(1+u)}p(J, (u+S_p)(u-S_p)^{-1}J)\\
p^\infty(H^{+>\infty}, (P-t)^{-1}J^{>\infty}) &= -(q+1)p(H, S_p(u-S_p)^{-1}J)\\
p^\infty(J^{+>\infty}, (P-t)^{-1}H^{>\infty}) &= (q+1)p(J, S_p(u-S_p)^{-1}H).
\end{aligned}$$

Recall from Subsection 3.3 that $\mathbb{H}_q$ is the set of u in $\mathbb{C}$ with $\Im u > 0$ and $|u| > \sqrt{q}$. In particular, the formulae make sense as Proposition 7.16 implies that S_p has no eigenvalue in $\mathbb{H}_q$.

8.1. **A formula for inverses.** The proof of Proposition 8.1 will rely on an explicit construction of the vector $(P-t)^{-1}H^{>\infty}$ in the completion of $\mathcal{H}_\infty$ with respect to p. This construction will be an analogue of the ones provided for the model operators. It will require us to show that certain series converge in $\mathcal{H}^p_{\infty,\mathbb{C}}$, which will be ensured by the following

Lemma 8.2. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and H be in $\mathcal{H}_k$. Then, for every $i \geq 0$, we have*

$$p(H^{+^{2i}}, H^{+^{2i}}) \leq q^{2i}p(H, H).$$

For brevity, we have denoted by p the scalar product obtained from p on $\mathcal{H}_{k+2i}$ by successive orthogonal extensions.

Proof. Recall the notation of Subsections 2.3 and 2.5. In particular, by Lemma 2.15, we have, if k is even,

$$H^{++>\infty} = H^{>\vee>\vee>\infty} = R(H^{>\vee>\infty}) = RS(H^{>\infty}).$$

If k is odd, we have $H^{++>\infty} = SR(H^{>\infty})$. By iterating, we get, for $i \geq 0$,

$$\begin{aligned} H^{+^{2i}>\infty} &= (RS)^i(H^{>\infty}) && \text{if } k \text{ is even} \\ &= (SR)^i(H^{>\infty}) && \text{if } k \text{ is odd.} \end{aligned}$$

To conclude, recall that the operators R and S are self-adjoint in $\mathcal{H}_\infty$ with respect to p. As $S^2 = 1$, S is an isometry, whereas, as the eigenvalues of R are q and -1, R has norm q. Thus, both RS and SR have norm $\leq q$ in $\mathcal{H}_\infty$. The Lemma follows. □

Thus, we can ensure that the series defined in the following Lemma converge.

Lemma 8.3. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and H be in $\mathcal{H}_k$. Fix u in $\mathbb{C}$ with $|u| > q$ and set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Then, for H in $\mathcal{H}_{k,+}$ and J in $\mathcal{H}_{k,-}$, the vectors $(P-t)^{-1}H^{>\infty}$ and $(P-t)^{-1}J^{>\infty}$ may be defined by the following absolutely converging series in $\mathcal{H}^p_{\infty,\mathbb{C}}$. If k is even,*

$$\begin{aligned} (P-t)^{-1}H^{>\infty} &= \frac{q+1}{1-u^2}\sum_{i=0}^{\infty} u^{-i}(uH^{+^{2i}>\infty} + H^{+^{2i+1}>\infty}) \\ (P-t)^{-1}J^{>\infty} &= \frac{q+1}{q^2-u^2}\sum_{i=0}^{\infty} u^{-i}(uJ^{+^{2i}>\infty} - qJ^{+^{2i+1}>\infty}). \end{aligned}$$

If k is odd,

$$\begin{aligned} (P-t)^{-1}H^{>\infty} &= \frac{q+1}{(q+u)(1-u)}\sum_{i=0}^{\infty} u^{-i}(uH^{+^{2i}>\infty} + H^{+^{2i+1}>\infty}) \\ (P-t)^{-1}J^{>\infty} &= \frac{q+1}{(q-u)(1+u)}\sum_{i=0}^{\infty} u^{-i}(uJ^{+^{2i}>\infty} - J^{+^{2i+1}>\infty}). \end{aligned}$$

Proof. The convergence of the series is a consequence of Lemma 8.2. The fact that they define $(P-t)^{-1}H^{>\infty}$ follows from Lemma 2.18 and the analogous results for the model operators in Lemmas 4.9, 4.19, 5.9 and 5.18. □

8.2. **Scalar product with large extensions.** Now that we have constructed $(P-t)^{-1}H^{>\infty}$ in Lemma 8.3, it remains to evaluate the quantity $p^\infty(H^{>\infty}, (P-t)^{-1}H^{>\infty})$. This will use the following Lemma which relates this computation with the simple transfer operator.

Lemma 8.4. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and H, J be in $\mathcal{H}_k$. For every $i \geq 0$, we have*

$$\begin{aligned} p(H^{+^{2i}}, J^{>^{2i}}) &= p(S_p^i H, J) \\ p(H^{+^{2i+1}}, J^{+>^{2i}}) &= p(S_p^i H, J) \qquad \text{if } k \text{ is even} \\ p(H^{+^{2i+1}}, J^{+>^{2i}} - (q-1)J^{>^{2i+1}}) &= qp(S_p^i H, J) \qquad \text{if } k \text{ is odd.} \end{aligned}$$

Proof. Let us prove the first formula. This follows from the properties of the operations on pseudofunctions and an induction argument. Indeed, assume the formula holds for $i \geq 0$ and let us still write S_p for the simple transfer operator of the double orthogonal extension p^{++}. Then, we have, by using Lemma 7.4,

$$\begin{aligned} p(H^{+^{2i+2}}, J^{>^{2i+2}}) &= p(S_p^i(H^{>\vee>\vee}), J^{>>}) = p((S_p^i H)^{>\vee>\vee}, J^{>>}) \\ &= p(S_p^i H, J^{>>\vee<_p\vee<_p}) = p(S_p^i H, J^{\vee>><_p\vee<_p}) = p(S_p^i H, J^{\vee>\vee<_p}) \\ &= p(S_p^i H, S_p^\dagger J) = p(S_p^{i+1} H, J). \end{aligned}$$

Now we prove the other two formulas. Assume k is even. By the first formula, we have

$$\begin{aligned} p(H^{+^{2i+1}}, J^{+>^{2i}}) &= p(S_p^i(H^{>\vee}), J^{>\vee}) = p((S_p^i H)^{>\vee}, J^{>\vee}) \\ &= p(S_p^i H, J^{>\vee\vee<_p}) = p(S_p^i H, J). \end{aligned}$$

In the same way, if k is odd,

$$\begin{aligned} p(H^{+^{2i+1}}, J^{+>^{2i}} - (q-1)J^{>^{2i+1}}) &= p(S_p^i H, (J^{>\vee} - (q-1)J^{>})^{\vee<_p}) \\ &= qp(S_p^i H, J). \end{aligned}$$

□

When J is an eigenvector of the $\vee$ operator, we can also get a formula for other extensions.

Corollary 8.5. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and H, J be in $\mathcal{H}_k$. For every $i \geq 0$, we have,*

$$\begin{aligned} p(H^{+^{2i+1}}, J^{>^{2i+1}}) &= \frac{1}{q} p(S_p^{i+1} H, J) \qquad && k \text{ even}, \ J^\vee = qJ \\ &= p(S_p^{i+1} H, J) && k \text{ odd}, \ J^\vee = J \\ &= -p(S_p^{i+1} H, J) && J^\vee = -J \end{aligned}$$

and also

$$\begin{aligned} p(H^{+^{2i}>}, J^{+>^{2i}}) &= \frac{1}{q} p(S_p^{i+1} H, J) && k \text{ even, } J^\vee = qJ \\ &= p(S_p^{i+1} H, J) && k \text{ odd, } J^\vee = J \\ &= -p(S_p^{i+1} H, J) && J^\vee = -J. \end{aligned}$$

Proof. By Lemma 7.4 and Lemma 8.4, we have

$$\begin{aligned} p(H^{+^{2i+1}}, J^{>^{2i+1}}) = p(S_p^i(H^{>\vee}), J^{>}) &= p((S_p^i H)^{>\vee}, J^{>}) \\ &= p((S_p^i H)^{>\vee<_p}, J). \end{aligned}$$

The first set of formulas easily follows.

Let us study the second set. If $i = 0$, we have

$$p(H^{>}, J^{+}) = p(H^{>}, J^{>\vee}) = p(H^{>\vee<_p}, J)$$

and the conclusion follows. If $i \geq 1$, by Lemma 7.2, Lemma 7.4 and Lemma 8.4, we have

$$\begin{aligned} p(H^{+^{2i}>}, J^{+>^{2i}}) &= p(H^{+^{2i}}, J^{+>^{2i-1}}) \\ &= p(S_p^{i-1}(H^{>\vee>\vee}), J^{>\vee>}) \\ &= p((S_p^{i-1} H)^{>\vee>\vee}, J^{>\vee>}) \\ &= p((S_p^{i-1} H)^{>\vee>\vee<_p\vee<_p}, J) \\ &= p((S_p^{i-1} H)^{>\vee<_p\vee>\vee<_p}, J) = p((S_p^i H)^{>\vee<_p}, J). \end{aligned}$$

The second set of formulas follows. □

8.3. **Geometric series and the resolvent formula.** We can now conclude the proof of the resolvent formula which essentially relies on summing a geometric series.

Proof of Proposition 8.1. We assume that k is even and we compute $p^\infty(H^{>^\infty}, (P-t)^{-1} H^{>^\infty})$. Take u in $\mathbb{C}$ with $|u| > q$ and set as usual $t = \frac{1}{q+1}(u + \frac{q}{u})$. By Lemma 8.3, Lemma 8.4 and Corollary 8.5, we get

$$p^\infty(H^{>^\infty}, (P-t)^{-1} H^{>^\infty}) = \\ \frac{q+1}{1-u^2} \sum_{i=0}^{\infty} u^{-i} \left(u p(H, S_p^i H) + \frac{1}{q} p(H, S_p^{i+1} H) \right).$$

By Lemma 7.8 and Proposition 7.9, S_p has spectral radius $< q$ in $\mathcal{H}_k$. Thus, the series $\sum_{i \geq 0} u^{-i} S_p^i$ converges absolutely in the space of endomorphisms of $\mathcal{H}_{k,\mathbb{C}}$. We get

$$\sum_{i=0}^{\infty} u^{-i}\left(uS_p^i + \frac{1}{q}S_p^{i+1}\right) = \frac{1}{q}(qu+S_p)(1-u^{-1}S_p)^{-1}$$
$$= \frac{u}{q}(qu+S_p)(u-S_p)^{-1}$$

hence,

$$p^\infty(H^{>\infty},(P-t)^{-1}H^{>\infty}) = \frac{q+1}{q}\frac{u}{1-u^2}p(H,(qu+S_p)(u-S_p)^{-1}H).$$

We have shown that the two hand-sides of the first formula in Proposition 8.1 are equal for $|u|>q$. Now, as P is self-adjoint, the left hand-side is a holomorphic function of t as t varies in $\mathbb{H}$. Therefore, by Lemma 3.6, it is a holomorphic function of u as u varies in $\mathbb{H}_q$. By Proposition 7.16, the right hand-side is a holomorphic function of u as u varies in $\mathbb{H}_q$. Hence, by analytic continuation, the equality holds for every u in $\mathbb{H}_q$. The computation of $p^\infty(J^{>\infty},(P-t)^{-1}J^{>\infty})$ is analogous.

Still assume k is even and let us now compute the complex number $p^\infty(H^{+>\infty},(P-t)^{-1}J^{>\infty})$. As above, we let t,u be in $\mathbb{C}$ with $|u|>q$ and $q+u^2=(q+1)tu$. Again by Lemma 8.3, Lemma 8.4 and Corollary 8.5, we have

$$p^\infty(H^{+>\infty},(P-t)^{-1}J^{>\infty}) =$$
$$\frac{q+1}{q^2-u^2}\sum_{i=0}^{\infty}u^{-i}\left(\frac{u}{q}p(H,S_p^{i+1}J) - qp(H,S_p^iJ)\right).$$

As $p(H,J)=0$, we get

$$p^\infty(H^{+>\infty},(P-t)^{-1}J^{>\infty}) = \frac{q+1}{q^2-u^2}\sum_{i=0}^{\infty}u^{-i}\left(\frac{u}{q}-\frac{q}{u}\right)p(H,S_p^{i+1}J)$$
$$= -\frac{q+1}{q}\sum_{i=1}^{\infty}u^{-i}p(H,S_p^iJ)$$
$$= -\frac{q+1}{q}p(H,S_p(u-S_p)^{-1}J)$$

and we conclude as in the first case.

The remaing cases can be dealt with similarly. □

Later on, we will deduce a Plancherel formula from Proposition 8.1. For now, we can already say that it allows to control the spectrum of the operator P with respect to the scalar product associated to p on $\mathcal{H}_\infty$. We denote by Σ_p the finite set

$$(8.1)\quad \Sigma_p = \left\{ \frac{1}{q+1}\left(u+\frac{q}{u}\right) \middle| u \in \mathbb{R}, |u| > \sqrt{q}, \ker(S_p - u) \neq \{0\} \right\}.$$

We call Σ_p the exceptional spectrum. Note that, if Γ is bipartite, we have $\Sigma_p = -\Sigma_p$ by Lemma 7.6.

Corollary 8.6. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Then, the spectrum of P in the completion $\mathcal{H}^p_\infty$ of $\mathcal{H}_\infty$ with respect to p is contained in the set*

$$\mathcal{I}_q \cup \Sigma_p \cup \{-1, 1\}.$$

We will prove later in Corollary 13.2 that equality actually holds. Recall from Subsection 3.3 that $\mathcal{I}_q$ stands for the critical interval $\mathcal{I}_q = [-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}]$.

Proof. By Lemma 3.6, the rational function $u \mapsto \frac{1}{q+1}(u + \frac{q}{u})$ induces a biholomorphism from the set $\{u \in \mathbb{C} | |u| > \sqrt{q}\}$ onto $\mathbb{C} \smallsetminus \mathcal{I}_q$. Thus, Lemma 7.8, Proposition 7.16 and Proposition 8.1 imply that for any $j \geq k$ and any H in $\mathcal{H}_j$, the resolvent function $t \mapsto p^\infty(H^{>\infty}, (P-t)^{-1}H^{>\infty})$ admits an analytic continuation to the set $\mathbb{C} \smallsetminus (\mathcal{I}_q \cup \Sigma_p \cup \{-1, 1\})$. By standard properties of self-adjoint operators, this implies the result. $\square$

In the sequel of the article, we will build three families of nonnegative symmetric bilinear forms on $\mathcal{H}^2_k$ parametrized by each of the three sets $\mathcal{I}_q$, Σ_p and $\{-1, 1\}$. These families will allow us to write a Plancherel formula for Euclidean fields in Section 13.

9. u-radical pseudofields and spectral quadratic forms

In this Section, we introduce new algebraic objects that will be needed in order to construct the above mentioned families of symmetric bilinear forms.

9.1. u-radical pseudofields.

We start by introducing new subspaces of the space of simple pseudofields. They are defined by an equation that will later turn out to be related to the formulas in Proposition 6.5. Since this relation relies on the spectral parametrization $u \mapsto t = \frac{1}{q+1}(u + \frac{q}{u})$ of Subsection 3.3, we will need to work with complex simple pseudofields: the space of complex simple pseudofields is the complexification of the space of simple pseudofields.

For $k \geq 1$, if k is even, $k = 2\ell$, $\ell \geq 1$, a complex k-simple pseudofield may be seen as a family $(s_{xy})_{(x,y)\in X_1}$ where, for $x \sim y$ in X, s_{xy} is a function $S^\ell(x) \to \mathbb{C}$ the sum of whose values is 0. If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, a complex k-simple pseudofield may be seen as a family $(s_{xy})_{(x,y)\in X_1}$ where, for $x \sim y$ in X, s_{xy} is a function $S^\ell(xy) \to \mathbb{C}$ the sum of whose values is 0. For $k = 0$, a complex 0-simple pseudofield may be seen as a function $X \to \mathbb{C}$.

Definition 9.1. Let $k \geq 1$, u be in $\mathbb{C}^*$, s be a complex k-simple pseudofield. We say that s is u-radical if $s^{\vee<\vee} = us^<$.

Remark 9.2. This definition is motivated by analogous notions previously introduced for quadratic pseudofields (see Definition II.3.7 and Definition II.6.17). It will turn out later that it is related to the formula that appears in Proposition 6.5 (see Lemma 10.3 below).

For $k \geq 1$, the space of Γ-invariant u-radical complex k-simple pseudofields is denoted by $\mathcal{S}^u_{k,\mathbb{C}}$. If u is real, the space of Γ-invariant u-radical k-simple pseudofields is denoted by $\mathcal{S}^u_k$.

Lemma 9.3. *Let $k \geq 2$, u be be in $\mathbb{C}^*$ and s be a u-radical complex k-simple pseudofield. Then $s^{\vee<}$ is u-radical. The map $s \mapsto s^{\vee<}$ sends $\mathcal{S}^u_{k,\mathbb{C}}$ onto $\mathcal{S}^u_{k-1,\mathbb{C}}$.*

Proof. Let s be as in the statement. By Lemma 6.9, we have
$$s^{\vee<\vee<\vee} = us^{<<\vee} = us^{\vee<<},$$
hence $s^{\vee<}$ is u-radical. Now, consider the linear map
$$s \mapsto s^{\vee<}, \mathcal{S}^u_{k,\mathbb{C}} \to \mathcal{S}^u_{k-1,\mathbb{C}}.$$
To show that it is surjective, we will show that the adjoint map is injective. Let H be in $\mathcal{H}_{k-1,\mathbb{C}}$ and assume that there exists J in $\mathcal{H}_{k-1,\mathbb{C}}$ with $H^{>\vee} = J^{\vee>\vee} - uJ^>$. Then, Lemma 2.8 says that there exists K in $\mathcal{H}_{k-2,\mathbb{C}}$ with $J^\vee - H = K^>$ and $uJ = K^{\vee>}$. We get $H = J^\vee - K^> = u^{-1}K^{\vee>\vee} - K^>$. Thus, we have shown that if $H^{>\vee}$ belongs to the orthogonal subspace to $\mathcal{S}^u_{k,\mathbb{C}}$ in $\mathcal{H}_{k,\mathbb{C}}$, then H belongs to the orthogonal subspace to $\mathcal{S}^u_{k-1,\mathbb{C}}$ in $\mathcal{H}_{k-1,\mathbb{C}}$, which is the desired statement. □

9.2. **The u-opposition map.** We describe a relation between u-radical pseudofiels and $\frac{q}{u}$-radical pseudofields.

Definition 9.4. Let $k \geq 0$, u be in $\mathbb{C}^*$ and s be complex k-simple pseudofied. We define the u-opposite $I_u s$ of s as follows. If k is even, we set
$$I_u s = (q - u^2)s^\vee + u^2(q-1)s.$$
If k is odd, we set
$$I_u s = (q - u^2)s^\vee + u(q-1)s.$$

Lemma 9.5. *Let u in $\mathbb{C}^*$, $k \geq 1$ and s be a u-radical complex k-simple pseudofield. Then $I_u s$ is $\frac{q}{u}$-radical.*

Proof. This is a straightforward computation. If k is even, we get

$$\begin{aligned}(I_u s)^{\vee<\vee} &= (q-u^2)s^{\vee\vee<\vee} + u^2(q-1)s^{\vee<\vee} \\ &= q(q-u^2)s^{<\vee} + (q-1)(q-u^2)s^{\vee<\vee} + u^2(q-1)s^{\vee<\vee} \\ &= q(q-u^2)s^{<\vee} + q(q-1)s^{\vee<\vee} \\ &= \frac{q}{u}(q-u^2)s^{\vee<} + q(q-1)us^{<} = \frac{q}{u}(I_u s)^{<}.\end{aligned}$$

In the same way, if k is odd,

$$\begin{aligned}(I_u s)^{\vee<\vee} &= (q-u^2)s^{\vee\vee<\vee} + u(q-1)s^{\vee<\vee} \\ &= (q-u^2)s^{<\vee} + u(q-1)s^{\vee<\vee} \\ &= \frac{1}{u}(q-u^2)s^{\vee<\vee\vee} + u(q-1)s^{\vee<\vee} \\ &= \frac{q}{u}(q-u^2)s^{\vee<} + \left(\frac{q-1}{u}(q-u^2) + u(q-1)\right)s^{\vee<\vee} \\ &= \frac{q}{u}(q-u^2)s^{\vee<} + \frac{q}{u}(q-1)s^{\vee<\vee} \\ &= \frac{q}{u}(q-u^2)s^{\vee<} + q(q-1)s^{<} = \frac{q}{u}(I_u s)^{<}.\end{aligned}$$

□

When u is not one of the special values $-q$, -1, 1 or q, the opposition maps I_u and $I_{\frac{q}{u}}$ are essentially inverse to each other.

Lemma 9.6. *Let u be in $\mathbb{C}^*$, $k \geq 0$ and s be a complex k-simple pseudofield. We have*

$$\begin{aligned}I_{\frac{q}{u}} I_u s &= \frac{q^2}{u^2}(q^2-u^2)(1-u^2) = (q+1)^2 q^2 (1-t^2) \qquad \text{if } k \text{ is even} \\ &= \frac{q}{u^2}(q^2-u^2)(1-u^2) = (q+1)^2 q (1-t^2) \qquad \text{if } k \text{ is odd,}\end{aligned}$$

where $t = \frac{1}{q+1}(u + \frac{q}{u})$.

Proof. If k is even, we have

$$\begin{aligned}
I_{\frac{q}{u}} I_u s &= \left(q - \frac{q^2}{u^2}\right)((q-u^2)s^{\vee\vee} + u^2(q-1)s^\vee) \\
&\quad + \frac{q^2}{u^2}(q-1)((q-u^2)s^\vee + u^2(q-1)s) \\
&= \frac{q}{u^2}(u^2-q)(q(q-u^2)s + q(q-1)s^\vee) \\
&\quad + \frac{q^2}{u^2}(q-1)(q-u^2)s^\vee + q^2(q-1)^2 s \\
&= -\frac{q^2}{u^2}(q-u^2)^2 s + q^2(q-1)^2 s
\end{aligned}$$

and

$$\begin{aligned}
u^2(q-1)^2 - (q-u^2)^2 &= (u(q-1) - q + u^2)(u(q-1) + q - u^2) \\
&= (q+u)(u-1)(q-u)(u+1) \qquad (9.1) \\
&= (q^2-u^2)(1-u^2) = \frac{(q+1)^2}{u^2}(1-t^2).
\end{aligned}$$

The conclusion follows.

If k is odd, we get

$$\begin{aligned}
I_{\frac{q}{u}} I_u s &= \left(q - \frac{q^2}{u^2}\right)((q-u^2)s^{\vee\vee} + u(q-1)s^\vee) \\
&\quad + \frac{q}{u}(q-1)((q-u^2)s^\vee + u(q-1)s) \\
&= -\frac{q}{u^2}(q-u^2)^2 s + \frac{q}{u}(u^2-q)(q-1)s^\vee \\
&\quad + \frac{q}{u}(q-1)(q-u^2)s^\vee + q(q-1)^2 s \\
&= q(q-1)^2 s - \frac{q}{u^2}(q-u^2)^2 s.
\end{aligned}$$

Again, the conclusion follows from (9.1). □

Corollary 9.7. *Let u be in $\mathbb{C} \smallsetminus \{-q, -1, 0, 1, q\}$. For $k \geq 1$, the u-opposition map I_u induces a linear isomorphism from the space of u-radical complex k-simple pseudofields onto the space of $\frac{q}{u}$-radical complex k-simple pseudofields.*

The opposition map is also compatible with restriction of u-radical pseudofields.

Lemma 9.8. *Let u in $\mathbb{C}^*$, $k \geq 1$ and s be a u-radical complex k-simple pseudofield. If k is even, we have*

$$(I_u s)^{\vee<} = \frac{q}{u} I_u(s^{\vee<}).$$

If k is odd, we have

$$(I_u s)^{\vee<} = \frac{1}{u} I_u(s^{\vee<}).$$

Proof. Again, this is a straightforward computation. For k even, we get

$$\begin{aligned}(I_u s)^{\vee<} &= (q-u^2)s^{\vee\vee<} + u^2(q-1)s^{\vee<}\\ &= q(q-u^2)s^{<} + q(q-1)s^{\vee<}\\ &= \frac{q}{u}(q-u^2)s^{\vee<\vee} + q(q-1)s^{\vee<} = \frac{q}{u} I_u(s^{\vee<}).\end{aligned}$$

For k odd, we get

$$\begin{aligned}(I_u s)^{\vee<} = (q-u^2)s^{<} + u(q-1)s^{\vee<} = \frac{1}{u}(q-u^2)s^{\vee<\vee} + u(q-1)s^{\vee<}\\ = \frac{1}{u} I_u(s^{\vee<}).\end{aligned}$$

□

9.3. **Adjoint operations and transfer operator on pseudofields.** We will now relate the theory of u-radical pseudofields to the spectral theory of the simple transfer operator S_p. To this aim, we introduce adjoint operations on pseudofields in analogy with the language of Subsection 7.1.

Let $k \geq 1$ and p be a Γ-invariant k-Euclidean field. We use p to define as usual a natural scalar product on $\mathcal{S}_k$. Let r, s be in $\mathcal{S}_k$. If k is even, $k = 2\ell$, $\ell \geq 1$, let, for every x in X, p_x^* be the bilinear form dual to p_x on $V_0^\ell(x)$. We set,

$$p^*(r,s) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} p_x^*(r_{xy}, s_{xy}).$$

In the same way, if k is odd, $k = 2\ell+1$, $\ell \geq 0$, let, for every $x \sim y$ in X, p_{xy}^* be the bilinear form dual to p_{xy} on $V_0^\ell(xy)$. We set,

$$p^*(r,s) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} p_{xy}^*(r_{xy}, s_{xy}).$$

Assume $k \geq 2$. We also introduce the adjoint operator of restriction of pseudofields. Let s be in $\mathcal{S}_{k-1}$ and let us define $s^{>_p}$ in $\mathcal{S}_k$. We use the notation of Subsection I.10.6 and Subsection II.6.2. If k is even,

$k = 2\ell$, $\ell \geq 1$, for $x \sim y$ in X, we set $s_{xy}^{>_p} = I_{xy}^{\ell-1,*\dagger p} s_{xy}$. If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for $x \sim y$ in X, we set $s_{xy}^{>_p} = J_{xy}^{\ell,*\dagger p} s_{xy}$. As usual, we get

Lemma 9.9. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For r in $\mathcal{S}_k$ and s in $\mathcal{S}_{k-1}$, we have $p^{-,*}(r^<, s) = p^*(r, s^{>_p})$.*

Still assume $k \geq 2$ and p is a Γ-invariant k-Euclidean field. In Subsection 7.1, we have defined the simple transfer operator S_p acting on the space of Γ-invariant $(k-1)$-pseudofunctions by the formula

$$S_p H = H^{>\vee<_p\vee}, \quad H \in \mathcal{H}_{k-1}.$$

The adjoint $S_p^\dagger$ of S_p with respect to the Euclidean structure may be defined by the formula

$$S_p^\dagger H = H^{\vee>\vee<_p}, \quad H \in \mathcal{H}_{k-1}.$$

Now, the adjoint of these operator with respect to the duality are linear endomorphisms of $\mathcal{S}_{k-1}$ which should formally be written as S_p^* and $S_p^{*\dagger} = S_p^{\dagger *}$. To avoid the latter heavy notation we write S_p instead. In other words, we set

$$S_p s = s^{>_p\vee<\vee}, \quad s \in \mathcal{S}_{k-1}.$$

The adjoint $S_p^\dagger$ of this operator with respect to the Euclidean structure on $\mathcal{S}_{k-1}$ is defined by

$$S_p^\dagger s = s^{\vee>_p\vee<}, \quad s \in \mathcal{S}_{k-1}.$$

Note that all these operators have the same spectrum and that we still have the relation

$$S_p(s^\vee) = (S_p^\dagger s)^\vee, \quad s \in \mathcal{S}_{k-1}.$$

Using these definitions, we get a direct relationship between the spectrum of S_p and the theory of u-radical pseudofields.

Lemma 9.10. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Fix u in $\mathbb{C}^*$ and s in $\mathcal{S}_{k-1,\mathbb{C}}$. Then, $s^{>_p}$ is u-radical if and only if $S_p s = us$.*

Proof. By construction, we have $s^{>_p<} = s$ and $s^{>_p\vee<\vee} = S_p s$. $\square$

As for pseudofunctions, we also denote by S_p and $S_p^\dagger$ the linear endomorphisms of $\mathcal{S}_{k,\mathbb{C}}$ defined by

$$S_p s = s^{<\vee>_p\vee} \text{ and } S_p^\dagger s = s^{\vee<\vee>_p}, \quad s \in \mathcal{S}_{k,\mathbb{C}},$$

which can be thought of as the transfer operator and adjoint transfer operator associated to the orthogonal extension p^+ of p. We directly get

Lemma 9.11. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For any s in $\mathcal{H}_k$, we have*

$$S_p^\dagger(s^<) = (S_p s)^< \text{ and } S_p(s^{<\vee}) = (S_p s)^{<\vee}.$$

9.4. **The dual spectral bilinear form.** We now introduce natural bilinear forms on spaces of u-radical simple pseudofields which are associated to the choice of a Euclidean field. These forms will play a key role in the statement of the Plancherel formula for Euclidean fields.

Definition 9.12. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For u in $\mathbb{C}^*$, we define the dual spectral bilinear form p_u^* associated to p on $\mathcal{S}_{k,\mathbb{C}}^u$ as the bilinear form

$$p_u^*(r,s) = p^*(r, I_u(s)) - p^{-,*}(r^<, I_u(s)^<), \quad r,s \in \mathcal{S}_{k,\mathbb{C}}^u.$$

This bilinear form is actually symmetric.

Lemma 9.13. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For u in $\mathbb{C}^*$, the dual spectral bilinear form p_u^* is symmetric on $\mathcal{S}_{k,\mathbb{C}}^u$.*

Proof. This is again a direct computation. Let r, s be in $\mathcal{S}_{k,\mathbb{C}}^u$. As the $\vee$ operator is symmetric, we have $p^*(r, I_u(s)) = p^*(I_u(r), s)$. Now, if k is even, we have $r^{\vee<} = ur^{<\vee}$ and $s^{\vee<} = us^{<\vee}$, hence

$$\begin{aligned} p^{-,*}(r^<, I_u(s)^<) &= p^{-,*}(r^<, (q-u^2)s^{\vee<} + u^2(q-1)s^<) \\ &= p^{-,*}(r^<, u(q-u^2)s^{<\vee} + u^2(q-1)s^<) \\ &= p^{-,*}(u(q-u^2)r^{<\vee} + u^2(q-1)r^<, s^<) \\ &= p^{-,*}((q-u^2)r^{\vee<} + u^2(q-1)r^<, s^<) \\ &= p^{-,*}(I_u(r)^<, s^<). \end{aligned}$$

If k is odd, we have

$$qs^{\vee<} + (q-1)s^{\vee<\vee} = s^{\vee<\vee\vee} = us^{<\vee},$$

hence $qs^{\vee<} = us^{<\vee} - (q-1)us^<$, which gives

$$I_u(s)^< = (q-u^2)q^{-1}us^{<\vee} + u(q-1)(1-q^{-1})s^<.$$

As in the even case, we get $p^{-,*}(r^<, I_u(s)^<) = p^{-,*}(I_u(r)^<, s^<)$. This shows that p_u is symmetric. □

We have a non degeneracy criterion for p_u^*.

Proposition 9.14. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For u in $\mathbb{C}^*$, the bilinear form*

$$(r,s) \mapsto p^*(r,s) - p^{-,*}(r^<, s^<) = p^*(r - r^{<>_p}, s - s^{<>_p}) \tag{9.2}$$

is non degenerate on $\mathcal{S}^u_{k,\mathbb{C}} \times \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$ *if and only if neither* u *nor* $\frac{q}{u}$ *are eigenvalues of the simple transfer operator* S_p. *Therefore, if this is the case and if in addition* $u \notin \{-q,-1,1,q\}$, *then* p^*_u *is non degenerate.*

To show non degeneracy, we shall use the following elementary

Lemma 9.15. *Let* V *be a finite-dimensional vector space over a field* $\mathbb{K}$ *with characteristic* $\neq 2$ *and* p *be a non degenerate symmetric bilinear form on* V. *Let* W *be a subspace of* V *and assume that the restriction of* p *to* W *is non degenerate, so that the* p*-orthogonal projection* $\pi : V \to W$ *is well-defined. Let* $U = W^{\perp_p}$ *be the* p*-orthogonal complement of* W *and* X *and* Y *be complementary subspaces of* U *in* V. *Then the bilinear form* $(x,y) \mapsto p(\pi(x),\pi(y))$ *is non degenerate on* $X \times Y$.

Proof of Proposition 9.14. Note that the second claim follows from the first and Corollary 9.7. We now prove the first claim.

First, assume for example that u is an eigenvalue of S_p. Then, by Lemma 9.10, there exists $s \neq 0$ in $\mathcal{S}_{k-1,\mathbb{C}}$ with $s^{>_p} \in \mathcal{S}^u_{k,\mathbb{C}}$. For r in $\mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$, we have, by Lemma 9.9, $p^*(r, s^{>_p}) = p^{-,*}(r^<, s)$, hence the bilinear form in (9.2) is degenerate.

To prove the converse, we will apply the criterion in Lemma 9.15. Indeed, the linear map

$$s \mapsto s^{<>_p}, \mathcal{S}_{k,\mathbb{C}} \to \mathcal{S}_{k,\mathbb{C}},$$

is the p-orthogonal projection onto the subspace $\mathcal{S}^{>_p}_{k-1,\mathbb{C}}$, so that the linear map

$$s \mapsto s - s^{<>_p}, \mathcal{S}_{k,\mathbb{C}} \to \mathcal{S}_{k,\mathbb{C}},$$

is the p-orthogonal projection onto the p-orthogonal complement of $\mathcal{S}^{>_p}_{k-1,\mathbb{C}}$. Thus, by Lemma 9.15, to conclude, it suffices to show that $\mathcal{S}^u_{k,\mathbb{C}}$ and $\mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$ are complentary subspaces to $\mathcal{S}^{>_p}_{k-1,\mathbb{C}}$ in $\mathcal{S}_{k,\mathbb{C}}$.

Indeed, as by assumption u is not an eigenvalue of S_p, Lemma 9.10 says that $\mathcal{S}^{>_p}_{k-1,\mathbb{C}} \cap \mathcal{S}^u_{k,\mathbb{C}} = \{0\}$. Now, by definition, $\mathcal{S}^u_{k,\mathbb{C}}$ is the null space of the linear map

$$s \mapsto s^{\vee<\vee} - us^<, \mathcal{S}_{k,\mathbb{C}} \to \mathcal{S}_{k-1,\mathbb{C}},$$

hence its codimension in $\mathcal{S}_{k,\mathbb{C}}$ is less than the dimension of $\mathcal{S}_{k-1,\mathbb{C}}$. Since it intersects $\mathcal{S}^{>_p}_{k-1,\mathbb{C}}$ trivially, this codimension is exactly equal to the dimension of $\mathcal{S}_{k-1,\mathbb{C}}$ and we get $\mathcal{S}_{k,\mathbb{C}} = \mathcal{S}^{>_p}_{k-1,\mathbb{C}} \oplus \mathcal{S}^u_{k,\mathbb{C}}$ as required. In the same way, as $\frac{q}{u}$ is not an eigenvalue of S_p, we get $\mathcal{S}_{k,\mathbb{C}} = \mathcal{S}^{>_p}_{k-1,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$ and the conclusion follows by Lemma 9.15. □

The definition and the properties of the opposition operators yield the following adjointness formulae.

Lemma 9.16. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be in $\mathbb{C}^*$. Set $t = \frac{1}{q+1}(u + \frac{q}{u})$. If r is in $\mathcal{S}^u_{k,\mathbb{C}}$ and s is in $\mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$, we have*

$$p^*_{\frac{q}{u}}(I_u r, s) = p^*_u(r, I_{\frac{q}{u}} s).$$

If k is even, the latter is equal to

$$q^2(q+1)^2(1-t^2)(p^*(r,s) - p^{-,*}(r^<, s^<)).$$

If k is odd, it is equal to

$$q(q+1)^2(1-t^2)(p(r^*,s) - p^{-,*}(r^<, s^<)).$$

Proof. Indeed, by Definition 9.12, we have

$$p^*_u(r, I_{\frac{q}{u}} s) = p^*(r, I_u I_{\frac{q}{u}} s) - p^{-,*}(r^<, (I_u I_{\frac{q}{u}} s)^<).$$

The conclusion follows from this identity and Lemma 9.6 and Lemma 9.13. □

9.5. **The spectral bilinear form.** We will now study the dual object to p^*_u, which will later turn out to be related to the resolvent formula for p obtained in Proposition 8.1.

Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Pick u in $\mathbb{C} \smallsetminus \{-q,-1,0,1,q\}$ such that neither u nor $\frac{q}{u}$ are eigenvalues of S_p. Then, by Proposition 9.14, the dual spectral bilinear form p^*_u is non degenerate on $\mathcal{S}^u_{k,\mathbb{C}}$. Therefore, it defines by duality a symmetric bilinear form p_u on $\mathcal{H}_{k,\mathbb{C}}$ whose null space is the range of the map

$$G \mapsto G^{\vee>\vee} - uG^>, \mathcal{H}_{k-1,\mathbb{C}} \to \mathcal{H}_{k,\mathbb{C}}. \tag{9.3}$$

For H, J in $\mathcal{H}_{k,\mathbb{C}}$, we have $p_u(H,J) = \langle r, J\rangle$ where r is the unique element of $\mathcal{S}^u_{k,\mathbb{C}}$ such that, for any s in $\mathcal{S}^u_{k,\mathbb{C}}$, one has $\langle s, H\rangle = p^*_u(r,s)$.

Definition 9.17. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For u as above, the symmetric bilinear form p_u on $\mathcal{H}_{k,\mathbb{C}}$ is called the spectral bilinear form associated to u.

The purpose of this Subsection is to give a formula for computing p_u on eigenspaces of the operator $\vee$.

Proposition 9.18. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field, H and J be in $\mathcal{H}_{k,\mathbb{C}}$ and u be in $\mathbb{C} \smallsetminus \{-q,-1,0,1,q\}$ such that neither u nor $\frac{q}{u}$ are eigenvalues of S_p. Set $t = \frac{1}{q+1}(u + \frac{q}{u})$ and*

$$H' = (u - S_p)^{-1}\left(\frac{q}{u} - S_p\right)^{-1} h = (q + S_p^2 - (q+1)tS_p)^{-1}H.$$

If k is even, we have

$$\begin{aligned} p_u(H,J) &= \frac{1}{q}\frac{1}{q^2-u^2}p((q^2-S_p^2)H',J) && H^\vee = qH, J^\vee = qJ \\ &= \frac{1}{qu}p(S_pH',J) && H^\vee = -H, J^\vee = qJ \\ &= -\frac{1}{u}p(S_pH',J) && H^\vee = qH, J^\vee = -J \\ &= \frac{1}{u^2-1}p((1-S_p^2)H',J) && H^\vee = -H, J^\vee = -J. \end{aligned}$$

If k is odd, we have

$$\begin{aligned} p_u(H,J) &= \frac{p((q-S_p)(1+S_p)H',J)}{(q-u)(u+1)} && H^\vee = H, J^\vee = J \\ &= \frac{1}{u}p(S_pH',J) && H^\vee = -H, J^\vee = J \\ &= -\frac{1}{u}p(S_pH',J) && H^\vee = H, J^\vee = -J \\ &= \frac{p((q+S_p)(1-S_p)H',J)}{(q+u)(u-1)} && H^\vee = -H, J^\vee = -J. \end{aligned}$$

This result relies on constructions in $\mathcal{S}_{k,\mathbb{C}}$. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be in $\mathbb{C}^*$ such that u and $\frac{q}{u}$ are not eigenvalues of S_p. Pick r in $\mathcal{S}_{k,\mathbb{C}}$. Then, by Proposition 9.14, there exists a unique element r_u in $\mathcal{S}^u_{k,\mathbb{C}}$ such that, for any s in $\mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$, one has

$$p^*(r,s) = p^*(r_u,s) - p^{-,*}(r^<_u, s^<).$$

In case r is an eigenvector of the $\vee$ operator, we can give a formula for defining r_u.

Lemma 9.19. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field, r be in $\mathcal{S}_{k,\mathbb{C}}$ and u be in $\mathbb{C}^*$ such that neither u nor $\frac{q}{u}$ are eigenvalues of S_p. Set*

$$c = (u - S_p^\dagger)^{-1}\left(\frac{q}{u} - S_p^\dagger\right)^{-1}(r^<) \in \mathcal{S}_{k-1,\mathbb{C}}.$$

Then, r_u may be written as $r_u = r + a^{\vee >_p} + b^{\vee >_p \vee}$ where $b = (u - S_p^\dagger)c$ and $a \in \mathcal{S}_{k-1,\mathbb{C}}$ is defined as follows:

$$
\begin{aligned}
a &= \left(\frac{q^2}{u} - S_p^\dagger\right) c, && k \text{ even}, r^\vee = qr \\
a &= q\left(S_p^\dagger - \frac{1}{u}\right) c, && k \text{ even}, r^\vee = -r \\
a &= \left(q - 1 + \frac{q}{u} - S_p^\dagger\right) c, && k \text{ odd}, r^\vee = r \\
a &= \left(q - 1 - \frac{q}{u} + S_p^\dagger\right) c, && k \text{ odd}, r^\vee = -r.
\end{aligned}
$$

Proof. We take a, b, c to be as in the statement and we set $s = r + a^{\vee >_p} + b^{\vee >_p \vee}$. We want to show that $s = r_u$. By definition, we need to show that s is u-radical and that $s - s^{<>_p} - r$ is p-orthogonal to $\mathcal{S}_{k,\mathbb{C}}^{\frac{q}{u}}$. First we show the latter. By construction, as $a^{\vee >_p <} = a^\vee$, we have

$$s - s^{<>_p} - r = b^{\vee >_p \vee} - r^{<>_p} - b^{\vee >_p \vee <>_p} = b^{\vee >_p \vee} - r^{<>_p} - (S_p^\dagger b)^{>_p}.$$

We have $b = (u - S_p^\dagger)c$, hence, $(\frac{q}{u} - S_p^\dagger)b = r^<$. This gives

$$s - s^{<>_p} - r = b^{\vee >_p \vee} - \frac{q}{u} b^{>_p},$$

hence $s - s^{<>_p} - r$ is p-orthogonal to $\mathcal{S}_{k,\mathbb{C}}^{\frac{q}{u}}$.

To conclude, we must prove that s is u-radical. First we compute

$$us^< = ur^< + ua^\vee + uS_p^\dagger b = ua^\vee + qb, \tag{9.4}$$

where we have again used the identity $(\frac{q}{u} - S_p^\dagger)b = r^<$. The computation of $s^{\vee<\vee}$ will depend on the parity of k.

If k is even, we have

$$
\begin{aligned}
s^{\vee<\vee} &= r^{\vee<\vee} + a^{\vee >_p \vee<\vee} + b^{\vee >_p \vee\vee<\vee} \\
&= r^{\vee<\vee} + (S_p^\dagger a)^\vee + qb + (q-1)(S_p^\dagger b)^\vee \\
&= r^{\vee<\vee} + (S_p^\dagger a)^\vee + qb + (q-1)\frac{q}{u} b^\vee - (q-1) r^{<\vee}.
\end{aligned}
$$

Together with (9.4), this gives

$$
\begin{aligned}
s^{\vee>} - us^{<\vee} &= (r^\vee - (q-1)r)^< + (S_p^\dagger - u)a + (q-1)\frac{q}{u} b \\
&= (r^\vee - (q-1)r)^< + (S_p^\dagger - u)\left(a - (q-1)\frac{q}{u} c\right).
\end{aligned}
$$

If $r^\vee = qr$, by using the relation $r^< = (u - S_p^\dagger)\left(\frac{q}{u} - S_p^\dagger\right)c$, we get

$$\begin{aligned} s^{\vee>} - us^{<\vee} &= r^< + (S_p^\dagger - u)\left(a - (q-1)\frac{q}{u}c\right) \\ &= (S_p^\dagger - u)\left(a + S_p^\dagger c - \frac{q^2}{u}c\right), \end{aligned}$$

which is 0 by assumption. In the same way, if $r^\vee = -r$, we get

$$\begin{aligned} s^{\vee>} - us^{<\vee} &= -qr^< + (S_p^\dagger - u)\left(a - (q-1)\frac{q}{u}c\right) \\ &= (S_p^\dagger - u)\left(a - qS_p^\dagger c + \frac{q}{u}c\right), \end{aligned}$$

which is also 0.

If k is odd, we have

$$s^{\vee<\vee} = r^{\vee<\vee} + (S_p^\dagger a)^\vee + qb + (q-1)b^\vee.$$

Thanks to (9.4), we get

$$s^{\vee<\vee} - us^< = \left(r^{\vee<} + (S_p^\dagger - u)a + (q-1)b\right)^\vee.$$

If $r^\vee = r$, we have

$$r^{\vee<} + (S_p^\dagger - u)a + (q-1)b = (S_p^\dagger - u)\left(a - \frac{q}{u}c + S_p^\dagger c - (q-1)c\right) = 0.$$

If $r^\vee = -r$, we have

$$r^{\vee<} + (S_p^\dagger - u)a + (q-1)b = (S_p^\dagger - u)\left(a + \frac{q}{u}c - S_p^\dagger c - (q-1)c\right) = 0.$$

The Lemma follows. □

We shall need a formula for computing certain values of the bilinear forms.

Lemma 9.20. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Let $f \in \mathbb{C}(x)$ be a rational function with no poles at 0 and on the spectrum of S_p. Then, for every r, s in $\mathcal{S}_{k,\mathbb{C}}$, we have*

$$p^{-,*}(f(S_p^\dagger)(r^<), s^{\vee<\vee}) = p^*(S_p f(S_p) r, s).$$

Proof. First note that, by Lemma 7.8, $f(S_p)$ is a well-defined endomorphism of $\mathcal{S}_k$. Besides, as all spaces are finite dimensional, we can assume that f is a polynomial function. Then, from Lemma 9.11, we get $f(S_p^\dagger)(r^<) = (f(S_p)r)^<$, hence

$$\begin{aligned} p^{-,*}(f(S_p^\dagger)(r^<), s^{\vee<\vee}) = p^*(f(S_p)r, s^{\vee<\vee>_p}) &= p^*(f(S_p)r, S_p^\dagger s) \\ &= p^*(S_p f(S_p) r, s). \end{aligned}$$

□

We can now write formulae for the quantity $p^*(r_u, s) = p^*(r, s_{\frac{q}{u}})$, for $r, s \in S_{k,\mathbb{C}}$. Unfortunately, the formulae depend on the parity of k and the eigenvalue of the $\vee$ operator.

Lemma 9.21. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field, r and s be in $\mathcal{S}_{k,\mathbb{C}}$ and u be in $\mathbb{C}^*$. Assume neither u nor $\frac{q}{u}$ are eigenvalues of S_p. Set $t = \frac{1}{q+1}(u + \frac{q}{u})$ and*

$$r' = (u - S_p)^{-1} \left(\frac{q}{u} - S_p\right)^{-1} r = (q + S_p^2 - (q+1)tS_p)^{-1} r.$$

If k is even, we have

$$\begin{aligned} p^*(r_u, s) &= \frac{1}{q} p^*((q^2 - S_p^2)r', s) & r^\vee = qr, s^\vee = qs \\ &= \frac{u^2 - 1}{u} p^*(S_p r', s) & r^\vee = -r, s^\vee = qs \\ &= \frac{u^2 - q^2}{u} p^*(S_p r', s) & r^\vee = qr, s^\vee = -s \\ &= qp^*((1 - S_p^2)r', s) & r^\vee = -r, s^\vee = -s. \end{aligned}$$

If k is odd, we have

$$\begin{aligned} p^*(r_u, s) &= p^*((q - S_p)(1 + S_p)r', s) & r^\vee = r, s^\vee = s \\ &= \frac{(u+q)(u-1)}{u} p^*(S_p r', s) & r^\vee = -r, s^\vee = s \\ &= \frac{(u-q)(u+1)}{u} p^*(S_p r', s) & r^\vee = r, s^\vee = -s \\ &= p^*((q + S_p)(1 - S_p)r', s) & r^\vee = -r, s^\vee = -s. \end{aligned}$$

Proof. We only establish the first two cases.

Assume k is even and $s^\vee = qs$. Let a, b, c be as in Lemma 9.19, so that we have $r_u = r + a^{\vee >_p} + b^{\vee >_p \vee}$. We write

$$\begin{aligned} p^*(r_u, s) &= p^*(r, s) + p^{-,*}(a, s^{<\vee}) + p^{-,*}(b, s^{\vee<\vee}) \\ &= p^*(r, s) + p^{-,*}\left(\frac{1}{q}a + b, s^{\vee<\vee}\right). \end{aligned} \tag{9.5}$$

If $r^\vee = qr$, Lemma 9.19 says that

$$\frac{1}{q}a + b = \left(u + \frac{q}{u} - \frac{q+1}{q} S_p^\dagger\right) c = \frac{q+1}{q}(qt - S_p)c.$$

By Lemma 9.20, we get

$$p^{-,*}\left(\frac{1}{q}a + b, s^{\vee<\vee}\right) = \frac{q+1}{q} p^*(S_p(qt - S_p)r', s).$$

We use the identity in $\mathbb{C}(x)$,

$$1+\frac{q+1}{q}\frac{x(qt-x)}{q+x^2-(q+1)tx}=\frac{1}{q}\frac{q^2-x^2}{q+x^2-(q+1)tx}$$

to deduce from (9.5) that $p^*(r_u,s)=\frac{1}{q}p^*((q^2-S_p^2)r',s)$.

If $r^\vee=-r$, note that $p^*(r,s)=0$, so that, from (9.5), we get $p^*(r_u,s)=p^{-,*}\left(\frac{1}{q}a+b,s^{\vee<\vee}\right)$. Now, Lemma 9.19 says that $\frac{1}{q}a+b=\frac{u^2-1}{u}c$. By Lemma 9.20, we get $p^*(r_u,s)=\frac{u^2-1}{u}p^*(S_pr',s)$ as required.

The other cases are obtained in the same way. $\square$

To conclude, we will need another set of formulae.

Lemma 9.22. *Let $k\geq 2$, p be a Γ-invariant k-Euclidean field and u be in $\mathbb{C}\smallsetminus\{-q,-1,0,1,q\}$. Pick r,s in $\mathcal{S}_{k,\mathbb{C}}$. Then, we have*

$$\begin{aligned}
p^*(I_u^{-1}r,s)&=\frac{1}{q^2-u^2}p^*(r,s) && k \text{ even, } s^\vee=qs\\
&=\frac{1}{q(u^2-1)}p^*(r,s) && k \text{ even, } s^\vee=-s\\
&=\frac{1}{(q-u)(u+1)}p^*(r,s) && k \text{ odd, } s^\vee=s\\
&=\frac{1}{(q+u)(u-1)}p^*(r,s) && k \text{ odd, } s^\vee=-s.
\end{aligned}$$

Proof. By Lemma 9.6, we have $I_u^{-1}r=\frac{u^2}{q^2(q^2-u^2)(u^2-1)}I_{\frac{q}{u}}r$ if k is even and $I_u^{-1}r=\frac{u^2}{q(q^2-u^2)(u^2-1)}I_{\frac{q}{u}}r$ if k is odd.

Assume for example k is even and $s^\vee=qs$. From the Definition 9.4 of the opposition map $I_{\frac{q}{u}}$ we have

$$\begin{aligned}
p^*(I_{\frac{q}{u}}r,s)&=p^*\left(r,\left(q-\frac{q^2}{u^2}\right)s^\vee+\frac{q^2}{u^2}(q-1)s\right)\\
&=\frac{1}{u^2}p^*(r,(q^2u^2-q^3+q^3-q^2)s)=\frac{q^2}{u^2}(u^2-1)p^*(r,s)
\end{aligned}$$

and the conclusion follows.

The other cases are dealt with in the same way. $\square$

We conclude the

Proof of Proposition 9.18. Take H and J in $\mathcal{H}_{k,\mathbb{C}}$ and let r and s be the elements of $\mathcal{S}_{k,\mathbb{C}}$ such that, for any G in $\mathcal{H}_{k,\mathbb{C}}$, one has $p(H,G)=\langle r,G\rangle$ and $p(J,G)=\langle s,G\rangle$. For a in $\mathcal{S}^u_{k,\mathbb{C}}$, we have

$$\langle a,H\rangle=p^*(r,a)=p^*(r_{\frac{q}{u}},a)-p^{-,*}(r^<_{\frac{q}{u}},a^<)=p^*_u(I_u^{-1}r_{\frac{q}{u}},a),$$

so that, by definition, we get $p_u(H,J) = p_u^*(I_u^{-1} r_{\frac{q}{u}}, s)$ and the Proposition follows from Lemma 9.21 and Lemma 9.22. □

9.6. **u-radical pseudofields in the bipartite case.** When Γ is bipartite, the twist operation of Subsection 2.6 and Subsection 6.7 interacts with the previous constructions. Let us write how this interaction works.

First, by the dual version of Lemma 2.22, we directly get,

Lemma 9.23. *Assume Γ is bipartite. Let $k \geq 1$, u be in $\mathbb{C}^*$ and s be in $\mathcal{S}_{k,\mathbb{C}}$. Then s is u-radical if and only if $s^\wr$ is $(-u)$-radical. Besides, we have*

$$(I_u s)^\wr = (-1)^k I_{(-u)}(s^\wr).$$

Using these properties, straightforward computations yield

Lemma 9.24. *Assume Γ is bipartite. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be in $\mathbb{C}^*$. For r, s in $\mathcal{S}_{k,\mathbb{C}}^u$, we have*

$$p_{(-u)}^*(r^\wr, s^\wr) = (-1)^k p_u^*(r,s).$$

If $u^2 \notin \{1, q^2\}$ and neither u nor $\frac{q}{u}$ is an eigenvalue of S_p, for H, J in $\mathcal{H}_{k,\mathbb{C}}$, we have

$$p_{(-u)}(H^\wr, J^\wr) = (-1)^k p_u(H,J).$$

Note that the condition on u in the latter statement is symmetric in view of Lemma 7.6.

10. t-radical pairs and full spectral quadratic forms

In this Section we continue the algebraic constructions that are necessary in order to state the Plancherel formula for Euclidean fields.

10.1. **t-radical pairs.** We start by introducing a new subspace of $\mathcal{S}_k^2$ and relate it to the range of the default map of Subsection 2.3 thanks to Proposition 6.5.

Definition 10.1. Let $k \geq 1$, t be in $\mathbb{C}$ and $s = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$ be a pair of complex k-simple pseudofields. Then, s is said to be t-radical if one has

$$\begin{pmatrix} s_0^{\vee<\vee} \\ s_1^{\vee<\vee} \end{pmatrix} = \begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} \begin{pmatrix} s_0^< \\ s_1^< \end{pmatrix}.$$

For $k \geq 1$, the space of t-radical pairs in $\mathcal{S}_{k,\mathbb{C}}^2$ is denoted by $\mathcal{S}_{k,\mathbb{C}}^{2,t}$. If t is real, the space of t-radical pairs in $\mathcal{S}_k^2$ is denoted by $\mathcal{S}_k^{2,t}$.

Remark 10.2. Note that the matrix $\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix}$ already appeared in the proof of the Ihara trace formula in Subsection 1.3.

We will relate the notion of a t-radical pair to the formulas in Proposition 6.5. To this aim, for $k \geq 0$, we introduce a natural duality between $\mathcal{S}^2_k$ and $\mathcal{H}^2_k$. For $s = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$ in $\mathcal{S}^2_k$ and $H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$ in $\mathcal{H}^2_k$, we set

$$\langle s, H \rangle = \langle s_0, H_0 \rangle + \langle s_1, H_1 \rangle.$$

Then, Proposition 6.5 translates into

Lemma 10.3. *Let $k \geq 1$, s be in $\mathcal{S}^2_k$ and t be in $\mathbb{R}$. Then s is t-radical if and only if, for every G in $\mathcal{H}^{(\mathbb{N})}_{k-1}$, one has $\langle s, \widehat{D_k G}(t) \rangle = 0$.*

Recall that D_k is the default map that was defined in Subsection 2.3.

Proof. Indeed, Proposition 6.5 gives, is k is even,

$$\langle s, \widehat{D_k G}(t) \rangle = \left\langle \begin{pmatrix} s_0^< \\ s_1^< \end{pmatrix} - \begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix}^{-1} \begin{pmatrix} s_0^{\vee<\vee} \\ s_1^{\vee<\vee} \end{pmatrix}, \breve{G}(t) \right\rangle$$

and if k is odd,

$$\langle s, \widehat{D_k G}(t) \rangle = \left\langle \begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} \begin{pmatrix} s_0^< \\ s_1^< \end{pmatrix} - \begin{pmatrix} s_0^{\vee<\vee} \\ s_1^{\vee<\vee} \end{pmatrix}, \breve{G}(t) \right\rangle.$$

The Proposition follows as, by Proposition 6.4, the linear map $G \mapsto \breve{G}(t)$ sends $\mathcal{H}^{(\mathbb{N})}_{k-1}$ onto $\mathcal{H}^2_{k-1}$. □

In Lemma 10.3, we have established a relation between t-radical pairs and the range of the polyextension map. Now, from Lemma 2.21, Proposition 6.3 and Proposition 6.4, we should see a link between t-radical pairs and the operators of (6.1) and (6.1). Note that the adjoint operators of these operators may be defined by the following formulae. For $s = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$ in $\mathcal{S}^2_k$, if k is even, we have

$$\mathfrak{S}^*_t s = \begin{pmatrix} s_1^\vee - (q-1)s_1 \\ q^{-1} s_0^\vee \end{pmatrix}$$
$$\mathfrak{R}^*_t s = \begin{pmatrix} s_0^\vee \\ q^{-1}(q+1)t s_0^\vee + (q-1)s_1 - s_1^\vee \end{pmatrix}.$$

If k is odd, we have

$$\mathfrak{R}_t^* s = \begin{pmatrix} q s_1^\vee \\ (q-1)s_1 + s_0^\vee \end{pmatrix}$$
$$\mathfrak{S}_t^* s = \begin{pmatrix} -s_0^\vee + (q+1)ts_1^\vee - (q-1)s_1 \\ s_1^\vee \end{pmatrix}.$$

Lemma 10.4. *Let $k \geq 1$, t be in $\mathbb{C}$ and s be a t-radical pair of complex k-simple pseudofields. The pairs $\mathfrak{R}_t^* s$ and $\mathfrak{S}_t^* s$ are t-radical.*

Proof. As mentioned above this can be seen as a consequence as Lemma 2.21, Proposition 6.3, Proposition 6.4 and Lemma 10.3. We can also get it by a straightforward computation. We shall use the fact that the inverse of the matrix $\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix}$ is the matrix $\begin{pmatrix} q^{-1}(q+1)t & -1 \\ q^{-1} & 0 \end{pmatrix}$. Assume k is even. We get

$$(\mathfrak{S}_t^* s)^{\vee<\vee} = \begin{pmatrix} q s_1^{<\vee} \\ s_0^{<\vee} + q^{-1}(q-1)s_0^{\vee<\vee} \end{pmatrix}$$
$$\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} (\mathfrak{S}_t^* s)^{<} = \begin{pmatrix} s_0^{\vee<} \\ (q-1)s_1^{<} - s_1^{\vee<} + q^{-1}(q+1)ts_0^{\vee<} \end{pmatrix}.$$

As s is t-radical, we have $s_0^{\vee<} = q s_1^{<\vee}$ and $-s_1^{\vee<} + q^{-1}(q+1)ts_0^{\vee<} = s_0^{<\vee}$. We get $(\mathfrak{S}_t^* s)^{\vee<\vee} = \begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} (\mathfrak{S}_t^* s)^{<}$ as required. In the same way, we have

$$(\mathfrak{R}_t^* s)^{\vee<\vee} = \begin{pmatrix} q s_0^{<\vee} + (q-1)s_0^{\vee<\vee} \\ (q+1)ts_0^{<\vee} + q^{-1}(q^2-1)ts_0^{\vee<\vee} - q s_1^{<\vee} \end{pmatrix}$$

and

$$\begin{aligned}
&\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} (\mathfrak{R}_t^* s)^{<} \\
&\quad = \begin{pmatrix} (q+1)ts_0^{\vee<} + q(q-1)s_1^{<} - q s_1^{\vee<} \\ -s_0^{\vee<} + q^{-1}(q+1)^2 t^2 s_0^{\vee<} + (q^2-1)ts_1^{<} - (q+1)ts_1^{\vee<} \end{pmatrix} \\
&\quad\quad = \begin{pmatrix} q s_0^{<\vee} + q(q-1)s_1^{<} \\ -s_0^{\vee<} + (q+1)ts_0^{<\vee} + (q^2-1)ts_1^{<} \end{pmatrix},
\end{aligned}$$

where we have used again the relation $-s_1^{\vee<} + q^{-1}(q+1)ts_0^{\vee<} = s_0^{<\vee}$. Still by using the relation $s_0^{\vee<} = q s_1^{<}$, we get $(\mathfrak{R}_t^* s)^{\vee<\vee} = \begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} (\mathfrak{R}_t^* s)^{<}$.

Assume now k is odd, so that we write

$$(\mathfrak{R}_t^* s)^{\vee<\vee} = \begin{pmatrix} q s_1^{<\vee} \\ (q-1)s_1^{\vee<\vee} + s_0^{<\vee} \end{pmatrix}$$

$$\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} (\mathfrak{R}_t^* s)^{<} = \begin{pmatrix} q(q-1)s_1^{<} + q s_0^{\vee<} \\ -q s_1^{\vee<} + (q^2-1)t s_1^{<} + (q+1)t s_0^{\vee<} \end{pmatrix}.$$

As $s_0^{\vee<\vee} = q s_1^{<}$, we get $s_0^{\vee<} = s_1^{<\vee} - (q-1)s_1^{<}$, which gives equality of the first lines. Besides, from $s_1^{\vee<\vee} = -s_0^{<} + (q+1)t s_1^{<}$, we get $q s_1^{\vee<} + (q-1)s_1^{\vee<\vee} = -s_0^{<\vee} + (q+1)t s_1^{<\vee}$ which leads to the equality of the second lines. It remains to compute

$$(\mathfrak{S}_t^* s)^{\vee<\vee} = \begin{pmatrix} -s_0^{<\vee} + (q+1)t s_1^{<\vee} - (q-1)s_1^{\vee<\vee} \\ s_1^{<\vee} \end{pmatrix}$$

$$\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} (\mathfrak{S}_t^* s)^{<} = \begin{pmatrix} q s_1^{\vee<} \\ s_0^{\vee<} + (q-1)s_1^{<} \end{pmatrix}.$$

The relation $s_0^{\vee<\vee} = q s_1^{<}$ implies the equality of the second lines, whereas the relation $s_1^{\vee<\vee} = -s_0^{<} + (q+1)t s_1^{<}$ implies the equality of the first lines. □

10.2. **t-radical pairs and u-radical pseudofields.** We will now relate t-radical pairs to the theory of Section 9. This relies on diagonalizing the matrix $\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix}$. Note that the eigenvalues of this matrix are the roots of the equation $q^2 + u = (q+1)tu$. This fact was already used in Subsection 1.3.

To proceed to the diagonalization, we introduce a change of variables in spaces of pairs. Let $k \geq 0$ and fix u, t in $\mathbb{C}$ with $q^2 + u = (q+1)tu$ and $u^2 \neq q$ or equivalently $(q+1)t^2 \neq 4q$. For $H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$ in $\mathcal{H}^2_{k,\mathbb{C}}$, we set

$$\delta_u H = \begin{pmatrix} q & u \\ q & \frac{q}{u} \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}. \tag{10.1}$$

As $u^2 \neq q$, this map is invertible and we get

$$\delta_u^{-1} H = \frac{1}{q-u^2} \begin{pmatrix} 1 & -q^{-1}u^2 \\ -u & u \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}. \tag{10.2}$$

The adjoint endomorphism δ_u^* of δ_u and its inverse $(\delta_u^*)^{-1} = (\delta_u^{-1})^*$ are defined by, for all $s = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$ in $\mathcal{S}^2_{k,\mathbb{C}}$,

$$\delta_u^* s = \begin{pmatrix} q & q \\ u & \frac{q}{u} \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \end{pmatrix} \text{ and } (\delta_u^*)^{-1} s = \frac{1}{q-u^2} \begin{pmatrix} 1 & -u \\ -q^{-1}u^2 & u \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}.$$

Lemma 10.5. *Let u be in $\mathbb{C}^*$, with $u^2 \neq q$, and set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Then, for $k \geq 1$, a pair $s = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$ of complex k-simple pseudofields is t-radical, if and only if the pseudofield $s_0 - us_1$ is u-radical and the pseudofield $s_0 - \frac{q}{u}s_1$ is $\frac{q}{u}$-radical. The map δ_u^* induces a linear isomorphism from $\mathcal{S}^u_{k,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$ onto $\mathcal{S}^{2,t}_{k,\mathbb{C}}$.*

Proof. Let T be the matrix $\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix}$ and U be the matrix $\begin{pmatrix} q & q \\ u & \frac{q}{u} \end{pmatrix}$. Then U is invertible and a direct computation shows that U diagonalizes T, that is, we have $U^{-1}TU = \begin{pmatrix} u & 0 \\ 0 & \frac{q}{u} \end{pmatrix}$. The conclusion follows. □

10.3. **Diagonalization map and natural operators.** Let us transport the operators of Lemma 10.4 via the diagonalization map δ_u. For u in $\mathbb{C}^*$, by abuse of notation, we still write I_u for the endomorphism of $\mathcal{H}_{k,\mathbb{C}}$ that is adjoint to the u-opposition map of Definition 9.4. When $u^2 \neq q$, we define two endomorphisms ρ_u and σ_u of $\mathcal{H}^2_{k,\mathbb{C}}$ which we write in the following matrix forms.

If k is even, we set

$$\rho_u = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u^2 & I_u \\ -q^{-1}u^2 I_{\frac{q}{u}} & q(q-1) \end{pmatrix} \tag{10.3}$$

$$\text{and } \sigma_u = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u & q^{-1}uI_u \\ -q^{-1}uI_{\frac{q}{u}} & u(q-1) \end{pmatrix}.$$

Their adjoint endomorphisms are defined by

$$\rho_u^* = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u^2 & -q^{-1}u^2 I_{\frac{q}{u}} \\ I_u & q(q-1) \end{pmatrix} \tag{10.4}$$

$$\text{and } \sigma_u^* = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u & -q^{-1}uI_{\frac{q}{u}} \\ q^{-1}uI_u & u(q-1) \end{pmatrix}.$$

If k is odd, we set

$$\rho_u = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u^2 & uI_u \\ -uI_{\frac{q}{u}} & q(q-1) \end{pmatrix} \tag{10.5}$$

$$\text{and } \sigma_u = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u & q^{-1}u^2 I_u \\ -I_{\frac{q}{u}} & u(q-1) \end{pmatrix}.$$

Their adjoint endomorphisms are defined by

$$\rho_u^* = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u^2 & -uI_{\frac{q}{u}} \\ uI_u & q(q-1) \end{pmatrix} \tag{10.6}$$
$$\text{and } \sigma_u^* = \frac{1}{q-u^2}\begin{pmatrix} -(q-1)u & -I_{\frac{q}{u}} \\ q^{-1}u^2 I_u & u(q-1) \end{pmatrix}.$$

By using Lemma 9.6, one can check that in both cases, one has $\rho_u^2 = q + (q-1)\rho_u$ and $\sigma_u^2 = 1$. Besides, by Lemma 9.5, both ρ_u^* and σ_u^* preserve the space $\mathcal{S}_{k,\mathbb{C}}^u \oplus \mathcal{S}_{k,\mathbb{C}}^{\frac{q}{u}}$.

Lemma 10.6. *Let $k \geq 0$, u be in $\mathbb{C} \smallsetminus \{-\sqrt{q}, 0, \sqrt{q}\}$ and $t = \frac{1}{q+1}(u + \frac{q}{u})$. One has $\delta_u \mathfrak{R}_t = \rho_u \delta_u$ and $\delta_u \mathfrak{S}_t = \sigma_u \delta_u$.*

Proof. Assume k is even and take H in $\mathcal{H}_{k,\mathbb{C}}$. We get

$$\delta_u \begin{pmatrix} H \\ -uH \end{pmatrix} = (q-u^2)\begin{pmatrix} H \\ 0 \end{pmatrix},$$

hence

$$\rho_u \delta_u \begin{pmatrix} H \\ -uH \end{pmatrix} = \begin{pmatrix} -(q-1)u^2 H \\ -q^{-1}u^2 I_{\frac{q}{u}} H \end{pmatrix} = \begin{pmatrix} -(q-1)u^2 H \\ (q-u^2)H^\vee - q(q-1)H \end{pmatrix},$$

where we have used Definition 9.4. On the other hand, by (6.1), we have

$$\mathfrak{R}_t \begin{pmatrix} H \\ -uH \end{pmatrix} = \begin{pmatrix} (1 - \frac{u}{q}(q+1)t)H^\vee \\ uH^\vee - (q-1)uH \end{pmatrix} = \begin{pmatrix} -\frac{u^2}{q}H^\vee \\ uH^\vee - (q-1)uH \end{pmatrix}$$

and therefore

$$\delta_u \mathfrak{R}_t \begin{pmatrix} H \\ -uH \end{pmatrix} = \begin{pmatrix} -(q-1)u^2 H \\ (q-u^2)H^\vee - q(q-1)H \end{pmatrix} = \rho_u \delta_u \begin{pmatrix} H \\ -uH \end{pmatrix}.$$

By symmetry, we also get $\rho_u \delta_u \begin{pmatrix} H \\ -\frac{q}{u}H \end{pmatrix} = \delta_u \mathfrak{R}_t \begin{pmatrix} H \\ -\frac{q}{u}H \end{pmatrix}$, hence, as $u^2 \neq q$, $\rho_u \delta_u = \delta_u \mathfrak{R}_t$. The other computations can be lead in the same way. □

10.4. **Doubling of quadratic forms.** We will now use the diagonalization map δ_u to construct symmetric bilinear forms on $\mathcal{H}_k^2$ with respect to which the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ are self adjoint.

Proposition 10.7. *Let $k \geq 0$ and t, u be in $\mathbb{C}$ with $q + u^2 = (q+1)tu$. Assume $(q+1)^2 t^2 \neq 4q$ and $t^2 \neq 1$. Define $\mathcal{Q}_{k,\mathbb{C}}^t$ as the space of complex symmetric bilinear forms π on $\mathcal{H}_{k,\mathbb{C}}^2$ such that the linear operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ are symmetric with respect to π.*

If k is even, for π in $\mathcal{Q}^t_{k,\mathbb{C}}$, set $\Phi_u\pi$ to be the symmetric bilinear form

$$(H,J)\mapsto \frac{1}{(q-u^2)^2}\pi\left(\begin{pmatrix}H\\-uH\end{pmatrix},\begin{pmatrix}J\\-uJ\end{pmatrix}\right)$$

on $\mathcal{H}_{k,\mathbb{C}}$.

If k is odd, for π in $\mathcal{Q}^t_{k,\mathbb{C}}$, set $\Phi_u\pi$ to be the symmetric bilinear form

$$(H,J)\mapsto \frac{u}{q}\frac{1}{(q-u^2)^2}\pi\left(\begin{pmatrix}H\\-uH\end{pmatrix},\begin{pmatrix}J\\-uJ\end{pmatrix}\right)$$

on $\mathcal{H}_{k,\mathbb{C}}$.

Then, the map Φ_u is a linear isomorphism from $\mathcal{Q}^t_{k,\mathbb{C}}$ onto the space of symmetric bilinear forms on $\mathcal{H}_{k,\mathbb{C}}$.

Proof. Let us prove that the map Φ_u is injective. Pick π in $\mathcal{Q}^t_{k,\mathbb{C}}$ and assume that $\Phi_u\pi$ is 0. Set $\chi=(\delta_u^{-1})^\star\pi$. Then χ is a symmetric bilinear form on $\mathcal{H}_k^2$ and, by Lemma 10.6, the operators ρ_u and σ_u are symmetric with respect to χ. By (10.2) and the assumption, χ has trivial restriction to $\mathcal{H}_k\times\{0\}$. Let us show that this implies $\chi=0$.

If k is even, from (10.3), we see that the endomorphism of $\mathcal{H}^2_{k,\mathbb{C}}$ with matrix $\begin{pmatrix}0 & I_u\\ 0 & q(q-1)\end{pmatrix}$ is symmetric with respect to χ. We get, for H,J in $\mathcal{H}_{k,\mathbb{C}}$, as $\begin{pmatrix}0 & I_u\\ 0 & q(q-1)\end{pmatrix}\begin{pmatrix}H\\0\end{pmatrix}=0$,

$$\chi\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}0 & I_u\\ 0 & q(q-1)\end{pmatrix}\begin{pmatrix}0\\J\end{pmatrix}\right)=0$$

hence $\chi\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}0\\J\end{pmatrix}\right)=0$. Still by (10.3), the endomorphism $\begin{pmatrix}q(q-1) & 0\\ I_{\frac{q}{u}} & 0\end{pmatrix}$ is symmetric with respect to χ, so that we get

$$\chi\left(\begin{pmatrix}q(q-1) & 0\\ I_{\frac{q}{u}} & 0\end{pmatrix}\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}0\\J\end{pmatrix}\right)=0,$$

hence $\chi\left(\begin{pmatrix}0\\ I_{\frac{q}{u}}H\end{pmatrix},\begin{pmatrix}0\\J\end{pmatrix}\right)=0$. Since by Lemma 9.6, $I_{\frac{q}{u}}$ is invertible, we get $\chi=0$ as required.

If k is odd, from (10.5), we see that the endomorphisms of $\mathcal{H}^2_{k,\mathbb{C}}$ with matrices

$$\begin{pmatrix}0 & uI_u\\ 0 & q(q-1)\end{pmatrix}\text{ and }\begin{pmatrix}q(q-1) & 0\\ \frac{q}{u}I_{\frac{q}{u}} & 0\end{pmatrix}$$

are symmetric with respect to χ and we proceed as in the previous case.

In both cases, we have shown that the map Φ_u is injective. Now, we will show that it surjective. We fix a symmetric bilinear form φ on $\mathcal{H}_{k,\mathbb{C}}$ and we set

$$\begin{aligned}\varphi' &= \frac{1}{q^2(q+1)^2(1-t^2)} I_u^\star \varphi && \text{if } k \text{ is even}\\ &= \frac{1}{q(q+1)^2(1-t^2)} I_u^\star \varphi && \text{if } k \text{ is odd.}\end{aligned}$$

By using Lemma 9.6, this definition ensures that, for H, J in $\mathcal{H}_{k,\mathbb{C}}$, one has

$$\varphi(H, I_u J) = \varphi'(I_{\frac{q}{u}} H, J).$$

We define a linear endomorphism α_u on $\mathcal{H}^2_{k,\mathbb{C}}$ which is written in matrix form as

$$\begin{aligned}\alpha_u &= \begin{pmatrix} 1 & -\frac{1}{q(q-1)} I_u \\ -\frac{1}{q(q-1)} I_{\frac{q}{u}} & 1 \end{pmatrix} && \text{if } k \text{ is even}\\ &= \begin{pmatrix} \frac{q}{u} & -\frac{1}{q-1} I_u \\ -\frac{1}{q-1} I_{\frac{q}{u}} & u \end{pmatrix} && \text{if } k \text{ is odd.}\end{aligned} \tag{10.7}$$

As α_u is self adjoint with respect to $\varphi \oplus \varphi'$, it defines a symmetric bilinear form χ on $\mathcal{H}^2_{k,\mathbb{C}}$, that is, for H, J in $\mathcal{H}^2_{k,\mathbb{C}}$,

$$\chi(H, J) = (\varphi \oplus \varphi')(H, \alpha_u J) = (\varphi \oplus \varphi')(\alpha_u H, J).$$

A direct computation using (10.3) and (10.5) shows that ρ_u and σ_u are symmetric with respect to χ. We set $\pi = \delta_u^\star \chi$, so that, by Lemma 10.6, $\mathfrak{R}_t$ and $\mathfrak{S}_t$ are symmetric with respect to π. By construction, we have $\Phi_u \pi = \varphi$. □

10.5. **The full dual spectral bilinear form.** Given $k \geq 2$ and p a Γ-invariant k-Euclidean field, in Definition 9.17, we have introduced the spectral bilinear forms p_u associated with p. By Proposition 10.7 these objects define symmetric bilinear forms on $\mathcal{H}^2_k$ with respect to which the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ are symmetric. In order to describe the null spaces of these forms, we start by constructing dual objects.

We keep using the notation (10.7) and we introduce a new operator $\beta_u : \mathcal{H}^2_{k,\mathbb{C}} \to \mathcal{H}^2_{k,\mathbb{C}}$. For u in $\mathbb{C}^*$, we set

$$\begin{aligned}\beta_u &= \begin{pmatrix} 1 & \frac{1}{q(q-1)} I_u \\ \frac{1}{q(q-1)} I_{\frac{q}{u}} & 1 \end{pmatrix} && \text{if } k \text{ is even}\\ &= \begin{pmatrix} u & \frac{1}{q-1} I_u \\ \frac{1}{q-1} I_{\frac{q}{u}} & \frac{q}{u} \end{pmatrix} && \text{if } k \text{ is odd.}\end{aligned}$$

For $u^2 \neq q$, this operator is a multiple of the inverse of the operator α_u in (10.7). Indeed, by using Lemma 9.6, we get

$$\text{(10.8)}\qquad \alpha_u\beta_u = \left(\frac{q-u^2}{u(q-1)}\right)^2 = \frac{(q+1)^2t^2-4q}{(q-1)^2} \qquad \text{if } k \text{ is even}$$
$$= q\left(\frac{q-u^2}{u(q-1)}\right)^2 = q\frac{(q+1)^2t^2-4q}{(q-1)^2} \qquad \text{if } k \text{ is odd.}$$

Besides, the adjoint operator β_u^* of β_u is defined by the matrix

$$\text{(10.9)}\qquad \beta_u^* = \begin{pmatrix} 1 & \frac{1}{q(q-1)}I_{\frac{q}{u}} \\ \frac{1}{q(q-1)}I_u & 1 \end{pmatrix} \qquad \text{if } k \text{ is even}$$
$$= \begin{pmatrix} u & \frac{1}{q-1}I_{\frac{q}{u}} \\ \frac{1}{q-1}I_u & \frac{q}{u} \end{pmatrix} \qquad \text{if } k \text{ is odd.}$$

Definition 10.8. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and t be in $\mathbb{C}^*$ with $(q+1)^2t^2 \neq 4q$. Choose u in $\mathbb{C}^*$ with $(q+1)tu = q+u^2$ and equip the space $\mathcal{S}^u_{k,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$ with the bilinear form $p_u^* \oplus p^*_{\frac{q}{u}}$. We set $p_t^{2,*}$ to be the symmetric bilinear form on $\mathcal{S}^{2,t}_{k,\mathbb{C}}$ such that the bilinear form $(\delta_u^*)^\star p_t^{2,*}$ is defined by the matrix β_u^* with respect to $p_u^* \oplus p^*_{\frac{q}{u}}$. We call $p_t^{2,*}$ the full dual spectral bilinear form associated with p on $\mathcal{S}^{2,t}_{k,\mathbb{C}}$.

In other words, for r, s in $\mathcal{S}^u_{k,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$, we have

$$p_t^{2,*}(\delta_u^* r, \delta_u^* s) = (p_u^* \oplus p^*_{\frac{q}{u}})(r, \beta_u^* s).$$

Note that symmetry of the matrices defining $p_t^{2,*}$ follows from Lemma 9.16.

Below, we summarize the properties of this bilinear form.

Proposition 10.9. *Let $k \geq 2$ be even, p be a Γ-invariant k-Euclidean field and t be in $\mathbb{C}$ with $(q+1)^2 \neq 4q$. Then, for every r, s in $\mathcal{S}^{2,t}_{k,\mathbb{C}}$, we have*

$$p_t^{2,*}(r, \mathfrak{R}_t^* s) = p_t^{2,*}(\mathfrak{R}_t^* r, s) \text{ and } p_t^{2,*}(r, \mathfrak{S}_t^* s) = p_t^{2,*}(\mathfrak{S}_t^* r, s).$$

If $t^2 \neq 1$ and none of the two roots of the equation $(q+1)tu = q+u^2$ is an eigenvalue of S_p, then the full dual spectral bilinear form $p_t^{2,*}$ is non degenerate on $\mathcal{S}^{2,t}_{k,\mathbb{C}}$.

If t is real, then $p_t^{2,*}$ is real.

If t belongs to the interior of the interval $\mathcal{I}_q$, then $p_t^{2,*}$ is real and positive definite on the real vector space $\mathcal{S}^{2,t}_k$.

If V is a real vector space and p is a complex symmetric bilinear form on the complexification $V_{\mathbb{C}}$ of V, then p is said to be real if $p(V,V) \subset \mathbb{R}$, that is, if p is obtained by complexification of a real symmetric bilinear form on V.

As in Subsection 3.3, we write $\mathcal{I}_q$ for the interval $[-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}]$.

To prove positivity, we shall use

Lemma 10.10. *Let V and W be finite dimensional complex vector spaces, equipped with Hermitian scalar products p and q. Let a, d be in $\mathbb{R}$ and $B : W \to V$ and $C : V \to W$ be complex linear maps which are adjoint to each other with respect to p and q. Then, on $V \oplus W$, the Hermitian form defined by the matrix*

$$\begin{pmatrix} a & B \\ C & d \end{pmatrix}$$

is positive if and only if a or d is positive and the endomorphism $ad - BC$ of V is positive.

Proof. Note that BC is a non-negative Hermitian endomorphism of V since $p(BCv, v') = q(Cv, Cv')$ for v, v' in V.

Assume the matrix $\begin{pmatrix} a & B \\ C & d \end{pmatrix}$ defines a positive Hermitian form r on $V \times W$. Then, necessarily, a and d are positive. Now, pick $v \neq 0$ in V and let us show that $adp(v,v) > p(BCv, v)$. If $p(BCv, v) = 0$, there is nothing to prove. If $p(BCv, v) > 0$, note that we have

$$r\left(\begin{pmatrix} v \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Cv \end{pmatrix}\right) = p(v, BCv).$$

Hence, by applying Cauchy-Schwarz inequality to the positive Hermitian form r, we get

$$\begin{aligned} p(v, BCv)^2 \leq r\left(\begin{pmatrix} v \\ 0 \end{pmatrix}, \begin{pmatrix} v \\ 0 \end{pmatrix}\right) r\left(\begin{pmatrix} 0 \\ Cv \end{pmatrix}, \begin{pmatrix} 0 \\ Cv \end{pmatrix}\right) \\ = ap(v,v)dq(Cv, Cv) = adp(v,v)p(v, BCv). \end{aligned}$$

As we have assumed $p(BCv, v) > 0$, we get $p(BCv, v) < adp(v,v)$ as required.

Conversely, assume a or d is positive and the Hermitian endomorphism $ad - BC$ is positive in V. Then, as BC is non-negative, necessarily, both a and d are positive. Now, take v in V and w in W which are not both 0. We must show the positivity of the following real number

$$s = \begin{pmatrix} v & w \end{pmatrix} \begin{pmatrix} a & B \\ C & d \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = ap(v,v) + 2q(Cv, w) + dq(w, w).$$

If $v = 0$, then $w \neq 0$ and $s = dq(w,w) > 0$. If $w = 0$, then $v \neq 0$ and $s = ap(v,v) > 0$. If $v \neq 0$ and $w \neq 0$, we have, by Cauchy-Schwarz inequality,

$$s \geq ap(v,v) - 2q(Cv,Cv)^{\frac{1}{2}} q(w,w)^{\frac{1}{2}} + dp(v,v).$$

As $ad - BC$ is positive, we have $q(Cv,Cv) = p(BCv,v) < adp(v,v)$ and we get

$$\begin{aligned} s > ap(v,v) - 2a^{\frac{1}{2}} d^{\frac{1}{2}} p(v,v)^{\frac{1}{2}} q(w,w)^{\frac{1}{2}} + dp(v,v) \\ = \left(a^{\frac{1}{2}} p(v,v)^{\frac{1}{2}} - d^{\frac{1}{2}} q(w,w)^{\frac{1}{2}}\right)^2 \geq 0. \end{aligned}$$

The conclusion follows. □

Proof of Proposition 10.9. We choose u to be one of the roots of the equation $u^2 + q = (q+1)tu$. Note that the assumption $(q+1)^2 t^2 \neq 4q$ amounts to $u^2 \neq q$. The adjointness properties rely on elementary matrix computations in $\mathcal{S}_{k,\mathbb{C}}^u \oplus \mathcal{S}_{k,\mathbb{C}}^{\frac{q}{u}}$ using (10.4) and (10.6) and Lemma 10.6.

Assume that $t^2 \neq 1$, so that $u^2 \notin \{1, q^2\}$ and also that neither u nor $\frac{q}{u}$ are eigenvalues of S_p. Then, by Proposition 9.14, the bilinear forms p_u^* and $p_{\frac{q}{u}}^*$ are non degenerate. Therefore, the symmetric bilinear form $(\delta_u^*)^\star p_t^{2,*}$, which is defined by an invertible matrix with respect to $p_u^* \oplus p_{\frac{q}{u}}^*$ is non degenerate. By Lemma 10.5 and Definition 10.8, $p_t^{2,*}$ is non degenerate.

Assume that t is real, and let us show that $p_t^{2,*}$ is real, which is to say that, for r, s in $\mathcal{S}_{k,\mathbb{C}}^{2,t}$, one has

$$\overline{p_t^{2,*}(r,s)} = p_t^{2,*}(\overline{r}, \overline{s}). \tag{10.10}$$

As by assumption $(q+1)^2 t^2 \neq 4q$, either t belongs to $\mathbb{R} \smallsetminus \mathcal{I}_q$ or t belongs to the interior of $\mathcal{I}_q$. In the first case, both u and $\frac{q}{u}$ are real numbers and all our previous constructions are real constructions, so that $p_t^{2,*}$ can easily be seen to be real.

It remains to deal with the case where t belongs to the interior of the interval $\mathcal{I}_q$, which we now assume. In particular, both u and $\frac{q}{u}$ are non real with modulus $\sqrt{q}$. Therefore, by Proposition 7.16, none of them is an eigenvalue of S_p, hence by the previous, $p_t^{2,*}$ is non degenerate. To show that the bilinear form $p_t^{2,*}$ is real, we will check the action of complex conjugation on all previous constructions. First, we note that, as t is real, one has $\overline{u} = \frac{q}{u}$. Therefore, we have $\overline{\mathcal{S}_{k,\mathbb{C}}^u} = \mathcal{S}_{k,\mathbb{C}}^{\frac{q}{u}}$ as subspaces of $\mathcal{S}_{k,\mathbb{C}}$. By the Definition 9.4 of the opposition map I_u, we have, for s

in $\mathcal{S}^u_{k,\mathbb{C}}$,

$$\overline{I_u s} = I_{\frac{q}{u}} \overline{s} \tag{10.11}$$

and, by the Definition 9.12 of the dual spectral bilinear form p^*_u, we have, for r, s in $\mathcal{S}^u_{k,\mathbb{C}}$,

$$\overline{p^*_u(r,s)} = p^*_{\frac{q}{u}}(\overline{r}, \overline{s}). \tag{10.12}$$

Finally, by the definition of the diagonalization map δ_u in (10.1), for r in $\mathcal{S}^u_{k,\mathbb{C}}$ and s in $\mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$,

$$\overline{\delta^*_u \begin{pmatrix} r \\ s \end{pmatrix}} = \delta^*_u \begin{pmatrix} \overline{s} \\ \overline{r} \end{pmatrix}. \tag{10.13}$$

Putting together (10.11), (10.12) and (10.13), we get (10.10) from the Definition 10.8 of $p^{2,*}_t$.

It remains to prove that, still for t in the interior of $\mathcal{I}_q$, the full dual spectral bilinear form $p^{2,*}_t$ is positive definite on the real vector space $\mathcal{S}^{2,t}_k$. We will show the equivalent statement that the Hermitian form $\tilde{p}^{2,*}_t : (r,s) \mapsto p^{2,*}_t(\overline{r}, s)$ is positive definite on the complex vector space $\mathcal{S}^{2,t}_{k,\mathbb{C}}$. Again, this will require us to have a closer look at the definition of all the objects. First, as the real bilinear form $(r,s) \mapsto p^*(r,s) - p^{-,*}(r^<, s^<)$ is non-negative on $\mathcal{S}_k$, the Hermitian form $(r,s) \mapsto p^*(\overline{r}, s) - p^{-,*}(\overline{r}^<, s^<)$ is non-negative on $\mathcal{S}_{k,\mathbb{C}}$. As by Proposition 7.16, neither u nor $\frac{q}{u}$ is an eigenvalue of S_p, by Proposition 9.14, the Hermitian form $(r,s) \mapsto p^*(\overline{r}, s) - p^{-,*}(\overline{r}^<, s^<)$ is positive definite on $\mathcal{S}^u_k$. We denote it by $\tilde{p}^*_u$. To conclude, we will consider the space $\mathcal{S}^u_{k,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$ which is equipped with the positive definite Hermitian form $\tilde{p}^*_u \oplus \tilde{p}^*_{\frac{q}{u}}$. On this space, there is a natural anti-automorphism γ_u which is $\gamma_u : (r,s) \mapsto (\overline{s}, \overline{r})$ and (10.13) can be rewritten as

$$\overline{\delta^*_u s} = \delta^*_u \gamma_u s \quad s \in \mathcal{S}^u_{k,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}. \tag{10.14}$$

By the Definition 9.12 of the dual spectral bilinear forms p^*_u and $p^*_{\frac{q}{u}}$, for r, r' in $\mathcal{S}^u_{k,\mathbb{C}}$ and s, s' in $\mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}}$, we have

$$(p^*_u \oplus p^*_{\frac{q}{u}}) \left(\begin{pmatrix} r \\ s \end{pmatrix}, \begin{pmatrix} r' \\ s' \end{pmatrix} \right) = (\tilde{p}^*_u \oplus \tilde{p}^*_{\frac{q}{u}}) \left(\gamma_u \begin{pmatrix} r \\ s \end{pmatrix}, \begin{pmatrix} I_{\frac{q}{u}} s' \\ I_u r' \end{pmatrix} \right).$$

This, together with the definition of $p^{2,*}_t$ and (10.14), implies that

$$\tilde{p}^{2,*}_t \left(\delta^*_u \begin{pmatrix} r \\ s \end{pmatrix}, \delta^*_u \begin{pmatrix} r' \\ s' \end{pmatrix} \right) = (\tilde{p}^*_u \oplus \tilde{p}^*_{\frac{q}{u}}) \left(\begin{pmatrix} r \\ s \end{pmatrix}, \begin{pmatrix} 0 & I_{\frac{q}{u}} \\ I_u & 0 \end{pmatrix} \beta^*_u \begin{pmatrix} s' \\ r' \end{pmatrix} \right). \tag{10.15}$$

To conclude, we will apply the criterion in Lemma 10.10.

If k is even, by Lemma 9.6 and (10.9), we have

$$\begin{pmatrix} 0 & I_{\frac{q}{u}} \\ I_u & 0 \end{pmatrix} \beta_u^* = \begin{pmatrix} q\frac{(q+1)^2}{q-1}(1-t^2) & I_{\frac{q}{u}} \\ I_u & q\frac{(q+1)^2}{q-1}(1-t^2) \end{pmatrix}.$$

As

$$\begin{aligned} q^2\frac{(q+1)^4}{(q-1)^2}&(1-t^2)^2 - I_{\frac{q}{u}} I_u \\ &= q^2\frac{(q+1)^2}{(q-1)^2}(1-t^2)((1-t^2)(q+1)^2-(q-1)^2) \\ &\qquad = q^2\frac{(q+1)^2}{(q-1)^2}(1-t^2)(4q-(q+1)^2t^2) > 0, \end{aligned}$$

by Lemma 10.10 and (10.15), the Hermitian form $\tilde{p}_t^{2,*}$ is positive definite.

If k is odd, by Lemma 9.6 and (10.9), we have

$$\begin{pmatrix} 0 & I_{\frac{q}{u}} \\ I_u & 0 \end{pmatrix} \beta_u^* = \begin{pmatrix} q\frac{(q+1)^2}{q-1}(1-t^2) & \frac{q}{u} I_{\frac{q}{u}} \\ u I_u & q\frac{(q+1)^2}{q-1}(1-t^2) \end{pmatrix}.$$

As

$$\begin{aligned} q^2\frac{(q+1)^4}{(q-1)^2}&(1-t^2)^2 - qI_{\frac{q}{u}} I_u \\ &= q^2\frac{(q+1)^2}{(q-1)^2}(1-t^2)((1-t^2)(q+1)^2-(q-1)^2) \\ &\qquad = q^2\frac{(q+1)^2}{(q-1)^2}(1-t^2)(4q-(q+1)^2t^2) > 0, \end{aligned}$$

again by Lemma 10.10 and (10.15), the Hermitian form $\tilde{p}_t^{2,*}$ is positive definite. □

10.6. **The full spectral bilinear form.** Given $k \geq 2$ and p a Γ-invariant k-Euclidean field, we now introduce the dual objects to the bilinear forms defined above.

Let t be a complex number such that $t^2 \notin \{1, \frac{4q}{(q+1)^2}\}$ and none of the roots of the equation $(q+1)tu = q + u^2$ is an eigenvalue of S_p. We have defined the full dual spectral bilinear form $p_t^{2,*}$ on the space $\mathcal{S}_{k,\mathbb{C}}^{2,t}$ of Γ-invariant t-radical pairs of complex k-simple pseudofields. In Proposition 10.9, we have shown that this bilinear form is non degenerate. Therefore, by duality, it defines a symmetric bilinear form p_t^2

on the space $\mathcal{H}^2_{k,\mathbb{C}}$ of Γ-invariant pairs of complex k-pseudofunctions whose null space is the range of the map
(10.16)

$$\mathcal{H}^2_{k-1,\mathbb{C}} \to \mathcal{H}^2_{k,\mathbb{C}}, G = \begin{pmatrix} G_0 \\ G_1 \end{pmatrix} \mapsto \begin{pmatrix} G_0^{\vee>\vee} \\ G_1^{\vee>\vee} \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix} \begin{pmatrix} G_0^{>} \\ G_1^{>} \end{pmatrix}.$$

For H, J in $\mathcal{H}^2_{k,\mathbb{C}}$, we have $p^2_t(H,J) = \langle r, J\rangle$ where r is the unique element of $\mathcal{S}^{2,t}_{k,\mathbb{C}}$ such that, for any s in $\mathcal{S}^{2,t}_{k,\mathbb{C}}$, one has $\langle s, H\rangle = p^{2,*}_t(r,s)$.

Definition 10.11. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For t as above, the symmetric bilinear form p^2_t on $\mathcal{H}^2_{k,\mathbb{C}}$ is called the full spectral bilinear form associated to t.

The full spectral bilinear form p^2_t will appear in the Plancherel formula for Euclidean fields on the continuous part of the spectrum of Corollary 8.6, which is the critical interval $\mathcal{I}_q$.

In Proposition 9.18, we have established formulae for computing the spectral bilinear form p_u, which are rather heavy to state. Fortunately, these formulae will be sufficient to characterize p^2_t.

Proposition 10.12. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Fix t a complex number with $t^2 \notin \{1, \frac{4q}{(q+1)^2}\}$ and choose u to be a root of the equation $(q+1)tu = q + u^2$. Assume neither u nor $\frac{q}{u}$ is an eigenvalue of S_p. Then p^2_t enjoys the following properties.*
(i) *For every H, J in $\mathcal{H}^2_{k,\mathbb{C}}$, we have,*

$$p^2_t(\mathfrak{R}_t H, J) = p^2_t(H, \mathfrak{R}_t J) \text{ and } p^2_t(\mathfrak{S}_t H, J) = p^2_t(H, \mathfrak{S}_t J).$$

(ii) *For every H, J in $\mathcal{H}_{k,\mathbb{C}}$, we have*

$$\begin{aligned} p^2_t\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} J \\ -uJ \end{pmatrix}\right) &= (q-1)^2 u^2 p_u(H,J) \text{ if } k \text{ is even} \\ &= (q-1)^2 u p_u(H,J) \text{ if } k \text{ is odd.} \end{aligned}$$

Conversely, p^2_t is the unique bilinear form on $\mathcal{H}^2_{k,\mathbb{C}}$ that satisfies the two properties above.

Proof. The uniqueness statement directly follows from Proposition 10.7.

(i) is the direct consequence of the analogous property established for $p^{2,*}_t$ in Proposition 10.9.

(ii) will follow from the explicit form of the matrices β_u in (10.9). Indeed, as $p^{2,*}_t$ is defined by the formula

$$p^{2,*}_t(\delta^*_u r, \delta^*_u s) = (p^*_u \oplus p^*_{\frac{q}{u}})(r, \beta^*_u s), \quad r, s \in \mathcal{S}^u_{k,\mathbb{C}} \oplus \mathcal{S}^{\frac{q}{u}}_{k,\mathbb{C}},$$

the bilinear form p_t^2 is defined by the dual formula

$$p_t^2(\delta_u^{-1}H, \delta_u^{-1}J) = (p_u \oplus p_{\frac{q}{u}})(H, \beta_u^{-1}J), \quad H, J \in \mathcal{H}^2_{k,\mathbb{C}}.$$

For H in $\mathcal{H}_{k,\mathbb{C}}$, we get, from (10.2),

$$\delta_u^{-1}\begin{pmatrix} H \\ 0 \end{pmatrix} = \frac{1}{q-u^2}\begin{pmatrix} H \\ -uH \end{pmatrix}$$

and, from (10.7) and (10.8),

$$\begin{aligned} \beta_u^{-1}\begin{pmatrix} H \\ 0 \end{pmatrix} &= \left(\frac{(q-1)u}{q-u^2}\right)^2 \begin{pmatrix} H \\ -q^{-1}(q-1)^{-1}I_{\frac{q}{u}}H \end{pmatrix} && \text{if } k \text{ is even} \\ &= \frac{1}{q}\left(\frac{(q-1)u}{q-u^2}\right)^2 \begin{pmatrix} \frac{q}{u}H \\ -(q-1)^{-1}I_{\frac{q}{u}}H \end{pmatrix} && \text{if } k \text{ is odd.} \end{aligned}$$

The conclusion follows. □

10.7. **t-radical pairs in the bipartite case.** As in Subsection 9.6, if Γ is bipartite, the twist operation of Subsection 2.6 and Subsection 6.7 induces a symmetry in the previous constructions.

In analogy with Lemma 9.23, one can show

Lemma 10.13. *Assume Γ is bipartite. Let $k \geq 1$, t be in $\mathbb{C}$ and $s = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$ be in $\mathcal{S}^2_{k,\mathbb{C}}$. Then s is t-radical if and only if $\begin{pmatrix} s_0^{\wr} \\ -s_1^{\wr} \end{pmatrix}$ is $(-t)$-radical.*

Then, by direct computations, we get

Lemma 10.14. *Assume Γ is bipartite. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and t be in $\mathbb{C}$, $t^2 \neq \frac{4q}{(q+1)^2}$. For r, s in $\mathcal{S}^{2,t}_{k,\mathbb{C}}$, we have*

$$p^{2,*}_{(-t)}\left(\begin{pmatrix} r_0^{\wr} \\ -r_1^{\wr} \end{pmatrix}, \begin{pmatrix} s_0^{\wr} \\ -s_1^{\wr} \end{pmatrix}\right) = p_t^{2,*}(r,s).$$

If moreover $t^2 \neq 1$ and none of the roots of the equation $q + u^2 = (q+1)tu$ is an eigenvalue of S_p, for H, J in $\mathcal{H}_{k,\mathbb{C}}$, we have

$$p^2_{(-t)}\left(\begin{pmatrix} H_0^{\wr} \\ -H_1^{\wr} \end{pmatrix}, \begin{pmatrix} H_0^{\wr} \\ -H_1^{\wr} \end{pmatrix}\right) = p_t^2(H,J).$$

The last property could also be seen as a consequence of Lemma 9.23, Lemma 9.24 and the uniqueness part in Proposition 10.12

11. Exceptional quadratic forms

For $k \geq 2$ and p a Γ-invariant k-Euclidean field, we have introduced the objects that will allow to write the Plancherel formula on the continuous part of the spectrum in Corollary 8.6, which is the critical interval $\mathcal{I}_q$. We now focus on the objects that will appear on the exceptional spectrum Σ_p defined in (8.1). Thus, we will now construct a family of quadratic forms parametrized by Σ_p.

11.1. **The exceptional spectral bilinear form.** Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Recall from Proposition 7.16 that, if u is an eigenvalue of the simple transfer operator S_p such that $|u| > \sqrt{q}$, then u is real and simple. We then write Π_p^u for the unique projection of $\mathcal{H}_k$ onto $\ker(S_p - u)$ that commutes with S_p. In other words, for H in $\mathcal{H}_k$, we have $S_p \Pi_p^u H = u \Pi_p^u H$ and $H - \Pi_p^u H \in (S_p - u)\mathcal{H}_k$.

Definition 11.1. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be a real eigenvalue of the simple transfer operator S_p with $|u| > \sqrt{q}$. We define the exceptional spectral bilinear form p_u^{ex} associated to p and u on $\mathcal{H}_k$ as the bilinear form

$$p_u^{\mathrm{ex}}(H, J) = p(\Pi_p^u H, J^{<_p \vee >}), \quad H, J \in \mathcal{H}_k.$$

By Lemma 2.22 and Lemma 7.6, we have

Lemma 11.2. *Assume Γ is bipartite. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be a real eigenvalue of the simple transfer operator S_p with $|u| > \sqrt{q}$. For H, J in $\mathcal{H}_k$, we have*

$$p_{(-u)}^{\mathrm{ex}}(H^\wr, J^\wr) = (-1)^{k+1} p_u^{\mathrm{ex}}(H, J).$$

The exceptional spectral bilinear form enjoys nonnegativity properties.

Proposition 11.3. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be a real eigenvalue of the simple transfer operator S_p with $|u| > \sqrt{q}$. The exceptional spectral bilinear form p_u^{ex} is symmetric. It is nonnegative if $u > \sqrt{q}$ or k is odd; it is nonpositive if $u < -\sqrt{q}$ and k is even. Its null space is exactly $(S_p - u)\mathcal{H}_k$. In particular, this null space contains the range of the map $\mathcal{H}_{k-1} \to \mathcal{H}_k, G \mapsto G^{\vee > \vee} - uG^{>}$.*

Note that the adjoint operator $\Pi_p^{u,\dagger}$ of Π_p^u with respect to p is the projection onto $\ker(S_p^\dagger - u)$ that commutes with $S_p^\dagger$. The symmetry of p_u^{ex} relies on

Lemma 11.4. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be a real eigenvalue of the simple transfer operator S_p with $|u| > \sqrt{q}$.*

For H in $\mathcal{H}_k$, we have

$$\Pi_p^{u,\dagger}(H^{<_p\vee>}) = (\Pi_p^u H)^{<_p\vee>}.$$

Proof. The definition of the operators in Subsection 7.1 gives $S_pH = H^{<_p\vee>\vee}$ and $S_p^\dagger H = H^{\vee<_p\vee>}$, hence

$$S_p^\dagger(H^{<_p\vee>}) = H^{<_p\vee>\vee<_p\vee>} = (S_pH)^{<_p\vee>}.$$

Therefore, if f is a polynomial function, we have $f(S_p^\dagger)(H^{<_p\vee>}) = (f(S_p)H)^{<_p\vee>}$. The conclusion follows as we can find a polynomial function f with $f(S_p) = \Pi_p^u$ and hence $f(S_p^\dagger) = \Pi_p^{u,\dagger}$. □

By abuse of notation, we also denote by Π_p^u and $\Pi_p^{u,\dagger}$ the projections of $\mathcal{H}_{k-1}$ onto $\ker(S_p - u)$ and $\ker(S_p^\dagger - u)$ that commute respectively with S_p and $S_p^\dagger$.

Lemma 11.5. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be a real eigenvalue of the simple transfer operator S_p with $|u| > \sqrt{q}$. For H in $\mathcal{H}_k$, we have*

$$\Pi_p^{u,\dagger}(H^{<_p}) = (\Pi_p^u H)^{<_p}$$

and this element is 0 if and only if H belongs to $(S_p - u)\mathcal{H}_k$.

Proof. Write $J = \Pi_p^u H$ and $H - J = S_pK - uK$ for some K in $\mathcal{H}_k$. Still by the definitions in Subsection 7.1, we have

$$H^{<_p} = J^{<_p} + K^{<_p\vee>\vee<_p} - uK^{<_p} = J^{<_p} + S_p^\dagger(K^{<_p}) - uK^{<_p}$$

and, in the same way,

$$S_p^\dagger(J^{<_p}) = J^{<_p\vee>\vee<_p} = (S_pJ)^{<_p} = uJ^{<_p}.$$

We get $\Pi_p^{u,\dagger}(H^{<_p}) = J^{<_p} = (\Pi_p^u H)^{<_p}$ as required.

Assume now $J^{<_p} = 0$. As we have $J^{<_p\vee>\vee} = S_pJ = uJ$ and $u \neq 0$, we get $J = 0$ and $H = S_pK - uK$, which should be proved. □

Proof of Proposition 11.3. Let us prove the symmetry of the exceptional spectral bilinear form. For H, J in $\mathcal{H}_k$, we have, by Lemma 11.4,

$$\begin{aligned} p_u^{\mathrm{ex}}(H, J) &= p(\Pi_p^u H, J^{<_p\vee>}) = p(H, \Pi_p^{u,\dagger}(J^{<_p\vee>})) \\ &= p(H, (\Pi_p^u J)^{<_p\vee>}) = p(H^{<_p\vee>}, \Pi_p^u J) = p_u^{\mathrm{ex}}(J, H). \end{aligned}$$

Now, for the other statements, recall from Proposition 7.23, that the symmetric bilinear form $(H, J) \mapsto p(H, J^\vee)$ is positive definite on

$\ker(S_p^\dagger - u)$ if $u > \sqrt{k}$ or k is odd and that it is negative definite if $u < -\sqrt{k}$ and k is even. For H, J in $\mathcal{H}_k$, we have, by Lemma 11.5,

$$\begin{aligned} p(\Pi_p^{u,\dagger}(H^{<_p}), \Pi_p^{u,\dagger}(J^{<_p})^\vee) &= p((\Pi_p^u H)^{<_p}, (\Pi_p^u J)^{<_p \vee}) \\ &= p(\Pi_p^u H, (\Pi_p^u J)^{<_p \vee >}) \\ &= p(H, \Pi_p^{u,\dagger}((\Pi_p^u J)^{<_p \vee >})) \\ &= p(H, (\Pi_p^u \Pi_p^u J)^{<_p \vee >}) \\ &= p(H, (\Pi_p^u J)^{<_p \vee >}) = p_u^{\mathrm{ex}}(H, J), \end{aligned}$$

where we have used Lemma 11.4 and the fact that $\Pi_p^u \Pi_p^u = \Pi_p^u$. The remainder of the Proposition is now a consequence of Proposition 7.23, Lemma 11.5 and the fact that, for G in $\mathcal{H}_{k-1}$, we have $G^{\vee > \vee} - uG^{>} = (S_p - u)(G^{>})$. ☐

11.2. **Values on $\mathcal{H}_{k,+}$ and $\mathcal{H}_{k,-}$.** In the study of the spectral theory of Euclidean fields, we will need more information on the behaviour of the exceptional spectral bilinear forms on eigenspaces of the $\vee$ operator. On these eigenspaces, we can rewrite the definition of the exceptional spectral bilinear forms.

Corollary 11.6. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field, u be a real eigenvalue of the simple transfer operator S_p with $|u| > \sqrt{q}$, H be in $\mathcal{H}_{k,+}$ and J be in $\mathcal{H}_{k,-}$. We have*

$$\begin{aligned} p_u^{\mathrm{ex}}(H, H) &= \frac{u}{q} p(\Pi_p^u H, H) && \text{if } k \text{ is even} \\ &= up(\Pi_p^u H, H) && \text{if } k \text{ is odd} \\ p_u^{\mathrm{ex}}(J, J) &= -up(\Pi_p^u J, J) \\ p_u^{\mathrm{ex}}(H, J) &= -up(\Pi_p^u H, J) \\ &= \frac{u}{q} p(H, \Pi_p^u J) && \text{if } k \text{ is even} \\ &= up(H, \Pi_p^u J) && \text{if } k \text{ is odd.} \end{aligned}$$

Proof. By the definitions of the simple transfer operator S_p and of its adjoint operator $S_p^\dagger$ in Subsection 7.1, and by Definition 11.1, we get

$$\begin{aligned} p_u^{\mathrm{ex}}(J, J) = p(\Pi_p^u J, J^{<_p \vee >}) = -p(\Pi_p^u J, J^{\vee <_p \vee >}) &= -p(\Pi_p^u J, S_p^\dagger J) \\ &= -p(S_p \Pi_p^u J, J) = -up(\Pi_p^u J, J) \end{aligned}$$

as required. The other formulae are obtained by the same technique. ☐

We also can show that the exceptional spectral bilinear forms are not 0 on the eigenspaces of the $\vee$ operator.

Corollary 11.7. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and u be a real eigenvalue of the simple transfer operator S_p with $|u| > \sqrt{q}$. Then, the restrictions of p_u^{ex} to $\mathcal{H}_{k,+}$ and $\mathcal{H}_{k,-}$ are not* 0.

Proof. Assume by contradiction that we have $p_u^{\mathrm{ex}}(H,J) = 0$ for any H, J in $\mathcal{H}_{k,+}$. Then, by Proposition 11.3, we have

$$\mathcal{H}_{k,+} \subset (S_p - u)\mathcal{H}_k.$$

As the p-orthogonal complement of $\mathcal{H}_{k,+}$ in $\mathcal{H}_k$ is $\mathcal{H}_{k,-}$, we get

$$\ker(S_p^\dagger - u) \subset \mathcal{H}_{k,-}.$$

Therefore, by assumption, we can find $H \neq 0$ in $\mathcal{H}_k$ with $S_p^\dagger H = uH$ and $H^\vee = -H$. As $S_p^\dagger H = H^{\vee <_p \vee >}$, we get $H^{<_p \vee >} = -uH$. Since $u \neq 0$, we have $H^{<_p} \neq 0$ and

$$H^{<_p \vee} = H^{<_p \vee > <_p} = -uH^{<_p}.$$

By Proposition 7.9 and the assumption, we have $\sqrt{q} < |u| < q$. Thus, $(-u)$ does not belong to the set of eigenvalues of the $\vee$ operator, a contradiction.

One proves that p_u^{ex} is non zero on $\mathcal{H}_{k,-}$ in the same way. $\square$

11.3. **The full exceptional spectral bilinear form.** We now apply the doubling procedure of Section 10.4 to the exceptional spectral bilinear forms. Recall from (8.1) that, for p in $\mathcal{P}_k$, Σ_p stands for the exceptional spectrum.

Definition 11.8. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and t be in Σ_p. In other words, we have $(q+1)^2 t^2 > 4q^2$ and the unique real number u such that $q + u^2 = (q+1)tu$ and $u^2 > q$ is an eigenvalue of the simple transfer operator S_p. We denote by $p_t^{2,\mathrm{ex}}$ the unique bilinear form on $\mathcal{H}_k^2$ such that
(i) The operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ of (6.1) and (6.2) are symmetric with respect to $p_t^{2,\mathrm{ex}}$.
(ii) For H, J in $\mathcal{H}_k$, one has

$$\begin{aligned} p_t^{2,\mathrm{ex}}\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} J \\ -uJ \end{pmatrix}\right) &= \left(u - \frac{q}{u}\right) p_u^{\mathrm{ex}}(H,J) \qquad \text{if } k \text{ is even} \\ &= \left(1 - \frac{q}{u^2}\right) p_u^{\mathrm{ex}}(H,J) \qquad \text{if } k \text{ is odd.} \end{aligned}$$

We call $p_t^{2,\mathrm{ex}}$ the full exceptional spectral bilinear form associated to p and t.

Note that the existence and uniqueness of $p_t^{2,\mathrm{ex}}$ are warranted by Proposition 10.7. Indeed, by the defintion of Σ_p in (8.1), we have $(q+1)^2t^2 > 4q$ and, by Proposition 7.9, we have $u^2 < q^2$, hence $t^2 < 1$.

From Lemma 11.2 and Definition 11.8, we directy get

Lemma 11.9. *Assume Γ is bipartite. Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and t be in Σ_p. For H, J in $\mathcal{H}_k^2$, we have*

$$p_{(-t)}^{\mathrm{ex}}\left(\begin{pmatrix} H_0^{\wr} \\ -H_1^{\wr} \end{pmatrix}, \begin{pmatrix} J_0^{\wr} \\ -J_1^{\wr} \end{pmatrix}\right) = p_t^{\mathrm{ex}}(H, J).$$

The full exceptional spectral bilinear is non-negative.

Proposition 11.10. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and t be in Σ_p. Then, the full exceptional spectral bilinear form $p_t^{2,\mathrm{ex}}$ is nonnegative.*

Proof. Let u be the root of the equation $q+u^2 = (q+1)tu$ with $|u| > \sqrt{q}$. The proof of the Proposition follows from a careful rereading of the proof of Proposition 10.7. Indeed, following this proof, we set, for H, J in $\mathcal{H}_k$,

$$\begin{aligned} p_{\frac{q}{u}}^{\mathrm{ex}}(H, J) &= \frac{1}{q^2(q+1)^2(1-t^2)} p_u^{\mathrm{ex}}(I_u H, I_u J) &\quad \text{if } k \text{ is even} \\ &= \frac{1}{q(q+1)^2(1-t^2)} p_u^{\mathrm{ex}}(I_u H, I_u J) &\quad \text{if } k \text{ is odd.} \end{aligned}$$

Since Proposition 7.9 implies $t^2 < 1$, the symmetric bilinear form $p_{\frac{q}{u}}^{\mathrm{ex}}$ on $\mathcal{H}_k$ enjoys the same sign properties as p_u^{ex}. Now, from the proof of Proposition 10.7, we see that, if k is even, the bilinear form $(\delta_u^{-1})^\star p_t^{2,\mathrm{ex}}$ is defined by the matrix $u^{-1}(u^2-q)^{-1}\alpha_u$ of (10.7) with respect to $p_u^{\mathrm{ex}} \oplus p_{\frac{q}{u}}^{\mathrm{ex}}$; if k is odd, it is defined by the matrix $q^{-1}u^{-1}(u^2-q)^{-1}\alpha_u$. Note that the diagonal coefficients of α_u are 1 if k is even, and have the same sign as u is k is odd. The conclusion follows from Lemma 9.6, the non-negativity criterion in Lemma 10.10 and Proposition 11.3. $\square$

12. Special quadratic forms

Given $k \geq 2$ and a Γ-invariant k-Euclidean field p, for t in $\mathbb{C}$, we have constructed in Section 10 the full spectral bilinear form p_t^2 which will be used to write the Plancherel formula on the continuous part $\mathcal{I}_q$ of the spectrum in Corollary 8.6. Then, in Section 11, we have constructed the full exceptional spectral bilinear form $p_t^{2,\mathrm{ex}}$ to write this formula at the points t in Σ_p. It remains to study the behaviour of the formula at the special points 1 and -1 of the spectrum: this is the purpose of this Section.

The constructions below rely on a direct analogy we the study of skew quadratic fields and skew dual kernels in Section II.6 and Section II.7.

12.1. **Skew fields.** We introduce a final algebraic object that is related to the behaviour of the theory of u-radical simple pseudofields at the degenerated values $u \in \{-q, -1, 1, q\}$. Its definition comes from a straightforward analogy with Subsection II.6.1.

Definition 12.1. Let $k \geq 1$.

A k-skew field is a k-simple pseudofield s such that $s^\vee = -s$ and $s^{<\vee} = -s^<$.

If k is even, a reverse k-skew field is a k-simple pseudofield s such that $s^\vee = -s$ and $s^{<\vee} = s^<$.

If k is odd, a reverse k-skew field is a k-simple pseudofield s such that $s^\vee = s$ and $s^{<\vee} = -s^<$.

The space of Γ-invariant k-skew fields is denoted by $\mathcal{G}_k^1$. The space of Γ-invariant reverse k-skew fields is denoted by $\mathcal{G}_k^{(-1)}$.

From the dual version of Lemma 2.22, we get

Lemma 12.2. *Assume Γ is bipartite. Let $k \geq 1$ and s be a k-simple pseudofield. Then s is a k-skew field if and only if $s^\wr$ is a reverse k-skew field.*

Let us now relate these new objects to the formerly introduced ones. First, note that skew fields are 1-radical. Indeed, the definitions directly give.

Lemma 12.3. *Let $k \geq 1$ and s be a k-simple pseudofield.*

If s is a k-skew field, then s is 1-radical.

If s is a reverse k-skew field, then s is (-1)-radical.

Skew fields can be built from q-radical simple pseudofields.

Lemma 12.4. *Let $k \geq 1$ and s be a k-simple pseudofield.*

Assume k is even. If s is q-radical, then $qs - s^\vee$ is a k-skew field. If s is $(-q)$-radical, then $qs - s^\vee$ is a reverse k-skew field.

Assume k is odd. If s is q-radical, then $s - s^\vee$ is a k-skew field. If s is $(-q)$-radical, then $s + s^\vee$ is a reverse k-skew field.

Proof. First, we assume that k is even. As $s^{\vee\vee} = qs + (q-1)s^\vee$, we have $(qs - s^\vee)^\vee = s^\vee - qs$. Besides, if s is q-radical, we have $s^{\vee<\vee} = qs^<$. We get

$$(qs - s^\vee)^{<\vee} = qs^{<\vee} - s^{\vee<\vee} = s^{\vee<} - qs^< = -(qs - s^\vee)^<,$$

that is, $qs - s^\vee$ is a k-skew field. If s is $(-q)$-radical, the same computation shows that $qs - s^\vee$ is a reverse k-skew field.

Now assume that k is odd. If s is q-radical, we have $s^{\vee<\vee} = qs^<$. As $s^{\vee<\vee\vee} = qs^{\vee<} + (q-1)s^{\vee<\vee}$, this gives $s^{<\vee} = s^{\vee<} + (q-1)s^<$. We get

$$(s - s^\vee)^{<\vee} = s^{<\vee} - s^{\vee<\vee} = s^{\vee<} + (q-1)s^< - qs^< = -(s - s^\vee)^<,$$

that is, $s - s^\vee$ is a k-skew field. Again, if s is $(-q)$-radical, the same computation shows that $s + s^\vee$ is a reverse k-skew field. □

Direct restriction preserves skew fields.

Lemma 12.5. *Let $k \geq 2$ and s be a k-simple pseudofield. If s is a k-skew field, then $s^<$ is a $(k-1)$-skew field. If s is a reverse k-skew field, then $s^<$ is a reverse $(k-1)$-skew field. Direct restriction maps $\mathcal{G}_k^1$ onto $\mathcal{G}_{k-1}^1$ and $\mathcal{G}_k^{(-1)}$ onto $\mathcal{G}_{k-1}^{(-1)}$.*

Proof. The fact that direct restriction preserves the space of k-skew fields and the one of reverse k-skew fields follows directly from the definitions and from Lemma 6.9.

For example, we show that the direct restriction maps $\mathcal{G}_k^1$ onto $\mathcal{G}_{k-1}^1$. As usual, we will prove that the adjoint map is injective. Thus, let H be in $\mathcal{H}_{k-1}$ and assume that $H^>$ is orthogonal to $\mathcal{G}_k^1$, that is, there exists J in $\mathcal{H}_{k,+}$ and K in $\mathcal{H}_{k-1,+}$ with $H^> = J + K^>$. If k is even (resp. odd), we get $(H-K)^{>\vee} = q(H-K)^>$ (resp. $(H-K)^{>\vee} = (H-K)^>$), hence, by Lemma 2.8, there exists L in $\mathcal{H}_{k-2}$ with $L^\vee = qL$ (resp. $L^\vee = L$) and $H = K + L^>$. Therefore, H is orthogonal to $\mathcal{G}_{k-1}^1$ as should be proved. The proof for reverse skew fields is analogous. □

Remark 12.6. By reasoning as in Proposition II.6.16, one could show that the projective limit of the projective system $(\mathcal{G}_k^1)_{k\geq 1}$ (resp. $(\mathcal{G}_k^{(-1)})_{k\geq 1}$ may be seen as the space of harmonic Γ-equivariant skew-symmetric (resp. symmetric) maps from X_1 to the space $\mathcal{D}_0(\partial X)$ of distributions θ on ∂X with $\theta(\mathbf{1}) = 0$.

Remark 12.7. In view of the analogy between the language of skew fields and the one of skew quadratic fields in Section II.6, it would be tempting to introduce the notions of a field and of a reverse field, by mimicking the language of Section I.4. Thus, if $k \geq 1$ is even, we could define a k-field (resp. a reverse k-field) as a k-simple pseudofield s with $s^\vee = qs$ and $s^{<\vee} = s^<$ (resp. $s^{<\vee} = -s^<$). If k is odd, a k-field (resp. a reverse k-field) would be a k-simple pseudofield s with $s^\vee = s$ (resp. $s^\vee = -s$) and $s^{<\vee} = qs^<$. But, by Lemma 6.10, for $k \geq 2$, direct restriction would induce a linear isomorphism from the space of k-fields (resp. reverse k-fields) onto the space of $(k-1)$-fields (resp. reverse

$(k-1)$-fields). Then, in view of the identification between 1-simple pseudofields and functions on X_1, the space of 1-fields (resp. reverse 1-fields), would just be the space of harmonic skew-symmetric (resp. symmetric) functions on X_1. Note that the Γ-invariant functions of this kind play a role in the proof of the Ihara trace formula in Theorem 1.4.

12.2. **The special dual spectral bilinear forms.** Given a Euclidean field, in analogy with the quadratic case (see Subsection II.7.5), we build scalar products on the spaces of skew fields and reverse skew fields.

Definition 12.8. (k even) Let $k \geq 2$ be even and p be a Γ-invariant k-Euclidean field. For r, s in $\mathcal{G}_k^1$, we set

$$p_1^{\mathrm{sp},*}(r,s) = \frac{q}{q+1} p^*(r,s) - \frac{1}{2} p^{-,*}(r^<, s^<).$$

For r, s in $\mathcal{G}_k^{(-1)}$, we also set

$$p_{(-1)}^{\mathrm{sp},*}(r,s) = \frac{q}{q+1} p^*(r,s) - \frac{1}{2} p^{-,*}(r^<, s^<).$$

We call $p_1^{\mathrm{sp},*}$ and $p_{(-1)}^{\mathrm{sp},*}$ the special dual spectral symmetric bilinear forms associated to p.

Definition 12.9. (k odd) Let $k \geq 2$ be odd and p be a Γ-invariant k-Euclidean field. For r, s in $\mathcal{G}_k^1$, we set

$$p_1^{\mathrm{sp},*}(r,s) = \frac{1}{2} p^*(r,s) - \frac{1}{q+1} p^{-,*}(r^<, s^<).$$

For r, s in $\mathcal{G}_k^{(-1)}$, we also set

$$p_{(-1)}^{\mathrm{sp},*}(r,s) = \frac{1}{2} p^*(r,s) - \frac{1}{q+1} p^{-,*}(r^<, s^<).$$

We call $p_1^{\mathrm{sp},*}$ and $p_{(-1)}^{\mathrm{sp},*}$ the special dual spectral symmetric bilinear forms associated to p.

These forms define scalar products.

Lemma 12.10. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. The special dual symmetric bilinear forms $p_1^{\mathrm{sp},*}$ and $p_{(-1)}^{\mathrm{sp},*}$ are positive definite on $\mathcal{G}_k^1$ and $\mathcal{G}_k^{(-1)}$.*

Proof. This directly follows from the fact that, as $q > 1$, we have $\frac{q}{q+1} > \frac{1}{2} > \frac{1}{q+1}$. □

In case Γ is bipartite, these forms are identified by the twist operator.

Lemma 12.11. *Assume Γ is bipartite. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For r, s in $\mathcal{G}_k^1$, we have*

$$p_{(-1)}^{\mathrm{sp},*}(r^\wr, s^\wr) = p_1^{\mathrm{sp},*}(r, s).$$

12.3. **The special spectral bilinear forms.** We now use duality to build quadratic forms on spaces of pseudofunctions.

Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Lemma 12.10 tells us that the special dual symmetric bilinear forms $p_1^{\mathrm{sp},*}$ and $p_{(-1)}^{\mathrm{sp},*}$ are non degenerate on $\mathcal{G}_k^1$ and $\mathcal{G}_k^{(-1)}$. Therefore, by duality, they define symmetric bilinear forms p_1^{sp} and $p_{(-1)}^{\mathrm{sp}}$ on $\mathcal{H}_k$. For H, J in $\mathcal{H}_k$, we have $p_1^{\mathrm{sp}}(H, J) = \langle r, J\rangle$ and $p_{(-1)}^{\mathrm{sp}}(H, J) = \langle s, J\rangle$ where r (resp. s) is the unique element of $\mathcal{G}_k^1$ (resp. $\mathcal{G}_k^{(-1)}$) such that $\langle a, H\rangle = p_1^{\mathrm{sp},*}(r, a)$ (resp. $\langle a, H\rangle = p_{(-1)}^{\mathrm{sp},*}(s, a)$) for any a in $\mathcal{G}_k^1$ (resp. $\mathcal{G}_k^{(-1)}$).

Definition 12.12. The symmetric bilinear forms p_1^{sp} and $p_{(-1)}^{\mathrm{sp}}$ are called the special symmetric bilinear forms associated to p.

We summarize the properties of p_1^{sp} and $p_{(-1)}^{\mathrm{sp}}$. Recall from Proposition 7.9 that the operators $(q - S_p)$ and $(q + S_p)$ are invertible.

Proposition 12.13. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. The special bilinear forms p_1^{sp} and $p_{(-1)}^{\mathrm{sp}}$ are non-negative on $\mathcal{H}_k$.*

The null space of p_1^{sp} is $\mathcal{H}_{k,+} + \mathcal{H}_{k-1,+}^{>}$. If k is even, the null space of $p_{(-1)}^{\mathrm{sp}}$ is $\mathcal{H}_{k,+} + \mathcal{H}_{k-1,-}^{>}$. If k is odd, the null space of $p_{(-1)}^{\mathrm{sp}}$ is $\mathcal{H}_{k,-} + \mathcal{H}_{k-1,+}^{>}$.

If k is even, for H, J in $\mathcal{H}_{k,-}$, one has

$$p_1^{\mathrm{sp}}(H, J) = (q+1)p((1 + S_p)(q - S_p)^{-1}H, J)$$
$$p_{(-1)}^{\mathrm{sp}}(H, J) = (q+1)p((1 - S_p)(q + S_p)^{-1}H, J).$$

If k is odd, for H, J in $\mathcal{H}_{k,-}$, one has

$$p_1^{\mathrm{sp}}(H, J) = 2p((q + S_p)(q - S_p)^{-1}H, J)$$

and, for H, J in $\mathcal{H}_{k,+}$, one has

$$p_{(-1)}^{\mathrm{sp}}(H, J) = 2p((q - S_p)(q + S_p)^{-1}H, J).$$

We will prove this statement in the same way as we proved Proposition 9.18 of which it can be seen as a degenerated version.

By Lemma 12.10, the dual special bilinear form $p_1^{\mathrm{sp},*}$ is non degenerate on $\mathcal{G}_k^1$. Therefore, for any r in $\mathcal{S}_{k,-}$, there exists r_1 in $\mathcal{G}_k^1$ such that, for any s in $\mathcal{G}_k^1$, one has

$$p^*(r, s) = p_1^{\mathrm{sp},*}(r_1, s).$$

In the same way, the dual special bilinear form $p^{\mathrm{sp},*}_{(-1)}$ is non degenerate on $\mathcal{G}_k^{(-1)}$. If k is even (resp. odd), for any r in $\mathcal{S}_{k,-}$ (resp. $\mathcal{S}_{k,+}$), there exists $r_{(-1)}$ in $\mathcal{G}_k^{(-1)}$ such that, for any s in $\mathcal{G}_k^{(-1)}$, one has

$$p^*(r,s) = p^{\mathrm{sp},*}_{(-1)}(r_{(-1)}, s).$$

Below, we give a formula to compute r_1 and $r_{(-1)}$. This is an analogue of Lemma 9.19.

Lemma 12.14. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field.*

If k is even, for r in $\mathcal{S}_{k,-}$, we have

$$r_1 = \frac{q+1}{q} r - q a^{>_p} + a^{>_p\vee} \text{ and } r_{(-1)} = \frac{q+1}{q} r + q b^{>_p} - b^{>_p\vee}$$

where

$$a = \frac{q+1}{q}(q - S_p)^{-1}(r^{<\vee}) \text{ and } b = \frac{q+1}{q}(q + S_p)^{-1}(r^{<\vee}).$$

If k is odd, for r in $\mathcal{S}_{k,-}$, we have

$$r_1 = 2r - a^{>_p} + a^{>_p\vee} \text{ where } a = 2(q - S_p)^{-1}(r^{<\vee}).$$

For r in $\mathcal{S}_{k,+}$, we have

$$r_{(-1)} = 2r - b^{>_p} - b^{>_p\vee} \text{ where } b = 2(q + S_p)^{-1}(r^{<\vee}).$$

Proof. We assume that k is even and we check the formula for r_1. Therefore, we set

$$a = \frac{q+1}{q}(q - S_p)^{-1}(r^{<\vee}) \text{ and } s = \frac{q+1}{q} r - q a^{>_p} + a^{>_p\vee}.$$

First we claim that s is a k-skew field. Indeed, on one hand, as $r^\vee = -r$, we have

$$s + s^\vee = -q a^{>_p} + a^{>_p\vee} - q a^{>_p\vee} + a^{>_p\vee\vee} = 0.$$

On the other hand, we have

$$\begin{aligned} s^< + s^{<\vee} &= \frac{q+1}{q} r^< - qa + a^{>_p\vee<} + \frac{q+1}{q} r^{<\vee} - q a^\vee + a^{>_p\vee<\vee} \\ &= \frac{q+1}{q} r^< - qa + (S_p a)^\vee + \frac{q+1}{q} r^{<\vee} - q a^\vee + S_p a. \end{aligned}$$

By definition, a satisfies the equation

$$S_p a + \frac{q+1}{q} r^{<\vee} = qa, \tag{12.1}$$

hence $s^< + s^{<\vee} = 0$.

Now, to conclude, we must show that $c = \frac{q}{q+1}s - \frac{1}{2}s^{<>_p} - r$ is p^*-orthogonal to $\mathcal{G}_k^1$. Let us compute this k-simple pseudofield. We get

$$\begin{aligned} c &= r - \frac{q^2}{q+1}a^{>_p} + \frac{q}{q+1}a^{>_p\vee} - \frac{q+1}{2q}r^{<>_p} + \frac{q}{2}a^{>_p} - \frac{1}{2}(S_pa)^{\vee>_p} - r \\ &= \frac{q}{q+1}a^{>_p\vee} - \frac{q+1}{2q}r^{<>_p} - \frac{q(q-1)}{2(q+1)}a^{>_p} - \frac{1}{2}(S_pa)^{\vee>_p}. \end{aligned}$$

Using (12.1) yields

$$c = \frac{q}{q+1}a^{>_p\vee} - \frac{q}{2}a^{\vee>_p} - \frac{q(q-1)}{2(q+1)}a^{>_p} = \frac{q}{q+1}(a^{>_p\vee} + a^{>_p}) - \frac{q}{2}(a^\vee + a)^{>_p},$$

hence c is indeed p^*-orthogonal to $\mathcal{G}_k^1$ as required and therefore $r_1 = s$. The formula for r_{-1} is obtained in the same way.

Assume now k is odd and let us again check the formula for r_1. Thus, we set

$$a = 2(q - S_p)^{-1}(r^{<\vee}) \text{ and } s = 2r - a^{>_p} + a^{>_p\vee}.$$

Again, we claim that s is a k-skew field. Indeed, as $r^\vee = -r$, we have $s^\vee = -s$. Besides,

$$s^< + s^{<\vee} = 2r^< - a + a^{>_p\vee<} + 2r^{<\vee} - a^\vee + a^{>_p\vee<\vee}.$$

As $S_pa = a^{>_p\vee<\vee}$, the definition of a yields

$$a^{>_p\vee<\vee} + 2r^{<\vee} = qa \text{ and } a^{>_p\vee<} + 2r^< = a^\vee - (q-1)a. \tag{12.2}$$

We get

$$s^< + s^{<\vee} = a^\vee - (q-1)a - a + qa - a^\vee = 0$$

as required.

As in the even case, to conclude, we need to show that the k-simple pseudofield c defined by $c = \frac{1}{2}s - \frac{1}{q+1}s^{<>_p} - r$ is p^*-orthogonal to $\mathcal{G}_k^1$. Indeed, the definitions of the objects give

$$\begin{aligned} c &= r - \frac{1}{2}a^{>_p} + \frac{1}{2}a^{>_p\vee} - \frac{2}{q+1}r^{<>_p} + \frac{1}{q+1}a^{>_p} - \frac{1}{q+1}a^{>_p\vee<>_p} - r \\ &= \frac{1}{2}a^{>_p\vee} - \frac{2}{q+1}r^{<>_p} - \frac{q-1}{2(q+1)}a^{>_p} - \frac{1}{q+1}a^{>_p\vee<>_p}. \end{aligned}$$

Thanks to (12.2), we have

$$\begin{aligned} c &= \frac{1}{2}a^{>_p\vee} - \frac{1}{q+1}(a^\vee - (q-1)a)^{>_p} - \frac{q-1}{2(q+1)}a^{>_p} \\ &= \frac{1}{2}a^{>_p\vee} - \frac{1}{q+1}a^{\vee>_p} + \frac{q-1}{2(q+1)}a^{>_p} \\ &= \frac{1}{2}(a^{>_p\vee} + a^{>_p}) - \frac{1}{q+1}(a^\vee + a)^{>_p}. \end{aligned}$$

Thus, c is p^*-orthogonal to $\mathcal{G}_k^1$ and hence $r_1 = s$. The same method yields the formula for r_{-1}. □

Proof of Proposition 12.13. By construction, the null space of p_1^{sp} is the orthogonal subspace of $\mathcal{G}_k^1$ in $\mathcal{H}_k$ and the null space of $p_{(-1)}^{\mathrm{sp}}$ is the orthogonal subspace of $\mathcal{G}_k^{(-1)}$ in $\mathcal{H}_k$. The description of these spaces in the Proposition directly follows from the definition of skew fields and reverse skew fields.

We now check the formulae. Assume that k is even and take r, s in $\mathcal{S}_{k,-}$. We keep the notation of Lemma 12.14. As $s^\vee = -s$, we get

$$p^*(r_1, s) = (q+1)p^*\left(\frac{1}{q}r + a^{>_p\vee}, s\right).$$

By Lemma 9.11 and Lemma 12.14, we have

$$a^{>_p\vee} = \frac{q+1}{q}((q - S_p)^{-1}r)^{<\vee>_p\vee} = \frac{q+1}{q}S_p(q - S_p)^{-1}r.$$

Now, we have

$$1 + (q+1)S_p(q - S_p)^{-1} = q(1 + S_p)(q - S_p)^{-1},$$

hence

$$p^*(r_1, s) = (q+1)p^*((1 + S_p)(q - S_p)^{-1}r, s)$$

and the first formula follows. The other three are obtained in the same way. □

As usual, we have a symmetry property in the bipartite case. From Lemma 12.11, we get

Lemma 12.15. *Assume Γ is bipartite and let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For H, J in $\mathcal{H}_k$, we have*

$$p_{(-1)}^{\mathrm{sp}}(H^\wr, J^\wr) = p_1^{\mathrm{sp}}(H, J).$$

12.4. **The full special spectral bilinear forms.** We now use the special bilinear forms to build bilinear forms on $\mathcal{H}_k^2$ in the spirit of the doubling technique of Proposition 10.7.

Definition 12.16. Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For $H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$ and $J = \begin{pmatrix} J_0 \\ J_1 \end{pmatrix}$ in $\mathcal{H}_k^2$, we set

$$p_1^{2,\mathrm{sp}}(H, J) = p_1^{\mathrm{sp}}(qH_0 + H_1, qJ_0 + J_1)$$
$$\text{and } p_{(-1)}^{2,\mathrm{sp}}(H, J) = p_{(-1)}^{\mathrm{sp}}(qH_0 - H_1, qJ_0 - J_1).$$

The bilinear forms $p_1^{2,\mathrm{sp}}$ and $p_{(-1)}^{2,\mathrm{sp}}$ are called the full special spectral bilinear forms associated to p.

We summarize the properties of the full special spectral bilinear forms.

Proposition 12.17. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. The full special spectral bilinear forms $p_1^{2,\mathrm{sp}}$ and $p_{(-1)}^{2,\mathrm{sp}}$ are non-negative on $\mathcal{H}_k^2$. The null space of $p_1^{2,\mathrm{sp}}$ is the space*

$$\mathcal{H}_{k,+}^2 + (\mathcal{H}_{k-1,+}^{>})^2 + \left\{ \begin{pmatrix} H \\ -qH \end{pmatrix} \middle| H \in \mathcal{H}_k \right\}.$$

If k is even, the null space of $p_{(-1)}^{2,\mathrm{sp}}$ is the space

$$\mathcal{H}_{k,+}^2 + (\mathcal{H}_{k-1,-}^{>})^2 + \left\{ \begin{pmatrix} H \\ qH \end{pmatrix} \middle| H \in \mathcal{H}_k \right\}.$$

If k is odd, the null space of $p_{(-1)}^{2,\mathrm{sp}}$ is the space

$$\mathcal{H}_{k,-}^2 + (\mathcal{H}_{k-1,+}^{>})^2 + \left\{ \begin{pmatrix} H \\ qH \end{pmatrix} \middle| H \in \mathcal{H}_k \right\}.$$

In both cases, the operators $\mathfrak{R}_1$ and $\mathfrak{S}_1$ are symmetric with respect to $p_1^{2,\mathrm{sp}}$ and the operators $\mathfrak{R}_{(-1)}$ and $\mathfrak{S}_{(-1)}$ are symmetric with respect to $p_{(-1)}^{2,\mathrm{sp}}$.

Proof. The non-negativity properties and the description of the null spaces directly follows from Proposition 12.13 and Definition 12.16.

Assume k is even and let us show that the operators $\mathfrak{R}_1$ and $\mathfrak{S}_1$ are symmetric with respect to $p_1^{2,\mathrm{sp}}$. Recall from (6.1) that, for $H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$ in $\mathcal{H}_k^2$, we have $\mathfrak{S}_1 H = \begin{pmatrix} q^{-1} H_1^\vee \\ H_0^\vee - (q-1)H_0 \end{pmatrix}$. Therefore, for $H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$ and $J = \begin{pmatrix} J_0 \\ J_1 \end{pmatrix}$ in $\mathcal{H}_k^2$,

$$p_1^{2,\mathrm{sp}}(\mathfrak{S}_1 H, J) = p_1^{\mathrm{sp}}(H_1^\vee + H_0^\vee - (q-1)H_0, qJ_0 + J_1).$$

By Proposition 12.13 $H_0 + H_0^\vee$ and $H_1 + H_1^\vee$ belong to the null space of p_1^{sp}. Thus, we get

$$p_1^{2,\mathrm{sp}}(\mathfrak{S}_1 H, J) = -p_1^{\mathrm{sp}}(qH_0 + H_1, qJ_0 + J_1).$$

By symmetry, we obtain $p_1^{2,\mathrm{sp}}(\mathfrak{S}_1 H, J) = p_1^{2,\mathrm{sp}}(H, \mathfrak{S}_1 J)$ as required. In the same way, still by (6.1), we have $\mathfrak{R}_1 H = \begin{pmatrix} H_0^\vee + q^{-1}(q+1)H_1^\vee \\ (q-1)H_1 - H_1^\vee \end{pmatrix}$, hence

$$p_1^{2,\mathrm{sp}}(\mathfrak{R}_1 H, J) = p_1^{\mathrm{sp}}(qH_0^\vee + qH_1^\vee + (q-1)H_1, qJ_0 + J_1) \\ = -p_1^{\mathrm{sp}}(qH_0 + H_1, qJ_0 + J_1).$$

Again, we get $p_1^{2,\mathrm{sp}}(\mathfrak{R}_1 H, J) = p_1^{2,\mathrm{sp}}(H, \mathfrak{R}_1 J)$.

The other cases may be obtained by the same method. □

In the bipartite case, Definition 12.16 and Lemma 12.15 yield

Lemma 12.18. *Assume Γ is bipartite and let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For H, J in $\mathcal{H}_k^2$, we have*

$$p_{(-1)}^{2,\mathrm{sp}}\left(\begin{pmatrix} H_0^\wr \\ -H_1^\wr \end{pmatrix}, \begin{pmatrix} J_0^\wr \\ -J_1^\wr \end{pmatrix}\right) = p_1^{\mathrm{sp}}(H, J).$$

13. A Plancherel formula for Euclidean fields

In this final Section, we state and prove the Plancherel formula for a Γ-invariant Euclidean field p. We use this formula to compute the spectrum of the natural geometric operator of Subsection 1.1, acting on the space of Γ-invariant maps $X \to H^p$.

13.1. **The formula and its spectral consequences.** Given $k \geq 2$ and a Γ-invariant k-Euclidean field p, we use the previously introduced notions to write the following Plancherel formula for the non negative symmetric bilinear form $E_k^\star p$ obtained by pulling back the natural scalar product on $\mathcal{H}_\infty$ by the polyextension map $E_k : \mathcal{H}_k^{(\mathbb{N})} \to \mathcal{H}_\infty$.

Theorem 13.1. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. For any H, J in $\mathcal{H}_k^{(\mathbb{N})}$, we have*

(13.1)
$$p^\infty(E_k H, E_k J) = \frac{q+1}{2\pi(q-1)} \int_{\mathcal{I}_q} p_t^2(\widehat{H}(t), \widehat{J}(t)) \sqrt{4q - (q+1)^2 t^2} \mathrm{d}t \\ + (q-1) \sum_{t \in \Sigma_p} p_t^{2,\mathrm{ex}}(\widehat{H}(t), \widehat{J}(t)) + \frac{q-1}{2(q+1)} p_1^{2,\mathrm{sp}}(\widehat{H}(1), \widehat{J}(1)) \\ + \frac{q-1}{2(q+1)} p_{(-1)}^{2,\mathrm{sp}}(\widehat{H}(-1), \widehat{J}(-1)).$$

The polyextension map E_k was defined in Subsection 2.3. The spectral transform $H \mapsto \widehat{H}, \mathcal{H}_k^{(\mathbb{N})} \to \mathcal{H}_k^2[t]$ was constructed in Proposition 6.3. For t in $\mathbb{C}$ with $(q+1)^2 t^2 \neq 4q$, the full spectral bilinear form p_t^2 on $\mathcal{H}_k^2$ was introduced in Definition 10.8. The exceptional spectrum Σ_p was introduced in (8.1) and, for t in Σ_p, we defined the full exceptional spectral bilinear form $p_t^{2,\mathrm{ex}}$ on $\mathcal{H}_k^2$ in Definition 11.8. Finally, for $t = 1$ or $t = -1$, we defined the full special spectral bilinear form $p_t^{2,\mathrm{sp}}$ on $\mathcal{H}_k^2$ in Definition 12.16.

Note that, if Γ is bipartite, the symmetry properties in the spectral part of the formula which come from Lemma 10.14, Lemma 11.9 and Lemma 12.18 are compatible with the equivariance property of the polyextension map established in Lemma 2.23 and the one of the spectral transform established in Lemma 6.18.

In analogy with the case of the model operators that we studied in Section 4 and Section 5, the proof of the formula in Theorem 13.1 will rely on the identification of the boundary values in the resolvent formula of Proposition 8.1.

Before beginning the proof, we use the formula to compute spectra of natural operators. First, we can complete Corollary 8.6.

Corollary 13.2. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Then, the spectrum of P in the completion $\mathcal{H}_\infty^p$ of $\mathcal{H}_\infty$ with respect to p is the set*

$$\mathcal{I}_q \cup \Sigma_p \cup \{-1, 1\}.$$

Proof. Note that, by Lemma 7.8, we have $\Sigma_{p^+} = \Sigma_p$. Hence, we can replace p by a large orthogonal extension. In particular, by Corollary 2.7, we can assume that we have $\mathcal{H}_{k,+} \neq \{0\}$ and $\mathcal{H}_{k,-} \neq \{0\}$.

By Lemma 2.18, for H in $\mathcal{H}_k^{(\mathbb{N})}$, we have $PE_kH = E_kPH$. By Lemma 6.1 and Proposition 6.3, for t in $\mathbb{R}$, we have $\widehat{PH}(t) = t\widehat{H}(t)$. By Proposition 10.12, for t in the interior of $\mathcal{I}_q$, the full spectral bilinear form p_t^2 is positive definite. By Proposition 11.3, for t in Σ_p, the full exceptional spectral bilinear form $p_t^{2,\mathrm{ex}}$ is non zero. By Proposition 12.17, the full special spectral bilinear forms $p_1^{2,\mathrm{sp}}$ and $p_{(-1)}^{2,\mathrm{sp}}$ are also non zero. Therefore, by Theorem 13.1, the spectrum of P in the closure of $E_k\mathcal{H}_k^{(\mathbb{N})}$ in $\mathcal{H}_\infty^p$ is $\mathcal{I}_q \cup \Sigma_p \cup \{-1, 1\}$. The conclusion follows by applying this property to all orthogonal extensions of p. □

We can also use the Plancherel formula to prove our first statement.

Proof of Corollary 1.3. This is obtained as in Corollary 4.10 where we retrieved the computation of the spectral measures of the operator Q acting on $\ell^2(X)$.

Indeed, as in the proof of Corollary 4.10, we consider the natural operator $L : \mathcal{F}(X, H^p)^\Gamma \to \mathcal{F}(X_1, H^p)^\Gamma = \mathcal{H}_\infty^p$ defined by, for $x \sim y$ in X and f in $\mathcal{F}(X, H^p)^\Gamma$, $Lf(x, y) = f(x)$. We still have $PL = LQ$ and a direct computation using Lemma I.9.11 shows that $p(Lf, Lf) =$

$(q+1)p(f,f)$. Therefore, by Corollary 13.2, the spectrum Σ_X of Q in $\mathcal{F}(X,H^p)^\Gamma$ is contained in $\mathcal{I}_q \cup \Sigma_p \cup \{-1,1\}$.

Conversely, by Proposition 10.12, for t in the interior of $\mathcal{I}_q$, the full spectral bilinear form p^2_t is positive definite, hence we have $\mathcal{I}_q \subset \Sigma_X$. Besides, note that, by construction, for every f in $\mathcal{F}(X,H^p)^\Gamma$, we have $RLf = qLf$. As by Proposition 12.17, the full special spectral bilinear forms $p_1^{2,\mathrm{sp}}$ and $p_{(-1)}^{2,\mathrm{sp}}$ are 0 on $\mathcal{H}^2_{j,+}$ for all even $j \geq k$, the special points (-1) and 1 do not belong to Σ_X. Finally, with no loss of generality, we can assume that k is even. For t in Σ_p, let u be the root of the equation $q + u^2 = (q+1)tu$ with $\sqrt{q} < |u| < q$. By Corollary 11.7, we know that the exceptional spectral bilinear form p_u^{ex} is non zero on $\mathcal{H}_{k,+}$. This implies that Σ_q is contained in Σ_X. The result follows. □

We now go back to proving Theorem 13.1.

13.2. **Particular values of spectral quadratic forms.** The resolvent formulae in Proposition 8.1 are obtained only for certain particular vectors. To deduce from these particular cases the general Plancherel formula in Theorem 13.1, we will need the following uniqueness criterion. Recall that the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ were introduced in (6.1) and (6.2).

Proposition 13.3. *Let $k \geq -1$, u be in $\mathbb{C}^*$ and set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Let ϖ be a symmetric bilinear form on $\mathcal{H}^2_{k,\mathbb{C}}$ such that the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ are symmetric with respect to ϖ.*

Then, if k is even, ϖ is entirely determined by the bilinear forms

$$(H,J) \mapsto \varpi\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} J \\ 0 \end{pmatrix}\right)$$

on $\mathcal{H}_{k,+,\mathbb{C}} \times \mathcal{H}_{k,+,\mathbb{C}}$ and $\mathcal{H}_{k,-,\mathbb{C}} \times \mathcal{H}_{k,-,\mathbb{C}}$ as well as the bilinear form

$$(H,J) \mapsto \varpi\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right)$$

on $\mathcal{H}_{k,+,\mathbb{C}} \times \mathcal{H}_{k,-,\mathbb{C}}$.

If k is odd, ϖ is entirely determined by the bilinear forms

$$(H,J) \mapsto \varpi\left(\begin{pmatrix} 0 \\ H \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right)$$

on $\mathcal{H}_{k,+,\mathbb{C}} \times \mathcal{H}_{k,+,\mathbb{C}}$ and $\mathcal{H}_{k,-,\mathbb{C}} \times \mathcal{H}_{k,-,\mathbb{C}}$ as well as the bilinear form

$$(H,J) \mapsto \varpi\left(\begin{pmatrix} 0 \\ H \end{pmatrix}, \begin{pmatrix} -J \\ (q-1)J \end{pmatrix}\right)$$

on $\mathcal{H}_{k,+,\mathbb{C}} \times \mathcal{H}_{k,-,\mathbb{C}}$.

In particular, we can compute the values of the bilinear forms that appear in our other uniqueness criterion, that is, Proposition 10.7.

Corollary 13.4. *Let $k \geq -1$, u be in $\mathbb{C}^*$ and set $t = \frac{1}{q+1}(u + \frac{q}{u})$. Let ϖ be a symmetric bilinear form on $\mathcal{H}^2_{k,\mathbb{C}}$ such that the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ are symmetric with respect to ϖ. Take H in $\mathcal{H}_{k,+,\mathbb{C}}$ and J in $\mathcal{H}_{k,-,\mathbb{C}}$.*

If k is even, we have

$$\begin{aligned}
\varpi\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} H \\ -uH \end{pmatrix}\right) &= \frac{q-1}{q+1}(u^2-1)\varpi\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} H \\ 0 \end{pmatrix}\right) \\
\varpi\left(\begin{pmatrix} J \\ -uJ \end{pmatrix}, \begin{pmatrix} J \\ -uJ \end{pmatrix}\right) &= \frac{1}{q^2}\frac{q-1}{q+1}(q^2-u^2)\varpi\left(\begin{pmatrix} J \\ 0 \end{pmatrix}, \begin{pmatrix} J \\ 0 \end{pmatrix}\right) \\
\varpi\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} J \\ -uJ \end{pmatrix}\right) &= (q-1)u\varpi\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right).
\end{aligned}$$

If k is odd, we have

$$\begin{aligned}
\varpi\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} H \\ -uH \end{pmatrix}\right) &= \frac{q-1}{2u}(q+u)(u-1)\varpi\left(\begin{pmatrix} 0 \\ H \end{pmatrix}, \begin{pmatrix} 0 \\ H \end{pmatrix}\right) \\
\varpi\left(\begin{pmatrix} J \\ -uJ \end{pmatrix}, \begin{pmatrix} J \\ -uJ \end{pmatrix}\right) &= \frac{q-1}{2u}(q-u)(u+1)\varpi\left(\begin{pmatrix} 0 \\ J \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right) \\
\varpi\left(\begin{pmatrix} H \\ -uH \end{pmatrix}, \begin{pmatrix} J \\ -uJ \end{pmatrix}\right) &= (q-1)\varpi\left(\begin{pmatrix} 0 \\ H \end{pmatrix}, \begin{pmatrix} -J \\ (q-1)J \end{pmatrix}\right).
\end{aligned}$$

The proofs rely on a case by case description of invariant bilinear forms.

Lemma 13.5. *Fix t, u in $\mathbb{C}$ with $q + u^2 = (q+1)tu$. In each of the following six systems of matrix equations, the space of solutions is a complex line.*

The space of solutions of the system

$$\begin{pmatrix} q & 0 \\ (q+1)t & -1 \end{pmatrix} A = A \begin{pmatrix} q & (q+1)t \\ 0 & -1 \end{pmatrix} \tag{13.2}$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the space $\mathbb{C}A_0$ where $A_0 = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$ and we have

$$\begin{pmatrix} 1 & -u \end{pmatrix} A_0 \begin{pmatrix} 1 \\ -u \end{pmatrix} = \frac{q-1}{q+1}(u^2-1).$$

The space of solutions of the system

$$\begin{pmatrix} -1 & 0 \\ -q^{-1}(q+1)t & q \end{pmatrix} B = B \begin{pmatrix} -1 & -q^{-1}(q+1)t \\ 0 & q \end{pmatrix} \tag{13.3}$$

$$\begin{pmatrix} 0 & -q \\ -q^{-1} & 0 \end{pmatrix} B = B \begin{pmatrix} 0 & -q^{-1} \\ -q & 0 \end{pmatrix}$$

is the space $\mathbb{C}B_0$ *where* $B_0 = \begin{pmatrix} q^2 & qt \\ qt & 1 \end{pmatrix}$ *and we have*

$$\begin{pmatrix} 1 & -u \end{pmatrix} B_0 \begin{pmatrix} 1 \\ -u \end{pmatrix} = \frac{q-1}{q+1}(q^2-u^2).$$

The space of solutions of the system

$$\begin{pmatrix} q & 0 \\ (q+1)t & -1 \end{pmatrix} C = C \begin{pmatrix} -1 & -q^{-1}(q+1)t \\ 0 & q \end{pmatrix} \tag{13.4}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} C = C \begin{pmatrix} 0 & -q^{-1} \\ -q & 0 \end{pmatrix}$$

is the space $\mathbb{C}C_0$ *where* $C_0 = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}$ *and we have*

$$\begin{pmatrix} 1 & -u \end{pmatrix} C_0 \begin{pmatrix} 1 \\ -u \end{pmatrix} = (q-1)u.$$

The space of solutions of the system

$$\begin{pmatrix} 0 & q \\ 1 & q-1 \end{pmatrix} D = D \begin{pmatrix} 0 & 1 \\ q & q-1 \end{pmatrix} \tag{13.5}$$

$$\begin{pmatrix} -1 & (q+1)t-(q-1) \\ 0 & 1 \end{pmatrix} D = D \begin{pmatrix} -1 & 0 \\ (q+1)t-(q-1) & 1 \end{pmatrix}$$

is the space $\mathbb{C}D_0$ *where* $D_0 = \begin{pmatrix} q^2+1-(q^2-1)t & (q+1)t-(q-1) \\ (q+1)t-(q-1) & 2 \end{pmatrix}$ *and we have*

$$\begin{pmatrix} 1 & -u \end{pmatrix} D_0 \begin{pmatrix} 1 \\ -u \end{pmatrix} = \frac{q-1}{u}(q+u)(u-1).$$

The space of solutions of the system

$$\begin{pmatrix} 0 & -q \\ -1 & q-1 \end{pmatrix} E = E \begin{pmatrix} 0 & -1 \\ -q & q-1 \end{pmatrix} \tag{13.6}$$

$$\begin{pmatrix} 1 & -(q+1)t-(q-1) \\ 0 & -1 \end{pmatrix} E = E \begin{pmatrix} 1 & 0 \\ -(q+1)t-(q-1) & -1 \end{pmatrix}$$

is the space $\mathbb{C}E_0$ *where* $E_0 = \begin{pmatrix} q^2+1+(q^2-1)t & (q+1)t+(q-1) \\ (q+1)t+(q-1) & 2 \end{pmatrix}$ *and we have*

$$\begin{pmatrix}1 & -u\end{pmatrix} E_0 \begin{pmatrix}1 \\ -u\end{pmatrix} = \frac{q-1}{u}(q-u)(u+1).$$

The space of solutions of the system

$$\begin{pmatrix}0 & q \\ 1 & q-1\end{pmatrix} F = F \begin{pmatrix}0 & -1 \\ -q & q-1\end{pmatrix} \tag{13.7}$$

$$\begin{pmatrix}-1 & (q+1)t-(q-1) \\ 0 & 1\end{pmatrix} F = F \begin{pmatrix}1 & 0 \\ -(q+1)t-(q-1) & -1\end{pmatrix}$$

is the space $\mathbb{C}F_0$ *where* $F_0 = \begin{pmatrix}q-1 & 1 \\ -1 & 0\end{pmatrix}$ *and we have*

$$\begin{pmatrix}1 & -u\end{pmatrix} F_0 \begin{pmatrix}1 \\ -u\end{pmatrix} = q-1.$$

Proof of Proposition 13.3. The proof is a direct consequence of the explicit formulae in Lemma 13.5. For example, assume k is even. Then, by (6.1), the action of $\mathfrak{R}_t$ on $\mathcal{H}^2_{k,+,\mathbb{C}}$ is given by the matrix $\begin{pmatrix}q & 0 \\ (q+1)t & -1\end{pmatrix}$ whereas the action of $\mathfrak{S}_t$ is given by the matrix $\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$. Therefore, as the space of solution to the system (13.2) in Lemma 13.5 has dimension 1 and is generated by the matrix A_0 whose top left coefficient is non zero, the restriction of ϖ to the space $\mathcal{H}^2_{k,+,\mathbb{C}} \times \mathcal{H}^2_{k,+,\mathbb{C}}$ is completely determined by the bilinear form

$$(H,J) \mapsto \varpi\left(\begin{pmatrix}H \\ 0\end{pmatrix}, \begin{pmatrix}J \\ 0\end{pmatrix}\right)$$

on $\mathcal{H}_{k,+,\mathbb{C}} \times \mathcal{H}_{k,+,\mathbb{C}}$. In the same way we use (13.3) and (13.4) to prove that the restrictions of ϖ to the spaces $\mathcal{H}^2_{k,-,\mathbb{C}} \times \mathcal{H}^2_{k,-,\mathbb{C}}$ and $\mathcal{H}^2_{k,+,\mathbb{C}} \times \mathcal{H}^2_{k,-,\mathbb{C}}$ are determined respectively by the bilinear forms

$$(H,J) \mapsto \varpi\left(\begin{pmatrix}H \\ 0\end{pmatrix}, \begin{pmatrix}J \\ 0\end{pmatrix}\right)$$

on $\mathcal{H}_{k,-,\mathbb{C}} \times \mathcal{H}_{k,-,\mathbb{C}}$ and

$$(H,J) \mapsto \varpi\left(\begin{pmatrix}H \\ 0\end{pmatrix}, \begin{pmatrix}0 \\ J\end{pmatrix}\right)$$

on $\mathcal{H}_{k,+,\mathbb{C}} \times \mathcal{H}_{k,-,\mathbb{C}}$.

We proceed similarly in the odd case by using (13.4), (13.5) and (13.6). ☐

Proof of Corollary 13.4. As above, if k is even, it follows from (13.2) in Lemma 13.5 that the restriction of ϖ to the space $\mathcal{H}^2_{k,+,\mathbb{C}}$ is given by a formula of the form

$$\varpi\left(\begin{pmatrix}H_0\\H_1\end{pmatrix},\begin{pmatrix}J_0\\J_1\end{pmatrix}\right)=\chi(H_0,J_0)+\chi(H_1,J_1)+t\chi(H_0,J_1)+t\chi(H_1,J_0),$$

for H_0, H_1, J_0, J_1 in $\mathcal{H}_{k,+,\mathbb{C}}$, where χ is a symmetric bilinear form on $\mathcal{H}_{k,+,\mathbb{C}}$. The formula for $\varpi\left(\begin{pmatrix}H\\-uH\end{pmatrix},\begin{pmatrix}H\\-uH\end{pmatrix}\right)$ follows. The other cases are dealt with in the same way. ☐

13.3. **Imaginary parts of rational functions.** As for the Plancherel formulas associated to the model operators (4.1), (4.8), (5.1) and (5.4), the proof of the Plancherel formula for Euclidean fields will rely on the identification of the boundary values of certain harmonic functions on $\mathbb{H}$. These harmonic functions are the imaginary parts of the resolvent functions which appear in Proposition 8.1. We will need purely algebraic formulae in order to compute these imaginary parts.

Lemma 13.6. *Let $\mathcal{A}$ be a real algebra and a be an element of $\mathcal{A}$. Let t be in $\mathcal{I}_q$ and u be in $\mathbb{C}^*$ with $q+u^2=(q+1)tu$, so that $|u|=\sqrt{q}$. Assume that $u-a$ is invertible in the complexification $\mathcal{A}_\mathbb{C}$ of $\mathcal{A}$. Define elements of $\mathcal{A}_\mathbb{C}$ by setting*

$$\begin{aligned}
b_{++} &= \frac{u}{1-u^2}(qu+a)(u-a)^{-1}\\
b_{+-} &= \frac{u}{q^2-u^2}(u+qa)(u-a)^{-1}\\
b_{-+} &= \frac{u}{(q+u)(1-u)}(u+a)(u-a)^{-1}\\
b_{--} &= \frac{u}{(q-u)(1+u)}(u+a)(u-a)^{-1}\\
c &= a(u-a)^{-1}.
\end{aligned}$$

Then, the imaginary parts of these elements are given by

$$\begin{aligned}
\Im b_{++} &= \frac{1}{q+1}\frac{\Im u}{1-t^2}(q^2-a^2)(q-(q+1)ta+a^2)^{-1}\\
\Im b_{+-} &= \frac{1}{q+1}\frac{\Im u}{1-t^2}(1-a^2)(q-(q+1)ta+a^2)^{-1}\\
\Im b_{-+} &= \frac{2}{(q+1)^2}\frac{\Im u}{1-t^2}(1+a)(q-a)(q-(q+1)ta+a^2)^{-1}\\
\Im b_{--} &= \frac{2}{(q+1)^2}\frac{\Im u}{1-t^2}(1-a)(q+a)(q-(q+1)ta+a^2)^{-1}\\
\Im c &= -\Im(u)a(q-(q+1)ta+a^2)^{-1}.
\end{aligned}$$

Proof. Note that the formulae make sense. Indeed, as a is real and $u-a$ is invertible, so is $\frac{q}{u}-a=\overline{u}-a$. Therefore, $q-(q+1)ta+a^2=(u-a)(\frac{q}{u}-a)$ is invertible.

Let us compute the imaginary part of b_{++}. We set $n_{++}=u(qu+a)$ and $d_{++}=(1-u^2)(u-a)$, so that $b_{++}=n_{++}d_{++}^{-1}$, hence

$$\Im b_{++}=\Im(n_{++}\overline{d_{++}})(d_{++}\overline{d_{++}})^{-1}.$$

By taking in account that $|u|=\sqrt{q}$, we get

$$\begin{aligned}
n_{++}\overline{d_{++}} &= u(qu+a)(1-\overline{u}^2)(\overline{u}-a)=(qu^2+ua-q^3-q\overline{u}a)(\overline{u}-a)\\
&= q^2u+qa-q^3\overline{u}-q\overline{u}^2a-qu^2a-ua^2+q^3a+q\overline{u}a^2,
\end{aligned}$$

which gives $\Im(n_{++}\overline{d_{++}})=(q+1)\Im(u)(q^2-a^2)$. Besides, we have

$$\begin{aligned}
d_{++}\overline{d_{++}} &= (1-u)(1-\overline{u})(1+u)(1+\overline{u})(u-a)(\overline{u}-a)\\
&= (q+1)^2(1-t^2)(q-(q+1)ta+a^2).
\end{aligned}$$

The conclusion follows. The other formulae are obtained in the same way. □

13.4. **Isolated singularities of harmonic functions.** To deal with the discrete part of the Plancherel formula in Theorem 13.1, we will need to recall standard properties of non-negative harmonic functions which can be found in [1, Chapter 7]. We use the notation of Subsection 3.3 for Poisson transforms.

Lemma 13.7. *Let μ be a positive Borel measure on $\mathbb{R}$ and $a<b$ be in $\mathbb{R}$. Assume that $\int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{1+t^2}<\infty$ so that the Poisson transform of μ is well-defined. Then, we have*

$$\int_a^b \mathcal{P}\mu(x+iy)\mathrm{d}x \xrightarrow[y\to 0]{} \frac{1}{2}\mu(\{a\})+\mu((a,b))+\frac{1}{2}\mu(\{b\}).$$

Proof. This is a straightforward computation. By the definition of the Poisson transform and Fubini Theorem, for $y > 0$, we have

$$\int_a^b \mathcal{P}\mu(x+iy)\mathrm{d}x = \frac{1}{\pi}\int_a^b \int_{\mathbb{R}} \frac{y\mathrm{d}\mu(t)}{(x-t)^2+y^2}\mathrm{d}x$$
$$= \frac{1}{\pi}\int_{\mathbb{R}}\int_a^b \frac{y\mathrm{d}x}{(x-t)^2+y^2}\mathrm{d}\mu(t).$$

We will apply the Dominated Convergence Theorem to determine the limit of the above integral as $y \to 0$. To show the domination, we note that, for $t \le a-1$ and $a \le x \le b$, we have $(t-x)^2 \ge (t-a)^2 \ge 1$. We get, for $y > 0$,

$$(t-a)^2 + 1 \le 2(t-x)^2 + 2y^2,$$

hence

$$\frac{y}{(x-t)^2+y^2} \le \frac{2y}{(t-a)^2+1}$$

and

$$\left|\int_a^b \frac{y\mathrm{d}x}{(x-t)^2+y^2}\right| \le \frac{2(b-a)y}{(t-a)^2+1}. \tag{13.8}$$

Similarly, for $t \ge b+1$ and $y > 0$,

$$\left|\int_a^b \frac{y\mathrm{d}x}{(x-t)^2+y^2}\right| \le \frac{2(b-a)y}{(t-b)^2+1}. \tag{13.9}$$

Besides, for any t in $\mathbb{R}$ and $y > 0$, we have

$$\int_a^b \frac{y\mathrm{d}x}{(x-t)^2+y^2} = \arctan\frac{b-t}{y} - \arctan\frac{a-t}{y}.$$

The latter quantity satisfies

$$\left|\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y}\right| \le \pi \tag{13.10}$$

and

$$\begin{aligned}
&\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y} \xrightarrow[y\to 0]{} 0 \qquad \text{if } t < a \text{ or } t > b\\
&\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y} \xrightarrow[y\to 0]{} \pi \qquad \text{if } a < t < b\\
&\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y} \xrightarrow[y\to 0]{} \frac{\pi}{2} \qquad \text{if } t = a \text{ or } t = b.
\end{aligned} \tag{13.11}$$

The bounds in (13.8), (13.9) and (13.10), together with the assumption that $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$, allow us to apply the Dominated Convergence Theorem to the convergence in (13.11). The conclusion follows. ☐

We shall use this property to remove isolated singularities of harmonic functions.

Corollary 13.8. *Let μ be a positive Borel measure on $\mathbb{R}$ with $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$. Fix s in $\mathbb{R}$ and assume that there exists a harmonic function F on $\mathbb{H}$ such that, for any z in $\mathbb{H}$,*

$$\mathcal{P}\mu(z) = \frac{1}{\pi}\Im\left(\frac{1}{s-z}\right) + F(z)$$

and that $F(x+iy) \xrightarrow[y\to 0]{} 0$ uniformly for x in a neighborhood of s in $\mathbb{R}$. Then, there exists a positive Borel measure ν on $\mathbb{R}$ whose support does not contain s such that $\mu = \delta_s + \nu$ and one has $F = \mathcal{P}\nu$.

Proof. Note that we have

$$\Im\left(\frac{1}{s-(x+iy)}\right) \xrightarrow[y\to 0]{} 0 \tag{13.12}$$

uniformly for x in a compact subset of $\mathbb{R}$ that does not contain s. Therefore, by the assumption and by Lemma 13.7, there exists a neighborhood N of s in $\mathbb{R}$ such that, for any closed interval $I \subset N$ with $s \notin I$, one has $\mu(I) = 0$. Hence, we can write $\mu = \alpha\delta_s + \nu$ where $\alpha \geq 0$ and ν is a positive Radon measure whose support does not contain t.

As $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$, we still have $\int_{\mathbb{R}} \frac{\mathrm{d}\nu(t)}{1+t^2} < \infty$. Saying that the support of ν does not contain s amounts to saying that there exists $\varepsilon > 0$ such that, for ν-almost any t in $\mathbb{R}$, one has $|s-t| \geq \varepsilon$. For $y > 0$, this gives

$$\Im\left(\frac{1}{s-(t+iy)}\right) = \frac{y}{(s-t)^2+y^2} \leq \frac{2y}{(s-t)^2+\varepsilon^2},$$

hence, by (13.12) and the Dominated Convergence Theorem,

$$\mathcal{P}\nu(s+iy) \xrightarrow[y\to 0]{} 0.$$

For $z \in \mathbb{H}$, we have

$$\frac{\alpha-1}{\pi}\Im\left(\frac{1}{s-z}\right) = F(z) - \mathcal{P}\nu(z).$$

By applying this identity to $z = s+iy$ and letting y go to 0, we get

$$\frac{\alpha-1}{y} \xrightarrow[y\to 0]{} 0,$$

hence $\alpha = 1$. The conclusion follows. □

By applying elementary properties of holomorphic functions, we get

Corollary 13.9. *Let μ be a positive Borel measure on $\mathbb{R}$ with $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$. Fix s in $\mathbb{R}$ and assume that there exists an open neighborhood U of s in $\mathbb{C}$ and a holomorphic function F on $U \smallsetminus \{s\}$, with a simple pole at s, such that*

$$F(U \cap \mathbb{R}) \subset \mathbb{R} \text{ and } \mathcal{P}\mu(z) = \frac{1}{\pi}\Im F(z), \quad z \in U \cap \mathbb{H}.$$

Then, we have $\mu = \nu - a\delta_s$ where a is the residue of F at s and ν is a positive Borel measure on $\mathbb{R}$ whose support does not contain s.

The reader should beware the sign of a in the formula $\mu = \nu - a\delta_s$. In particular, the assumption imply $a \leq 0$.

13.5. **Residue computations.** When applying Corollary 13.9, we will need to compute the residues of certain meromorphic functions. They will always be given by the following elementary

Lemma 13.10. *Let V be a finite-dimensional complex vector space and T be an endomorphism of V. Let s be a simple eigenvalue of T and Π be the projection on $\ker(T - s)$ with null space $(T - s)V$. Let $F \in \mathbb{C}[z, t]$ be a polynomial function. Then, the meromorphic operator valued function on $\mathbb{C}$*

$$G : z \mapsto F(z, T)(z - T)^{-1}$$

has at most a simple pole at s, with residue $F(s, s)\Pi$.

Proof. Write $\Xi = 1 - \Pi$. As $T\Pi = s\Pi$, we get, for any z in $\mathbb{C}$ that is not an eigenvalue of T,

$$\begin{aligned} G(z) = F(z, T)(z - T)^{-1}\Pi + F(z, T)(z - T)^{-1}\Xi \\ = \frac{F(z, s)}{z - s}\Pi + F(z, T)(z - T)^{-1}\Xi. \end{aligned}$$

On one hand, by assumption, s is not an eigenvalue of T in $\Xi V = (T - s)V$. Thus, the meromorphic function $z \mapsto F(z, T)(z - T)^{-1}\Xi$ is holomorphic at s. On the other hand, the meromorphic function $z \mapsto \frac{F(z,s)}{z-s}\Pi$ has at most a simple pole at s, with residue $F(s, s)\Pi$. The conclusion follows. □

We will also need to use the change of coordinates of Subsection 3.3 in residue computations. This is possible thanks to an easy formula of complex analysis.

Lemma 13.11. *Let U and V be open subsets of $\mathbb{C}$ and $\varphi : U \to V$ be a biholomorphism. Let w be in V and F be a holomorphic function on $V \smallsetminus \{w\}$ with a simple pole at w, with residue a. Set $z = \varphi^{-1}(w)$. Then, $F \circ \varphi$ has a simple pole at z, with residue $\varphi'(z)^{-1}a$.*

13.6. **Proof of the Plancherel formula when $\sqrt{q}$ and $-\sqrt{q}$ are not eigenvalues of S_p.** When $\sqrt{q}$ or $-\sqrt{q}$ is an eigenvalue of the simple transfer operator S_p, the resolvent formulae in Proposition 8.1 have no continuous extension in the neighborhood of the boundary points of the critical interval $\mathcal{I}_q$ in $\mathbb{C}$ and the proof of Theorem 13.1 is slightly more complicated. Therefore, we start by assuming that this is not the case.

Proof of Theorem 13.1 when $\sqrt{q}$ and $-\sqrt{q}$ are not eigenvalues of S_p. Recall that the definition of the scalar product on $\mathcal{H}_\infty$ in (7.1), as well as Lemma I.9.11, imply that the operators R and S defined on $\mathcal{H}_\infty$ in Subsection 2.5 are self-adjoint with respect to this scalar product p^∞. Therefore, by Lemma 2.18, the natural operators R and S of Definition 2.16 and Definition 2.17, which act on $\mathcal{H}_k^{(\mathbb{N})}$, are symmetric with respect to the pull-back of p^∞ under the polyextension map, that is, for H, J in $\mathcal{H}_k^{(\mathbb{N})}$, we have

$$\begin{aligned} p^\infty(E_kRH, E_kJ) &= p^\infty(RE_kH, E_kJ) \\ &= p^\infty(E_kH, RE_kJ) = p^\infty(E_kH, E_kRJ) \end{aligned}$$

and, in the same way, $p^\infty(E_kSH, J) = p^\infty(E_kH, E_kSJ)$.

We use the spectral transform constructed in Proposition 6.3 to identify $\mathcal{H}_k^{(\mathbb{N})}$ with the space $\mathcal{H}_k^2[t]$ of polynomial functions $\mathbb{R} \to \mathcal{H}_k^2$. By Lemma 6.1, for t in $\mathbb{R}$, the operators of (6.1) and (6.2) satisfy

$$\mathfrak{R}_t\mathfrak{S}_t + \mathfrak{S}_t\mathfrak{R}_t - (q-1)\mathfrak{S}_t = (q+1)t. \tag{13.13}$$

Therefore, by using Lemma 3.4 as in the proof of Corollary 3.5, we get that there exists a Radon measure μ on $\mathbb{R}$ with support in $[-1, 1]$ and an integrable Borel map $t \mapsto \varpi_t, \mathbb{R} \to \mathcal{Q}_+(\mathcal{H}_k^2)$ from $\mathbb{R}$ to the space of non-negative symmetric bilinear forms on $\mathcal{H}_k^2$ such that
(i) for μ-almost any t in $\mathbb{R}$, the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ of (6.1) and (6.2) are symmetric with respect to ϖ_t.
(ii) for any H, J in $\mathcal{H}_k^{(\mathbb{N})}$, one has

$$p^\infty(E_kH, E_kJ) = \int_{\mathbb{R}} \varpi_t(\widehat{H}(t), \widehat{J}(t))\mathrm{d}\mu(t). \tag{13.14}$$

As we did for the model operators in Section 4 and Section 5, we will now use the resolvent formulae of Proposition 8.1 to compute these spectral invariants. More precisely, we will relate the bilinear forms ϖ_t to the families of bilinear forms constructed in Sections 9, 10, 11 and 12.

Indeed, the properties of the spectral transform in Proposition 6.3, together with (13.13) and Lemma 3.7, imply that, for H in $\mathcal{H}_k^{(\mathbb{N})}$, the

function
$$z \mapsto \frac{1}{\pi}\Im p^{\infty}((P-z)^{-1}E_k H, E_k H)$$
on $\mathbb{H}$ is the Poisson transform of the measure $\varpi_t(\widehat{H}(t), \widehat{H}(t))\mathrm{d}\mu(t)$ on $\mathbb{R}$.

Let us have a closer look at the expression for the resolvent given in Proposition 8.1. First, we consider the right hand-side of these formulae as meromorphic functions of u in $\mathbb{C}$ with $|u| > \sqrt{q}$. By Proposition 7.16, all the eigenvalues of S_p in the set $\{u \in \mathbb{C} | \, |u| > \sqrt{q}\}$ are real and simple. By Proposition 7.9, all these eigenvalues have absolute value $< q$. Therefore, the right hand-side of the resolvent formulae define meromorphic functions of u, $|u| > \sqrt{q}$, with at most simple poles in the finite set
$$\{u \in \mathbb{R} | \, |u| > \sqrt{q}, \ker(S_p - u) \neq \{0\}\} \cup \{-q, q\}$$
and these functions are real on the real line. Now, consider the same formulae as meromorphic functions of t. Recall from Lemma 3.6 that the function $u \mapsto t = \frac{1}{q+1}(u + \frac{q}{u})$ induces a biholomorphism from the set $\{u \in \mathbb{C} | \, |u| > \sqrt{q}\}$ onto the set $\mathbb{C} \smallsetminus \mathcal{I}_q$ and that this biholomorphism maps the real set $\{u \in \mathbb{R} | \, |u| > \sqrt{q}\}$ onto $\mathbb{R} \smallsetminus \mathcal{I}_q$. Hence, the formulae in Proposition 8.1 define meromorphic functions of t in $\mathbb{C} \smallsetminus \mathcal{I}_q$ with at most simple poles in the set $\Sigma_p \cup \{-1, 1\}$, where Σ_p is defined in (8.1), and these functions are real on $\mathbb{R} \smallsetminus \mathcal{I}_q$.

Assume temporarily that k is even. By Proposition 6.3, for H in $\mathcal{H}_k$, the sequences $H\mathbf{1}_0$ and $H\mathbf{1}_1$ admit respectively as spectral transforms the constant functions with value $\begin{pmatrix} q^{-1}H^{\vee} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ H \end{pmatrix}$. Besides, the Definition 2.11 of the polyextension map gives $E_k(H\mathbf{1}_0) = H^{>\infty}$ and $E_k(H\mathbf{1}_1) = H^{+>\infty}$. Thus, from the discussion above on the consequences of the formulae of Proposition 8.1, for H in $\mathcal{H}_{k,+}$ and J in $\mathcal{H}_{k,-}$, the Poisson transforms of the positive measures
$$\varpi_t\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} H \\ 0 \end{pmatrix}\right)\mathrm{d}\mu(t) \text{ and } \varpi_t\left(\begin{pmatrix} J \\ 0 \end{pmatrix}, \begin{pmatrix} J \\ 0 \end{pmatrix}\right)\mathrm{d}\mu(t),$$
as well as the Poisson transform of the signed measure
$$\varpi_t\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J \end{pmatrix}\right)\mathrm{d}\mu(t),$$
are the imaginary parts of meromorphic functions of t in $\mathbb{C} \smallsetminus \mathcal{I}_q$ with at most simple poles in the set $\Sigma_p \cup \{-1, 1\}$, and which take real values on $\mathbb{R} \smallsetminus \mathcal{I}_q$. By Proposition 13.3, Lemma 13.7 and Corollary 13.9, this

tells us that, after maybe changing the normalization, we can assume that there exists a Radon measure ν with support in $\mathcal{I}_q$ such that

$$\mu = \nu + \sum_{t\in\Sigma_p} \delta_t + \delta_1 + \delta_{-1}.$$

If k is odd, the analogous arguments imply the same conclusion.

In both cases, it remains to compute the values of ϖ_t, for t in $\Sigma_p \cup \{-1, 1\}$, and of the bilinear forms valued measure $\varpi_t \mathrm{d}\nu(t)$.

We first compute the values on Σ_p. As in (13.1), we will show that they are related to the full exceptional spectral bilinear forms of Section 11. By the reasoning above and by Corollary 13.9, it suffices to compute the residues of the meromorphic functions of Proposition 8.1 at their poles in the set $\Sigma_p \cup \{-1, 1\}$. For example, assume that k is even. Take H in $\mathcal{H}_{k,+}$. Again, Definition 2.11, Proposition 6.3 and Proposition 8.1 tell us that the Poisson transform of the positive measure

$$\varpi_t\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} H \\ 0 \end{pmatrix}\right) \mathrm{d}\mu(t)$$

is the imaginary part of the function on $\mathbb{H}$,

$$t \mapsto \frac{1}{\pi}\frac{q+1}{q}\frac{u}{1-u^2} p(H, (qu + S_p)(u - S_p)^{-1}H), \tag{13.15}$$

where $\Im u > 0$ and $q + u^2 = (q + 1)tu$. Note that the right hand-side of the above is a meromorphic function of u, $|u| > \sqrt{q}$, with at most simple poles on the spectral values of S_p. Fix s in Σ_p and let v be the spectral value of S_p with $q+v^2 = (q+1)sv$ and $v > \sqrt{q}$. By Proposition 7.16 and Lemma 13.10, the residue of the right hand-side of (13.15) at v is

$$\frac{1}{\pi}\frac{q+1}{q}\frac{v}{1-v^2} p(H, (qv + S_p)\Pi_p^v H) = \frac{1}{\pi}\frac{(q+1)^2}{q}\frac{v^2}{1-v^2} p(H, \Pi_p^v H),$$

where, as in Subsection 11.1, Π_p^v stands for the projection of $\mathcal{H}_k$ onto $\ker(S_p - v)$ with null space $(S_p - v)\mathcal{H}_k$. The derivative of the function $u \mapsto t = \frac{1}{q+1}(u + \frac{q}{u})$ is the function $u \mapsto t = \frac{1}{q+1}\frac{u^2-q}{u^2}$. Therefore, by Lemma 13.11, the residue at s of the right hand-side of (13.15), viewed as a function of t, is

$$\frac{1}{q+1}\frac{v^2-q}{v^2}\frac{1}{\pi}\frac{(q+1)^2}{q}\frac{v^2}{1-v^2} p(H, \Pi_p^v H) = \frac{1}{\pi}\frac{q+1}{q}\frac{v^2-q}{1-v^2} p(H, \Pi_p^v H).$$

By Corollary 13.9, we get

$$\varpi_s\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} H \\ 0 \end{pmatrix}\right) = \frac{q+1}{q}\frac{v^2-q}{v^2-1} p(H, \Pi_p^v H). \tag{13.16}$$

Take now J in $\mathcal{H}_{k,-}$. Still by Definition 2.11, Proposition 6.3 and Proposition 8.1, the Poisson transform of the positive measure

$$\varpi_t\left(\begin{pmatrix}J\\0\end{pmatrix},\begin{pmatrix}J\\0\end{pmatrix}\right)\mathrm{d}\mu(t)$$

is the imaginary part of the function on $\mathbb{H}$,

$$t\mapsto\frac{1}{\pi}q^2(q+1)\frac{u}{q^2-u^2}p(J,(u+qS_p)(u-S_p)^{-1}J). \tag{13.17}$$

By Proposition 7.9, Proposition 7.16 and Lemma 13.10, the residue at v of the above, viewed as a function of u, is

$$\frac{1}{\pi}q^2(q+1)^2\frac{v^2}{q^2-v^2}p(J,\Pi_p^vJ),$$

so that by Corollary 13.9 and Lemma 13.11,

$$\varpi_s\left(\begin{pmatrix}J\\0\end{pmatrix},\begin{pmatrix}J\\0\end{pmatrix}\right)=q^2(q+1)\frac{q-v^2}{q^2-v^2}p(J,\Pi_p^vJ). \tag{13.18}$$

Finally, again by Definition 2.11, Proposition 6.3 and Proposition 8.1, the Poisson transform of the signed measure

$$\varpi_t\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}0\\J\end{pmatrix}\right)\mathrm{d}\mu(t)$$

is the imaginary part of the function on $\mathbb{H}$,

$$t\mapsto\frac{q+1}{\pi}p(J,S_p(u-S_p)^{-1}H). \tag{13.19}$$

By Proposition 7.16 and Lemma 13.10, the residue at v of the above, viewed as a function of u, is

$$\frac{q+1}{\pi}vp(J,\Pi_p^vH),$$

so that by Corollary 13.9 and Lemma 13.11,

$$\varpi_s\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}0\\J\end{pmatrix}\right)=\frac{q-v^2}{v}p(J,\Pi_p^vH). \tag{13.20}$$

Recall that, by construction, the operators $\mathfrak{R}_s$ and $\mathfrak{S}_s$ are symmetric with respect to the symmetric bilinear form ϖ_s on $\mathcal{H}_k^2$. Therefore, Corollary 13.4, together with (13.16), (13.18) and (13.20), gives

$$\varpi_s\left(\begin{pmatrix}H\\-vH\end{pmatrix},\begin{pmatrix}H\\-vH\end{pmatrix}\right)=\frac{q-1}{q}(v^2-q)p(H,\Pi_p^v H)$$
$$\varpi_s\left(\begin{pmatrix}J\\-vJ\end{pmatrix},\begin{pmatrix}J\\-vJ\end{pmatrix}\right)=(q-1)(q-v^2)p(J,\Pi_p^v J)$$
$$\varpi_s\left(\begin{pmatrix}H\\-vH\end{pmatrix},\begin{pmatrix}J\\-vJ\end{pmatrix}\right)=(q-1)(q-v^2)p(J,\Pi_p^v H).$$

By comparing the latter with the formulae in Corollary 11.6, we get, from Definition 11.8, $\varpi_s=(q-1)p_s^{2,\mathrm{ex}}$ as in the statement of the Plancherel formula (13.1). The case where k is odd can be obtained by analogous computations.

Now, we compute the values of ϖ_1 and ϖ_{-1} which will be related to the full special spectral bilinear forms of Section 12. To simplify, we assume that k is even and we compute ϖ_1. We still let H be in $\mathcal{H}_{k,+}$ and J be in $\mathcal{H}_{k,-}$. By Proposition 7.9, the left hand-side of (13.15) and (13.19) is holomorphic at $u=q$. Thus, by Definition 2.11, Proposition 6.3, Proposition 8.1 and Lemma 13.7, we get

$$\varpi_1\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}H\\0\end{pmatrix}\right)=\varpi_1\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}0\\J\end{pmatrix}\right)=0. \tag{13.21}$$

Besides, the residue at q of the right hand-side of (13.17), viewed as a function of u, is

$$-\frac{1}{2\pi}q^3(q+1)p(J,(S_p+1)(q-S_p)^{-1}J).$$

By the same arguments as above, we get

$$\varpi_1\left(\begin{pmatrix}J\\0\end{pmatrix},\begin{pmatrix}J\\0\end{pmatrix}\right)=\frac{1}{2}q^2(q-1)p(J,(S_p+1)(q-S_p)^{-1}J). \tag{13.22}$$

By comparing (13.21) and (13.22) with the formulae in Proposition 12.13 and the Definition 12.16 of the full special spectral quadratic forms, and lastly, by using the uniqueness result in Proposition 13.3, we get $\varpi_1=\frac{q-1}{2(q+1)}p_1^{2,\mathrm{ex}}$ as required in (13.1). In the same way, one can show that $\varpi_{(-1)}=\frac{q-1}{2(q+1)}p_{(-1)}^{2,\mathrm{ex}}$. The case where k is odd can be dealt with analogously.

So far, we have determined the behaviour of the Plancherel formula (13.1) on the finite set $\Sigma_p\cup\{-1,1\}$. To conclude, we will investigate its behaviour on the critical interval $\mathcal{I}_q$. As above, we assume for example that k is even and we pick H in $\mathcal{H}_{k,+}$ and J in $\mathcal{H}_{k,-}$. As usual, for t in $\mathbb{H}$, write u for the unique solution of the equation $q+u^2=(q+1)tu$ with $\Im u>0$. Recall that we assume for the moment that neither $\sqrt{q}$ nor $-\sqrt{q}$ is a spectral value of the simple transfer operator S_p. Therefore, Lemma 3.6, Proposition 7.16 and our former residue computations ensure that the function

$$t \mapsto \frac{q+1}{q}\frac{u}{1-u^2}p(H,(qu+S_p)(u-S_p)^{-1}H) - (q-1)\sum_{s\in\Sigma_q}\frac{1}{s-t}p_s^{2,\mathrm{ex}}\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}H\\0\end{pmatrix}\right) \tag{13.23}$$

is holomorphic on $\mathbb{H}$ and admits a holomorphic continuation in the neighborhood of $\overline{\mathbb{H}}$. Besides this function goes to 0 as t goes to ∞, so that it is bounded on $\overline{\mathbb{H}}$. As the imaginary part of this holomorphic function is the Poisson transform of the positive measure

$$\pi\varpi_t\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}H\\0\end{pmatrix}\right)\mathrm{d}\nu(t),$$

by [1, Theorem 7.5], this measure is absolutely continuous with respect to the Lebesgue measure and its density function is given by the imaginary part of the continuous extension of the right hand-side of (13.23). Since all the residue terms in this right hand-side are real, by the imaginary part computations in Lemma 13.6, this density function is the function

$$t \mapsto \frac{1}{2q}\frac{1}{1-t^2}p(H,(q^2-S_p^2)(q-(q+1)tS_p+S_p^2)^{-1}H)\sqrt{4q-(q+1)^2t^2}$$

on the interval $\mathcal{I}_q$. Reasoning in the same way and using the meromorphic function in (13.17), one shows that the positive measure

$$\pi\varpi_t\left(\begin{pmatrix}J\\0\end{pmatrix},\begin{pmatrix}J\\0\end{pmatrix}\right)\mathrm{d}\nu(t),$$

is absolutely continuous with respect to the Lebesgue measure with density function

$$t \mapsto \frac{q^2}{2}\frac{1}{1-t^2}p(J,(1-S_p^2)(q-(q+1)tS_p+S_p^2)^{-1}J)\sqrt{4q-(q+1)^2t^2}$$

on $\mathcal{I}_q$. Finally, the same arguments imply, by using the the meromorphic function in (13.19), that the signed measure

$$\pi\varpi_t\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}0\\J\end{pmatrix}\right)\mathrm{d}\nu(t),$$

is absolutely continuous with respect to the Lebesgue measure with density function

$$t \mapsto -\frac{q+1}{2}p(J,S_p(q-(q+1)tS_p+S_p^2)^{-1}H)\sqrt{4q-(q+1)^2t^2}$$

on $\mathcal{I}_q$. By Proposition 13.3, up to a change of normalization, we can assume that ν is the measure $\sqrt{4q-(q+1)^2t^2}\mathrm{d}t$ on the interval $\mathcal{I}_q$ and that the bilinear forms ϖ_t, $t \in \mathcal{I}_q$, satisfy the relations:

$$\begin{aligned}\varpi_t\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}H\\0\end{pmatrix}\right) &= \frac{1}{2\pi q}\frac{1}{1-t^2}p(H,(q^2-S_p^2)H_t)\\ \varpi_t\left(\begin{pmatrix}J\\0\end{pmatrix},\begin{pmatrix}J\\0\end{pmatrix}\right) &= \frac{q^2}{2\pi}\frac{1}{1-t^2}p(J,(1-S_p^2)J_t)\\ \varpi_t\left(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}0\\J\end{pmatrix}\right) &= -\frac{q+1}{2\pi}p(J,S_pH_t),\end{aligned}\tag{13.24}$$

where we have set

$$H_t = (q-(q+1)tS_p+S_p^2)^{-1}H \text{ and } J_t = (q-(q+1)tS_p+S_p^2)^{-1}J.$$

Let us show that these formulae allow to relate these bilinear forms to the full spectral bilinear forms of Section 10. Take t to be an interior point of $\mathcal{I}_q$ and choose a root u of the equation $q+u^2=(q+1)tu$. As by construction, the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$ are symmetric with respect to ϖ_t, we get from (13.24), by Corollary 13.4,

$$\begin{aligned}\varpi_t\left(\begin{pmatrix}H\\-uH\end{pmatrix},\begin{pmatrix}H\\-uH\end{pmatrix}\right) &= \frac{q^2-1}{2\pi q}\frac{u^2}{q^2-u^2}p(H,(q^2-S_p^2)H_t)\\ \varpi_t\left(\begin{pmatrix}J\\-uJ\end{pmatrix},\begin{pmatrix}J\\-uJ\end{pmatrix}\right) &= \frac{q^2-1}{2\pi}\frac{u^2}{u^2-1}p(J,(1-S_p^2)J_t)\\ \varpi_t\left(\begin{pmatrix}H\\-uH\end{pmatrix},\begin{pmatrix}J\\-uJ\end{pmatrix}\right) &= -\frac{q^2-1}{2\pi}up(J,S_pH_t),\end{aligned}\tag{13.25}$$

where we used the relation

$$(q^2-u^2)(u^2-1) = (q+1)^2u^2(1-t^2).$$

By comparing (13.25) with the formulae for the spectral bilinear form p_u in Proposition 9.18, the uniqueness result in Proposition 10.12 gives $\varpi_t = \frac{1}{2\pi}\frac{q+1}{q-1}p_t^2$ as required. The same computations work in case k is odd. □

13.7. **Proof of the Plancherel formula in the general case.** If $\sqrt{q}$ or $-\sqrt{q}$ is a spectral value of the simple transfer operator S_p, we must modify the end of the argument of the preceding proof. Indeed, holomorphic functions as the one in (13.23) do not admit an analytic continuation to a neighborhood of $\overline{\mathbb{H}}$ since they have a singularity at at least one of the endpoints of the interval $\mathcal{I}_q$. However, it will turn out that these singularities are compensated by the vanishing at these endpoints of the coefficient $\sqrt{4q-(q+1)^2t^2}$ in (13.1).

To show this precisely, we start by describing some model harmonic functions.

Lemma 13.12. *For z in $\mathbb{H}$, let w be the unique element of $\mathbb{H}$ with $\frac{1}{2}(w+w^{-1}) = z$ and set $F(z) = \Im(1-w)^{-1}$. Then the harmonic function F is the Poisson transform of the function $f : t \mapsto \frac{1}{2}\sqrt{\frac{1+t}{1-t}}$ on $[-1,1]$, that is, for $z = x+iy$ in $\mathbb{H}$, we have*

$$F(z) = \frac{1}{2\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{y\mathrm{d}t}{(x-t)^2+y^2}. \tag{13.26}$$

Proof. As F is a non-negative harmonic function, by [1, Theorem 7.24], there exists a positive Borel measure μ on $\mathbb{R}$ with $\int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$ and $c \geq 0$ such that, for $z = x+iy$ in $\mathbb{H}$, one has $F(z) = \mathcal{P}\mu(z) + cy$. As $F(z) \xrightarrow[z\to\infty]{} 0$, one has $c = 0$. Besides, one has $F(x+iy) \xrightarrow[y\to 0]{} 0$ uniformly for x in a compact set of $\mathbb{R} \smallsetminus [-1,1]$, so that, by Lemma 13.7, the measure μ has support in $[-1,1]$. A direct computation gives, that, uniformly for t in a compact set of $[-1,1)$, one has

$$F(t+iy) \xrightarrow[y\to 0]{} \frac{1}{2}\sqrt{\frac{1+t}{1-t}}.$$

Therefore, by Lemma 13.7, there exists $d \geq 0$ such that $\mu = \frac{1}{2}\sqrt{\frac{1+t}{1-t}}\mathbf{1}_{|t|\leq 1}\mathrm{d}t + d\delta_1$. Let G be the Poisson transform of the measure $\frac{1}{2}\sqrt{\frac{1+t}{1-t}}\mathbf{1}_{|t|\leq 1}\mathrm{d}t$, so that G is given by the right hand-side of (13.26). Then, G is a non-negative harmonic function and, for z in $\mathbb{H}$, we have

$$F(z) = G(z) + \frac{d}{\pi}\Im\left(\frac{1}{1-z}\right). \tag{13.27}$$

We claim that $d = 0$, which implies the Lemma.

Indeed, fix $s > 0$ and set $w = 1 + s(1+i)$ and $z = \frac{1}{2}(w+w^{-1})$. On one hand, we have

$$F(z) = \Im\left(\frac{1}{1-w}\right) = \frac{1}{2s}. \tag{13.28}$$

On the other hand, we have

$$2(1-z) = 1 - s(1+i) - \frac{1}{1+s(1+i)} = -\frac{s^2(1+i)^2}{1+s(1+i)} = -\frac{2is^2}{1+s(1+i)},$$

hence

$$\Im\left(\frac{1}{1-z}\right) = \frac{1+s}{s^2}. \tag{13.29}$$

Now, as $G \geq 0$, (13.27), (13.28) and (13.29) give

$$\frac{1}{2s} \geq \frac{d}{\pi}\frac{1+s}{s^2}.$$

Letting s go to 0 gives $d = 0$ as required. □

From this computation, we deduce a statement that will be directly applicable to the proof of Theorem 13.1.

Corollary 13.13. *Let ν be a finite positive Borel measure on $\mathcal{I}_q$. Let F be the holomorphic function on $\mathbb{H}_q$ with $F(u) \xrightarrow[u\to\infty]{} 0$ such that, for any u in $\mathbb{H}_q$, one has*

$$\mathcal{P}\nu(t) = \Im F(u),$$

where $t = \frac{1}{q+1}(u + \frac{q}{u})$. Assume that F admits a meromorphic extension to a neighborhood of $\overline{\mathbb{H}}_q$ with at most simple poles with real residues at $\sqrt{q}$ and $-\sqrt{q}$. Then the measure ν is absolutely continuous with respect to the Lebesgue measure, with density function

$$t \mapsto \Im F(u) = \Im F\left(\frac{q+1}{2}t + \frac{i}{2}\sqrt{4q - (q+1)^2t^2}\right)$$

on $\mathcal{I}_q$.

Proof. By assumption, there exist real numbers a and b and a holomorphic function G on a neighborhood of $\overline{\mathbb{H}}_q$ such that, for any u in $\mathbb{H}_q$, one has

$$F(u) = G(u) + \frac{a}{u - \sqrt{q}} + \frac{b}{u + \sqrt{q}}.$$

As G admits a continuous extension to the boundary and $G(u) \xrightarrow[u\to\infty]{} 0$, by [1, Theorem 7.5], the function $t \mapsto \Im G(u)$ on $\mathbb{H}$ is the Poisson transform of its restriction to $\mathbb{R}$, whereas, after a change of variable, Lemma 13.13 implies that the functions

$$t \mapsto \Im\left(\frac{1}{u - \sqrt{q}}\right) \text{ and } t \mapsto \Im\left(\frac{1}{u + \sqrt{q}}\right)$$

are the Poisson transforms of their restrictions to the interior of $\mathcal{I}_q$. The conclusion follows. □

Proof of Theorem 13.1 in the general case. The proof can be lead exactly as in the case where $\sqrt{q}$ and $\sqrt{-q}$ are not eigenvalues of S_p. Indeed, Proposition 7.16 implies that $\sqrt{q}$ and $\sqrt{-q}$ are at most simple roots of the minimal polynomial of S_p and hence that we can apply Corollary 13.13 to functions such as the one in (13.23). □

References

[1] S. Axler, P. Bourdon, W. Ramey, *Harmonic function theory*, Graduate Texts in Mathematics 137, Springer-Verlag, New York, 1992.

[2] N. Dunford, J. T. Schwartz, *Linear operators. Part II. Spectral theory. Self adjoint operators in Hilbert space*, John Wiley & Sons, New York, 1988.

[3] A. Figà-Talamanca, C. Nebbia, *Harmonic analysis and representation theory for groups acting on homogeneous trees*, London Mathematical Society Lecture Note Series 162, Cambridge University Press, Cambridge, 1991.

[4] Y. Ihara, On discrete subgroups of the two by two projective linear group over $\mathfrak{p}$-adic fields, *J. Math. Soc. Japan* **18** (1966), 219-235.

[5] H. Kesten, Symmetric random walks on groups, *Trans. Amer. Math. Soc.* **92** (1959), 336-354.

[6] J.-F. Quint, Additive representations of tree lattices 1. Quadratic fields and dual kernels, preprint, 2020.

[7] J.-F. Quint, Additive representations of tree lattices 2. Radical pseudofields and limit metrics, preprint, 2021.

[8] W. Rudin, *Functional analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1991.