ADDITIVE REPRESENTATIONS OF TREE LATTICES 4. FINITENESS OF GENUINE SPECTRAL OBSTRUCTIONS

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ABSTRACT. We continue the systematic study of unitary representations of tree lattices from [7], [8] and [9] whose goal is to describe the spectral theory of such representations.

The spectral transform of [9], which is a concrete version of the spectral theorem, allows to define a map from the set of all such representations to spaces of measures on $[-1,1]$ with values in the set of non-negative bilinear forms on some finite dimensional vector spaces. In this paper, we describe the subspace spanned by the range of this map up to a finite dimensional space.

CONTENTS

1. Introduction

1.1. **Objective of the article.** Let $q \geq 2$ be an integer and X be a homogeneous tree of degree $q+1$. We equip X with a proper action of a discrete group Γ such that the quotient set $\Gamma\backslash X$ is finite. We write ∂X for the boundary of X. This is a totally discontinuous compact topological space and the natural action of Γ on ∂X is minimal. We write $\mathcal{D}(\partial X)$ for the space of locally constant real valued functions on ∂X and $\overline{\mathcal{D}}(\partial X)$ for the quotient space of $\mathcal{D}(\partial X)$ by the line of constant functions.

Our purpose in this paper, which is the sequel of [7], [8] and [9], is to study Γ-invariant non-negative symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$. The space of all such bilinear forms is a convex cone $\mathcal{Q}_+(\overline{\mathcal{D}}(\partial X))^\Gamma \subset \mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$, where the latter stands for the space of all Γ-invariant symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$.

The completion of $\overline{\mathcal{D}}(\partial X)$ with respect to such a non-negative bilinear form is a unitary representation of Γ. Conversely, any unitary representation of Γ which admits a cyclic and harmonic first cohomology class may be obtained in such a way. This point of view is explained in the independent Appendix A (see in particular Remark A.9). Thus, studying Γ-invariant non-negative symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$ amounts to studying a wide class of unitary representations of Γ.

For these representations, in particular for the extremal ones, we aim at a better understanding of the spectral problem stated in [9, Subsection 1.1]. In other words, we want to describe the spectral theory of (the completion of) the operator Q defined on the space of all Γ-equivariant maps $f : X \to \overline{\mathcal{D}}(\partial X)$ by

$$Qf(x) = \frac{1}{q+1} \sum_{y \sim x} f(y), \quad x \in X.$$

As in [9], for technical reasons, it will turn out to be more convenient to work instead in the space of all Γ-equivariant maps $f : X_1 \to \overline{\mathcal{D}}(\partial X)$ where X_1 is the set of oriented edges of X. Following the terminology of [9], we will call such maps Γ-invariant ∞-pseudofunctions and denote the space of all Γ-invariant ∞-pseudofunctions by $\mathcal{H}_\infty$. For H in $\mathcal{H}_\infty$ and (x, y) in X_1, we write $H_{xy} \in \overline{\mathcal{D}}(\partial X)$ for the value of H at (x, y).

The space $\mathcal{H}_\infty$ comes with two natural operators R and S given by

$$(RH)_{xy} = \sum_{\substack{z \sim x \\ z \neq y}} H_{xz} \text{ and } (SH)_{xy} = H_{yx}, \quad H \in \mathcal{H}_\infty, \quad (x, y) \in X_1.$$

From the relations $R^2 = q + (q-1)R$ and $S^2 = 1$, one deduces that the operator

$$P = \frac{1}{q+1}(RS + SR - (q-1)S)$$

is central in the algebra $\mathcal{A}$ spanned by R and S.

Denote by $\mathcal{Q}(\mathcal{H}_\infty)^{R,S}$ the space of all symmetric bilinear forms on $\mathcal{H}_\infty$ for which the operators R and S are symmetric. If p is in $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$, we associate to p an element of $\mathcal{Q}(\mathcal{H}_\infty)^{R,S}$, which we still denote by p, by setting, for H, J in $\mathcal{H}_\infty$,

$$p(H, J) = \sum_{(x,y) \in \Gamma \backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} p(H_{xy}, J_{xy})$$

(this construction is already used in [9]). This defines an injective linear map

$$\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma \to \mathcal{Q}(\mathcal{H}_\infty)^{R,S}, \tag{1.1}$$

and an element of $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ is non-negative if and only if the associated bilinear form on $\mathcal{H}_\infty$ is non-negative. The spectral transform of [9] (whose main properties will be recalled below), together with the standard spectral theorem, give a full description of non-negative elements in $\mathcal{Q}(\mathcal{H}_\infty)^{R,S}$.

The linear map in (1.1) is not surjective and hence this description can not be carried back directly to $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$. The purpose of this article is to bound the default of injectivity in (1.1) in the following way.

In [9, Section 2], the space $\mathcal{H}_\infty$ is written as a union $\mathcal{H}_\infty = \bigcup_{k\geq -1} \mathcal{H}_k$ of finite-dimensional spaces, where for $k \geq -1$, $\mathcal{H}_k$ is the space of Γ-invariant k-pseudofunctions. By restriction of (1.1), for $k \geq -1$, we get a natural map

$$\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma \to \mathcal{Q}(\mathcal{A}\mathcal{H}_k)^{R,S} \tag{1.2}$$

where $\mathcal{A}\mathcal{H}_k$ is the (infinite-dimensional) subspace of $\mathcal{H}_\infty$ spanned by the images of $\mathcal{H}_k$ under the elements of the algebra $\mathcal{A}$ and $\mathcal{Q}(\mathcal{A}\mathcal{H}_k)^{R,S}$ is the space of all symmetric bilinear forms on $\mathcal{A}\mathcal{H}_k$ for which the operators R and S are symmetric. The main result of this article yields

Corollary 1.1. *For $k \geq 0$, the image of $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ in $\mathcal{Q}(\mathcal{A}\mathcal{H}_k)^{R,S}$ has finite codimension.*

1.2. **Strategy of the proof.** We will deduce Corollary 1.1 above from a dual injectivity statement.

Indeed, as in [7, Subsection 2.1], denote by $\mathscr{S}$ the space of parametrized geodesic lines of X and by $T : \mathscr{S} \to \mathscr{S}$ and $\iota : \mathscr{S} \to \mathscr{S}$ the time shift and the time reversal. Recall from [8, Subsection 5.3] that the space $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ may be identified with the space $\mathcal{D}^*(\Gamma\backslash\mathscr{S})^{\iota,T}$ of distributions on $\Gamma\backslash\mathscr{S}$ that are invariant by both ι and T. In other words, this space may be seen as the dual space of the space of cohomology classes of smooth ι-invariant functions on $\Gamma\backslash\mathscr{S}$.

Our strategy for proving Corollary 1.1 will be to write the linear map in (1.1) as the adjoint of a map with values into the space of cohomology classes of smooth ι-invariant functions on $\Gamma\backslash\mathscr{S}$. To this aim, we need to construct a space whose dual space may be identified with $\mathcal{Q}(\mathcal{A}\mathcal{H}_k)^{R,S}$. This will use the spectral transform of [6].

Indeed, with the aim of diagonalizing the operator P, in [9, Section 6], we have constructed the spectral transform. For $k \geq 0$, this map induces a linear isomorphism between the space $\mathcal{A}\mathcal{H}_k \subset \mathcal{H}_\infty$ and the quotient space of the space $\mathcal{H}^2_k[t]$ of all polynomial functions with values in $\mathcal{H}^2_k$ by the space of functions of the form

$$G(t)^{\vee>\vee} - \begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix} G(t)^{>},$$

where $G(t)$ is in $\mathcal{H}^2_{k-1}[t]$ (see [9, Subsection 2.2] for the $\vee$ and $>$ notation). The spectral transform intertwines the action of R and S in $\mathcal{H}_\infty$ with the action of certain polynomial matrix operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$

on $\mathcal{H}_k^2[t]$. In particular, it intertwines the action of P and that of the multiplication by t, that is, it diagonalizes the action of P.

These properties allow to identify $\mathcal{Q}(\mathcal{AH}_k)^{R,S}$ with the dual space of a certain explicit quotient of the space $\otimes^2\mathcal{H}_k^2[t]$ of polynomial functions with values in the tensor square $\otimes^2\mathcal{H}_k^2$ of $\mathcal{H}_k^2$.

Therefore, to prove Corollary 1.1, we will construct a linear map $\otimes^2\mathcal{H}_k^2[t] \to \mathcal{D}(\Gamma\backslash\mathscr{S})$. Actually, by using an elementary reduction, we will only have to consider a certain linear map $\Omega_k : \otimes^2\mathcal{H}_k[t] \to \mathcal{D}(\Gamma\backslash\mathscr{S})$, which we will call the ultraweight map, since it shares some relations with the weight map of [7, Section 8]. Thanks to explicit formulas for the ultraweight map, we will be able to describe, up to a finite-dimensional subspace, the set of polynomial tensors H in $\otimes^2\mathcal{H}_k[t]$ such that $\Omega_k(H)$ is a coboundary with respect to the dynamics T on $\Gamma\backslash\mathscr{S}$. This is our main result, which in turn will lead to Corollary 1.1.

1.3. **Structure of the article.** References to [7], [8] and [9] are indicated with I, II and III.

In Section 2, we introduce Hölder continuous functions and the cohomology equivalence relation among them on $\Gamma\backslash\mathscr{S}$. We show a version of the Livšic Theorem. This is mostly a translation from the language of subshifts of finite type (see [6, Chapter 1]). The space $\Gamma\backslash\mathscr{S}$ plays the role of a two-sided shift, whereas the space $\Gamma\backslash\mathscr{S}_+$ plays the one of a one-sided shift. We introduce a transfer operator on functions on $\Gamma\backslash\mathscr{S}_+$. We use its spectral properties to write a decomposition of every Hölder continuous function on $\Gamma\backslash\mathscr{S}$ as the sum of a coboundary, a constant and a function depending only on the future (that is defined on $\Gamma\backslash\mathscr{S}_+$) that is killed by the transfer operator.

In Section 3, we study certain classes of functions on $\Gamma\backslash\mathscr{S}$ which are defined by sums. We call these sums endpoints series. By using the decomposition of functions from Section 2 and a mixing property of the action of T, we describe the space of endpoints series which are coboundaries. Later, this criterion will be applied to endpoints series associated to the ultraweight map.

In Section 4, we write a Plancherel formula for functions on X_1. This formula is essentially equivalent to the one for functions on X that is established in [3]. We will later need this version on X_1 when studying the ultraweight map.

In Section 5, we introduce the fundamental bilinear map. This construction is dual to the identification between the spaces $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ and $\mathcal{D}^*(\Gamma\backslash\mathscr{S})^{\iota,T}$ in Subsection II.5.3. The fundamental bilinear map Φ sends $\mathcal{H}_\infty \times \mathcal{H}_\infty$ to $\mathcal{D}(\Gamma\backslash\mathscr{S})$. If p is in $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ and θ is the associated distribution in $\mathcal{D}^*(\Gamma\backslash\mathscr{S})^{\iota,T}$, for H, J in $\mathcal{H}_\infty$, we have

$$p(H,J) = \langle\theta, \Phi(H,J)\rangle, \tag{1.3}$$

where we use the same letter to denote p and the associated bilinear form on $\mathcal{H}_\infty$. We also introduce spectral fundamental bilinear maps which send $\mathcal{H}_\infty \times \mathcal{H}_\infty$ to the space of Hölder continuous functions on $\Gamma\backslash\mathscr{S}$. Thanks to the Plancherel formula of Section 4, we decompose the fundamental bilinear map by means of the spectral bilinear maps. We use this decomposition to describe the spectral theory of the representation of Γ associated with a (ι, T)-invariant probability measure by the correspondance in II.5.3.

In Section 6, we use the explicit definition of the spectral bilinear maps to write them as the sum of a coboundary and an endpoints series as in Section 3. For $k \geq 1$, $j \geq 0$, the j-th coefficient of this endpoints series (when restricted to $\mathcal{H}_k \times \mathcal{H}_k \subset \mathcal{H}_\infty \times \mathcal{H}_\infty$) is defined by a bilinear map

$$\kappa_{j,k} : \mathcal{H}_k \times \mathcal{H}_k \to V_{j+k},$$

where, for $h \geq 0$, V_h is the space of all Γ-invariant functions on

$$X_h = \{(x,y) \in X | d(x,y) = h\}.$$

We build a linear map $\omega_k : \mathcal{H}_k \to V_k$ such that, if $j \geq k-1$, for H, J in $\mathcal{H}_k$, $\kappa_{j,k}(H,J)$ is cohomologous to the function on X_{j+k},

$$(x,y) \mapsto -\omega_k(H)(xx_k)\omega_k(J)(yy_k) - \omega_k(J)(xx_k)\omega_k(H)(yy_k), \tag{1.4}$$

where x_k and y_k are the elements of $[xy]$ at distance k from x and y. The map ω_k is called the weight map of k-pseudofunction. The study of the weight map will play an important role in our next results.

In Section 7 we give some elementary properties of the weight map ω_k. This weight map of pseudofunctions shares some analogies with the weight map W_k of k-dual kernels of Section I.8, but the weight map W_k maps $\mathcal{K}_k$ onto the space of cohomology classes of symmetric Γ-invariant functions on X_k, whereas the range of the weight map ω_k is far away from being all of V_k. Nevertheless, we show that for $k \geq 1$, a Γ-invariant k-pseudofunction H may be written as $G^{\vee>} - G^{>\vee}$ for some G in $\mathcal{H}_{k-1}$ if and only if the Γ-invariant function $\omega_k(H)$ on X_k is a coboundary. We extend this result to sequences by showing that, for $(H_j)_{j\geq 1}$ a finitely supported sequence in $\mathcal{H}_k$, there exists a finitely supported sequence $(v_j)_{j\geq 0}$ in V_{k-1} such that, for (a,b) in X_k and $j \geq 1$,

$$w_j(a,b) = v_j(a,b_1) - v_{j-1}(a_1,b), \tag{1.5}$$

if and only if there exists a finitely supported sequence $(G_j)_{j\geq 0}$ in $\mathcal{H}_{k-1}$ such that, for $j \geq 1$, one has

$$
\begin{aligned}
H_j &= G_j^{>\vee} - G_{j-1}^{\vee>} && \text{if } k \text{ is even} \\
&= G_j^{\vee>} - G_{j-1}^{>\vee} && \text{if } k \text{ is odd.}
\end{aligned} \tag{1.6}
$$

This result will not be used directly, but its proof serves as a model for an analogue result for functions on spaces $X_k \times X_k$, Proposition 9.3. The latter statement will be our main tool for transfering the general cohomology criterion for endpoints series of Section 3 to the language of pseudofunctions by means of the endpoints formulas of Section 6. The proof of the technical Proposition 9.3 will occupy the next two sections.

In Section 8, we introduce some algebraic formalism which will be used to solve certain algebraic equations on spaces of tensors.

In Section 9, we state and prove Proposition 9.3 by using the previously introduced formalism. This result focuses on the map $\otimes^2\omega_k : \otimes^2\mathcal{H}_k \to \otimes^2 V_k$. Given a finitely supported sequence $(H_j)_{j\geq 1}$ in $\otimes^2\mathcal{H}_k$, it says that, whenever the images by $\otimes^2\omega_k$ of the H_j satisfy analogues of (1.5) for functions of two variables, then the H_j may be defined from sequences in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$ and $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$ through analogues of (1.6).

In Section 10, we introduce the ultraweight map Ω_k, which is a linear map from $\otimes^2\mathcal{H}_k[t]$ to the space of Hölder continuous functions on $\Gamma\backslash\mathscr{S}$. Its definition uses the Plancherel formula of Section 4 and the spectral maps of Section 5, which allows to relate it to the fundamental bilinear maps (and hence to the duality formula (1.3)). Although the ultraweight is a priori defined as a Hölder continuous function, this relation shows that it is cohomologous to a smooth function. We state Proposition 10.14 which is a description, up to a finite-dimensional space, of the space of those H in $\otimes^2\mathcal{H}_k[t]$ such that $\Omega_k(H)$ is a coboundary. To prepare the proof, we use the formulas from Section 6 to write the ultraweight as the sum of an endpoints series and a coboundary.

In Section 11, we finish the proof of Proposition 10.14. Given H in $\otimes^2\mathcal{H}_k[t]$ such that $\Omega_k(H)$ is a coboundary, we apply the criterion of Section 3 to the above mentioned endpoints formula for the ultraweight. Thanks to (1.4), this tells us that the images by $\otimes^2\omega_k$ of the coefficients of high degree of H in a certain basis of $\otimes^2\mathcal{H}_k[t]$ satisfy certain equations. Proposition 9.3 precisely allows to transfer these equations to relations in $\otimes^2\mathcal{H}_k$. We use these relations to conclude.

Finally, in Section 12, we state and prove the main result of the article, which describes, up to a finite-dimensional space, the null space of a certain linear map from $\otimes^2\mathcal{H}_k^2[t]$ to the space of cohomology classes of smooth functions of $\Gamma\backslash\mathscr{S}$. This map is defined thanks to the ultra-weight Ω_{k+1} and to a linear map $I_k : \mathcal{H}_k^2 \to \mathcal{H}_{k+1}$ (which is essentially the converse of the spectral transform when restricted to constant polynomials in $\mathcal{H}_k^2[t]$). By duality and the crucial formula (1.3), this yields Corollary 1.1.

In the independent Appendix A, we explain how the space $\overline{\mathcal{D}}(\partial X)$ can be considered as a universal model for representations equipped with a cyclic harmonic first cohomology class. We recall the basic definitions of cohomology in degree 1 and we explain how cohomology of Γ-modules can be defined by means of Γ-equivariant maps on X and X_1. We use this point of view to introduce harmonic cohomology classes and we show that, in a unitary representation, a harmonic class is associated with a unique harmonic cocycle. We define unitary representations with a spectral gap and we show that, for these representations, all cohomology classes are harmonic. All this Appendix is built up from material borrowed from [1, 4, 5, 10].

1.4. **Notation.** We freely use the notation of I, II and III.

If G is a group acting on a set A, we identify G-invariant functions on A with functions on the quotient space $G\backslash A$.

For $k \geq 0$, we denote by X_k the set of pairs (x, y) in X with $d(x, y) = k$. When there is no confusion, we often write xy instead of (x, y) to denote an element of X_k. The set X_0 is identified with X.

For $0 \leq h \leq k$ and (x, y) in X_k, when no confusion is possible, the element of the segment $[xy]$ which is at distance h from x is denoted by x_h.

2. Hölder continuous cohomology classes

The space $\mathscr{S}$ of parametrized geodesic lines in X comes with the action of the geodesic shift map T. So far, we have only considered the cohomology relation among smooth functions on $\Gamma\backslash\mathscr{S}$ (see for example Subsection I.2.3, Subsection I.3.3, Subsection I.11.1, Subsection II.5.2). We will now need to develop the language of cohomology classes for Hölder continuous functions, as in hyperbolic dynamics (see [6, Chapter 1]).

2.1. **Hölder continuous functions and cohomology.** We introduce the language of Hölder continuous function and the cohomology equivalence relation.

Recall from Subsection 2.1 that an element σ of $\mathscr{S}$ is a sequence $(\sigma_i)_{i\in\mathbb{Z}}$, where, for any i in $\mathbb{Z}$, we have $\sigma_{i+1} \sim \sigma_i$ and $\sigma_{i+1} \neq \sigma_{i-1}$. Given $0 < \alpha < 1$, we shall say that a function $f : \mathscr{S} \to \mathbb{R}$ is α-Hölder continuous if there exists $C > 0$ such that, for every σ and σ' in $\mathscr{S}$, and every $h \geq 0$, we have

$$(\forall i \quad |i| \leq h \quad \sigma_i = \sigma'_i) \Rightarrow |f(\sigma) - f(\sigma')| \leq C\alpha^h. \tag{2.1}$$

If there exists α in $(0,1)$ such that f is α-Hölder continuous, we simply say that f is Hölder continuous.

Two Γ-invariant Hölder continuous functions f and g are said to be cohomologuous if there exists a Γ-invariant Hölder continuous function h on $\mathscr{S}$ such that $f - g = h - h \circ T$. We shall see later in Corollary 2.7 that, when f and g are smooth, then h must be smooth. Therefore, the cohomology relation on smooth functions is the same as the one defined in Subsection I.2.3. A function is called a coboundary if it is cohomologuous to 0. We will sometimes write $f \equiv g$ to say that f is cohomologous to g.

The following Livšic Theorem characterizes cohomology:

Proposition 2.1. *Let f be a Hölder continuous function on $\Gamma\backslash\mathscr{S}$. Then the following are equivalent:*
(i) *f is a coboundary.*
(ii) *for any σ in $\mathscr{S}$ and $h \geq 1$ with $T^h\sigma \in \Gamma\sigma$, one has*

$$\sum_{i=0}^{h-1} f(T^i\sigma) = 0.$$

(iii) *for any T-invariant Borel probability measure μ on $\Gamma\backslash\mathscr{S}$, one has*

$$\int_{\Gamma\backslash\mathscr{S}} f \mathrm{d}\mu = 0.$$

(iv) *There exists a continuous function h on $\Gamma\backslash\mathscr{S}$ with $f = h - h \circ T$.*

Proof. This is an adaptation of the classical argument from hyperbolic dynamics (see [2, Theorem 19.2.1]).

Note that the implications *(i)⇒(iv)*, *(iv)⇒(iii)* and *(iii)⇒(ii)* are obvious. We will now show *(ii)⇒(i)*.

Let f be a Hölder continuous function on $\mathscr{S}$ which satisfies *(ii)*. Fix C, α as in (2.1). We will build a Hölder continuous function h on $\mathscr{S}$ with $f = h - h \circ T$. By Proposition I.2.3, there exists σ in $\mathscr{S}$ whose orbit under T is dense in $\Gamma\backslash\mathscr{S}$. In particular, for any $i \neq j \in \mathbb{Z}$, we have $T^i\sigma \neq T^j\sigma$. We start by defining h on $T^{\mathbb{Z}}\sigma$ by setting, for $j \geq 0$,

$$h(T^j\sigma) = -\sum_{i=0}^{j-1} f(T^i\sigma) \text{ and } h(T^{-j}\sigma) = \sum_{i=0}^{j-1} f(T^{i-j}\sigma).$$

We get $f = h - h \circ T$ on $T^{\mathbb{Z}}\sigma$. To conclude, we will show that h may be extended by continuity to all of $\Gamma\backslash\mathscr{S}$. By standard arguments of topology, it suffices to prove that h satisfies a uniform continuity property.

Therefore, we choose integers $i < j$ and we assume that $T^i\sigma$ and $T^j\sigma$ are closed in $\Gamma\backslash\mathscr{S}$. In other words, there exists $\ell \geq 1$ and γ in Γ with $\sigma_{j+k} = \gamma\sigma_{i+k}$ for all $k \in \mathbb{Z}$ with $|k| \leq \ell$. We defined a family $\tau = (\tau_k)_{k\in\mathbb{Z}}$ of points in X by setting, for k in $\mathbb{Z}$,

$$\tau_k = \gamma^q\sigma_{i+r},$$

where $0 \leq r < j - i$ and $k = q(j - i) + r$.

We claim that τ is a parametrized geodesic line. Indeed, since σ is a parametrized geodesic line, for every k in $\mathbb{Z} \smallsetminus (j - i)\mathbb{Z}$, τ_{k-1} and τ_{k+1} are different neighbours of τ_k. If $k = q(j - i)$ for q in $\mathbb{Z}$, this is still the case: indeed, we have

$$\begin{aligned}\tau_{k-1} &= \gamma^{q-1}\sigma_{j-1}\\ \tau_k &= \gamma^q\sigma_i = \gamma^{q-1}\sigma_j\\ \tau_{k+1} &= \gamma^q\sigma_{i+1} = \gamma^{q-1}\sigma_{j+1},\end{aligned}$$

where we have used the fact that $\sigma_j = \gamma\sigma_i$ and $\sigma_{j+1} = \gamma\sigma_{i+1}$.

Let us use this construction and the assumption on f to show that $h(T^i\sigma)$ and $h(T^j\sigma)$ are close to each other. Indeed, by definition, we have

$$h(T^i\sigma) - h(T^j\sigma) = \sum_{k=0}^{j-i-1} f(T^{i+k}\sigma).$$

Assume first that $\ell < j - i$. Then, by construction, for any $-\ell \leq k \leq j - i + \ell$, we have $\sigma_{i+k} = \tau_k$. This and the assumption that f is Hölder continuous, tell us that, for $0 \leq k \leq j - i - 1$, we have

$$\left|f(T^{i+k}\sigma) - f(T^k\tau)\right| \leq C\alpha^{\min(k,j-i-k)+\ell}.$$

Besides, as $T^{j-i}\tau = \tau$, by the assumption on f, we have

$$\sum_{k=0}^{j-i-1} f(T^k\tau) = 0.$$

We get

$$\left|h(T^i\sigma) - h(T^j\sigma)\right| \leq C\alpha^\ell \sum_{k=0}^{j-i-1} \alpha^{\min(k,j-i-k)} \leq 2C\alpha^\ell \sum_{k=0}^{\infty} \alpha^k = \frac{2C}{1-\alpha}\alpha^\ell.$$

If $\ell \geq j - i$, as the orbit of σ under T is dense in $\Gamma\backslash\mathscr{S}$, we can find an integer g with $|g - i| > \ell$ and $|g - j| > \ell$ and an element γ' in Γ

such that, for any k with $|k| \leq \ell$, one has $\sigma_{g+k} = \gamma' \sigma_{i+k}$, and hence also $\sigma_{g+k} = \gamma' \gamma^{-1} \sigma_{j+k}$. By applying the previous case, we get

$$\left|h(T^i\sigma) - h(T^g\sigma)\right| \leq \frac{2C}{1-\alpha}\alpha^\ell \text{ and } \left|h(T^j\sigma) - h(T^g\sigma)\right| \leq \frac{2C}{1-\alpha}\alpha^\ell.$$

Hence

$$\left|h(T^i\sigma) - h(T^j\sigma)\right| \leq \frac{4C}{1-\alpha}\alpha^\ell.$$

Standard arguments of topology tell us that h admits a Hölder continuous extension to all of $\Gamma \backslash \mathscr{S}$. □

2.2. **Functions on $\mathscr{S}_+$.** In Section 3, we will establish a criterion for a certain type of Hölder continuous functions to be coboundaries. To this aim, we will need to study functions only depending on the future. We start by showing that every Hölder continuous function is cohomologuous to such a function.

As in Subsection I.2.3, say that a function f on $\mathscr{S}$ is M-invariant if, for any σ and σ' in $\mathscr{S}$, if $\sigma_h = \sigma'_h$ for any $h \geq 0$, then $f(\sigma) = f(\sigma')$. Write $\mathscr{S}_+ = X \times \partial X$ and $\varpi : \mathscr{S} \to \mathscr{S}_+, \sigma \mapsto (\sigma_0, \sigma^+)$. An M-invariant function is a function of the form $f_+ \circ \varpi$, where f_+ is a function on $\mathscr{S}_+$.

We have an analogue of a classical result of hyperbolic dynamics.

Lemma 2.2. *Let f be a Hölder continuous function on $\Gamma \backslash \mathscr{S}$. Then f is cohomologous to an M-invariant Hölder continuous function.*

If f is smooth, this is Lemma I.2.14.

Proof. If Γ is torsion free, this can be directly deduced from the analogue statement for subshifts of finite type (see [6, Proposition 1.2]). We prove the general case by following the same lines.

We start by choosing a section of the natural map $\mathscr{S} \to \mathscr{S}_+$ as follows. We pick a system of representatives $D \subset X_1$ for the action of Γ, that is, we have $X_1 = \Gamma D$ and $\Gamma(x,y) \cap D = \{(x,y)\}$ for any (x,y) in D. Then, we choose any map $\psi : D \to \partial X$ such that, for (x,y) in X_1, we have $y \notin [x\psi(x,y))$. We extend ψ to all X_1: for (x,y) in $X_1 \smallsetminus D$, we set $\psi(x,y) = \gamma\psi(\gamma^{-1}(x,y))$, where γ is an arbitrary element of Γ with $\gamma(x,y) \in D$. The map ψ is not Γ-equivariant in general, but it satisfies the following properties which will be enough for our purposes:

$$y \notin [x\psi(x,y)) \text{ and } \psi(\Gamma(x,y)) \subset \Gamma\psi(x,y), \quad (x,y) \in X_1.$$

For (x,ξ) in $\mathscr{S}_+ = X \times \partial X$, we set $\rho(x,\xi)$ to be the parametrized geodesic line σ with $\sigma_0 = x$, $\sigma^+ = \xi$ and $\sigma^- = \psi(x,y)$, where $y = \sigma_1$ is the neighbour of x on $[x\xi)$.

Let σ be in $\mathscr{S}$. The element $\rho(\sigma_0, \sigma^+)$ has the same future as σ, that is $\varpi\rho(\sigma_0, \sigma^+) = \varpi(\sigma)$. We set

$$h(\sigma) = \sum_{k\geq 0} f(T^k\sigma) - f(T^k\rho(\sigma_0, \sigma^+)).$$

The series converges as f is Hölder continuous and the function h is Γ-invariant. We have

$$h(\sigma) - h(T\sigma) - f(\sigma) = \\ - f(\rho(\sigma_0, \sigma^+)) + \sum_{k\geq 0} f(T^k\rho(\sigma_1, \sigma^+)) - f(T^{k+1}\rho(\sigma_0, \sigma^+)).$$

As the right hand side of the latter is clearly M-invariant, it only remains to prove that h is Hölder continuous.

Indeed, let C and α be as in (2.1). Pick $\ell \geq 1$ and σ and τ in $\mathscr{S}$ with $\sigma_i = \tau_i$ for any $|i| \leq \ell$. Note that by construction, the same property holds when σ and τ are replaced by $\rho(\sigma_0, \sigma^+)$ and $\rho(\tau_0, \tau^+)$. We write

$$h(\sigma) - h(\tau) = \\ \sum_{k=0}^{m} f(T^k\sigma) - f(T^k\tau) - \sum_{k=0}^{m} f(T^k\rho(\sigma_0, \sigma^+)) - f(T^k\rho(\tau_0, \tau^+)) \\ + \sum_{k=m+1}^{\infty} f(T^k\sigma) - f(T^k\rho(\sigma_0, \sigma^+)) - \sum_{k=m+1}^{\infty} f(T^k\tau) - f(T^k\rho(\tau_0, \tau^+)),$$

where m is the floor integer of $\frac{\ell}{2}$. This gives

$$|h(\sigma) - h(\tau)| \leq 2C \sum_{k=0}^{m} \alpha^{\ell-k} + 2C \sum_{k=m+1}^{\infty} \alpha^k \leq \frac{4C}{1-\alpha}\alpha^m$$

and the conclusion follows. □

2.3. **Transfer operator.** We introduce a transfer operator acting on functions on $\Gamma\backslash\mathscr{S}_+$. The properties of this operator will allow us to develop criteria for Hölder continuous functions to be coboundaries.

Let f be a function on $\mathscr{S}_+ = X \times \partial X$. By analogy with [6, Chapter 2], for (x, ξ) in $\mathscr{S}_+$, we set

$$\mathscr{L} f(x, \xi) = \frac{1}{q} \sum_{\substack{y \sim x \\ y \notin [x\xi)}} f(y, \xi). \tag{2.2}$$

We call $\mathscr{L}$ the transfer operator (in the language of [6], we should say the transfer operator associated to the constant potential $\log q$). Note that $\mathscr{L}\mathbf{1} = \mathbf{1}$.

We will establish some standard spectral properties of the operator $\mathscr{L}$. Later, they will help us to solve cohomological equations.

First, we introduce a family of Banach spaces. Given α in $(0,1)$ and f a function on $\mathscr{S}_+$, we say that f is α-Hölder continuous if the function $f \circ \varpi$ on $\mathscr{S}$ is α-Hölder continuous. Note from (2.1), that the function f is α-Hölder continuous if and only if there exists $C > 0$ such that, for every $h \geq 0$, x in X and ξ, η in ∂X, we have

$$|[x\xi) \cap [x\eta)| \geq h + 1 \Rightarrow |f(x,\xi) - f(x,\eta)| \leq C\alpha^h.$$

The infimum of all C satisfying the above inequality is then called the α-Hölder constant of f and is denoted by $C_\alpha(f)$.

For α in $(0,1)$, we let $\mathscr{H}_\alpha^+$ denote the space of all Γ-invariant α-Hölder continuous functions on $\mathscr{S}_+$. The space $\mathscr{H}_\alpha^+$ is a Banach space with respect to the natural Hölder norm defined by, for any f in $\mathscr{H}_\alpha^+$,

$$\|f\|_\alpha = \sup |f| + C_\alpha(f).$$

As mentioned above, we have $\mathscr{L}\mathbf{1} = \mathbf{1}$. If Γ is not bipartite (see Subsection III.2.1), we set $\overline{\mathscr{H}}_\alpha^+ = \mathscr{H}_\alpha^+ / \mathbb{R}\mathbf{1}$ to be the quotient of $\mathscr{H}_\alpha^+$ by the line of constant functions.

If Γ is bipartite, let v be a function on X that is constant on neighbours and w be the opposite of v, that is, for any $x \sim y$ in X, we have $v(x) = w(y)$. Set f_v to be the function $(x,\xi) \mapsto v(x)$ on $\mathscr{S}_+$. Then, we have $\mathscr{L} f_v = f_w$. We let $\overline{\mathscr{H}}_\alpha^+$ denote the quotient space of $\mathscr{H}_\alpha^+$ by the 2-plane of functions of the form f_v where v is as above.

The transfer operator $\mathscr{L}$ has a spectral gap in $\overline{\mathscr{H}}_\alpha^+$.

Proposition 2.3. *Let α be in $(0,1)$. The operator $\mathscr{L}$ has spectral radius < 1 in the quotient space $\overline{\mathscr{H}}_\alpha^+$. In particular, any function f in $\mathscr{H}_\alpha^+$ may be written as $f = g - \mathscr{L}g + c$ for some constant function c and some g in $\mathscr{H}_\alpha^+$. The constant c is uniquely defined by f and g is unique up to the addition of a constant function.*

In the course of the proof, we will need elementary facts from abstract functional analysis.

Lemma 2.4. *Let V be a Banach space and $T : V \to V$ be a bounded linear operator.*
(i) *Assume W is a finite dimensional subspace of V with $TW \subset W$. Then the spectrum of T in V is the union of the spectrum of T in W and the spectrum of T in V/W.*
(ii) *Let $v \neq 0$ be a vector of V with $Tv = v$. Assume 1 is not a spectral value of T in $V/\mathbb{R}v$. Then $(1-T)V$ is a closed subspace of V and $V = \mathbb{R}v \oplus (1-T)V$.*

Proof. (i) For $\lambda \in \mathbb{C}$, assume λ is not a spectral value of T in V. Then, λ is not an eigenvalue of T in V, hence not in W. As W is finite dimensional, the restriction of $\lambda - T$ to W is invertible, which means that $(\lambda - T)^{-1}$ preserves W. Then, it induces an endomorphism in V/W, which is the inverse of the one induced by $\lambda - T$, so that $\lambda - T$ is invertible in V/W.

Conversely, assume λ is neither a spectral value of T in W nor in V/W. As V is finite-dimensional, it admits a closed complementary subspace X and the decomposition $V = W \oplus X$ is an isomorphism of Banach spaces. In this decomposition, T may be written as a matrix $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A is an endomorphism of W, B is a bounded linear map $X \to W$ and C is a bounded endomorphism of X. By assumption, both $\lambda - A$ and $\lambda - C$ are invertible. Then, $\lambda - T$ is invertible with inverse in matrix form

$$(\lambda - T)^{-1} = \begin{pmatrix} (\lambda - A)^{-1} & (\lambda - A)^{-1} B (\lambda - C)^{-1} \\ 0 & (\lambda - C)^{-1} \end{pmatrix}.$$

(ii) As above, we write $V = \mathbb{R}v \oplus X$, where X is a closed hyperplane. The endomorphism T may be written as

$$T(tv + x) = (t + \varphi(x))v + Sx, \quad t \in \mathbb{R}, \quad x \in X,$$

where S is a bounded endomoprhism of X and φ is a continuous linear functional on X. By assumption, 1 is not a spectral value of S. We set $\psi = \varphi \circ (S - 1)^{-1}$, so that ψ is also a continuous linear functional of X. For x in X, we have

$$T(\psi(x)v + x) = (\psi(x) + \varphi(x))v + Sx = \psi(Sx) + Sx,$$

hence the space $Y = \{\psi(x)v + x | x \in X\}$ is stable under T. One easily checks that Y is closed, that $V = \mathbb{R}v \oplus Y$ and that $(T - 1)V = Y$. $\square$

Proof of Proposition 2.3. We let $V \subset \mathscr{H}_\alpha^+$ be the space of functions on $\Gamma \backslash \mathscr{S}_+$ which are of the form $(x, \xi) \mapsto v(x, x_1)$, where x_1 is the neighbour of x on $[x\xi)$ and v is a Γ-invariant function on X_1. Then V is stable under $\mathscr{L}$ and, by Corollary II.5.6, the operator $\mathscr{L}$ has spectral radius < 1 on the image of V in $\overline{\mathscr{H}}_\alpha^+$.

To conclude, we will show that $\mathscr{L}$ has spectral radius $\leq \alpha$ in $\mathscr{H}_\alpha^+/V$. To this aim, we change slightly the definition of the Hölder constant of a function: for f in $\mathscr{H}_\alpha^+$, we set

$$C_\alpha^1(f) = \sup_{\substack{(x,\xi,\eta) \in X \times \partial X \times \partial X \\ h \geq 1 \\ |[x\xi) \cap [x\eta)| \geq h+1}} \alpha^{-h} |f(x, \xi) - f(x, \eta)|,$$

that is, we consider the above supremum only when the intersection $[x\xi) \cap [x\eta)$ contains at least two points. One easily checks that the seminorm C^1_α induces a norm on $\mathscr{H}^+_\alpha/V$ which is equivalent to the quotient norm of the Hölder norm. Now, a direct computation shows that, for f in $\mathscr{H}^+_\alpha$, one has

$$C^1_\alpha(\mathscr{L}f) \leq \alpha C^1_\alpha(f),$$

hence $\mathscr{L}$ has spectral radius $\leq \alpha$ in $\mathscr{H}^+_\alpha/V$.

It follows from the first part of Lemma 2.4 that $\mathscr{L}$ has spectral radius < 1 in $\overline{\mathscr{H}}^+_\alpha$. The rest of the statement is a consequence of the second part of Lemma 2.4. □

2.4. **Decomposition of Hölder continuous functions.** Thanks to the spectral properties of the transfer operator $\mathscr{L}$, we get a way of decomposing a Hölder continuous function.

Corollary 2.5. *Let f be a Hölder continuous function on $\Gamma\backslash\mathscr{S}$. Then f may be written as*

$$g = h - h \circ T + g \circ \varpi + c,$$

where c is a constant function, h is a Hölder continuous function on $\Gamma\backslash\mathscr{S}$, g is a Hölder continuous function on $\Gamma\backslash\mathscr{S}_+$ and $\mathscr{L}g = 0$. The constant c and the function g are uniquely determined by f; the function h is uniquely determined up to the addition of a constant function.

Proof. First, we prove the existence of the decomposition. By Lemma 2.2, we can find a Hölder continuous function h_1 on $\Gamma\backslash\mathscr{S}$ and a Hölder continuous function f_1 on $\Gamma\backslash\mathscr{S}_+$ with

$$f = h_1 - h_1 \circ T + f_1 \circ \varpi.$$

By Proposition 2.3, there exists a Hölder continuous function h_2 on $\Gamma\backslash\mathscr{S}_+$ and a constant c such that

$$\mathscr{L}f_1 = \mathscr{L}h_2 - h_2 + c.$$

Write $T_+ : \mathscr{S}_+ \to \mathscr{S}_+$ for the natural transformation, that is, for (x, ξ) in $\mathscr{S}_+$, $T_+(x, \xi) = (x_1, \xi)$, where x_1 is the neighbour of x on $[x\xi)$. We have $\varpi T = T_+\varpi$ and, for any function φ on $\mathscr{S}_+$, $\mathscr{L}(\varphi \circ T_+) = \varphi$. Thus, if we set $g = f_1 - h_2 + h_2 \circ T_+ - c$, we get $\mathscr{L}g = 0$ and

$$f = g \circ \varpi + (h_1 + h_2 \circ \varpi) - (h_1 + h_2 \circ \varpi) \circ T + c$$

and the existence of the decomposition follows.

As for the uniqueness, let c be a constant, h be a Hölder continuous function on $\Gamma\backslash\mathscr{S}$ and g be a Hölder continuous function on $\Gamma\backslash\mathscr{S}_+$ with

$\mathscr{L}g = 0$. Assume that we have

$$g \circ \varpi + h - h \circ T + c = 0.$$

First, we claim that h is M-invariant, that is $h = h_1 \circ \varpi$ for some function h_1 on $\Gamma\backslash\mathscr{S}_+$. Indeed, let σ and τ be in $\mathscr{S}$ with $\varpi(\sigma) = \varpi(\tau)$. Then, as $h \circ T - h = g \circ \varpi + c$, we have

$$h(\sigma) - h(\tau) = h(T\sigma) - h(T\tau).$$

By iterating this identity, we get, for $k \geq 0$,

$$h(\sigma) - h(\tau) = h(T^k\sigma) - h(T^k\tau).$$

As h is continuous on the compact set $\Gamma\backslash\mathscr{S}$ and the parametrized geodesic lines $T^k\sigma$ and $T^k\tau$ get closer and closer to each other as $k \to \infty$, we get $h(\sigma) = h(\tau)$, hence $h = h_1 \circ \varpi$ for some Hölder continuous function h_1 on $\Gamma\backslash\mathscr{S}_+$.

Now, we have $g + h_1 - h_1 \circ T_+ + c = 0$. By applying the operator $\mathscr{L}$ to this identity, we get

$$\mathscr{L}h_1 - h_1 + c = 0.$$

By Proposition 2.3, h_1 is a constant function. In particular, we have $h_1 = h_1 \circ T$, hence $g + c = 0$. As $\mathscr{L}g = 0$, this gives $c = 0$, hence also $g = 0$, which should be proved. □

2.5. **Solving the cohomological equation in subspaces.** Later, we shall use the proof of Corollary 2.5 through the following *ad hoc* formulation.

Corollary 2.6. *Let V be a Banach space, $\Lambda : V \to V$ be a bounded operator and u be a non zero vector of V. Assume $\Lambda u = u$ and $1 - \Lambda$ is invertible in the quotient space $E/\mathbb{R}u$. Fix $0 < \alpha < 1$ and suppose we are given a bounded linear map $\Theta : E \to \mathscr{H}_\alpha^+$ with $\Theta\Lambda = \mathscr{L}\Theta$ and $\Theta u = \mathbf{1}$. Let v be in E and assume that the Hölder continuous function $\Theta(v) \circ \varpi$ is a coboundary on $\Gamma\backslash\mathscr{S}$, that is, there exists a Hölder continuous function h on $\Gamma\backslash\mathscr{S}$ with $\Theta(v) \circ \varpi = h - h \circ T$. Then, we must have*

$$h = \Theta(\Lambda w) \circ \varpi$$

for some w in V.

Proof. Set $f = \Theta(v) \circ \varpi$ so that we have $f \circ \varpi = h - h \circ T$. As in the proof of Corollary 2.5, this tells us that $h = h_1 \circ \varpi$ for some Hölder continuous function h_1 on $\Gamma\backslash\mathscr{S}_+$. We have $f = h_1 - h_1 \circ T_+$, hence, still as in the proof of Corollary 2.5,

$$\mathscr{L}f = \mathscr{L}h_1 - h_1.$$

Now, Lemma 2.4 and the assumption imply that there exists a real number c and a w in V with $v = \Lambda w - w + cu$. We have $\Lambda v = \Lambda^2 w - \Lambda w + cu$, hence, by applying Θ to this identity, we get

$$\mathscr{L} f = \mathscr{L}\Theta(\Lambda w) - \Theta(\Lambda w) + c.$$

From the uniqueness statement in Proposition 2.3, we get

$$h_1 - \Theta(\Lambda w) \in \mathbb{R}\mathbf{1} = \Theta(\mathbb{R}u),$$

and the conclusion follows. □

Thanks to this technical result, we can prove that smooth functions are cohomologous when viewed as Hölder continuous functions, if and only if they are cohomologous in the sense of Subsection I.2.3 (see also Lemma I.3.12). For $k \geq 1$, we let V_k stand for the space of Γ-invariant functions on X_k.

Corollary 2.7. *Let f be a smooth function on $\Gamma\backslash\mathscr{S}$. Assume that there exists a Hölder continuous function h on $\Gamma\backslash\mathscr{S}$ with $f = h - h \circ T$. Then, h is smooth.*

More precisely, given $k \geq 1$ and w in V_k, the smooth function $\sigma \mapsto w(\sigma_0\sigma_k)$ on $\mathscr{S}$ is a coboundary if and only if there exists v in V_{k-1} such that, for any xy in X_k, one has

$$w(xy) = v(xy_1) - v(x_1y),$$

where x_1 and y_1 are the neigbours of x and y on $[xy]$.

In the sequel, as in I, for $k \geq 1$, we shall say that two elements w and w' in V_k are cohomologous if there exists v in V_{k-1} such that, for any xy in X_k, one has

$$w(xy) - w'(xy) = v(xy_1) - v(x_1y).$$

Proof. Note that the second part of the statement follows from the first and Lemma I.2.14. Thus, we only need to prove the second part. We will obtain it by using Corollary 2.6.

For $k \geq 1$, define a map Θ_k from V_k to the space of smooth functions on $\Gamma\backslash\mathscr{S}_+$ by setting, for w in V_k and (x,ξ) in $\mathscr{S}_+$,

$$\Theta_k w(x,\xi) = w(x, x_k),$$

where x_k is the unique element of $[x\xi)$ with $d(x,x_k) = k$. By compactness, the space of smooth functions on $\Gamma\backslash\mathscr{S}_+$ is $\bigcup_{k\geq 1} \Theta_k(V_k)$.

We define an endomorphism Λ_k of V_k by setting, for w in V_k and (x,y) in X_k,

$$\Lambda_k w(x,y) = \frac{1}{q} \sum_{\substack{z\sim x \\ z\neq x_1}} w(z, y_1),$$

where x_1 and y_1 are the neighbours of x and y in $[xy]$. Note that we have $\mathscr{L}\Theta_k = \Theta_k\Lambda_k$.

Let $I_k : V_k \to V_{k+1}$ be the injection given by, for w in V_k and (x, y) in X_{k+1},

$$I_k w(x, y) = w(x, y_1),$$

where as above, y_1 is the neighbour of y in $[xy]$. We have $\Lambda_{k+1}I_k = I_k\Lambda_k$ and $\Theta_{k+1}I_k = \Theta_k$. Besides, we have $\Lambda_{k+1}V_{k+1} \subset I_kV_k$. Thus, for $k \geq 2$, the spectrum of Λ_k in V_k is the union of $\{0\}$ and the spectrum of Λ_1 in V_1. By using Corollary II.5.6, we get that $1 - \Lambda_k$ is invertible in $V_k/\mathbb{R}\mathbf{1}$ for any $k \geq 1$. Hence, the assumption of Corollary 2.6 is satisfied.

Assume $k \geq 2$ and let w be in V_k such that the function $\Theta_k w$ is a coboundary when viewed as a Hölder continuous function $\Gamma\backslash\mathscr{S}$. By Corollary 2.6, the solutions h of the equation $\Theta_k w = h - h \circ T$ belong to the space

$$\Theta_k\Lambda_kV_k \subset \Theta_kI_{k-1}V_{k-1} = \Theta_{k-1}V_{k-1}.$$

The conclusion follows.

It remains to deal with the case where $k = 1$. Then, let still w be in V_1. If the smooth function $\sigma \mapsto w(\sigma_0\sigma_1)$ on $\mathscr{S}$ is a coboundary, the above in case $k = 2$ tells us that there exists v in V_1 such that, for any xy in X_1 and any neighbour $z \neq x$ of y, one has

$$w(xy) = v(xy) - v(yz).$$

In particular, for y in X, let z, z' be two neighbours of y. As $q \geq 2$, the vertex y admits a neighbour x that is neither z nor z', and we get

$$v(yz) = v(xy) - w(xy) = v(yz'),$$

hence, there exists u in V_0 such that

$$v(yz) = u(y), \quad yz \in X_1.$$

For xy in X_1, we get

$$w(xy) = u(x) - u(y)$$

as required. □

Remark 2.8. Let f be a Hölder continuous function on $\Gamma\backslash\mathscr{S}$ and θ be a T-invariant distribution. Assume that f is cohomologous to some smooth function g. Then, it follows from Corollary 2.7 above that $\langle\theta, g\rangle$ is independent on the choice of g. In the sequel, we shall write this number as $\langle\theta, f\rangle$.

3. The cohomological equation for endpoints series

We will use the cohomology theory of Hölder continuous functions to give a criterion for certain functions defined by series to be coboundaries.

3.1. **Endpoints series.** We define a particular construction of smooth functions on $\Gamma\backslash\mathscr{S}$. This type of functions will later appear naturally in the spectral theory of the action of Γ on $\overline{\mathcal{D}}(\partial X)$.

In the sequel, we will be concerned with the following finite-dimensional spaces. For $k \geq 0$, we let as above V_k be the space of all functions on $\Gamma\backslash X_k$. We shall also think to V_k as the space of all functions on X_k which are Γ-invariant. In the same way, for $h, k \geq 1$, we let $W_{h,k}$ be the space of all functions on $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$. Equivalently, we shall also see $W_{h,k}$ as the space of functions on $X_h \times X_k$ which are invariant under the product action of $\Gamma \times \Gamma$. When $h = k$, we will write W_k for $W_{k,k}$.

For $\ell \geq h, k \geq 1$, we define the space $\mathscr{W}_{h,k,\ell}$ as follows: an element w of $\mathscr{W}_{h,k,\ell,\alpha}$ is a family $(w_j)_{j\geq\ell}$ where w_ℓ belongs to V_ℓ and $(w_j)_{j\geq\ell+1}$ is a finitely supported sequence in $W_{h,k}$.

Still for $\ell \geq h, k \geq 1$, let us construct a linear map $\Theta_{h,k,\ell} : \mathscr{W}_{h,k,\ell} \to \mathcal{D}(\Gamma\backslash\mathscr{S}_+)$. For $w = (w_j)_{j\geq\ell}$ in $\mathscr{W}_{h,k,\ell}$, we define the associated endpoints series $\Theta_{h,k,\ell}w$ as the function such that, for (x, ξ) in $\mathscr{S}_+$, one has

$$\Theta_{h,k,\ell}w(x, \xi) = w_\ell(x_0x_\ell) + \sum_{j=\ell+1}^{\infty} w_j(x_0x_h, x_{j-k}x_j), \tag{3.1}$$

where $(x_j)_{j\geq 0}$ is the parametrization of the geodesic ray $[x\xi)$.

3.2. **Accessible pairs.** We will give a criterion for an endpoints series to be a coboundary in the sense of Section 2. To state this criterion, we need to introduce new subsets of $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$, $h, k \geq 1$.

Definition 3.1. Let $j \geq h, k \geq 1$. A pair $(\Gamma ab, \Gamma xy)$ in $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$ is said to be j-accessible if there exists pq in X_j such that $pp_h \in \Gamma ab$ and $q_kp \in \Gamma xy$, where p_h and q_k are the elements of $[pq]$ wich lie at distance h from p and k from q. The set of j-accessible pairs in $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$ is denoted by $((\Gamma\backslash X_h) \times (\Gamma\backslash X_k))_j$.

In other words, the pair $(\Gamma ab, \Gamma xy)$ is j-accessible if there exits a path of length j from Γab to Γxy.

When j is large, we can describe the set $((\Gamma\backslash X_h) \times (\Gamma\backslash X_k))_j$.

Lemma 3.2. *Assume Γ is not bipartite. Then, there exists an integer n such that, for every $h, k \geq 1$, and every $j \geq h + k + n$, every pair in $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$ is j-accessible.*

Assume Γ is bipartite. Then, there exists an integer n such that, for every $h, k \geq 1$, and every $j \geq h+k+n$, a pair (ab, xy) in $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$ is j-accessible if and only if the integral number $d(b, x)$ has the same parity as $j-h-k$.

Proof. This is a consequence of the equidistribution statement in Corollary II.5.6. We keep the notation of this result. Note that we have $\rho = q$ and that u and $u^\vee$ may be chosen to be the constant function $\mathbf{1}$. For ab in X_1, let v_{ab} denote the Γ-invariant function defined by

$$v_{ab}(xy) = \sum_{\gamma\in\Gamma} \mathbf{1}_{ab=\gamma(xy)}.$$

By Lemma II.3.25, we have

$$\langle v_{ab}, \mathbf{1}\rangle = \sum_{xy\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} v_{ab}(xy) = 1.$$

If Γ is not bipartite, by Corollary II.5.6, for ab in X_1, we have

$$\frac{1}{q^n} R^n v_{ab} \xrightarrow[n\to\infty]{} \frac{1}{\sum_{xy\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x\cap\Gamma_y|}}.$$

Therefore, there exists n such that, for every $m \geq n$ and every ab and xy in X_1, we can find a geodesic path $b_{-1} = a, b_0 = b, b_1, \dots, b_m$ in X and γ in Γ with $b_{m-1} = \gamma x$ and $b_m = \gamma y$. In particular, the pair (ab, xy) is j-accessible in $(\Gamma\backslash X_1) \times (\Gamma\backslash X_1)$ for every $j \geq n+1$.

Still when Γ is not bipartite, if (ab, xy) is in $(\Gamma\backslash X_h)\times(\Gamma\backslash X_k)$ for some $h, k \geq 1$, we claim that (ab, xy) is j-accessible for every $j \geq h+k+n-1$. Indeed, for such a j, we have $i = j-h-k+1 \geq n$, hence, by the previous case, there exists pq in X_{i+1} and γ in Γ with $pp_1 = b_1 b$ and $q_1 q = \gamma(xx_1)$, where as usual, b_1, x_1, p_1 and q_1 are the neighbours of b, x, p and q on $[ab]$, $[xy]$, $[pq]$ and $[pq]$. Then, we have $d(a, \gamma y) = j$ and $b, \gamma x \in [a(\gamma y)]$, so that the pair (ab, xy) is j-accessible in $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$.

The proof in the non bipartite case follows the same lines by keeping in mind that the action of Γ preserves the classes of the even distance equivalence relation on X. $\square$

By using the notion of an accessible pair, we can formulate the main result of this Section. This is a criterion for an endpoints series to be a coboundary.

Proposition 3.3. *Let $\ell \geq h, k \geq 2$ and $w = (w_j)_{j\geq\ell}$ be in $\mathcal{W}_{h,k,\ell}$. Then the following are equivalent.*
(i) *The endpoints series $\Theta_{h,k,\ell} w$ is a coboundary, when wiewed as a smooth function on $\Gamma\backslash\mathscr{S}$.*

(ii) *There exists t in $V_{\ell-1}$ and finitely supported sequences $(u_j)_{j\geq\ell}$ in $W_{h,k-1}$ and $(v_j)_{j\geq\ell}$ in $W_{h-1,k}$ such that, for every ab in X_ℓ,*

$$w_\ell(ab) = t(ab_1) - t(a_1b) + u_\ell(aa_h, b_{k-1}b) + v_\ell(aa_{h-1}, b_kb)$$

and, for every $j \geq \ell+1$ and every j-accessible pair (ab, xy) in $(\Gamma\backslash X_h)\times(\Gamma\backslash X_k)$,

$$w_j(ab, xy) = u_j(ab, x_1y) + v_j(ab_1, xy) - u_{j-1}(ab, xy_1) - v_{j-1}(a_1b, xy).$$

In the statement above, for a, b in X and $i \leq d(a,b)$, we have denoted by a_i the point of the segment $[ab]$ that lies as distance i from a.

First part of the proof. We prove the easy case of the Proposition, that is, *(ii)⇒(i)*. Let u and v be as as in the statement. Fix σ in $\mathscr{S}$. By definition, for every $j \geq \ell+1$, the pair $(\sigma_0\sigma_h, \sigma_{j-k}\sigma_j)$ is j-accessible in $(\Gamma\backslash X_h)\times(\Gamma\backslash X_k)$. Therefore, by (3.1) and the assumption, we have

$$\begin{aligned}\Theta_{h,k,\ell}w(\sigma) &= t(\sigma_0\sigma_{\ell-1}) - t(\sigma_1\sigma_\ell) + u_\ell(\sigma_0\sigma_h, \sigma_{\ell-k+1}\sigma_\ell)\\ &+ v_\ell(\sigma_0\sigma_{h-1}, \sigma_{\ell-k}\sigma_\ell) + \sum_{j\geq\ell+1} u_j(\sigma_0\sigma_h, \sigma_{j-k+1}\sigma_j) + v_j(\sigma_0\sigma_{h-1}, \sigma_{j-k}\sigma_j)\\ &\qquad - u_{j-1}(\sigma_0\sigma_h, \sigma_{j-k}\sigma_{j-1}) - v_{j-1}(\sigma_1\sigma_h, \sigma_{j-k}\sigma_j).\end{aligned}$$

By using the cancellation of the telescoping series, we get

$$\begin{aligned}\Theta_{h,k}w(\sigma) &= t(\sigma_0\sigma_{\ell-1}) - t(\sigma_1\sigma_\ell)\\ &+ \sum_{j\geq\ell} v_j(\sigma_0\sigma_{h-1}, \sigma_{j-k}\sigma_j) - v_j(\sigma_1\sigma_h, \sigma_{j+1-k}\sigma_{j+1}) = h(\sigma) - h(T\sigma),\end{aligned}$$

where

$$h(\sigma) = t(\sigma_0\sigma_{\ell-1}) + \sum_{j\geq\ell} v_j(\sigma_0\sigma_{h-1}, \sigma_{j-k}\sigma_j).$$

The result follows. □

The proof of the converse statement will last until the end of the Section.

3.3. **Vanishing endpoints series.** We begin by determining the null space of the endpoints series operator.

Lemma 3.4. *Let $\ell \geq h, k \geq 1$ and w be in $\mathcal{W}_{h,k,\ell}$. Assume the endpoints series function $\Theta_{h,k,\ell}w$ vanishes on $\Gamma\backslash\mathscr{S}$. Then, there exists a finitely supported sequence $u = (u_j)_{j\geq\ell}$ in $W_{h,k-1}$ such that, for every ab in X_h,*

$$w_\ell(ab) = u_\ell(aa_h, b_{k-1}b)$$

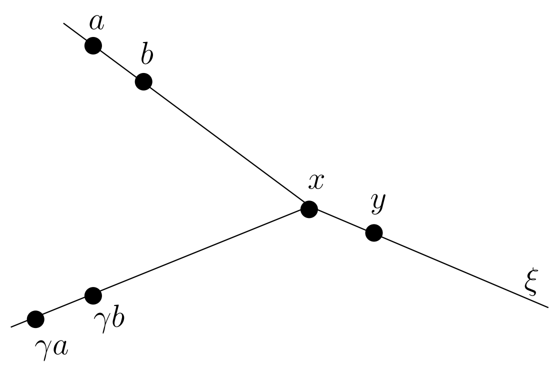

FIGURE 1. Proof of Lemma 3.4

and, for every $j \geq \ell+1$ and every j-accessible pair (ab, xy) in $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$,

$$w_j(ab, xy) = u_j(ab, x_1y) - u_{j-1}(ab, xy_1).$$

Note that the converse is true, in view of the first part of the proof of Proposition 3.3.

Proof. The idea of the proof is that u may be defined by integrating w along certain paths. The fact that the integral does not depend on the path is warranted by the vanishing of the function $\Theta_{h,k,\ell}w$. Let us write this precisely.

For $j \geq h+k-1$, write $(X_h \times X_{k-1})_j$ for the set of pairs (ab, xy) such that $d(a, y) = j$ and b and x belong to $[ay]$. If $j \geq \ell$, we define a function u_j on $(X_h \times X_{k-1})_j$ by setting

$$u_j(ab, xy) = w_\ell(p_0p_\ell) + \sum_{i=\ell+1}^{j} w_i(ab, p_{i-k}p_i), \quad (ab, xy) \in (X_h \times X_{k-1})_j,$$

where $a = p_0, p_1, \ldots, p_j = y$ is the geodesic parametrization of the segment $[ay]$ (in particular, we have $p_h = b$ and $p_{j-k+1} = x$).

The function u_j is invariant under the diagonal action of Γ on $(X_h \times X_{k-1})_j$. We claim that the vanishing assumption on $\Theta_{h,k,\ell}w$ implies that u_j satisfies the following additional invariance property: for every (ab, xy) in $(X_h \times X_{k-1})_j$ and every γ in Γ, if $(\gamma(ab), xy)$ also belongs to $(X_h \times X_{k-1})_j$, then

$$u_j(ab, xy) = u_j(\gamma(ab), xy)$$

(see Figure 1).

Indeed, for such ab, xy and γ, let $a = p_0, p_1, \ldots, p_j = y$ be the geodesic parametrization of the segment $[ay]$ and choose a ξ in ∂X

such that $[y\xi) \cap [ay] = [y\xi) \cap [(\gamma a)y] = \emptyset$. Finally, let $x_0 = x, x_1, \dots$ be the geodesic parametrization of the geodesic ray $[x\xi)$. We get

$$\begin{aligned}\Theta_{h,k,\ell} w(a,\xi) = \\ & w_\ell(p_0 p_\ell) + \sum_{i=\ell+1}^{j} w_i(ab, p_{i-k}p_i) + \sum_{i=j+1}^{\infty} w_i(ab, x_{i-j-1}x_{i-j+k-1}) \\ & = u_j(ab, xy) + \sum_{i=j+1}^{\infty} w_i(ab, x_{i-j-1}x_{i-j+k-1})\end{aligned}$$

and in the same way,

$$\Theta_{h,k,\ell} w(\gamma a, \xi) = u_j(\gamma(ab), xy) + \sum_{i=j+1}^{\infty} w_i(\gamma(ab), x_{i-j-1}x_{i-j+k-1}).$$

As $\Theta_{h,k,\ell} w = 0$ and the functions w_i, $i \geq \ell$, are invariant under the product action of $\Gamma \times \Gamma$ on $X_h \times X_k$, we obtain

$$u_j(\gamma(ab), xy) = -\sum_{i=j+1}^{\infty} w_i(ab, x_{i-j-1}x_{i-j+k-1}) = u_j(ab, xy)$$

as required.

Note that by definition, the image of $(X_h \times X_{k-1})_j$ in $(\Gamma \backslash X_h) \times (\Gamma \backslash X_{k-1})$ is the set $((\Gamma \backslash X_h) \times (\Gamma \backslash X_{k-1}))_j$ of j-accessible pairs. The previous tells us that we can consider u_j as a function on $((\Gamma \backslash X_h) \times (\Gamma \backslash X_{k-1}))_j$. We extend it as a function defined everywhere on $(\Gamma \backslash X_h) \times (\Gamma \backslash X_{k-1})$ by setting $u_j(ab, xy) = 0$ for any (ab, xy) in the complement of $((\Gamma \backslash X_h) \times (\Gamma \backslash X_{k-1}))_j$ in $(\Gamma \backslash X_h) \times (\Gamma \backslash X_{k-1})$. Note that as above, for (ab, xy) in $((\Gamma \backslash X_h) \times (\Gamma \backslash X_{k-1}))_j$, we have

$$u_j(ab, xy) = -\sum_{i=j+1}^{\infty} w_i(ab, x_{i-j-1}x_{i-j+k-1}),$$

where $(x_i)_{i \geq 0}$ is the parametrization of a geodesic ray $[x\xi)$ with $[x\xi) \cap [ay] = [xy]$. Therefore, since the sequence $(w_j)_{j \geq \ell+1}$ is finitely supported, the sequence $(u_j)_{j \geq \ell}$ is also finitely supported.

Finally, on one hand, for ab in X_ℓ, we have

$$w_\ell(ab) = u_\ell(aa_h, b_{k-1}b).$$

On the other hand, if $j \geq \ell + 1$ and (ab, xy) is a j-accessible pair in $(\Gamma \backslash X_h) \times (\Gamma \backslash X_k)$, we can find γ in Γ such that $d(a, \gamma y) = j$ and b and γx belong to $[a(\gamma y)]$. Then, we have

$$(ab, \gamma(x_1 y)) \in (X_k \times X_{k-1})_j \text{ and } (ab, \gamma(x y_1)) \in (X_k \times X_{k-1})_{j-1}.$$

We get

$$\begin{aligned} w_j(ab,xy) = w_j(ab,\gamma(xy)) &= u_j(ab,\gamma(x_1y)) - u_{j-1}(ab,\gamma(xy_1)) \\ &= u_j(ab,x_1y) - u_{j-1}(ab,xy_1) \end{aligned}$$

as required. □

3.4. **A symbolic transfer operator.** To prove Proposition 3.3, we will apply Corollary 3.4 to the endpoints operator $\Theta_{h,k,\ell}$. To this aim, we will introduce a new operator $\Lambda_{h,k,\ell}$ acting on sequences spaces $\mathcal{W}_{h,k,\ell}$, $\ell \geq h, k \geq 2$, which will be semiconjugate to the transfer operator $\mathscr{L}$ via $\Theta_{h,k,\ell}$.

Let $\ell \geq h, k \geq 1$ and $w = (w_j)_{j \geq \ell}$ be a sequence where w_ℓ is a function on $\Gamma\backslash X_\ell$, and, for $j \geq \ell+1$, w_j is a function on $(\Gamma\backslash X_h) \times (\Gamma\backslash X_k)$. We define a new sequence $\Lambda_{h,k}w$ by setting, for ab in X_ℓ,

$$(\Lambda_{h,k,\ell}w)_\ell(ab) = \frac{1}{q}\sum_{\substack{c\sim a\\ c\neq a_1}}(w_\ell(cb_1) + w_{\ell+1}(ca_{h-1}, b_kb)) \tag{3.2}$$

and, for $j \geq \ell+1$ and (ab,xy) in $X_h \times X_k$,

$$(\Lambda_{h,k,\ell}w)_j(ab,xy) = \frac{1}{q}\sum_{\substack{c\sim a\\ c\neq a_1}} w_{j+1}(cb_1,xy) \tag{3.3}$$

The definition of the transfer operator $\mathscr{L}$ in (2.2) and the one if the endpoints operator $\Theta_{h,k,\ell}$ in (3.1) directly give

Lemma 3.5. *Let $\ell \geq h, k \geq 1$. For any w in $\mathscr{W}_{h,k,\ell}$, we have*

$$\mathscr{L}\Theta_{h,k,\ell}w = \Theta_{h,k,\ell}\Lambda_{h,k,\ell}w.$$

In order to apply Corollary 3.4, we will prove

Lemma 3.6. *Let $\ell \geq h, k \geq 1$. Let u be the element of $\mathscr{W}_{h,k,\ell}$ such that $u_j = 0$ for $j \geq \ell+1$ and that $u_\ell(ab) = 1$ for ab in X_ℓ. Then, we have $\Lambda_{h,k,\ell}u = u$ and $\Lambda_{h,k,\ell} - 1$ is invertible in $\mathscr{W}_{h,k,\ell}/\mathbb{R}u$.*

Proof. For $i \geq 0$, let $\mathscr{W}^i_{h,k,\ell}$ be the space of sequences w in $\mathscr{W}_{h,k,\ell,\alpha}$ with $w_j = 0$ for any $j \geq \ell+i+1$, so that $\mathscr{W}_{h,k,\ell} = \bigcup_{i\geq 0}\mathscr{W}^i_{h,k,\ell}$. By (3.2) and (3.3), the space $\mathscr{W}^0_{h,k,\ell}$ is stable under $\Lambda_{h,k}$ and, for $i \geq 1$, we have $\Lambda_{h,k}\mathscr{W}^i_{h,k,\ell} \subset \mathscr{W}^{i-1}_{h,k,\ell}$. The latter implies in particular that $\Lambda_{h,k,\ell} - 1$ is invertible in the quotient space $\mathscr{W}_{h,k,\ell}/\mathscr{W}^0_{h,k,\ell}$.

Now, still by (3.2), we may identify $\mathscr{W}^0_{h,k,\ell}$ with V_ℓ and the restriction of $\Lambda_{h,k,\ell}$ with the operator L defined by, for w in V_ℓ and ab in X_ℓ,

$$Lw(ab) = \frac{1}{q}\sum_{\substack{c\sim a_1\\ c\neq a_1}} w(cb_1).$$

From the proof of Corollary 2.7, we know that $L-1$ is invertible in $V_\ell/\mathbb{R}\mathbf{1}$. Thus, $\Lambda_{h,k,\ell}-1$ is invertible in $\mathscr{W}^0_{k,\alpha}/\mathbb{R}u$. Therefore, $\Lambda_{h,k,\ell}-1$ is invertible in $\mathscr{W}_{h,k,\ell,\alpha}/\mathbb{R}u$ as required. □

3.5. **Solving the cohomological equation.** Thanks to the study of the symbolic transfer operator, we can apply Corollary 2.6 and transport the cohomological equation back to an equation in sequences spaces.

Second part of the proof of Proposition 3.3. We now prove the difficult case of the Proposition, that is, *(ii)⇒(i)*. Thus, we assume that we are given w in $\mathscr{W}_{h,k,\ell}$ such that the endpoint series $\Theta_{h,k,\ell}w$ is a coboundary, when viewed as a smooth function on $\Gamma\backslash\mathscr{S}$.

By Lemma 3.5 and Lemma 3.6, we can apply Corollary 2.6. This tells us that the solutions h of the equation

$$\Theta_{h,k,\ell}w = h - h\circ T \tag{3.4}$$

belong to $\Theta_{h,k,\ell}\Lambda_{h,k,\ell}\mathscr{W}_{h,k,\ell,\alpha}$.

As we have assumed $h\geq 2$, in view of the range of the operator $\Lambda_{h,k,\ell}$ in $\mathscr{W}_{h,k,\ell,\alpha}$ as described from (3.2) and (3.3), we can assume that there exists a function t on $\Gamma\backslash X_{\ell-1}$ and a finitely supported sequence $v=(v_j)_{j\geq\ell}$ of functions on $(\Gamma\backslash X_{h-1})\times(\Gamma\backslash X_k)$ such that, for any σ in $\mathscr{S}$,

$$h(\sigma) = t(\sigma_0\sigma_{\ell-1}) + \sum_{j\geq\ell} v_j(\sigma_0\sigma_{h-1},\sigma_{j-k}\sigma_j).$$

Thus, from (3.4), we get

$$\begin{aligned}\Theta_{h,k,\ell}w(\sigma) &= t(\sigma_0\sigma_{\ell-1}) - t(\sigma_1\sigma_\ell)\\ &\quad + \sum_{j\geq\ell} v_j(\sigma_0\sigma_{h-1},\sigma_{j-k}\sigma_j) - v_j(\sigma_1\sigma_h,\sigma_{j-k+1}\sigma_{j+1})\\ &= t(\sigma_0\sigma_{\ell-1}) - t(\sigma_1\sigma_\ell) + v_\ell(\sigma_0\sigma_{h-1},\sigma_{\ell-k}\sigma_\ell)\\ &\quad + \sum_{j\geq\ell+1} v_j(\sigma_0\sigma_{h-1},\sigma_{j-k}\sigma_j) - v_{j-1}(\sigma_1\sigma_h,\sigma_{j-k}\sigma_j)\\ &= \Theta_{h,k,\ell}w'(\sigma),\end{aligned}$$

where $w'=(w'_j)_{j\geq\ell}$ is the element of $\mathscr{W}_{h,k,\ell,\alpha}$ defined by

$$w'_\ell(ab) = t(ab_1) - t(a_1b) + v_\ell(aa_{h-1},b_kb),\quad ab\in X_\ell$$

and, for $j\geq\ell+1$,

$$w'_j(ab,xy) = v_j(ab_1,xy) - v_{j-1}(a_1b,xy),\quad ab\in X_h,\quad xy\in X_k.$$

The result now follows from Lemma 3.4. □

4. The Plancherel formula on X_1

The purpose of this Section is to establish a Plancherel formula for functions on X_1. For functions on X, such a formula is obtained in [3]. Our formula is closely related to the latter.

4.1. **Polynomial functions of the spectral parameter.**

We start by introducing remarkable polynomial functions which will play a role all along the article. We use the notation of Subsection III.3.3 for objects related to spectral analysis, so that below, $t = \frac{1}{q+1}(u + \frac{q}{u})$ will be the spectral parameter.

Let t be in $\mathbb{C}$ and write $(q+1)t = u + \frac{q}{u}$ for some u in $\mathbb{C}^*$. For $(q+1)^2t^2 \neq 4q$ (that is, $u^2 \neq q$), we set

$$\begin{aligned} A_0(t) &= 1, \\ A_j(t) &= u^j + \left(\frac{q}{u}\right)^j && j \geq 1, \\ B_j(t) &= \frac{u^j - \left(\frac{q}{u}\right)^j}{u - \frac{q}{u}} && j \geq 0. \end{aligned}$$

The reason for the choice of the value A_0 will become clear later.

These functions play a role in the spectral formulas of [3]. They are actually regular.

Lemma 4.1. *For $j \geq 0$, the functions A_j and B_j are polynomial functions. For t in $\mathbb{C}$ and $j \geq 1$, the following relations hold:*

$$(q+1)tB_j(t) = B_{j+1}(t) + qB_{j-1}(t) \text{ and } B_{j+1}(t) = A_j(t) + qB_{j-1}(t).$$

Proof. Let t, u be in $\mathbb{C}$ with $(q+1)tu = u^2 + q$. For $(q+1)^2t^2 \neq 4q$, we have $B_0(t) = 0$ and $B_1(t) = 1$. Take $j \geq 1$. We have

$$\begin{aligned} (q+1)tB_j(t) &= \left(\frac{u + \frac{q}{u}}{u - \frac{q}{u}}\right)\left(u^j - \left(\frac{q}{u}\right)^j\right), \\ &= \frac{1}{u - \frac{q}{u}}\left(u^{j+1} - q\left(\frac{q}{u}\right)^{j-1} + qu^{j-1} - \left(\frac{q}{u}\right)^{j+1}\right) \\ &= B_{j+1}(t) + qB_{j-1}(t). \end{aligned}$$

In particular, an easy induction argument shows that B_j is a polynomial for any $j \geq 0$.

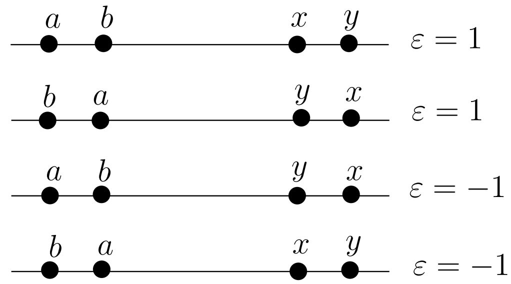

FIGURE 2. Construction of ε

Besides, still for $j \geq 1$, we have

$$A_j(t) + qB_{j-1}(t) = u^j + \left(\frac{q}{u}\right)^j + q\frac{u^{j-1} - \left(\frac{q}{u}\right)^{j-1}}{u - \frac{q}{u}}$$

$$= \frac{1}{u - \frac{q}{u}}\left(u^{j+1} - qu^{j-1} + q\left(\frac{q}{u}\right)^{j-1} - \left(\frac{q}{u}\right)^{j+1} + qu^{j-1} - q\left(\frac{q}{u}\right)^{j-1}\right)$$
$$= B_{j+1}(t).$$

As B_{j+1} and B_{j-1} are polynomials, so is A_j. $\square$

4.2. **Geometric functions on $X_1 \times X_1$.** We will use the above polynomial functions to define bilinear forms on the space $\mathcal{D}(X_1)$ of finitely supported functions on X_1. This definition will also require us to use some notation to describe the respective positions of two edges.

First, we introduce a notion of the distance between two elements of X_1. For ab and xy in X_1, we set

$$\delta(ab, xy) = \max(d(a,x), d(b,x), d(a,y), d(b,y)) - 1.$$

This number is non-negative and satisfies the relations

$$\delta(ab, xy) = \delta(xy, ab) = \delta(ba, xy).$$

One can check that δ actually defines a distance on the set of non oriented edges, that is, the quotient of X_1 by the involution $ab \mapsto ba$. We shall not use this fact.

We also introduce a function ε that checks whether two edges have compatible orientations or not. For ab in X_1, we set $\varepsilon(ab, ab) = 1$, $\varepsilon(ab, ba) = -1$ and, for xy in X_1 with $\delta(ab, xy) \geq 1$,

$$\varepsilon(ab, xy) = 1 \qquad \text{if } b, x \in [ay] \text{ or } a, y \in [bx],$$
$$\varepsilon(ab, xy) = -1 \qquad \text{if } b, y \in [ax] \text{ or } a, x \in [by]$$

(see Figure 2). We have the relations

$$\varepsilon(ab, xy) = \varepsilon(xy, ab) = -\varepsilon(ba, xy).$$

4.3. **Spectral bilinear forms on $\mathcal{D}(X_1)$.** We now begin the construction of the elements of the Plancherel formula.

For t in $\mathbb{C}$, we define a function χ_t on $X_1 \times X_1$ as follows. For ab in X_1, we set

$$\chi_t(ab, ab) = 1 \text{ and } \chi_t(ab, ba) = 0.$$

For ab and xy in X_1 with $j = \delta(ab, xy) \geq 1$, we set

$$\begin{aligned} \chi_t(ab, xy) &= \frac{1}{2q^j} A_j(t) && \text{if } \varepsilon(ab, xy) = 1 \\ \chi_t(ab, xy) &= \frac{q-1}{2q^j} B_j(t) && \text{if } \varepsilon(ab, xy) = -1. \end{aligned}$$

Later, we shall need the following easy bound:

Lemma 4.2. *Let ab and xy be in X_1 and t, u be in $\mathbb{C}$ with $(q+1)tu = q + u^2$. We have*

$$\chi_t(ab, xy) \leq (\delta(ab, xy) + 1) \max \left(|u|^{-\delta(ab,xy)}, \left| \frac{q}{u} \right|^{-\delta(ab,xy)} \right).$$

Proof. Indeed, for $j \geq 1$, we have

$$q^{-j} |A_j(t)| = \left| u^{-j} + \left(\frac{q}{u} \right)^{-j} \right| \leq 2 \max \left(|u|^{-j}, \left| \frac{q}{u} \right|^{-j} \right).$$

For $j \geq 0$, we have

$$q^{-j} |B_j(t)| = q^{-j} \left| \sum_{h=0}^{j-1} u^h \left(\frac{q}{u} \right)^{j-1-h} \right| \leq \frac{j}{q} \max \left(|u|^{-j}, \left| \frac{q}{u} \right|^{-j} \right).$$

□

By abuse of notation, we still write χ_t for the symmetric bilinear form on $\mathcal{D}(X_1)$ defined by

$$\chi_t(f, g) = \sum_{ab, xy \in X_1} \chi_t(ab, xy) f(ab) g(xy), \quad f, g \in \mathcal{D}(X_1).$$

This bilinear form will be used to describe the continuous part of the spectrum in the Plancherel formula. We first relate it to natural operations on functions on X_1.

For f a function on X_1 and xy in X_1, we write

$$Rf(xy) = \sum_{\substack{z \sim x \\ z \neq y}} f(xz) \text{ and } Sf(xy) = f(yx). \tag{4.1}$$

We have $R^2 = q + (q-1)R$ and $S^2 = 1$, so that we also set

$$P = \frac{1}{q+1}(RS + SR - (q-1)S),$$

as in Subsection III.3.1. Then, P commutes with both R and S. A direct computation shows

Lemma 4.3. *Let f, g be in $\mathcal{D}(X_1)$. For t in $\mathbb{C}$, we have*

$$\begin{aligned} \chi_t(Rf, g) &= \chi_t(f, Rg), \\ \chi_t(Sf, g) &= \chi_t(f, Sg) \\ \textit{and } \chi_t(Pf, g) &= t\chi_t(f, g). \end{aligned}$$

4.4. **Special spectral bilinear forms on $\mathcal{D}(X_1)$.** To describe the discrete part of the spectrum in the Plancherel formula, we introduce two bilinear forms that are related to the special representations of the group of automorphisms of X (see [3]).

For ab, xy in X_1, we set

$$\begin{aligned} \chi_1^{\mathrm{sp}}(ab, xy) &= \varepsilon(ab, xy) q^{-\delta(ab,xy)}, \\ \chi_{(-1)}^{\mathrm{sp}}(ab, xy) &= (-q)^{-\delta(ab,xy)}. \end{aligned}$$

Again, for f, g in $\mathcal{D}(X_1)$, we write

$$\begin{aligned} \chi_1^{\mathrm{sp}}(f, g) &= \sum_{ab, xy \in X_1} \chi_1^{\mathrm{sp}}(ab, xy) f(ab) g(xy), \\ \text{and } \chi_{(-1)}^{\mathrm{sp}}(f, g) &= \sum_{ab, xy \in X_1} \chi_{(-1)}^{\mathrm{sp}}(ab, xy) f(ab) g(xy). \end{aligned}$$

We get

Lemma 4.4. *Let f, g be in $\mathcal{D}(X_1)$. We have*

$$\begin{aligned} \chi_1^{\mathrm{sp}}(Rf, g) &= -\chi_1^{\mathrm{sp}}(f, g), & \chi_{(-1)}^{\mathrm{sp}}(Rf, g) &= -\chi_{(-1)}^{\mathrm{sp}}(f, g), \\ \chi_1^{\mathrm{sp}}(Sf, g) &= -\chi_1^{\mathrm{sp}}(f, g), & \chi_{(-1)}^{\mathrm{sp}}(Sf, g) &= \chi_{(-1)}^{\mathrm{sp}}(f, g), \\ \chi_1^{\mathrm{sp}}(Pf, g) &= \chi_1^{\mathrm{sp}}(f, g) & \textit{and } \chi_{(-1)}^{\mathrm{sp}}(Pf, g) &= -\chi_{(-1)}^{\mathrm{sp}}(f, g). \end{aligned}$$

4.5. **Statement of the formula.** Denote by $\langle ., . \rangle_2$ the standard scalar product on $\mathcal{D}(X_1)$, that is,

$$\langle f, g \rangle_2 = \sum_{ab \in X_1} f(ab) g(ab), \quad f, g \in \mathcal{D}(X_1).$$

The Plancherel formula reads as

Proposition 4.5. *For any t in $\mathcal{I}_q$, the symmetric bilinear form χ_t is non-negative on $\mathcal{D}(X_1)$. So are the symmetric bilinear forms χ_1^{sp} and $\chi_{(-1)}^{\mathrm{sp}}$.*

Let f, g be in $\mathcal{D}(X_1)$. We have

$$\begin{aligned}(4.2)\quad & \langle f, g\rangle_2 = \\ & \frac{2}{q+1}\int_{\mathcal{I}_q}\chi_t(f,g)\mathrm{d}\mu_q(t) + \frac{q-1}{2(q+1)}\chi_1^{\mathrm{sp}}(f,g) + \frac{q-1}{2(q+1)}\chi_{(-1)}^{\mathrm{sp}}(f,g).\end{aligned}$$

As in Subsection III.3.3, we wrote $\mathcal{I}_q = \left[-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}\right]$ and we let μ_q be the Borel probability measure on $\mathcal{I}_q$ which is absolutely continuous with respect to Lebesgue measure, with density $t \mapsto \frac{q+1}{2\pi}\frac{\sqrt{4q-(q+1)^2t^2}}{1-t^2}$.

Note that, by Lemma 4.1, for f, g in $\mathcal{D}(X_1)$, the function $t \mapsto \chi_t(f,g)$ is polynomial, so that the integral in (4.2) makes sense.

4.6. **Functional relations.** The proof of Proposition 4.5 essentially relies on the computation of the integrals of the functions A_j and B_j, $j \geq 0$, with respect to the measure μ_q. We will actually establish other related facts about them that we will need later in the article.

We begin by defining a third family of polynomial functions. We set

$$C_0 = 1 \text{ and } C_j = B_{j+1} - B_{j-1}, \quad j \geq 1. \tag{4.3}$$

Lemma 4.6. *For any $j \geq 0$, C_j has degree j and the familly $(C_j)_{j\geq 0}$ is an orthogonal basis of $\mathbb{R}[t]$ with respect to the scalar product of the Lebesgue space $\mathrm{L}^2(\mu_q)$. For any $j \geq 1$, we have*

$$\int_{\mathcal{I}_q} C_j(t)^2 \mathrm{d}\mu_q(t) = (q+1)q^{j-1}.$$

This directly follows from the study of the spherical transform of X in [3]. Below, we give a direct proof in our language.

Proof. We have $B_0 = 0$ and $B_1 = 1$. This, together with the formula $(q+1)tB_j(t) = B_{j+1}(t) + qB_{j-1}(t)$, $j \geq 1$, $t \in \mathbb{R}$, from Lemma 4.1 shows that B_{j+1} has degree j for any $j \geq 0$. This implies that C_j also has degree j. In particular $(C_j)_{j\geq 0}$ is a basis of the vector space $\mathbb{R}[t]$.

To check that this basis is actually orthogonal and to compute the $\int_{\mathcal{I}_q} C_j(t)^2\mathrm{d}\mu_q(t)$, $j \geq 1$, we will use the Plancherel formula for the model operators established in Proposition III.4.2.

We adopt temporarily the notation of Subsection III.4.1 and we claim that, for $j \geq 0$, the spectral transforms of the sequences $\mathbf{1}_{2j}$ and $\mathbf{1}_{2j+1}$

in $\mathbb{R}^{(\mathbb{N})}$ are defined by

$$\widehat{\mathbf{1}_{2j}}(t) = \begin{pmatrix} B_{j+1}(t) \\ -B_j(t) \end{pmatrix} \text{ and } \widehat{\mathbf{1}_{2j+1}}(t) = \begin{pmatrix} -B_j(t) \\ B_{j+1}(t) \end{pmatrix}, \quad t \in \mathbb{R}. \tag{4.4}$$

This, we show by induction on j. For $j = 0$, by Proposition III.4.2, we have $\widehat{\mathbf{1}}_0(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and, as $\mathbf{1}_1 = S_{++}\mathbf{1}_0$,

$$\widehat{\mathbf{1}}_1(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now, if $j \geq 1$ and (4.4) holds for $j-1$, as $\mathbf{1}_{2j} = R_{++}\mathbf{1}_{2j-1}$, Proposition III.4.2 gives

$$\begin{aligned} \widehat{\mathbf{1}_{2j}}(t) = \begin{pmatrix} q & (q+1)t \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -B_{j-1}(t) \\ B_j(t) \end{pmatrix} &= \begin{pmatrix} (q+1)tB_j(t) - qB_{j-1}(t) \\ -B_j(t) \end{pmatrix} \\ &= \begin{pmatrix} B_{j+1}(t) \\ -B_j(t), \end{pmatrix} \end{aligned}$$

where the latter identity follows from Lemma 4.1. Also, as $\mathbf{1}_{2j+1} = S_{++}\mathbf{1}_{2j}$, we get $\widehat{\mathbf{1}_{2j+1}}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \widehat{\mathbf{1}_{2j}}(t)$ and hence (4.4) holds for j. Therefore, it holds for any j.

Set $x_0 = \mathbf{1}_0$ and $x_j = \mathbf{1}_{2j} + \mathbf{1}_{2j-1}$, $j \geq 1$. From (4.3) and (4.4), we get, for any $j \geq 0$,

$$\widehat{x_j}(t) = \begin{pmatrix} C_j(t) \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Besides, for the scalar product of Subsection III.3.4, the family $(x_j)_{j\geq 0}$ is orthogonal and we have

$$\langle x_0, x_0 \rangle_+ = 1 \text{ and } \langle x_j, x_j \rangle_+ = (q+1)q^{j-1}, \quad j \geq 1.$$

The conclusion now follows from the Plancherel formula of Proposition III.5.2 and the definition of the matrix $a_{++}(t)$, $t \in \mathbb{R}$, in Subsection III.3.1. □

We can express the families $(A_j)_{j\geq 0}$ and $(B_j)_{j\geq 0}$ in the orthogonal basis $(C_j)_{j\geq 0}$.

Corollary 4.7. *For any $j \geq 0$, we have*

$$A_j = C_j - (q-1) \sum_{\substack{0 \leq h < j \\ j-h even}} C_h$$

$$B_j = \sum_{\substack{0 \leq h < j \\ j-h odd}} C_h.$$

Proof. The formula for $(B_j)_{j \geq 0}$ directly follows from the definition of $(C_j)_{j \geq 0}$. The formula for $(A_j)_{j \geq 0}$ is obtained from the latter and the relation $A_j = B_{j+1} - qB_{j-1}$ in Lemma 4.1. □

This implies the following integral computation that will be used in the proof of Proposition 4.5.

Corollary 4.8. *Let $j \geq 2$ be even. We have*

$$\int_{\mathcal{I}_q} A_j \mathrm{d}\mu_q = -(q-1) \text{ and } \int_{\mathcal{I}_q} B_j \mathrm{d}\mu_q = 0.$$

Let $j \geq 1$ be odd. We have

$$\int_{\mathcal{I}_q} A_j \mathrm{d}\mu_q = 0 \text{ and } \int_{\mathcal{I}_q} B_j \mathrm{d}\mu_q = 1.$$

We can now give the

Proof of Proposition 4.5. We will check that, for every ab, xy in X_1, we have

$$\frac{2}{q+1} \int_{\mathcal{I}_q} \chi_t(ab, xy) \mathrm{d}\mu_q(t) + \frac{q-1}{2(q+1)} \chi_1^{\mathrm{sp}}(ab, xy) + \frac{q-1}{2(q+1)} \chi_{(-1)}^{\mathrm{sp}}(ab, xy) = \mathbf{1}_{ab=xy}. \tag{4.5}$$

Indeed, assume first $ab = xy$. Then, by construction, for any t in $\mathbb{R}$, we have $\chi_t(ab, ab) = 1$ and also $\chi_1^{\mathrm{sp}}(ab, ab) = 1 = \chi_{(-1)}^{\mathrm{sp}}(ab, ab)$. We get equality in (4.5).

Assume $ab = yx$. We have $\chi_t(ab, ba) = 0$ for any t in $\mathbb{R}$ whereas $\chi_1^{\mathrm{sp}}(ab, ab) = 1$ and $\chi_{(-1)}^{\mathrm{sp}}(ab, ab) = -1$. Again, (4.5) holds.

We now check the cases where $j = \delta(ab, xy) \geq 1$.

If $\varepsilon(ab, xy) = 1$, we have $\chi_t(ab, xy) = \frac{1}{2} q^{-j} A_j(t)$, hence, by Corollary 4.8,

$$\int_{\mathcal{I}_q} \chi_t(ab, xy) \mathrm{d}\mu_q(t) = -\frac{1}{4} q^{-j} (q-1)((-1)^j + 1).$$

The definitions of χ_1^{sp} and $\chi_{(-1)}^{\mathrm{sp}}$ give

$$\chi_1^{\mathrm{sp}}(ab,xy) = q^{-j} \text{ and } \chi_{(-1)}^{\mathrm{sp}}(ab,xy) = (-q)^j,$$

hence the left hand-side of (4.5) vanishes.

Finally, if $\varepsilon(ab,xy) = -1$, we have $\chi_t(ab,xy) = \frac{1}{2}(q-1)q^{-j}B_j(t)$, hence, by Corollary 4.8,

$$\int_{\mathcal{I}_q} \chi_t(ab,xy)\mathrm{d}\mu_q(t) = \frac{1}{4}q^{-j}(q-1)(1-(-1)^j).$$

By definition,

$$\chi_1^{\mathrm{sp}}(ab,xy) = -q^{-j} \text{ and } \chi_{(-1)}^{\mathrm{sp}}(ab,xy) = (-q)^j,$$

and (4.5) is valid.

Therefore, we have shown that (4.5) holds for any value of ab and xy, that is, equivalently, (4.2) holds for any functions f,g in $\mathcal{D}(X_1)$. It only remains to prove that the bilinear forms χ_1^{sp}, $\chi_{(-1)}^{\mathrm{sp}}$ and χ_t, $t \in \mathcal{I}_q$, are non-negative. Fix f in $\mathcal{D}(X_1)$. Let φ be in $\mathbb{R}[t]$, a polynomial function. By (4.2), Lemma 4.3 and Lemma 4.4, we have

$$\frac{2}{q+1}\int_{\mathcal{I}_q} \varphi(t)^2\chi_t(f,f)\mathrm{d}\mu_q(t) + \frac{q-1}{2(q+1)}\varphi(1)^2\chi_1^{\mathrm{sp}}(f,f) \\ + \frac{q-1}{2(q+1)}\varphi(-1)^2\chi_{(-1)}^{\mathrm{sp}}(f,f) = \langle \varphi(P)f, \varphi(P)f\rangle_2 \geq 0.$$

Elementary real analysis arguments show that we have $\chi_1^{\mathrm{sp}}(f,f) \geq 0$, $\chi_{(-1)}^{\mathrm{sp}}(f,f) \geq 0$ and $\chi_t(f,f) \geq 0$ for μ_q-almost any t in $\mathcal{I}_q$. The conclusion follows as $\chi_t(f,f)$ depends continuously on t. $\square$

5. Fundamental bilinear maps

In Subsection II.5.3, we have established a natural correspondance between Γ-invariant bilinear forms on $\overline{\mathcal{D}}(\partial X)$ and (ι, T)-invariant distributions on the space $\Gamma\backslash\mathscr{S}$, where $\mathscr{S}$ is the space of parametrized geodesic lines of X, T is the time shift and ι the natural involution.

We will now construct a dual object that is a bilinear map from the space $\mathcal{H}_\infty$ of Γ-invariant ∞-pseudofunctions towards the space of smooth functions on $\Gamma\backslash\mathscr{S}$. We will then use the Plancherel formula for X_1, Proposition 4.5, to split this bilinear maps into spectral components.

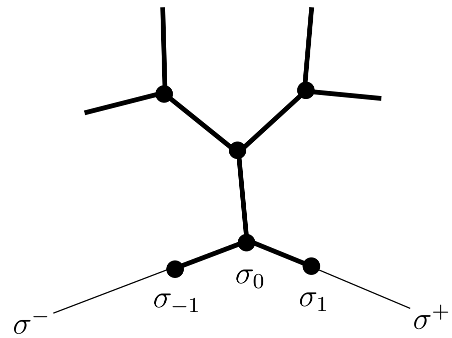

FIGURE 3. The set X_1^σ

5.1. **Global fundamental bilinear map.** We define the global fundamental bilinear map $\mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathcal{D}(\Gamma\backslash\mathscr{S})$, where $\mathcal{H}_\infty$ is the space of Γ-invariant ∞-pseudofunctions introduced in Subsection III.2.3.

Let $\sigma = (\sigma_h)_{h\in\mathbb{Z}}$ be a parametrized geodesic line (see Subsection I.2.1). We shall always denote by σ^+ and σ^- the endpoints of σ in ∂X. More precisely, we will write σ^+ to be the endpoint of the geodesic ray $(\sigma_h)_{h\geq 0}$ and σ^- to be the endpoint of the geodesic ray $(\sigma_{-h})_{h\geq 0}$. We also write $\langle\sigma\rangle$ for the set $\{\sigma_h | h \in \mathbb{Z}\}$, where we forget the parametrization. Thus, we have $\langle\sigma\rangle = (\sigma^-\sigma^+)$.

We let X_1^σ be the subset of X_1 defined by

$$X_1^\sigma = \{xy \in X_1 | [x, \sigma_0] \cap \langle\sigma\rangle = \{\sigma_0\}\}$$

(see Figure 3). The main interest of this set is that it allows to define partitions of X_1 associated to the orbit of σ under the time shift T. Indeed, we have

Lemma 5.1. *Let σ be a parametrized geodesic line. We have*

$$X_1 = \bigsqcup_{h\in\mathbb{Z}} X_1^{T^h\sigma},$$

that is, any xy in X_1 belongs to exactly one of the sets $X_1^{T^h\sigma}$, $h \in \mathbb{Z}$.

Proof. Indeed, writing $\sigma = (\sigma_h)_{h\in\mathbb{Z}}$, we let h be the unique element of $\mathbb{Z}$ with

$$d(x, \sigma_h) = \min_{j\in\mathbb{Z}} d(x, \sigma_j).$$

$\square$

Let H be an ∞-pseudofunction as in Subsection III.2.3, that is, H is a map $xy \mapsto H_{xy}$ from X_1 to the space $\overline{\mathcal{D}}(\partial X)$. Thus, if xy is in

X_1, H_{xy} is a smooth function on ∂X which is defined up to an additive constant. In particular, for any ξ, η in ∂X, the number

$$\Delta H_{xy}(\xi, \eta) = H_{xy}(\xi) - H_{xy}(\eta)$$

is well defined. If H is a k-pseudofunction for some $k \geq -1$, we write $\Delta H_{xy}(\xi, \eta)$ for $\Delta(H^{>\infty})(\xi, \eta)$ (see Subsection III.2.3 for the notation).

Now, note that, as Γ has finitely many orbits in X_1, we have $\mathcal{H}_\infty = \bigcup_{k \geq -1} \mathcal{H}_k^{>\infty}$. In particular, if H is a Γ-invariant ∞-pseudofunction, there exists $\ell \geq 0$ such that, for any $\xi \neq \eta$ in ∂X, we have $\Delta H_{xy}(\xi, \eta) = 0$ for any xy in X_1 with $d(x, (\xi\eta)) \geq \ell$. This justifies the following definition.

Definition 5.2. We define the global fundamental bilinear map

$$\Phi : \mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathcal{D}(\Gamma \backslash \mathscr{S})$$

as follows. For any H, J in $\mathcal{H}_\infty$ and any σ in $\mathscr{S}$, we set

$$\Phi(H, J)(\sigma) = \frac{1}{2} \sum_{xy \in X_1^\sigma} \Delta H_{xy}(\sigma^+, \sigma^-) \Delta J_{xy}(\sigma^+, \sigma^-).$$

Note that, for H, J as above, the function $\Phi(H, J)$ is ι-invariant.

The fundamental bilinear map has to be thought of as a map towards the space of cohomology classes of smooth functions on $\Gamma \backslash \mathscr{S}$, where cohomology classes were defined in Subsection I.2.3 and Subsection 2.1 (both definitions being compatible by Corollary 2.7). In this sense, the next lemma says that the natural operators of the space of ∞-pseudofunctions are symmetric with respect to Φ.

Lemma 5.3. *Let H, J be in $\mathcal{H}_\infty$. Then, we have $\Phi(RH, J) = \Phi(H, RJ)$ and the smooth functions $\Phi(SH, J)$ and $\Phi(H, SJ)$ are cohomologuous. So are the smooth functions $\Phi(PH, J)$ and $\Phi(H, PJ)$*

See Subsection III.2.5 for the definition of R and S. As usual, we write $P = \frac{1}{q+1}(RS + SR - (q-1)S)$.

Proof. In view of the definition of P, it suffices to prove the statements for R and S. Fix σ in $\mathscr{S}$. The fact that an edge xy of X_1 belongs to X_1^σ only depends on x. This directly implies that $\Phi(RH, J)(\sigma) = \Phi(H, RJ)(\sigma)$. Besides, we have

$$X_1^\sigma \smallsetminus \{yx | xy \in X_1^\sigma\} = \{\sigma_0\sigma_1, \sigma_0\sigma_{-1}\},$$

which implies

$$\begin{aligned}2\Phi(SH,J)(\sigma)-2\Phi(H,SJ)(\sigma) &= \Delta H_{\sigma_1\sigma_0}(\sigma^+,\sigma^-)\Delta J_{\sigma_0\sigma_1}(\sigma^+,\sigma^-)\\ &\quad+\Delta H_{\sigma_{-1}\sigma_0}(\sigma^+,\sigma^-)\Delta J_{\sigma_0\sigma_{-1}}(\sigma^+,\sigma^-)\\ &\quad-\Delta H_{\sigma_0\sigma_1}(\sigma^+,\sigma^-)\Delta J_{\sigma_1\sigma_0}(\sigma^+,\sigma^-)\\ &\quad-\Delta H_{\sigma_0\sigma_{-1}}(\sigma^+,\sigma^-)\Delta J_{\sigma_{-1}\sigma_0}(\sigma^+,\sigma^-)\\ &=\Psi(H,J)(T\sigma)-\Psi(H,J)(\sigma),\end{aligned}$$

where

$$\begin{aligned}\Psi(H,J)(\sigma)=\Delta H_{\sigma_0\sigma_{-1}}(\sigma^+,\sigma^-)&\Delta J_{\sigma_{-1}\sigma_0}(\sigma^+,\sigma^-)\\ &-\Delta H_{\sigma_{-1}\sigma_0}(\sigma^+,\sigma^-)\Delta J_{\sigma_0\sigma_{-1}}(\sigma^+,\sigma^-).\end{aligned}$$

The Lemma follows. □

5.2. **Invariant distributions.** We will now relate the fundamental bilinear map Φ to the constructions of Subsection II.5.3. Let p be a Γ-invariant symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$. As in Section III.7, we still write p for the symmetric bilinear form on $\mathcal{H}_\infty$ defined by, for any (H,J) in $\mathcal{H}_\infty$,

$$p(H,J)=\sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}p(H_{xy},J_{xy}).$$

We use the same notation for the associated bilinear forms on $\mathcal{H}_k$, $k\geq -1$.

Proposition 5.4. *Let p be a Γ-invariant symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$ and θ be the associated (ι,T)-invariant distribution on $\Gamma\backslash\mathscr{S}$. For any (H,J) in $\mathcal{H}_\infty$, we have*

$$p(H,J)=\langle\theta,\Phi(H,J)\rangle.$$

The proof is a consequence of the discussion in Subsection II.5.3 and of the following purely combinatorial

Lemma 5.5. *Let G be a discrete group, A be a discrete set and U be a totally discontinuous locally compact topological space. Assume we are given actions of G on A and U with the following properties: the action of G on A has finite stabilizers and finitely many orbits; the action of G on U is proper and cocompact.*

Pick a distribution θ on $G\backslash U$ and denote by $\widetilde{\theta}$ the associated G-invariant distribution on U. Let $\varphi : A\times U\to\mathbb{R}$ be a G-invariant function which is locally constant in the following uniform way: for every u in U, there exists a neighborhood V of u in U such that $\varphi(a,.)$ is constant on V for every a in A. Assume for every a in A, $\varphi(a,.)$ has *compact support in U and, for every u in U, $\varphi(.,u)$ has finite support in A. Then, the function $u \mapsto \sum_{a\in A}\varphi(a,u)$ is locally constant on U and we have*

$$\sum_{a\in G\backslash A}\frac{1}{|G_a|}\left\langle\widetilde{\theta},\varphi(a,.)\right\rangle=\left\langle\theta,\sum_{a\in A}\varphi(a,.)\right\rangle.$$

Proof. Recall from Subsection II.5.3 that $\widetilde{\theta}$ and θ are related as follows: if ψ is in $\mathcal{D}(U)$ and $\overline{\psi}(u)=\sum_{g\in G}\psi(gu)$ for u in U, then we have $\langle\widetilde{\theta},\psi\rangle=\langle\theta,\overline{\psi}\rangle$.

Fix a system of representatives S for the action of G on A, that is, $S\subset A$ is such that $A=GS$ and $Ga\cap S=\{a\}$ for every a in S. By definition, we have

$$\sum_{a\in G\backslash A}\frac{1}{|G_a|}\left\langle\widetilde{\theta},\varphi(a,.)\right\rangle=\sum_{a\in S}\frac{1}{|G_a|}\left\langle\widetilde{\theta},\varphi(a,.)\right\rangle=\left\langle\widetilde{\theta},\sum_{a\in S}\frac{1}{|G_a|}\varphi(a,.)\right\rangle.$$

For u in U, set $\psi(u)=\sum_{a\in S}\frac{1}{|G_a|}\varphi(a,u)$. The assumption on φ implies that ψ is in $\mathcal{D}(U)$. As S is a system of representatives for the action of G on A, for u in U, we have

$$\overline{\psi}(u)=\sum_{g\in G}\sum_{a\in S}\frac{1}{|G_a|}\varphi(ga,u)=\sum_{a\in A}\varphi(a,u).$$

The Lemma follows. □

Proof of Proposition 5.4. Still as in Subsection II.5.3, we associate to θ a Γ-invariant and (ι,T)-invariant distribution $\widetilde{\theta}$ on $\mathscr{S}$. In the same way, as the quotient of $\mathscr{S}$ by the time shift T may be identified with ∂^2X, we associate to $\widetilde{\theta}$ a distribution θ_{∂^2X} on ∂^2X. Note that $\widetilde{\theta}$ and θ_{∂^2X} are Γ-invariant and symmetric.

Let H,J be in $\mathcal{H}_\infty$ and xy be in X_1. By Lemma II.5.9, we have

$$p(H_{xy},J_{xy})=\frac{1}{2}\theta_{\partial^2X}((\xi,\eta)\mapsto\Delta H_{xy}(\xi,\eta)\Delta J_{xy}(\xi,\eta)).$$

By Lemma 5.1, we get

$$p(H_{xy},J_{xy})=\frac{1}{2}\widetilde{\theta}\left(\sigma\mapsto\Delta H_{xy}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)\mathbf{1}_{xy\in X_1^\sigma}\right).$$

Thus, we obtain

$$p(H,J)=\\ \frac{1}{2}\sum_{xy\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\widetilde{\theta}\left(\sigma\mapsto\Delta H_{xy}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)\mathbf{1}_{xy\in X_1^\sigma}\right).$$

The Proposition follows by applying Lemma 5.5 to the action of Γ on $A = X_1$ and $U = \mathscr{S}$ and to the function

$$\varphi : X_1 \times \mathscr{S} \to \mathbb{R}, (xy, \sigma) \mapsto \Delta H_{xy}(\sigma^+, \sigma^-)\Delta J_{xy}(\sigma^+, \sigma^-)\mathbf{1}_{xy \in X_1^\sigma}.$$

□

5.3. **Spectral fundamental bilinear maps.** We will use the Plancherel formula for X_1, Proposition 4.5, to decompose the global fundamental bilinear map Φ into spectral components. We start by introducing those components.

The spectral components will depend from a complex parameter t. We will only define these components as t ranges in the domain delimited by a certain ellipse in $\mathbb{C}$. We define $\mathcal{E}_q \subset \mathbb{C}$ as the set of those t in $\mathbb{C}$ such that

$$(\Re t)^2 + \left(\frac{q+1}{q-1}\Im t\right)^2 < 1.$$

Note that $\mathcal{E}_q \cap \mathbb{R} = (-1, 1)$. Elementary computations show

Lemma 5.6. *Let t be in $\mathbb{C}$. Then t belongs to $\mathcal{E}_q$ if and only if the solutions of the equation $(q+1)tu = q + u^2$ satisfy $1 < |u| < q$.*

Proof. For u in $\mathbb{C}^*$, write $u = \rho e^{i\theta}$ for some $\rho > 0$ and θ in $\mathbb{R}$. Then, we have

$$t = \frac{1}{q+1}\left(u + \frac{q}{u}\right) = \frac{\rho + \frac{q}{\rho}}{q+1}\cos\theta + \frac{\rho - \frac{q}{\rho}}{q+1}\sin\theta.$$

Thus, if $\rho = \sqrt{q}$, when u ranges in the circle $\{|u| = \sqrt{q}\}$, t ranges in the interval $\mathcal{I}_q$; if $\rho \neq \sqrt{q}$ and u ranges in the circle $\{|u| = \rho\}$, t ranges in the ellipse

$$\left(\frac{q+1}{\rho + \frac{q}{\rho}}\Re t\right)^2 + \left(\frac{q+1}{\rho - \frac{q}{\rho}}\Im t\right)^2 = 1.$$

The conclusion follows. □

Corollary 5.7. *Let H be in $\mathcal{H}_\infty$ and K be a compact subset of $\mathcal{E}_q$. We have*

$$\sup_{\substack{(\xi,\eta)\in\partial^2 X\\ ab\in X_1\\ t\in K}} \sum_{xy\in X_1} |\chi_t(ab, xy)\Delta H_{xy}(\xi, \eta)| < \infty.$$

Proof. By Lemma 5.6, we can find $\beta > 0$ such that, for any t in K and u in $\mathbb{C}$ with $(q+1)tu = q + u^2$, we have $1 + \beta \leq |u| \leq q - \beta$. We fix such t, u. As $\mathcal{H}_\infty = \bigcup_{\ell \geq 1} \mathcal{H}_{2\ell}^{>\infty}$, we can assume that H belongs to

$\mathcal{H}_{2\ell}^{>\infty}$ for some $\ell \geq 1$. Let (ξ, η) be in $\partial^2 X$. For xy in X_1, we have $\Delta H_{xy}(\xi, \eta) = 0$ as soon as $d(x, (\xi\eta)) \geq \ell$. Note that we have

$$\sup_{j \geq 0} |\{xy \in X_1 | \delta(ab, xy) = j, d(x, (\xi\eta)) \leq \ell\}| < \infty.$$

Besides, as H is Γ-invariant, we have

$$\sup_{\substack{(\xi,\eta) \in \partial^2 X \\ xy \in X_1}} |\Delta H_{xy}(\xi, \eta)| < \infty.$$

By Lemma 4.2, we get, for some $C > 0$, for any (ξ, η) in $\partial^2 X$ and ab in X_1,

$$\sum_{xy \in X_1} |\chi_t(ab, xy) \Delta H_{xy}(\xi, \eta)| \leq C \sum_{j \geq 0} (j+1) \max\left(|u|^{-j}, \left|\frac{q}{u}\right|^{-j} \right).$$

As $1 + \beta \leq |u| \leq q - \beta$, the latter sum is uniformly bounded as t ranges in K. $\square$

The convergence of these sums justifies the following

Definition 5.8. Let t be in $\mathcal{E}_q$. For any H, J in $\mathcal{H}_\infty$ and any σ in $\mathscr{S}$, we set

$$\Phi_t(H, J)(\sigma) = \frac{1}{2} \sum_{\substack{ab \in X_1^\sigma \\ xy \in X_1}} \chi_t(ab, xy) \Delta H_{ab}(\sigma^+, \sigma^-) \Delta J_{xy}(\sigma^+, \sigma^-).$$

We call Φ_t the spectral fundamental bilinear map associated to t.

Note that, for t and H, J as above, the function $\Phi_t(H, J)$ is ι-invariant.

The spectral fundamental bilinear map creates Hölder continuous functions. For $0 < \alpha < 1$, we let $\mathscr{H}_\alpha$ denote the space of α-Hölder continuous functions on $\Gamma \backslash \mathscr{S}$, equipped with the natural norm

$$\|f\|_\alpha = \sup |f| + \sup_{\substack{h \geq 0 \\ \sigma, \tau \in \mathscr{S} \\ \forall |i| \leq h \quad \sigma_i = \tau_i}} \alpha^{-h} |f(\sigma) - f(\tau)|, \quad f \in \mathscr{H}_\alpha.$$

Lemma 5.9. *Let H, J be in $\mathcal{H}_\infty$. For any t in $\mathcal{E}_q$, $\Phi_t(H, J)$ is a Hölder continuous function. More precisely, given an open subset Ω of $\mathbb{C}$, whose closure in $\mathbb{C}$ is contained in $\mathcal{E}_q$, there exists $0 < \alpha < 1$ such that $\Phi_t(H, J)$ belongs to $\mathscr{H}_\alpha$ for any t in Ω and the map $t \mapsto \Phi_t(H, J), \Omega \to \mathscr{H}_\alpha$ is analytic.*

Proof. Let Ω be as in the statement. Then, by Lemma 5.6, we can find $\alpha < 1$ such that, for any t in Ω and u in $\mathbb{C}$ with $(q+1)ty = q + u^2$, we have $|u|^{-1} < \alpha$ and $\left|\frac{u}{q}\right| < \alpha$. The statement then follows by using Lemma 4.2 as in the proof of Corollary 5.7. □

As for the global fundamental bilinear map, we shall think to the spectral fundamental bilinear maps as maps towards the space of cohomology classes of Hölder continuous functions. We get, by analogy with Lemma 4.3 and Lemma 5.3,

Lemma 5.10. *Let H, J be in $\mathcal{H}_\infty$ and t be in $\mathcal{E}_q$. Then we have*

$$\Phi_t(RH, J) = \Phi_t(H, RJ)$$

and the Hölder continuous functions

$$\begin{aligned} \Phi_t(H, J) - \Phi_t(J, H),\\ \Phi_t(SH, J) - \Phi_t(H, SJ)\\ \textit{and } \Phi_t(PH, J) - t\Phi_t(H, J) \end{aligned}$$

are coboundaries.

To avoid lenghty dominations in converging sums, we shall use

Lemma 5.11. *Let f be a Hölder continuous function on $\Gamma\backslash\mathscr{S}$. Assume that there exists a family $(h_n)_{n\in\mathbb{Z}}$ of continuous functions on $\Gamma\backslash\mathscr{S}$ such that*

$$\sup_{\sigma\in\mathscr{S}} \sum_{n\in\mathbb{Z}} |h_n(\sigma)| < \infty \textit{ and } \sup_{\sigma\in\mathscr{S}} \sum_{n\in\mathbb{Z}} |h_n(T^n\sigma)| < \infty$$

and such that $f = \sum_{n\in\mathbb{Z}} h_n \circ T^n - h_n$. Then, f is a coboundary.

Proof. This is a consequence of Livšic Theorem, Proposition 2.1. Indeed, let μ be a T-invariant Borel probability measure on $\Gamma\backslash\mathscr{S}$. By the Dominated Convergence Theorem, the series $\sum_{n\in\mathbb{Z}} h_n \circ T^n - h_n$ converges in $\mathrm{L}^1(\Gamma\backslash\mathscr{S}, \mu)$. As $\int_{\Gamma\backslash\mathscr{S}} (h_n \circ T^n - h_n)\mathrm{d}\mu = 0$ for every n in $\mathbb{Z}$, we get $\int_{\Gamma\backslash\mathscr{S}} f\mathrm{d}\mu = 0$. The conclusion follows from Proposition 2.1. □

Proof of Lemma 5.10. The fact that $\Phi_t(RH, J) = \Phi_t(H, RJ)$ is obtained as in the proof of Lemma 5.3, by using the corresponding property of χ_t from Lemma 4.3.

Let us prove that $\Phi_t(H, J) - \Phi_t(J, H)$ is a coboundary. From Lemma 5.1, we get, for σ in $\mathscr{S}$,

$$\Phi_t(H, J)(\sigma) = \frac{1}{2} \sum_{h\in\mathbb{Z}} \sum_{\substack{ab\in X_1^\sigma\\ xy\in X_1^{T^h\sigma}}} \chi_t(ab, xy)\Delta H_{ab}(\sigma^+, \sigma^-)\Delta J_{xy}(\sigma^+, \sigma^-).$$

By using the symmetry of χ_t, we get

$$\Phi_t(H,J)(\sigma) - \Phi_t(J,H)(\sigma) = \sum_{h\in\mathbb{Z}} \Xi_{t,h}(H,J)(T^h\sigma) - \Xi_{t,h}(H,J)(\sigma),$$

where, for h in $\mathbb{Z}$,

$$\Xi_{t,h}(H,J)(\sigma) = \frac{1}{2} \sum_{\substack{ab\in X_1^{T^{-h}\sigma} \\ xy\in X_1^{\sigma}}} \chi_t(ab,xy)\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-).$$

Due to Corollary 5.7, we have

$$\sup_{\sigma\in\mathscr{S}} \sum_{h\in\mathbb{Z}} |\Xi_{t,h}(H,J)(\sigma)| < \infty \text{ and } \sup_{\sigma\in\mathscr{S}} \sum_{h\in\mathbb{Z}} \left|\Xi_{t,h}(H,J)(T^h\sigma)\right| < \infty.$$

Therefore, by Lemma 5.11, $\Phi_t(H,J) - \Phi_t(J,H)$ is a coboundary.

We now study $\Phi_t(SH,J) - \Phi_t(H,SJ)$. By using the relation

$$\chi_t(ba,xy) = \chi_t(ab,yx),$$

for ab, xy in X_1, and by reasoning as in the proof of Lemma 5.3, we obtain, for σ in $\mathscr{S}$,

$$\Phi_t(SH,J) - \Phi_t(H,SJ) = \Psi_t(H,J)(\sigma) - \Psi_t(H,J)(T\sigma),$$

with

$$\begin{aligned}\Psi_t(H,J)(\sigma) = \frac{1}{2} \sum_{xy\in X_1} \big(&\Delta H_{\sigma_{-1}\sigma_0}(\sigma^+,\sigma^-)\chi_t(\sigma_{-1}\sigma_0,xy) \\ &-\Delta H_{\sigma_0\sigma_{-1}}(\sigma^+,\sigma^-)\chi_t(\sigma_0\sigma_{-1},xy))\big)\,\Delta J_{xy}(\sigma^+,\sigma^-).\end{aligned}$$

By reasoning as in the proof of Lemma 5.9, one can show that the function $\Psi_t(H,J)$ is Hölder continuous on $\Gamma\backslash\mathscr{S}$. The conclusion follows.

Finally, notice from Lemma 4.3 that we have $\Phi_t(J,PH) = t\Phi(H,J)$. As the above implies that $\Phi_t(PH,J)$ is cohomologous to $\Phi_t(J,PH)$, the last statement follows. □

5.4. **Special spectral fundamental bilinear maps.** We still aim at using the Plancherel formula of X_1 from Proposition 4.5 in order to decompose the global fundamental bilinear map Φ. Thus, we need to introduce special components.

Recall the definition of δ in Subsection 4.2. The convergence in the formulae defining these special components is warranted by

Lemma 5.12. *Let H be in $\mathcal{H}_\infty$. We have*

$$\sup_{\substack{(\xi,\eta)\in\partial^2 X \\ ab\in X_1}} \sum_{xy\in X_1} q^{-\delta(ab,xy)}|\Delta H_{xy}(\xi,\eta)| < \infty.$$

The proof is the same as the one of Corollary 5.7.

Definition 5.13. For any H, J in $\mathcal{H}_\infty$ in $\mathcal{H}_\infty$, and any σ in $\mathscr{S}$, we set

$$\Phi_1^{\mathrm{sp}}(H,J)(\sigma) = \frac{1}{2}\sum_{\substack{ab\in X_1^\sigma\\ xy\in X_1}} \chi_1^{\mathrm{sp}}(ab,xy)\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)$$
$$\Phi_{(-1)}^{\mathrm{sp}}(H,J)(\sigma) = \frac{1}{2}\sum_{\substack{ab\in X_1^\sigma\\ xy\in X_1}} \chi_{(-1)}^{\mathrm{sp}}(ab,xy)\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)$$

We call Φ_1^{sp} and $\Phi_{(-1)}^{\mathrm{sp}}$ the special spectral fundamental bilinear maps.

Again, for H, J as above, the functions $\Phi_1^{\mathrm{sp}}(H,J)$ and $\Phi_{(-1)}^{\mathrm{sp}}(H,J)$ are ι-invariant.

These bilinear maps send $\mathcal{H}_\infty$ to $\mathscr{H}_{q^{-1}}$.

Lemma 5.14. *Let H, J be in $\mathcal{H}_\infty$. Then, the functions $\Phi_1^{\mathrm{sp}}(H,J)$ and $\Phi_{(-1)}^{\mathrm{sp}}(H,J)$ are q^{-1}-Hölder continuous on $\Gamma\backslash\mathscr{S}$.*

Proof. This directly follows from the definition of χ_1^{sp} and $\chi_{(-1)}^{\mathrm{sp}}$ in Subsection 4.4 and from the fact that, for H in $\mathcal{H}_\infty$, xy in X_1 and (ξ,η) in $\partial^2 X$, $\Delta H_{xy}(\xi,\eta)$ is 0 when xy is far enough from the geodesic line $(\xi\eta)$. □

In the same way as for Lemma 5.10, we show

Lemma 5.15. *Let H, J be in $\mathcal{H}_\infty$. Then we have*

$$\Phi_1^{\mathrm{sp}}(RH,J) = -\Phi_1^{\mathrm{sp}}(H,J) \text{ and } \Phi_{(-1)}^{\mathrm{sp}}(RH,J) = -\Phi_{(-1)}^{\mathrm{sp}}(H,J)$$

and the Hölder continuous functions

$$\begin{aligned}
\Phi_1^{\mathrm{sp}}(H,J) - \Phi_1^{\mathrm{sp}}(J,H), &\qquad \Phi_{(-1)}^{\mathrm{sp}}(H,J) - \Phi_{(-1)}^{\mathrm{sp}}(J,H),\\
\Phi_1^{\mathrm{sp}}(SH,J) + \Phi_1^{\mathrm{sp}}(H,J), &\qquad \Phi_{(-1)}^{\mathrm{sp}}(SH,J) - \Phi_{(-1)}^{\mathrm{sp}}(H,J),\\
\Phi_1^{\mathrm{sp}}(PH,J) - \Phi_1^{\mathrm{sp}}(H,J), &\qquad \Phi_{(-1)}^{\mathrm{sp}}(PH,J) + \Phi_{(-1)}^{\mathrm{sp}}(H,J),
\end{aligned}$$

are coboundaries.

5.5. **The Plancherel formula for fundamental bilinear maps.** From the Plancherel formula in X_1, we can decompose the global fundamental bilinear map.

Proposition 5.16. *Let H, J be in $\mathcal{H}_\infty$. For σ in $\mathscr{S}$, we have*

$$\Phi(H,J)(\sigma) = \frac{2}{q+1}\int_{\mathcal{I}_q} \Phi_t(H,J)(\sigma)\mathrm{d}\mu_q(t) \\ + \frac{q-1}{2(q+1)}\Phi_1^{\mathrm{sp}}(H,J)(\sigma) + \frac{q-1}{2(q+1)}\Phi_{(-1)}^{\mathrm{sp}}(H,J)(\sigma).$$

Proof. For ab in X_1 and $(\xi,\eta)\in\partial^2 X$, Corollary 5.7 and Lemma 5.12, together with the Dominated Convergence Theorem give, by the same computation as in the proof of Proposition 4.5,

$$\Delta J_{ab}(\xi,\eta) = \frac{2}{q+1}\int_{\mathcal{I}_q}\sum_{xy\in X_1}\chi_t(ab,xy)\Delta J_{xy}(\xi,\eta)\mathrm{d}\mu_q(t) \\ + \frac{q-1}{2(q+1)}\sum_{xy\in X_1}\chi_1^{\mathrm{sp}}(ab,xy)\Delta J_{xy}(\xi,\eta) \\ + \frac{q-1}{2(q+1)}\sum_{xy\in X_1}\chi_{(-1)}^{\mathrm{sp}}(ab,xy)\Delta J_{xy}(\xi,\eta).$$

The conclusion then follows from the definition of the fundamental bilinear maps. $\square$

Our goal is to use Proposition 5.16 to get a better understanding of the spectral theory of completions of $\overline{\mathcal{D}}(\partial X)$ with respect to Γ-invariant non-negative symmetric bilinear forms. We can already manage the case of representations associated to Radon measures. Recall from Proposition II.5.14 that if ν is a finite (ι,T)-invariant Borel measure on $\Gamma\backslash\mathscr{S}$, the associated Γ-invariant symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$ is non-negative.

Corollary 5.17. *Let ν be a finite (ι,T)-invariant Borel measure on $\Gamma\backslash\mathscr{S}$ and p be the associated Γ-invariant symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$. Then, in the completion of $\mathcal{H}_\infty$ with respect to p, the spectrum of the operator P is $\mathcal{I}_q\cup\{-1,1\}$. The associated spectral measures are absolutely continuous with respect to the Lebesgue measure on $\mathcal{I}_q$.*

Proof. Let H, J be in $\mathcal{H}_\infty$. By Proposition 5.4, we have

$$p(H,J) = \int_{\Gamma\backslash\mathscr{S}}\Phi(H,J)\mathrm{d}\nu.$$

By Lemma 5.9, we can apply Fubini Theorem in the Plancherel formula of Proposition 5.16; this gives

$$\begin{aligned}p(H,J)=&\frac{2}{q+1}\int_{\mathcal{I}_q}\int_{\Gamma\backslash\mathscr{S}}\Phi_t(H,J)\mathrm{d}\nu\mathrm{d}\mu_q(t)\\&+\frac{q-1}{2(q+1)}\int_{\Gamma\backslash\mathscr{S}}\Phi_1^{\mathrm{sp}}(H,J)\mathrm{d}\nu+\frac{q-1}{2(q+1)}\int_{\Gamma\backslash\mathscr{S}}\Phi_{(-1)}^{\mathrm{sp}}(H,J)\mathrm{d}\nu.\end{aligned}$$

As ν is T-invariant, by Lemma 5.10 and Lemma 5.15, for any polynomial function φ in $\mathbb{R}[t]$, we get

$$\begin{aligned}p(\varphi(P)H,J)=&\frac{2}{q+1}\int_{\mathcal{I}_q}\varphi(t)\int_{\Gamma\backslash\mathscr{S}}\Phi_t(H,J)\mathrm{d}\nu\mathrm{d}\mu_q(t)\\&+\frac{q-1}{2(q+1)}\varphi(1)\int_{\Gamma\backslash\mathscr{S}}\Phi_1^{\mathrm{sp}}(H,J)\mathrm{d}\nu\\&+\frac{q-1}{2(q+1)}\varphi(-1)\int_{\Gamma\backslash\mathscr{S}}\Phi_{(-1)}^{\mathrm{sp}}(H,J)\mathrm{d}\nu.\end{aligned}$$

The conclusion follows by standard properties of spectral analysis of self-adjoint operators. □

6. Endpoints series and fundamental bilinear maps

In order to be able to study the spectral fundamental maps, we will now show that, up to a coboundary, they can be written as an endpoints series as in Subsection 3.1.

6.1. Weight of pseudofunctions.

In this Subsection, for $k\geq 0$, we associate to every k-pseudofunction a function on X_k. This construction will later allow us to rewrite the definition of the spectral fundamental maps.

Definition 6.1. Let $k\geq 0$ be an integer and H be a k-pseudofunction. We define the weight $\omega_k(H)$, which is a function on X_k, as follows. Let ab be in X_k and $a_0=a,a_1,\dots,a_k=b$ be the geodesic parametrization of the segment $[ab]$. For c in X, $c\neq b$ we write c_- for the neighbour of c on $[bc]$.

If $k=0$ and H is the 0-pseudofunction associated to the function u on X, we set $\omega_0(H)=u$.

If k is even, $k=2\ell$, $\ell\geq 1$, we set

$$\omega_k(H)(ab)=\Delta H_{a_\ell a_{\ell+1}}(b,a)+\sum_{i=1}^{\ell-1}\sum_{\substack{c\in X\\ [a_{\ell+i}c]\cap[ab]=\{a_{\ell+i}\}\\ d(c,a_{\ell+i})=i}}\Delta H_{cc_-}(b,a_{2i}).$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, we set

$$\omega_k(H)(ab) = \Delta H_{a_\ell a_{\ell+1}}(b, a) + \sum_{i=1}^{\ell} \sum_{\substack{c \in X \\ [a_{\ell+i}c] \cap [ab] = \{a_{\ell+i}\} \\ d(c, a_{\ell+i}) = i}} \Delta H_{cc_-}(b, a_{2i-1}).$$

Later, in Section 6, we will study the weight map ω_k. For the moment we show how it appears naturally when handling fundamental bilinear maps.

6.2. **Initial bilinear maps.** Here, we introduce a notation in order to reformulate the definition of the fundamental bilinear maps and we relate this notation with the weight defined above.

For σ in $\mathscr{S}$, we set

$$X^\sigma = \{x \in X | [x, \sigma_0] \cap \langle \sigma \rangle = \{\sigma_0\}\},$$

so that $X_1^\sigma = \{xy \in X_1 | x \in X^\sigma\}$.

Recall from Subsection 3.1 that, for $k \geq 1$, we let V_k stand for the space of Γ-invariant functions on X_k.

Assume k is even, $k = 2\ell$, $\ell \geq 1$. If H is a k-pseudofunction, for σ in $\mathscr{S}$ and ab in X_1^σ with $d(a, \sigma_0) = r$, the quantity $\Delta H_{ab}^{>\infty}(\sigma^+, \sigma^-)$ is zero if $r \geq \ell$. If $r < \ell$, this quantity only depends on the segment $[\sigma_{-(\ell-r)}, \sigma_{\ell-r}]$. Therefore, for $j \geq 0$, there exists a symmetric bilinear map

$$\kappa_{j,k} : \mathcal{H}_k \times \mathcal{H}_k \to V_{j+k}$$

such that, for any H, J in $\mathcal{H}_k$ and σ in $\mathscr{S}$, one has

$$\sum_{ab \in X_1^\sigma} \Delta H_{ab}^{>\infty}(\sigma^+, \sigma^-) \Delta J_{ab}^{>\infty}(\sigma^+, \sigma^-) = \kappa_{0,k}(H, J)(\sigma_{-\ell}, \sigma_\ell)$$

if $j = 0$ and

$$\begin{aligned}
&\sum_{\substack{a, x \in X^\sigma \\ d(a,x) = j}} \Delta H_{aa_1}^{>\infty}(\sigma^+, \sigma^-) \Delta J_{xx_1}^{>\infty}(\sigma^+, \sigma^-) \\
&+ \sum_{h=1}^{j} \sum_{\substack{a \in X^\sigma \\ x \in X^{T^h \sigma} \\ d(a,x) = j}} (\Delta H_{aa_1}^{>\infty}(\sigma^+, \sigma^-) \Delta J_{xx_1}^{>\infty}(\sigma^+, \sigma^-) + \Delta H_{xx_1}^{>\infty}(\sigma^+, \sigma^-) \Delta J_{aa_1}^{>\infty}(\sigma^+, \sigma^-)) \\
&\qquad = \kappa_{j,k}(H, J)(\sigma_{-\ell}, \sigma_{j+\ell})
\end{aligned}$$

if $j \geq 1$ (where as usual, for a, x as above, a_1 and x_1 are the neighbours of a and x on $[ax]$).

Assume now k is odd, $k = 2\ell+1$, $\ell \geq 0$. To deal with the symmetries associated with edges instead of vertices we introduce a new family of subsets of X_1. For σ in $\mathscr{S}$, we set

$$X_1^{\sigma,\sharp} = X_1^\sigma \cup \{\sigma_1\sigma_0\} \smallsetminus \{\sigma_0\sigma_{-1}\}.$$

If H is a k-pseudofunction, for σ in $\mathscr{S}$ and ab in $X_1^{\sigma,\sharp}$, $ab \notin \{\sigma_0\sigma_1, \sigma_1\sigma_0\}$, the quantity $\Delta H_{ab}^{>\infty}(\sigma^+, \sigma^-)$ is zero if $r = \min(d(a, \langle\sigma\rangle), d(b, \langle\sigma\rangle)) \geq \ell$. If $r < \ell$, this quantity only depends on the segment $[\sigma_{-(\ell-r)}, \sigma_{\ell-r}]$. Besides, the quantity $\Delta H_{\sigma_0\sigma_1}^{>\infty}(\sigma^+, \sigma^-)$ only depends on the segment $[\sigma_{-\ell}, \sigma_{\ell+1}]$. Therefore, for $j \geq 0$, there exists a symmetric bilinear map

$$\kappa_{j,k} : \mathcal{H}_k \times \mathcal{H}_k \to V_{j+k}$$

such that, for any H, J in $\mathcal{H}_k$ and σ in $\mathscr{S}$, one has

$$\sum_{ab \in X_1^{\sigma,\sharp}} \Delta H_{ab}^{>\infty}(\sigma^+, \sigma^-) \Delta J_{ab}^{>\infty}(\sigma^+, \sigma^-) = 2\kappa_{0,k}(H, J)(\sigma_{-\ell}, \sigma_{\ell+1})$$

if $j = 0$ and

$$\sum_{\substack{ab, xy \in X_1^{\sigma,\sharp} \\ \delta(ab,xy)=j \\ b,y \in [ax]}} \Delta H_{ab}^{>\infty}(\sigma^+, \sigma^-) \Delta J_{xy}^{>\infty}(\sigma^+, \sigma^-)$$

$$+\sum_{h=1}^{j} \sum_{\substack{ab \in X_1^{\sigma,\sharp} \\ xy \in X_1^{T^h\sigma,\sharp} \\ \delta(ab,xy)=j \\ b,y \in [ax]}} (\Delta H_{ab}^{>\infty}(\sigma^+, \sigma^-) \Delta J_{xy}^{>\infty}(\sigma^+, \sigma^-) + \Delta H_{xy}^{>\infty}(\sigma^+, \sigma^-) \Delta J_{ab}^{>\infty}(\sigma^+, \sigma^-))$$

$$= 2\kappa_{j,k}(H, J)(\sigma_{-\ell}, \sigma_{j+\ell+1})$$

if $j \geq 1$.

When $j \geq k-1$, we have a better formula for these bilinear maps which will help us to write the fundamental bilinear maps as endpoints series.

Lemma 6.2. *Let $k \geq 1$, $j \geq k-1$ and H, J be in $\mathcal{H}_k$. Then the function $\kappa_{j,k}(H, J)$ is cohomologous to the function*

$$X_{j+k} \to \mathbb{R}, xy \mapsto -\omega_k(H)(xx_k)\omega_k(J)(yy_k) - \omega_k(J)(xx_k)\omega_k(H)(yy_k),$$

where x_k and y_k are the elements of $[xy]$ at distance k from x and y.

Proof. Assume k is even, $k = 2\ell$, $\ell \geq 1$. Take σ in $\mathscr{S}$. As mentioned above, for a in X^σ with $d(a, \langle\sigma\rangle) \geq \ell$ and $b \sim a$, we have

$\Delta H_{ab}^{>\infty}(\sigma^+,\sigma^-)=0$. Therefore, for $h\geq 0$, a in X^σ and x in $X^{T^h\sigma}$, if $d(a,x)=j$ and $\Delta H_{aa_1}^{>\infty}(\sigma^+,\sigma^-)\Delta J_{xx_1}^{>\infty}(\sigma^+,\sigma^-)\neq 0$, we have

$$h\geq j-2(\ell-1)=j-k+2\geq 1.$$

More precisely, we have

$$\begin{aligned}&\kappa_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell})=\\&\sum_{0\leq r,s\leq \ell-1}\sum_{\substack{a\in S^r(\sigma_0)\cap X^\sigma\\ x\in S^s(\sigma_{j-r-s})\cap X^{T^{j-r-s}\sigma}}}\Delta H_{aa_1}(\sigma_{\ell-r},\sigma_{-(\ell-r)})\Delta J_{xx_1}(\sigma_{j+\ell-r-2s},\sigma_{j-r-\ell})\\&\qquad\qquad+\Delta H_{xx_1}(\sigma_{j+\ell-r-2s},\sigma_{j-r-\ell})\Delta J_{aa_1}(\sigma_{\ell-r},\sigma_{-(\ell-r)}).\end{aligned}$$

For r,s as above, the smooth function

$$\sigma\mapsto\sum_{\substack{a\in S^r(\sigma_0)\cap X^\sigma\\ x\in S^s(\sigma_{j-r-s})\cap X^{T^{j-r-s}\sigma}}}\Delta H_{aa_1}(\sigma_{\ell-r},\sigma_{-(\ell-r)})\Delta J_{xx_1}(\sigma_{j+\ell-r-2s},\sigma_{j-r-\ell})$$

is cohomologous to

$$\sigma\mapsto\sum_{\substack{a\in S^r(\sigma_{\ell+r})\cap X^{T^{\ell+r}\sigma}\\ x\in S^s(\sigma_{j+\ell-s})\cap X^{T^{j+\ell-s}\sigma}}}\Delta H_{aa_1}(\sigma_{2\ell},\sigma_{2r})\Delta J_{xx_1}(\sigma_{j+2\ell-2s},\sigma_j).$$

The conclusion follows by using Definition 6.1.

Assume k is odd, $k=2\ell+1$, $\ell\geq 0$. Now, for ab in X_1 and σ in $\mathscr{S}$, we have $\Delta H_{ab}(\sigma^+,\sigma^-)=0$ as soon as $\min(d(a,\langle\sigma\rangle),d(b,\langle\sigma\rangle))\geq\ell$. For $j\geq k-1$, we will split the sum defining $\kappa_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell+1})$ according to whether $a=\sigma_0$ or not and whether $x=\sigma_{h+1}$ or not. Thus, we write

$$\kappa_{j,k}(H,J)=\kappa_{j,k}^{00}(H,J)+\kappa_{j,k}^{01}(H,J)+\kappa_{j,k}^{10}(H,J)+\kappa_{j,k}^{11}(H,J),$$

where, first,

$$\begin{aligned}\kappa_{j,k}^{00}(H,J)(\sigma_{-\ell}\sigma_{j+\ell+1})=\ &\Delta H_{\sigma_0\sigma_1}(\sigma_{\ell+1},\sigma_{-\ell})\Delta J_{\sigma_{j+1}\sigma_j}(\sigma_{j+\ell+1},\sigma_{j-\ell})\\&+\Delta H_{\sigma_{j+1}\sigma_j}(\sigma_{j+\ell+1},\sigma_{j-\ell})\Delta J_{\sigma_0\sigma_1}(\sigma_{\ell+1},\sigma_{-\ell}),\end{aligned}$$

which is cohomologous to the function

$$\begin{aligned}\sigma\mapsto\ &\Delta H_{\sigma_\ell\sigma_{\ell+1}}(\sigma_k,\sigma_0)\Delta J_{\sigma_{j+\ell+1}\sigma_{j+\ell}}(\sigma_{j+k},\sigma_j)\\&+\Delta H_{\sigma_{j+\ell+1}\sigma_{j+\ell}}(\sigma_{j+k},\sigma_j)\Delta J_{\sigma_\ell\sigma_{\ell+1}}(\sigma_k,\sigma_0);\end{aligned}$$

second,

$$\kappa^{01}_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell+1}) = \sum_{1\le s\le \ell}\ \sum_{x\in S^s(\sigma_{j+1-s})\cap X^{T^{j+1-s}\sigma}} \Delta H_{\sigma_0\sigma_1}(\sigma_{\ell+1},\sigma_{-\ell})\Delta J_{xx_-}(\sigma_{j+\ell+2-2s},\sigma_{j-\ell}) + \Delta H_{xx_-}(\sigma_{j+\ell+2-2s},\sigma_{j-\ell})\Delta J_{\sigma_0\sigma_1}(\sigma_{\ell+1},\sigma_{-\ell}),$$

which is cohomologous to the function

$$\sigma \mapsto \sum_{1\le s\le \ell}\ \sum_{x\in S^s(\sigma_{j+\ell+1-s})\cap X^{T^{j+\ell+1-s}\sigma}} \Delta H_{\sigma_\ell\sigma_{\ell+1}}(\sigma_k,\sigma_0)\Delta J_{xx_-}(\sigma_{j+k+1-2s},\sigma_j) + \Delta H_{xx_-}(\sigma_{j+k+1-2s},\sigma_j)\Delta J_{\sigma_\ell\sigma_{\ell+1}}(\sigma_k,\sigma_0);$$

third,

$$\kappa^{10}_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell+1}) = \sum_{1\le r\le \ell}\ \sum_{a\in S^r(\sigma_0)\cap X^{\sigma}} \Delta H_{aa_-}(\sigma_{\ell-r+1},\sigma_{-\ell+r-1})\Delta J_{\sigma_{j+1-r}\sigma_{j-r}}(\sigma_{j+\ell+1-r},\sigma_{j-\ell-r}) + \Delta H_{\sigma_{j+1-r}\sigma_{j-r}}(\sigma_{j+\ell+1-r},\sigma_{j-\ell-r})\Delta J_{aa_-}(\sigma_{\ell-r+1},\sigma_{-\ell+r-1}),$$

which is cohomologous to the function

$$\sigma \mapsto \sum_{1\le r\le \ell}\ \sum_{a\in S^r(\sigma_{\ell+r})\cap X^{T^{\ell+r}\sigma}} \Delta H_{aa_-}(\sigma_k,\sigma_{2r-1})\Delta J_{\sigma_{j+\ell+1}\sigma_{j+\ell}}(\sigma_{j+k},\sigma_j) + \Delta H_{\sigma_{j+\ell+1}\sigma_{j+\ell}}(\sigma_{j+k},\sigma_j)\Delta J_{aa_-}(\sigma_k,\sigma_{2r-1});$$

fourth

$$\kappa^{11}_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell+1}) = \sum_{1\le r,s\le \ell}\ \sum_{\substack{a\in S^r(\sigma_0)\cap X^{\sigma}\\ x\in S^s(\sigma_{j+1-r-s})\cap X^{T^{j+1-r-s}\sigma}}} \Delta H_{aa_-}(\sigma_{\ell-r+1},\sigma_{-\ell+r-1})\Delta J_{xx_-}(\sigma_{j+\ell+2-r-2s},\sigma_{j-\ell-r}) + \Delta H_{xx_-}(\sigma_{j+\ell+2-r-2s},\sigma_{j-\ell-r})\Delta J_{aa_-}(\sigma_{\ell-r+1},\sigma_{-\ell+r-1}),$$

which is cohomologous to

$$\sigma \mapsto \sum_{1\le r,s\le \ell}\ \sum_{\substack{a\in S^r(\sigma_{\ell+r})\cap X^{T^{\ell+r}\sigma}\\ x\in S^s(\sigma_{j+\ell+1-s})\cap X^{T^{j+\ell+1-s}\sigma}}} \Delta H_{aa_-}(\sigma_k,\sigma_{2r-1})\Delta J_{xx_-}(\sigma_{j+k+1-2s},\sigma_j) + \Delta H_{xx_-}(\sigma_{j+k+1-2s},\sigma_j)\Delta J_{aa_-}(\sigma_k,\sigma_{2r-1}).$$

Using Definition 6.1 yields the conclusion. $\square$

6.3. **Endpoints series formulas for the spectral fundamental bilinear maps.** In this section, we use the previous constructions to give an alternative formula for defining the spectral fundamental maps by means of an endpoints series as in Subsection 3.1.

We will need new families of polynomial functions besides the $(A_j)_{j\geq 0}$ and the $(B_j)_{j\geq 0}$ from Subsection 4.1 and the $(C_j)_{j\geq 0}$ from Subsection 4.6. For $j \geq 0$, we set

$$D_j = A_j - (q-1)B_j = C_j - (q-1)\sum_{0\leq h<j} C_h$$

$$E_j = A_j + (q-1)B_j = C_j - (q-1)\sum_{0\leq h<j} (-1)^{j-h} C_h$$

$$F_j = qA_j - (q-1)B_{j+1} = C_j - (q^2-1)\sum_{\substack{0\leq h<j\\ j-h \text{ even}}} C_h,$$

where the equalities follow from Corollary 4.7.

As in Subsection III.2.1, for $k \geq -1$, we write $\mathcal{H}_k = \mathcal{H}_{k,+} \oplus \mathcal{H}_{k,-}$ the decomposition into eigenspaces of the $\vee$ operator.

Proposition 6.3. *Let $k \geq 2$ and t be in $\mathcal{E}_q$.*

Assume k is even. Then, for any H, J in $\mathcal{H}_{k,+}$, the Hölder continuous function $\Phi_t\left(H^{>\infty}, J^{>\infty}\right)$ is cohomologous to the Hölder continuous function $\Phi^+_{t,k}(H,J)$ defined by, for σ in $\mathscr{S}$,

$$\Phi^+_{t,k}(H,J)(\sigma) = \frac{q+1}{4}\kappa_{0,k}(H,J)(\sigma_0,\sigma_k) + \frac{q(q+1)}{4}\sum_{j=1}^{k-2} q^{-j} C_j(t)\kappa_{j,k}(H,J)(\sigma_0\sigma_{j+k})$$

$$-\frac{q(q+1)}{4}\sum_{j=k-1}^{\infty} q^{-j}C_j(t)(\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j) + \omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j)).$$

For any H, J in $\mathcal{H}_{k,-}$, the Hölder continuous function $\Phi_t\left(H^{>\infty}, J^{>\infty}\right)$ is cohomologous to the Hölder continuous function $\Phi^-_{t,k}(H,J)$ defined by, for σ in $\mathscr{S}$,

$$\Phi^-_{t,k}(H,J)(\sigma) = \frac{q+1}{4q}\kappa_{0,k}(H,J)(\sigma_0,\sigma_k) - \frac{q+1}{4q}\sum_{j=1}^{k-2} q^{-j} F_j(t)\kappa_{j,k}(H,J)(\sigma_0\sigma_{j+k})$$

$$+\frac{q+1}{4q}\sum_{j=k-1}^{\infty} q^{-j}F_j(t)(\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j) + \omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j)).$$

Assume k is odd. Then, for any H, J in $\mathcal{H}_{k,+}$, the Hölder continuous function $\Phi_t\left(H^{>\infty}, J^{>\infty}\right)$ is cohomologous to the Hölder continuous *function* $\Phi^+_{t,k}(H,J)$ *defined by, for* σ *in* $\mathscr{S}$,

$$\Phi^+_{t,k}(H,J)(\sigma) = \sum_{j=0}^{k-2} q^{-j} E_j(t) \kappa_{j,k}(H,J)(\sigma_0\sigma_{j+k})$$

$$- \sum_{j=k-1}^{\infty} q^{-j} E_j(t) (\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j) + \omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j)).$$

For any H, J *in* $\mathcal{H}_{k,-}$, *the Hölder continuous function* $\Phi_t\left(H^{>\infty}, J^{>\infty}\right)$ *is cohomologous to the Hölder continuous function* $\Phi^-_{t,k}(H,J)$ *defined by, for* σ *in* $\mathscr{S}$,

$$\Phi^-_{t,k}(H,J)(\sigma) = \kappa_{0,k}(H,J)(\sigma_0\sigma_k) - \sum_{j=1}^{k-2} q^{-j} D_j(t) \kappa_{j,k}(H,J)(\sigma_0\sigma_{j+k})$$

$$+ \sum_{j=k-1}^{\infty} q^{-j} D_j(t) (\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j) + \omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j)).$$

The proof is a consequence of the formulas below. These allow to compute the spectral bilinear forms of Subsection 4.3 in eigenspaces of the operators R and S by means of our new families of polynomial functions.

Lemma 6.4. *Fix* t *in* $\mathbb{R}$.

For a, x *in* X *and* $j = d(a,x)$, *we have*

$$\sum_{\substack{b\sim a \\ y\sim x}} \chi_t(ab, xy) = \frac{(q+1)^2}{2} \qquad j = 0$$

$$= \frac{q+1}{2q^{j-1}} C_j(t) \qquad j \geq 1.$$

For a *in* X *anb* $b, c \sim a$, $b \neq c$, *we have*

$$\chi_t(ab, ab) - \chi_t(ab, ac) = \frac{q+1}{2q}.$$

For a, x *in* X *with* $j = d(a,x) \geq 1$ *and* $b \sim a$, $y \sim x$ *with* $b, y \notin [ax]$,

$$\chi_t(ab, xy) - \chi_t(aa_1, xy) - \chi_t(ab, xx_1) + \chi_t(aa_1, xx_1) = -\frac{q+1}{2q^{j+1}} F_j(t).$$

For ab, xy in X_1 and $j = \delta(ab, xy)$, we have

$$\begin{aligned}
\chi_t(ab, xy) - \chi_t(ab, yx) &= \varepsilon(ab, xy) && j = 0\\
&= \frac{1}{2q^j} D_j(t)\varepsilon(ab, xy) && j \geq 1\\
\chi_t(ab, xy) + \chi_t(ab, yx) &= 1 && j = 0\\
&= \frac{1}{2q^j} E_j(t) && j \geq 1.
\end{aligned}$$

The proofs directly follow from the definition of χ_t in Subsection 4.3.

Proof of Proposition 6.3 in case k is even. For t in $\mathcal{E}_q$, H, J in $\mathcal{H}_\infty$ and σ in $\mathscr{S}$, we set

$$\begin{aligned}
(6.1)\quad \Phi'_t(H, J)(\sigma) = \frac{1}{2} \sum_{ab, xy \in X_1^\sigma} \chi_t(ab, xy) \Delta H_{ab}(\sigma^+, \sigma^-) \Delta J_{xy}(\sigma^+, \sigma^-)\\
+ \frac{1}{2} \sum_{h \geq 1} \sum_{\substack{ab \in X_1^\sigma \\ xy \in X_1^{T^h\sigma}}} \chi_t(ab, xy)(\Delta H_{ab}(\sigma^+, \sigma^-)\Delta J_{xy}(\sigma^+, \sigma^-)\\
+ \Delta H_{xy}(\sigma^+, \sigma^-)\Delta J_{ab}(\sigma^+, \sigma^-)).
\end{aligned}$$

As in Corollary 5.9, one shows that $\Phi'_t(H, J)$ is a Hölder continuous function. In view of Definition 5.8, by Corollary 5.7 and Lemma 5.11, $\Phi'_t(H, J)$ and $\Phi_t(H, J)$ are cohomologous.

Assume $RH = qH$ and $RJ = qJ$. As, for $x \sim y$ in X, the elements H_{xy} and J_{xy} of $\overline{\mathcal{D}}(\partial X)$ only depend on x, we will write H_x and J_x for them below. In particular, from Lemma 6.4, we get

$$\begin{aligned}
\Phi'_t(H, J)(\sigma) = \frac{(q+1)^2}{4} \sum_{a \in X^\sigma} \Delta H_a(\sigma^+, \sigma^-)\Delta J_a(\sigma^+, \sigma^-)\\
+ \frac{q+1}{4} \sum_{a \neq x \in X^\sigma} \frac{C_{d(a,x)}(t)}{q^{d(a,x)-1}} \Delta H_a(\sigma^+, \sigma^-)\Delta J_x(\sigma^+, \sigma^-)\\
+ \frac{q+1}{4} \sum_{h \geq 1} \sum_{\substack{a \in X^\sigma \\ x \in X^{T^h\sigma}}} \frac{C_{d(a,x)}(t)}{q^{d(a,x)-1}} (\Delta H_a(\sigma^+, \sigma^-)\Delta J_x(\sigma^+, \sigma^-)\\
+ \Delta H_x(\sigma^+, \sigma^-)\Delta J_a(\sigma^+, \sigma^-)).
\end{aligned}$$

Recall that $k \geq 2$ is an even integer, $k = 2\ell$, $\ell \geq 1$. Take H, J in $\mathcal{H}_{k,+}$. From the formula above, we write

$$\begin{aligned}\Phi'_t(H^{>\infty}, J^{>\infty})(\sigma_{-\ell}\sigma_\ell) &= \frac{q+1}{4}\kappa_{0,k}(H,J)(\sigma) \\ &\quad + \frac{q(q+1)}{4}\sum_{j=1}^{\infty} q^{-j} C_j(t)\kappa_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell}).\end{aligned}$$

The first case now follows from Lemma 6.2.

We now address the second case. We will follow the same lines as above. We start by noticing that, by Lemma 6.4, if H, J are in $\mathcal{H}_\infty$ with $RH = -H$ and $RJ = -J$, for a, x in X with $j = d(a,x) \geq 1$ and ξ, η in ∂X, we have

$$\begin{aligned}&\sum_{\substack{b\sim a\\ y\sim x}} \chi_t(ab, xy)\Delta H_{ab}(\xi,\eta)\Delta J_{xy}(\xi,\eta) \\ &\qquad = -\frac{q+1}{2q^{j+1}} F_j(t)\Delta H_{aa_1}(\xi,\eta)\Delta J_{xx_1}(\xi,\eta).\end{aligned}$$

When $a = x$, we get

$$\sum_{b,c\sim a} \chi_t(ab, ac)\Delta H_{ab}(\xi,\eta)\Delta J_{ac}(\xi,\eta) = \frac{q+1}{2q}\sum_{b\sim a}\Delta H_{ab}(\xi,\eta)\Delta J_{ab}(\xi,\eta).$$

Thus, when $RH = -H$ and $RJ = -J$, we may rewrite (6.1) as

$$\begin{aligned}\Phi'_t(H,J)(\sigma) &= \frac{q+1}{4q}\sum_{ab\in X_1^\sigma}\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{ab}(\sigma^+,\sigma^-) \\ &\quad - \frac{q+1}{4}\sum_{a\neq x\in X^\sigma}\frac{F_{d(a,x)}(t)}{q^{d(a,x)+1}}\Delta H_{aa_1}(\sigma^+,\sigma^-)\Delta J_{xx_1}(\sigma^+,\sigma^-) \\ &\quad - \frac{q+1}{4}\sum_{h\geq 1}\sum_{\substack{a\in X^\sigma\\ x\in X^{T^h\sigma}}}\frac{F_{d(a,x)}(t)}{q^{d(a,x)+1}}(\Delta H_{aa_1}(\sigma^+,\sigma^-)\Delta J_{xx_1}(\sigma^+,\sigma^-) \\ &\qquad\qquad + \Delta H_{xx_1}(\sigma^+,\sigma^-)\Delta J_{aa_1}(\sigma^+,\sigma^-)).\end{aligned}$$

For H, J in $\mathcal{H}_{k,-}$, we get

$$\begin{aligned}\Phi'_t(H^{>\infty}, J^{>\infty})(\sigma) &= \frac{q+1}{4q}\kappa_{0,k}(H,K)(\sigma_{-\ell},\sigma_\ell) \\ &\quad - \frac{q+1}{4q}\sum_{j=0}^{\infty} q^{-j}F_j(t)\kappa_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell}).\end{aligned}$$

Again, the conclusion follows from Lemma 6.2. $\square$

Proof of Proposition 6.3 in case k is odd. The odd case will be dealt with in an analogue way. Given H, J in $\mathcal{H}_\infty$, we define a new Hölder continuous function that is cohomologous to $\Phi_t(H, J)$. For σ in $\mathscr{S}$, we set

$$\Phi''_t(H,J)(\sigma) = \frac{1}{2} \sum_{ab,xy \in X_1^{\sigma,\sharp}} \chi_t(ab,xy) \Delta H_{ab}(\sigma^+,\sigma^-) \Delta J_{xy}(\sigma^+,\sigma^-) \tag{6.2}$$
$$+ \frac{1}{2} \sum_{h\geq 1} \sum_{\substack{ab \in X_1^{\sigma,\sharp} \\ xy \in X_1^{T^h\sigma,\sharp}}} \chi_t(ab,xy)(\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)$$
$$+ \Delta H_{xy}(\sigma^+,\sigma^-)\Delta J_{ab}(\sigma^+,\sigma^-)).$$

By (6.1), $\Phi''_t(H, J)$ and $\Phi'_t(H, J)$ are cohomologous.

Assume $SH = H$ and $SJ = J$. In view of Lemma 6.4, we have

$$\Phi''_t(H,J)(\sigma) = \frac{1}{2} \sum_{ab \in X_1^{\sigma,\sharp}} \Delta H_{ab}(\sigma^+,\sigma^-) \Delta J_{ab}(\sigma^+,\sigma^-)$$
$$+ \frac{1}{2} \sum_{\substack{ab,xy \in X_1^{\sigma,\sharp} \\ \delta(ab,xy)\geq 1 \\ b,y \in [ac]}} \frac{E_{\delta(ab,xy)}(t)}{q^{\delta(ab,xy)}} \Delta H_{ab}(\sigma^+,\sigma^-) \Delta J_{xy}(\sigma^+,\sigma^-)$$
$$+ \frac{1}{2} \sum_{h\geq 1} \sum_{\substack{ab \in X_1^{\sigma,\sharp} \\ xy \in X_1^{T^h\sigma,\sharp} \\ b,y \in [ac]}} \frac{E_{\delta(ab,xy)}(t)}{q^{\delta(ab,xy)}} (\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-)$$
$$+ \Delta H_{xy}(\sigma^+,\sigma^-)\Delta J_{ab}(\sigma^+,\sigma^-)).$$

Recall that $k \geq 1$ is an odd integer, $k = 2\ell + 1$, $\ell \geq 0$. For H, J in $\mathcal{H}_{k,+}$, we get

$$\Phi''_t(H^{>\infty}, J^{>\infty})(\sigma) = \sum_{j\geq 0} q^{-j} E_j(t) \kappa_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell+1}).$$

Lemma 6.2 yields the first case of the Proposition.

For the second case, given H, J in $\mathcal{H}_\infty$ with $SH = -H$ and $SJ = -J$, we use Lemma 6.4 and (6.2) to write, for σ in $\mathscr{S}$,

$$\begin{aligned}\Phi''_t(H,J)(\sigma) &= \frac{1}{2}\sum_{ab\in X_1^{\sigma,\sharp}} \Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{ab}(\sigma^+,\sigma^-) \\ &\quad - \frac{1}{2}\sum_{\substack{ab,xy\in X_1^{\sigma,\sharp}\\ \delta(ab,xy)\geq 1\\ b,y\in[ac]}} \frac{D_{\delta(ab,xy)}(t)}{q^{\delta(ab,xy)}}\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-) \\ &\quad - \frac{1}{2}\sum_{h\geq 1}\sum_{\substack{ab\in X_1^{\sigma,\sharp}\\ xy\in X_1^{T^h\sigma,\sharp}\\ b,y\in[ac]}} \frac{D_{\delta(ab,xy)}(t)}{q^{\delta(ab,xy)}}(\Delta H_{ab}(\sigma^+,\sigma^-)\Delta J_{xy}(\sigma^+,\sigma^-) \\ &\qquad\qquad + \Delta H_{xy}(\sigma^+,\sigma^-)\Delta J_{ab}(\sigma^+,\sigma^-)).\end{aligned}$$

For H, J in $\mathcal{H}_{k,-}$, we obtain

$$\begin{aligned}\Phi''_t(H^{>\infty}, J^{>\infty})(\sigma) &= \kappa_{0,k}(H,J)(\sigma_{-\ell}\sigma_{\ell+1}) \\ &\quad - \sum_{j\geq 1} q^{-j} D_j(t)\kappa_{j,k}(H,J)(\sigma_{-\ell}\sigma_{j+\ell+1}).\end{aligned}$$

Still by Lemma 6.2, the last case of the Proposition follows. □

6.4. **An endpoints series formula for the special spectral fundamental maps.** In the case of the special spectral fundamental maps, following the same strategy as above yields

Proposition 6.5. *Let $k \geq 1$.*

Assume k is even. Then, for any H, J in $\mathcal{H}_{k,+}$, the Hölder continuous functions $\Phi^{\rm sp}_{(-1)}\left(H^{>\infty}, J^{>\infty}\right)$ and $\Phi^{\rm sp}_{1}\left(H^{>\infty}, J^{>\infty}\right)$ are coboundaries. For any H, J in $\mathcal{H}_{k,-}$, the Hölder continuous functions $\Phi^{\rm sp}_{(-1)}\left(H^{>\infty}, J^{>\infty}\right)$ and $\Phi^{\rm sp}_{1}\left(H^{>\infty}, J^{>\infty}\right)$ are cohomologous to the Hölder continuous functions $\Phi^{\rm sp,-}_{(-1),k}(H,J)$ and $\Phi^{\rm sp,-}_{1,k}(H,J)$ defined by, for σ in $\mathscr{S}$,

$$\Phi^{\rm sp,-}_{(-1),k}(H,J)(\sigma) = \frac{q+1}{2q}\kappa_{0,k}(H,J)(\sigma_0,\sigma_k) - \frac{(q+1)^2}{2q}\sum_{j=1}^{k-2}(-q)^{-j}\kappa_{j,k}(H,J)(\sigma_0\sigma_{j+k})$$

$$+\frac{(q+1)^2}{2q}\sum_{j=k-1}^{\infty}(-q)^{-j}(\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j)+\omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j))$$

and

$$\Phi^{\mathrm{sp},-}_{1,k}(H,J)(\sigma)=\frac{q+1}{2q}\kappa_{0,k}(H,J)(\sigma_0,\sigma_k)-\frac{(q+1)^2}{2q}\sum_{j=1}^{k-2}q^{-j}\kappa_{j,k}(H,J)(\sigma_0\sigma_{j+k})$$

$$+\frac{(q+1)^2}{2q}\sum_{j=k-1}^{\infty}q^{-j}(\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j)+\omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j)).$$

Assume k is odd. Then, for any H,J in $\mathcal{H}_{k,+}$, the Hölder continuous function $\Phi^{\mathrm{sp}}_1\left(H^{>\infty},J^{>\infty}\right)$ is a coboundary; the Hölder continuous function $\Phi^{\mathrm{sp}}_{(-1)}\left(H^{>\infty},J^{>\infty}\right)$ is cohomologous to the Hölder continuous function $\Phi^{\mathrm{sp},+}_{(-1),k}(H,J)$ defined by, for σ in $\mathscr{S}$,

$$\Phi^{\mathrm{sp},+}_{(-1),k}(H,J)(\sigma)=2\kappa_{0,k}(H,J)(\sigma_0,\sigma_k)+4\sum_{j=1}^{k-2}(-q)^{-j}\kappa_{j,k}(H,J)(\sigma_0\sigma_{j+k})$$

$$-4\sum_{j=k-1}^{\infty}(-q)^{-j}(\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j)+\omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j)).$$

For any H,J in $\mathcal{H}_{k,-}$, the Hölder continuous function $\Phi^{\mathrm{sp}}_{(-1)}\left(H^{>\infty},J^{>\infty}\right)$ is a coboundary; the Hölder continuous function $\Phi^{\mathrm{sp}}_1\left(H^{>\infty},J^{>\infty}\right)$ is cohomologous to the Hölder continuous function $\Phi^{\mathrm{sp},-}_{1,k}(H,J)$ defined by, for σ in $\mathscr{S}$,

$$\Phi^{\mathrm{sp},-}_{1,k}(H,J)(\sigma)=2\kappa_{0,k}(H,J)(\sigma_0,\sigma_k)-4\sum_{j=1}^{k-2}q^{-j}\kappa_{j,k}(H,J)(\sigma_0\sigma_{j+k})$$

$$+4\sum_{j=k-1}^{\infty}q^{-j}(\omega_k(H)(\sigma_0\sigma_k)\omega_k(J)(\sigma_{j+k}\sigma_j)+\omega_k(J)(\sigma_0\sigma_k)\omega_k(H)(\sigma_{j+k}\sigma_j)).$$

The proof is analogue to that of Proposition 6.5, by replacing Lemma 6.4 with

Lemma 6.6. *For a,x in X and $j=d(a,x)$, we have*

$$\sum_{\substack{b\sim a\\ y\sim x}}\chi^{\mathrm{sp}}_1(ab,xy)=0=\sum_{\substack{b\sim a\\ y\sim x}}\chi^{\mathrm{sp}}_{(-1)}(ab,xy).$$

For a in X anb $b,c\sim a$, $b\neq c$, we have

$$\chi^{\mathrm{sp}}_1(ab,ab)-\chi^{\mathrm{sp}}_1(ab,ac)=\chi^{\mathrm{sp}}_{(-1)}(ab,ab)-\chi^{\mathrm{sp}}_{(-1)}(ab,ac)=\frac{q+1}{q}.$$

For a, x in X with $j = d(a,x) \geq 1$ and $b \sim a$, $y \sim x$ with $b, y \notin [ax]$, we have

$$\chi_1^{\mathrm{sp}}(ab, xy) - \chi_1^{\mathrm{sp}}(aa_1, xy) - \chi_1^{\mathrm{sp}}(ab, xx_1) + \chi_1^{\mathrm{sp}}(aa_1, xx_1) = -\frac{(q+1)^2}{q^{j+1}}$$

$$\chi_{(-1)}^{\mathrm{sp}}(ab, xy) - \chi_{(-1)}^{\mathrm{sp}}(aa_1, xy) - \chi_{(-1)}^{\mathrm{sp}}(ab, xx_1) + \chi_{(-1)}^{\mathrm{sp}}(aa_1, xx_1) = \frac{(q+1)^2}{(-q)^{j+1}}.$$

For ab, xy in X_1, we have

$$\chi_1^{\mathrm{sp}}(ab, yx) = -\chi_1^{\mathrm{sp}}(ab, xy)$$
$$\chi_{(-1)}^{\mathrm{sp}}(ab, yx) = \chi_1^{\mathrm{sp}}(ab, xy).$$

7. The weight of pseudofunctions

In the sequel of the article, we will study the consequences of Proposition 3.3, which describes under which conditions an endpoints series is a coboundary, when applied to the endpoints series which appear in Proposition 6.3 and Proposition 6.5. To this aim, we will need a better understanding of the weight map of pseudofunctions.

7.1. **Weight and natural operations.** First, we relate the weight construction to the natural operations on pseudofunctions of Subsection III.2.2.

Lemma 7.1. *Let $k \geq 0$, H be a k-pseudofunction, ab be in X_{k+1} and a_1 and b_1 be the neighbours of a and b on $[ab]$. If k is even, we have*

$$\omega_{k+1}(H^{>})(ab) = \omega_k(H)(ab_1) \text{ and } \omega_{k+1}(H^{>\vee})(ab) = \omega_k(H^{\vee})(a_1 b).$$

If k is odd, we have

$$\omega_{k+1}(H^{>})(ab) = \omega_k(H)(a_1 b) \text{ and } \omega_{k+1}(H^{>\vee})(ab) = \omega_k(H^{\vee})(ab_1).$$

The weight map ω_k was introduced in Definition 6.1.

Proof. If $k = 0$, and H is the 0-pseudofunction associated to the function u on X (see Subsection III.2.1), we have

$$\omega_1(H^{>})(ab) = \Delta H_{ab}^{>}(b,a) = u(a)(\mathbf{1}_b(b) - \mathbf{1}_b(a)) = u(a) = \omega_0(H)(a)$$

and

$$\begin{aligned}\omega_1(H^{>\vee})(ab) &= \Delta H_{ab}^{>\vee}(b,a) = \Delta H_{ba}^{>}(b,a) \\ &= u(b)(\mathbf{1}_a(b) - \mathbf{1}_a(a)) = -u(b) = \omega_0(H^{\vee})(b).\end{aligned}$$

Assume now $k \geq 1$ and let us write as usual $a_0 = a, a_1, \ldots, a_{k+1} = b$ for the geodesic parametrization on the segment $[ab]$.

If k is even, $k=2\ell$, $\ell\geq 1$, we have

$$\begin{aligned}\omega_{k+1}(H^{>})(ab) &= \Delta H^{>}_{a_\ell a_{\ell+1}}(b,a)+\sum_{i=1}^{\ell}\sum_{\substack{c\in X\\ [a_{\ell+i}c]\cap[ab]=\{a_{\ell+i}\}\\ d(c,a_{\ell+i})=i}}\Delta H^{>}_{cc_-}(b,a_{2i-1})\\ &= \Delta H_{a_\ell a_{\ell+1}}(b_1,a)+\sum_{i=1}^{\ell}\sum_{\substack{c\in X\\ [a_{\ell+i}c]\cap[ab]=\{a_{\ell+i}\}\\ d(c,a_{\ell+i})=i}}\Delta H_{cc_-}(b_1,a_{2i}).\end{aligned}$$

As $b_1=a_{2\ell}$, the last term of the sum vanishes and, as required, we get $\omega_{k+1}(H^{>})(ab)=\omega_k(H)(a)$. Besides,

$$\begin{aligned}\omega_{k+1}(H^{>\vee})(ab) &= \Delta H^{>}_{a_{\ell+1}a_\ell}(b,a)+\sum_{i=1}^{\ell}\sum_{\substack{c\in X\\ [a_{\ell+i}c]\cap[ab]=\{a_{\ell+i}\}\\ d(c,a_{\ell+i})=i}}\Delta H^{>}_{c_-c}(b,a_{2i-1})\\ &= \Delta H_{a_{\ell+1}a_\ell}(b,a_1)+\sum_{i=1}^{\ell}\sum_{\substack{c\in X\\ [a_{\ell+i}c]\cap[ab]=\{a_{\ell+i}\}\\ d(c,a_{\ell+i})=i}}\Delta H_{c_-c}(b,a_{2i-1}).\end{aligned}$$

Now, on one hand,

$$\Delta H_{a_{\ell+1}a_\ell}(b,a_1)+\sum_{\substack{c\sim a_{\ell+1}\\ c\notin\{a_\ell,a_{\ell+2}\}}}\Delta H_{a_{\ell+1}c}(b,a_1)=\Delta H^{\vee}_{a_{\ell+1}a_{\ell+2}}(b,a_1),$$

whereas, on the other hand, for $2\leq i\leq\ell$, by setting $d=c_-$ in the sums below, we get

$$\sum_{\substack{c\in X\\ [a_{\ell+i}c]\cap[ab]=\{a_{\ell+i}\}\\ d(c,a_{\ell+i})=i}}\Delta H_{c_-c}(b,a_{2i-1})=\sum_{\substack{d\in X\\ [a_{\ell+i}d]\cap[ab]=\{a_{\ell+i}\}\\ d(d,a_{\ell+i})=i-1}}\Delta H^{\vee}_{dd_-}(b,a_{2i-1}).$$

We obtain indeed $\omega_{k+1}(H^{>\vee})(ab)=\omega_k(H^{\vee})(a_1b)$.

If k is odd, $k=2\ell+1$, $\ell\geq 0$, we have

$$\begin{aligned}\omega_{k+1}(H^{>})(ab) &= \Delta H^{>}_{a_{\ell+1}a_{\ell+2}}(b,a)+\sum_{i=1}^{\ell}\sum_{\substack{c\in X\\ [a_{\ell+i+1}c]\cap[ab]=\{a_{\ell+i+1}\}\\ d(c,a_{\ell+i+1})=i}}\Delta H^{>}_{cc_-}(b,a_{2i})\\ &= \Delta H_{a_{\ell+1}a_{\ell+2}}(b,a_1)+\sum_{i=1}^{\ell}\sum_{\substack{c\in X\\ [a_{\ell+i+1}c]\cap[ab]=\{a_{\ell+i+1}\}\\ d(c,a_{\ell+i+1})=i}}\Delta H_{cc_-}(b,a_{2i})\\ &= \omega_k(H)(a_1b)\end{aligned}$$

and in the same way,

$$\omega_{k+1}(H^{>\vee})(ab)=\Delta H^{>\vee}_{a_{\ell+1}a_{\ell+2}}(b,a)+\sum_{i=1}^{\ell}\sum_{\substack{c\in X\\ [a_{\ell+i+1}c]\cap[ab]=\{a_{\ell+i+1}\}\\ d(c,a_{\ell+i+1})=i}}\Delta H^{>\vee}_{cc_-}(b,a_{2i}).$$

First, we have

$$\Delta H^{>\vee}_{a_{\ell+1}a_{\ell+2}}(b,a)=\Delta H_{a_{\ell+1}a_{\ell}}(b_1,a)+\sum_{\substack{c\sim a_{\ell+1}\\ c\notin\{a_\ell,a_{\ell+2}\}}}\Delta H_{a_{\ell+1}c}(b_1,a_1);$$

second, for $1\leq i\leq \ell$, we have

$$\sum_{\substack{c\in X\\ [a_{\ell+i+1}c]\cap[ab]=\{a_{\ell+i+1}\}\\ d(c,a_{\ell+i+1})=i}}\Delta H^{>\vee}_{cc_-}(b,a_{2i})=\sum_{\substack{d\in X\\ [a_{\ell+i+1}d]\cap[ab]=\{a_{\ell+i+1}\}\\ d(d,a_{\ell+i+1})=i+1}}\Delta H_{d_-d}(b_1,a_{2i+1});$$

in particular, for $i=\ell$, this vanishes as $b_1=a_k=a_{2\ell+1}$. We get $\omega_{k+1}(H^{>\vee})(ab)=\omega_k(H^{\vee})(ab_1)$. $\square$

7.2. **Injectivity properties of the weight.** We will show that pseudofunctions are determined by their weight. We start by stating a converse to Lemma 7.1.

Lemma 7.2. *Let $k\geq 1$ and H be a k-pseudofunction. Assume there exists a function v on X_{k-1} such that, for any ab in X_k, one has $\omega_k(H)(ab)=v(a_1b)$. Then, if k is even, there exists a $(k-1)$-pseudofunction G with $H=G^{>}$ and $v=\omega_{k-1}(G)$. If k is odd, there exists a $(k-1)$-pseudofunction G with $H=G^{>\vee}$ and $v=\omega_{k-1}(G^{\vee})$*

Proof. If $k=1$, for $a\sim b$ in X_1, we have $\omega_1(H)(ab)=\Delta H_{ab}(b,a)=v(b)$, which means that $H_{ab}=v(b)\mathbf{1}_b=-v(b)\mathbf{1}_a$ in $\overline{V}^0(ab)$. We get $H=G^{>\vee}$ where G is the 0-pseudofunction associated with $-v$ and hence $v = -\omega_0(G) = \omega_0(G^\vee)$ (see Subsection III.2.1 and Subsection III.2.2).

Suppose k is even, $k = 2\ell$, $\ell \geq 1$. Then, saying that there exists a $(k-1)$-pseudofunction G with $H = G^>$ is saying that, for every $x \sim y$ in X and every a, a' in $S^\ell(x)$ such that $d(a, a') = 2$ and $y \notin [ax] \cup [a'x]$, we have $\Delta H_{xy}(a, a') = 0$. Indeed, assume the latter holds. Then, pick b in $S^\ell(x)$ with $y \in [bx]$. By assumption, we have $d(a, b) = d(a', b) = k$. Let a_1 be the neighbour of a on $[ax]$, which is also the neighbour of a' on $[a'x]$ since $d(a, a') = 2$, and write $a_1, a_2, \dots, a_k = b$ for the geodesic parametrization of the segment $[a_1 b]$. Then, by Definition 6.1 and the asusmption, we have

$$\Delta H_{xy}(b, a) = v(a_1 b) - \sum_{i=1}^{\ell-1} \sum_{\substack{c \in X \\ [a_{\ell+i} c] \cap [ab] = \{a_{\ell+i}\} \\ d(c, a_{\ell+i}) = i}} \Delta H_{cc_-}(b, a_{2i}) = \Delta H_{xy}(b, a').$$

Hence $\Delta H_{xy}(a, a') = \Delta H_{xy}(b, a') - \Delta H_{xy}(b, a) = 0$ and therefore, there exists a $(k-1)$-pseudofunction G with $H = G^>$. We have $v = \omega_{k-1}(G)$ by Lemma 7.1.

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, we proceed in the same way. We first show that there exists a $(k-1)$-pseudofunction G with $H = G^{>\vee}$. Indeed this amounts to saying that, for every $x \sim y$ in X and every a, a' in $S^\ell(xy)$ with $d(a, a') = 2$ and $x \in [ay]$, one has $\Delta H_{xy}(a, a')$, which is warranted by Definition 6.1 and the assumption. Then, Lemma 7.1 ensures that $v = \omega_{k-1}(G^\vee)$. □

From this, we can deduce that the weight determines the pseudo-function.

Corollary 7.3. *Let $k \geq 0$ and H be a k-pseudofunction. If $\omega_k(H) = 0$, then $H = 0$.*

In general, for $k \geq 3$, the weight map does not map $\mathcal{H}_k$ onto the space of Γ-invariant functions on X_k.

Proof. For $k = 0, 1$, the proof is immediate. The general case follows by Lemma 7.2 and a straightforward induction argument. □

7.3. **Weights and cohomology.** Now, we describe under which condition a weight function is a coboundary.

Proposition 7.4. *Let $k \geq 1$ and H be in $\mathcal{H}_k$. Then the following are equivalent:*
(i) *The weight function $\omega_k(H)$ is a coboundary.*
(ii) *There exists G in $\mathcal{H}_{k-1}$ such that $H = G^{\vee>} - G^{>\vee}$.*

This statement is closely related to Theorem I.8.32. The equivalence of the different notions of cohomology among elements of V_k is established in Corollary 2.7.

Formally, Proposition 7.4 will not be used later in the article. Nevertheless, its statement and its proof serve as a model for those of Proposition 7.12 and Proposition 9.3 below. The latter result will play a crucial role in applying the cohomology criterion of Proposition 3.3 to endpoints series as in Proposition 6.3 or Proposition 6.5.

First part of the proof. The direction *(ii)⇒(i)* is easy. Indeed, assume $H = G^{\vee>} - G^{>\vee}$ for some G in $\mathcal{H}_{k-1}$. Then, by Lemma 7.1, for ab in X_k we have,

$$\begin{aligned}\omega_k(H)(ab) &= \omega_{k-1}(G^\vee)(a_1b) - \omega_{k-1}(G^\vee)(ab_1) \qquad &&\text{if } k \text{ is even}\\ &= \omega_{k-1}(G^\vee)(ab_1) - \omega_{k-1}(G^\vee)(a_1b) \qquad &&\text{if } k \text{ is odd.}\end{aligned}$$

The conclusion follows. □

Here is the difficulty for proving the converse statement. Assuming that *(i)* holds, we know from Corollary 2.7 that there exists a function v on X_{k-1} such that, for ab in X_k, one has $\omega_k(H)(ab) = v(ab_1) - v(a_1b)$. But we don't know whether v is the weight function of some $(k-1)$-pseudofunction. It turns out that this is the case, but this requires some work to be proved.

We will need to introduce a new object. For $k \geq 1$, we define a complete k-pseudofunction as a family $(H_{xy})_{xy\in X_1}$ such that, for xy in X_1, ik k is even, $k = 2\ell$, $\ell \geq 1$, H_{xy} is an element of $V^\ell(x)$; if k is odd, $k = 2\ell + 1$, $\ell \geq 0$, H_{xy} is an element of $V^\ell(xy)$. Thus, the definition is the same as the one of a k-pseudofunction, except that we don't kill the constant functions in the spaces $V^\ell(x)$ and $V^\ell(xy)$.

Starting from a complete k-pseudofunction, one can obtain a k-pseudofunction by killing the constant part. In particular, if $\widetilde{\mathcal{H}}_k$ is the space of Γ-invariant complete k-pseudofunctions, we have a natural map $\widetilde{\mathcal{H}}_k \to \mathcal{H}_k$.

Lemma 7.5. *Let $k \geq 1$. The natural map $\widetilde{\mathcal{H}}_k \to \mathcal{H}_k$ is surjective.*

Since the stabilizers of the elements of X in Γ are finite, the proof is a direct consequence of the following classical phenomenon in group theory:

Lemma 7.6. *Let V be a real vector space, equipped with an action of a finite group G. Let W be a G-invariant subspace. Then the natural map $V \to V/W$ maps the space V^G of G-invariant elements of V onto $(V/W)^G$.*

The advantage of dealing with complete pseudofunctions is that for them, we can replace the weight by a function that is defined on smaller segments. Indeed, for $k \geq 1$ and H a complete k-pseudofunction, we define the pseudoweight $\rho_k(H)$ of H as follows. If k is even, $k = 2\ell$, $\ell \geq 1$, for ab in $X_{\ell+1}$, we set

$$\rho_k(H)(ab) = \sum_{\substack{c\in X\\ d(b,c)=\ell\\ b_1\in[bc]}} H_{cc_-}(b) - \sum_{i=1}^{\ell-1} \sum_{\substack{c\in X\\ d(b_1,c)=i\\ b,b_2\notin[b_1c]}} H_{cc_-}(a_i) - H_{b_1b}(a). \tag{7.1}$$

If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, for ab in $X_{\ell+1}$, we set

$$\rho_k(H)(ab) = \sum_{\substack{c\in X\\ d(b,c)=\ell+1\\ b_1\in[bc]}} H_{cc_-}(b) - \sum_{i=1}^{\ell} \sum_{\substack{c\in X\\ d(b_1,c)=i\\ b,b_2\notin[b_1c]}} H_{cc_-}(a_{i-1}) - H_{b_1b}(a). \tag{7.2}$$

From Definition 6.1, we directly get

Lemma 7.7. *Let $k \geq 1$ and H be in $\widetilde{H}_k$. The pseudoweight $\rho_k(H)$ is cohomologous to the weight $\omega_k(H)$.*

By abuse of language, we have denoted by $\omega_k(H)$ the weight of the image of H in $\mathcal{H}_k$.

We define the natural operations $H \mapsto H^>$ and $H \mapsto H^\vee$ for complete pseudofunctions as for pseudofunctions (see Subsection III.2.2).

As in Subsection I.8.7, for $k \geq 1$, we say that a Γ-invariant function w on X_k is split if we can find Γ-invariant functions u and v on X_{k-1} such that, for xy in X_k, one has

$$w(xy) = u(xy_1) + v(x_1y).$$

We have a criterion for a complete k-pseudofunction to be obtained through complete $(k-1)$-pseudofunctions.

Lemma 7.8. *Let $k \geq 2$ and H be in $\widetilde{\mathcal{H}}_k$.*

If k is odd, there exists F, G in $\widetilde{\mathcal{H}}_{k-1}$ with $H = F^> + G^{>\vee}$.

If k is even, $k = 2\ell$, $\ell \geq 1$, then the following are equivalent:

(i) *there exists F, G in $\widetilde{\mathcal{H}}_{k-1}$ with $H = F^> + G^{>\vee}$.*

(ii) *the function $xy \mapsto H_{y_1y}(x)$ is split on $X_{\ell+1}$.*

(iii) *the pseudoweight $\rho_k(H)$ is split on $X_{\ell+1}$.*

Proof. Assume k is odd, $k = 2\ell + 1$, $\ell \geq 1$, and recall from Proposition I.4.6 that, for xy in X_1, we have $V^\ell(xy) = J^\ell_{xy}V^\ell(x) + J^\ell_{yx}V^\ell(y)$. Choose a system of representatives $S \subset X_1$. for the action of Γ on X_1. Then, for xy in S, H_{xy} is a $(\Gamma_x \cap \Gamma_y)$-invariant element of $V^\ell(xy)$. By Lemma

7.6, there exists $(\Gamma_x \cap \Gamma_y)$-invariant elements f_{xy} in $V^\ell(x)$ and g_{yx} in $V^\ell(y)$ with

$$H_{xy} = J^\ell_{xy} f_{xy} + J^\ell_{yx} g_{yx}.$$

As f_{xy} and g_{yx} are $\Gamma_x \cap \Gamma_y$-invariant, there exists unique F, G in $\widetilde{\mathcal{H}}_{k-1}$ such that $F_{xy} = f_{xy}$ and $G_{yx} = g_{yx}$ for xy in S. By construction, we have $H = F^{>} + G^{>\vee}$.

Assume now k is even, $k = 2\ell$, $\ell \geq 1$ and first note that, in view of the definition of the pseudoweight $\rho_k(H)$ in (7.1), the function $xy \mapsto \rho_k(H)(xy) + H_{y_1 y}(x)$ is split on $X_{\ell+1}$, so that *(ii)* is equivalent to *(iii)*.

Let us prove *(i)⇒(ii)*. Suppose we may write $H = F^{>} + G^{>\vee}$ with F, G in $\widetilde{\mathcal{H}}_{k-1}$. Then, a direct computation gives, for xy in $X_{\ell+1}$,

$$\begin{aligned} H_{y_1 y}(x) &= F^{>}_{y_1 y}(x) + \sum_{\substack{z \sim y_1 \\ z \neq y}} G^{>}_{y_1 z}(x) \\ &= F_{y_1 y}(x_1) + \sum_{\substack{z \sim y_1 \\ z \notin \{y, y_2\}}} G_{y_1 z}(x_1) + G_{y_1 y_2}(x) \\ &= F_{y_1 y}(x_1) + \sum_{\substack{z \sim y_1 \\ z \neq y_2}} G_{y_1 z}(x_1) - G_{y_1 y}(x_1) + G_{y_1 y_2}(x) \end{aligned}$$

(the last step only being necessary when $k = 2$). As required, the function $xy \mapsto H_{y_1 y}(x)$ is split.

Conversely, we now prove *(ii)⇒(i)*. Thus, assume we may find Γ-invariant functions v and w on X_ℓ such that, for xy in $X_{\ell+1}$, one has

$$H_{y_1 y}(x) = v(xy_1) + w(x_1 y).$$

We define F and G in $\widetilde{\mathcal{H}}_{k-1}$ as follows: for xy in X_ℓ, we set

$$\begin{aligned} F_{yy_1}(x) &= H_{yy_1}(x) & \qquad F_{y_1 y}(x) &= w(xy) \\ G_{yy_1}(x) &= v(xy) & \qquad G_{y_1 y}(x) &= 0. \end{aligned}$$

A direct computation then shows that $H = F^{>} + G^{>\vee}$. □

Second part of the proof of Proposition 7.4. We prove *(i)⇒(ii)* by induction on $k \geq 1$. Let H be in $\mathcal{H}_k$ and assume that $\omega_k(H)$ is a coboundary. By Corollary 2.7, we may find a Γ-invariant function v on X_{k-1} such that, for any xy in X_k, one has

$$\omega_k(H)(xy) = v(xy_1) - v(x_1 y).$$

If $k = 1$, we let G be the 0-pseudofunction associated with the function $-v$ on $X_0 = X$. Then, Definition 6.1 and Lemma 7.1 imply that we have $\omega_1(H - G^{\vee>} + G^{>\vee}) = 0$, hence $H = G^{\vee>} - G^{>\vee}$ by Corollary 7.3.

If $k = 2$, we let G be the 1-pseudofunction associated with the function $xy \mapsto v(yx)$ on X_1. As above, we get $H = G^{\vee>} - G^{>\vee}$.

Suppose $k \geq 3$ and the result holds for $k-1$; let us show that it also holds for k.

If k is odd, by Lemma 7.5 and Lemma 7.8, we may find J, K in $\mathcal{H}_{k-1}$ with

$$H = J^{>} + K^{>\vee} = (J + K^{\vee})^{>} - K^{\vee>} + K^{>\vee} = L^{>} - K^{\vee>} + K^{>\vee}, \tag{7.3}$$

where $L = J + K^{\vee}$. Then the first part of the proof and the assumption ensure that the weight $\omega_k(L^{>})$ is a coboundary. By Lemma 7.1, the weight $\omega_{k-1}(L)$ is a coboundary. Thus, by the induction assumption, we may find M in $\mathcal{H}_{k-2}$ with $L = M^{\vee>} - M^{>\vee}$. By (7.3), we get

$$\begin{aligned} H = L^{>} - K^{\vee>} + K^{>\vee} &= M^{\vee>>} - M^{>\vee>} - K^{\vee>} + K^{>\vee} \\ &= (M^{>} + K)^{>\vee} - (M^{>} + K)^{\vee>} \end{aligned}$$

(where we have used Lemma III.2.6). The conclusion follows.

If k is even, $k = 2\ell$, $\ell \geq 1$, we will proceed in the same way, by showing that the assumption of Lemma 7.8 is satisfied. Indeed, by Lemma 7.5, we may assume that H is a Γ-invariant complete k-pseudofunction (which we still denote by H by abuse of language). Then, by the assumption and Lemma 7.7, the pseudoweight $\rho_k(H)$ is a coboundary. By Corollary 2.7, this means that we may find a Γ-invariant function v on X_ℓ such that, for any ab in $X_{\ell+1}$, one has

$$\rho_k(H)(ab) = v(ab_1) - v(a_1 b).$$

In particular, $\rho_k(H)$ is split and, by Lemma 7.8, there exists J, K in $\mathcal{H}_{k-1}$ with $H = J^{>} + K^{>\vee}$. We conclude as in the odd case. □

7.4. **Sequences of pseudofunctions.** We will now prove a statement for sequences of weights that may be seen as a generalization of Proposition 7.4. The proof will rely on some improvements of the techniques used above. It will also serve as a model for the proof of Proposition 9.3 below, which will be a further generalization that will play a crucial role in translating the conclusion of Proposition 3.3 in the language of pseudofunctions.

We start with a definition that is inspired by the language of Proposition 3.3.

Definition 7.9. Let $k \geq 1$. We say that a finitely supported sequence $(w_j)_{j\geq 1}$ in V_k is cohomologically trivial if there exists a finitely supported sequence $(v_j)_{j\geq 1}$ in V_{k-1} such that, for $j \geq 1$ and ab in X_k, one has

$$w_j(ab) = v_j(ab_1) - v_{j-1}(a_1 b).$$

This notion is invariant under some shifts.

Lemma 7.10. *Let $h \geq k \geq 1$ and $(w_j)_{j\geq 1}$ be a finitely supported sequence in V_k. For $j \geq 1$ and ab in X_h, set $w'_j(ab) = w_j(aa_k)$. Assume the sequence $(w'_j)_{j\geq 1}$ is cohomologically trivial in V_h. Then, the sequence $(w_j)_{j\geq 1}$ is cohomologically trivial in V_k.*

The proof will use the easy

Lemma 7.11. *Let $k \geq 1$ and u, v be Γ-invariant functions on X_k. Assume that, for any ab in X_{k+1}, we have*

$$u(ab_1) = v(a_1 b).$$

Then, there exists a Γ-invariant function w on X_{k-1} such that, for every ab on X_k, one has

$$u(ab) = w(a_1 b) \text{ and } v(ab) = w(ab_1).$$

Proof of Lemma 7.10. It suffices to prove the statement when $h = k+1$, the general case following by an easy induction. If $h = k+1$, we can find a finitely supported sequence $(v_j)_{j\geq 0}$ in V_k such that, for $j \geq 0$ and ab in X_{k+1}, we have

$$w_{j+1}(ab_1) - v_{j+1}(ab_1) = -v_j(a_1 b).$$

By Lemma 7.11, there exists a function u_j on X_{k-1} such that, for ab in X_{k-1}, we have

$$w_{j+1}(ab) - v_{j+1}(ab) = -u_j(a_1 b) \text{ and } v_j(ab) = u_j(ab_1).$$

Thus, for $j \geq 1$, we get

$$w_j(ab) = u_j(ab_1) - u_{j-1}(a_1 b)$$

as required. □

The objective of the remainder of the Section is to show

Proposition 7.12. *Let $k \geq 1$ and $(H_j)_{j\geq 1}$ be a finitely supported sequence of elements of $\mathcal{H}_k$. Assume that the sequence $(\omega(H_j))_{j\geq 1}$ is cohomologically trivial in V_k. Then, there exists a finitely supported sequence $(G_j)_{j\geq 0}$ of elements of $\mathcal{H}_{k-1}$ such that, for $j \geq 1$, one has*

$$\begin{aligned} H_j &= G_j^{>\vee} - G_{j-1}^{\vee>} && \text{if } k \text{ is even} \\ &= G_j^{\vee>} - G_{j-1}^{>\vee} && \text{if } k \text{ is odd.} \end{aligned}$$

Note that the converse is also true by Lemma 7.1.

7.5. **Splitting sequences.** The proof of Proposition 7.12 will follow the same lines as the one of Proposition 7.4. In particular, we will show that the assumption implies that we can apply the criterion of Lemma 7.8 to the pseudofunctions H_j, $j \geq 1$. This is achieved in the following

Lemma 7.13. *Let $k \geq 2$ be an even integer, $k = 2\ell$, $\ell \geq 1$, and $(H_j)_{j\geq 1}$ be a finitely supported sequence of elements of $\mathcal{H}_k$. Assume that the sequence $(\omega(H_j))_{j\geq 0}$ is cohomologically trivial in V_k. Then, for every $j \geq 1$, there exists F_j and G_j in $\mathcal{H}_{k-1}$ with*

$$H_j = F_j^{>} + G_j^{>\vee}.$$

We introduce new notation for the proof. We keep the language of Subsection 7.3. For $k \geq 4$ an even integer, $k = 2\ell$, $\ell \geq 2$, and H a complete k-pseudofunction, we define a family of functions on segments of different sizes.

For ab in $X_{\ell+1}$, we set

$$\rho_k^0(H)(ab) = -H_{b_1 b}(a).$$

For $1 \leq i \leq \ell - 2$ and ab in $X_{\ell-i+1}$, we set

$$\rho_k^i(H)(ab) = -\sum_{\substack{c\in X\\ [b_1 c]\cap[ab]=\{b_1\}\\ d(c,b_1)=i}} H_{cc_-}(a).$$

And lastly, for $i = \ell - 1$ and ab in X_2, we set

$$\rho_k^{\ell-1}(H)(ab) = \sum_{\substack{c\in X\\ b_1\in[bc]\\ d(c,b)=\ell}} H_{cc_-}(b) - \sum_{\substack{c\in X\\ [b_1 c]\cap[ab]=\{b_1\}\\ d(c,b_1)=\ell-1}} H_{cc_-}(a).$$

Thus, Definition 6.1 can be rewritten as, for ab in X_k,

$$\omega_k(H)(ab) = \sum_{i=0}^{\ell-1} \rho_k^i(H)(a_{2i} a_{\ell+i+1}). \tag{7.4}$$

If $k = 2$, we set $\rho_2^0(H) = \omega_2(H)$, so that (7.4) still holds.

Proof of Lemma 7.13. Note that, for $k = 2$, the statement directly follows from Definition 6.1, Lemma 7.5 and Lemma 7.8. Assume $k \geq 4$ and let $(v_j)_{j\geq 0}$ be a finitely supported sequence in V_{k-1} such that, for ab in X_k and $j \geq 1$, we have

$$\omega(H_j)(ab) = v_j(ab_1) - v_{j-1}(a_1 b).$$

By Lemma 7.5, we can assume that $(H_j)_{j\geq 1}$ is a finitely supported sequence of elements of $\widetilde{\mathcal{H}}_k$. Then, by (7.4), we get

$$v_j(ab_1) - v_{j-1}(a_1b) = \sum_{i=0}^{\ell-1} \rho_k^i(H_j)(a_{2i}a_{\ell+i+1})$$

$$= \sum_{i=0}^{\ell-1} \rho_k^i(H_{i+j})(a_ia_{\ell+1}) - \sum_{i=1}^{\ell-1}(\rho_k^i(H_{i+j})(a_ia_{\ell+1}) - \rho_k^i(H_j)(a_{2i}a_{\ell+i+1}))$$

$$= \sum_{i=0}^{\ell-1} \rho_k^i(H_{i+j})(a_ia_{\ell+1}) - v'_j(ab_1) + v'_{j-1}(a_1b),$$

where, for $j \geq 0$ and ab in X_{k-1}, we have set

$$v'_j(ab) = \sum_{i=1}^{\ell-1}\sum_{h=0}^{i-1} \rho_k^i(H_{i+j-h})(a_{h+i}a_{h+\ell+1}).$$

By Lemma 7.10, there exists a finitely supported sequence $(v''_j)_{j\geq 0}$ of Γ-invariant functions on $X_{\ell+1}$ such that, for any $j \geq 1$ and ab in $X_{\ell+1}$, one has

$$\sum_{i=0}^{\ell-1} \rho_k^i(H_{i+j})(a_ia_{\ell+1}) = v''_j(ab_1) - v''_{j-1}(a_1b).$$

In particular, the function $\rho_k^0(H_j)$ is split on $X_{\ell+1}$ and the conclusion follows from Lemma 7.8. □

We can now conclude by using an induction argument.

Proof of Proposition 7.12. As for Proposition 7.4, we prove this statement by induction on $k \geq 1$.

Assume $k = 1$. For $j \geq 0$, we let G_j be the 0-pseudofunction associated with the function v_j on X_0 (see Subsection III.2.1). Then, Definition 6.1, Lemma 7.1 and Corollary 7.3 imply that, for $j \geq 1$, we have $H_j = G_{j-1}^{>\vee} - G_j^{\vee>}$.

Assume $k = 2$. For $j \geq 0$, we let G_j be the 1-pseudofunction associated with the function $xy \mapsto v_j(yx)$ on X_1 (see again Subsection III.2.1). Then, as above, for $j \geq 1$, we get $H_j = G_{j-1}^{\vee>} - G_j^{>\vee}$.

Suppose now $k \geq 3$ and the statement holds for $k-1$. If k is odd, we know from Lemma 7.5 and Lemma 7.8 that we may find sequences $(J_j)_{j\geq 1}$ and $(K_j)_{j\geq 1}$ in $\mathcal{H}_{k-1}$ such that, for $j \geq 1$, we have

$$H_j = J_j^> + K_j^{>\vee} = (J_j + K_{j+1}^\vee)^> + K_j^{>\vee} - K_{j+1}^{\vee>}.$$

As $(H_j)_{j\geq 1}$ is finitely supported, we may assume that $(J_j)_{j\geq 1}$ and $(K_j)_{j\geq 1}$ also are.

For $j \geq 1$, we set $L_j = J_j + K_{j+1}^{\vee}$. Then, by Lemma 7.1, for ab in X_k, we have

$$\begin{aligned}\omega_{k-1}(L_j)(ab_1) &= \omega_k(L_j^{>})(ab)\\ &= \omega_k(H_j)(ab) - \omega_k(K_j^{>\vee})(ab) + \omega_k(K_{j+1}^{\vee>})(ab)\\ &= \omega_k(H_j)(ab) - \omega_{k-1}(K_j^{\vee})(a_1b) + \omega_{k-1}(K_{j+1}^{\vee})(ab_1).\end{aligned}$$

Therefore, by Lemma 7.10, the sequence $(\omega_{k-1}(L_j))_{j\geq 1}$ is cohomologically trivial in V_{k-1}. By the induction assumption, there exists a finitely supported sequence $(M_j)_{j\geq 0}$ of elements of $\mathcal{H}_{k-2}$ such that, for $j \geq 1$, one has

$$L_j = M_j^{>\vee} - M_{j-1}^{\vee>}.$$

By using Lemma III.2.6, we get

$$H_j = L_j^{>} + K_j^{>\vee} - K_{j+1}^{\vee>} = (M_j^{>} - K_{j+1})^{\vee>} - (M_{j-1}^{>} - K_j)^{>\vee}$$

and we are done.

Suppose now k is even. By Lemma 7.13, there exist sequences $(J_j)_{j\geq 1}$ and $(K_j)_{j\geq 1}$ in $\mathcal{H}_{k-1}$, which we may assume to be fintely supported, such that, for $j \geq 1$, we have

$$H_j = J_j^{>} + K_j^{>\vee} = (J_{j+1}^{\vee} + K_j)^{>\vee} + J_j^{>} - J_{j+1}^{\vee>\vee}.$$

We now set $L_j = J_{j+1} + K_j^{\vee}$, so that Lemma 7.1 gives, for ab in X_k,

$$\begin{aligned}\omega_{k-1}(L_j)(ab_1) &= \omega_k(L_j^{\vee>\vee})(ab)\\ &= \omega_k(H_j)(ab) - \omega_k(J_j^{>})(ab) + \omega_k(J_{j+1}^{\vee>\vee})(ab)\\ &= \omega_k(H_j)(ab) - \omega_{k-1}(J_j)(a_1b) + \omega_{k-1}(J_{j+1})(ab_1).\end{aligned}$$

By Lemma 7.10, the sequence $(\omega_{k-1}(L_j))_{j\geq 1}$ is cohomologically trivial in V_{k-1}. By the induction assumption, there exists a finitely supported sequence $(M_j)_{j\geq 0}$ of elements of $\mathcal{H}_{k-2}$ such that, for $j \geq 1$, one has

$$L_j = M_j^{\vee>} - M_{j-1}^{>\vee}.$$

By using again Lemma III.2.6, we obtain

$$H_j = L_j^{\vee>\vee} + J_j^{>} - J_{j+1}^{\vee>\vee} = (M_j^{\vee>\vee} - J_{j+1}^{\vee})^{>\vee} - (M_{j-1}^{\vee>\vee} - J_j^{\vee})^{\vee>}$$

as required. □

8. Simplification schemes

Our objective in the next two Sections is to show Proposition 9.3, which is a generalization of Proposition 7.12 for functions of two sets of variables. Later, it will be used to check the consequences of Proposition 3.3 when it is applied to the objects appearing in the Plancherel formula in Proposition 5.16.

In the present Section, as a preliminary, we study certain linear equations on functions on two variables. As analogous equations will also appear later on tensor products of spaces of pseudofunctions, we regroup those two studies in a common abstract formalism, which we call the language of simplication schemes.

8.1. **Definition and examples.** We now introduce precisely this language and we relate it to our two examples.

Definition 8.1. A simplification scheme is a family

$$(V_-, V, V_+, L, R, L_+, R_+)$$

where V_-, V and V_+ are real vector spaces and L and R are injective linear maps $V_- \to V$ and L_+ and R_+ are injective linear maps $V \to V_+$ with the following property: we have $L_+R = R_+L$ and, for every v and w in V, if

$$L_+v = R_+w,$$

there exists u in V with $v = Ru$ and $w = Lu$.

We set this definition in order to encompass the following two examples.

Example 8.2. For $k \geq 0$, we set $V_- = \mathcal{H}_{k-1}$, $V = \mathcal{H}_k$ and $V_+ = \mathcal{H}_{k+1}$. Then, for G in V_- and H in V, we set

$$LG = G^{>} \qquad RG = G^{\vee>}$$
$$L_+H = H^{>} \qquad R_+H = H^{>\vee}.$$

This defines a simplification scheme as follows from Lemma III.2.6 and Lemma III.2.8. The same construction works for pseudokernels instead of pseudofunctions, by Lemma II.2.4 and Lemma II.2.5.

Example 8.3. For $k \geq 1$, we set $V_- = V_{k-1}$, $V = V_k$ and $V_+ = V_{k+1}$. Then, for f in V_- and ab in X_k, we set

$$Lf(ab) = f(a_1b) \text{ and } Rf(ab) = f(ab_1).$$

In the same way, for g in V_+ and ab in X_{k+1}, we set

$$L_+g(ab) = g(a_1b) \text{ and } R_+g(ab) = g(ab_1).$$

This defines a simplification scheme by Lemma 7.11.

We extend this definition for $k = 0$ in the following way. The space V_0 is the space of Γ-invariant functions on $X_0 = X$. We let V_{-1} be the space of Γ-invariant functions on X which are constant on neighbours: this space is the line of constant functions if Γ is not bipartite; it has dimension 2 else. If f is in V_{-1}, we set Lf to be f, viewed as an element of V_0, and Rf to be the opposite of f, that is, for $a \sim b$ in X, $Rf(a) = f(b)$. If g is in V_0, for $a \sim b$ in X, we set

$$L_+ g(ab) = g(b) \text{ and } R_+ g(ab) = g(a).$$

Again, one easily checks that this defines a simplification scheme.

To avoid heavy notation, in this sequel, we will only write (V_-, V, V_+) to mean a simplification scheme and we will use the letters L and R to design the associated linear maps for every simplification scheme. We will also simply write L instead of L_+ and R instead of R_+.

8.2. **Tensor products.** In the rest of the Section, we will describe how the simplification rule of simplification schemes behaves in tensor products. Here, we start by introducing precisely our notation for tensor products.

Let V and W be vector spaces. We write $V \otimes W$ for the algebraic tensor product of V with W. If $V = W$, we write $\otimes^2 V$ for $V \otimes V$. If X is another vector space and $\varphi : V \times W \to X$ is a bilinear map, we still write $\varphi : V \otimes W \to X$ for the linear map such that $\varphi(v \otimes w) = \varphi(v, w)$, for v in V and w in W. If V' and W' are other vector spaces and $\chi : V \to V'$ and $\psi : W \to W'$ are linear maps, we denote by $u \mapsto \chi u \psi$ the natural associated linear map $V \otimes W \to V' \otimes W'$, so that, for v in V and w in W, one has $\chi(v \otimes w)\psi = (\chi v) \otimes (\psi w)$.

The following is standard:

Lemma 8.4. *Let V, V' and W be vector spaces and $\varphi : V \to V'$ be a linear map. Then the linear map $u \mapsto \varphi u, V \otimes W \to V' \otimes W$ has kernel $(\ker \varphi) \otimes W$ and range $(\varphi V) \otimes W$.*

As in Section 3, for $h, k \geq 1$, we write V_k for the space of Γ-invariant functions on X_k and $W_{h,k}$ for the space of $(\Gamma \times \Gamma)$-invariant functions on $X_h \times X_k$. We identify $W_{h,k}$ with the tensor product $V_h \times V_k$ in the standard way. More precisely, for v in V_h and w in V_k, we consider $v \otimes w$ as the function on $X_h \times X_k$ defined by

$$(v \otimes w)(ab, xy) = v(ab)w(xy), \quad ab, xy \in X_k.$$

8.3. **Tensor products of simplification schemes.** In the abstract framework of simplification schemes, we establish the following result that will allow us to solve functional equations with two variables.

Proposition 8.5. *Let (V_-, V, V_+) and (W_-, W, W_+) be simplification schemes. Take g, h in $V \otimes W_+$ and j, k in $V_+ \otimes W$. Assume we have*

$$Lg + Rh + jL + kR = 0.$$

Then, there exist a in $V_- \otimes W_+$, b in $V_+ \otimes W_-$ and c, d, e, f in $V \otimes W$ such that

$$g = Ra + cL + dR \qquad h = -La + eL + fR$$
$$j = bR - Lc - Re \qquad k = -Lb - Ld - Rf.$$

We will split the proof into several steps.

Lemma 8.6. *Let (V_-, V, V_+) and (W_-, W, W_+) be simplification schemes. Asssume f is in $V_- \otimes W_+$ and g, h are in $V \otimes W$ and we have*

$$Lf = gL + hR.$$

Then, there exist a in $V \otimes W_-$ and b, c in $V_- \otimes W$ with

$$f = bL + cR$$
$$g = aR + Lb$$
$$h = -aL + Lc.$$

Proof. The relation $Lf = gL + hR$ implies that $gL + hR$ has trivial image in $(V/LV_-) \otimes W_+$. Therefore, by Definition 8.1 and Lemma 8.4, there exists a in $V \otimes W_-$ such that $g - aR$ and $h + aL$ both belong to $LV_- \otimes W$. In other words, we can find b, c in $V_- \otimes W$ with

$$g = aR + Lb \text{ and } h = -aL + Lc.$$

We get

$$Lf = gL + hR = LbL + LcR$$

and the conclusion follows. □

Lemma 8.7. *Let (V_-, V, V_+) and (W_-, W, W_+) be simplification schemes and f, g, h, j be in $V \otimes W$. Assume we have*

$$LfL + RgL + LhR + RjR = 0.$$

Then, there exist a, d in $V_- \otimes W$ and b, c in $V \otimes W_-$ such that

$$f = Ra + bR \qquad g = -La + cR$$
$$h = Rd - bL \qquad j = -Ld - cL.$$

Proof. We write the starting equation as

$$(Lf + Rg)L + (Lh + Rj)R = 0.$$

By Definition 8.1 and Lemma 8.4, there exists k in $V_+ \otimes W_-$ such that

$$Lg + Rg = kR \text{ and } Lh + Rj = -kL.$$

By Lemma 8.6, we can find u, x in $V_- \otimes W$ and v, w, y, z in $V \otimes W_-$ with

$$\begin{aligned} k &= Lv + Rw & -k &= Ly + Rz \\ f &= Ru + vR & h &= Rx + yL \\ g &= -Lu + wR & j &= -Lx + zL. \end{aligned}$$

By looking at the two equations above which involve k, we get

$$Lv + Rw + Ly + Rz = 0.$$

By Definition 8.1 and Lemma 8.4, we know that there exists l in $V_- \otimes W_-$ with

$$v + y = Rl \text{ and } w + z = -Ll,$$

which yields

$$\begin{aligned} f &= Ru - yR + RlR & h &= Rx + yL \\ g &= -Lu - zR - LlR & j &= -Lx + zL. \end{aligned}$$

The result folllows with $a = u + lR$, $b = -y$, $c = -z$ and $d = x$. □

Proof of Proposition 8.5. The assumption implies that $Lg + Rh$ has trivial image in $V_+ \otimes (W_+/(LW + RW))$. Therefore, by Definition 8.1 and Lemma 8.4, there exists l in $V_- \otimes W_+$ such that $g - Rl$ and $h + Ll$ both belong to $V \otimes (LW + RW)$. We choose g_0, g_1, h_0, h_1 in $V \otimes W$ with

$$g = Rl + g_0 L + g_1 R \text{ and } h = -Ll + h_0 L + h_1 R.$$

In the same way, we can find m in $V_+ \otimes W_-$ and j_0, j_1, k_0, k_1 in $V \otimes W$ satisfying

$$j = mR + Lj_0 + Rj_1 \text{ and } k = -mL + Lk_0 + Rk_1.$$

The assumption now reads as

$$Lg_0L + Lg_1R + Rh_0L + Rh_1R + Lj_0L + Rj_1L + Lk_0R + Rk_1R = 0,$$

which we rewrite as

$$L(g_0 + j_0)L + R(h_0 + j_1)L + L(g_1 + k_0)R + R(h_1 + k_1)R = 0.$$

We can therefore apply Lemma 8.7. This tells us that we may find u, x in $V_- \otimes W$ and v, w in $V \otimes W_-$ with

$$\begin{aligned} g_0 + j_0 &= Ru + vR & h_0 + j_1 &= -Lu + wR \\ g_1 + k_0 &= Rx - vL & h_1 + k_1 &= -Lx - wL. \end{aligned}$$

We get

$$\begin{aligned} j = mR + Lj_0 + Rj_1 &= mR - Lg_0 + LRu + LvR - Rh_0 - RLu + RwR \\ &= mR - Lg_0 + LvR - Rh_0 + RwR \end{aligned}$$

and in the same way,

$$\begin{aligned}k = -mL+Lk_0+Rk_1 &= -mL-Lg_1+LRx-LvL-Rh_1-RLx-RwL\\ &= -mL - Lg_1 - LvL - Rh_1 - RwL.\end{aligned}$$

Since $g = Rl + g_0L + g_1R$ and $h = -Ll + h_0L + h_1R$, the result follows with

$$\begin{array}{lll} a = l & c = g_0 & d = g_1 \\ b = m + Lv + Rw & e = h_0 & f = h_1. \end{array}$$

□

8.4. **The case of pseudofunctions.** We now translate Proposition 8.5 for our concrete examples and we add the description of the boundary cases. In case of pseudofunctions, as in Example 8.2, we get

Corollary 8.8. *Let $h, k \geq -1$, G, H be in $\mathcal{H}_h \otimes \mathcal{H}_{k+1}$ and J, K be in $\mathcal{H}_{h+1} \otimes \mathcal{H}_k$. Assume we have*

$$^{>}G + {}^{\vee>}H + J^{>} + K^{>\vee} = 0.$$

Then, if h and k are both ≥ 0, there exist A in $\mathcal{H}_{h-1} \otimes \mathcal{H}_{k+1}$, B in $\mathcal{H}_{h+1} \otimes \mathcal{H}_{k-1}$ and C, D, E, F in $\mathcal{H}_h \otimes \mathcal{H}_k$ such that

$$\begin{array}{ll} G = {}^{>\vee}A + C^{>} + D^{>\vee} & H = -{}^{>}A + E^{>} + F^{>\vee} \\ J = B^{\vee>} - {}^{>}C - {}^{\vee>}E & K = -B^{>} - {}^{>}D - {}^{\vee>}F. \end{array}$$

If $h \geq 0$ and $k = -1$, there exist A in $\mathcal{H}_{h-1} \otimes \mathcal{H}_0$ and B, C in $\mathcal{H}_h \otimes \mathcal{H}_{-1}$ with

$$\begin{aligned} K - J &= {}^{>}B + {}^{\vee>}C \\ G &= {}^{\vee>}A + B^{>} \\ H &= -{}^{>}A + C^{>}. \end{aligned}$$

If $h = k = -1$, there exists A in $\mathcal{H}_{-1} \otimes \mathcal{H}_{-1}$ with

$$G - H = A^{>} \text{ and } K - J = {}^{>}A.$$

Proof. In case $h, k \geq 0$, this is Proposition 8.5. In case $h \geq 0$ and $k = -1$, the equation reads as

$$^{>}G + {}^{\vee>}H = (K - J)^{>}$$

and the conclusion follows from Lemma 8.6. Finally, if $h = k = -1$, we have

$$^{>}(G - H) = (K - J)^{>}$$

and the conclusion is obvious. □

We also include the translation of Lemma 8.6 which we shall need later.

Corollary 8.9. *Let $h, k \geq 0$. Asssume F is in $\mathcal{H}_{h-1} \otimes \mathcal{H}_{k+1}$ and G, H are in $\mathcal{H}_h \otimes \mathcal{H}_k$ and we have*

$$^{>}F = G^{>} + H^{>\vee}.$$

Then, there exist A in $\mathcal{H}_h \otimes \mathcal{H}_{k-1}$ and B, C in $\mathcal{H}_{h-1} \otimes \mathcal{H}_k$ with

$$\begin{aligned} F &= B^{>} + C^{>\vee} \\ G &= A^{\vee>} + {}^{>}B \\ H &= -A^{>} + {}^{>}C. \end{aligned}$$

8.5. **The case of functions on segments.** In the case of Example 8.3, we get

Corollary 8.10. *Let $h, k \geq 0$, g, h be in $W_{h,k+1}$ and j, k be in $W_{h+1,k}$. Assume that, for pq in X_{h+1} and xy in X_{k+1}, we have*

$$g(p_1q, xy) + h(pq_1, xy) + j(pq, x_1y) + k(pq, xy_1) = 0.$$

Then, if h and k are both ≥ 1, there exist a in $W_{h-1,k+1}$, b in $W_{h+1,k-1}$ and c, d, e, f in $W_{h,k}$ such that, for pq in X_h and xy in X_{k+1},

$$\begin{aligned} g(pq, xy) &= a(pq_1, xy) + c(pq, x_1y) + d(pq, xy_1) \\ h(pq, xy) &= -a(p_1q, xy) + e(pq, x_1y) + f(pq, xy_1) \end{aligned}$$

and, for pq in X_{h+1} and xy in X_k,

$$\begin{aligned} j(pq, xy) &= b(pq, xy_1) - c(p_1q, xy) - e(pq_1, xy) \\ k(pq, xy) &= -b(pq, x_1y) - d(p_1q, xy) - f(pq_1, xy). \end{aligned}$$

If $h \geq 1$ and $k = 0$, there exist a in $W_{h-1,1}$, b in $W_{h+1,-1}$ and c, d, e, f in $W_{h,0}$ such that, for pq in X_h and X_1,

$$\begin{aligned} g(pq, xy) &= a(pq_1, xy) + c(pq, y) + d(pq, x) \\ h(pq, xy) &= -a(p_1q, xy) + e(pq, y) + f(pq, x) \end{aligned}$$

and, for pq in X_{h+1} and xy in X_1,

$$\begin{aligned} j(pq, x) &= b(pq, y) - c(p_1q, x) - e(pq_1, x) \\ k(pq, x) &= -b(pq, x) - d(p_1q, x) - f(pq_1, x). \end{aligned}$$

If $h = k = 0$, there exist a in $W_{-1,1}$, b in $W_{1,-1}$ and c, d, e, f in $W_{0,0}$ such that, for pq, xy in X_1,

$$\begin{aligned} g(p, xy) &= a(q, xy) + c(p, y) + d(p, x) \\ h(pq, xy) &= -a(p, xy) + e(p, y) + f(p, x) \\ j(pq, x) &= b(pq, y) - c(q, x) - e(p, x) \\ k(pq, x) &= -b(pq, x) - d(q, x) - f(p, x). \end{aligned}$$

We also state the results in degenerated cases.

Corollary 8.11. *Let $k \geq 0$, g, h be in $W_{k,0}$ and j be in $W_{k+1,-1}$. Assume that, for pq in X_{k+1} and x in X, we have*

$$g(p_1q, x) + h(pq_1, x) + j(pq, x) = 0.$$

Then, if $k \geq 1$, there exist a in $W_{k-1,0}$ and b, c in $W_{k,-1}$ such that, for pq in X_k and x in X,

$$\begin{aligned} g(pq, x) &= a(pq_1, x) + b(pq, x) \\ h(pq, x) &= -a(p_1q, x) + c(pq, x) \end{aligned}$$

and, for pq in X_{k+1} and x in X,

$$j(pq, x) = -b(p_1q, x) - c(pq_1, x).$$

If $k = 0$, there exist a in $W_{-1,0}$ and b, c in $W_{0,-1}$ such that, for pq in X_1 and x in X,

$$\begin{aligned} g(p, x) &= a(q, x) + b(pq, x) \\ h(pq, x) &= -a(p, x) + c(pq, x) \\ j(pq, x) &= -b(q, x) - c(p, x). \end{aligned}$$

Proof. This is a direct consequence of Lemma 8.6. □

9. Tensors products of pseudofunctions

In this Section, we state and prove Proposition 9.3, which will be our main tool for translating the result of Proposition 3.3 in the language of pseudofunctions.

9.1. **Sequences of tensors.** We define the double weight of tensors, which is obtained directly from the weight construction. We introduce a notion of cohomological triviality for sequences of elements of $W_{h,k}$, $h, k \geq 0$, that is inspired by the language of Proposition 3.3. Then, we state an analogue of Proposition 7.12.

For $k \geq 0$ and v in V_k, we set $v^\vee$ to be the function $ab \mapsto v(ba)$ on X_k. Let still ω_k be the weight of pseudofunctions from Definition 6.1.

Definition 9.1. For $h,k \geq 0$, and H in $\mathcal{H}_h \otimes \mathcal{H}_k$, we define the double weight $\varpi_{h,k}(H)$ as the element $\omega_h H \omega_k^\vee$ of $W_{h,k}$. In other words, for any J in $\mathcal{H}_h$, K in $\mathcal{H}_k$, ab in X_h and xy in X_k, one has

$$\varpi_{h,k}(J \otimes K)(ab, xy) = \omega_h(J)(ab)\omega_k(K)(yx).$$

When $h = k$, we write ϖ_k for $\varpi_{k,k}$.

Definition 9.2. Let $h,k \geq 1$. We say that a finitely supported sequence $(w_j)_{j\geq 1}$ in $W_{h,k}$ is cohomologically trivial if there exist finitely supported sequences $(u_j)_{j\geq 0}$ in $W_{h,k-1}$ and $(v_j)_{j\geq 0}$ in $W_{h-1,k}$ such that, for any $j \geq 1$ and ab in X_h and xy in X_k, one has

$$w_j(ab, xy) = u_j(ab, x_1 y) - u_{j-1}(ab, xy_1) + v_j(ab_1, xy) - v_{j-1}(a_1 b, xy).$$

The following statement is a tensor analogue of Proposition 7.12. Its proof will last until the end of the Section.

Proposition 9.3. *Let $k \geq 1$ and $(H_j)_{j\geq 1}$ be a finitely supported sequence of elements of $\otimes^2 \mathcal{H}_k$. Assume that the sequence $(\varpi_k(H_j))_{j\geq 1}$ is cohomologically trivial in W_k. Then, there exist finitely supported sequences $(F_j)_{j\geq 0}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$ and $(G_j)_{j\geq 0}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$ such that, for $j \geq 1$, one has*

$$\begin{aligned} H_j &= F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1} && \text{if } k \text{ is even} \\ &= F_j^{\vee>} - F_{j-1}^{>\vee} + {}^{>\vee}G_j - {}^{\vee>}G_{j-1} && \text{if } k \text{ is odd.} \end{aligned}$$

Note that the converse is also true by Lemma 7.1.

9.2. **Shortening the dependence.** We will prove Proposition 9.3 by following the same lines as for proving Proposition 7.12. In particular, we will need the following analogue of Lemma 7.10:

Proposition 9.4. *Let $h' \geq h \geq 1$, $k' \geq k \geq 1$ and $(w_j)_{j\geq 1}$ be a finitely supported sequence in $W_{h,k}$. For $j \geq 1$, ab in $X_{h'}$ and xy in $X_{k'}$, set $w'_j(ab, xy) = w_j(aa_h, y_k y)$. Assume that the sequence $(w'_j)_{j\geq 1}$ is cohomologically trivial in $W_{h',k'}$. Then, the sequence $(w_j)_{j\geq 1}$ is cohomologically trivial in $W_{h,k}$.*

This result will follow from several applications of Corollary 8.10 and Corollary 8.11. We summarize them in the technical

Lemma 9.5. *Let $h \geq 1$, $k \geq 0$ and $(u_j)_{j\geq 0}$ be a finitely supported sequence in $W_{h,k}$. Assume that there exist a finitely supported sequence $(u'_j)_{j\geq 1}$ in $W_{h,k-1}$ and finitely supported sequences $(\alpha_j)_{j\geq 1}$, $(\beta_j)_{j\geq 1}$, $(\gamma_j)_{j\geq 1}$ and $(\delta_j)_{j\geq 1}$ in $W_{h-1,k}$ such that, for any $j \geq 1$, ab in X_h and* *xy in X_k, one has*

$$u_j(ab, xy) = u'_j(ab, xy_1) + \alpha_j(a_1b, xy) + \beta_j(ab_1, xy) \tag{9.1}$$
$$u_{j-1}(ab, xy) = u'_j(ab, x_1y) + \gamma_j(a_1b, xy) + \delta_j(ab_1, xy).$$

Then, there exist finitely supported sequences $(\varphi_j)_{j\geq 1}$ in $W_{h-1,k}$ and $(\psi_j)_{j\geq 1}$ in $W_{h-2,k}$ such that, for any $j \geq 2$, ab in X_{h-1} and xy in X_{k+1}, one has

$$\alpha_j(ab, x_1y) - \gamma_j(ab, xy_1) + \beta_{j-1}(ab, x_1y) - \delta_{j-1}(ab, xy_1) =$$
$$\varphi_j(ab, x_1y) - \varphi_{j-1}(ab, xy_1) + \psi_j(ab_1, x_1y) - \psi_{j-1}(a_1b, x_1y).$$

Note that, when $h = 1$ or $k = 0$, there is a slight abuse of notation in (9.1) which should be understood by means of the language of Example 8.3.

Proof. We fix $h \geq 1$ and we prove the result by induction on $k \geq 0$.

For $k = 0$, (9.1) says that, for any $j \geq 1$, ab in X_h and xy in X_1, one has

$$u_j(ab, x) = u'_j(ab, x) + \alpha_j(a_1b, x) + \beta_j(ab_1, x)$$
$$u_{j-1}(ab, x) = u'_j(ab, y) + \gamma_j(a_1b, x) + \delta_j(ab_1, x).$$

If $j \geq 2$, we get

$$u'_{j-1}(ab, x) + \alpha_{j-1}(a_1b, x) + \beta_{j-1}(ab_1, x)$$
$$= u'_j(ab, y) + \gamma_j(a_1b, x) + \delta_j(ab_1, x).$$

Thus, Corollary 8.11 says that we may find v_j in $W_{h-2,0}$ and α'_j and β'_j in $W_{h-1,-1}$ such that, for ab in X_{h-1} and x in X, one has

$$\alpha_{j-1}(ab, x) - \gamma_j(ab, x) = v_j(ab_1, x) + \alpha'_j(ab, x)$$
$$\beta_{j-1}(ab, x) - \delta_j(ab, x) = -v_j(a_1b, x) + \beta'_j(ab, x).$$

If j is large, we assume $v_j = 0$ and $\alpha'_j = \beta'_j = 0$. For $j \geq 2$, ab in X_{h-1} and xy in X_1, we get

$$\alpha_j(ab, y) - \gamma_j(ab, x) + \beta_{j-1}(ab, y) - \delta_{j-1}(ab, x) = \tag{9.2}$$
$$\gamma_{j+1}(ab, y) + v_{j+1}(ab_1, y) + \alpha'_{j+1}(ab, y) - \gamma_j(ab, x)$$
$$+ \delta_j(ab, y) - v_j(a_1b, y) + \beta'_j(ab, y) - \delta_{j-1}(ab, x).$$

We define a finitely supported sequence $(\varepsilon_j)_{j\geq 1}$ in $W_{h-1,-1}$ as follows. For $j \geq 1$, ab in X_{h-1} and x in X, we set

$$\varepsilon_j(ab,x) = -\sum_{\substack{i\geq j+1\\ i-j \text{ even}}} (\alpha'_{i+1}(ab,x) + \beta'_i(ab,x)) \\ - \sum_{\substack{i\geq j+1\\ i-j \text{ odd}}} (\alpha'_{i+1}(ab,y) + \beta'_i(ab,y)),$$

where y is any neighbour of x. Thus, for $j \geq 2$, ab in X_{h-1} and xy in X_1, we get

$$\alpha'_{j+1}(ab,y) + \beta'_j(ab,y) = \varepsilon_j(ab,y) - \varepsilon_{j-1}(ab,x),$$

hence, from (9.2),

$$\begin{aligned}\alpha_j(ab,y) - \gamma_j(ab,x) + \beta_{j-1}(ab,y) - \delta_{j-1}(ab,x) = \\ \gamma_{j+1}(ab,y) - \gamma_j(ab,x) + \delta_j(ab,y) - \delta_{j-1}(ab,x) \\ + v_{j+1}(ab_1,y) - v_j(a_1b,y) + \varepsilon_j(ab,y) - \varepsilon_{j-1}(ab,x)\end{aligned}$$

and the conclusion follows by setting, for $j \geq 1$ and x in X,

$$\begin{aligned}\varphi_j(ab,x) &= \gamma_{j+1}(ab,x) + \delta_j(ab,x) + \varepsilon_j(ab,x), \qquad && ab \in X_{h-1},\\ \psi_j(ab,x) &= v_{j+1}(ab,x), && ab \in X_{h-2}.\end{aligned}$$

We now deal with the case where $k \geq 1$. To avoid using the same abuse of notation as in the statement, we separate the cases $k = 1$ and $k \geq 2$.

For $k = 1$, (9.1) says that, for any $j \geq 1$, ab in X_h and xy in X_1, one has

$$\begin{aligned}u_j(ab,xy) &= u'_j(ab,x) + \alpha_j(a_1b,xy) + \beta_j(ab_1,xy)\\ u_{j-1}(ab,xy) &= u'_j(ab,y) + \gamma_j(a_1b,xy) + \delta_j(ab_1,xy).\end{aligned}$$

If $j \geq 2$, we get

$$\begin{aligned}u'_{j-1}(ab,x) + \alpha_{j-1}(a_1b,xy) + \beta_{j-1}(ab_1,xy)\\ = u'_j(ab,y) + \gamma_j(a_1b,xy) + \delta_j(ab_1,xy).\end{aligned}$$

Thus, Corollary 8.10 says that we may find u''_j in $W_{h,-1}$, v_j in $W_{h-2,1}$ and $\alpha'_j, \beta'_j, \gamma'_j, \delta'_j$ in $W_{h-1,0}$ such that, for ab in X_{h-1} and xy in X_1, one has

$$\begin{aligned}\alpha_{j-1}(ab,xy) - \gamma_j(ab,xy) &= v_j(ab_1,xy) + \alpha'_j(ab,y) - \gamma'_j(ab,x)\\ \beta_{j-1}(ab,xy) - \delta_j(ab,xy) &= -v_j(a_1b,xy) + \beta'_j(ab,y) - \delta'_j(ab,x)\end{aligned} \tag{9.3}$$

and, for ab in X_h and xy in X_1, one has

$$u'_j(ab,x) = u''_j(ab,x) + \alpha'_j(a_1b,x) + \beta'_j(ab_1,x)$$
$$u'_{j-1}(ab,x) = u''_j(ab,y) + \gamma'_j(a_1b,x) + \delta'_j(ab_1,x).$$

If j is large, we assume $u''_j = 0$, $v_j = 0$ and $\alpha'_j = \beta'_j = \gamma'_j = \delta'_j = 0$. Then, in view of the first case, we may find finitely supported sequences $(\varphi'_j)_{j\geq 1}$ in $W_{h-1,0}$ and $(\psi'_j)_{j\geq 1}$ in $W_{h-2,0}$ such that, for any $j \geq 2$, ab in X_{h-1} and xy in X_1, one has

$$\begin{aligned}\alpha'_j(ab,y) - \gamma'_j(ab,x) + \beta'_{j-1}(ab,y) - \delta'_{j-1}(ab,x) = \\ \varphi'_j(ab,y) - \varphi'_{j-1}(ab,x) + \psi'_j(ab_1,y) - \psi'_{j-1}(a_1b,y).\end{aligned}$$

Besides, for $j \geq 2$, ab in X_{h-1} and xy in X_2, we get, from (9.3),

$$\begin{aligned}&\alpha_j(ab,x_1y) - \gamma_j(ab,xy_1) + \beta_{j-1}(ab,x_1y) - \delta_{j-1}(ab,xy_1) = \\ &\gamma_{j+1}(ab,x_1y) + v_{j+1}(ab_1,x_1y) + \alpha'_{j+1}(ab,y) - \gamma'_{j+1}(ab,x_1) - \gamma_j(ab,xy_1) \\ &\quad + \delta_j(ab,x_1y) - v_j(a_1b,x_1y) + \beta'_j(ab,y) - \delta'_j(ab,x_1) - \delta_{j-1}(ab,xy_1).\end{aligned}$$

The conclusion follows by setting, for $j \geq 1$ and xy in X_1,

$$\begin{aligned}\varphi_j(ab,xy) &= \gamma_{j+1}(ab,xy) + \delta_j(ab,xy) + \varphi'_{j+1}(ab,y), \quad & ab \in X_{h-1}, \\ \psi_j(ab,xy) &= v_{j+1}(ab,xy) + \psi'_{j+1}(ab,y), & ab \in X_{h-2}.\end{aligned}$$

Now, we assume that $k \geq 2$ and the result holds for $k-1$. Let us show that it also holds for k. This is analogous to the case above. Indeed, (9.1) says that, for any $j \geq 1$, ab in X_h and xy in X_k, one has

$$u_j(ab,xy) = u'_j(ab,xy_1) + \alpha_j(a_1b,xy) + \beta_j(ab_1,xy)$$
$$u_{j-1}(ab,xy) = u'_j(ab,x_1y) + \gamma_j(a_1b,xy) + \delta_j(ab_1,xy).$$

If $j \geq 2$, we get

$$\begin{aligned}u'_{j-1}(ab,xy_1) + \alpha_{j-1}(a_1b,xy) + \beta_{j-1}(ab_1,xy) \\ = u'_j(ab,x_1y) + \gamma_j(a_1b,xy) + \delta_j(ab_1,xy).\end{aligned}$$

Thus, Corollary 8.11 says that we may find u''_j in $W_{h,k-2}$, v_j in $W_{h-2,k}$ and $\alpha'_j, \beta'_j, \gamma'_j, \delta'_j$ in $W_{h-1,k-1}$ such that, for ab in X_{h-1} and xy in X_k, one has

(9.4)
$$\alpha_{j-1}(ab,xy) - \gamma_j(ab,xy) = v_j(ab_1,xy) + \alpha'_j(ab,x_1y) - \gamma'_j(ab,xy_1)$$
$$\beta_{j-1}(ab,xy) - \delta_j(ab,xy) = -v_j(a_1b,xy) + \beta'_j(ab,x_1y) - \delta'_j(ab,xy_1)$$

and, for ab in X_h and xy in X_{k-1}, one has

$$\begin{aligned} u'_j(ab,xy) &= u''_j(ab,xy_1) + \alpha'_j(a_1b,xy) + \beta'_j(ab_1,xy)\\ u'_{j-1}(ab,xy) &= u''_j(ab,x_1y) + \gamma'_j(a_1b,xy) + \delta'_j(ab_1,xy). \end{aligned}$$

If j is large, we assume $u''_j = 0$, $v_j = 0$ and $\alpha'_j = \beta'_j = \gamma'_j = \delta'_j = 0$. Then, in view of the induction assumption, we may find finitely supported sequences $(\varphi'_j)_{j\geq 1}$ in $W_{h-1,k-1}$ and $(\psi'_j)_{j\geq 1}$ in $W_{h-2,k-1}$ such that, for any $j \geq 2$, ab in X_{h-1} and xy in X_k, one has

$$\begin{aligned} &\alpha'_j(ab,x_1y) - \gamma'_j(ab,xy_1) + \beta'_{j-1}(ab,x_1y) - \delta'_{j-1}(ab,xy_1) =\\ &\qquad \varphi'_j(ab,x_1y) - \varphi'_{j-1}(ab,xy_1) + \psi'_j(ab_1,x_1y) - \psi'_{j-1}(a_1b,x_1y). \end{aligned}$$

Besides, for $j \geq 2$, ab in X_{h-1} and xy in X_{k+1}, we get, from (9.4),

$$\begin{aligned} &\alpha_j(ab,x_1y) - \gamma_j(ab,xy_1) + \beta_{j-1}(ab,x_1y) - \delta_{j-1}(ab,xy_1) =\\ &\gamma_{j+1}(ab,x_1y) + v_{j+1}(ab_1,x_1y) + \alpha'_{j+1}(a_1b,x_2y) - \gamma'_{j+1}(ab,x_1y_1) - \gamma_j(ab,xy_1)\\ &+ \delta_j(ab,x_1y) - v_j(a_1b,x_1y) + \beta'_j(ab,x_2y) - \delta'_j(ab,x_1y_1) - \delta_{j-1}(ab,xy_1). \end{aligned}$$

The conclusion follows by setting, for $j \geq 1$ and xy in X_k,

$$\begin{aligned} \varphi_j(ab,xy) &= \gamma_{j+1}(ab,xy) + \delta_j(ab,xy) + \varphi'_{j+1}(ab,x_1y), && ab \in X_{h-1},\\ \psi_j(ab,xy) &= v_{j+1}(ab,xy) + \psi'_{j+1}(ab,x_1y), && ab \in X_{h-2}. \end{aligned}$$

□

Proof of Proposition 9.4. We first deal with the case where $h' = h+1$ and $k' = k$. Then, by Definition 9.2 and the assumption, there exists finitely supported sequences $(u_j)_{j\geq 0}$ in $W_{h+1,k-1}$ and $(v_j)_{j\geq 0}$ in $W_{h,k}$ such that, for any $j \geq 1$, ab in X_{h+1} and xy in X_k, one has

$$w_j(ab_1,xy) = u_j(ab,x_1y) - u_{j-1}(ab,xy_1) + v_j(ab_1,xy) - v_{j-1}(a_1b,xy).$$

Then, Corollary 8.10 says that we may find α_j in $W_{h-1,k}$, β_j in $W_{h+1,k-2}$ and $\gamma_j, \delta_j, \varepsilon_j, \zeta_j$ in $W_{h,k-1}$ such that, for ab in X_h and xy in X_k,

$$\begin{aligned} v_{j-1}(ab,xy) &= \alpha_j(ab_1,xy) + \gamma_j(ab,x_1y) + \delta_j(ab,xy_1)\\ w_j(ab,xy) - v_j(ab,xy) &= -\alpha_j(a_1b,xy) + \varepsilon_j(ab,x_1y) + \zeta_j(ab,xy_1) \end{aligned} \tag{9.5}$$

and, for ab in X_{h+1} and xy in X_{k-1},

$$\begin{aligned} -u_j(ab,xy) &= \beta_j(ab,xy_1) - \gamma_j(a_1b,xy) - \varepsilon_j(ab_1,xy)\\ u_{j-1}(ab,xy) &= -\beta_j(ab,x_1y) - \delta_j(a_1b,xy) - \zeta_j(ab_1,xy). \end{aligned}$$

By Lemma 9.4, the latter tells us that we may find finitely supported sequences $(\varphi_j)_{j\geq 1}$ in $W_{h,k-1}$ and $(\psi_j)_{j\geq 1}$ in $W_{h-1,k-1}$ such that, for any

$j \geq 2$, ab in X_h and xy in X_k, one has

$$(9.6)\quad \gamma_j(ab, x_1y) + \delta_j(ab, xy_1) + \varepsilon_{j-1}(ab, x_1y) + \zeta_{j-1}(ab, xy_1) = \\ \varphi_j(ab, x_1y) - \varphi_{j-1}(ab, xy_1) + \psi_j(ab_1, x_1y) - \psi_{j-1}(a_1b, x_1y).$$

Besides, by using (9.5), we get, for $j \geq 1$,

$$w_j(ab, xy) = \alpha_{j+1}(ab_1, xy) + \gamma_{j+1}(ab, x_1y) + \delta_{j+1}(ab, xy_1) \\ - \alpha_j(a_1b, xy) + \varepsilon_j(ab, x_1y) + \zeta_j(ab, xy_1).$$

By using (9.6), this gives

$$w_j(ab, xy) = \alpha_{j+1}(ab_1, xy) - \alpha_j(a_1b, xy) + \varphi_{j+1}(ab, x_1y) - \varphi_j(ab, xy_1) \\ + \psi_{j+1}(ab_1, x_1y) - \psi_j(a_1b, x_1y)$$

as required.

Now, we can obtain the case where $h' = h$ and $k' = k + 1$ by symmetry. Indeed, in view of Definition 9.2, a sequence $(w_j)_{j\geq 1}$ in $W_{h,k}$ is cohomologically trivial if and only if the sequence of functions in $W_{k,h}$

$$(ab, xy) \mapsto w_j(yx, ba), \quad j \geq 1,$$

is cohomologically trivial. The general case follows by an easy induction. □

9.3. **Splitting sequences.** We pursue the proof of Proposition 9.3 by proving an intermediate result, which will play the role of Lemma 7.13 in the proof of Proposition 7.12.

To state it, we introduce notation. For $i \geq 0$ an integer, we set $r(i) = 0$ if $i = 0$, $r(i) = 1$ if $i = 1$ and $r(i) = 2(i-1)$ if $i \geq 1$. As in Section III.2.2, if H is a pseudofunction we write H^+ for $H^{>\vee}$. The purpose of this Subsection is to establish the following

Lemma 9.6. *Let $k \geq 4$ be an even integer, $k = 2\ell$, $\ell \geq 2$. Suppose, for any $0 \leq i \leq \ell + 1$, we are given a finitely supported sequence $(H_{i,j})_{j\geq 1}$ in $\mathcal{H}_{r(\ell+1-i)} \otimes \mathcal{H}_{r(i)}$. For $j \geq 1$, we set*

$$H_j = H_{0,j}^{+^k} + {}^{+^2}H_{1,j}{}^{+^{k-1}} + \sum_{i=2}^{\ell-1} {}^{+^{2i}}H_{i,j}{}^{+^{k-2(i-1)}} + {}^{+^{k-1}}H_{\ell,j}{}^{+^2} + {}^{+^k}H_{\ell+1,j},$$

which is an element of $\otimes^2\mathcal{H}_k$. Assume that the sequence $(\varpi(H_j))_{j\geq 1}$ is cohomologically trivial in W_k. Then, for any $0 \leq i \leq h$, there exist finitely supported sequences $(D_{i,j})_{j\geq 1}$ and $(E_{i,j})_{j\geq 1}$ in $\mathcal{H}_{r(\ell+1-i)-1} \otimes \mathcal{H}_{r(i)}$ and $(F_{i,j})_{j\geq 1}$ and $(G_{i,j})_{j\geq 1}$ in $\mathcal{H}_{r(\ell+1-i)} \otimes \mathcal{H}_{r(i)-1}$ such that, for $j \geq 1$, one has

$$H_{i,j} = {}^{>}D_{i,j} + {}^{\vee>}E_{i,j} + F_{i,j}^{>} + G_{i,j}^{>\vee}.$$

To prove this, we will aim at applying the tensor version of the criterion of Lemma 7.8. To this purpose, we introduce the notion of a split element of $W_{h,k}$ for $h,k \geq 0$ (see Subsection I.8.7 and Subsection 7.3 for the notion of a split elements of V_k, $k \geq 0$). In the language of Example 8.3, we say that an element u in $W_{h,k}$ is split if it belongs to $LW_{h-1,k}+RW_{h-1,k}+W_{h,k-1}L+W_{h,k-1}R$. More concretely, for example, if $h,k \geq 1$, this means that there exists v_0, v_1 in $W_{h-1,k}$ and w_0, w_1 in $W_{h,k-1}$ such that, for ab in X_h and xy in X_k,

$$u(ab,xy) = v_0(ab_1,xy) + v_1(a_1b,xy) + w_0(ab,xy_1) + w_1(ab,x_1y).$$

If $h=0$ and $k \geq 1$, this means that there exists v in $W_{-1,k}$ and w_0, w_1 in $W_{0,k-1}$ such that, for ab in X_h and xy in X_k,

$$u(a,xy) = v(a,xy) + w_0(a,xy_1) + w_1(a,x_1y).$$

To study triangular families of tensors as in Lemma 9.6, we shall use

Lemma 9.7. *Let $k \geq 1$ and, for $0 \leq i \leq k$, let $(w_{i,j})_{j\geq 1}$ be a finitely supported sequence in $W_{k-i,i}$. For $j \geq 1$ and ab, xy in X_k, we set*

$$w_j(ab,xy) = \sum_{i=0}^{k} w_{i,j}(aa_{k-i}, y_i y).$$

Assume that the sequence $(w_j)_{j\geq 1}$ is cohomologically trivial in W_k. Then, for every $j \geq 1$ and $0 \leq i \leq k$, the function $w_{i,j}$ is split in $W_{k-i,i}$.

Proof. Let $0 \leq h \leq k+1$ be the least integer such that, for all $h \leq i \leq k$, for all $j \geq 1$, $w_{i,j} = 0$. We will show the statement by induction on h. For $h = 0$, there is nothing to prove.

Assume $h = 1$. By assumption, the sequence of functions in $W_{k,k}$,

$$(ab,xy) \mapsto w_{0,j}(ab,y), \quad j \geq 1,$$

is cohomologically trivial. By Propositon 9.4, the sequence $(w_{0,j})_{j\geq 1}$ is cohomologically trivial in $W_{k,0}$. In other words, by Definition 9.2, there exist finitely supported sequences $(u_j)_{j\geq 0}$ in $W_{k,-1}$ and $(v_j)_{j\geq 0}$ in $W_{k-1,0}$ such that, for any $j \geq 1$, ab in X_k and xy in X_1, one has

$$w_{0,j}(ab,y) = u_j(ab,y) - u_{j-1}(ab,x) + v_j(ab_1,y) - v_{j-1}(a_1b,y).$$

Thus $w_{0,j}$ is split in $W_{k,0}$ as required.

Assume $h \geq 2$ and the result is true for $h-1$. By assumption, the sequence of functions in $W_{k,k}$,

$$(ab,xy) \mapsto \sum_{i=0}^{h-1} w_{i,j}(aa_{k-i}, y_i y), \quad j \geq 1,$$

is cohomologically trivial. By Definition 9.2 and Propositon 9.4, there exist finitely supported sequences $(u_j)_{j\geq 0}$ in $W_{k,h-2}$ and $(v_j)_{j\geq 0}$ in $W_{k-1,h-1}$ such that, for any $j \geq 1$, ab in X_k and xy in X_{h-1}, one has

$$\sum_{i=0}^{h-1} w_{i,j}(aa_{k-i}, y_i y) = \\ u_j(ab, x_1 y) - u_{j-1}(ab, xy_1) + v_j(ab_1, xy) - v_{j-1}(a_1 b, xy). \tag{9.7}$$

We temporarily consider each $w_{i,j}$, $0 \leq i \leq h-1$, as a function from X_{k-i} towards V_i. From (9.7), we get, for ab in X_k,

$$w_{h-1,j}(aa_{k-h+1}) - v_j(ab_1) + v_{j-1}(a_1 b) \in LV_{h-2} + RV_{h-2}$$

(where we have used the notation of Example 8.3). Thus, Lemma 7.10 and Lemma 8.4 say that there exist a finitely supported sequence $(v'_j)_{j\geq 0}$ in $W_{k-h,h-1}$ such that for any $j \geq 1$ and ab in X_{k-h+1}, one has

$$w_{h-1,j}(ab) - v'_j(ab_1) + v'_{j-1}(a_1 b) \in LV_{h-2} + RV_{h-2}.$$

In other words, there exist finitely supported sequences $(\alpha_j)_{j\geq 1}$ and $(\beta_j)_{j\geq 1}$ in $W_{k-h+1,h-2}$ such that, for $j \geq 1$, ab in X_{k-h+1} and xy in X_{h-1}, one has

$$w_{h-1,j}(ab, xy) = v'_j(ab_1, xy) - v'_{j-1}(a_1 b, xy) + \alpha_j(ab, xy_1) + \beta_j(ab, x_1 y)$$

and the function $w_{h-1,j}$ is split in $W_{k-h+1,h-1}$.

Besides, we set $\gamma_j = \alpha_{j+1} + \beta_j$ and we rewrite the latter as, for $j \geq 1$, ab in X_{k-h+1} and xy in X_{h-1},

$$w_{h-1,j}(ab, xy) = v'_j(ab_1, xy) - v'_{j-1}(a_1 b, xy) + \alpha_j(ab, xy_1) - \alpha_{j+1}(ab, x_1 y) \\ + \gamma_j(ab, x_1 y).$$

From (9.7), we get, for $j \geq 1$ and ab, xy in X_k,

$$\sum_{i=0}^{h-2} w_{i,j}(aa_{k-i}, y_i y) + \gamma_j(aa_{k-h+1}, y_{h-2} y) = u_j(ab, y_{h-2} y) \\ - u_{j-1}(ab, y_{h-1} y_1) + v_j(ab_1, y_{h-1} y) - v_{j-1}(a_1 b, y_{h-1} y) \\ - v'_j(aa_{k-h}, y_{h-1} y) + v'_{j-1}(a_1 a_{k-h+1}, y_{h-1} y) - \alpha_j(aa_{k-h+1}, y_{h-1} y_1) \\ + \alpha_{j+1}(aa_{k-h+1}, y_{h-2} y),$$

so that the sequence in $W_{k,k}$,

$$(ab, xy) \mapsto \sum_{i=0}^{h-2} w_{i,j}(aa_{k-i}, y_i y) + \gamma_j(aa_{k-h+1}, y_{h-2} y), \quad j \geq 1,$$

is cohomologically trivial. From the induction assumption, it follows that the functions $w_{h-2,j}+\gamma_j R$ and $w_{i,j}$, $0\leq i\leq h-3$ are split. The conclusion follows. □

To compute the weights of large orthogonal extensions, we shall use

Lemma 9.8. *Let $k\geq 0$ and H be in $\mathcal{H}_k$. Fix $h\geq 0$ even.*
If k is even, for ab in X_{h+k}, we have

$$\omega_{h+k}\left(H^{+^h}\right)(ab)=\omega_k(H)(ab_h).$$

If k is odd, for ab in X_{h+k+1}, we have

$$\omega_{h+k+1}\left(H^{+^{h+1}}\right)(ab)=\omega_k(H^\vee)(ab_{h+1}).$$

Proof. This is a direct consequence of Lemma 7.1. □

We now define the tensor squares of the objects introduced in Subsection 7.5. There, given $k\geq 2$ even, $k=2\ell$, $\ell\geq 1$, and a complete k-pseudofunction H, we have defined functions $\rho_k^i(H)$ on $X_{\ell-i+1}$ for any $0\leq i\leq \ell-1$.

Now, for $k,m\geq 2$ even, $k=2\ell$, $\ell\geq 1$, $m=2n$, $n\geq 1$, $0\leq h\leq \ell-1$ and $0\leq i\leq n-1$, we let $\sigma_{k,m}^{h,i}:\widetilde{\mathcal{H}}_k\otimes\widetilde{\mathcal{H}}_m\to W_{\ell-h+1,n-i+1}$ be the linear map such that, for H in $\widetilde{\mathcal{H}}_k$, J in $\widetilde{\mathcal{H}}_m$, ab in $X_{\ell-h+1}$ and xy in X_{n-i+1}, one has

$$\sigma_{k,m}^{h,i}(H\otimes J)(ab,xy)=\rho_k^h(H)(ab)\rho_m^i(H)(yx).$$

From (7.4) and Definition 9.1, we get, for H in $\widetilde{\mathcal{H}}_k\otimes\widetilde{\mathcal{H}}_m$, ab in X_k and xy in X_m,

$$\varpi_{k,m}(H)(ab,xy)=\sum_{\substack{0\leq h\leq \ell-1\\ 0\leq i\leq n-1}}\sigma_{k,m}^{h,i}(H)(a_{2h}a_{\ell+h+1},y_{n+i+1}y_{2i}). \tag{9.8}$$

As in the proof of Lemma 7.13, this yields

Lemma 9.9. *Let $k,m\geq 2$ be even, $k=2\ell$, $\ell\geq 1$, $m=2n$, $n\geq 1$. Let $(H_j)_{j\geq 1}$ be a finitely supported sequence in $\widetilde{\mathcal{H}}_k\otimes\widetilde{\mathcal{H}}_m$. For $j\geq 1$, ab in X_k and xy in X_m, set*

$$w_j(ab,xy)=\sum_{\substack{0\leq h\leq \ell-1\\ 0\leq i\leq n-1}}\sigma_{k,m}^{h,i}(H_{h+i+j})(a_ha_{\ell+1},y_{n+1}y_i)-\varpi_{k,m}(H)(ab,xy).$$

Then, the sequence $(w_j)_{j\geq 1}$ is cohomologically trivial in $W_{k,m}$.

Proof. This is a direct computation. Indeed, by (9.8), for $j \geq 1$, we have, for ab in X_k and xy in X_m,

$$\sum_{\substack{0\leq h\leq \ell-1\\ 0\leq i\leq n-1}} \sigma^{h,i}_{k,m}(H_{h+i+j})(a_h a_{\ell+1}, y_{n+1}y_i) - \varpi_{k,m}(H)(ab,xy) =$$

$$\sum_{\substack{0\leq h\leq \ell-1\\ 0\leq i\leq n-1}} \sigma^{h,i}_{k,m}(H_{h+i+j})(a_h a_{\ell+1}, y_{\ell+1}y_i) - \sigma^{h,i}_{k,m}(H_j)(a_{2h}a_{\ell+h+1}, y_{n+i+1}y_{2i}).$$

For h, i as above, we write

$$\sigma^{h,i}_{k,m}(H_{h+i+j})(a_h a_{\ell+1}, y_{n+1}y_i) - \sigma^{h,i}_{k,m}(H_j)(a_{2h}a_{\ell+h+1}, y_{n+i+1}y_{2i}) =$$

$$\sum_{p=0}^{h-1} \sigma^{h,i}_{k,m}(H_{h+i+j-p})(a_{h+p}a_{\ell+p+1}, y_{n+1}y_i) - \sigma^{h,i}_{k,m}(H_{h+i+j-p-1})(a_{h+p+1}a_{\ell+p+2}, y_{n+1}y_i)$$

$$+\sum_{q=0}^{i-1} \sigma^{h,i}_{k,m}(H_{i+j-q})(a_{2h}a_{\ell+h+1}, y_{n+q+1}y_{i+q}) - \sigma^{h,i}_{k,m}(H_{i+j-q-1})(a_{2h}a_{\ell+h+1}, y_{n+q+2}y_{i+q+1}).$$

The conclusion follows by Definition 9.2. $\square$

We will need an adapted version of Lemma 9.9 to deal with the boundary terms appearing in Lemma 9.6.

For $k \geq 2$, $k = 2\ell$, $\ell \geq 1$, $0 \leq i \leq \ell - 1$, and $m \in \{0, 1\}$, we let $\theta^i_{k,m} : \widetilde{\mathcal{H}}_k \otimes \mathcal{H}_m \to W_{\ell+1-i,m}$ and $\theta^i_{m,k} : \mathcal{H}_m \otimes \widetilde{\mathcal{H}}_k \to W_{m,\ell+1-i}$ be the linear maps defined by, for H in $\widetilde{\mathcal{H}}_k$, J in $\mathcal{H}_m$, ab in $X_{\ell+1-i}$,

$$\theta^i_{k,m}(H \otimes J)(ab, xy) = \rho^i_k(H)(ab)\omega_m(J)(xy) = \theta^i_{m,k}(J \otimes H)(xy, ab).$$

If H is in $\widetilde{\mathcal{H}}_k \otimes \mathcal{H}_m$, from (7.4), we now get, for ab in X_k and xy in X_m,

$$\varpi_{k,m}(H)(ab, xy) = \sum_{i=0}^{\ell-1} \theta^i_{k,m}(a_{2i}a_{\ell+i+1}, xy),$$

which, as above, yields

Lemma 9.10. *Let $k \geq 2$ be even, $k = 2\ell$, $\ell \geq 1$, and m be in $\{0, 1\}$. Let $(H_j)_{j\geq 1}$ be a finitely supported sequence in $\widetilde{\mathcal{H}}_k \otimes \mathcal{H}_m$. There exists a finitely supported sequence $(v_j)_{j\geq 0}$ in $W_{k-1,m}$ such that, for any $j \geq 1$ and ab in X_k and xy in X_m, one has*

$$\sum_{i=0}^{\ell-1} \theta^i_{k,m}(H_{i+j})(a_i a_{\ell+1}, xy) - \varpi_{k,m}(H)(ab, xy) =$$
$$v_j(ab_1, xy) - v_{j-1}(a_1 b, xy).$$

Finally, we will need to complete the information given by Lemma 7.8 in case $k = 0, 1$.

Lemma 9.11. *Let H be in $\mathcal{H}_0$. Then $\omega_0(H)$ is split in V_0 if and only if there exists G in $\mathcal{H}_{-1}$ with $H = G^{>}$.*

Let H be in $\mathcal{H}_1$. Then $\omega_1(H)$ is split in V_1 if and only if there exists F, G in $\mathcal{H}_{-1}$ with $H = F^{>} + G^{>\vee}$.

Proof. Recall the conventions on pseudofunctions with low degree from Subsection III.2.1 and Subsection III.2.2. If H in $\mathcal{H}_0$, then H is the 0-pseudofunction associated with the function $\omega_0(H)$ on X_0. Then saying that H is in $\mathcal{H}_{-1}^{>}$ is saying that $\omega_0(H)$ is constant on neighbours.

If H is in $\mathcal{H}_1$ and $H = F^{>}+G^{>\vee}$ for F, G in $\mathcal{H}_0$, then $\omega_1(H)$ is split by Lemma 7.1. Conversely, let u, v be in V_0 with $\omega_1(H)(ab) = u(a)+v(b)$, $ab \in X_1$. Then, still by Lemma 7.1, we have $H = F^{>} - G^{>\vee}$, where F and G are the 0-pseudofunctions associated with u and v. $\square$

We can now conclude the

Proof of Lemma 9.6. By Lemma 9.8, for $j \geq 1$ and ab, xy in X_k, we have

$$\begin{aligned}\varpi_k(H_j)(ab, xy) &= \varpi_{k,0}(H_{0,j})(ab, y) + \varpi_{k-2,1}(H_{1,j}^{\vee})(ab_2, y_1 y)\\ &+ \sum_{i=2}^{\ell-1} \varpi_{2(\ell-i),2(i-1)}(H_{i,j})(ab_{2i}, x_{2(\ell-i+1)}y) + \varpi_{1,k-2}({}^{\vee}H_{\ell,j})(aa_1, x_2 y)\\ &\qquad + \varpi_{0,k}(H_{\ell+1,j})(a, xy).\end{aligned}$$

By Lemma 7.5, we can assume that, for $j \geq 1$, we have

$$\begin{aligned} H_{0,j} &\in \widetilde{\mathcal{H}}_k \otimes \mathcal{H}_0 \\ H_{1,j} &\in \widetilde{\mathcal{H}}_{k-2} \otimes \mathcal{H}_1 \\ H_{i,j} &\in \widetilde{\mathcal{H}}_{k-2i} \otimes \widetilde{\mathcal{H}}_{k-2(i-1)} && 2 \leq i \leq \ell - 1 \\ H_{\ell,j} &\in \mathcal{H}_1 \otimes \widetilde{\mathcal{H}}_{k-2} \\ H_{\ell+1,j} &\in \mathcal{H}_0 \otimes \widetilde{\mathcal{H}}_k. \end{aligned}$$

By the assumption, Proposition 9.4, Lemma 9.7 and Lemma 9.10, we can find finitely supported sequences $(w_{i,j})_{j\geq 1}$ in $W_{\ell+1-i,i}$, $0 \leq i \leq \ell+1$, such that, on one hand, the sequence in $W_{\ell+1,\ell+1}$,

$$(ab, xy) \mapsto \sum_{i=0}^{\ell+1} w_j(ab_i, x_{\ell+1-i}y), \quad j \geq 1,$$

is cohomologically trivial and, on the other hand, for $j \geq 1$, the functions

$$\begin{array}{lll} & \theta^0_{k,0}(H_{0,j}) - w_{0,j} & i = 0 \\ & \theta^0_{k-2,1}(H^{\vee}_{1,j}) - w_{1,j} & i = 1 \\ (9.9) & \sigma^0_{2(\ell-i),2(i-1)}(H_{i,j}) - w_{i,j} & 2 \leq i \leq \ell - 1 \\ & \theta^0_{1,k-2}({}^{\vee}H_{\ell,j}) - w_{\ell,j} & i = \ell \\ & \theta^0_{0,k}(H_{\ell+1,j}) - w_{\ell+1,j} & i = \ell + 1 \end{array}$$

are split in $W_{\ell+1-i,i}$ for all $0 \leq i \leq \ell + 1$. Therefore, by Lemma 9.7, for any $0 \leq i \leq \ell + 1$, the function $w_{i,j}$ is split in $W_{\ell+1-i,i}$. The conclusion follows by Lemma 7.8, Lemma 9.11 and (9.9). □

9.4. **Triangular sequences.** In this final Subsection, we will use the previous constructions to prove Proposition 9.3. We will split it into several steps, which will eventually allow us to reduce the question to the study of sequences of the form appearing in Lemma 9.6.

In the first step, we show

Lemma 9.12. *Let $k \geq 4$ be even, $k = 2\ell$, $\ell \geq 2$, and $(H_j)_{j\geq 1}$ be a finitely supported sequence in $\otimes^2\mathcal{H}_k$. Assume that the sequence $(\varpi_k(H_j))_{j\geq 1}$ is cohomologically trivial in W_k. Then, there exist finitely supported sequences $(F_j)_{j\geq 0}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$, $(G_j)_{j\geq 0}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$, and, for $0 \leq i \leq \ell - 1$, $(H_{i,j})_{j\geq 1}$ in $\mathcal{H}_{k-2i} \otimes \mathcal{H}_{2(i+1)}$ such that, for $j \geq 1$, one has*

$$H_j = \sum_{i=0}^{\ell-1} {}^{+^{2i}}H_{i,j}{}^{+^{2(\ell-i-1)}} + F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1}.$$

In the course of the proof, we shall need some easy properties of the natural operations on pseudofunctions.

Lemma 9.13. *Let $h \geq 0$ and $k \geq -1$.*

If k is odd, for any H in $\mathcal{H}_k$, we have

$$(H^{>\vee})^{+^{2h}} = (H^{+^{2h}})^{>\vee} \text{ and } (H^{\vee>})^{+^{2h}} = (H^{+^{2h}})^{\vee>}.$$

If $k = 0$, for any H in $\mathcal{H}_0$, we have

$$(H^{>\vee})^{+^{2h+1}} = (H^{+^{2h+1}})^{>\vee} \text{ and } (H^{\vee>})^{+^{2h+1}} = (H^{+^{2h+1}})^{\vee>}.$$

Proof. In both cases, the first equality is obvious since by definition $H^+ = H^{>\vee}$.

Assume first k is odd. Then, using Lemma III.2.6 and the fact that $H^{\vee\vee} = H$, we get

$$H^{\vee>++} = H^{\vee>>\vee>\vee} = H^{>>>\vee} = H^{>\vee>>} = H^{>\vee>\vee\vee>} = H^{++\vee>}.$$

The conclusion follows by a straightforward induction.

Assume now $k = 0$. Then, by definition, we have $H^\vee = -H$. Using again Lemma III.2.6, we get

$$H^{\vee>+} = -H^{>>\vee} = -H^{\vee>>} = H^{>>} = H^{>\vee\vee>} = H^{+\vee>}.$$

The conclusion follows by the first case. □

Proof of Lemma 9.12. We will actually prove by induction on $0 \leq h \leq \ell - 1$ that there exist finitely supported sequences $(F_j)_{j\geq 0}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$, $(G_j)_{j\geq 0}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$, and, for $0 \leq i \leq h$, $(H_{i,j})_{j\geq 1}$ in $\mathcal{H}_{k-2i} \otimes \mathcal{H}_{k-2(h-i)}$ such that, for $j \geq 1$, one has

$$H_j = \sum_{i=0}^{h} {}^{+^{2i}}H_{i,j}{}^{+^{2(h-i)}} + F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1}. \tag{9.10}$$

For $h = 0$, there is nothing to prove. Assume $0 \leq h \leq \ell - 2$ and (9.10) holds for h. Let us show that it also holds for $h + 1$.

We write $k' = 2(k - h + 1)$ and we will apply Lemma 9.6 in $\otimes^2 \mathcal{H}_{k'}$. Indeed, for $j \geq 1$, we set

$$H'_j = \sum_{i=0}^{h} {}^{+^{2i}}H_{i,j}{}^{+^{2(h-i)}}.$$

Then, Lemma 7.1, (9.10) and the assumption imply that the sequence $(\varpi_k(H'_j))_{j\geq 1}$ is cohomologically trivial in W_k. Therefore, if we now set,

$$H''_j = {}^{+^{k-2h+2}}(H'_j)^{+^{k-2h+2}} = \sum_{i=0}^{h} {}^{+^{k-2(h-i-1)}}H_{i,j}{}^{+^{k-2(i-1)}},$$

then, by Lemma 9.8, the sequence $(\varpi_{k'}(H''_j))_{j\geq 1}$ is cohomologically trivial in $W_{k'}$. Thus, the assumption of Lemma 9.6 is satisfied. Hence, for $0 \leq i \leq h$, we may find finitely supported sequences $(J_{i,j})_{j\geq 1}$ and $(K_{i,j})_{j\geq 1}$ in $\mathcal{H}_{k-2i} \otimes \mathcal{H}_{k-2(h-i)-1}$ and $(L_{i,j})_{j\geq 1}$ and $(M_{i,j})_{j\geq 1}$ in $\mathcal{H}_{k-2i-1} \otimes \mathcal{H}_{k-2(h-i)}$ such that, for $j \geq 1$, one has

$$\begin{aligned} H_{i,j} &= J_{i,j}^{>} + K_{i,j}^{>\vee} + {}^{>}L_{i,j} + {}^{\vee>}M_{i,j} \\ &= N_{i,j}^{\vee>\vee} + {}^{\vee>\vee}P_{i,j} + J_{i,j}^{>} - J_{i,j+1}^{\vee>\vee} + {}^{>}L_{i,j} - {}^{\vee>\vee}L_{i,j+1}, \end{aligned}$$

where $N_{i,j} = K_{i,j}^{\vee} + J_{i,j+1}$ and $P_{i,j} = {}^{\vee}M_{i,j} + L_{i,j+1}$.

Note that, since $h \leq \ell - 2$, for $0 \leq i \leq h$, we have

$$k - 2i - 1 \geq k - 2h - 1 \geq 3.$$

Therefore, by Lemma 7.8 and Lemma 8.4, we can find finitely supported sequences $(U_{i,j})_{j\geq 1}$ and $(V_{i,j})_{j\geq 1}$ in $\mathcal{H}_{k-2i} \otimes \mathcal{H}_{k-2(h-i)-2}$ and $(W_{i,j})_{j\geq 1}$

and $(X_{i,j})_{j\geq 1}$ in $\mathcal{H}_{k-2i-2}\otimes\mathcal{H}_{k-2(h-i)}$ with, for $j\geq 1$,

$$\begin{aligned}N_{i,j} &= U_{i,j}^{>} + V_{i,j}^{>\vee} = Y_{i,j}^{>} + V_{i,j}^{>\vee} - V_{i,j+1}^{\vee>}\\ P_{i,j} &= {}^{>}W_{i,j} + {}^{\vee>}X_{i,j} = {}^{>}Z_{i,j} + {}^{\vee>}X_{i,j} - {}^{>\vee}X_{i,j+1},\end{aligned}\tag{9.11}$$

with $Y_{i,j} = U_{i,j} + V_{i,j+1}^{\vee}$ and $Z_{i,j} = W_{i,j} + {}^{\vee}X_{i,j+1}$. For $j\geq 0$, we set $A_{i,j} = -J_{i,j+1}^{\vee} - V_{i,j+1}^{\vee>\vee}$ and $B_{i,j} = -{}^{\vee}L_{i,j+1} - {}^{\vee>\vee}X_{i,j+1}$ and we get, by using Lemma III.2.6, for $j\geq 1$,

$$H_{i,j} = Y_{i,j}^{++} + {}^{++}Z_{i,j} + A_{i,j}^{>\vee} - A_{i,j-1}^{\vee>} + {}^{\vee>}B_{i,j} - {}^{>\vee}B_{i,j-1}.$$

Plugging this into (9.10) and using Lemma 9.13 yields, for $j\geq 1$,

$$\begin{aligned}H_j = Y_{0,j}^{+^{2(h+1)}} + \sum_{i=1}^{h} {}^{+^{2i}}(Z_{i-1,j} + Y_{i,j})^{+^{2(h+1-i)}} + {}^{+^{2(h+1)}}Z_{h,j}&\\ + C_j^{>\vee} - C_{j-1}^{\vee>} + {}^{\vee>}D_j - {}^{>\vee}D_{j-1}&.\end{aligned}$$

with, for $j\geq 0$,

$$C_j = \sum_{i=0}^{h} {}^{+^{2i}}A_{i,j}{}^{+^{2(h-i)}} + F_j \text{ and } D_j = \sum_{i=0}^{h} {}^{+^{2i}}B_{i,j}{}^{+^{2(h-i)}} + G_j.$$

Thus, (9.10) holds also for $h+1$ and we are done. $\square$

By using the same method, we can go one step further, but we have to take into account the fact that $\mathcal{H}_1$ is in general not equal to $\mathcal{H}_0^{>} + \mathcal{H}_0^{>\vee}$.

Corollary 9.14. *Let $k\geq 4$ be even, $k = 2\ell$, $\ell\geq 2$, and $(H_j)_{j\geq 1}$ be a finitely supported sequence in $\otimes^2\mathcal{H}_k$. Assume that the sequence $(\varpi_k(H_j))_{j\geq 1}$ is cohomologically trivial in W_k. Then, there exist finitely supported sequences $(F_j)_{j\geq 0}$ in $\mathcal{H}_k\otimes\mathcal{H}_{k-1}$, $(G_j)_{j\geq 0}$ in $\mathcal{H}_{k-1}\otimes\mathcal{H}_k$, $(H_{0,j})_{j\geq 1}$ in $\mathcal{H}_k\otimes\mathcal{H}_1$, $(H_{\ell,j})_{j\geq 1}$ in $\mathcal{H}_1\otimes\mathcal{H}_k$ and, for $1\leq i\leq \ell-1$, $(H_{i,j})_{j\geq 1}$ in $\mathcal{H}_{k-2i}\otimes\mathcal{H}_{2i}$ such that, for $j\geq 1$, one has*

$$H_j = H_{0,j}^{+^{k-1}} + \sum_{i=1}^{\ell-1} {}^{+^{2i}}H_{i,j}{}^{+^{k-2i}} + {}^{+^{k-1}}H_{\ell,j} + F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1}.\tag{9.12}$$

Proof. By Lemma 9.12, we get, for $j\geq 1$,

$$H_j = \sum_{i=0}^{\ell-1} {}^{+^{2i}}H_{i,j}{}^{+^{k-2(i+1)}} + F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1},\tag{9.13}$$

where $(F_j)_{j\geq 0}$ is a sequence in $\mathcal{H}_k\otimes\mathcal{H}_{k-1}$, $(G_j)_{j\geq 0}$ is a sequence in $\mathcal{H}_{k-1}\otimes\mathcal{H}_k$, and, for $0\leq i\leq \ell-1$, $(H_{i,j})_{j\geq 1}$ is a sequence in $\mathcal{H}_{k-2i}\otimes\mathcal{H}_{2(i+1)}$.

We apply the same procedure as in the proof of Lemma 9.12 by using Lemma 9.6 in $\otimes^2\mathcal{H}_{k+4}$. We keep the same notation. The only difference is that for $j \geq 1$ and $i = 0$, the tensor $N_{0,j}$ belongs to $\mathcal{H}_k \otimes \mathcal{H}_1$ and that, for $i = \ell - 1$, the tensor $P_{\ell-1,j}$ belongs to $\mathcal{H}_1 \otimes \mathcal{H}_k$. Therefore, we can not apply Lemma 7.8 in order to split them as in (9.11). Thus, the elements $Y_{i,j}$ and $V_{i,j}$ are only defined for $1 \leq i \leq \ell-1$ and the $Z_{i,j}$ and $X_{i,j}$ are only defined for $0 \leq i \leq \ell - 2$.

We now get, for $j \geq 1$,

$$H_{0,j} = N_{0,j}^{\vee>\vee} + {}^{++}Z_{0,j} + J_{0,j}^{>} - J_{0,j+1}^{\vee>\vee} + {}^{\vee>}B_{0,j} - {}^{>\vee}B_{0,j-1}$$
$$H_{\ell-1,j} = Y_{\ell-1,j}^{++} + {}^{\vee>\vee}P_{\ell-1,j} + A_{\ell-1,j}^{>\vee} - A_{\ell-1,j-1}^{\vee>} + {}^{>}L_{\ell-1,j} - {}^{\vee>\vee}L_{\ell-1,j+1}$$

and, for $1 \leq i \leq \ell - 2$,

$$H_{i,j} = Y_{i,j}^{++} + {}^{++}Z_{i,j} + A_{i,j}^{>\vee} - A_{i,j-1}^{\vee>} + {}^{\vee>}B_{i,j} - {}^{>\vee}B_{i,j-1}.$$

Using this in (9.13) and applying Lemma 9.13 gives, for $j \geq 1$,

$$H_j = N_{0,j}^{\vee+^{k-1}} + \sum_{i=1}^{\ell-1} {}^{+^{2i}}(Z_{i-1,j} + Y_{i,j})^{+^{k-2i}} + {}^{+^{k-1}\vee}P_{\ell-1,j}$$
$$+ C_j^{>\vee} - C_{j-1}^{\vee>} + {}^{\vee>}D_j - {}^{>\vee}D_{j-1}.$$

with, for $j \geq 0$,

$$C_j = -J_{0,j+1}^{\vee+^{k-2}} + \sum_{i=1}^{\ell-1} {}^{+^{2i}}A_{i,j}{}^{+^{k-2(i+1)}} + F_j$$
$$\text{and } D_j = \sum_{i=0}^{\ell-2} {}^{+^{2i}}B_{i,j}{}^{+^{k-2(i+1)}} - {}^{+^{k-2}\vee}L_{\ell-1,j+1} + G_j.$$

Thus, (9.12) holds as required. □

In the same way, we can get a last step. Recall that we write $r(0) = 0$, $r(1) = 1$ and $r(i) = 2(i-1)$ for $i \geq 2$.

Corollary 9.15. *Let $k \geq 4$ be even, $k = 2\ell$, $\ell \geq 2$, and $(H_j)_{j\geq 1}$ be a finitely supported sequence in $\otimes^2\mathcal{H}_k$. Assume that the sequence $(\varpi_k(H_j))_{j\geq 1}$ is cohomologically trivial in W_k. Then, there exists finitely supported sequences $(F_j)_{j\geq 0}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$, $(G_j)_{j\geq 0}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$ and, for $0 \leq i \leq \ell + 1$, $(H_{i,j})_{j\geq 1}$ in $\mathcal{H}_{r(\ell+1-i)} \otimes \mathcal{H}_{r(i)}$ such that, for $j \geq 1$,* *one has*

(9.14)

$$H_j = H_{0,j}^{+^k} + {}^{+^2}H_{1,j}{}^{+^{k-1}} + \sum_{i=2}^{\ell-1} {}^{+^{2i}}H_{i,j}{}^{+^{k-2(i-1)}} + {}^{+^{k-1}}H_{\ell,j}{}^{+^2} + {}^{+^k}H_{\ell+1,j} + F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1}.$$

Proof. We start with the decomposition (9.12) given by Corollary 9.15. Again, we apply the induction procedure of the proof of Lemma 9.12 which now relies on the use of Lemma 9.6 in $\otimes^2\mathcal{H}_{k+2}$. We keep the same notation, so that we first write, for $j \geq 1$ and $0 \leq i \leq \ell$,

$$H_{i,j} = N_{i,j}^{\vee>\vee} + {}^{\vee>\vee}P_{i,j} + J_{i,j}^{>} - J_{i,j+1}^{\vee>\vee} + {}^{>}L_{i,j} - {}^{\vee>\vee}L_{i,j+1},$$

where

$$\begin{array}{lll} N_{0,j}, J_{0,j} \in \mathcal{H}_k \otimes \mathcal{H}_0 & P_{0,j}, L_{0,j} \in \mathcal{H}_{k-1} \otimes \mathcal{H}_1 & \\ N_{i,j}, J_{i,j} \in \mathcal{H}_{k-2i} \otimes \mathcal{H}_{2i-1} & P_{i,j}, L_{i,j} \in \mathcal{H}_{k-2i-1} \otimes \mathcal{H}_{2i} & 1 \leq i \leq \ell-1 \\ N_{\ell,j}, J_{\ell,j} \in \mathcal{H}_1 \otimes \mathcal{H}_{k-1} & P_{\ell,j}, L_{\ell,j} \in \mathcal{H}_0 \otimes \mathcal{H}_k. & \end{array}$$

Now, the tensors $Y_{i,j}$ and $V_{i,j}$ may be defined for $2 \leq i \leq \ell$ and the tensors $Z_{i,j}$ and $X_{i,j}$ may be defined defined for $0 \leq i \leq \ell - 2$.

If $\ell \geq 3$, we get, for $\epsilon \in \{0, 1\}$,

$$\begin{aligned} H_{\epsilon,j} &= N_{\epsilon,j}^{\vee>\vee} + {}^{++}Z_{\epsilon,j} + J_{\epsilon,j}^{>} - J_{\epsilon,j+1}^{\vee>\vee} + {}^{\vee>}B_{\epsilon,j} - {}^{>\vee}B_{\epsilon,j-1} \\ H_{\ell-\epsilon,j} &= Y_{\ell-\epsilon,j}^{++} + {}^{\vee>\vee}P_{\ell-\epsilon,j} + A_{\ell-\epsilon,j}^{>\vee} - A_{\ell-\epsilon,j-1}^{\vee>} + {}^{>}L_{i,j} - {}^{\vee>\vee}L_{i,j+1} \end{aligned}$$

and, for $2 \leq i \leq \ell - 2$,

$$H_{i,j} = Y_{i,j}^{++} + {}^{++}Z_{i,j} + A_{i,j}^{>\vee} - A_{i,j-1}^{\vee>} + {}^{\vee>}B_{i,j} - {}^{>\vee}B_{i,j-1}.$$

If $\ell = 2$, we have

$$\begin{aligned} H_{0,j} &= N_{0,j}^{\vee>\vee} + {}^{++}Z_{0,j} + J_{0,j}^{>} - J_{0,j+1}^{\vee>\vee} + {}^{\vee>}B_{0,j} - {}^{>\vee}B_{0,j-1} \\ H_{1,j} &= N_{1,j}^{\vee>\vee} + {}^{\vee>\vee}P_{1,j} + J_{1,j}^{>} - J_{1,j+1}^{\vee>\vee} + {}^{>}L_{1,j} - {}^{\vee>\vee}L_{1,j+1} \\ H_{2,j} &= Y_{2,j}^{++} + {}^{\vee>\vee}P_{2,j} + A_{2,j}^{>\vee} - A_{2,j-1}^{\vee>} + {}^{>}L_{2,j} - {}^{\vee>\vee}L_{2,j+1} \end{aligned}$$

In both cases, using this in (9.12) and applying Lemma 9.13 gives, for $j \geq 1$,

$$\begin{aligned} H_j = N_{0,j}^{\vee+^k} &+ {}^{+^2}(Z_{0,j} + N_{1,j}^{\vee})^{+^{k-1}} + \sum_{i=2}^{\ell-1} {}^{+^{2i}}(Z_{i-1,j} + Y_{i,j})^{+^{k-2(i-1)}} \\ &+ {}^{+^{k-1}}({}^{\vee}P_{\ell-1,j} + Y_{\ell,j})^{+^2} + {}^{+^k\vee}P_{\ell,j} + C_j^{>\vee} - C_{j-1}^{\vee>} \\ &+ {}^{\vee>}D_j - {}^{>\vee}D_{j-1}. \end{aligned}$$

with, for $j \geq 0$,

$$C_j = -J_{0,j+1}^{\vee +^{k-1}} - {}^{+^2}J_{1,j+1}{}^{\vee +^{k-2}} + \sum_{i=2}^{\ell-1} {}^{+^{2i}}A_{i,j}{}^{+^{k-2i}} + {}^{+^{k-1}}A_{\ell,j} + F_j \text{ and}$$

$$D_j = B_{0,j}^{+^{k-1}} + \sum_{i=1}^{\ell-2} {}^{+^{2i}}B_{i,j}{}^{+^{k-2i}} - {}^{+^{k-2}\vee}L_{\ell-1,j+1}{}^{+^2} - {}^{+^{k-1}\vee}L_{\ell,j+1} + G_j.$$

Thus, we have established a decomposition of the form in (9.14) as required. □

We can now conclude.

Proof of Proposition 9.3. As for Proposition 7.12, this relies on an induction argument.

For $k = 1, 2$, the statement is a direct consequence of the definition of the objects and of Lemma 7.1.

We now prove that if $k \geq 4$ is even, $k = 2\ell$, $\ell \geq 2$, and if the statement holds for $k-2$, it also holds for k. We apply Corollary 9.15. Therefore, we may write, for $j \geq 1$,

$$\begin{aligned} H_j = H_{0,j}^{+^k} + {}^{+^2}H_{1,j}{}^{+^{k-1}} + \sum_{i=2}^{\ell-1} {}^{+^{2i}}H_{i,j}{}^{+^{k-2(i-1)}} + {}^{+^{k-1}}H_{\ell,j}{}^{+^2} + {}^{+^k}H_{\ell+1,j} \\ + F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1}, \end{aligned}$$

where the $(F_j)_{j\geq 0}$ are in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$, the $(G_j)_{j\geq 0}$ are in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$ and, for $0 \leq i \leq \ell+1$, the $(H_{i,j})_{j\geq 1}$ are in $\mathcal{H}_{r(\ell+1-i)} \otimes \mathcal{H}_{r(i)}$ and all these sequences are finitely supported.

Since the sequence $(\varpi_k(H_j))_{j\geq 1}$ is cohomologically trivial in W_k, by Lemma 7.1, the assumption of Lemma 9.6 is satisfied. In particular, the conclusion of this result for $i = 0$ says that we may find finitely supported sequences $(P_j)_{j\geq 1}$ in $\mathcal{H}_k \otimes \mathcal{H}_{-1}$ and $(Q_j)_{j\geq 1}$ and $(R_j)_{j\geq 1}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_0$ such that, for $j \geq 1$,

$$H_{0,j} = P_j^{>} + {}^{>}Q_j + {}^{\vee>}R_j = P_j^{>} + {}^{\vee>\vee}S_j + {}^{>}Q_j - {}^{\vee>\vee}Q_{j+1},$$

where $S_j = Q_{j+1} + {}^{\vee}R_j$. As $k \geq 4$, by Lemma 7.8 and Lemma 8.4, we can find finitely supported sequences $(T_j)_{j\geq 1}$ and $(U_j)_{j\geq 1}$ in $\mathcal{H}_{k-2} \otimes \mathcal{H}_0$ such that, for $j \geq 1$,

$$S_j = {}^{>}T_j + {}^{\vee>}U_j = {}^{>}V_j + {}^{\vee>}U_j - {}^{>\vee}U_{j+1},$$

with $V_j = T_j + {}^{\vee}U_{j+1}$. We get, for $j \geq 1$,

$$H_{0,j} = P_j^{>} + {}^{++}V_j + {}^{\vee>}W_j - {}^{>\vee}W_{j-1},$$

where, for $j \geq 0$, $W_j = -{}^{\vee}Q_{j+1} - {}^{\vee>\vee}U_{j+1}$. Now, for $j \geq 0$, we set

$$X_j = \sum_{\substack{i \geq j+1 \\ i-j \text{ even}}} P_i - \sum_{\substack{i \geq j+1 \\ i-j \text{ odd}}} P_i^{\vee},$$

so that, for $j \geq 1$, we get $P_j = -X_j - X_{j-1}^{\vee}$, hence

$$H_{0,j} = {}^{++}V_j + {}^{\vee>}W_j - {}^{>\vee}W_{j-1} + X_j^{>\vee} - X_{j-1}^{\vee>}.$$

Reasoning in the same way from the conclusion of Lemma 9.6 for $i = \ell+1$, we obtain finitely supported sequences $(A_j)_{j\geq 1}$ in $\mathcal{H}_0 \otimes \mathcal{H}_{k-2}$, $(B_j)_{j\geq 0}$ in $\mathcal{H}_0 \otimes \mathcal{H}_{k-1}$ and $(C_j)_{j\geq 0}$ in $\mathcal{H}_{-1} \otimes \mathcal{H}_k$ such that, for $j \geq 1$,

$$H_{\ell-1,j} = A_j^{++} + B_j^{>\vee} - B_{j-1}^{\vee>} + {}^{\vee>}C_j - {}^{>\vee}C_{j-1}.$$

We set

$$J_j = V_j^{+^{k-2}} + H_{1,j}^{+^{k-3}} + \sum_{i=2}^{\ell-1} {}^{+^{2(i-1)}}H_{i,j}{}^{+^{k-2i}} + {}^{+^{k-3}}H_{\ell,j} + {}^{+^{k-2}}A_j,$$

which is an element of $\otimes^2\mathcal{H}_{k-2}$. By using Lemma 9.13, we can write

$$H_j = {}^{++}J_j{}^{++} + K_j^{>\vee} - K_{j-1}^{\vee>} + {}^{\vee>}L_j - {}^{>\vee}L_{j-1},$$

with

$$K_j = F_j + X_j^{+^k} + {}^{+^k}B_j \text{ and } L_j = G_j + W_j^{+^k} + {}^{+^k}C_j.$$

Therefore, as the sequence $(\varpi_k(H_j))_{j\geq 1}$ is cohomologically trivial in W_k, by Lemma 7.1, Propositon 9.4 and Lemma 9.8, the sequence $(\varpi_{k-2}(J_j))_{j\geq 1}$ is cohomologically trivial in W_{k-2}. By induction, we may find finitely supported sequences $(M_j)_{j\geq 0}$ in $\mathcal{H}_{k-2} \otimes \mathcal{H}_{k-3}$ and $(N_j)_{j\geq 0}$ in $\mathcal{H}_{k-3} \otimes \mathcal{H}_{k-2}$ such that, for $j \geq 1$,

$$J_j = M_j^{>\vee} - M_{j-1}^{\vee>} + {}^{\vee>}N_j - {}^{>\vee}N_{j-1}.$$

By using Lemma 9.13, we get

$$\begin{aligned} H_j = (K_j + {}^{++}M_j{}^{++})^{>\vee} &- (K_{j-1} + {}^{++}M_{j-1}{}^{++})^{\vee>} \\ &+ {}^{\vee>}(L_j + {}^{++}N_j{}^{++}) - {}^{>\vee}(L_{j-1} + {}^{++}N_{j-1}{}^{++}) \end{aligned}$$

and the conclusion follows.

So far, we have proved the Proposition for $k = 1$ and for $k \geq 2$ even. It remains to prove it for $k \geq 3$ odd. For such a k, by Lemma 7.5 and Lemma 7.8, we can find finitely supported sequences $(J_j)_{j\geq 1}$ and $(K_j)_{j\geq 1}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$ and $(L_j)_{j\geq 1}$ and $(M_j)_{j\geq 1}$ in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$ such that, for $j \geq 1$,

$$\begin{aligned} H_j &= J_j^{>} + K_j^{>\vee} + {}^{>}L_j + {}^{\vee>}M_j \\ &= N_j^{>} + {}^{>}P_j + K_j^{>\vee} - K_{j+1}^{\vee>} + {}^{\vee>}M_j - {}^{>\vee}M_{j+1}, \end{aligned}$$

where $N_j = J_j + K_{j+1}^{\vee}$ and $P_j = L_j + {}^{\vee}M_{j+1}$. Applying again Lemma 7.8, we can find finitely supported sequences $(U_j)_{j\geq 1}$, $(V_j)_{j\geq 1}$, $(W_j)_{j\geq 1}$ and $(X_j)_{j\geq 1}$ in $\mathcal{H}_{k-1}\otimes\mathcal{H}_{k-1}$ with, for $j\geq 1$,

$$\begin{aligned} N_j &= {}^{>}U_j + {}^{\vee>}V_j = {}^{>}Y_j + {}^{\vee>}V_j - {}^{>\vee}V_{j+1} \\ P_j &= W_j^{>} + X_j^{>\vee} = Z_j^{>} + X_j^{>\vee} - X_{j+1}^{\vee>}, \end{aligned}$$

where $Y_j = {}^{>}U_j + {}^{\vee}V_{j+1}$ and $Z_j = W_j + X_{j+1}^{\vee}$. We set $A_j = Y_j + Z_j$, $B_j = K_{j+1} + {}^{>}X_{j+1}$ and $C_j = M_{j+1} + V_{j+1}^{>}$ and we get

$$H_j = {}^{>}A_j{}^{>} + B_{j-1}^{>\vee} - B_j^{\vee>} + {}^{\vee>}C_{j-1} - {}^{>\vee}C_j.$$

As the sequence $(\varpi_k(H_j))_{j\geq 1}$ is cohomologically trivial in W_k, Lemma 7.1 and Proposition 9.4 say that the sequence $(\varpi_{k-1}(A_j))_{j\geq 1}$ is cohomologically trivial in W_{k-1}. Therefore, as the Proposition is true for $k-1$, we can find finitely supported sequences $(D_j)_{j\geq 0}$ and $(E_j)_{j\geq 0}$ in in $\mathcal{H}_{k-1}\otimes\mathcal{H}_{k-2}$ and $\mathcal{H}_{k-1}\otimes\mathcal{H}_{k-2}$ such that, for $j\geq 1$,

$$A_j = D_j^{>\vee} - D_{j-1}^{\vee>} + {}^{\vee>}E_j - {}^{>\vee}E_{j-1}.$$

By using Lemma III.2.6, we get

$$\begin{aligned} H_j = ({}^{>}D_j{}^{>} - B_j)^{\vee>} - ({}^{>}D_{j-1}{}^{>} - B_{j-1})^{>\vee} + {}^{>\vee}({}^{>}E_j{}^{>} - C_j) \\ - {}^{\vee>}({}^{>}E_{j-1}{}^{>} - C_{j-1}) \end{aligned}$$

and the conclusion follows. $\square$

10. The ultraweight map

We now come back to the study of the Plancherel formula of Proposition 5.16, which, as we have seen in the proof of Corollary 5.17, should be thought of as a universal tool for defining the spectral theory of non-negative Γ-invariant bilinear forms on $\overline{\mathcal{D}}(\partial X)$.

In the present Section, for $k\geq -1$, we will use this Plancherel formula to construct a linear map Ω_k, from the space $\otimes^2\mathcal{H}_k[t]$ of polynomial functions with values in $\otimes^2\mathcal{H}_k$, towards the space of (cohomology classes of) Hölder continuous functions on $\Gamma\backslash\mathscr{S}$. We will call Ω_k the ultraweight map. Later, in Section 12 we will use the ultraweight map to describe the bilinear forms on $\mathcal{H}_k^2[t]$ that are obtained from Γ-invariant symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$ through the spectral transform of Section III.6.

10.1. **Plancherel formula and the ultraweight map.** We start by defining a new object that implicitely appears in the proof of Corollary 5.17.

The following definition is inspired by Proposition 5.16 and the latter proof.

Definition 10.1. Let $k \geq -1$ and H be in $\otimes^2 \mathcal{H}_k[t]$, that is, H is a polynomial function with values in $\otimes^2 \mathcal{H}_k$. We defined the ultraweight $\Omega_k(H)$ as the Hölder continuous function on $\Gamma \backslash \mathscr{S}$ given by, for σ in $\mathscr{S}$,

$$\Omega_k(H)(\sigma) = \frac{2}{q+1} \int_{\mathcal{I}_q} \Phi_t \left({}^{>\infty} H(t)^{>\infty} \right) (\sigma) \mathrm{d}\mu_q(t)$$
$$+ \frac{q-1}{2(q+1)} \Phi_1^{\mathrm{sp}} \left({}^{>\infty} H(1)^{>\infty} \right) (\sigma) + \frac{q-1}{2(q+1)} \Phi_{(-1)}^{\mathrm{sp}} \left({}^{>\infty} H(-1)^{>\infty} \right) (\sigma).$$

The fact that the formula makes sense and defines a Hölder continuous function on $\Gamma \backslash \mathscr{S}$ is a consequence of Lemma 5.9.

As for the fundamental bilinear maps, we will think to the ultraweight map as taking its values in the space of cohomology classes of Hölder continuous functions on $\Gamma \backslash \mathscr{S}$. In this sense, our purpose will now be to describe the null space of the ultraweight map: for $k \geq 1$, we let Θ_k be the space of those H in $\otimes^2 \mathcal{H}_k[t]$ such that $\Omega_k(H)$ is a coboundary in the space of Hölder continuous functions on $\Gamma \backslash \mathscr{S}$ (in the sense of Subsection 2.1). We call the elements of Θ_k the k-coboundary polynomial tensors. The purpose of the remainder of the Section is to use the previously introduced tools to describe the space Θ_k (up to a finite-dimensional subspace).

10.2. **The twist operator.** We first construct an operator that preserves the ultraweight in case Γ is bipartite.

If V is a vector space and A is an algebra acting on V, the algebra $A \otimes A$ acts on $V \otimes V$ in a natural way: for v, w in V and a, b in A, we have $(a \otimes b)(v \otimes w) = (av) \otimes (bw)$. In our situation, we fix $k \geq 1$. Then, the space $\mathcal{H}_k$ is equipped with an action of the algebra A_1 of Γ-invariant functions on X_1. Indeed, if H is in $\mathcal{H}_k$ and u is in A_1, we let uH be the k-pseudofunction defined by

$$(uH)_{xy} = u(xy) H_{xy}, \quad xy \in X_1.$$

The algebra $A_1 \otimes A_1$ may be identified naturally with the algebra of $(\Gamma \times \Gamma)$-invariant functions on $X_1 \times X_1$, so that every such function defines an endomorphism of $\otimes^2 \mathcal{H}_k$.

Let δ and ε be as in Subsection 4.2. We have a characterization of bipartite actions:

Lemma 10.2. *The action of Γ on X is bipartite if and only if the function $\varepsilon(-1)^\delta$ on $X_1 \times X_1$ is $(\Gamma \times \Gamma)$-invariant.*

Proof. Recall from Subsection III.2.1 that saying that Γ is bipartite amounts to saying that the function $(a,x) \mapsto (-1)^{d(a,x)}$ is $(\Gamma \times \Gamma)$-invariant on $X \times X$. Now, in view of the definitions, for ab and xy in X_1, we have $(-1)^{d(a,x)} = \varepsilon(ab,xy)(-1)^{\delta(ab,xy)}$. The conclusion follows. $\square$

Let the twist operation $\wr$ on pseudofunctions be defined as in Subsection III.2.6. From the definitions, one directly gets

Lemma 10.3. *Let $h,k \geq 1$ and H be in $\mathcal{H}_h \otimes \mathcal{H}_k$. We have ${}^{\wr}H^{\wr} = \varepsilon(-1)^{\delta}H$.*

By abuse of notation, for any $h,k \geq -1$, we will write $\varepsilon(-1)^{\delta}H$ instead of ${}^{\wr}H^{\wr}$ for H in $\mathcal{H}_h \otimes \mathcal{H}_k$.

We can relate multiplication by $\varepsilon(-1)^{\delta}$ and the natural operations.

Lemma 10.4. *Assume Γ is bipartite. Let $h,k \geq -1$ and H be in $\mathcal{H}_h \otimes \mathcal{H}_k$. We have*

$$^{\vee}(\varepsilon(-1)^{\delta}H) = (-1)^{k}\varepsilon(-1)^{\delta}(^{\vee}H) \text{ and } (\varepsilon(-1)^{\delta}H)^{\vee} = (-1)^{h}\varepsilon(-1)^{\delta}(H^{\vee})$$

as well as

$$^{>}(\varepsilon(-1)^{\delta}H) = \varepsilon(-1)^{\delta}(^{>}H) \text{ and } (\varepsilon(-1)^{\delta}H)^{>} = \varepsilon(-1)^{\delta}(H^{>}).$$

Proof. This can be obtained by a direct computation or by applying Lemma IIII.2.22 and Lemma 10.3. $\square$

Multiplication by $\varepsilon(-1)^{\delta}$ also behaves well with respect to the double weight map of Definition 9.1.

Lemma 10.5. *Assume Γ is bipartite. Let $k \geq 1$, H be in $\otimes^2\mathcal{H}_k$ and ab, xy be in X_k. We have*

$$\varpi_k(\varepsilon(-1)^{\delta}H)(ab,xy) = (-1)^{d(a,y)}\varpi_k(H)(ab,xy).$$

Proof. It suffices to prove the claim when H is of the form $J \otimes K$ for J, K in $\mathcal{H}_k$.

Assume k is even, $k = 2\ell$, $\ell \geq 1$. Then, by Definition 6.1 and Definition 9.1, we have

$$\begin{aligned}
&\varpi_k(\varepsilon(-1)^\delta H)(ab, xy) = (-1)^{d(a_\ell, y_\ell)} \Delta J_{a_\ell a_{\ell+1}}(b, a) \Delta K_{y_\ell y_{\ell+1}}(x, y) \\
&\quad + \sum_{1 \leq j \leq \ell-1} \sum_{\substack{z \in X \\ [y_{\ell+j} z] \cap [xy] = \{y_{\ell+j}\} \\ d(z, y_{\ell+j}) = j}} (-1)^{d(a_\ell, z)} \Delta J_{a_\ell a_{\ell+1}}(b, a) \Delta K_{zz_-}(x, y_{2i}) \\
&\quad + \sum_{1 \leq i \leq \ell-1} \sum_{\substack{c \in X \\ [a_{\ell+i} c] \cap [ab] = \{a_{\ell+i}\} \\ d(c, a_{\ell+i}) = i}} (-1)^{d(c, y_\ell)} \Delta J_{cc_-}(b, a_{2i}) \Delta K_{y_\ell y_{\ell+1}}(x, y) \\
&\quad + \sum_{1 \leq i,j \leq \ell-1} \sum_{\substack{c,z \in X \\ [a_{\ell+i} c] \cap [ab] = \{a_{\ell+i}\} \\ d(c, a_{\ell+i}) = i \\ [y_{\ell+j} z] \cap [xy] = \{y_{\ell+j}\} \\ d(z, y_{\ell+j}) = j}} (-1)^{d(c, z)} \Delta J_{cc_-}(b, a_{2i}) \Delta K_{zz_-}(x, y_{2i}).
\end{aligned}$$

For $1 \leq i, j \leq \ell - 1$ and c, z as above, we have

$$d(a_\ell, c) = 2i \text{ and } d(y_\ell, z) = 2j,$$

hence

$$(-1)^{d(a_\ell, z)} = (-1)^{d(c, y_\ell)} = (-1)^{d(c, z)} = (-1)^{d(a_\ell, y_\ell)} = (-1)^{d(a, y)}$$

and the conclusion follows.

The proof is the same in the odd case. □

Assume Γ is bipartite. In view of Lemma 10.2, we can construct an operator as follows. For $k \geq 1$ and H in $\otimes^2 \mathcal{H}_k[t]$, we define the twist $\widetilde{H}$ of H as the polynomial tensor defined by

$$\widetilde{H}(t) = \varepsilon(-1)^\delta H(-t).$$

This operator preserves the ultraweight.

Lemma 10.6. *Assume Γ is bipartite. Let $k \geq 1$ and H be in $\otimes^2 \mathcal{H}_k[t]$. We have*

$$\Omega_k(\widetilde{H}) = \Omega_k(H).$$

This statement is a direct consequence of the parity properties of the polynomial functions A_j and B_j, $j \geq 0$, of Subsection 4.1.

Proof. For $j \geq 0$, the definitions of A_j and B_j give $A_j(-t) = (-1)^j A_j(t)$ and $B_j(t) = (-1)^{j+1} B_j(-t)$, hence $\chi_{-t} = \varepsilon(-1)^\delta \chi_t$. Also, we have $\chi_{-1}^{\mathrm{sp}} = \varepsilon(-1)^\delta \chi_1^{\mathrm{sp}}$. The conclusion now follows from the definition of Ω_k and the fact that the measure μ_q on $\mathbb{R}$ is symmetric. □

10.3. **Trivial coboundary polynomial tensors.** We now give an a priori list of polynomial tensors which are annihilated by the ultra-weight map.

Definition 10.7. (k even) Let $k \geq 2$ be an even integer. Then, we define the space of k-trivial coboundary polynomial tensors $\Theta_k^0 \subset \otimes^2 \mathcal{H}_k[t]$ as the subspace of $\otimes^2 \mathcal{H}_k[t]$ spanned by the following polynomial tensors:

$$\begin{array}{rl} J \otimes K - K \otimes J & \quad J, K \in \mathcal{H}_k[t] \\ H & \quad H \in (\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-})[t] \\ {}^{>}H^{>} & \quad H \in (\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,-})[t] \end{array}$$

as well as

$$(q+1)t\,{}^{>}H^{>} + (q-1)\,{}^{>}H^{\vee>} - 2\,{}^{>}H^{\vee>\vee} \qquad H \in (\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,+} \oplus \mathcal{H}_{k-1,-} \otimes \mathcal{H}_{k-1,-})[t]$$

and

$$(q+1)tH^{>>} + (q-1)H^{>\vee>} - H^{\vee>\vee>} - H^{>\vee>\vee} \qquad H \in (\mathcal{H}_k \otimes \mathcal{H}_{k-2})[t]$$

and, if Γ is bipartite,

$$\widetilde{H} - H, \quad H \in \otimes^2 \mathcal{H}_k[t].$$

Definition 10.8. (k odd) Let $k \geq 1$ be an odd integer. Then, we define the space of k-trivial coboundary polynomial tensors $\Theta_k^0 \subset \otimes^2 \mathcal{H}_k[t]$ as the subspace of $\otimes^2 \mathcal{H}_k[t]$ spanned by the following polynomial tensors:

$$\begin{array}{rl} J \otimes K - K \otimes J & \quad J, K \in \mathcal{H}_k[t] \\ H & \quad H \in (\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-})[t] \\ {}^{>}H^{>} & \quad H \in (\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,-})[t] \end{array}$$

as well as

$$(q+1)t\,{}^{>}H^{>} + (q-1)\,{}^{>}H^{>\vee} - 2\,{}^{>}H^{\vee>\vee} \qquad H \in (\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,+} \oplus \mathcal{H}_{k-1,-} \otimes \mathcal{H}_{k-1,-})[t]$$

and

$$(q+1)tH^{>>} + (q-1)H^{\vee>>} - H^{\vee>\vee>} - H^{>\vee>\vee} \qquad H \in (\mathcal{H}_k \otimes \mathcal{H}_{k-2})[t]$$

and, if Γ is bipartite,

$$\widetilde{H} - H, \quad H \in \otimes^2 \mathcal{H}_k[t].$$

Note that the two definitions are identical except in the fifth case.

These uncomfortable definitions are justified by the following

Lemma 10.9. *Let $k \geq 1$ and H be a trivial coboundary polynomial tensor in $\otimes^2 \mathcal{H}_k[t]$. Then, the Hölder continuous function $\Omega_k(H)$ is a coboundary.*

This is a consequence of the properties established in Lemma 5.10 and Lemma 5.15. To prove this precisely, we will need

Lemma 10.10. *Let $0 < \alpha < 1$, $I \subset \mathbb{R}$ be a closed interval and $\varphi : I \to \mathscr{H}_\alpha$ be a continuous function such that, for any t in I, $\varphi(t)$ is a coboundary. Then the Hölder continuous function $\int_I \varphi(t)\mathrm{d}t$ is a coboundary.*

Proof. Indeed, it follows from Livšic Theorem, Proposition 2.1, that the space of coboundaries is a closed subspace of $\mathscr{H}_\alpha$. □

In the context of the Plancherel formula for the fundamental bilinear maps, we will use the previous under the following form.

Corollary 10.11. *Let $h \geq 0$ and $k \geq -1$ be integers. For every H, J in $\mathcal{H}_k$, the ultraweight $\Omega_k(t^h H \otimes J)$ of the polynomial tensor $t^h H \otimes J$ is cohomologous to $\Phi(P^h H^{>\infty}, J^{>\infty})$.*

Proof. Indeed, it follows from Lemma 5.9, Lemma 5.10 and Lemma 10.10 that the function

$$\int_{\mathcal{I}_q} \left(t^h \Phi_t\left(H^{>\infty}, J^{>\infty}\right) - \Phi_t\left(P^h H^{>\infty}, J^{>\infty}\right)\right) \mathrm{d}\mu_q(t)$$

is a coboundary. Then, the claim follows from Lemma 5.15 and Proposition 5.16. □

We note that this yields

Corollary 10.12. *Let $k \geq -1$ and H be a polynomial tensor in $\otimes^2 \mathcal{H}_k[t]$. Then the ultraweight $\Omega_k(H)$ is cohomologous to a smooth function on $\Gamma \backslash \mathscr{S}$.*

Remark 10.13. In particular, if θ is a T-invariant distribution on $\Gamma \backslash \mathscr{S}$, for H in $\otimes^2 \mathcal{H}_k[t]$, we can define $\langle \theta, \Omega_k(H) \rangle$ by means of the convention of Remark 2.8.

Proof of Lemma 10.9. We will only deal with the case where k is even, the odd case being analogous. We will check that in each of the five first cases of Definition 10.7, the ultraweight is a coboundary, as in the sixth case, the ultraweight is actually 0 by Lemma 10.6.

Let $h \geq 0$ be an integer.

Let J, K be in $\mathcal{H}_k$. By Corollary 10.11, $\Omega_k(t^h J \otimes K - t^h K \otimes J)$ is cohomologous to $\Phi(P^h J^{>\infty}, K^{>\infty}) - \Phi(P^h K^{>\infty}, J^{>\infty})$, which is a coboundary by Lemma 5.3.

In particular, if J is in $\mathcal{H}_{k,+}$ and K is in $\mathcal{H}_{k,-}$, then, as above $\Omega_k(t^h J \otimes K)$ is cohomologous to $\Phi(P^h J^{>\infty}, K^{>\infty})$. As $J^\vee = qJ$ and $K^\vee = -K$ and P commutes to R and S, we get, by Lemma 5.3,

$$\begin{aligned} \Phi(P^h J^{>\infty}, K^{>\infty}) &= \frac{1}{q}\Phi(P^h J^{\vee>\infty}, K^{>\infty}) = \frac{1}{q}\Phi(P^h R J^{>\infty}, K^{>\infty}) \\ = \frac{1}{q}\Phi(R P^h J^{>\infty}, K^{>\infty}) &\equiv \frac{1}{q}\Phi(P^h J^{>\infty}, R K^{>\infty}) = \frac{1}{q}\Phi(P^h J^{>\infty}, K^{\vee>\infty}) \\ &= -\frac{1}{q}\Phi(P^h J^{>\infty}, K^{>\infty}) \end{aligned}$$

(where we have written $\equiv$ for the cohomology equivalence relation). This gives $\Omega_k(t^h J \otimes K) \equiv 0$ as required.

In the same way, if J is in $\mathcal{H}_{k-1,+}$ and K is in $\mathcal{H}_{k-1,-}$, since $\Omega_k(t^h J^> \otimes K^>) = \Omega_{k-1}(t^h J \otimes K)$, we get $\Omega_k(t^h J^> \otimes K^>) \equiv 0$.

Let now J, K be both in $\mathcal{H}_{k-1,+}$ or both in $\mathcal{H}_{k-1,-}$. Still by Corollary 10.11, we have

$$\begin{aligned} &\Omega_k(t^h((q+1)tJ^> \otimes K^> + (q-1)J^> \otimes K^{\vee>} - 2J^> \otimes K^{\vee>\vee})) \\ &\qquad \equiv (q+1)\Phi(P^{h+1}J^{>\infty}, K^{>\infty}) + (q-1)\Phi(P^h J^{>\infty}, SK^{>\infty}) \\ &\qquad\qquad - 2\Phi(P^h J^{>\infty}, RSK^{>\infty}). \end{aligned}$$

As $(q+1)P = RS + SR - (q-1)S$, using Lemma 5.3, we obtain

$$\begin{aligned} &\Omega_k(t^h((q+1)tJ^> \otimes K^> + (q-1)J^> \otimes K^{\vee>} - 2J^> \otimes K^{\vee>\vee})) \\ &\qquad \equiv \Phi(P^h RSJ^{>\infty}, K^{>\infty}) - \Phi(P^h J^{>\infty}, RSK^{>\infty}) \end{aligned}$$

If J, K are in $\mathcal{H}_{k-1,+}$, we have $SJ^{>\infty} = J^{>\infty}$ and $SK^{>\infty} = K^{>\infty}$ and the latter is a coboundary. If they are in $\mathcal{H}_{k-1,-}$, we have $SJ^{>\infty} = -J^{>\infty}$ and $SK^{>\infty} = -K^{>\infty}$ and the same holds. Thus, in both cases, we get

$$\Omega_k(t^h((q+1)tJ^> \otimes K^> + (q-1)J^> \otimes K^{\vee>} - 2J^> \otimes K^{\vee>\vee})) \equiv 0$$

as required.

Finally, we take J in $\mathcal{H}_k$ and K in $\mathcal{H}_{k-2}$. By Corollary 10.11, we have

$$\begin{aligned} &\Omega_k(t^h((q+1)tJ \otimes K^{>>} + (q-1)J \otimes K^{>\vee>} - J \otimes K^{\vee>\vee>} - J \otimes K^{>\vee>\vee})) \\ &\qquad \equiv \Phi((q+1)P^{h+1}J^{>\infty}, K^{>\infty}) \\ &\qquad\quad + \Phi(P^h J^{>\infty}, (q-1)SK^{>\infty} - SRK^{>\infty} - RSK^{>\infty}). \end{aligned}$$

As $(q+1)P = RS + SR - (q-1)S$, the conclusion follows from Lemma 5.3. $\square$

Recall that, for $k \geq 1$, we let $\Theta_k \subset \otimes^2 \mathcal{H}_k[t]$ be the set of those polynomial tensors H in $\otimes^2 \mathcal{H}_k[t]$ such that the Hölder continuous function $\Omega_k(H)$ on $\Gamma \backslash \mathscr{S}$ is a coboundary. We have just shown that we have $\Theta_k^0 \subset \Theta_k$. The next statement says that the reverse inclusion is true up to a finite-dimensional subspace.

Proposition 10.14. *There exists an integer $n \geq 0$ such that, for any $k \geq 2$, we have $\Theta_k \subset \Theta_k^0 + (\otimes^2 \mathcal{H}_k)_{k+n}[t]$, that is, for every H in $\otimes^2 \mathcal{H}_k[t]$, if $\Omega_k(H)$ is a coboundary, we may find a trivial coboundary tensor J in Θ_k^0 such that $H - J$ has degree $\leq k+n$ in t.*

The proof of this statement will rely on Proposition 3.3. It will last until the end of the next Section.

10.4. **Endpoints series formulas for the ultraweight.** In order to be able to use Proposition 3.3 to analyse the vanishing of the ultraweight in cohomology, we translate the formulas of Section 6 in the language of the ultraweight.

Let V be a vector space. Recall that a tensor in $\otimes^2 V$ is said to be symmetric if it is invariant by the natural involution of $\otimes^2 V$ that maps $v_1 \otimes v_2$ to $v_2 \otimes v_1$ for v_1, v_2 in V. The space of symmetric tensors is denoted by $\mathrm{S}^2 V \subset \otimes^2 V$.

We still let $(C_j)_{j \geq 0}$ be the family of orthogonal polynomials of Subsection 4.6. From Proposition 6.3 and Proposition 6.4, we get

Proposition 10.15. *Let $k \geq 1$ and H be a symmetric polynomial tensor in $\mathrm{S}^2 \mathcal{H}_k[t]$. For $j \geq 0$, let H_j be the element of $\mathrm{S}^2 \mathcal{H}_k$ defined by*

$$H_j = \int_{\mathcal{I}_q} C_j(t) H(t) \mathrm{d}\mu_q(t).$$

Assume k is even. Then, if H is in $\mathrm{S}^2 \mathcal{H}_{k,+}[t]$, the ultraweight $\Omega_k(H)$ is cohomologous to the function

$$\sigma \mapsto \frac{1}{2} \kappa_{0,k}(H_0) + \frac{1}{2} \sum_{j=1}^{k-2} q^{-(j-1)} \kappa_{j,k}(H_j)(\sigma_0 \sigma_{j+k}) \\ - \sum_{j=k-1}^{\infty} q^{-(j-1)} \varpi_k(H_j)(\sigma_0 \sigma_k, \sigma_j \sigma_{j+k}).$$

If H is in $\mathrm{S}^2\mathcal{H}_{k,-}[t]$, the ultraweight $\Omega_k(H)$ is cohomologous to the function

$$\begin{aligned}\sigma \mapsto \frac{1}{2q}\kappa_{0,k}&\left(qH_0 + (q-1)\sum_{\substack{h>0\\ h \text{ even}}} H_h\right)\\ &- \frac{1}{2}\sum_{j=1}^{k-2} q^{-(j+1)}\kappa_{j,k}\left(q^2H_j + (q^2-1)\sum_{\substack{h>j\\ j-h \text{ even}}} H_h\right)(\sigma_0\sigma_{j+k})\\ &+ \sum_{j=k-1}^{\infty} q^{-(j+1)}\varpi_k\left(q^2H_j + (q^2-1)\sum_{\substack{h>j\\ j-h \text{ even}}} H_h\right)(\sigma_0\sigma_k, \sigma_j\sigma_{j+k}).\end{aligned}$$

Assume k is odd. Then, if H is in $\mathrm{S}^2\mathcal{H}_{k,+}[t]$, the ultraweight $\Omega_k(H)$ is cohomologous to the function

$$\begin{aligned}\sigma \mapsto \frac{1}{q+1}\kappa_{0,k}&\left((q+1)H_0 + (q-1)\sum_{h>0}(-1)^h H_h\right)\\ &+ \frac{2}{q+1}\sum_{j=1}^{k-2} q^{-j}\kappa_{j,k}\left(qH_j + (q-1)\sum_{h>j}(-1)^{j-h} H_h\right)(\sigma_0\sigma_{j+k})\\ - \frac{1}{q+1}&\sum_{j=k-1}^{\infty} q^{-j}\varpi_k\left(qH_j + (q-1)\sum_{h>j}(-1)^{j-h} H_h\right)(\sigma_0\sigma_k, \sigma_j\sigma_{j+k}).\end{aligned}$$

If H is in $\mathrm{S}^2\mathcal{H}_{k,-}[t]$, the ultraweight $\Omega_k(H)$ is cohomologous to the function

$$\begin{aligned}\sigma \mapsto \frac{1}{q+1}\kappa_{0,k}&\left((q+1)H_0 + (q-1)\sum_{h>0} H_h\right)\\ &- \frac{2}{q+1}\sum_{j=1}^{k-2} q^{-j}\kappa_{j,k}\left(qH_j + (q-1)\sum_{h>j} H_h\right)(\sigma_0\sigma_{j+k})\\ &+ \frac{1}{q+1}\sum_{j=k-1}^{\infty} q^{-j}\varpi_k\left(qH_j + (q-1)\sum_{h>j} H_h\right)(\sigma_0\sigma_k, \sigma_j\sigma_{j+k}).\end{aligned}$$

Note that, by Lemma 4.6, we have $H_j = 0$ for j large, so that all the sums above are finite and define smooth functions on $\Gamma\backslash\mathscr{S}$.

Remark 10.16. The formulas above only concern symmetric polynomial tensors in $\otimes^2\mathcal{H}_{k,+} \oplus \otimes^2\mathcal{H}_{k,-}$. Indeed, due to the form of trivial polynomial tensors in Definition 10.7 and Definition 10.8, and to Lemma 10.9, the ultraweight of skew symmetric polynomial tensors and of polynomial tensors in $\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-} \oplus \mathcal{H}_{k,-} \otimes \mathcal{H}_{k,+}$ is a coboundary. Therefore, for proving Proposition 10.14, we will only need to deal with symmetric polynomial tensors H such that $H^\vee = {}^\vee H$.

The proof uses the following formula for computing the values of a polynomial function at 1 and (-1) by means of its components in the basis $(C_j)_{j\geq 0}$.

Lemma 10.17. *Let f in $\mathbb{R}[t]$ be a polynomial function. For $j \geq 0$, set $f_j = \int_{\mathcal{I}_q} f(t)C_j(t)\mathrm{d}\mu_q(t)$. Then we have*

$$f(1) = \sum_{j\geq 0} f_j \text{ and } f(-1) = \sum_{j\geq 0}(-1)^j f_j.$$

Proof. Since by Lemma 4.6, the $(C_j)_{j\geq 0}$ form a basis of $\mathbb{R}[t]$, it suffices to check the formulas when f is one of them.

Now, the definition of these polynomials gives $C_0 = 1$ and, for $j \geq 1$,

$$\begin{aligned} C_j(1) = B_{j+1}(1) - B_{j-1}(1) = \frac{q^{j+1}-1}{q-1} - \frac{q^{j-1}-1}{q-1} &= \frac{q^2-1}{q-1}q^{j-1} \\ &= (q+1)q^{j-1} = \int_{\mathcal{I}_q} C_j(t)^2\mathrm{d}\mu_q(t), \end{aligned}$$

where the last equality follows from Lemma 4.6. Thus, the first formula holds. The second is obtained in the same way. □

Proof of Proposition 10.15. This is a direct computation. We start with the endpoints formulas obtained in Proposition 6.3 and Proposition 6.5, and we apply the construction of the ultraweight in Definition 10.1, the fact that integrals of coboundary are coboundaries which was shown in Lemma 10.10, and Lemma 10.17 above which allows to compute the values at (-1) and 1 of the polynomial tensors. □

11. Building trivial coboundary tensors

In the present Section, we will prove Proposition 10.14. Thus, for $k \geq 1$, we are given a polynomial tensor H in $\otimes^2\mathcal{H}_k[t]$ whose ultraweight $\Omega_k(H)$ is a coboundary and we want to build a trivial coboundary tensor J such that $H - J$ has degree $\leq k + n$ for some fixed n.

Our strategy is to apply Proposition 3.3 to the formulas in Proposition 10.15. Then, we will use Proposition 9.3 to say more about the form of sequences related to the coefficients $H_j = \int_{\mathcal{I}_q} C_j(t)H(t)\mathrm{d}\mu_q(t)$ and relate them to the analogue sequences for trivial coboundary tensors.

11.1. **Cohomological equations and eigenvectors.** First, we analyze precisely the fact that the sequences of polynomial tensors we will need to study from Proposition 10.15 have values in $\otimes^2\mathcal{H}_{k,+}\oplus\otimes^2\mathcal{H}_{k,-}$, $k\geq 1$ (see Remark 10.16).

Proposition 11.1. *Let $k\geq 1$ and $(F_j)_{j\geq 0}$ and $(G_j)_{j\geq 0}$ be finitely supported sequences in $\mathcal{H}_k\otimes\mathcal{H}_{k-1}$ and $\mathcal{H}_{k-1}\otimes H_k$. For $j\geq 1$, we set*

$$\begin{aligned} H_j &= F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1} && \text{if } k \text{ is even}\\ &= F_j^{\vee>} - F_{j-1}^{>\vee} + {}^{>\vee}G_j - {}^{\vee>}G_{j-1} && \text{if } k \text{ is odd}\end{aligned}$$

and we assume that H_j belongs to $\otimes^2\mathcal{H}_{k,+}\oplus\otimes^2\mathcal{H}_{k,-}$, that is, ${}^{\vee}H_j = H_j^{\vee}$.

Then, if k is even, there exist finitely supported sequences $(P_j)_{j\geq 1}$ in $\mathcal{H}_{k-2}\otimes\mathcal{H}_k$, $(Q_j)_{j\geq 1}$ in $\mathcal{H}_k\otimes\mathcal{H}_{k-2}$ and $(X_j)_{j\geq 1}$ and $(Y_j)_{j\geq 1}$ in $\mathcal{H}_{k-1}\otimes\mathcal{H}_{k-1}$ such that for $j\geq 1$, one has $P_j^{\vee} = {}^{\vee}P_j$, $Q_j^{\vee} = {}^{\vee}Q_j$ and

$$\begin{aligned} {}^{\vee}F_j - (q-1)F_j + F_{j-1}^{\vee} &= Q_j^{>} - (q-1)\,{}^{>}X_j + {}^{\vee>}(X_j+Y_j)\\ G_j^{\vee} - (q-1)G_j + {}^{\vee}G_{j-1} &= {}^{>}P_j - (q-1)Y_j^{>} + (X_j+Y_j)^{>\vee}.\end{aligned}$$

If k is odd, there exist finitely supported sequences $(P_j)_{j\geq 1}$ in $\mathcal{H}_{k-2}\otimes\mathcal{H}_k$, $(Q_j)_{j\geq 1}$ in $\mathcal{H}_k\otimes\mathcal{H}_{k-2}$ and $(Y_j)_{j\geq 1}$ and $(Z_j)_{j\geq 1}$ in $\mathcal{H}_{k-1}\otimes\mathcal{H}_{k-1}$ such that for $j\geq 1$, one has $P_j^{\vee} = {}^{\vee}P_j$, $Q_j^{\vee} = {}^{\vee}Q_j$ and

$$\begin{aligned} F_j^{\vee} + {}^{\vee}F_{j-1} &= Q_j^{>} + {}^{>}Y_j - {}^{\vee>}Z_j\\ {}^{\vee}G_j + G_{j-1}^{\vee} &= {}^{>}P_j - Y_j^{>} - Z_j^{>\vee}.\end{aligned}$$

The proof is a straightforward consequence of the results of Subsection 8.4 on the simplification of tensor equations. We split it according to the parity of k.

Lemma 11.2. *Let $k\geq 2$ be even, F_0, F_1 be in $\mathcal{H}_k\otimes\mathcal{H}_{k-1}$ and G_0, G_1 be in $\mathcal{H}_{k-1}\otimes\mathcal{H}_k$. We set*

$$H = F_1^{>\vee} - F_0^{\vee>} + {}^{\vee>}G_1 - {}^{>\vee}G_0.$$

Assume that we have ${}^{\vee}H = H^{\vee}$, that is, H belongs to $\otimes^2\mathcal{H}_{k,+}\oplus\otimes^2\mathcal{H}_{k,-}$. Then, there exist P in $\mathcal{H}_{k-2}\otimes\mathcal{H}_k$, Q in $\mathcal{H}_k\otimes\mathcal{H}_{k-2}$ and X, Y in $\mathcal{H}_{k-1}\otimes\mathcal{H}_{k-1}$ such that $P^{\vee} = {}^{\vee}P$, $Q^{\vee} = {}^{\vee}Q$ and

$$\begin{aligned} {}^{\vee}F_1 - (q-1)F_1 + F_0^{\vee} &= Q^{>} - (q-1)\,{}^{>}X + {}^{\vee>}(X+Y)\\ G_1^{\vee} - (q-1)G_1 + {}^{\vee}G_0 &= {}^{>}P - (q-1)Y^{>} + (X+Y)^{>\vee}.\end{aligned}$$

Proof. We set

$$J = {}^{\vee}F_1 - (q-1)F_1 + F_0^{\vee}$$
$$K = G_1^{\vee} - (q-1)G_1 + {}^{\vee}G_0.$$

The relation ${}^{\vee}H = H^{\vee}$ gives

$$\begin{aligned}{}^{\vee}F_1^{>\vee} - {}^{\vee}F_0^{\vee>} + q\,{}^{>}G_1 + (q-1)\,{}^{\vee>}G_1 - {}^{\vee>\vee}G_0 = \\ qF_1^{>} + (q-1)F_1^{>\vee} - F_0^{\vee>\vee} + {}^{\vee>}G_1^{\vee} - {}^{>\vee}G_0^{\vee}\end{aligned}$$

which we rewrite as

$${}^{\vee}J^{>} - J^{>\vee} = {}^{>}K^{\vee} - {}^{\vee>}K.$$

By Corollary 8.8, we may find U in $\mathcal{H}_{k-2} \otimes \mathcal{H}_k$, V in $\mathcal{H}_k \otimes \mathcal{H}_{k-2}$ and W, X, Y, Z in $\mathcal{H}_{k-1} \otimes \mathcal{H}_{k-1}$, such that

$$\begin{aligned} K^{\vee} &= {}^{>\vee}U + W^{>} + X^{>\vee} \qquad & K &= {}^{>}U - Y^{>} - Z^{>\vee} \\ {}^{\vee}J &= V^{\vee>} + {}^{>}W + {}^{\vee>}Y \qquad & J &= V^{>} - {}^{>}X - {}^{\vee>}Z. \end{aligned} \tag{11.1}$$

By comparing the above two expressions for J and K, we get

$${}^{>\vee}U + W^{>} + X^{>\vee} = {}^{>}U^{\vee} - qZ^{>} - (Y + (q-1)Z)^{>\vee}$$
$$V^{\vee>} + {}^{>}W + {}^{\vee>}Y = {}^{\vee}V^{>} - q\,{}^{>}Z - {}^{\vee>}(X + (q-1)Z).$$

By Corollary 8.9, we may find A, E, F in $\mathcal{H}_{k-1} \otimes \mathcal{H}_{k-2}$ and B, C, D in $\mathcal{H}_{k-2} \otimes \mathcal{H}_{k-1}$ with

$$\begin{aligned} W + qZ &= A^{\vee>} + {}^{>}B = {}^{>\vee}D + E^{>} \\ X + Y + (q-1)Z &= -A^{>} + {}^{>}C = -\,{}^{>}D + F^{>} \\ U^{\vee} - {}^{\vee}U &= B^{>} + C^{>\vee} \\ {}^{\vee}V - V^{\vee} &= {}^{>}E + {}^{\vee>}F. \end{aligned} \tag{11.2}$$

From the above, we get in particular

$$(A^{\vee} - E)^{>} = {}^{>}({}^{\vee}D - B) \text{ and } (A + F)^{>} = {}^{>}(C + D).$$

Thus, by Lemma 8.4, we can find L, M in $\mathcal{H}_{k-2} \otimes \mathcal{H}_{k-2}$ with

$$\begin{aligned} E &= A^{\vee} - {}^{>}L \qquad & B &= {}^{\vee}D - L^{>} \\ F &= {}^{>}M - A \qquad & C &= M^{>} - D. \end{aligned}$$

Using these relations in (11.2) gives

$$\begin{aligned} W + qZ &= A^{\vee>} + {}^{>\vee}D - {}^{>}L^{>} \\ X + Y + (q-1)Z &= -A^{>} - {}^{>}D + {}^{>}M^{>} \\ U^{\vee} - {}^{\vee}U &= {}^{\vee}D^{>} - D^{>\vee} - L^{>>} + M^{\vee>>} \\ {}^{\vee}V - V^{\vee} &= {}^{>}A^{\vee} - {}^{\vee>}A - {}^{>>}L + {}^{>>\vee}M. \end{aligned} \tag{11.3}$$

The last two relations imply that we may find P in $\mathcal{H}_{k-2} \otimes \mathcal{H}_k$, Q in $\mathcal{H}_k \otimes \mathcal{H}_{k-2}$ and R, S in $\mathcal{H}_{k-2} \otimes \mathcal{H}_{k-2}$ such that

$$\begin{aligned} P^\vee &= {}^\vee P & Q^\vee &= {}^\vee Q \\ U + D^> &= P + R^{>>} & V + {}^>A &= Q + {}^{>>}S \\ R^\vee - {}^\vee R &= -L + M^\vee & {}^\vee S - S^\vee &= -L + {}^\vee M. \end{aligned} \tag{11.4}$$

We set $T = R + S - M$. Note that the last two relations give $T^\vee = {}^\vee T$. To simplify the next expressions, we set

$$\begin{aligned} P_1 &= P + \frac{1}{q-1} T^{>>\vee} \\ Q_1 &= Q + \frac{1}{q-1} {}^{\vee>>}T \\ X_1 &= X + A^> - {}^>S^> \\ Y_1 &= Y + {}^>D - {}^>R^>. \end{aligned}$$

In particular, by (11.3), we get

$$X_1 + Y_1 + (q-1)Z + {}^>T^> = 0$$

and, by (11.1), (11.2) and (11.4),

$$\begin{aligned} J = Q^> - {}^>X_1 - {}^{\vee>}Z &= Q_1^> - {}^>X_1 + \frac{1}{q-1} {}^{\vee>}(X_1 + Y_1) \\ K = {}^>P - Y_1^> - Z^{>\vee} &= {}^>P_1 - Y_1^> + \frac{1}{q-1}(X_1 + Y_1)^{>\vee} \end{aligned}$$

as required. □

In the odd case, the same technique yields

Lemma 11.3. *Let $k \geq 1$ be odd, F_0, F_1 be in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$ and G_0, G_1 be in $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$. We set*

$$H = F_1^{\vee>} - F_0^{>\vee} + {}^{>\vee}G_1 - {}^{\vee>}G_0.$$

Assume that we have ${}^\vee H = H^\vee$, that is, H belongs to $\otimes^2\mathcal{H}_{k,+} \oplus \otimes^2\mathcal{H}_{k,-}$. Then, there exist P in $\mathcal{H}_{k-2} \otimes \mathcal{H}_k$, Q in $\mathcal{H}_k \otimes \mathcal{H}_{k-2}$ and Y, Z in $\mathcal{H}_{k-1} \otimes \mathcal{H}_{k-1}$ such that $P^\vee = {}^\vee P$, $Q^\vee = {}^\vee Q$ and

$$\begin{aligned} F_1^\vee + {}^\vee F_0 &= Q^> + {}^>Y - {}^{\vee>}Z \\ {}^\vee G_1 + G_0^\vee &= {}^>P - Y^> - Z^{>\vee}. \end{aligned}$$

Proof. The proof follows the same lines as the previous one. We set

$$J = F_1^\vee + {}^\vee F_0 \text{ and } K = {}^\vee G_1 + G_0^\vee.$$

As above, the relation ${}^\vee H = H^\vee$ gives

$${}^\vee J^> - J^{>\vee} = {}^>K^\vee - {}^{\vee>}K.$$

By Corollary 8.8, we get U in $\mathcal{H}_{k-2}\otimes\mathcal{H}_k$, V in $\mathcal{H}_k\otimes\mathcal{H}_{k-2}$ and W, X, Y, Z in $\mathcal{H}_{k-1}\otimes\mathcal{H}_{k-1}$, such that

$$\begin{aligned} K^{\vee} &= {}^{>\vee}U + W^{>} + X^{>\vee} & K &= {}^{>}U - Y^{>} - Z^{>\vee} \\ {}^{\vee}J &= V^{\vee>} + {}^{>}W + {}^{\vee>}Y & J &= V^{>} - {}^{>}X - {}^{\vee>}Z. \end{aligned} \tag{11.5}$$

We compare the two expressions for J and K and we get

$$\begin{aligned} {}^{>\vee}U + W^{>} + X^{>\vee} &= {}^{>}U^{\vee} - Z^{>} - Y^{>\vee} \\ V^{\vee>} + {}^{>}W + {}^{\vee>}Y &= {}^{\vee}V^{>} - {}^{>}Z - {}^{\vee>}X. \end{aligned}$$

By Corollary 8.9, we may find A, E, F in $\mathcal{H}_{k-1}\otimes\mathcal{H}_{k-2}$ and B, C, D in $\mathcal{H}_{k-2}\otimes\mathcal{H}_{k-1}$ with

$$\begin{aligned} W + Z &= A^{\vee>} + {}^{>}B = {}^{>\vee}D + E^{>} \\ X + Y &= -A^{>} + {}^{>}C = -{}^{>}D + F^{>} \\ U^{\vee} - {}^{\vee}U &= B^{>} + C^{>\vee} \\ {}^{\vee}V - V^{\vee} &= {}^{>}E + {}^{\vee>}F. \end{aligned} \tag{11.6}$$

From the above, we get in particular

$$(A^{\vee} - E)^{>} = {}^{>}({}^{\vee}D - B) \text{ and } (A + F)^{>} = {}^{>}(C + D).$$

Thus, by Lemma 8.4, we can find L, M in $\mathcal{H}_{k-2}\otimes\mathcal{H}_{k-2}$ with

$$\begin{aligned} E &= A^{\vee} - {}^{>}L & B &= {}^{\vee}D - L^{>} \\ F &= {}^{>}M - A & C &= M^{>} - D. \end{aligned}$$

Using these relations in (11.6) gives

$$\begin{aligned} W + Z &= A^{\vee>} + {}^{>\vee}D - {}^{>}L^{>} \\ X + Y &= -A^{>} - {}^{>}D + {}^{>}M^{>} \\ U^{\vee} - {}^{\vee}U &= {}^{\vee}D^{>} - D^{>\vee} - L^{>>} + M^{\vee>>} \\ {}^{\vee}V - V^{\vee} &= {}^{>}A^{\vee} - {}^{\vee>}A - {}^{>>}L + {}^{>>\vee}M. \end{aligned} \tag{11.7}$$

The last two relations imply that we may find P in $\mathcal{H}_{k-2}\otimes\mathcal{H}_k$, Q in $\mathcal{H}_k\otimes\mathcal{H}_{k-2}$ and R, S in $\mathcal{H}_{k-2}\otimes\mathcal{H}_{k-2}$ such that

$$\begin{aligned} P^{\vee} &= {}^{\vee}P & Q^{\vee} &= {}^{\vee}Q \\ U + D^{>} &= P + R^{>>} & V + {}^{>}A &= Q + {}^{>>}S \\ R^{\vee} - {}^{\vee}R &= -L + M^{\vee} & {}^{\vee}S - S^{\vee} &= -L + {}^{\vee}M. \end{aligned} \tag{11.8}$$

We set $T = R + S - M$. Note that the last two relations give $T^{\vee} = {}^{\vee}T$. To simplify the next expressions, we set

$$X_1 = X + A^{>} - {}^{>}S^{>} \text{ and } Y_1 = Y + {}^{>}D - {}^{>}R^{>},$$

so that we get, by (11.7)

$$X_1 + Y_1 + {}^{>}T^{>} = 0$$

and, by (11.5), (11.6) and (11.8),

$$J = Q^{>} - {}^{>}X_1 - {}^{\vee>}Z = (Q + {}^{>>}T)^{>} + {}^{>}Y_1 - {}^{\vee>}Z$$
$$K = {}^{>}P - Y_1^{>} - Z^{>\vee}$$

as required. □

Proof of Proposition 11.1. This directly follows from Lemma 11.2 and 11.3 that we apply for each $j \geq 1$. □

11.2. **Building trivial coboundary tensors from $\mathcal{H}_k \otimes \mathcal{H}_{k-2}$.** We are still aiming at proving Proposition 10.14. We will now use the particular form of the sequences appearing in Proposition 11.1 to relate them to the trivial coboundary polynomial tensors of Definition 10.7 and Definition 10.8. Unfortunately, this will require us to check different cases separately. In the present Subsection, we show

Lemma 11.4. *Let $k \geq 1$ and $(K_j)_{j\geq 1}$ and $(L_j)_{j\geq 1}$ be finitely supported sequences in $\mathcal{H}_{k,+} \otimes \mathcal{H}_{k-2,+}$ and $\mathcal{H}_{k,-} \otimes \mathcal{H}_{k-2,-}$. Let $(F_j)_{j\geq 0}$ and $(G_j)_{j\geq 0}$ be the unique finitely supported sequences in $\mathcal{H}_k \otimes \mathcal{H}_{k-1}$ such that, for $j \geq 1$, one has, if k is even,*

$$(11.9)\quad {}^{\vee}F_j - (q-1)F_j + F_{j-1}^{\vee} = K_j^{>} \text{ and } {}^{\vee}G_j - (q-1)G_j + G_{j-1}^{\vee} = L_j^{>}$$

and, if k is odd,

$$(11.10)\qquad F_j^{\vee} + {}^{\vee}F_{j-1} = K_j^{>} \text{ and } G_j^{\vee} + {}^{\vee}G_{j-1} = L_j^{>}.$$

Then, there exist trivial coboundary polynomial tensors H in $\otimes^2\mathcal{H}_{k,+}[t]$ and J in $\otimes^2\mathcal{H}_{k,-}[t]$ such that, for $j \geq 1$, one has, if k is even,

$$F_j^{>\vee} - F_{j-1}^{\vee>} = q^{-j}H_j$$
$$G_j^{>\vee} - G_{j-1}^{\vee>} = q^{-j}J_j + q^{-j}(1-q^{-2}) \sum_{\substack{h>j \\ j-h \text{ even}}} J_h$$

and, if k is odd,

$$F_j^{\vee>} - F_{j-1}^{>\vee} = q^{-j}H_j + q^{-j}(1-q^{-1}) \sum_{h>j} (-1)^{j-h} H_h$$
$$G_j^{\vee>} - G_{j-1}^{>\vee} = q^{-j}J_j + q^{-j}(1-q^{-1}) \sum_{h>j} J_h,$$

where

$$H_j = \int_{\mathcal{I}_q} C_j(t)H(t)\mathrm{d}\mu_q(t) \text{ and } J_j = \int_{\mathcal{I}_q} C_j(t)J(t)\mathrm{d}\mu_q(t).$$

Note that the sequences $(F_j)_{j\geq 0}$ and $(G_j)_{j\geq 0}$ are uniquely defined by (11.9) and (11.10) due to the easy

Lemma 11.5. *Let V be a vector space and $\theta : V \to V$ be an endomorphism. Then, for any finitely supported sequence $(x_j)_{j\geq 1}$ in V, there exists a unique finitely supported sequence $(y_j)_{j\geq 0}$ in V such that, for $j \geq 1$, one has $y_{j-1} = x_j + \theta y_j$.*

Proof. Indeed, for $h \geq j \geq 0$, one has necessarily

$$y_j = \sum_{i=j+1}^{h} \theta^{i-j-1} x_i + \theta^{h-j} y_h.$$

Since both sequences are finitely supported, this gives

$$y_j = \sum_{i=j+1}^{\infty} \theta^{i-j-1} x_i,$$

hence the uniqueness. The existence follows from the fact that the latter formula actually defines a finitely supported sequence. $\square$

We shall use the following formulas which can be seen as a consequence of the fact that the polynomial functions $(C_j)_{j\geq 0}$ of Subsection 4.6 are the spherical transforms of the spheres of the tree X in the language of [3].

Lemma 11.6. *For $j \geq 0$, the polynomial function $(q+1)tC_j(t)$ may be written as*

$$\begin{aligned}
&(q+1)tC_0(t) = C_1(t) \\
&(q+1)tC_1(t) = C_2(t) + (q+1)C_0(t) \\
&(q+1)tC_j(t) = C_{j+1}(t) + qC_{j-1}(t) \qquad\qquad j \geq 2.
\end{aligned}$$

Proof. This directly follows from the definitions in Subsection 4.6. The first relation is obvious. The third relation follows from the definition of C_j, $j \geq 1$, and the analogue property of B_j, $j \geq 1$ in Lemma 4.1. Finally, for proving the second relation, we write $(q+1)t = \frac{q}{u} + u$, hence

$$\begin{aligned}
C_2(t) = B_3(t) - B_1(t) = \frac{q^2}{u^2} + q + u^2 - 1 &= (q+1)^2t^2 - (q+1) \\
&= (q+1)tC_1(t) - (q+1)C_0(t)
\end{aligned}$$

as required. $\square$

We now start the proof of the main Lemma, which we split according to the parity of k.

Proof of Lemma 11.4 in case k is even. We first construct H. We look for H to be of the form

$$H(t) = (q+1)tP(t)^{>>} - P(t)^{>\vee>} - P(t)^{>\vee>\vee},$$

where P is in $\mathcal{H}_{k,+} \otimes \mathcal{H}_{k-2,+}[t]$. Note that H is then a trivial coboundary polynomial tensor in view of Definition 10.7. For $j \geq 0$, we set $P_j = \int_{\mathcal{I}_q} C_j(t)P(t)\mathrm{d}\mu_q(t)$. By Lemma 11.6, we have

$$\begin{aligned}
H_0 &= P_1^{>>} - P_0^{>\vee>} - P_0^{>\vee>\vee} \\
H_1 &= P_2^{>>} + (q+1)P_0^{>>} - P_1^{>\vee>} - P_1^{>\vee>\vee} \\
H_j &= P_{j+1}^{>>} + qP_{j-1}^{>>} - P_j^{>\vee>} - P_j^{>\vee>\vee}, && j \geq 2.
\end{aligned}$$

We set

$$\begin{aligned}
M_0 &= q^{-1}P_1^{>} - q^{-1}(q+1)P_0^{>\vee} \\
M_j &= q^{-j-1}P_{j+1}^{>} - q^{-j}P_j^{>\vee} && j \geq 1,
\end{aligned}$$

so that, for $j \geq 1$, as ${}^{\vee}P_j = P_j^{\vee} = qP_j$, we have

$$q^{-j}H_j = M_j^{>\vee} - M_{j-1}^{\vee>}.$$

To conclude, it suffices to choose P in order to have $M_j = F_j$ for any $j \geq 0$. In view of the definition of the $(F_j)_{j\geq 0}$ in the statement, we compute

$$\begin{aligned}
{}^{\vee}M_1 - (q-1)M_1 + M_0^{\vee} &= q^{-2}P_2^{>} - q^{-1}(q+1)P_0^{>} \\
{}^{\vee}M_j - (q-1)M_j + M_{j-1}^{\vee} &= q^{-j-1}P_{j+1}^{>} - q^{1-j}P_{j-1}^{>} && j \geq 2,
\end{aligned}$$

where we have used again the assumption that ${}^{\vee}P_j = qP_j$, $j \geq 0$. Thus, for the conclusion of the Lemma to hold, it suffices to have

$$q^{-2}P_2 - q^{-1}(q+1)P_0 = K_1 \text{ and } q^{-j-1}P_{j+1} - q^{1-j}P_{j-1} = K_j, \quad j \geq 2.$$

By Lemma 11.5, these equations uniquely define the finitely supported sequence $(P_j)_{j\geq 0}$, hence the polynomial tensor $P(t)$. The conclusion follows.

We now construct J, which we will seek to be of the form

$$J(t) = (q+1)tQ(t)^{>>} + qQ(t)^{>\vee>} - Q(t)^{>\vee>\vee},$$

where Q is in $\mathcal{H}_{k,-} \otimes \mathcal{H}_{k-2,-}[t]$. As above, J is then trivial in view of Definition 10.7. For $j \geq 0$, we set $Q_j = \int_{\mathcal{I}_q} C_j(t)Q(t)\mathrm{d}\mu_q(t)$. By Lemma 11.6, we have

$$\begin{aligned}
J_0 &= Q_1^{>>} + qQ_0^{>\vee>} - Q_0^{>\vee>\vee} \\
J_1 &= Q_2^{>>} + (q+1)Q_0^{>>} + qQ_1^{>\vee>} - Q_1^{>\vee>\vee} \\
J_j &= Q_{j+1}^{>>} + qQ_{j-1}^{>>} + qQ_j^{>\vee>} - Q_j^{>\vee>\vee}, && j \geq 2.
\end{aligned}$$

Now, we set

$$N_0 = -Q_1^{>} - (1+q^{-1})Q_0^{>\vee} - (1-q^{-2}) \sum_{\substack{h>0 \\ h \text{ even}}} Q_{h+1}^{>} + Q_h^{>\vee}$$

and, for $j \geq 1$,

$$N_j = -q^{-j}(Q_{j+1}^{>} + Q_j^{>\vee}) - q^{-j}(1-q^{-2}) \sum_{\substack{h>j \\ j-h \text{ even}}} Q_{h+1}^{>} + Q_h^{>\vee}.$$

A direct computation gives, for $j \geq 1$, as ${}^{\vee}Q_j = Q_j^{\vee} = -Q_j$,

$$q^{-j}J_j + q^{-j}(1-q^{-2}) \sum_{\substack{h>j \\ j-h \text{ even}}} J_h = N_j^{>\vee} - N_{j-1}^{\vee>}.$$

We will choose Q in order to have $N_j = G_j$ for any $j \geq 0$. By using again the assumption that ${}^{\vee}Q_j = -Q_j$, $j \geq 0$, we get

$$\begin{aligned} {}^{\vee}N_1 - (q-1)N_1 + N_0^{\vee} &= q^{-2}Q_2^{>} - q^{-1}(q+1)Q_0^{>} \\ {}^{\vee}N_j - (q-1)N_j + N_{j-1}^{\vee} &= q^{-j-1}Q_{j+1}^{>} - q^{1-j}Q_{j-1}^{>} \qquad j \geq 2. \end{aligned}$$

By Lemma 11.5, the equations

$$q^{-2}Q_2 - q^{-1}(q+1)Q_0 = L_1 \text{ and } q^{-j-1}Q_{j+1} - q^{1-j}Q_{j-1} = L_j, \quad j \geq 2.$$

uniquely define the finitely supported sequence $(Q_j)_{j\geq 0}$, hence the polynomial tensor $Q(t)$ and we are done. □

Proof of Lemma 11.4 in case k is odd. Now, we look for H to be of the form

$$H(t) = ((q+1)t + (q-1))P(t)^{>>} - P(t)^{>\vee>} - P(t)^{>\vee>\vee},$$

where P is in $\mathcal{H}_{k,+} \otimes \mathcal{H}_{k-2,+}[t]$. By Definition 10.8, this is a trivial polynomial tensor. For $j \geq 0$, we set $P_j = \int_{\mathcal{I}_q} C_j(t)P(t)\mathrm{d}\mu_q(t)$. By Lemma 11.6, we have

$$\begin{aligned} H_0 &= P_1^{>>} + (q-1)P_0^{>>} - P_0^{>\vee>} - P_0^{>\vee>\vee} \\ H_1 &= P_2^{>>} + (q-1)P_1^{>>} + (q+1)P_0^{>>} - P_1^{>\vee>} - P_1^{>\vee>\vee} \\ H_j &= P_{j+1}^{>>} + (q-1)P_j^{>>} + qP_{j-1}^{>>} - P_j^{>\vee>} - P_j^{>\vee>\vee}, \qquad j \geq 2. \end{aligned}$$

A tedious computation shows that, by setting

$$M_0 = -(1+q^{-1})P_0^{>} + q^{-1}P_1^{>\vee} - q^{-1}(1-q^{-1}) \sum_{h>1} (-1)^h (P_h^{>\vee} + P_h^{>})$$

and, for $j \geq 1$,

$$M_j = -q^{-j}P_j^{>} + q^{-j-1}P_{j+1}^{>\vee} - q^{-j-1}(1-q^{-1}) \sum_{h>j+1} (-1)^{j-h}(P_h^{>\vee} + P_h^{>}),$$

we get, for $j \geq 1$,

$$q^{-j}H_j + q^{-j}(1-q^{-1})\sum_{h>j}(-1)^{j-h}H_h = M_j^{\vee >} - M_{j-1}^{>\vee}.$$

As above, we want to get $M_j = F_j$ for any $j \geq 0$, so that we compute

$$\begin{aligned} M_1^{\vee} + {}^{\vee}M_0 &= q^{-2}P_2^{>} - q^{-1}(q+1)P_0^{>} \\ M_j^{\vee} + {}^{\vee}M_{j-1} &= q^{-j-1}P_{j+1}^{>} - q^{1-j}P_{j-1}^{>} \qquad j \geq 2. \end{aligned}$$

Using Lemma 11.5, we define the finitely supported sequence $(P_j)_{j\geq 0}$, hence the polynomial tensor $P(t)$, by

$$q^{-2}P_2 - q^{-1}(q+1)P_0 = K_1 \text{ and } q^{-j-1}P_{j+1} - q^{1-j}P_{j-1} = K_j, \quad j \geq 2$$

and we are done.

Finally we search for a J of the form

$$J(t) = ((q+1)t - (q-1))Q(t)^{>>} + Q(t)^{>\vee>} - Q(t)^{>\vee>\vee},$$

where Q is in $\mathcal{H}_{k,-} \otimes \mathcal{H}_{k-2,-}[t]$. Still by Definition 10.8, the polynomial tensor J is then trivial. For $j \geq 0$, we set $Q_j = \int_{\mathcal{I}_q} C_j(t)Q(t)\mathrm{d}\mu_q(t)$. By Lemma 11.6, we have

$$\begin{aligned} J_0 &= Q_1^{>>} - (q-1)Q_0^{>>} + Q_0^{>\vee>} - Q_0^{>\vee>\vee} \\ J_1 &= Q_2^{>>} - (q-1)Q_1^{>>} + (q+1)Q_0^{>>} + Q_1^{>\vee>} - Q_1^{>\vee>\vee} \\ J_j &= Q_{j+1}^{>>} - (q-1)Q_j^{>>} + qQ_{j-1}^{>>} + Q_j^{>\vee>} - Q_j^{>\vee>\vee}, \qquad j \geq 2. \end{aligned}$$

In this case, we set

$$N_0 = (1+q^{-1})Q_0^{>} + q^{-1}Q_1^{>\vee} + q^{-1}(1-q^{-1})\sum_{h>1} Q_h^{>\vee} + Q_h^{>}$$

and, for $j \geq 1$,

$$N_j = q^{-j}Q_j^{>} + q^{-j-1}Q_{j+1}^{>\vee} + q^{-j-1}(1-q^{-1})\sum_{h>j+1} Q_h^{>\vee} + Q_h^{>},$$

and we get, for $j \geq 1$,

$$q^{-j}J_j + q^{-j}(1-q^{-1})\sum_{h>j} J_h = N_j^{\vee >} - N_{j-1}^{>\vee}.$$

In order to get $N_j = G_j$ for any $j \geq 0$, we compute

$$\begin{aligned} N_1^{\vee} + {}^{\vee}N_0 &= q^{-2}Q_2^{>} - q^{-1}(q+1)Q_0^{>} \\ N_j^{\vee} + {}^{\vee}N_{j-1} &= q^{-j-1}Q_{j+1}^{>} - q^{1-j}Q_{j-1}^{>} \qquad j \geq 2. \end{aligned}$$

As above we use Lemma 11.5, to define the finitely supported sequence $(Q_j)_{j\geq 0}$, hence the polynomial tensor $Q(t)$, by

$$q^{-2}Q_2 - q^{-1}(q+1)Q_0 = L_1 \text{ and } q^{-j-1}Q_{j+1} - q^{1-j}Q_{j-1} = L_j, \quad j \geq 2.$$

The result follows. ☐

11.3. **Building trivial coboundary tensors from $\otimes^2\mathcal{H}_{k-1,+}\oplus\otimes^2\mathcal{H}_{k-1,-}$.** We continue preparing the proof of Proposition 10.14. Now, for $k \geq 1$, still in order to reconstruct the formulas from Proposition 11.1, we will use a second kind of trivial polynomial tensors in $\otimes^2\mathcal{H}_k[t]$, namely the ones coming from $\otimes^2\mathcal{H}_{k-1}[t]$ in Definition 10.7 and Definition 10.8. We split the statements according to the parity of k.

Lemma 11.7. *Let $k \geq 2$ be even and $(X_j)_{j\geq 1}$ and $(Y_j)_{j\geq 1}$ be finitely supported sequences in $\otimes^2\mathcal{H}_{k-1}$. Let $(F_j)_{j\geq 0}$ and $(G_j)_{j\geq 0}$ be the unique finitely supported sequences in $\mathcal{H}_k\otimes\mathcal{H}_{k-1}$ and $\mathcal{H}_{k-1}\otimes\mathcal{H}_k$ such that, for $j \geq 1$, one has,*

$$\begin{aligned}{}^{\vee}F_j-(q-1)F_j+F_{j-1}^{\vee}&=-(q-1)\,{}^{>}X_j+{}^{\vee>}(X_j+Y_j)\\ G_j^{\vee}-(q-1)G_j+{}^{\vee}G_{j-1}&=-(q-1)Y_j^{>}+(X_j+Y_j)^{>\vee}.\end{aligned}$$

Then, there exist a trivial coboundary polynomial tensor H in $\otimes^2\mathcal{H}_k[t]$ such that, for $j \geq 2$, one has,

$$\begin{aligned}q^j(F_j^{>\vee}-F_{j-1}^{\vee>}+{}^{\vee>}G_j-{}^{>\vee}G_{j-1})&=(q+1)({}^{\vee}H_j+H_j^{\vee})\\ -(q^2-1)H_j-(1-q^{-2})&\sum_{\substack{h>j\\ j-h\text{ even}}}{}^{\vee}H_h^{\vee}-q({}^{\vee}H_h+H_h^{\vee})+q^2H_h,\end{aligned}$$

where $H_j=\int_{\mathcal{I}_q}C_j(t)H(t)\mathrm{d}\mu_q(t)$.

Lemma 11.8. *Let $k \geq 1$ be odd and $(Y_j)_{j\geq 1}$ and $(Z_j)_{j\geq 1}$ be finitely supported sequences in $\otimes^2\mathcal{H}_{k-1}$. Let $(F_j)_{j\geq 0}$ and $(G_j)_{j\geq 0}$ be the unique finitely supported sequences in $\mathcal{H}_k\otimes\mathcal{H}_{k-1}$ and $\mathcal{H}_{k-1}\otimes\mathcal{H}_k$ such that, for $j \geq 1$, one has,*

$$\begin{aligned}F_j^{\vee}+{}^{\vee}F_{j-1}&={}^{>}Y_j-{}^{\vee>}Z_j\\ {}^{\vee}G_j+G_{j-1}^{\vee}&=-Y_j^{>}-Z_j^{>\vee}.\end{aligned}$$

Then, there exist a trivial coboundary polynomial tensor H in $\otimes^2\mathcal{H}_k[t]$ such that, for $j \geq 2$, one has,

$$\begin{aligned}&q^j(F_j^{\vee>}-F_{j-1}^{>\vee}+{}^{>\vee}G_j-{}^{\vee>}G_{j-1})=\\ &\qquad{}^{\vee}H_j+H_j^{\vee}+(1-q^{-1})\sum_{\substack{h>j\\ j-h\text{ even}}}{}^{\vee}H_h+H_h^{\vee}-{}^{\vee}H_{h-1}^{\vee}-H_{h-1},\end{aligned}$$

where $H_j=\int_{\mathcal{I}_q}C_j(t)H(t)\mathrm{d}\mu_q(t)$.

The complicated structure of the formulas above comes from the need of applying them to the sequences appearing in Proposition 10.15.

To prove Lemma 11.7 and Lemma 11.8, we will need to use the fact that in Proposition 9.3, the sequences $(F_j)_{j\geq 0}$ and $(G_j)_{j\geq 0}$ are not uniquely determined by the sequence $(H_j)_{j\geq 1}$.

Lemma 11.9. *Let $k \geq 0$ and $(B_j)_{j\geq 0}$ be a finitely supported sequence in $\otimes^2\mathcal{H}_k$.*

If k is odd, for $j \geq 0$, we set

$$F_j = {}^{>\vee}B_j - {}^{\vee>}B_{j+1} \text{ and } G_j = B_{j+1}^{>\vee} - B_j^{\vee>}.$$

Then, for $j \geq 1$, we have

$$F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1} = 0$$

and

$$\begin{aligned}{}^{\vee}F_j - (q-1)F_j + F_{j-1}^{\vee} = -\,{}^{>}((q-1)\,{}^{\vee}B_j + qB_{j+1} - {}^{\vee}B_{j-1}^{\vee}) \\ + {}^{\vee>}({}^{\vee}B_j - B_j^{\vee})\end{aligned}$$

as well as

$$\begin{aligned}G_j^{\vee} - (q-1)G_j + {}^{\vee}G_{j-1} = ((q-1)B_j^{\vee} + qB_{j+1} - {}^{\vee}B_{j-1}^{\vee})^{>} \\ + ({}^{\vee}B_j - B_j^{\vee})^{>\vee}.\end{aligned}$$

If k is even, for $j \geq 0$, we set

$$F_j = {}^{\vee>}B_j - {}^{>\vee}B_{j+1} \text{ and } G_j = B_{j+1}^{\vee>} - B_j^{>\vee}.$$

Then, for $j \geq 1$, we have

$$F_j^{\vee>} - F_{j-1}^{>\vee} + {}^{>\vee}G_j - {}^{\vee>}G_{j-1} = 0$$

and

$$\begin{aligned}F_j^{\vee} + {}^{\vee}F_{j-1} &= {}^{>}(B_{j-1} - {}^{\vee}B_{j+1}^{\vee}) - {}^{\vee>}({}^{\vee}B_j - B_j^{\vee}) \\ {}^{\vee}G_j + G_{j-1}^{\vee} &= -(B_{j-1} - {}^{\vee}B_{j+1}^{\vee})^{>} - ({}^{\vee}B_j - B_j^{\vee})^{>\vee}.\end{aligned}$$

Proof. These are direct computations. □

We will split the proofs of Lemma 11.7 and 11.8 according to whether the considered sequences belong to certain eigenspaces of the natural operators.

Proof of Lemma 11.7 when ${}^{\vee}X_j = X_j^{\vee}$ and ${}^{\vee}Y_j = Y_j^{\vee}$, $j \geq 1$. First assume that $X_j + Y_j = 0$ for $j \geq 1$. By Lemma 11.5, we know that there exists a unique finitely supported sequence $(B_j)_{j\geq 0}$ in $\otimes^2\mathcal{H}_{k-1,+}\oplus\otimes^2\mathcal{H}_{k-1,-}$ such that, for $j\geq 1$, one has

$$-\frac{1}{q-1}B_{j-1}+{}^{\vee}B_j+\frac{q}{q-1}B_{j+1}=X_j.$$

Then, the conclusion directly follows from Lemma 11.9.

Now, assume that $X_j=Y_j$ for $j\geq 1$. We will seek for $H(t)$ to be of the form

$$H(t)=(q+1)t\,{}^{>}A(t)^{>}+(q-1)\,{}^{>}A(t)^{\vee>}-2\,{}^{>}A(t)^{\vee>\vee}$$

where $A(t)$ is a polynomial tensor in $(\otimes^2\mathcal{H}_{k-1,+}\oplus\otimes^2\mathcal{H}_{k-1,-})[t]$. This is a trivial polynomial tensor in view of Definition 10.7. For $j\geq 0$, we set $A_j=\int_{\mathcal{I}_q}C_j(t)A(t)\mathrm{d}\mu_q(t)$ so that Lemma 11.6 gives, for $j\geq 2$,

$$H_j={}^{>}A_{j+1}^{>}+q\,{}^{>}A_{j-1}^{>}+(q-1)\,{}^{>}A_j^{\vee>}-2\,{}^{>}A_j^{\vee>\vee}.$$

For $j\geq 2$, we set

$$\begin{aligned}J_j=(q+1)({}^{\vee}H_j+H_j^{\vee})-(q^2-1)H_j\\ -(1-q^{-2})\sum_{\substack{h>j\\ j-h\text{ even}}}{}^{\vee}H_h^{\vee}-q({}^{\vee}H_h+H_h^{\vee})+q^2H_h.\end{aligned}$$

A direct computation gives

$$q^{-j}J_j=F_j^{>\vee}-F_{j-1}^{\vee>}+{}^{\vee>}G_j-{}^{>\vee}G_{j-1},$$

where, for $j\geq 0$,

$$\begin{aligned}&q^jF_j=\\ &\frac{q^2-1}{2}\,{}^{>}A_j^{\vee}-(q+1)\,{}^{\vee>}A_j^{\vee}+\frac{(q+1)(q^2+1)}{2q}\,{}^{>}A_{j+1}-\frac{q^2-1}{2q}\,{}^{\vee>}A_{j+1}\\ &\qquad+\frac{(q-1)(q+1)^2}{2q}\sum_{\substack{h>j\\ j-h\text{ even}}}q\,{}^{>}A_h^{\vee}-{}^{\vee>}A_h^{\vee}+q\,{}^{>}A_{h+1}-{}^{\vee>}A_{h+1}\end{aligned}$$

and

$$\begin{aligned}&q^jG_j=\\ &\frac{q^2-1}{2}\,{}^{\vee}A_j^{>}-(q+1)\,{}^{\vee}A_j^{>\vee}+\frac{(q+1)(q^2+1)}{2q}A_{j+1}^{>}-\frac{q^2-1}{2q}A_{j+1}^{>\vee}\\ &\qquad+\frac{(q-1)(q+1)^2}{2q}\sum_{\substack{h>j\\ j-h\text{ even}}}q\,{}^{\vee}A_h^{>}-{}^{\vee}A_h^{>\vee}+qA_{h+1}^{>}-A_{h+1}^{>\vee}.\end{aligned}$$

In particular, this gives, for $j \geq 2$,

$$q^j({}^\vee F_j - (q-1)F_j + F_{j-1}^\vee) = \frac{q(q^2-1)}{2}\,{}^>A_{j-1} - q(q+1)\,{}^{\vee>}A_{j-1} - \frac{q^2-1}{2q}\,{}^>A_{j+1} + \frac{q+1}{q}\,{}^{\vee>}A_{j+1}$$

and the symmetric relation

$$q^j(G_j^\vee - (q-1)G_j + {}^\vee G_{j-1}) = \frac{q(q^2-1)}{2}A_{j-1}^> - q(q+1)A_{j-1}^{>\vee} - \frac{q^2-1}{2q}A_{j+1}^> + \frac{q+1}{q}A_{j+1}^{>\vee}.$$

Thus, to conclude, it suffices to ensure that, for $j \geq 2$, one has

$$q^j X_j = \frac{q+1}{2q}A_{j+1} - \frac{q(q+1)}{2}A_{j-1},$$

which is possible by Lemma 11.5. □

We manage the odd case in an analogue way

Proof of Lemma 11.8 when ${}^\vee Y_j = Y_j^\vee$ *and* ${}^\vee Z_j = Z_j^\vee$, $j \geq 1$. If $Z_j = 0$ for any $j \geq 1$, then, by Lemma 11.5, there exists a unique finitely supported sequence $(B_j)_{j\geq 0}$ in $\otimes^2\mathcal{H}_{k-1,+} \oplus \otimes^2\mathcal{H}_{k-1,-}$ such that, for $j \geq 1$, one has

$$B_{j-1} - {}^\vee B_{j+1}^\vee = Y_j$$

and as above, the conclusion directly follows from Lemma 11.9.

We now assume $Y_j = 0$ for any $j \geq 1$ and we will construct $H(t)$ of the form

$$H(t) = (q+1)t\,{}^>A(t)^> + (q-1)\,{}^>A(t)^{>\vee} - 2\,{}^>A(t)^{\vee>\vee}$$

where $A(t)$ is a polynomial tensor in $(\otimes^2\mathcal{H}_{k-1,+} \oplus \otimes^2\mathcal{H}_{k-1,-})[t]$. This is a trivial polynomial tensor in view of Definition 10.8. As usual, for $j \geq 0$, we set $A_j = \int_{\mathcal{I}_q} C_j(t)A(t)\mathrm{d}\mu_q(t)$. By Lemma 11.6, we get, for $j \geq 2$,

$$H_j = {}^>A_{j+1}^> + q\,{}^>A_{j-1}^> + (q-1)\,{}^>A_j^{>\vee} - 2\,{}^>A_j^{\vee>\vee}.$$

For $j \geq 2$, we set

$$J_j = {}^\vee H_j + H_j^\vee + (1-q^{-1}) \sum_{\substack{h>j \\ j-h \text{ even}}} {}^\vee H_h + H_h^\vee - {}^\vee H_{h-1}^\vee - H_{h-1}.$$

A direct computation gives

$$q^{-j}J_j = F_j^{\vee>} - F_{j-1}^{>\vee} + {}^{>\vee}G_j - {}^{\vee>}G_{j-1},$$

where, for $j \geq 0$,

$$q^j F_j = -{}^{>}A_j + \frac{1}{q}{}^{\vee>}A^{\vee}_{j+1} + \frac{q-1}{q^2} \sum_{\substack{h>j \\ j-h \text{ even}}} -{}^{>}A_h - {}^{>}A^{\vee}_h + {}^{\vee>}A_{h+1} + {}^{\vee>}A^{\vee}_{h+1}$$

and

$$q^j G_j = -A^{>}_j + \frac{1}{q}{}^{\vee}A^{>\vee}_{j+1} + \frac{q-1}{q^2} \sum_{\substack{h>j \\ j-h \text{ even}}} -A^{>}_h - {}^{\vee}A^{>}_h + A^{>\vee}_{h+1} + {}^{\vee}A^{>\vee}_{h+1}.$$

For $j \geq 2$, we obtain

$$q^j(F^{\vee}_j + {}^{\vee}F_{j-1}) = -q\,{}^{\vee>}A_{j-1} + \frac{1}{q}{}^{\vee>}A_{j+1}$$
$$q^j({}^{\vee}G_j + G^{\vee}_{j-1}) = -qA^{>\vee}_{j-1} + \frac{1}{q}A^{>\vee}_{j+1}.$$

By Lemma 11.5, we may choose the $(A_j)_{j\geq 0}$ in order to get, for $j \geq 1$,

$$q^{-(j-1)}A_{j-1} - q^{-(j+1)}A_{j+1} = Z_j$$

and the conclusion follows. □

11.4. **Building trivial coboundary tensors from $\mathcal{H}_{k-1,+}\otimes\mathcal{H}_{k-1,-}\oplus \mathcal{H}_{k-1,-}\otimes\mathcal{H}_{k-1,+}$.** We now aim at proving Lemma 11.7 and Lemma 11.8 for sequences in $\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,-} \oplus \mathcal{H}_{k-1,-} \otimes \mathcal{H}_{k-1,+}$. We will follow the same method as above, which consists into simplifying the proof by removing cases where one can apply Lemma 11.9. Unfortunately, the formulas in the remaining cases are less easy to handle than before. To compensate this, we show

Lemma 11.10. *Let $k \geq 1$ be odd and $(X_j)_{j\geq 1}$ and $(Y_j)_{j\geq 1}$ be finitely supported sequences in $\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-} \oplus \mathcal{H}_{k,-} \otimes \mathcal{H}_{k,+}$. Then, there exist finitely supported sequences $(A_j)_{j\geq 0}$ and $(B_j)_{j\geq 0}$ in $\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-}\oplus\mathcal{H}_{k,-}\otimes \mathcal{H}_{k,+}$ such that, for $j \geq 1$, one has*

$$q^j(X_j + Y_j) = (q+1)A_j + \frac{q^2-1}{q} \sum_{\substack{h>j \\ j-h \text{ even}}} A_h + 2q^j\,{}^{\vee}B_j$$

and

$$q^j(X_j - Y_j) = 2\frac{q(q+1)}{q-1}{}^\vee A_{j-1} + \frac{(q+1)^2}{q}\sum_{\substack{h>j\\ j-h \text{ even}}} {}^\vee A_{h-1} + \frac{2q}{q-1}q^{j-1}B_{j-1} + \frac{2}{q-1}q^{j+1}B_{j+1}.$$

Proof. We set $X_0 = 0 = Y_0$. By Lemma 11.5, there exist unique finitely supported sequences $(U_j)_{j\geq -1}$ and $(V_j)_{j\geq -1}$ in $\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-}\oplus\mathcal{H}_{k,-}\otimes\mathcal{H}_{k,+}$ such that, for $j \geq 0$, one has

$$\begin{aligned}\frac{1}{q}U_{j+1} - qU_{j-1} &= \frac{q-1}{2(q+1)}q^j(X_j + Y_j)\\ \frac{1}{q}V_{j+1} - qV_{j-1} &= \frac{q-1}{2(q+1)}q^j(X_j - Y_j).\end{aligned}$$

For $j \geq 0$, we set

$$A_j = \frac{2q}{q-1}U_{j-1} + \frac{2}{q-1}U_{j+1} - 2\,{}^\vee V_j$$

and

$$q^jB_j = -2\frac{q(q+1)}{q-1}{}^\vee U_{j-1} - \frac{(q+1)^2}{q}\sum_{\substack{h>j\\ j-h \text{ even}}} {}^\vee U_{h-1} + (q+1)V_j + \frac{q^2-1}{q}\sum_{\substack{h>j\\ j-h \text{ even}}} V_h.$$

Straightforward computations show that the conclusion holds. □

In the even case, we have

Lemma 11.11. *Let $k \geq 0$ be even and $(Y_j)_{j\geq 1}$ and $(Z_j)_{j\geq 1}$ be finitely supported sequences in $\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-}\oplus\mathcal{H}_{k,-}\otimes\mathcal{H}_{k,+}$. Then, there exist finitely supported sequences $(A_j)_{j\geq 0}$ and $(B_j)_{j\geq 0}$ in $\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-}\oplus\mathcal{H}_{k,-}\otimes\mathcal{H}_{k,+}$ such that, for $j \geq 1$, one has*

$$q^{j+1}Y_j = \frac{q-1}{2}A_{j+1} - A_{j+1}^\vee + (1-q^{-1})\sum_{\substack{h>j\\ j-h \text{ even}}} \frac{q-1}{2}A_{h+1} - A_{h+1}^\vee + q^{j+1}B_{j-1} - q^{j+1}B_{j+1}$$

and

$$q^{j+1}Z_j = qA_j + \frac{q^2-1}{2q} \sum_{\substack{h>j \\ j-h \text{ even}}} A_h + q^{j+1}((q-1)B_j - 2B_j^\vee).$$

Proof. We set $Y_0 = 0 = Z_0$. By Lemma 11.5, there exist unique finitely supported sequences $(V_j)_{j\geq -1}$ and $(W_j)_{j\geq -1}$ in $\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-}\oplus\mathcal{H}_{k,-}\otimes\mathcal{H}_{k,+}$ such that, for $j \geq 0$, one has

$$V_{j+1} - q^2V_{j-1} = q^{j+1}Y_j \text{ and } W_{j+1} - q^2W_{j-1} = q^{j+1}Z_j.$$

For $j \geq 0$, we set

$$A_j = (q-1)V_j - 2V_j^\vee - qW_{j-1} - W_{j+1}$$

and

$$\begin{aligned} q^{j+1}B_j = -qV_j - \frac{q^2-1}{2q} \sum_{\substack{h>j \\ j-h \text{ even}}} V_h \\ + \frac{q-1}{2}W_{j+1} - W_{j+1}^\vee + (1-q^{-1}) \sum_{\substack{h>j \\ j-h \text{ even}}} \frac{q-1}{2}W_{h+1} - W_{h+1}^\vee. \end{aligned}$$

Again, the result follows by direct computations. $\square$

We use the above decompositions to finish the proofs Lemma 11.7 and Lemma 11.8.

Proof of Lemma 11.7 when ${}^\vee X_j = -X_j^\vee$ and ${}^\vee Y_j = -Y_j^\vee$, $j \geq 1$. We split the proof according to the decomposition given by Lemma 11.10.

First, assume that there exists a finitely supported sequence $(B_j)_{j\geq 0}$ in $\mathcal{H}_{k-1,+}\otimes\mathcal{H}_{k-1,-}\oplus\mathcal{H}_{k-1,-}\otimes\mathcal{H}_{k-1,+}$ such that, for $j \geq 1$, one has

$$\begin{aligned} X_j &= \frac{1}{q-1}B_{j-1} + {}^\vee B_j + \frac{q}{q-1}B_{j+1} \\ \text{and } Y_j &= -\frac{1}{q-1}B_{j-1} + {}^\vee B_j - \frac{q}{q-1}B_{j+1}. \end{aligned}$$

In that case, the conclusion directly follows from Lemma 11.9.

Thus, by Lemma 11.10, we are reduced to deal with the case when there exists a finitely supported sequence $(A_j)_{j\geq 0}$ in $\mathcal{H}_{k-1,+}\otimes\mathcal{H}_{k-1,-}\oplus$

$\mathcal{H}_{k-1,-} \otimes \mathcal{H}_{k-1,+}$ such that, for $j \geq 1$, one has

$$
\begin{aligned}
(11.11) \qquad q^j(X_j + Y_j) &= (q+1)A_j + \frac{q^2-1}{q} \sum_{\substack{h>j \\ j-h \text{ even}}} A_h \\
q^j(X_j - Y_j) &= 2\frac{q(q+1)}{q-1}\, {}^{\vee}A_{j-1} + \frac{(q+1)^2}{q} \sum_{\substack{h>j \\ j-h \text{ even}}} {}^{\vee}A_{h-1}.
\end{aligned}
$$

Then, we let $A(t)$ be the polynomial tensor defined by

$$\int_{\mathcal{I}_q} C_j(t)A(t)\mathrm{d}\mu_q(t) = A_j, \quad j \geq 0.$$

As this polynomial tensor belongs to $(\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-} \oplus \mathcal{H}_{k,-} \otimes \mathcal{H}_{k,+})[t]$, the polynomial tensor $H(t) = {}^{>}A(t)^{>}$ is trivial in view of Definition 10.7. For $j \geq 0$, we set

$$
\begin{aligned}
J_j &= (q+1)({}^{\vee}H_j + H_j^{\vee}) \\
&\quad - (q^2-1)H_j - (1-q^{-2}) \sum_{\substack{h>j \\ j-h \text{ even}}} {}^{\vee}H_h^{\vee} - q({}^{\vee}H_h + H_h^{\vee}) + q^2 H_h.
\end{aligned}
$$

A direct computation gives

$$q^{-j}J_j = F_j^{>\vee} - F_{j-1}^{\vee>} + {}^{\vee>}G_j - {}^{>\vee}G_{j-1},$$

where, for $j \geq 0$,

$$
\begin{aligned}
q^j F_j &= (q+1)\,{}^{>}A_j + (1-q^{-2}) \sum_{\substack{h>j \\ j-h \text{ even}}} q\,{}^{>}A_h - \frac{1}{2}\,{}^{\vee>}A_h - \frac{q}{2}\,{}^{>\vee}A_{h-1} \\
q^j G_j &= (q+1)A_j^{>} + (1-q^{-2}) \sum_{\substack{h>j \\ j-h \text{ even}}} qA_h^{>} - \frac{1}{2}A_h^{>\vee} - \frac{q}{2}A_{h-1}^{\vee>}.
\end{aligned}
$$

Using (11.11) yields the conclusion. □

Proof of Lemma 11.8 when ${}^{\vee}Y_j{}^{\vee} = -qY_j$ *and* ${}^{\vee}Z_j{}^{\vee} = -qZ_j$, $j \geq 1$. We now split the proof according to the decomposition given by Lemma 11.11.

We first assume there exists a finitely supported sequence $(B_j)_{j\geq 0}$ in $\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,-} \oplus \mathcal{H}_{k-1,-} \otimes \mathcal{H}_{k-1,+}$ such that, for $j \geq 1$, one has

$$Y_j = B_{j-1} - B_{j+1} \text{ and } Z_j = (q-1)B_j - 2B_j^{\vee}.$$

Then, the conclusion directly follows from Lemma 11.9.

By Lemma 11.11, it remains to manage the case when there exists a finitely supported sequence $(A_j)_{j\geq 0}$ in $\mathcal{H}_{k-1,+}\otimes\mathcal{H}_{k-1,-}\oplus\mathcal{H}_{k-1,-}\otimes\mathcal{H}_{k-1,+}$ such that, for $j\geq 1$, one has

(11.12)
$$q^{j+1}Y_j = \frac{q-1}{2}A_{j+1} - A_{j+1}^{\vee} + (1-q^{-1})\sum_{\substack{h>j\\ j-h \text{ even}}} \frac{q-1}{2}A_{h+1} - A_{h+1}^{\vee}$$
$$q^{j+1}Z_j = qA_j + \frac{q^2-1}{2q}\sum_{\substack{h>j\\ j-h \text{ even}}} A_h.$$

As above, we let $A(t)$ be the polynomial tensor defined by
$$\int_{\mathcal{I}_q} C_j(t)A(t)\mathrm{d}\mu_q(t) = A_j, \quad j\geq 0.$$

By Definition 10.8, the polynomial tensor $H(t) = {}^{>}A(t)^{>}$ is trivial. For $j\geq 0$, we set
$$J_j = {}^{\vee}H_j + H_j^{\vee} + (1-q^{-1})\sum_{\substack{h>j\\ j-h \text{ even}}} {}^{\vee}H_h + H_h^{\vee} - {}^{\vee}H_{h-1}^{\vee} - H_{h-1}.$$

We get
$$q^{-j}J_j = F_j^{\vee>} - F_{j-1}^{>\vee} + {}^{>\vee}G_j - {}^{\vee>}G_{j-1},$$
where, for $j\geq 0$,
$$q^{j+1}F_j = -{}^{>}A_{j+1} + (1-q^{-1})\sum_{\substack{h>j\\ j-h \text{ even}}} \frac{1}{2}{}^{\vee>}A_{h-1} - {}^{>}A_h$$
$$q^{j}G_j = -A_{j+1}^{>} + (1-q^{-1})\sum_{\substack{h>j\\ j-h \text{ even}}} \frac{1}{2}A_{h-1}^{>\vee} - A_h^{>}.$$

The conclusion follows by using (11.12). $\square$

11.5. **Endpoints equations and trivial coboundary tensors.** We will now use the previous constructions to finish the proof of Proposition 10.14. We will need the following description of natural projections in tensor spaces.

Lemma 11.12. *Let $k\geq -1$ and H be in $\otimes^2\mathcal{H}_k$. Write $H = J+K+L$ with J in $\otimes^2\mathcal{H}_{k,+}$, K in $\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-}\oplus\mathcal{H}_{k,-}\otimes\mathcal{H}_{k,+}$ and L in $\otimes^2\mathcal{H}_{k,-}$.*

If k is even, we have

$$\begin{aligned}
J &= \frac{1}{(q+1)^2}({}^\vee H^\vee + {}^\vee H + H^\vee + H)\\
K &= \frac{1}{(q+1)^2}(-2\,{}^\vee H^\vee + (q-1)\,{}^\vee H + (q-1)H^\vee + 2qH)\\
L &= \frac{1}{(q+1)^2}({}^\vee H^\vee - q\,{}^\vee H - qH^\vee + q^2 H).
\end{aligned}$$

If k is odd, we have

$$\begin{aligned}
J &= \frac{1}{4}({}^\vee H^\vee + {}^\vee H + H^\vee + H)\\
K &= \frac{1}{2}(-\,{}^\vee H^\vee + H)\\
L &= \frac{1}{4}({}^\vee H^\vee - {}^\vee H - H^\vee + H).
\end{aligned}$$

Proof. This is a direct computation. □

Proof of Proposition 10.14. Let H be in $\otimes^2\mathcal{H}_k[t]$ and assume that the ultraweight $\Omega_k(H)$ is a coboundary. Let n be as in Proposition 3.3. We will show that there exists a trivial coboundary tensor K such that $H - K$ has degree $\leq k+n$.

By Definition 10.7 and Definition 10.8, we may assume that H is a symmetric tensor and that it belongs to $\otimes^2\mathcal{H}_{k,+} \oplus \otimes^2\mathcal{H}_{k,-}$, that is, we have ${}^\vee H = H^\vee$. If Γ is bipartite, we can also assume that we have $\widetilde{H} = H$, that is, H is invariant by the twist operator of Subsection 10.2. As usual, for $j \geq 0$, we set $H_j = \int_{\mathcal{I}_q} C_j(t)H(t)\mathrm{d}\mu_q(t)$.

Suppose k is even. Then, by Lemma 11.12, the component of H in $\otimes^2\mathcal{H}_{k,+}[t]$ is $(q+1)^{-1}(H + H^\vee)$ and its component in $\otimes^2\mathcal{H}_{k,-}[t]$ is $(q+1)^{-1}(qH - H^\vee)$. For $j \geq 0$, we set

$$\begin{aligned}
J_j &= -(H_j + H_j^\vee) + (qH_j - H_j^\vee) + (1-q^{-2}) \sum_{\substack{h>j\\ j-h \text{ even}}} qH_h - H_h^\vee\\
&= (q-1)H_j - 2H_j^\vee + (1-q^{-2}) \sum_{\substack{h>j\\ j-h \text{ even}}} qH_h - H_h^\vee.
\end{aligned}$$

Assume Γ is not bipartite. Then, by Lemma 3.2, Proposition 3.3 and Proposition 10.15, there exist finitely supported sequences $(u_j)_{j\geq 0}$ in $W_{k,k-1}$ and $(v_j)_{j\geq 0}$ in $W_{k-1,k}$ such that, for every $j \geq k+n$ and ab, xy

in X_k, we have

$$(11.13)\quad q^{-j}\varpi_k(J_j)(ab,xy) = \\ u_j(ab,x_1y)+v_j(ab_1,xy)-u_{j-1}(ab,xy_1)-v_{j-1}(a_1b,xy).$$

Assume Γ is bipartite. As above, by Lemma 3.2, Proposition 3.3 and Proposition 10.15, there exist finitely supported sequences $u=(u_j)_{j\geq 0}$ in $W_{k,k-1}$ and $v=(v_j)_{j\geq 0}$ in $W_{k-1,k}$ such that, for every $j\geq k+n$, (11.13) holds for any ab, xy in X_k such that $j+d(a,x)$ is even (recall that k is even). Note that in these relations, for $j\geq 0$, we only use the values of the function u_j on the set

$$\{(ab,xy)\in X_k\times X_{k-1}|j+d(a,x) \text{ is odd}\}.$$

Thus, we can assume that, for every (ab,xy) in $X_k\times X_{k-1}$ we have $u_j(ab,xy)=0$ if $j+d(a,x)$ is even. In the same way, we also assume that, for every (ab,xy) in $X_{k-1}\times X_k$ we have $v_j(ab,xy)=0$ if $j+d(a,x)$ is even. Besides, recall that $\widetilde{H}=H$, so that, for $j\geq 0$, we have by the definition of the twist operator in Subsection 10.2, $\varepsilon(-1)^\delta H_j=(-1)^jH_j$, hence, by Lemma 10.4, $\varepsilon(-1)^\delta J_j=(-1)^jJ_j$. By Lemma 10.5, for every $j\geq 0$ and ab,xy in X_k such that $j+d(a,x)$ is odd, we get $\varpi_j(J_j)(ab,xy)=0$ so that (11.13) also holds for such pairs (ab,xy).

Hence, in both cases, we can assume that (11.13) is valid for any $j\geq k+n$ and any ab,xy in X_k, that is, by Definition 9.2, the sequence $(q^{-j}\varpi_k(J_j))_{j\geq k+n}$ is cohomologically trivial in W_k. By Proposition 9.3, we can find finitely supported sequences $(F_j)_{j\geq k+n-1}$ in $\mathcal{H}_k\otimes\mathcal{H}_{k-1}$ and $(G_j)_{j\geq k+n-1}$ in $\mathcal{H}_{k-1}\otimes\mathcal{H}_k$ such that, for $j\geq k+n$, one has

$$q^{-j}J_j=F_j^{>\vee}-F_{j-1}^{\vee>}+{}^{\vee>}G_j-{}^{>\vee}G_{j-1}.$$

Recall that we have ${}^\vee J_j{=}J_j^\vee$, $j\geq 0$. Therefore, by Proposition 11.1, we may find finitely supported sequences $(P_j)_{j\geq 1}$ in $\mathcal{H}_{k-2}\otimes\mathcal{H}_k$, $(Q_j)_{j\geq 1}$ in $\mathcal{H}_k\otimes\mathcal{H}_{k-2}$ and $(X_j)_{j\geq 1}$ and $(Y_j)_{j\geq 1}$ in $\mathcal{H}_{k-1}\otimes\mathcal{H}_{k-1}$ such that for $j\geq 1$, one has $P_j^\vee={}^\vee P_j$, $Q_j^\vee={}^\vee Q_j$ and, for $j\geq k+n$,

$$\begin{gathered}{}^\vee F_j-(q-1)F_j+F_{j-1}^\vee=Q_j^>-(q-1)\,{}^>X_j+{}^{\vee>}(X_j+Y_j)\\ G_j^\vee-(q-1)G_j+{}^\vee G_{j-1}={}^>P_j-(q-1)Y_j^>+(X_j+Y_j)^{>\vee}.\end{gathered}$$

Thanks to Lemma 11.5, we can extend the definition of F_j and G_j to all $j\geq 0$ by using the above relations. Now, Lemma 11.4, Lemma 11.7 and Lemma 11.12 precisely tell us that we may find a trivial coboundary polynomial tensor K such that, for any $j\geq k+n$, we have

$$H_j-K_j\in\mathcal{H}_{k,+}\otimes\mathcal{H}_{k,-}\oplus\mathcal{H}_{k,-}\otimes\mathcal{H}_{k,+},$$

where $K_j = \int_{\mathcal{I}_q} C_j(t)K(t)\mathrm{d}\mu_q(t)$. By Definition 10.7, the component of $H - K$ on $(\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-} \oplus \mathcal{H}_{k,-} \otimes \mathcal{H}_{k,+})[t]$ is a trivial coboundary tensor. The conclusion follows in case k is even.

Suppose now k is odd. We just sketch the proof. For $j \geq 0$, we set

$$J_j = H_j^\vee + (1 - q^{-1}) \sum_{\substack{h>j \\ j-h \text{ even}}} H_h^\vee - H_{h-1}.$$

Using Lemma 3.2, Proposition 3.3 and Proposition 10.15, we get (11.13) for all $j \geq k+n$ and ab, xy in X_k when Γ is not bipartite. When Γ is bipartite, this is only true when $d(a,x)+j$ is odd. But as we have assumed that $\widetilde{H} = H$, Lemma 10.4 says that $\varepsilon(-1)^\delta J_j = (-1)^{j+1} J_j$, $j \geq 0$. Reasoning as above, we show that we can again assume (11.13) to hold for all $j \geq k+n$ and ab, xy in X_k.

We conclude as in the even case by using Lemma 11.4, Lemma 11.8 and Lemma 11.12. $\square$

12. Spectral obstructions

In Section III.6, we have introduced the spectral transform of pseudofunctions. For $k \geq 0$, the spectral transform essentially allows to diagonalize the action of the natural operators on the subspace of $\mathcal{H}_\infty$ spanned by the image of $(\mathcal{H}_k)^{>\infty}$. This construction leads to the description of the spectral theory of orthogonal extension.

Now, as in Subsection 5.2, every Γ-invariant symmetric bilinear form of $\overline{\mathcal{D}}(\partial X)$ defines in a natural way a symmetric bilinear form on $\mathcal{H}_\infty$ for which the action of the operators R and S are symmetric. By pulling back this bilinear form under the spectral transform (and the polyextension map of Subsection III.2.3), we get a symmetric bilinear form on $\mathcal{H}_k^2[t]$ for which the polynomial operators of Subsection III.6.1 are symmetric and which vanishes on the range of the map described in Proposition III.6.5.

In the present Section, we will use Proposition 10.14 to show that the last two properties characterize the range of this map $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma \to \mathcal{Q}(\mathcal{H}_k^2[t])$ up to a finite dimensional subspace.

12.1. The first step extension.

For $k \geq -1$, we introduce a map $\mathcal{H}_k^2 \to \mathcal{H}_{k+1}$ that will allow us to pull-back the result of Proposition 10.14 under the spectral transform.

Definition 12.1. Let $k \geq -1$. We define the first step extension map $I_k : \mathcal{H}_k^2 \to \mathcal{H}_{k+1}$ as follows. For $H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$ in $\mathcal{H}_k^2$, we set

$$\begin{aligned} I_k H &= H_0^{\vee >} - (q-1)H_0^{>} + H_1^{>\vee} && \text{if } k \text{ is even} \\ &= H_0^{\vee>\vee} - (q-1)H_0^{\vee >} + H_1^{>} && \text{if } k \text{ is odd.} \end{aligned}$$

We shall see below in Lemma 12.6 that the first step extension map allows to describe the behaviour of the converse of the spectral transform on constant vectors. Before showing this, we establish some basic properties of this map.

There is a compatibility of first step extension with some other extension maps. A direct computation (using as usual Lemma III.2.6) yields

Lemma 12.2. *Let $k \geq 0$ and H be in $\mathcal{H}_k^2$. We have*

$$\begin{aligned} (I_{k-1}H)^{>} &= I_k(H^{\vee >}) && \textit{if } k \textit{ is even} \\ &= I_k(H^{>\vee}) && \textit{if } k \textit{ is odd.} \end{aligned}$$

Using Lemma III.2.8 allows to determine the null space of the map I_k:

Lemma 12.3. *Let $k \geq 0$ and H be in $\mathcal{H}_{k-1}^2$. Then $I_k H = 0$ if and only if there exists G in $\mathcal{H}_{k-1}$ with $H = \begin{pmatrix} G^{\vee>\vee} \\ -qG^{>} \end{pmatrix}$.*

Proof. Assume k is even. Then, if $I_k H = 0$, by Definition 12.1, we have

$$H_0^{\vee >} - (q-1)H_0^{>} = -H_1^{>\vee},$$

hence, by Lemma III.2.6, there exists G in $\mathcal{H}_{k-1}$ with

$$H_0^{\vee} - (q-1)H_0 = G^{\vee >} \text{ and } H_1 = -G^{>}.$$

The first relation amounts to $H_0 = q^{-1}G^{\vee>\vee}$. Therefore, we obtain $H = q^{-1}\begin{pmatrix} G^{\vee>\vee} \\ -qG^{>} \end{pmatrix}$ as required. Conversely, if H may be written in this way, a straightforward computation using Lemma III.2.6 shows that $I_k H = 0$.

The proof is analogue in the odd case. $\square$

In the bipartite case, the first step extension behaves well with respect to the operations introduced in Subsection III.2.6. Lemma III.2.22 and Definition 12.1 give

Lemma 12.4. *Assume Γ is bipartite. Let $k \geq -1$ and H be in $\mathcal{H}_k^2$. Then, we have*
$$I_k \begin{pmatrix} H_0^{\wr} \\ -H_1^{\wr} \end{pmatrix} = (-1)^k (I_k H)^{\wr}.$$

Now, by using the first step extension map, we can relate the pull back of the natural bilinear forms of Subsection 5.2 with the ultraweight.

Proposition 12.5. *Let p be a Γ-invariant symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$ and θ be the associated (ι, T)-invariant distribution of $\Gamma \backslash \mathscr{S}$. For $k \geq 0$ and H, J in $\mathcal{H}_k^{(\mathbb{N})}$. We have*
$$p(E_k H, E_k J) = \langle \theta, \Omega_{k+1}(I_k \widehat{H}(t) \otimes I_k \widehat{J}(t)) \rangle.$$

The polyextension map E_k was introduced in Definition III.2.11. The spectral transform $H \mapsto \widehat{H}$ was constructed in Proposition III.6.3. The ultraweight Ω_k was introduced in Definition 10.1.

Note that, in the formula above, the ultraweight $\Omega_{k+1}(I_k \widehat{H}(t) \otimes I_k \widehat{J}(t))$ is not a priori a smooth function. Nevertheless, by Corollary 10.12, we know that it is cohomologous to a smooth function, so that the formula makes sense, thanks to the convention introduced in Remark 2.8.

The proof relies on the next lemma which tells us that the first extension map is essentially defined by studying the converse of the spectral transform on constant vectors in $\mathcal{H}_k^2[t]$.

Lemma 12.6. *Let $k \geq -1$ and H be in $\mathcal{H}_k^2$. Define G in $\mathcal{H}_k^{(\mathbb{N})}$ by setting $G_i = 0$ for $i \geq 2$ and*
$$\begin{aligned} G_0 &= H_0^{\vee} - (q-1)H_0 & \quad G_1 &= H_1 & \quad &\text{if } k \text{ is even} \\ G_0 &= H_1 - (q-1)H_0^{\vee} & \quad G_1 &= H_0^{\vee} & \quad &\text{if } k \text{ is even.} \end{aligned}$$
Then we have
$$\widehat{G}(t) = H \text{ and } E_k G = (I_k H)^{>\infty}.$$

Thanks to this Lemma, we could also have recovered Lemma 12.3 as a consequence of Proposition III.6.5.

Proof. Note first that Definition III.2.11 gives in both cases
$$E_k G = (G_0^{>} + G_1^{>\vee})^{>\infty} = (I_k H)^{>\infty}.$$

Assume k is even. By Definition III.2.16, we have
$$G = G_0 \mathbf{1}_0 + G_1 \mathbf{1}_1 = G_0 \mathbf{1}_0 + S(G_1 \mathbf{1}_0).$$

Using (III.6.1) and Proposition III.6.3, we obtain

$$\widehat{G}(t) = \begin{pmatrix} q^{-1}G_0^\vee \\ 0 \end{pmatrix} + \mathfrak{S}_t \begin{pmatrix} q^{-1}G_1^\vee \\ 0 \end{pmatrix} = \begin{pmatrix} q^{-1}G_0^\vee \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ G_1 \end{pmatrix} = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$$

as required.

Assume k is odd. By Definition III.2.17, we have

$$G = G_0\mathbf{1}_0 + G_1\mathbf{1}_1 = G_0\mathbf{1}_0 + R(G_1\mathbf{1}_0).$$

Using (III.6.2) and Proposition III.6.3, we obtain

$$\widehat{G}(t) = \begin{pmatrix} 0 \\ G_0 \end{pmatrix} + \mathfrak{R}_t \begin{pmatrix} 0 \\ G_1 \end{pmatrix} = \begin{pmatrix} 0 \\ G_0 \end{pmatrix} + \begin{pmatrix} G_1^\vee \\ (q-1)G_1 \end{pmatrix} = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$$

as required. □

Proof of Proposition 12.5. We fix A, B in $\mathcal{H}_k^2$ and we let K and L be the elements of $\mathcal{H}_k^{(\mathbb{N})}$ given by Lemma 12.6 so that

$$\begin{aligned} \widehat{K}(t) &= A & E_k K &= (I_k A)^{>\infty} \\ \widehat{L}(t) &= B & E_k L &= (I_k B)^{>\infty}. \end{aligned}$$

Let $a, b \geq 0$ be integers. By Proposition III.6.3, the spectral transforms of $P^a K$ and $P^b L$ are given by

$$\widehat{P^a K}(t) = t^a A \text{ and } \widehat{P^b L}(t) = t^b B.$$

By Lemma III.2.18, Lemma 5.3, Proposition 5.4 and Corollary 10.11, we obtain

$$\begin{aligned} p(E_k(P^a K), E_k(P^b L)) &= p(P^a E_k K, P^b E_k L) \\ &= \langle \theta, \Phi(P^a E_k K, P^b E_k L)\rangle \\ &= \langle \theta, \Phi(P^{a+b} E_k K, E_k L)\rangle \\ &= \langle \theta, \Phi(P^{a+b}(I_k A)^{>\infty}, (I_k B)^{>\infty})\rangle \\ &= \langle \theta, \Omega_{k+1}(t^{a+b}(I_k A) \otimes (I_k B))\rangle \\ &= \langle \theta, \Omega_{k+1}(I_k \widehat{P^a K} \otimes I_k \widehat{P^b L})\rangle. \end{aligned}$$

The conclusion follows when H and J are of the form $P^a K$ and $P^b L$. This is sufficient as, by Proposition III.6.3, the elements of this form span the space $\mathcal{H}_k^{(\mathbb{N})}$. □

12.2. **Tensors with trivial coboundary first step extension.** We aim at translating the result of Proposition 10.14 in the spectral representation obtained through the spectral transform. To this aim, we now describe the inverse image, under the first extension map, of the space of trivial coboundary tensors of Definition 10.7 and Definition 10.8.

We first define a notational convention which complements the notation of Subsection 8.2. If V and W are vector spaces, the elements of the tensor product $V^2 \otimes W^2$ will be written as matrices $u = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}$ whose coefficients are elements of $V \otimes W$. The natural bilinear map $V^2 \times W^2 \to V^2 \otimes W^2$ will be defined by

$$\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \otimes \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} v_0 \otimes w_0 & v_0 \otimes w_1 \\ v_1 \otimes w_0 & v_1 \otimes w_1 \end{pmatrix}, \quad v_0, v_1 \in V, \quad w_0, w_1 \in V.$$

With this matrix convention, we can describe the action of linear maps as follows. Assume V' and W' are other vector spaces and $\chi : V^2 \to (V')^2$ and $\psi : W^2 \to (W')^2$ are linear maps. We may write them as matrices

$$\chi = \begin{pmatrix} \chi_{00} & \chi_{01} \\ \chi_{10} & \chi_{11} \end{pmatrix} \text{ and } \psi = \begin{pmatrix} \psi_{00} & \psi_{01} \\ \psi_{10} & \psi_{11} \end{pmatrix}$$

whose coefficients are respectiely linear maps $V \to V'$ and $W \to W'$. Then, the associated linear maps $u \mapsto \chi u, V^2 \otimes W^2 \to (V')^2 \otimes W^2$ and $u \mapsto u\psi, V^2 \otimes W^2 \to V^2 \otimes (W')^2$ are given by the following matrix multiplications: for u in $V^2 \otimes W^2$, we have

$$\chi u = \begin{pmatrix} \chi_{00} & \chi_{01} \\ \chi_{10} & \chi_{11} \end{pmatrix} \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix} \text{ and } u\psi = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix} \begin{pmatrix} \psi_{00} & \psi_{10} \\ \psi_{01} & \psi_{11} \end{pmatrix}.$$

The reader should beware that, for this computation rule to hold, the antidiagonal coefficients of the latter matrix have to be exchanged.

Coming back to spaces of pseudofunctions, if Γ is bipartite, we extend the definition of the twist operator of Subsection 10.2 as follows. For $k \geq 0$ and $H(t) = \begin{pmatrix} H_{00}(t) & H_{01}(t) \\ H_{10}(t) & H_{11}(t) \end{pmatrix}$ an element of $\otimes^2 \mathcal{H}_k^2[t]$, we will define the twist $\widetilde{H}$ of H as the element

$$\widetilde{H}(t) = \begin{pmatrix} \widetilde{H}_{00}(t) & -\widetilde{H}_{01}(t) \\ -\widetilde{H}_{10}(t) & \widetilde{H}_{11}(t) \end{pmatrix}.$$

This definition allows to get

Lemma 12.7. *Assume Γ is bipartite. Let $k \geq 0$ and H be in $\otimes^2 \mathcal{H}_k^2[t]$. Then we have $I_k(\widetilde{H})I_k = \widetilde{I_k H I_k}$.*

Proof. This is a direct consequence the definition of the twist operator in Subsection 10.2 as well as of Lemma 10.3 and Lemma 12.4. □

Now we introduce our candidates for playing the roles of trivial coboundary tensors in $\otimes^2\mathcal{H}_k^2[t]$.

Definition 12.8. Let $k \geq 0$. We define the space of trivial spectral obstructions in $\otimes^2\mathcal{H}_k^2[t]$ as the subspace Σ_k^0 spanned by the polynomial tensors

$$\begin{aligned}
J \otimes K - K \otimes J & \qquad J, K \in \mathcal{H}_k^2[t] \\
\mathfrak{R}_t H - H\mathfrak{R}_t & \qquad H \in \otimes^2\mathcal{H}_k^2[t] \\
\mathfrak{S}_t H - H\mathfrak{S}_t & \qquad H \in \otimes^2\mathcal{H}_k^2[t] \\
H^{\vee>\vee} - H^{>}\begin{pmatrix} 0 & q \\ -1 & (q+1)t \end{pmatrix} & \qquad H \in (\mathcal{H}_k^2 \otimes \mathcal{H}_{k-1}^2)[t]
\end{aligned}$$

and, if Γ is bipartite,

$$\widetilde{H} - H, \quad H \in \otimes^2\mathcal{H}_k[t].$$

In this Definition, we have used the notation of Subsection III.6.1 for the operators $\mathfrak{R}_t$ and $\mathfrak{S}_t$.

Proposition 12.9. *Let $k \geq 1$ and $H(t)$ be in $\otimes^2\mathcal{H}_k^2[t]$. Then the element $I_kH(t)I_k$ of $\otimes^2\mathcal{H}_{k+1}[t]$ belongs to Θ_{k+1}^0 if and only if $H(t)$ belongs to Σ_k^0.*

The spaces of trivial coboundary polynomial tensors Θ_k°, $k \geq 0$, were defined in Definition 10.7 and Definition 10.8.

In the proof, we will need to compute explicitely the values of the double of the first step extension operator on the tensors that appear in Definition 12.8. Note that, when k is even, the operator $\mathfrak{S}_t$ of Subsection III.6.1 does not depend on t. In the formulas below, we simply denote it by $\mathfrak{S}$. In the same way, when k is odd, the operator $\mathfrak{R}_t$ is denoted by $\mathfrak{R}$.

Lemma 12.10. *Let $k \geq 1$. The action of the double first step extension operator on the generators of the space of trivial spectral obstructions in $\otimes^2\mathcal{H}_k^2[t]$ may be computed by using the following formulas.*

If k is even, for H in $\otimes^2\mathcal{H}_k^2$, we have

$$I_k(\mathfrak{S}H - H\mathfrak{S})I_k = {}^{\vee}(I_kHI_k) - (I_kHI_k)^{\vee}.$$

For $H(t) = \begin{pmatrix} H_{00}(t) & H_{01}(t) \\ H_{10}(t) & H_{11}(t) \end{pmatrix}$ *in* $\otimes^2 \mathcal{H}_k^2[t]$ *and* $J(t) = \begin{pmatrix} H_{00}(t) & {}^\vee H_{01}(t) \\ H_{10}(t)^\vee & H_{11}(t) \end{pmatrix}$, *we have*

$$\begin{aligned} I_k(\mathfrak{R}_t J(t) - J(t)\mathfrak{R}_t)I_k = q\,{}^>(H_{00}(t)^\vee - {}^\vee H_{00}(t))^> \\ +q(q+1)t\,{}^>H_{10}(t)^> + q(q-1)\,{}^{\vee>}H_{10}(t)^> - q\,{}^{\vee>\vee}H_{10}(t)^> - q\,{}^{\vee>}H_{10}(t)^{\vee>} \\ -q(q+1)t\,{}^>H_{01}(t)^> - q(q-1)\,{}^>H_{01}(t)^{>\vee} + q\,{}^>H_{01}(t)^{\vee>\vee} + q\,{}^{>\vee}H_{01}(t)^{>\vee} \\ + (q+1)t({}^>H_{11}(t)^{>\vee} - {}^{\vee>}H_{11}(t)^>) - {}^{\vee>}({}^\vee H_{11}(t) - H_{11}(t)^\vee)^{>\vee}. \end{aligned}$$

For $H(t)$ *in* $\mathcal{H}_{k-1}[t]$, *we have*

$$\begin{aligned} I_k \begin{pmatrix} H_0(t)^{\vee>\vee} \\ H_1(t)^{\vee>\vee} \end{pmatrix} - I_k \begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix} \begin{pmatrix} H_0(t)^> \\ H_1(t)^> \end{pmatrix} = H_1(t)^{\vee>\vee>\vee} \\ - (q-1)H_1(t)^{>>} + H_1(t)^{>\vee>} - (q+1)tH_1(t)^{>>\vee}. \end{aligned}$$

If k *is odd, for* H *in* $\otimes^2\mathcal{H}_k^2$, *we have*

$$I_k(\mathfrak{R}H - H\mathfrak{R})I_k = {}^\vee(I_k H I_k) - (I_k H I_k)^\vee.$$

For $H(t) = \begin{pmatrix} H_{00}(t) & H_{01}(t) \\ H_{10}(t) & H_{11}(t) \end{pmatrix}$ *in* $\otimes^2 \mathcal{H}_k^2[t]$, *we have*

$$\begin{aligned} I_k(\mathfrak{S}_t J(t) - J(t)\mathfrak{S}_t)I_k = -\,{}^{\vee>}H_{00}(t)^{\vee>\vee} + {}^{\vee>\vee}H_{00}(t)^{>\vee} \\ +(q-1)\,{}^{\vee>}H_{00}(t)^{\vee>} - (q-1)\,{}^{>\vee}H_{00}(t)^{>\vee} + (q+1)t({}^{>\vee}H_{00}(t)^{\vee>\vee} - {}^{\vee>\vee}H_{00}(t)^{\vee>}) \\ + {}^{>\vee}H_{10}(t)^{\vee>\vee} - (q-1)\,{}^{>\vee}H_{10}(t)^{\vee>} + {}^>H_{10}(t)^{>\vee} - (q+1)t\,{}^>H_{10}(t)^{\vee>} \\ -\,{}^{\vee>\vee}H_{01}(t)^{\vee>} + (q-1)\,{}^{>\vee}H_{01}(t)^{\vee>} - {}^{\vee>}H_{01}(t)^> + (q+1)t\,{}^{>\vee}H_{01}(t)^{\vee>} \\ + {}^>({}^\vee H_{11}(t) - H_{11}(t)^\vee)^> \end{aligned}$$

For $H(t)$ *in* $\mathcal{H}_{k-1}[t]$, *we have*

$$\begin{aligned} I_k \begin{pmatrix} H_0(t)^{\vee>\vee} \\ H_1(t)^{\vee>\vee} \end{pmatrix} - I_k \begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix} \begin{pmatrix} H_0(t)^> \\ H_1(t)^> \end{pmatrix} = H_1(t)^{\vee>\vee>} \\ + H_1(t)^{>\vee>\vee} - (q-1)H_1(t)^{>\vee>} - (q+1)tH_1(t)^{>>}. \end{aligned}$$

Proof. These are straightforward computations. $\square$

In case $k \geq 2$ is even, Lemma 7.5 and Lemma 7.8 imply that the first step extension I_k maps $\mathcal{H}_k$ onto $\mathcal{H}_{k+1}$. This fact will make the proof of Proposition 12.9 easier, so that we start with this case.

Proof of Proposition 12.9 when k is even. By Lemma 7.5 and Lemma 7.8, we have $I_k\mathcal{H}_k = \mathcal{H}_{k+1}$. By comparing the formulas in Definition 10.8 and the ones in Lemma 12.10 (and by using Lemma 12.7 in case Γ is bipartite), we get $I_k\Sigma_k^0 I_k = \Theta_{k+1}^0$. By Lemma 8.4 and Lemma 12.3, the null space of the map $H(t) \mapsto I_k H(t), \mathcal{H}_k^2[t] \to \mathcal{H}_{k+1}[t]$ is the space of polynomial of the form

$$\begin{pmatrix} G(t)^{\vee>\vee} \\ -qG(t) \end{pmatrix} = \begin{pmatrix} G(t)^{\vee>\vee} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ q & (q+1)t \end{pmatrix} \begin{pmatrix} G(t)^{>} \\ 0 \end{pmatrix}$$

where $G(t)$ is in $\mathcal{H}_{k-1}[t]$. Therefore, in view of Definition 12.8, the null space of the map $H(t) \mapsto I_k H(t) I_k, \otimes^2\mathcal{H}_k^2[t] \to \otimes^2\mathcal{H}_{k+1}[t]$ is contained in Σ_k^0. The conclusion follows. □

In case k is odd, we will need to give a set of generators for the space $(I_k \otimes^2 \mathcal{H}_k^2[t] I_k) \cap \Theta_{k+1}^0$. This will use

Lemma 12.11. *Let $k \geq -1$, H be in $\mathcal{H}_k$ and t be in $\mathbb{R}$, $t^2 \neq 1$. Assume that we have*

$$\begin{aligned} H^{\vee>\vee>} + H^{>\vee>\vee} &= (q+1)tH^{>>} + (q-1)H^{>\vee>} && \text{if } k \text{ is even} \\ &= (q+1)tH^{>>} + (q-1)H^{\vee>>} && \text{if } k \text{ is odd.} \end{aligned}$$

Then $H = 0$.

Proof. We prove the statement by induction on k.

For $k = -1$, as the $\vee$ operator is -1 on 0-pseudofunctions (see Subsection III.2.2), the assumption reads as $tH + H^{\vee} = 0$, hence $H = 0$ since $t^2 \neq 1$.

Assume now $k \geq 0$ and the Lemma holds for $k-1$.

If k is even, Lemma III.2.8 says that we may find J in $\mathcal{H}_k$ with

$$(q+1)tH^{>} + (q-1)H^{>\vee} - H^{\vee>\vee} = J^{\vee>} \text{ and } H^{>\vee} = J^{>}.$$

Applying again Lemma III.2.8 to the latter, we get K in $\mathcal{H}_{k-1}$ with

$$J = K^{\vee>} \text{ and } H = K^{>}.$$

The assumption now reads as

$$K^{>\vee>\vee} + K^{\vee>\vee>} = (q+1)tK^{>>} + (q-1)K^{\vee>>} = 0,$$

and the conclusion follows from the induction assumption.

The proof is analogue in the odd case. □

We can now give a new version of Definition 10.7.

Lemma 12.12. *Let $k \geq 2$ be an even integer. Then, the space*

$$(I_{k-1}(\otimes^2\mathcal{H}_{k-1}^2[t])I_{k-1}) \cap \Theta_k^0$$

is spanned by the following polynomial tensors:

$$\begin{aligned} J \otimes K - K \otimes J \qquad & J, K \in I_{k-1}\mathcal{H}_{k-1}^2[t] \\ H \qquad & H \in (\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-})[t] \cap I_{k-1}(\otimes^2\mathcal{H}_{k-1}^2[t])I_{k-1} \\ {}^{>}H^{>} \qquad & H \in (\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,-})[t] \end{aligned}$$

as well as

$$(q+1)t\,{}^{>}H^{>} + (q-1)\,{}^{>}H^{\vee>} - 2\,{}^{>}H^{\vee>\vee}$$
$$H \in (\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,+} \oplus \mathcal{H}_{k-1,-} \otimes \mathcal{H}_{k-1,-})[t]$$

and

$$(q+1)tH^{>>} + (q-1)H^{>\vee>} - H^{\vee>\vee>} - H^{>\vee>\vee}$$
$$H \in ((I_{k-1}\mathcal{H}^2_{k-1}) \otimes \mathcal{H}_{k-2})[t]$$

and, if Γ *is bipartite,*

$$\widetilde{H} - H, \quad H \in I_{k-1}(\otimes^2\mathcal{H}^2_{k-1}[t])I_{k-1}.$$

Proof. The main difficulty is to show that the component of an element of $(I_{k-1}\otimes^2\mathcal{H}^2_{k-1}[t]I_{k-1})\cap\Theta^0_k$ corresponding to the fifth case of Definition 10.7 can be assumed to belong to $(I_{k-1}\mathcal{H}^2_{k-1} \otimes \mathcal{H}_{k-2})[t]$.

We first claim that, in this fifth case, in full generality, it suffices to assume that the element $H(t)$ belongs to $(\mathcal{H}_{k,+} \otimes \mathcal{H}_{k-2,+} \oplus \mathcal{H}_{k,-} \otimes \mathcal{H}_{k-2,-})[t]$ (a fact that was already implicitely used in Subsection 11.2). Indeed, fix $H(t)$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-2}[t]$ and set

$$J(t) = (q+1)tH(t)^{>>} + (q-1)H(t)^{>\vee>} - H(t)^{\vee>\vee>} - H(t)^{>\vee>\vee}.$$

When $H(t)$ belongs to $(\mathcal{H}_{k,+} \otimes \mathcal{H}_{k-2,-} \oplus \mathcal{H}_{k,-} \otimes \mathcal{H}_{k-2,+})[t]$, we have $H(t)^{\vee} + {}^{\vee}H(t) = (q-1)H(t)$, hence

$$J(t) = (q+1)tH(t)^{>>} + {}^{\vee}H(t)^{>\vee>} - H(t)^{>\vee>\vee}$$

and the latter belongs to $(\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-} \oplus \mathcal{H}_{k,-} \otimes \mathcal{H}_{k,+})[t]$ so that he first two cases of Definition 10.7 already warrant that it is a trivial coboundary polynomial tensor. Notice in particular, that, if $H(t)$ is in $(\mathcal{H}_{k,+} \otimes \mathcal{H}_{k-2,+} \oplus \mathcal{H}_{k,-} \otimes \mathcal{H}_{k-2,-})[t]$, then

$$J(t) = (q+1)tH(t)^{>>} + (q-1)H(t)^{>\vee>} - {}^{\vee}H(t)^{>\vee>} - H(t)^{>\vee>\vee},$$

so that $J(t)$ belongs to $(\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,+} \oplus \mathcal{H}_{k,-} \otimes \mathcal{H}_{k,-})[t]$.

In the same way, if Γ is bipartite, by Lemma 10.4, for $H(t)$ in $\mathcal{H}_k \otimes \mathcal{H}_{k-2}[t]$, we have

$$\widetilde{J}(t) = -(q+1)t\widetilde{H}(t)^{>>} - (q-1)\widetilde{H}(t)^{>\vee>} + \widetilde{H}(t)^{\vee>\vee>} + \widetilde{H}(t)^{>\vee>\vee}.$$

Thus, we can also assume that $\widetilde{H}(t) = -H(t)$ and hence $\widetilde{J}(t) = J(t)$.

Now, let $G(t)$ be in $\otimes^2\mathcal{H}^2_{k-1}[t]$ and assume that $I_{k-1}G(t)I_{k-1}$ is in Θ^0_k. In view of the discussion above, we can assume that the component of $I_{k-1}G(t)I_{k-1}$ corresponding to the fifth case of Definition 10.7 is of the form

$$J(t) = (q+1)tH(t)^{>>} + (q-1)H(t)^{>\vee>} - H(t)^{\vee>\vee>} - H(t)^{>\vee>\vee},$$

for some H in $(\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,+} \oplus \mathcal{H}_{k,-} \otimes \mathcal{H}_{k,-})[t]$, with $\widetilde{H}(t) = -H(t)$ if Γ is bipartite. In particular, we have ${}^{\vee}J = J^{\vee}$ and $\widetilde{J} = \widetilde{J}$. Now, we notice that the space $I_{k-1}(\otimes^2\mathcal{H}^2_{k-1}[t])I_{k-1} \subset \otimes^2\mathcal{H}_k[t]$ is invariant under the symmetrization of tensors, the map $F(t) \mapsto {}^{\vee}F(t)^{\vee}$ and the twist operator if Γ is bipartite (the latter by Lemma 12.7). Moreover, these maps commute to each other. Therefore, in view of Definition 10.7, the polynomial tensor $J(t)$ may be written as the sum of a skew-symmetric polynomial tensor and an element of $I_{k-1}(\otimes^2\mathcal{H}^2_{k-1}[t])I_{k-1}$. In other words, the symmetrization of $J(t)$ belongs to $I_{k-1}(\otimes^2\mathcal{H}^2_{k-1}[t])I_{k-1}$. As $J(t)$ belongs to $(\mathcal{H}_k \otimes I_{k-1}\mathcal{H}^2_{k-1})[t]$, this tells us that $J(t)$ itself belongs to $I_{k-1}(\otimes^2\mathcal{H}^2_{k-1}[t])I_{k-1}$. By applying Lemma 8.4 and Lemma 12.11, we obtain that $H(t)$ belongs to $(I_{k-1}\mathcal{H}^2_{k-1} \otimes \mathcal{H}_{k-2})[t]$.

Therefore, we have shown that the space $(I_{k-1}(\otimes^2\mathcal{H}^2_{k-1}[t])I_{k-1}) \cap \Theta^0_k$ is the intersection of $I_{k-1}(\otimes^2\mathcal{H}^2_{k-1}[t])I_{k-1}$ with the subspace of $\otimes^2\mathcal{H}_k[t]$ spanned by the polynomial tensors

$$\begin{array}{rl} J \otimes K - K \otimes J \qquad & J, K \in \mathcal{H}_k[t] \\ H \qquad & H \in (\mathcal{H}_{k,+} \otimes \mathcal{H}_{k,-})[t] \\ {}^{>}H^{>} \qquad & H \in (\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,-})[t] \end{array}$$

as well as

$$(q+1)t\,{}^{>}H^{>} + (q-1)\,{}^{>}H^{\vee>} - 2\,{}^{>}H^{\vee>\vee} \qquad H \in (\mathcal{H}_{k-1,+} \otimes \mathcal{H}_{k-1,+} \oplus \mathcal{H}_{k-1,-} \otimes \mathcal{H}_{k-1,-})[t]$$

and

$$(q+1)tH^{>>} + (q-1)H^{>\vee>} - H^{\vee>\vee>} - H^{>\vee>\vee} \qquad H \in (I_{k-1}\mathcal{H}^2_{k-1} \otimes \mathcal{H}_{k-2})[t]$$

and, if Γ is bipartite,

$$\widetilde{H} - H, \quad H \in I_{k-1} \otimes^2 \mathcal{H}^2_{k-1}[t]I_{k-1}.$$

The conclusion follows as $I_{k-1}(\otimes^2\mathcal{H}^2_{k-1}[t])I_{k-1}$ is stable under the symmetrization operator, the map $F(t) \mapsto {}^{\vee}F(t)^{\vee}$ and the twist operator if Γ is bipartite. □

Proof of Proposition 12.9 when k is odd. This is analogue to the proof in case k is even, by using Lemma 12.12 instead of Definition 10.7. □

12.3. **Finiteness of genuine spectral obstructions.** We now use Proposition 12.9 to translate Proposition 10.14 into a result that only leaves in the spectral world.

Let p be a Γ-invariant symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$. For $k \geq 0$, we let $\widehat{p}_k$ be the symmetric bilinear form on $\mathcal{H}_k^2[t]$ such that, for every H, J in $\mathcal{H}_k^{(\mathbb{N})}$, one has

$$\widehat{p}_k(\widehat{H}, \widehat{J}) = p(E_k H, E_k J),$$

where E_k is the polyextension map of Definition III.2.11 and, as in Subsection 5.2, we still write p for the symmetric bilinear form on $\mathcal{H}_\infty$ associated with p.

The existence and uniqueness of $\widehat{p}_k$ are warranted by Proposition III.6.3.

Now, we will associate to $\widehat{p}_k$ a linear functional on $\otimes^2 \mathcal{H}_k^2[t]$ by the following construction which may be seen as an abstract form of the spectral theorem.

By construction in Subsection 5.2 (and by using Lemma I.9.11), for H, J in $\mathcal{H}_\infty$, we have

$$p(RH, J) = p(H, RJ) \text{ and } p(SH, J) = p(H, SJ) \tag{12.1}$$

where R and S are the natural operators on ∞-pseudofunctions defined in Subsection III.2.5. This gives $p(PH, J) = p(H, PJ)$, hence, by Proposition III.6.3, for $H(t), J(t)$ in $\mathcal{H}_k^2[t]$,

$$\widehat{p}_k(tH(t), J(t)) = \widehat{p}_k(H(t), tJ(t)). \tag{12.2}$$

Let V, W be vector spaces and r, s be inderminates. The product map

$$V[r] \times W[s] \to (V \otimes W)[r, s], (v(r), w(s)) \mapsto v(r) \otimes w(s)$$

defines an isomorphism between $V[r] \otimes W[s]$ and $(V \otimes W)[r, s]$. Under this isomorphism, the subspace of $V[r] \otimes W[s]$ spanned by the tensors of the form

$$(rv(r)) \otimes w(s) - v(r) \otimes (sw(s)), \quad v(r) \in V[r], \quad w(s) \in W[s],$$

may be identified with the space $(r - s)(V \otimes W)[r, s]$. Thus, if $\varphi : V[r] \otimes W[s] \to \mathbb{R}$ is a bilinear form such that, for every $v(r)$ in $V[r]$ and $w(s)$ in $W[s]$, we have

$$\varphi(rv(r), w(s)) = \varphi(v(r), sw(s)),$$

we may consider φ as a linear functional on $(V \otimes W)[r, s]$ which vanishes on the space $(r - s)(V \otimes W)[r, s]$. Now, if U a vector space, we have a natural map $\delta : U[r, s] \to U[t]$ defined by letting r and s take the value t, that is

$$\delta u(t) = u(t, t), \quad u(r, s) \in U[r, s].$$

Elementary algebraic considerations show that the null space of δ is exactly the space $(r-s)U[r,s]$. Therefore, if φ is as above, we may consider φ as a linear functional on $(V \otimes W)[t]$.

By applying this construction to the bilinear form $\widehat{p}_k$, since (12.2) holds, we will now consider $\widehat{p}_k$ as a linear functional on $\otimes^2 \mathcal{H}_k^2[t]$. This linear functional may also be computed as follows:

Lemma 12.13. *Let p be a Γ-invariant symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$ and θ be the associated (ι, T)-invariant distribution on $\Gamma \backslash \mathscr{S}$. For $k \geq 0$ and H in $\mathcal{H}_k^2[t]$, we have*

$$\widehat{p}_k(H) = \langle \theta, \Omega_{k+1}(I_k H I_k) \rangle.$$

Proof. This is a direct consequence of the definition of $\widehat{p}_k$ and of Proposition 12.5. □

We shall now study the range of the map $p \mapsto \widehat{p}_k$.

Definition 12.14. Let $k \geq 1$. We define the space of spectral obstructions as the subspace Σ_k of all H in $\otimes^2 \mathcal{H}_k^2[t]$ such that, for any Γ-invariant symmetric bilinear form p on $\overline{\mathcal{D}}(\partial X)$, one has $\widehat{p}_k(H) = 0$.

We have an alternative definition of Σ_k by means of the trivial coboundary tensors of Definition 10.7 and Definition 10.8.

Lemma 12.15. *Let $k \geq 1$. The space $\Sigma_k \subset \otimes^2 \mathcal{H}_k^2[t]$ is the inverse image of the space $\Theta_{k+1} \subset \otimes^2 \mathcal{H}_{k+1}[t]$ under the map $H \mapsto I_k H I_k$. In particular, we have $\Sigma_k^0 \subset \Sigma_k$.*

In other words, all the polynomial tensors that appear in Definition 12.8 are killed by any linear functional $\widehat{p}_k$ as above.

Proof. The equivalence between the two definitions of Σ_k is a direct consequence of Proposition 2.1, Corollary 2.7 and Lemma 12.13. The inclusion $\Sigma_k^0 \subset \Sigma_k$ then follows by Lemma 10.9 and Proposition 12.9. □

We can use Proposition 10.14 to give a partial converse to Lemma 12.15.

Theorem 12.16. *For any $k \geq 1$, the space Σ_k^0 has finite codimension in Σ_k. More precisely, there exists an integer $n \geq 0$ such that, for any $k \geq 1$, we have $\Sigma_k \subset \Sigma_k^0 + (\otimes^2 \mathcal{H}_k^2)_{k+n}[t]$, that is, every spectral obstruction may be written as the sum of a trivial spectral obstruction and a polynomial tensor of degree $\leq k+n$.*

As for Proposition 12.9, the proof of this statement will be easier in the even case. In the odd case, we shall need

Lemma 12.17. *Let $k \geq -1$ and $(F_j)_{j\geq 0}$ be a finitely supported sequence in $\mathcal{H}_k$. Assume that, for every $j \geq 1$, we have*

$$F_j^{\vee >} = F_{j-1}^{>\vee} \qquad \text{if } k \text{ is even}$$
$$F_j^{>\vee} = F_{j-1}^{\vee >} \qquad \text{if } k \text{ is odd.}$$

Then $F_j = 0$ for any $j \geq 0$.

Proof. As usual, we prove this statement by induction on k.

If $k = -1$, the assumptions says that, for any $j \geq 1$, we have $F_j + F_{j-1}^{\vee} = 0$. The conclusion follows as the sequence $(F_j)_{j\geq 0}$ is finitely supported.

Suppose now $k \geq 0$ and the result is true for $k-1$.

Assume k is even. From the assumption and Lemma III.2.8, we know that there exists a sequence $(G_j)_{j\geq 0}$ in $\mathcal{H}_{k-1}$ such that, for any $j \geq 1$, we have

$$F_j^{\vee} = G_{j-1}^{\vee >} \text{ and } F_{j-1} = G_{j-1}^{>}.$$

This gives $G_j^{>\vee} = G_{j-1}^{\vee >}$. Besides, as $(F_j)_{j\geq 0}$ is finitely supported, so is $(G_j)_{j\geq 0}$. The conclusion follows by induction.

The proof in the odd case is analogue. □

Using the latter, we can show that the solutions to certain cohomological equations lie in smaller subspaces.

Corollary 12.18. *Let $k \geq -1$ be odd and $(H_j)_{j\geq 1}$ be a finitely supported sequence in $I_k(\otimes^2 \mathcal{H}_k^2)I_k$. Assume that there exist finitely supported sequences $(F_j)_{j\geq 0}$ in $\mathcal{H}_{k+1} \otimes \mathcal{H}_k$ and $(G_j)_{j\geq 0}$ in $\mathcal{H}_k \otimes \mathcal{H}_{k+1}$ such that, for $j \geq 1$, one has*

$$H_j = F_j^{>\vee} - F_{j-1}^{\vee >} + {}^{\vee >}G_j - {}^{>\vee}G_{j-1}.$$

Then, for all $j \geq 0$, we have

$$F_j \in (I_k \mathcal{H}_k^2) \otimes \mathcal{H}_k \text{ and } G_j \in \mathcal{H}_k \otimes (I_k \mathcal{H}_k^2).$$

Proof. For $j \geq 1$, we have

$$F_j^{>\vee} - F_{j-1}^{\vee >} = H_j - {}^{\vee >}G_j + {}^{>\vee}G_{j-1} \in (I_k \mathcal{H}_k^2) \otimes \mathcal{H}_{k+1}.$$

By Lemma 8.4 and Lemma 12.17, F_j belongs to $(I_k \mathcal{H}_k^2) \otimes \mathcal{H}_k$ for all $j \geq 0$. The proof of the other case is symmetric. □

Proof of Theorem 12.16. Let n be as in Proposition 10.14. Take H in Σ_k. By Lemma 12.15, $I_k H I_k$ belongs to Θ_{k+1} and Proposition 10.14 says that there exists J in Θ_{k+1}^0 such that $H - J$ has degree $\leq k+n+1$.

If k is even, by Lemma 7.5 and Lemma 7.8, I_k maps $\mathcal{H}_k$ onto $\mathcal{H}_{k+1}$ hence J belongs to $I_k(\otimes^2 \mathcal{H}_k^2[t])I_k$ and Proposition 12.9 implies that we can find K in Σ_k^0 such that $H - K$ has degree $\leq k+n+1$.

If k is odd, we claim that we can actually choose J to belong to $I_k(\otimes^2\mathcal{H}_k^2[t])I_k$. Indeed, due to Corollary 12.18, all the constructions in the proof of Proposition 10.14, when applied to a tensor H in $\Theta_{k+1}^0 \cap I_k(\otimes^2\mathcal{H}_k^2[t])I_k$, provide trivial coboundary tensors also living in $I_k(\otimes^2\mathcal{H}_k^2[t])I_k$ (see in particular Subsection 11.1 and Subsection 11.2). We then conclude as in the even case by applying Proposition 12.9. $\square$

12.4. **The spectral projective limit.** In Section I.4, we have introduced the space $\mathcal{F}_k$ of Γ-invariant k-quadratic fields, $k \geq 2$. This is a finite-dimensional space and the reduction map $p \mapsto p^-$ sends $\mathcal{F}_{k+1}$ onto $\mathcal{F}_k$. The projective limit of the system $(\mathcal{F}_k)_{k\geq 2}$ is naturally identified with the space $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ of Γ-invariant symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$. We will now give an alternate construction of a projective system whose limit may be identified with $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$. The advantage of this new system is that it will keep track of the spectral theory of non-negative elements in $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$. The drawback is that it will be constructed through infinite-dimensional spaces.

We first note that we have a natural embedding $\otimes^2\mathcal{H}_k^2[t]/\Sigma_k \hookrightarrow \otimes^2\mathcal{H}_{k+1}^2[t]/\Sigma_{k+1}$.

Lemma 12.19. *Let $k \geq 1$ and H be in $\otimes^2\mathcal{H}_{k+1}^2[t]$. Then, if k is even, H is in Σ_k if and only if ${}^{\vee>}H^{>\vee}$ is in Σ_{k+1}. If k is odd, H is in Σ_k if and only if ${}^{>\vee}H^{\vee>}$ is in Σ_{k+1}.*

Proof. Assume k is even. Then from Definition 10.1, Lemma 12.2 and Lemma 12.15, we get

$$\begin{aligned} H \in \Sigma_k &\Leftrightarrow \Omega_{k+1}(I_k H I_k) \text{ is a coboundary} \\ &\Leftrightarrow \Omega_{k+2}({}^{>}I_k H I_k{}^{>}) \text{ is a coboundary} \\ &\Leftrightarrow \Omega_{k+2}(I_{k+1}\,{}^{\vee>}H^{>\vee} I_{k+1}) \text{ is a coboundary} \\ &\Leftrightarrow {}^{\vee>}H^{>\vee} \in \Sigma_{k+1}. \end{aligned}$$

The proof in the odd case is analogue. $\square$

Thus, for any $k \geq 1$, we have defined an injective map $\otimes^2\mathcal{H}_k^2[t]/\Sigma_k \hookrightarrow \otimes^2\mathcal{H}_{k+1}^2[t]/\Sigma_{k+1}$. We denote by

$$\Pi_k : (\otimes^2\mathcal{H}_{k+1}^2[t]/\Sigma_{k+1})^* \to (\otimes^2\mathcal{H}_k^2[t]/\Sigma_k)^*$$

the dual surjective map.

Proposition 12.20. *The maps*

$$p \mapsto \widehat{p}_k : \mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma \to (\otimes^2\mathcal{H}_k^2[t]/\Sigma_k)^*, \quad k \geq 1,$$

define a linear isomorphism between $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ and the projective limit of the projective system $((\otimes^2\mathcal{H}_k^2[t]/\Sigma_k)^*, \Pi_k)_{k\geq 1}$.

The main ingredient of the proof is the following injectivity property:

Lemma 12.21. *Let p be in $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ and assume that the associated symmetric bilinear form on $\mathcal{H}_\infty$ is zero. Then $p = 0$.*

Proof. First assume that $\Gamma_{xy} = \{e\}$ for some xy in X_1. In this case, if f, g are in $\overline{\mathcal{D}}(\partial X)$, we can find Γ-invariant ∞-pseudofunctions F, G with $F_{xy} = f$, $G_{xy} = g$ and $F_{ab} = G_{ab} = 0$ for any ab in $X_1 \smallsetminus \Gamma(xy)$. Then, by definition, we have $p(f, g) = p(F, G) = 0$, hence $p = 0$.

If all edges admit non trivial stabilizers, then reasoning as above shows that, for any xy in X_1, for any Γ_{xy}-invariant functions f, g in $\overline{\mathcal{D}}(\partial X)$, we have $p(f, g) = 0$. For xy in X_1, we let U_{xy} be as usual

$$U_{xy} = \{\xi \in \partial X | y \in [x\xi)\}.$$

Then, as p is Γ-invariant and U_{xy} is Γ_{xy}-invariant, for any f in $\overline{\mathcal{D}}(\partial X)$, we have

$$|\Gamma_{xy}| p(\mathbf{1}_{U_{xy}}, f) = \sum_{\gamma \in \Gamma_{xy}} p(\mathbf{1}_{U_{xy}}, \gamma f) = p\left(\mathbf{1}_{U_{xy}}, \sum_{\gamma \in \Gamma_{xy}} \gamma f\right) = 0.$$

Since the functions $\mathbf{1}_{U_{xy}}$ span $\overline{\mathcal{D}}(\partial X)$ when xy runs in X_1, we get $p = 0$ as required. $\square$

Proof of Proposition 12.20. Let U be the quotient by the cohomology equivalence relation of the space of all Hölder continuous functions on $\Gamma \backslash \mathscr{S}$ which are cohomologuous to a smooth ι-invariant function. Then, in view of Remark 2.8, we may identify $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ with the dual space of U.

By Lemma 12.15, for $k \geq 1$, the map $H \mapsto \Omega_{k+1}(I_k H I_k)$ induces an embedding of $\otimes^2 \mathcal{H}_k^2[t]/\Sigma_k$ into U, whose range we denote by U_k. Then, Lemma 12.2 warrants that we have $U_k \subset U_{k+1}$ and Proposition 5.4 and Lemma 12.21 warrant that we have $U = \bigcup_{k \geq 1} U_k$. The conclusion follows since, by Lemma 12.13, the restriction to U_k of the distribution associated to some element p of $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ may be identified with $\widehat{p}_k$. $\square$

Thanks to these constructions, Theorem 12.16 yields the

Proof of Corollary 1.1. Fix $k \geq 0$. We claim that the space $\mathcal{Q}(\mathcal{A}\mathcal{H}_k)^{R,S}$ may be identified with the dual space of the space $\otimes^2 \mathcal{H}_k^2[t]/\Sigma_k^\circ$. Indeed, if φ is some symmetric bilinear form on $\mathcal{A}\mathcal{H}_k$ such that the operators R and S are symmetric with respect to φ, we let $\widehat{\varphi}$ be the bilinear form on $\mathcal{H}_k^2[t]$ such that, for every H, J in $\mathcal{H}_k^{(\mathbb{N})}$, one has

$$\widehat{\varphi}(\widehat{H}, \widehat{J}) = \varphi(E_k H, E_k J),$$

where E_k is the polyextension map of Definition III.2.11. As in Subsection 12.3, the existence and uniqueness of $\widehat{\varphi}$ are warranted by Proposition III.6.3.

By Lemma III.6.1 and Proposition III.6.3, multiplication by t is a symmetric endomorphism of $\mathcal{H}_k^2[t]$ with respect to $\widehat{\varphi}$. Therefore, still as in Subsection 12.3, we can consider $\widehat{\varphi}$ as an element of the dual space of $\otimes^2\mathcal{H}_k^2[t]$. It follows from Corollary III.2.14, Proposition III.6.3 and Proposition III.6.5 as well as Definition 12.8 that the map $\varphi \mapsto \widehat{\varphi}$ establishes a linear isomorphism between $\mathcal{Q}(\mathcal{AH}_k)^{R,S}$ and the space of those elements in the dual space of $\otimes^2\mathcal{H}_k^2[t]$ which vanish on the space Σ_k° of trivial spectral obstructions.

Besides, Proposition 12.20 shows that this map sends the image of $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ in $\mathcal{Q}(\mathcal{AH}_k)^{R,S}$ onto the space of those elements in the dual space of $\otimes^2\mathcal{H}_k^2[t]$ which vanish on the space $\Sigma_k \supset \Sigma_k^\circ$ of all spectral obstructions. The conclusion follows as, by Theorem 12.16, the quotient space Σ_k/Σ_k° has finite dimension. $\square$

Appendix A. Harmonic cocycles

The purpose of this Appendix is to explain how the study of nonnegative Γ-invariant symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$ can be considered as the study of wide class of unitary representations of Γ, namely the ones admitting a cyclic harmonic first cohomology class.

A.1. **Geometric cocycles.** We start by recalling the basic definitions of 1-cohomology. We also introduce a geometric version of these definitions. Later, we will show that both versions define the same notion of cohomology.

Let G be a group with a linear representation on a real vector space V. A 1-cocycle of G in V is a map $\sigma : G \to V$ such that, for any g_1, g_2 in G, one has

$$\sigma(g_1g_2) = \sigma(g_1) + g_1\sigma(g_2).$$

This cocycle is said to be a coboundary if there exists some v in V such that, for g in G, one has

$$\sigma(g) = gv - v.$$

The space of 1-cocycles is denoted by $\mathrm{Z}^1(G, V)$ and the one of 1-coboundaries by $\mathrm{B}^1(G, V)$. The latter may be identified with the quotient space V/V^G of V by the space of G-invariant elements in V. Two cocycles are said to be cohomologous if their difference is a coboundary. The quotient space $\mathrm{H}^1(G, V) = \mathrm{Z}^1(G, V)/\mathrm{B}^1(G, V)$ is called the first cohomology group of G in V.

Now, assume G is our tree lattice Γ. In this case, the group $\mathrm{H}^1(\Gamma, V)$ may be computed in a different way. Define a geometric cocycle of Γ in V as a Γ-equivariant map $\sigma : X_1 \to V$ which is skew-symmetric, that is,

$$\sigma(y, x) + \sigma(x, y) = 0, \quad (x, y) \in X_1.$$

If $\varphi : X \to V$ is any map, we denote by $\mathrm{d}\varphi : X_1 \to V$ the map defined by

$$\mathrm{d}\varphi(x, y) = \varphi(y) - \varphi(x), \quad (x, y) \in X_1.$$

We shall say that a geometric cocycle σ is a geometric coboundary if there exists a Γ-equivariant map $\varphi : X \to V$ with $\sigma = \mathrm{d}\varphi$. Write $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$ for the space of geometric cocycles and $\mathrm{B}^1_{\mathrm{geom}}(\Gamma, V)$ for the one of geometric coboundaries. Again, two geometric coycles are said to be cohomologous if their difference is a geometric coboundary.

A.2. **Loops and integration.** We aim at showing that the quotient space $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)/\mathrm{B}^1_{\mathrm{geom}}(\Gamma, V)$ may be identified with the first cohomology group $\mathrm{H}^1(\Gamma, V)$. This identification will rely on an integration procedure of cocycles that we will now introduce.

For x, y in X, we define a path from x to y as a sequence $x_0 = x, x_1, \dots, x_n = y$ in X with $x_k \sim x_{k-1}$, $1 \leq k \leq n$. As in subsection I.2.1, we shall say that this path is geodesic if moreover, we have $x_{k-1} \neq x_{k+1}$, $1 \leq k \leq n-1$. By assumption, there exists a unique geodesic path from x to y.

Lemma A.1. *Let x be in X and $x_0 = x, x_1, \dots, x_n = x$ be a path from x to itself. Then, n is even and, if $n > 0$ and x does not belong to the set $\{x_1, \dots, x_{n-1}\}$, we have $x_1 = x_{n-1}$.*

Proof. We show this statement by induction on n. For $n = 0$, there is nothing to prove. For $n = 1$, there is no path of length 1 from x to itself.

Assume $n \geq 2$ and the statement holds for any $n' < n$. Let $x_0 = x, x_1, \dots, x_n = x$ be a path from x to itself. Then, by uniqueness, the path is not geodesic, that is, there exists $1 \leq k \leq n-1$ with $x_{k-1} = x_{k+1}$. For $0 \leq j \leq n-2$, we set

$$\begin{aligned} y_j &= x_j && \text{if } j \leq k-1 \\ y_j &= x_{j+2} && \text{if } j \geq k. \end{aligned}$$

Then, $y_0, \dots, y_{n-2}$ is a path from x to itself. In particular, the induction assumption implies that n is even.

Now, assume x does not belong to the set $\{x_1, \dots, x_{n-1}\}$. If $n = 2$, we have $x_1 = x_{n-1}$ as required. If $n \geq 4$, then, necessarily, we have $2 \leq k \leq n-2$, $y_1 = x_1$ and $y_{n-3} = x_{n-1}$. As x does not belong to the set $\{y_1, \dots, y_{n-3}\} \subset \{x_1, \dots, x_{n-1}\}$, the induction assumption says that we must have $y_1 = y_{n-3}$, hence $x_1 = x_{n-1}$. The conclusion follows. $\square$

Using this result, we can show that the sum of a skew-symmetric map along a path only depends on the endpoints.

Corollary A.2. *Let V be a real vector space and $\sigma : X_1 \to V$ be a skew-symmetric map. Then, for any x, y in X, the element of V*

$$\sum_{k=1}^{n} \sigma(x_{k-1}, x_k)$$

does not depend on the choice a path $x_0 = x, x_1, \dots, x_n = y$ from x to y.

In the sequel, for $\sigma : X_1 \to V$ a skew-symmetric map and x, y in X, we set

$$\sum_{x}^{y} \sigma = \sum_{k=1}^{n} \sigma(x_{k-1}, x_k)$$

where $x_0 = x, x_1, \dots, x_n = y$ is a path x to y. Note that, for x, y, z in X, we have the chain rule

$$\sum_{x}^{y} \sigma + \sum_{y}^{z} \sigma = \sum_{x}^{z} \sigma. \tag{A.1}$$

Proof of Corollary A.2. Fix x in X. We claim that, given a path $x_0 = x, x_1, \dots, x_{2n} = x$ from x to itself, we have $\sum_{k=1}^{2n} \sigma(x_{k-1}, x_k) = 0$. We show this statement by induction on $n \geq 0$.

If $n = 0$, there is nothing to prove. Assume $n \geq 1$ and the statement holds for all $n' < n$. Let $x_0 = x, x_1, \dots, x_{2n} = x$ be a path from x to itself. If there exits $1 \leq j \leq 2n - 1$ with $x_j = x$, then the sequences $x_0, \dots, x_j$ and $x_j, \dots, x_{2n}$ are paths from x to itself and, by the induction assumption, we have

$$\sum_{k=1}^{2n} \sigma(x_{k-1}, x_k) = \sum_{k=1}^{j} \sigma(x_{k-1}, x_k) + \sum_{k=j+1}^{2n} \sigma(x_{k-1}, x_k) = 0.$$

If there exists no such j, by Lemma A.1, we have $x_1 = x_{2n-1}$, hence, the sequence $x_1, \dots, x_{2n-1}$ is a path from x_1 to itself. By the induction assumption, we get

$$\sum_{k=1}^{2n} \sigma(x_{k-1}, x_k) = \sigma(x, x_1) + \sum_{k=2}^{2n-1} \sigma(x_{k-1}, x_k) + \sigma(x_{2n-1}, x)$$
$$= \sigma(x, x_1) + \sigma(x_1, x) = 0,$$

as σ is skew-symmetric. The claim follows.

Now assume x, y are in X and $u_0 = x, \ldots, u_m = y$ and $v_0 = x, \ldots, v_n = y$ are two paths from x to y. Since $u_0, \ldots, u_n, v_{m-1}, \ldots, v_0$ is a path from x to itself, we get

$$\sum_{k=1}^{m} \sigma(u_{k-1}, u_k) + \sum_{k=1}^{n} \sigma(v_k, v_{k-1}) = 0.$$

As σ is skew-symmetric, this gives

$$\sum_{k=1}^{m} \sigma(u_{k-1}, u_k) = \sum_{k=1}^{n} \sigma(v_{k-1}, v_k)$$

as required. □

A.3. **Geometric representation of cohomology.** We will use the constructions above to build an isomorphism between the quotient space $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)/\mathrm{B}^1_{\mathrm{geom}}(\Gamma, V)$ and the cohomology group $\mathrm{H}^1(\Gamma, V)$. This is an explicit version of [10, Proposition II.13].

Note that, for σ in $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$, we have the following equivariance property of path summation:

$$\sum_{\gamma x}^{\gamma y} \sigma = \gamma \sum_{x}^{y} \sigma, \quad \gamma \in \Gamma, \quad x, y \in X. \tag{A.2}$$

This together with (A.1) will be instrumental in proving

Proposition A.3. *Let V be a real vector space equipped with a linear action of Γ. Let σ be in $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$. For x in X and γ in Γ, set*

$$\sigma_x(\gamma) = \sum_{x}^{\gamma x} \sigma.$$

Then σ_x is a 1-cocycle of Γ in V.

If y is another element of X, the cocycles σ_x and σ_y are cohomologous. If θ is a geometric cocycle that is cohomologous to σ, the coycles σ_x and θ_x are cohomologous.

The linear map $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)/\mathrm{B}^1_{\mathrm{geom}}(\Gamma, V) \to \mathrm{H}^1(\Gamma, V)$ associated with this construction is an isomorphism.

Proof. Fix σ in $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$. For x in X and γ_1, γ_2 in Γ, we have, by (A.1) and (A.2),

$$\sigma_x(\gamma_1\gamma_2) = \sum_x^{\gamma_1 x} \sigma + \sum_{\gamma_1 x}^{\gamma_1\gamma_2 x} \sigma = \sum_x^{\gamma_1 x} \sigma + \gamma_1 \sum_x^{\gamma_2 x} \sigma = \sigma_x(\gamma_1) + \gamma_1 \sigma_x(\gamma_2).$$

Thus, σ_x is a 1-cocycle of Γ in V.

Besides, for x, y in X and γ in Γ, still by (A.1) and (A.2), we have

$$\sigma_x(\gamma) = \sum_x^{y} \sigma + \sum_y^{\gamma y} \sigma + \sum_{\gamma y}^{\gamma x} \sigma = \sum_x^{y} \sigma + \sigma_y(\gamma) - \gamma \sum_x^{y} \sigma,$$

hence $\sigma_x - \sigma_y$ is a coboundary.

Assume that σ is a coboundary in $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$. Then, there exists a Γ-equivariant map $\varphi : X \to V$ with $\sigma = \mathrm{d}\varphi$. For x in X and γ in Γ, we get

$$\sigma_x(\gamma) = \varphi(\gamma x) - \varphi(x) = \gamma\varphi(x) - \varphi(x),$$

hence σ_x is a coboundary in $\mathrm{Z}^1(\Gamma, V)$.

Conversely, fix x in X and suppose σ_x is a coboundary in $\mathrm{Z}^1(\Gamma, V)$. Choose v in V with

$$\sigma_x(\gamma) = \gamma v - v \quad \gamma \in \Gamma.$$

For y in X, we set

$$\varphi(y) = \sum_x^{y} \sigma + v.$$

We claim that $\varphi : X \to V$ is Γ-equivariant. Indeed, by (A.1) and (A.2), for γ in Γ, we have

$$\varphi(\gamma y) = \sum_x^{\gamma x} \sigma + \sum_{\gamma x}^{\gamma y} \sigma + v = \sigma_x(\gamma) + \gamma \sum_x^{y} \sigma + v = \gamma v + \gamma \sum_x^{y} \sigma = \gamma\varphi(y).$$

Again by (A.1), for $y \sim z$ in X, we have $\mathrm{d}\varphi(y, z) = \varphi(z) - \varphi(y) = \sigma(y, z)$, hence σ is a geometric coboundary.

So far, we have shown that the map $\sigma \mapsto \sigma_x$ defines an injective linear map $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)/\mathrm{B}^1_{\mathrm{geom}}(\Gamma, V) \to \mathrm{H}^1(\Gamma, V)$ that does not depend on x. To conclude, it remains to prove that this map is surjective. Therefore, we fix θ in $\mathrm{Z}^1(\Gamma, V)$ and we will build σ in $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$ such that, for x in X, θ and σ_x are cohomologous.

First, we use a standard trick of finite groups theory to eliminate the difficulties associated with stabilizers of vertices. For x in X, we set

$$v_x = \frac{1}{|\Gamma_x|} \sum_{\gamma \in \Gamma_x} \theta(\gamma).$$

We notice that, by the cocycle property, for γ in Γ_x,

$$\gamma v_x - v_x = \frac{1}{|\Gamma_x|}\sum_{\eta\in\Gamma_x}(\gamma\theta(\eta) - \theta(\gamma\eta)) = -\theta(\gamma).$$

The latter implies the following property: if γ and η are in Γ and $\gamma x = \eta x$, we have

$$\theta(\gamma) + \gamma v_x = \theta(\eta) + \eta v_x. \tag{A.3}$$

Indeed, we write $\eta = \gamma\zeta$ where ζ belongs to Γ_x and we get, from the cocycle property,

$$\theta(\eta) + \eta v_x - \theta(\gamma) - \gamma v_x = \gamma\theta(\zeta) + \gamma\zeta v_x - \gamma v_x = 0.$$

Now, we fix a system of representatives $S \subset X$ for the action of Γ on X. In other words, we have $X = \Gamma S$ and, for x in S, $S \cap \Gamma x = \{x\}$. To build σ, we will first build a map $\varphi : X \to V$ which will play the role of a primitive of σ, that is, we will have $\sigma = \mathrm{d}\varphi$. More precisely, for x in X, we choose γ in Γ with $\gamma^{-1}x \in S$ and we set

$$\varphi(x) = \theta(\gamma) + \gamma v_{\gamma^{-1}x}.$$

Due to (A.3), this does not depend on the choice of γ. The map $\varphi : X \to V$ is not Γ-equivariant in general. But, for x in X and γ in Γ, we have, by the construction and the cocycle property of θ,

$$\varphi(\gamma x) - \gamma\varphi(x) = \theta(\gamma).$$

Therefore, if we set $\sigma = \mathrm{d}\varphi$, the map $\sigma : X_1 \to V$ is Γ-equivariant. Thus, σ is a geometric cocycle. To conclude, we compute, for x in S and γ in Γ,

$$\sigma_x(\gamma) = \varphi(\gamma x) - \varphi(x) = \theta(\gamma) + \gamma v_x - v_x.$$

Therefore, σ_x and θ are cohomologous as required. □

A.4. **Harmonic cohomology classes.** In the sequel, we use Proposition A.3 to identify the spaces $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)/\mathrm{B}^1_{\mathrm{geom}}(\Gamma, V)$ and $\mathrm{H}^1(\Gamma, V)$. We will now introduce a notion of a harmonic cohomology class that is inspired by Hodge theory and is essentially the same as the one in [1, 5].

Let V be a vector space. A skew-symmetric map $\sigma : X_1 \to V$ is said to be harmonic if, for x in X, one has

$$\sum_{y\sim x}\sigma(x, y) = 0.$$

Example A.4. Let V be $\overline{\mathcal{D}}(\partial X)$. For $x \sim y$ in X_1, set $\sigma(x,y) = \mathbf{1}_{U_{xy}}$ where, as usual,

$$U_{xy} = \{\xi \in \partial X | y \in [x\xi)\}.$$

Then, σ is a harmonic skew-symmetric map.

This example is universal in the following sense:

Lemma A.5. *Let V be a real vector space and let $\sigma : X_1 \to V$ be a harmonic skew-symmetric map. Then, there exists a unique linear map $\rho : \overline{\mathcal{D}}(\partial X) \to V$ such that, for $x \sim y$ in X, one has*

$$\sigma(x,y) = \rho(\mathbf{1}_{U_{xy}}).$$

If V is equipped with an action of Γ and σ is Γ-equivariant (that is, σ is a harmonic geometric cocycle), then ρ is also Γ-equivariant.

Proof. In case $V = \mathbb{R}$ the existence and uniqueness of ρ are established in Lemma I.3.4. The general case can be obtained in the same way. The equivariance property for cocycles directly follows from uniqueness. □

A map $\varphi : X \to V$ will be said to be harmonic if, for x in X, one has

$$\frac{1}{q+1}\sum_{y\sim x}\varphi(y) = \varphi(x).$$

Let V be a vector space with an action of Γ. We will say that a cohomology class in $\mathrm{H}^1(\Gamma, V)$ is harmonic if it admits a harmonic representative in $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$. We can describe the obstruction for this representative to be unique.

Lemma A.6. *Let V be a real vector space equipped with a linear action of Γ. Then, the harmonic representatives of the trivial cohomology class in $\mathrm{H}^1(\Gamma, V)$ are the geometric 1-cocycles of the form $\mathrm{d}\varphi$, where $\varphi : X \to V$ is a harmonic Γ-equivariant map.*

The proof is immediate.

A.5. **Unitary representations.** We now gather all the previous constructions to describe the space of harmonic cohomology classes of a unitary representation.

If Γ acts on a vector space V, write $\mathrm{Hom}_\Gamma(\overline{\mathcal{D}}(\partial X), V)$ for the space of all Γ-equivariant linear maps $\overline{\mathcal{D}}(\partial X) \to V$. In Lemma A.5, we have defined a natural isomorphism between the spaces $\mathrm{Hom}_\Gamma(\overline{\mathcal{D}}(\partial X), V)$ and the space of harmonic cocycles in $\mathrm{Z}^1_{\mathrm{geom}}(\Gamma, V)$.

Proposition A.7. *Let V be a Hilbert space equipped with a unitary action of Γ. The natural map*

$$\mathrm{Hom}_\Gamma(\overline{\mathcal{D}}(\partial X), V) \to \mathrm{H}^1(\Gamma, V)$$

is an isomorphism onto the space of harmonic cohomology classes.

The proof uses the following generalization of Appollonius Theorem which gives a strong convexity property of the balls in Hilbert spaces.

Lemma A.8. *Let V be a Hilbert space, $v_1, \dots, v_n$ be vectors in V and $t_1, \dots, t_n$ be non-negative real numbers with $\sum_{i=1}^n t_i = 1$. We have*

$$\left\| \sum_{1\leq i\leq n} t_i v_i \right\|^2 + \frac{1}{2} \sum_{1\leq i,j\leq n} t_i t_j \left\| v_i - v_j \right\|^2 = \sum_{1\leq i\leq n} t_i \left\| v_i \right\|^2 .$$

The proof is immediate.

Proof of Proposition A.7. By Lemma A.5, it only remains to prove that every harmonic cohomology class admits a unique harmonic representative. This, we will show by using the criterion in Lemma A.6.

To this aim, we study harmonic Γ-equivariant maps $\varphi : X \to V$. We claim that any such φ is constant, with values in the space V^Γ of Γ-invariant vectors of V. This will follow from the maximum principle. Indeed, the function $x \mapsto \|\varphi(x)\|$ is Γ-invariant on X. As Γ has finitely many orbits in X, this function reaches its maximum value, that is, the set

$$E = \{x \in X | \, \|\varphi(x)\| = \sup_{y\in X} \|\varphi(y)\|\}$$

is not empty. For x in E, since φ is harmonic, we have

$$\varphi(x) = \frac{1}{q+1} \sum_{y\sim x} \varphi(y).$$

Note that all the $\varphi(y)$, $y \sim x$, have norm $\leq \|\varphi(x)\|$. By applying Lemma A.8 to the vectors $\varphi(y)$, $y \sim x$, with constants coefficients, we obtain that all these vectors are equal to each other and hence to $\varphi(x)$. Therefore $E = X$ and φ is constant with value some $v \in V$. Since φ is Γ-equivariant, the vector v is Γ-invariant.

In particular, the harmonic cocycles defined in Lemma A.6 are all 0, which amounts to say that a harmonic cohomology class admits a unique harmonic representative. □

Remark A.9. Given a harmonic geometric cocycle σ let us say that the cohomology class of σ is cyclic if the range of $\sigma : X_1 \to V$ spans a dense subspace of V. This amounts to saying that the linear map $\rho : \overline{\mathcal{D}}(\partial X) \to V$ associated with σ has dense image. Then, the pull back of the scalar product of V under ρ is a Γ-invariant non-negative symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$. In that sense, the study of these bilinear forms may be understood as the study of all unitary representations of Γ which admit a cyclic harmonic cohomology class.

A.6. **Spectral gap.** To conclude, we will give a sufficient criterion for all cohomology classes of a unitary representation to be harmonic. Before stating it, we discuss the properties of unitary representations that are far away from the trivial representations.

Let V be a Hilbert space equipped with a unitary action of Γ. We denote by $\mathcal{F}(X,V)^\Gamma$ and $\mathcal{F}(X_1,V)^\Gamma$ the spaces of Γ-equivariant maps $X \to V$ and $X_1 \to V$, which we equip with the natural Hilbert spaces structures defined in Subsection III.1.1. In particular, the operator Q on $\mathcal{F}(X,V)^\Gamma$ defined by

$$Q\varphi(x) = \frac{1}{q+1}\sum_{y\sim x}\varphi(y), \quad \varphi \in \mathcal{F}(X,V)^\Gamma, \quad x \in X,$$

is self-adjoint.

Recall that Γ is said to have almost invariant vectors in V if, for any $\varepsilon > 0$ and any finite subset F of Γ, there exists v in V with $\|v\| = 1$ such that one has

$$\|\gamma v - v\| \leq \varepsilon, \quad \gamma \in F.$$

The next result may be seen as a reformulation of Kesten's criterion for amenability [4].

Proposition A.10. *Let V be a Hilbert space equipped with a unitary action of Γ. The following are equivalent*
(i) *the group Γ has almost invariant vectors in V.*
(ii) *the number 1 belongs to the spectrum of Q in $\mathcal{F}(X,V)^\Gamma$.*

If these conditions are not satisfied, we shall say that the representation of Γ in V has a spectral gap.

The main difficulty of the proof of Proposition A.10 is to establish the following technical statement which may be seen as an effective version of the proof of Proposition A.7.

Lemma A.11. *There exists a non decreasing sequence $(\alpha_n)_{n\geq 0}$ of continuous non-negative functions on $[0,\infty)$ such that $\alpha_n(0) = 0$ for any $n \geq 0$ and with the following property. Let V be a Hilbert space equipped with a unitary action of Γ, $\varepsilon > 0$ and φ be in $\mathcal{F}(X,V)^\Gamma$ with $\|\varphi\| = 1$ and $\|Q\varphi - \varphi\| \leq \varepsilon$. Pick x in X with $\|\varphi(x)\| = \sup_{y\in X}\|\varphi(y)\|$. Then, for any y in X with $d(x,y) = n$, one has*

$$\|\varphi(x) - \varphi(y)\| \leq \alpha_n(\varepsilon).$$

Proof. Set $C = \sup_{x \in X} |\Gamma_x|^{\frac{1}{2}}$ and, for $\varepsilon \geq 0$, define by induction

$$\alpha_0(\varepsilon) = 0 \tag{A.4}$$
$$\alpha_{n+1}(\varepsilon) = \alpha_n(\varepsilon) + 2(qC(\alpha_n(\varepsilon) + C\varepsilon))^{\frac{1}{2}} + C\varepsilon, \quad n \geq 0.$$

The sequence is non decreasing and the functions $(\alpha_n)_{n \geq 0}$ are all continuous with value 0 at 0.

Now, let V, $\varepsilon > 0$ and φ be as in the statement. Set

$$M = \sup_{x \in X} \|\varphi(x)\|.$$

Note that the definition of the norm on $\mathcal{F}(X, V)^\Gamma$ gives

$$1 = \|\varphi\|^2 = \sum_{x \in \Gamma \backslash X} \frac{1}{|\Gamma_x|} \|\varphi(x)\|^2,$$

hence $M \leq C$. Besides, we have

$$\sum_{x \in \Gamma \backslash X} \frac{1}{|\Gamma_x|} \|Q\varphi(x) - \varphi(x)\|^2 \leq \varepsilon^2,$$

hence, for all x in X,

$$\|Q\varphi(x) - \varphi(x)\| \leq C\varepsilon. \tag{A.5}$$

Fix x in X with $\|\varphi(x)\| = M$. We will show by induction on $n \geq 0$ that, for every y in X with $d(x, y) = n$, one has $\|\varphi(x) - \varphi(y)\| \leq \alpha_n(\varepsilon)$. For $n \geq 0$, there is nothing to prove. Assume the statement holds for $n \geq 0$. Fix y in X with $d(x, y) = n$. By applying the formula in Lemma A.8 to the vectors $\varphi(z)$, $z \sim y$, we obtain

$$\frac{1}{2(q+1)^2} \sum_{w,z \sim y} \|\varphi(w) - \varphi(z)\|^2 \leq M^2 - \|Q\varphi(y)\|^2.$$

From (A.5) and the induction assumption, we get

$$\|Q\varphi(y)\| \geq \|\varphi(y)\| - C\varepsilon \geq M - (\alpha_n(\varepsilon) + C\varepsilon),$$

hence

$$\frac{1}{2(q+1)^2} \sum_{w,z \sim y} \|\varphi(w) - \varphi(z)\|^2 \leq M^2 - (M - (\alpha_n(\varepsilon) + C\varepsilon))^2 \tag{A.6}$$
$$\leq 2C(\alpha_n(\varepsilon) + C\varepsilon).$$

Fix $z \sim y$. By Cauchy-Schwarz inequality, we have

$$\|\varphi(z) - Q\varphi(y)\| \leq \frac{1}{q+1} \sum_{\substack{w \sim y \\ w \neq z}} \|\varphi(z) - \varphi(w)\|$$

$$\leq \frac{q^{\frac{1}{2}}}{q+1} \left(\sum_{\substack{w \sim y \\ w \neq z}} \|\varphi(z) - \varphi(w)\|^2 \right)^{\frac{1}{2}}.$$

Applying (A.6) yields

$$\|\varphi(z) - Q\varphi(y)\| \leq 2(qC(\alpha_n(\varepsilon) + C\varepsilon))^{\frac{1}{2}},$$

hence, from (A.5),

$$\|\varphi(z) - \varphi(y)\| \leq 2(qC(\alpha_n(\varepsilon) + C\varepsilon))^{\frac{1}{2}} + C\varepsilon.$$

Therefore, by (A.4) and the induction assumption, we get

$$\|\varphi(x) - \varphi(z)\| \leq \alpha_{n+1}(\varepsilon)$$

as required. □

Proof of Proposition A.10. (i)⇒(ii) First assume that Γ has almost invariant vectors in V. Fix a system of representatives $S \subset X$ for the action of Γ and set

$$F = \{\gamma \in \Gamma | \exists x, x' \in S \quad x \sim \gamma x'\} \cup \bigcup_{x \in S} \Gamma_x. \tag{A.7}$$

Since the action of Γ on X is proper, the set F is finite.

Fix $0 < \varepsilon < 1$ and a unit vector v in V such that $\|\gamma v - v\| \leq \varepsilon$ for any γ in F. We will use v to build φ in $\mathcal{F}(X, V)^\Gamma$ with $Q\varphi$ close to φ. For x in S, set

$$v_x = \frac{1}{|\Gamma_x|} \sum_{\gamma \in \Gamma_x} \gamma v.$$

We have $\|v_x - v\| \leq \varepsilon$, hence $\|v_x\| \geq 1 - \varepsilon$. Let φ be the unique element of $\mathcal{F}(X, V)^\Gamma$ such that $\varphi(x) = v_x$ for x in S. By definition, we have

$$\|\varphi\|^2 = \sum_{x \in S} \frac{1}{|\Gamma_x|} \|v_x\|^2,$$

hence $\|\varphi\| \geq (1 - \varepsilon)c$, where $c = \left(\sum_{x \in S} |\Gamma_x|^{-1}\right)^{\frac{1}{2}}$.

We claim that, for every x in S and $y \sim x$, we have $\|\varphi(y) - \varphi(x)\| \leq 3\varepsilon$. Indeed, for such x, y, choose γ in Γ with $z = \gamma^{-1} y \in S$. Then, by (A.7), γ belongs to F and we get

$$\|\varphi(y) - \varphi(x)\| = \|\gamma v_z - v_x\| \leq \|\gamma v_z - \gamma v\| + \|\gamma v - v\| + \|v - v_x\| \leq 3\varepsilon.$$

Since by definition, we have

$$\|Q\varphi - \varphi\|^2 = \sum_{x \in S} \frac{1}{|\Gamma_x|} \left\| \frac{1}{q+1} \sum_{y \sim x} (\varphi(y) - \varphi(x)) \right\|^2,$$

we obtain $\|Q\varphi - \varphi\| \leq 3c\varepsilon$ and hence $\|Q\varphi - \varphi\| \leq 3\varepsilon(1-\varepsilon)^{-1} \|\varphi\|$. As ε is arbitrary, 1 is a spectral value of Q in $\mathcal{F}(X, V)^\Gamma$.

(i)⇒(ii) Assume now that 1 is a spectral value of Q in $\mathcal{F}(X, V)^\Gamma$. We need to show that V admits almost invariant vectors. Let $(\alpha_n)_{n \geq 0}$ be as in Lemma A.11 and fix $\varepsilon > 0$ and a finite subset F of Γ. We still take $S \subset X$ to be a set of representatives for the action of Γ and we set

$$n = \sup_{\substack{x \in S \\ \gamma \in F}} d(\gamma x, x).$$

As the function α_n is continuous at 0, we can find $\eta > 0$ such that $\alpha_n(\eta) \leq \varepsilon$. As 1 is a spectral value of the self-adjoint operator Q, we can find φ in $\mathcal{F}(X, V)^\Gamma$ with $\|\varphi\| = 1$ and $\|Q\varphi - \varphi\| \leq \eta$. Choose x in S with $\|\varphi(x)\| = \sup_{y \in X} \|\varphi(y)\|$ and set $v = \varphi(x)$. Since

$$1 = \|\varphi\|^2 = \sum_{y \in S} \frac{1}{|\Gamma_y|} \|\varphi(y)\|^2,$$

we have $\|v\| \geq c^{-1}$, where, as above, $c = \left(\sum_{x \in S} |\Gamma_x|^{-1}\right)^{\frac{1}{2}}$. Besides, Lemma A.11 yields, for γ in F,

$$\|\gamma v - v\| = \|\varphi(\gamma x) - \varphi(x)\| \leq \alpha_n(\eta) \leq \varepsilon.$$

The conclusion follows. □

A.7. **Spectral gap and harmonic cocycles.** We now show that, when the representation has a spectral gap, all cohomology classes are harmonic. The following is mostly a translation from [1, 5].

Proposition A.12. *Let V be a Hilbert space equipped with a unitary action of Γ. If Γ has a spectral gap in V, any 1-cohomology class in $\mathrm{H}^1(\Gamma, V)$ is harmonic. In other words, the natural map*

$$\mathrm{Hom}_\Gamma(\overline{\mathcal{D}}(\partial X), V) \to \mathrm{H}^1(\Gamma, V)$$

is an isomorphism.

Proof. Note that the two statements are equivalent by Proposition A.7.

Consider the operator $\mathrm{d} : \mathcal{F}(X,V)^\Gamma \to \mathcal{F}(X_1,V)^\Gamma$. For an element φ in $\mathcal{F}(X,V)^\Gamma$, we get, by using Lemma I.9.11,

$$\begin{aligned}\|\mathrm{d}\varphi\|^2 &= \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \|\varphi(y)-\varphi(x)\|^2 \\ &= (q+1)^2 \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|} \|Q\varphi(x)-\varphi(x)\|^2 = (q+1)^2 \|Q\varphi-\varphi\|^2 .\end{aligned}$$

By Proposition A.10 and the assumption, 1 is not a spectral value of Q. Since Q is self-adjoint in $\mathcal{F}(X,V)^\Gamma$, this tells us that we may find $\varepsilon > 0$ such that, for φ in $\mathcal{F}(X,V)^\Gamma$, we have $\|Q\varphi - \varphi\| \geq \varepsilon \|\varphi\|$. Therefore, we get $\|\mathrm{d}\varphi\| \geq (q+1)\varepsilon \|\varphi\|$ and the operator d has closed range in $\mathcal{F}(X_1,V)^\Gamma$. Note that the orthogonal complement of $\mathrm{d}\mathcal{F}(X,V)^\Gamma$ in $\mathcal{F}(X_1,V)^\Gamma$ is the kernel of the adjoint operator $\mathrm{d}^\dagger$ of d. Thus, every element σ in $\mathcal{F}(X_1,V)^\Gamma$ may be written as

$$\sigma = \mathrm{d}\varphi + \theta, \tag{A.8}$$

with φ in $\mathcal{F}(X,V)^\Gamma$, θ in $\mathcal{F}(X_1,V)^\Gamma$ and $\mathrm{d}^\dagger\theta = 0$.

We claim that, for θ in $\mathcal{F}(X_1,V)^\Gamma$ and x in X, we have

$$\mathrm{d}^\dagger\theta(x) = \sum_{y\sim x} \theta(y,x) - \theta(x,y).$$

Indeed, for φ in $\mathcal{F}(X,V)^\Gamma$, by applying again Lemma I.9.11, we obtain

$$\begin{aligned}\langle \mathrm{d}\varphi, \theta\rangle &= \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle \varphi(y)-\varphi(x), \theta(x,y)\rangle \\ &= \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|} \langle \varphi(x), \sum_{y\sim x} \theta(y,x)-\theta(x,y)\rangle.\end{aligned}$$

Now, let σ be a geometric coycle of Γ in V, so that σ is a skew-symmetric element of $\mathcal{F}(X_1,V)^\Gamma$. Decompose σ as in (A.8). As σ and $\mathrm{d}\varphi$ are skew-symmetric, so is θ, so that θ is also a geometric cocycle. In particular, for x in X, we get

$$0 = \mathrm{d}^\dagger\theta(x) = \sum_{y\sim x} \theta(y,x) - \theta(x,y) = -2\sum_{y\sim x} \theta(x,y).$$

Hence θ is harmonic as required. □

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