ABOUT THE TRIMMED AND THE POINCARÉ-DULAC NORMAL FORM OF DIFFEOMORPHISMS

by

Jacky CRESSON & Jasmin RAISSY

Abstract. — In this paper, we give a self-contained introduction to the mould formalism of J. Écalle. We provide a dictionary between moulds and the classical Lie algebraic formalism using non-commutative formal power series. We review results by J. Écalle and B. Vallet about the Trimmed form of local analytic diffeomorphisms of $\mathbb{C}^\nu$, for which we provide full proofs and details. This allows us to discuss a mould approach to the classical Poincaré-Dulac normal form for diffeomorphisms.

1. Introduction

Let $\nu \in \mathbb{N}^*$, $x = (x_1, \dots, x_\nu) \in \mathbb{C}^\nu$ and $f : \mathbb{C}^\nu \to \mathbb{C}^\nu$ be a local analytic diffeomorphism of $\mathbb{C}^\nu$ such that $f(0) = 0$ and given by

$$f(c) = f_{\text{lin}}(x) + r(x), \tag{1.1}$$

where f_{lin} is the linear part of f and $r(x) = (r_1(x), \dots, r_\nu(x))$ consists in terms of order at least two, *i.e.,* for $i = 1, \dots, \nu$

$$r_i(x) = \sum_{N \in \mathbb{N}^\nu, |N| \geq 2} a_{i,n} x^N, \quad a_{i,n} \in \mathbb{C}, \tag{1.2}$$

with $N = (n_1, \dots, n_\nu)$, $|N| = n_1 + \dots + n_\nu$ and $x^N = x_1^{n_1} \dots x_\nu^{n_\nu}$.

The dynamics of f around 0 can be studied using *normal form theory* (see [**1**]). The basic idea is to look for *changes of variables* of the form $y = h(x)$, that are tangent to identity and such that f in this new coordinates system, denoted by f_{norm}, has a simpler form. The two objects are related by the conjugacy equation

$$f_{\text{norm}} \circ h = h \circ f. \tag{1.3}$$

Poincaré Theorem [**1**] asserts that if the linear part is non-resonant, *i.e.,* if the set of eigenvalues $\mu_1, \ldots, \mu_\nu$ of f_{lin} does not satisfy relations, called *resonances*, of the form

$$\mu_i = \mu^N, \text{ for some } N \in \mathbb{N}^\nu, \quad |N| \geq 2, \text{ and } i \in \{1, \ldots, \nu\} \tag{1.4}$$

where $\mu = (\mu_1, \ldots, \mu_\nu)$, then we can *linearize* f, *i.e.,* we can find a *formal* change of coordinates such that

$$f_{\text{norm}} := f_{\text{lin}}. \tag{1.5}$$

In the general case, Poincaré-Dulac Theorem [**1**] asserts that we can find a formal change of coordinates such that f_{norm} takes the form

$$f_{\text{norm}} = f_{\text{lin}} + f_{\text{res}}, \tag{1.6}$$

where f_{res} contains only *resonant monomials*, that is monomials $x^n e_i$ of $\mathbb{C}^\nu$, where $e_1 = (1, 0, \ldots, 0), \ldots, e_\nu = (0, \ldots, 0, 1)$ is the canonical basis of $\mathbb{C}^\nu$, with $N \in \mathbb{N}^\nu$ such that $\mu_i = \mu^N$ and $|N| \geq 2$.

A form like (1.6) is called a *resonant normal form* or a *prenormal form* by J. Écalle [**6**]. These forms are not unique. In order to obtain uniqueness, we must look for a prenormal form containing the minimal number of resonant terms and with formal invariants as coefficients. Such a form always exists [**2**] and is called *the normal form* by J. Écalle [**6**]. Although a normal form can be considered as the simplest prenormal form, it is not in general possible to compute it. Even if an *algorithmic* procedure can be obtained [**2**], its exact shape is related to the vanishing of certain quantities depending polynomially on the Taylor coefficients of the diffeomorphisms. This cannot be decided by a computer.

We look for *calculable* prenormal forms, *i.e.,* prenormal forms which can be obtained using a procedure which is *algorithmic* and *implementable.* As an example of such prenormal forms, we study *continuous prenormal forms* as defined by J. Écalle [**7**].

In this paper, we mainly focus on two particular continuous prenormal forms, one introduced by J. Écalle and B. Vallet [**8**] called the *Trimmed form* and the classical *Poincaré-Dulac normal form.*

The paper is organized as follow:

In the first part, we give a self-contained introduction to the mould formalism which is the natural framework for continuous prenormalization. We then describe the general problem of prenormalization for diffeomorphisms and define the notion of continuous prenormalization following Écalle.

In the second part, we review results by J. Écalle and B. Vallet [**8**] about the Trimmed form. We provide complete proofs and details for the computations of the different moulds associated to the Trimmed form. We also give *closed* formulae for these moulds using a different initial alphabet.

We then discuss the *Poincaré-Dulac normal form* in the mould framework and compared to the Trimmed form. We obtain two universal moulds $\mathrm{Poin}^{\bullet}$ and $\mathrm{Dulac}^{\bullet}$. These two universal moulds are associated to the Poincaré normalization procedure and the Poincaré-Dulac normal form. It seems very difficult to obtain such objects using the existing methods of perturbation theory. The mould formalism provides a direct and algorithmic way to capture the universal features of a normalization procedure.

2. Diffeomorphisms, automorphisms and continuous prenormalization

We consider local analytic diffeomorphisms of $\mathbb{C}^{\nu}$ with 0 as a fixed point and *diagonalizable* linear part. We work in a given analytic chart where the linear part is assumed to be in diagonal form. In such a case, the diffeomorphism is called in *prepared form* by J. Écalle.

Let $f : \mathbb{C}^{\nu} \to \mathbb{C}^{\nu}$, $\nu \in \mathbb{N}$ be defined by

$$f(x_1, \dots, x_{\nu}) = (e^{\lambda_1} x_1, \dots, e^{\lambda_{\nu}} x_{\nu}) + h(x_1, \dots, x_{\nu}), \tag{2.1}$$

with $f(0) = 0$, and $h = (h_1, \dots, h_{\nu})$, $h_i \in \mathbb{C}\{x\}$ for all $i = 1, \dots, \nu$. We denote by f_{lin} the linear part of f, *i.e.*, $f_{\mathrm{lin}}(x_1, \dots, x_{\nu}) = (e^{\lambda_1} x_1, \dots, e^{\lambda_{\nu}} x_{\nu})$.

J. Écalle looks for the *substitution operator* associated to f, denoted by F and defined by

$$\mathrm{F} : \begin{array}{lcl} \mathbb{C}\{x\} & \to & \mathbb{C}\{x\}, \\ \phi & \mapsto & \phi \circ f, \end{array} \tag{2.2}$$

where $\circ$ is the usual composition of functions.

As f is a diffeomorphism, the substitution operator F is an automorphism of $(\mathbb{C}\{x\}, \cdot)$ where $\cdot$ is the usual product of functions on $\mathbb{C}\{x\}$, *i.e.*, for all $\phi, \psi \in \mathbb{C}\{x\}$, we have

$$\mathrm{F}(\phi \cdot \psi) = \mathrm{F}\phi \cdot \mathrm{F}\psi, \tag{2.3}$$

and $\mathrm{F}^{-1}(\phi) = \phi \circ f^{-1}$.

J. Écalle proves the following result (see [**6**] Section 4) , which is a direct consequence of the Taylor expansion Theorem:

Lemma 1. — *Let f be an analytic diffeomorphism of $\mathbb{C}^\nu$ in prepared form and* F *its associated substitution operator. There exist a decomposition of* F *as*

$$\mathrm{F} = \mathrm{F}_{\mathrm{lin}} \left(\mathrm{Id} + \sum_{n \in A(\mathrm{F})} B_n \right), \tag{2.4}$$

where $A(\mathrm{F})$ is an infinite set of indices $n \in \mathbb{Z}^\nu$, $\mathrm{F}_{\mathrm{lin}}$ *the substitution operator associated to f_{lin}, and for all $n \in A(\mathrm{F})$, B_n is a homogeneous differential operator of degree n, i.e., for all $m \in \mathbb{N}^\nu$,*

$$B_n(x^m) = \beta_{n,m} x^{n+m}, \quad \beta_{n,m} \in \mathbb{C}. \tag{2.5}$$

In the following, we work essentially with the substitution operator F. In order to simplify our statements, we call *diffeo(s)* the automorphism F associated to a given diffeomorphism f.

Definition 1. — *Let* F *and* $\mathrm{F}_{\mathrm{conj}}$ *be two local analytic diffeos of $\mathbb{C}^\nu$. The diffeo* $\mathrm{F}_{\mathrm{conj}}$ *is said conjugated to* F *if there exists a tangent to the identity change of variables h of $\mathbb{C}^\nu$ such that the associated substitution operator denoted by Θ satisfies*

$$\mathrm{F}_{\mathrm{conj}} = \Theta \circ \mathrm{F} \circ \Theta^{-1}. \tag{2.6}$$

The substitution operator Θ is called the *normalizator* in the following. When the change of variables h is of class formal, C^k or C^ω, we speak of a formal, C^k or analytic normalization.

Definition 2. — *Let* F *be an analytic diffeo of $\mathbb{C}^\nu$ in prepared form. A prenormal form for* F*, denoted by* $\mathrm{F}_{\mathrm{pren}}$*, is an automorphism of* $\mathbb{C}\{x\}$ *conjugated to* F *such that*

$$\mathrm{F}_{\mathrm{pren}} \circ \mathrm{F}_{\mathrm{lin}} = \mathrm{F}_{\mathrm{lin}} \circ \mathrm{F}_{\mathrm{pren}}. \tag{2.7}$$

We can verify that this definition is coherent with the classical one. Indeed, we have $\mathrm{F}_{\mathrm{lin}}(\mathrm{Id}) = f_{\mathrm{lin}}$, and if we denote by $f_{pren} = \mathrm{F}_{\mathrm{pren}}(\mathrm{Id})$ we obtain $f_{\mathrm{pren}} \circ f_{\mathrm{lin}} = f_{\mathrm{lin}} \circ f_{\mathrm{pren}}$. As $f_{\mathrm{pren}} = f_{\mathrm{lin}} + r$, this equation induces the following relation

$$f_{\mathrm{lin}} \circ r = r \circ f_{\mathrm{lin}}. \tag{2.8}$$

Denoting $r(x) = (r_1(x), \dots, r_\nu(x))$, $r_i(x) = \sum_{N \in \mathbb{N}^\nu,\ |N| \geq 2} r_{i,N} x^N$, we obtain

$$e^{\lambda_i} \sum_{N \in \mathbb{N}^\nu,\ |N| \geq 2} r_{i,n} x^n = \sum_{N \in \mathbb{N}^\nu, \| N| \geq 2} r_{i,N} e^{\langle \lambda, N \rangle} x^N, \tag{2.9}$$

where $\langle \lambda, N \rangle := \sum_{j=1}^{\nu} n_j \lambda_j$ is the canonical scalar product. Denoting $\mu_i = e^{\lambda_i}$ the eigenvalues of f, we have for all $N \in \mathbb{N}^\nu$, $|N| \geq 2$

$$\mu_i r_{i,N} = \mu^N r_{i,N}. \tag{2.10}$$

If $\mu_i \neq \mu^N$ then $r_{i,N} = 0$. As a consequence, the commutation with $\mathrm{F}_{\mathrm{lin}}$ is equivalent to impose that f_{pren} contains only resonant terms.

J. Écalle introduced in [**7**] and extensively studied in [**8**] a very particular class of prenormal forms called *continuous prenormal forms.*

Definition 3. — *Let* F *be a diffeo of* $\mathbb{C}^\nu$ *in prepared form given by*

$$\mathrm{F} = \mathrm{F}_{\mathrm{lin}} \left(\mathrm{Id} + \sum_{n \in A(\mathrm{F})} B_n \right).$$

A continuous prenormal form $\mathrm{F}_{\mathrm{pren}}$ *is an automorphism of* $\mathbb{C}\{x\}$ *conjugated to of the form*

$$\mathrm{F}_{\mathrm{pren}} = \mathrm{F}_{\mathrm{lin}} \left(\sum_{\mathbf{n} \in A(\mathrm{F})^*} \mathrm{Pren}^{\mathbf{n}} B_{\mathbf{n}} \right), \tag{2.11}$$

where $A(\mathrm{F})^*$ *is the set of sequences* $\mathbf{n} = (n_1, \dots, n_r)$, $n_i \in A(F)$, $r \geq 0$, $\mathrm{Pren}^{\mathbf{n}} \in \mathbb{C}$ *satisfying*

$$\mathrm{Pren}^{\mathbf{n}} = 0 \quad \text{if} \quad \langle \|\mathbf{n}\|, \lambda \rangle \notin 2\pi i \mathbb{Z}, \tag{2.12}$$

with $\lambda = (\lambda_1, \dots, \lambda_\nu) \in \mathbb{C}^\nu$, $\|\mathbf{n}\| = n_1 + \cdots + n_r \in \mathbb{Z}^\nu$ *for all* $\mathbf{n} \in A(F)^*$, *and* $B_{\mathbf{n}} = B_{n_1} \dots B_{n_r}$ *with the usual composition of differential operators.*

These forms are calculable using the formalism of *moulds* developed by J. Écalle since 1970 in relation with his Resurgence theory (see [**5**]).

3. Moulds and prenormalization

3.1. Reminder about moulds. — We provide a self-contained introduction to the formalism of moulds and we refer to the articles of J. Écalle or to the surveys [**3**],[**4**] for more details.

3.1.1. Moulds and non-commutative formal power series. — We denote by A an alphabet, finite or not. A letter of A is denoted by a. Let A^* denotes the set of *words* constructed on A, *i.e.,* the finite sequences $a_1 \dots a_r$, $r \geq 0$, with $a_i \in A$, with the convention that for $r = 0$ we have the *empty-word* denoted by $\emptyset$. We denote a word of A^* with bold letter $\mathbf{a}$. We have a natural operation on A^* provided by the usual *concatenation* of two words $\mathbf{a}$, $\mathbf{b} \in A^*$, which glues the words $\mathbf{a}$ to $\mathbf{b}$, *i.e.,* $\mathbf{ab}$.

Definition 4. — *Let $\mathbb{K}$ be a ring (or a field) and A a given alphabet. A $\mathbb{K}$-valued mould on A is a map from A^* to $\mathbb{K}$, denoted by $\mathrm{M}^\bullet$. The set of $\mathbb{K}$-valued moulds on A is denoted by $\mathcal{M}_{\mathbb{K}}(A)$.*

The evaluation of $\mathrm{M}^\bullet$ on a word $\mathbf{a} \in A^*$ is denoted by $\mathrm{M}^{\mathbf{a}}$

We can define a $\mathbb{C}$-valued mould on $A(F)$ by

$$\begin{array}{rlcl} \mathrm{Pren}^\bullet : & A(F)^* & \longrightarrow & \mathbb{C} \\ & \mathbf{n} & \longmapsto & \mathrm{Pren}^{\mathbf{n}}. \end{array} \tag{3.1}$$

The mould $\mathrm{Pren}^\bullet$ is obtained collecting the coefficients of a formal power series $\sum_{\mathbf{n} \in A(F)^*} \mathrm{Pren}^{\mathbf{n}} B_{\mathbf{n}}$. There exists a one-to-one correspondence between moulds and formal power series.

For $r \geq 0$, we denote by A_r^* the set of words of length r, with the convention that $A_0^* = \{\emptyset\}$. We denote by $\mathbb{K}\langle A\rangle$ the set of finite $\mathbb{K}$-linear combinations of elements of A^*, *i.e., non-commutative* polynomials on A with coefficients in $\mathbb{K}$, and by $\mathbb{K}_r\langle A\rangle$ the set of $\mathbb{K}$-linear combination of elements of A_r^*, *i.e.,* the set of non-commutative homogeneous polynomials of degree r. We have a natural *graduation* on $\mathbb{K}\langle A\rangle$ by the length of words:

$$\mathbb{K}\langle A\rangle = \bigoplus_{r=0}^{\infty} \mathbb{K}_r\langle A\rangle. \tag{3.2}$$

The completion of $\mathbb{K}\langle A\rangle$ with respect to the graduation by length, denoted by $\mathbb{K}\langle\langle A\rangle\rangle$, is the set of formal power series with coefficients in $\mathbb{K}$. An element of $\mathbb{K}\langle\langle A\rangle\rangle$ is denoted by

$$\sum_{\mathbf{a}\in A^*} M^{\mathbf{a}}\mathbf{a},\quad M^{\mathbf{a}}\in\mathbb{K}, \tag{3.3}$$

where this sum must be understood as

$$\sum_{r\geq 0}\left(\sum_{\mathbf{a}\in A_r^*} M^{\mathbf{a}}\mathbf{a}\right), \tag{3.4}$$

ans so we have a mould. Conversely, let $\mathrm{M}^\bullet$ be a $\mathbb{K}$-valued mould on A, its generating series, denoted by Φ_M, belongs to $\mathbb{K}\langle\langle A\rangle\rangle$ and is defined by

$$\Phi_M = \sum_{\mathbf{a}\in A^*} M^{\mathbf{a}}\mathbf{a}, \tag{3.5}$$

or in a condensed way as $\sum_{\bullet} M^{\bullet}\bullet$. This correspondence provide a *one-to-one mapping* from the set $\mathcal{M}_{\mathbb{K}}(A)$ of $\mathbb{K}$-valued moulds on A and $\mathbb{K}\langle\langle A\rangle\rangle$.

3.1.2. Moulds algebra. — The set of moulds $\mathcal{M}_{\mathbb{K}}(A)$ inherits a *structure of algebra* from $\mathbb{K}\langle\langle A\rangle\rangle$. We recall here the definition of sum and product of two moulds $\mathrm{M}^\bullet$ and $\mathrm{N}^\bullet$, that are denoted respectively by $\mathrm{M}^\bullet+\mathrm{N}^\bullet$ and $\mathrm{M}^\bullet\cdot\mathrm{N}^\bullet$, and defined by

$$\begin{array}{rcl} (\mathrm{M}^\bullet+\mathrm{N}^\bullet)^{\mathbf{a}} & = & M^{\mathbf{a}}+N^{\mathbf{a}}, \\ (\mathrm{M}^\bullet\cdot\mathrm{N}^\bullet)^{\mathbf{a}} & = & \displaystyle\sum_{\mathbf{a}^1\mathbf{a}^2=\mathbf{a}} M^{\mathbf{a}^1}N^{\mathbf{a}^2}, \end{array} \tag{3.6}$$

for all $\mathbf{a}\in A^*$ where the sum corresponds to all the partition of $\mathbf{a}$ as a concatenation of two words $\mathbf{a}^1$ and $\mathbf{a}^2$ of A^*.

It is easy to check that the product of moulds is analogous to the composition of operators, and hence of maps.

The neutral element for the mould product is denoted by $1^\bullet$ and defined by

$$1^\bullet = \begin{cases} 1 & \text{if } \bullet=\emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{3.7}$$

Let $\mathrm{M}^\bullet$ be a mould. We denote by $\cancel{\mathrm{M}}^\bullet$ the inverse of $\mathrm{M}^\bullet$ for the mould product when it exists, *i.e.,* the solution of the mould equation:

$$\mathrm{M}^\bullet\cdot\cancel{\mathrm{M}}^\bullet = \cancel{\mathrm{M}}^\bullet\cdot\mathrm{M}^\bullet = 1^\bullet. \tag{3.8}$$

3.1.3. Composition of moulds. — Assuming that A possesses a *semi-group* structure, we can define a non-commutative version of the classical operation of *substitution* of formal power series.

We denote by $\star$ an internal law on A, such that $(A,\star)$ is a semi-group. We denote by $\|\cdot\|_\star$ the mapping from A^* to A defined by

$$\begin{array}{rlcl} \|\cdot\|_\star : & A^* & \longrightarrow & A, \\ & \mathbf{a}=a_1\dots a_r & \longmapsto & a_1\star\dots\star a_r. \end{array} \tag{3.9}$$

The $\star$ will be omitted when clear from the context.

The set $\mathbb{K}\langle\langle A\rangle\rangle$ is graded by $\|\,\|_\star$. A *homogeneous component* of degree $a\in A$ of a non-commutative series $\Phi_M=\sum_{\mathbf{a}\in A^*} M^{\mathbf{a}}\mathbf{a}$ is the quantity

$$\Phi_M^a=\sum_{\mathbf{a}\in A^*,\ \|\mathbf{a}\|_\star=a} M^{\mathbf{a}}\mathbf{a}. \tag{3.10}$$

We have by definition

$$\Phi_M=\sum_{a\in A}\Phi_M^a. \tag{3.11}$$

***Definition 5* (Composition).** — *Let $(A,\star)$ be a semi-group structure. Let $\mathrm{M}^\bullet$ and $\mathrm{N}^\bullet$ be two moulds on $\mathcal{M}_{\mathbb{K}}(A)$ and Φ_M, Φ_N their associated generating series. The substitution of Φ_N in Φ_M, denoted by $\Phi_M\circ\Phi_N$ is defined by*

$$\Phi_M\circ\Phi_N=\sum_{\mathbf{a}\in A^*} M^{\mathbf{a}}\Phi_N^{\mathbf{a}}, \tag{3.12}$$

where $\Phi_N^{\mathbf{a}}$ is given by $\Phi_N^{a_1}\dots\Phi_N^{a_r}$ for $\mathbf{a}=a_1\dots a_r$.

We denote by $\mathrm{M}^\bullet\circ\mathrm{N}^\bullet$ the mould of $\mathcal{M}_{\mathbb{K}}(A)$ such that

$$\Phi_M\circ\Phi_N=\sum_{\mathbf{a}\in A^*}(\mathrm{M}^\bullet\circ\mathrm{N}^\bullet)^{\mathbf{a}}\mathbf{a}. \tag{3.13}$$

Equation (3.13) define a natural operation on moulds denoted $\circ$ and called *composition.* Using $\|\,\|_\star$ we can give a closed formula for the composition of two moulds.

Lemma 2. — *Let $(A,\star)$ be a semi-group and $\mathrm{M}^\bullet$, $\mathrm{N}^\bullet$ be two moulds of $\mathcal{M}_{\mathbb{K}}(A)$. We have for all $\mathbf{a}\in A^*$,*

$$(\mathrm{M}^\bullet\circ\mathrm{N}^\bullet)^{\mathbf{a}}=\sum_{k=1}^{l(\mathbf{a})}\ \sum^{*}_{\mathbf{a}^1\dots\mathbf{a}^k=\mathbf{a}} M^{\|\mathbf{a}^1\|_\star\dots\|\mathbf{a}^k\|_\star}N^{\mathbf{a}^1}\dots N^{\mathbf{a}^k}, \tag{3.14}$$

where $l(\bullet)$ *denotes the length of a word of* A^**, and by* $\sum^*$ *we mean the sum restricted to the partitions* $\mathbf{a}^1 \dots \mathbf{a}^k = \mathbf{a}$ *with non-empty elements, that is such that* $\mathbf{a}^i \neq \emptyset$*,* $i = 1, \dots, k$*.*

Proof. — Equation (3.12) is equivalent to

(3.15)
$$\Phi_M \circ \Phi_N = \sum_{r \geq 0} \sum_{\mathbf{b}=b_1\dots b_r \in A_r^*} M^{b_1\dots b_r} \left(\sum_{\mathbf{a}^1 \in A^*,\ \|\mathbf{a}^1\|_\star = b_1} N^{\mathbf{a}^1} \mathbf{a}^1 \right) \dots \left(\sum_{\mathbf{a}^r \in A^*,\ \|\mathbf{a}^r\|_\star = b_r} N^{\mathbf{a}^r} \mathbf{a}^r \right).$$

Let $\mathbf{a} \in A^*$ be a given word of A^*. Each partition of $\mathbf{a}$ of the form $\mathbf{a} = \mathbf{a}^1 \dots \mathbf{a}^k$, $k = 1, \dots, l(\mathbf{a})$, occurs in the sum (3.15) with a coefficient given by

$$M^{b_1\dots b_r} N^{\mathbf{a}^1} \dots N^{\mathbf{a}^k}, \tag{3.16}$$

where $b_i = \|\mathbf{a}^i\|_\star$. Collecting all these coefficients, we obtain the formula (3.14) for the coefficient of $\mathbf{a}$ in $\Phi_M \circ \Phi_N$. □

The neutral element for the mould composition is denoted by $\mathrm{I}^\bullet$ and defined by

$$\mathrm{I}^\bullet = \begin{cases} 1 & \text{if } l(\bullet) = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.17}$$

where $l(\bullet)$ is the length of a word of A^*.

3.1.4. Exponential and logarithm of moulds. — We denote by $(\mathbb{K}\langle\langle A \rangle\rangle)_*$ the set of formal power series without a constant term. We define the *exponential* of an element $x \in (\mathbb{K}\langle\langle A \rangle\rangle)_*$, denoted by $\exp(x)$ using the classical formula

$$\exp(x) = \sum_{n \geq 0} \frac{x^n}{n!}. \tag{3.18}$$

The *logarithm* of an element $1 + x \in 1 + (\mathbb{K}\langle\langle A \rangle\rangle)_*$ is denoted by $\log(1 + x)$ and defined by

$$\log(1 + x) = \sum_{n \geq 1} (-1)^{n+1} \frac{x^n}{n}. \tag{3.19}$$

These two applications have their natural counterpart in $\mathcal{M}_{\mathbb{K}}(A)$.

Definition 6. — *Let* $\mathrm{M}^\bullet$ *be a mould of* $\mathcal{M}_{\mathbb{K}}(A)$ *and* Φ_M *the associated generating series. Assume that* $\exp(\Phi_M)$ *is defined. We denote by* $\mathrm{ExpM}^\bullet$ *the mould satisfying the equality*

$$\exp\left(\sum_\bullet \mathrm{M}^\bullet \bullet \right) = \sum_\bullet \mathrm{ExpM}^\bullet \bullet . \tag{3.20}$$

Simple computations lead to the following direct definition of Exp on moulds:

$$\mathrm{ExpM}^\bullet = \sum_{n \geq 0} \frac{[\mathrm{M}^\bullet]_{(\times n)}}{n!}, \tag{3.21}$$

where $[\mathrm{M}^\bullet]_{(\times n)}$, $n \in \mathbb{N}$, stands for

$$[\mathrm{M}^\bullet]_{(\times n)} = \underbrace{\mathrm{M}^\bullet \cdots \mathrm{M}^\bullet}_{n \text{ times}}. \tag{3.22}$$

The same procedure can be applied to define the logarithm of a mould.

Definition 7. — *Let* $\mathrm{M}^\bullet$ *be a mould of* $\mathcal{M}_{\mathbb{K}}(A)$ *and* Φ_M *the associated generating series. Assume that* $\log(1+\Phi_M)$ *is defined. We denote by* $\mathrm{LogM}^\bullet$ *the mould satisfying the equality*

$$\log\left(1 + \sum_\bullet \mathrm{M}^\bullet \bullet\right) = \sum_\bullet \mathrm{LogM}^\bullet \bullet\,. \tag{3.23}$$

A direct definition of Log is then given by

$$\mathrm{LogM}^\bullet = \sum_{n\geq 1} (-1)^{n+1} \frac{[\mathrm{M}^\bullet]_{(\times n)}}{n}. \tag{3.24}$$

As exp and log satisfy $\exp(\log(1+x)) = 1+x$ and $\log(1+\exp(x)-1) = x$, we have

$$\mathrm{Exp}\left(\mathrm{LogM}^\bullet\right) = 1 + \mathrm{M}^\bullet \quad \text{and} \quad \mathrm{Log}\left(\mathrm{ExpM}^\bullet - 1\right) = \mathrm{M}^\bullet, \tag{3.25}$$

for all moulds $\mathrm{M}^\bullet$ with $\mathrm{M}^\emptyset = 0$.

3.1.5. A technical lemma. — In this section, we derive simple results for the exponential and logarithm of moulds with non-zero components only on words of length 1.

Lemma 3. — *Let us denote by* $\mathrm{Z}^\bullet$ *a mould of* $\mathcal{M}_{\mathbb{K}}(A)$ *such that* $\mathrm{Z}^\bullet = 0$ *for all* $\bullet$ *of length different from* 1. *For all* $\mathbf{a} \in A^*$, $r \geq 1$, *we have*

$$[\mathrm{Z}^\bullet]^{\mathbf{a}}_{\times r} = \begin{cases} \mathrm{Z}^{a_1} \ldots \mathrm{Z}^{a_r} \text{ if } \ l(\mathbf{a}) = r,\ \mathbf{a} = a_1 \ldots a_r, \\ 0 \ \text{ otherwise.} \end{cases} \tag{3.26}$$

$$[\mathrm{ExpZ}^\bullet]^{\mathbf{a}} = 1^{\mathbf{a}} + \frac{1}{l(\mathbf{a})!}\, [\mathrm{Z}^\bullet]^{\mathbf{a}}_{(\times l(\mathbf{a}))}\,, \tag{3.27}$$

$$[\mathrm{LogZ}^\bullet]^{\mathbf{a}} = \frac{(-1)^{l(\mathbf{a})+1}}{l(\mathbf{a})}\, [\mathrm{Z}^\bullet]^{\mathbf{a}}_{(\times l(\mathbf{a}))}\,. \tag{3.28}$$

Proof. — We first remark that equations (3.27) and (3.28) easily follow from equation (3.26).

The proof of equation (3.26) is done by induction on r. Formula (3.26) is trivially true for $r = 1$. Assume that formula (3.26) is true for $r \geq 1$. By definition, we have

$$[\mathrm{Z}^\bullet]_{(\times r+1)} = \mathrm{Z}^\bullet \cdot [\mathrm{Z}^\bullet]_{(\times r)}\,. \tag{3.29}$$

Let $\mathbf{a} = a\mathbf{b}$, then by assumption on $\mathrm{Z}^\bullet$ we obtain

$$[\mathrm{Z}^\bullet]^{a\mathbf{b}}_{(\times r+1)} = \mathrm{Z}^a \, [\mathrm{Z}^\bullet]^{\mathbf{b}}_{(\times r)} \, . \tag{3.30}$$

As the mould $[\mathrm{Z}^\bullet]_{(\times r)}$ is non-trivial only on words of length r, we deduce that the mould $[\mathrm{Z}^\bullet]_{(\times r+1)}$ is non-trivial only on words of length $r+1$.

Moreover, using the fact that $[\mathrm{Z}^\bullet]^{a_1\ldots a_r}_{(\times r)} = \mathrm{Z}^{a_1}\ldots\mathrm{Z}^{a_r}$ for all $a_i \in A$, we also deduce that $[\mathrm{Z}^\bullet]^{a_1\ldots a_{r+1}}_{(\times r+1)} = \mathrm{Z}^{a_1}\ldots\mathrm{Z}^{a_{r+1}}$. This concludes the proof. □

3.2. Prenormalization. — We can associate

Let F be a diffeo in prepared form given by

$$F = \mathrm{F}_{\mathrm{lin}} \left(\mathrm{Id} + \sum_{n \in A(F)} B_n \right) .$$

Let Φ_Θ be an automorphism of $\mathbb{C}\{x\}$ of the form

$$\Phi_\Theta = \sum_{\mathbf{n} \in A(F)^*} \Theta^{\mathbf{n}} B_{\mathbf{n}}, \tag{3.31}$$

where $\Theta^{\mathbf{n}} \in \mathbb{C}$ for all $\mathbf{n} \in A(F)^*$, *i.e.,* $\Phi_\Theta \in \mathbb{C}\langle\langle \mathbf{B} \rangle\rangle$, where $\mathbf{B} = \{B_n\}_{n\in A(F)}$ and $\Theta^\bullet \in \mathcal{M}_{\mathbb{C}}(A(F))$.

Using the moulds $1^\bullet$ and $\mathrm{I}^\bullet$ we write $\mathrm{Id} + \sum_{n\in A(F)} B_n$ as an element of $\mathbb{C}\langle\langle \mathbf{B} \rangle\rangle$:

$$\mathrm{Id} + \sum_{n \in A(F)} B_n = \sum_\bullet (1^\bullet + \mathrm{I}^\bullet) \mathrm{B}_\bullet . \tag{3.32}$$

We assume that F is conjugated to an automorphism F_{conj} via Φ_Θ. Equation (2.6) is then given by

$$F_{\mathrm{conj}} = \Phi_\Theta \cdot F \cdot \Phi_\Theta^{-1} . \tag{3.33}$$

The automorphism F_{conj} can be written as

$$F_{\mathrm{conj}} = \mathrm{F}_{\mathrm{lin}} \left(\sum_\bullet \mathrm{C}^\bullet \mathrm{B}_\bullet \right) . \tag{3.34}$$

Equation (3.33) is then equivalent to

$$\mathrm{F}_{\mathrm{lin}} \left(\sum_\bullet \mathrm{C}^\bullet \mathrm{B}_\bullet \right) = \left(\sum_\bullet \Theta^\bullet \mathrm{B}_\bullet \right) \mathrm{F}_{\mathrm{lin}} \left(\sum_\bullet (1^\bullet + \mathrm{I}^\bullet) \mathrm{B}_\bullet \right) \left(\sum_\bullet \cancel{\Theta} \mathrm{B}_\bullet \right) , \tag{3.35}$$

where $\cancel{\Theta}$ is such that $\cancel{\Theta} \cdot \Theta^\bullet = \Theta^\bullet \cdot \cancel{\Theta} = 1^\bullet$, *i.e.,* $\Phi_\Theta^{-1} = \sum_\bullet \cancel{\Theta} \mathrm{B}_\bullet$.

In order to explicit $\mathrm{C}^\bullet$ we need to understand the action of a formal power series of $\mathbb{C}\langle\langle\mathbf{B}\rangle\rangle$ on $\mathrm{F}_{\mathrm{lin}}$. We have the following fundamental lemma:

Lemma 4. — *Let* $\mathrm{M}^\bullet \in \mathcal{M}_{\mathbb{C}}(A(F))$. *We have*

$$\left(\sum_\bullet \mathrm{M}^\bullet \mathrm{B}_\bullet\right) \mathrm{F}_{\mathrm{lin}} = \mathrm{F}_{\mathrm{lin}} \left(\sum_\bullet \mathrm{e}^\Delta \left(\mathrm{M}^\bullet\right)^\bullet \mathrm{B}_\bullet\right), \tag{3.36}$$

where e^Δ *is a map from* $\mathcal{M}_{\mathbb{C}}(A(F))$ *to* $\mathcal{M}_{\mathbb{C}}(A(F))$ *defined by*

$$\mathrm{e}^\Delta \left(\mathrm{M}^\bullet\right)^{\mathbf{n}} = e^{-\langle\lambda, \|\mathbf{n}\|\rangle} M^{\mathbf{n}} \text{ for all } \mathbf{n} \in A(F)^*. \tag{3.37}$$

Proof. — Let $B_{\mathbf{n}} = B_{n_1 \dots n_r}$ such that $B_{n_i}(x^m) = \beta_m^{n_i} x^{m+n_i}$, $\beta_m^{n_i} \in \mathbb{C}$, $i = 1, \dots, r$, for all $m \in \mathbb{N}^\nu$. We have

$$B_{\mathbf{n}}(x^m) = \beta^{n_1}_{m+n_r+\dots+n_2} \beta^{n_2}_{m+n_r+\dots+n_3} \dots \beta^{n_r}_m x^{m+n_1+\dots+n_r}. \tag{3.38}$$

As $\mathrm{F}_{\mathrm{lin}}(x^m) = e^{\langle\lambda, m\rangle} x^m$ we obtain

$$\begin{array}{rcl} B_{\mathbf{n}}\left(\mathrm{F}_{\mathrm{lin}}(x^m)\right) & = & e^{\langle\lambda, m\rangle} B_{\mathbf{n}}(x^m), \\ & = & e^{-\langle\lambda, n_1+\dots+n_r\rangle} e^{\langle\lambda, m+n_1+\dots+n_r\rangle} B_{\mathbf{n}}(x^m), \\ & = & e^{-\langle\lambda, n_1+\dots+n_r\rangle} \mathrm{F}_{\mathrm{lin}}\left(B_{\mathbf{n}}(x^m)\right), \\ & = & \mathrm{F}_{\mathrm{lin}}\left(e^{-\langle\lambda, n_1+\dots+n_r\rangle} B_{\mathbf{n}}(x^m)\right). \end{array} \tag{3.39}$$

This concludes the proof. □

Next lemma gives an explicit formula to compute the mould $\mathrm{C}^\bullet$ assuming that the mould $\Theta^\bullet$ is known.

Lemma 5. — *Equation (3.35) is equivalent to the mould equation*

$$\mathrm{C}^\bullet = \mathrm{e}^\Delta \left(\Theta^\bullet\right) \cdot \left(1^\bullet + \mathrm{I}^\bullet\right) \cdot \cancel{\Theta}. \tag{3.40}$$

Proof. — Using Lemma 4, we have

$$\begin{array}{rcl} \mathrm{F}_{\mathrm{lin}} \left(\sum_\bullet \mathrm{C}^\bullet \mathrm{B}_\bullet\right) & = & \left(\sum_\bullet \Theta^\bullet \mathrm{B}_\bullet\right) \mathrm{F}_{\mathrm{lin}} \left(\sum_\bullet (1^\bullet + \mathrm{I}^\bullet) \mathrm{B}_\bullet\right) \left(\sum_\bullet \cancel{\Theta} \mathrm{B}_\bullet\right), \\ & = & \mathrm{F}_{\mathrm{lin}} \left(\sum_\bullet \mathrm{e}^\Delta \left(\Theta^\bullet\right) \mathrm{B}_\bullet\right) \left(\sum_\bullet (1^\bullet + \mathrm{I}^\bullet) \mathrm{B}_\bullet\right) \left(\sum_\bullet \cancel{\Theta} \mathrm{B}_\bullet\right), \\ & = & \mathrm{F}_{\mathrm{lin}} \left(\sum_\bullet \left(\mathrm{e}^\Delta \left(\Theta^\bullet\right) \cdot \left(1^\bullet + \mathrm{I}^\bullet\right) \cdot \cancel{\Theta}\right) \mathrm{B}_\bullet\right). \end{array} \tag{3.41}$$

This concludes the proof. □

As a consequence, choosing carefully the normalizator Φ_Θ, we can obtain an inductive expression for the mould of normalization $\mathrm{C}^\bullet$.

We will give explicit formulae for $\mathrm{C}^\bullet$ using specific moulds for $\Theta^\bullet$ in the next section.

3.3. Universality of moulds and prenormalization. — Lemma 5 gives an important feature of the mould formalism in the context of continuous prenormalization. Formula (3.40) is valid whatever is the underlying alphabet $A(F)$. We then obtain a *universal* object underlying the prenormalization problem which is studied.

For example, in the context of *linearization*, *i.e.,* $F_{\mathrm{conj}} = \mathrm{F}_{\mathrm{lin}}$, the universal mould of linearization which defined the linearizing change of variables is given as follow (see [**3**] Chap. III for more details):

Theorem 1. — *Let* $\mathbf{L} = \{L_r\}_{r\geq 1}$, $r \in \mathbb{N}$, *be the set of* $\mathbb{C}$*-valued functions* $L_r : \mathbb{C}^r \to \mathbb{C}$ *defined by*

$$L_r(x_1, \dots, x_r) = \left[\left(e^{-(x_1+\cdots+x_r)} - 1\right)\left(e^{-(x_2+\cdots+x_r)} - 1\right)\dots\left(e^{-x_r} - 1\right)\right]^{-1}, \tag{3.42}$$

for all $(x_1, \dots, x_r) \in \mathbb{C}^r \setminus S_r$ *where the singular set* S_r *is given by*

$$S_r = \{x_r \in 2\pi i\mathbb{Z}\}\bigcup\{x_r + x_{r-1} \in 2\pi i\mathbb{Z}\}\bigcup\cdots\bigcup\{x_1 + \cdots + x_r \in 2\pi i\mathbb{Z}\}. \tag{3.43}$$

If F *possesses a non-resonant linear part* λ, *the mould of formal linearization is given for all* $\mathbf{n} \in A(F)^*$, $\mathbf{n} = n_1, \dots, n_r$, *by*

$$\Theta^{n_1\dots n_r} = L_r(\omega_1, \dots, \omega_r), \tag{3.44}$$

where $\omega_i = \langle n_i, \lambda\rangle$ *for* $i = 1, \dots, r$.

This result cannot be obtained using other existing formalisms. It is well-known that an expression like (3.44) is the important quantity entering the linearization problem. However, the previous result associates universal coefficients from which one can compute the desired linearization map for a given particular diffeo F by posing

$$\Phi_\Theta = \sum_{\mathbf{n}\in A(F)^*} \Theta^{\mathbf{n}} B_{\mathbf{n}}.$$

4. The Trimmed form

In this section, we give detailed proofs for results, presented in [**8**] with a sketch of proof, concerning the *Trimmed form* defined by J. Écalle and B. Vallet.

4.1. Cancelling non-resonant terms. — In this section, we give a mould approach to the classical problem of cancellation of non-resonant terms.

4.1.1. Around the Baker-Campbell-Hausdorff formula. — Let F be a diffeo in prepared form given by (2.4). The operator $\mathrm{Id} + \sum_{n\in A(F)} B_n$ is an automorphism of $\mathbb{C}\{x\}$ which can be viewed as the exponential of a vector field, *i.e.,*

$$\mathrm{Id} + \sum_{n\in A(F)} B_n = \exp\left(\sum_{m\in\mathcal{A}(F)} D_m\right), \tag{4.1}$$

where D_m is a homogeneous differential operator of degree m and order 1, *i.e.,* a derivation on $\mathbb{C}\{x\}$, $m = (m_1, \dots, m_\nu) \in \mathbb{Z}^\nu$, with all $m_i \in \mathbb{N}$, $i = 1, \dots, \nu$ except at most one which can be -1, and $\mathcal{A}(F)$ the set of degrees coming in the decomposition.

We look for an automorphism given by the exponential of a vector field $\mathbf{V}$ given by

$$\mathbf{V} = \sum_{\mathbf{n}\in A(F)^*} \mathrm{dem}^{\mathbf{n}} B_{\mathbf{n}}, \tag{4.2}$$

or equivalently given on the alphabet $\mathcal{A}(F)^*$ by

$$\mathbf{V} = \sum_{\mathbf{m}\in\mathcal{A}(F)^*} \mathrm{Dem}^{\mathbf{m}} D_{\mathbf{m}}, \tag{4.3}$$

where $\mathbf{m} = m_1 \cdots m_r$ and $D_{\mathbf{m}} = D_{m_1} D_{m_2} \cdots D_{m_r}$, with the usual composition of differential operators.

The action of $\exp \mathbf{V}$ on F is given by

$$\exp \mathbf{V} \cdot F \cdot \exp(-\mathbf{V}) \tag{4.4}$$

Equation (4.4) can be analyzed using the moulds expression of $\mathbf{V}$ and F with respect to the alphabet $\mathcal{A}(F)$. We have the following lemma:

Lemma 6. — *Equation (4.4) is equal to*

$$\exp \mathbf{V} \cdot F \cdot \exp(-\mathbf{V}) = \mathrm{F}_{\mathrm{lin}} \exp\left(\widetilde{\mathbf{V}} + \mathbf{D} - \mathbf{V} + \dots\right), \tag{4.5}$$

where the $\dots$ *stands for a formal power series beginning with words of length at least* 2, *and* $\mathbf{D}$ *and* $\widetilde{\mathbf{V}}$ *are vector fields defined by* $\mathbf{D} = \sum_{m\in\mathcal{A}(F)} D_m$ *and*

$$\widetilde{\mathbf{V}} = \sum_{\mathbf{m}\in\mathcal{A}(F)^*} e^{-\langle\lambda, \|\mathbf{m}\|\rangle} \mathrm{Dem}^{\mathbf{m}} D_{\mathbf{m}}, \tag{4.6}$$

respectively.

Proof. — Using the Baker-Campbell-Hausdorff formula (see [**10**], Theorem II.4.29), we obtain

$$\begin{aligned}\exp \mathbf{D}\cdot\exp(-\mathbf{V}) &= \exp\left(\mathbf{D}-\mathbf{V}+\frac{1}{2}[\mathbf{D},-\mathbf{V}]+\frac{1}{12}[\mathbf{D},[\mathbf{D},-\mathbf{V}]]-\frac{1}{12}[-\mathbf{V},[\mathbf{D},-\mathbf{V}]]+\dots\right),\\ &= \exp\left(\mathbf{D}-\mathbf{V}+\text{h.o.t.}\right),\end{aligned}$$

where h.o.t. stands for *higher order terms.*

Using Lemma 4, we have

$$\exp \mathbf{V}\cdot \mathrm{F}_{\mathrm{lin}} = \mathrm{F}_{\mathrm{lin}}\cdot \exp\widetilde{\mathbf{V}}, \tag{4.7}$$

where $\widetilde{\mathbf{V}}$ is given by

$$\widetilde{\mathbf{V}} = \sum_{\mathbf{m}\in\mathcal{A}(F)^*} e^{-\langle\lambda,\|\mathbf{m}\|\rangle}\mathrm{Dem}^{\mathbf{m}} D_{\mathbf{m}}. \tag{4.8}$$

As a consequence, applying again the Baker-Campbell-Hausdorff formula we obtain

$$\exp\widetilde{\mathbf{V}}\cdot\exp\mathbf{D}\cdot\exp(-\mathbf{V}) = \exp\left(\widetilde{\mathbf{V}}+\mathbf{D}-\mathbf{V}+\dots\right),$$

where the $\dots$ stand for a formal power series beginning with words of length at least 2. This concludes the proof. □

4.1.2. The simplified form and the moulds $\mathrm{dem}^\bullet$ *and* $\mathrm{Dem}^\bullet$. — The main consequence of Lemma 6 is that we can cancel the non-resonant terms of $\mathbf{D}$ using a simple vector field $\mathbf{V}$.

Definition 8. — *Let* $\mathbf{V}$ *be the vector field defined by the mould*

$$\mathrm{Dem}^\bullet = \begin{cases} \dfrac{I^{\mathbf{m}}}{1-e^{\langle\|\mathbf{m}\|,\lambda\rangle}} & \text{for } \mathbf{m}\in\mathcal{A}(F)^*\setminus\mathcal{R}(F),\\ 0 & \text{otherwise,}\end{cases} \tag{4.9}$$

where $\mathcal{R}(F)$ *is the set of resonant words of* $\mathcal{A}(F)^*$*, i.e.,* $\mathbf{m}\in\mathcal{R}(F)$ *if and only if* $\langle\|\mathbf{m}\|,\lambda\rangle\in 2\pi i\mathbb{Z}$*. We denote by* $\mathrm{dem}^\bullet$ *the associated mould on* $\mathcal{M}_{\mathbb{C}}(A(F))$*, i.e.,*

$$\mathbf{V} = \sum_\bullet \mathrm{Dem}^\bullet \mathrm{D}_\bullet = \sum_\bullet \mathrm{dem}^\bullet \mathrm{B}_\bullet. \tag{4.10}$$

We call simplified form of F *and we denote by* $\mathrm{F}_{\mathrm{Sem}}$ *the automorphism obtained from* F *under the action of* $\exp\mathbf{V}$.

***Theorem 2* (Simplified form).** — *Let* $\mathbf{V}$ *be the vector field defined by the mould in* (4.9)*, and let* $\mathrm{F}_{\mathrm{Sem}}$ *be the simplified form of* F *under the action of* $\exp \mathbf{V}$*. We have*

$$
\begin{aligned}
\mathrm{F}_{\mathrm{Sem}} &= \mathrm{F}_{\mathrm{lin}} \left(\sum_{\mathbf{m}\in\mathcal{A}(F)^*} \mathrm{Sem}^{\mathbf{m}} D_{\mathbf{m}} \right), \\
&= \mathrm{F}_{\mathrm{lin}} \left(\sum_{\mathbf{n}\in A(F)^*} \mathrm{sem}^{\mathbf{n}} B_{\mathbf{n}} \right)
\end{aligned} \tag{4.11}
$$

with the mould $\mathrm{Sem}^\bullet$ *given by*

$$
\mathrm{Sem}^\bullet = \mathrm{e}^{\Delta} \left(\mathrm{Exp}(\mathrm{Dem}^\bullet)\right) \cdot \mathrm{Exp}(\mathrm{I}^\bullet) \cdot \mathrm{Exp}(-\mathrm{Dem}^\bullet), \tag{4.12}
$$

and the mould $\mathrm{sem}^\bullet$ *given by*

$$
\mathrm{sem}^\bullet = \mathrm{e}^{\Delta} \left(\mathrm{Exp}(\mathrm{dem}^\bullet)\right) \cdot (1^\bullet + \mathrm{I}^\bullet) \cdot \mathrm{Exp}(-\mathrm{dem}^\bullet). \tag{4.13}
$$

Proof. — We have $\mathrm{F}_{\mathrm{Sem}} = \exp \mathbf{V} \cdot F \cdot \exp(-\mathbf{V})$ with $\mathbf{V} = \sum_{\mathbf{n}\in A(F)^*} \mathrm{dem}^{\mathbf{n}} B_{\mathbf{n}}$. As a consequence, we have $\exp \mathbf{V} = \sum_{\mathbf{n}\in A(F)^*} \left(\mathrm{Exp}\,\mathrm{dem}^\bullet\right)^{\mathbf{n}} B_{\mathbf{n}}$ and the formula for $\mathrm{sem}^\bullet$ follows from Lemma 5 using $\Theta^\bullet = \mathrm{Exp}(\mathrm{dem}^\bullet)$.

For $\mathrm{Sem}^\bullet$, we first use Lemma 4 to obtain

$$
\exp \mathbf{V} \mathrm{F}_{\mathrm{lin}} = \mathrm{F}_{\mathrm{lin}} \left(\sum_{\mathbf{m}\in\mathcal{A}(F)^*} \left[\mathrm{e}^{\Delta} \left(\mathrm{ExpDem}^\bullet\right)\right]^{\mathbf{m}} D_{\mathbf{m}} \right). \tag{4.14}
$$

As a consequence, the conjugacy equation is equivalent to

$$
\begin{aligned}
\mathrm{F}_{\mathrm{Sem}} &= \exp \mathbf{V} \cdot F \cdot \exp(-\mathbf{V}), \\
&= \mathrm{F}_{\mathrm{lin}} \left(\sum_{\bullet} \mathrm{e}^{\Delta} \left(\mathrm{ExpDem}^\bullet\right) \mathrm{D}_\bullet \right) \left(\sum_{\bullet} \mathrm{ExpI}^\bullet \mathrm{D}_\bullet \right) \left(\sum_{\bullet} \mathrm{Exp}(-\mathrm{Dem}^\bullet) \mathrm{D}_\bullet \right), \\
&= \mathrm{F}_{\mathrm{lin}} \left(\sum_{\bullet} \left[\mathrm{e}^{\Delta} \left(\mathrm{ExpDem}^\bullet\right) \cdot \mathrm{ExpI}^\bullet \cdot \mathrm{Exp}(-\mathrm{Dem}^\bullet)\right]^\bullet \mathrm{D}_\bullet \right).
\end{aligned}
$$

This concludes the proof. □

The mould $\mathrm{Sem}^\bullet$ can be computed explicitly. We first introduce some convenient notations.

Let $\mathbf{m} = m_1 \dots m_r$ be a word of length r, $r \geq 1$. We denote by $\mathbf{m}^{\leq i}$ and $\mathbf{m}^{>i}$ the words

$$\mathbf{m}^{\leq i} = m_1 \dots m_i, \quad \mathbf{m}^{>i} = m_{i+1} \dots m_r, \tag{4.15}$$

and analogously for $\mathbf{m}^{<i}$ and $\mathbf{m}^{\geq i}$. Moreover, we denote by $d(\mathbf{m})$ the index of the last m_i in $\mathbf{m} = m_1 \dots m_r$ such that $\langle \lambda, m_i \rangle \in 2\pi i\mathbb{Z}$, and we denote by $q(\mathbf{m})$ the last index just before of the first resonance $\omega_j = \langle \lambda, m_j \rangle$. We have $q(\mathbf{m}) < d(\mathbf{m})$ unless $\langle \lambda, m_i \rangle \notin 2\pi i\mathbb{Z}$ for all i, when one instead has $d(\mathbf{m}) = 0$ and $q(\mathbf{m}) = l(\mathbf{m})$.

Theorem 3. — *For all* $\mathbf{m} \in \mathcal{A}(F)^*$, *we have* $\mathrm{Sem}^{\mathbf{m}} = 1$ *if* $l(\mathbf{m}) = 0, 1$, *and*

$$\begin{aligned}
\mathrm{Sem}^{\mathbf{m}} = {} & \frac{(-1)^{l(\mathbf{m})}}{l(\mathbf{m})!} [\mathrm{Dem}^\bullet]^{\mathbf{m}}_{(\times l(\mathbf{m}))} + \frac{1}{l(\mathbf{m})!} + \sum_{j=\max(d(\mathbf{m})+1,2)}^{l(\mathbf{m})} \frac{(-1)^{l(\mathbf{m}^{>j})} [\mathrm{Dem}^\bullet]^{\mathbf{m}^{\geq j}}_{(\times l(\mathbf{m}^{\geq j}))}}{l(\mathbf{m}^{<j})! l(\mathbf{m}^{\geq j})!} + e^{-\langle \lambda, \|\mathbf{m}\| \rangle} 1^{\mathbf{m}} \\
& + \frac{e^{-\langle \lambda, \|\mathbf{m}\| \rangle}}{l(\mathbf{m})!} [\mathrm{Dem}^\bullet]^{\mathbf{m}}_{(\times l(\mathbf{m}))} \sum_{i=1}^{\min(q(\mathbf{m}), l(\mathbf{m})-1)} \frac{e^{-\langle \lambda, \|\mathbf{m}^{\leq i}\| \rangle}}{l(\mathbf{m}^{\leq i})!} [\mathrm{Dem}^\bullet]^{\mathbf{m}^{\leq i}}_{(\times l(\mathbf{m}^{\leq i}))} \times \\
& \left(\frac{(-1)^{l(\mathbf{m})}}{l(\mathbf{m})!} [\mathrm{Dem}^\bullet]^{\mathbf{m}^{>i}}_{(\times l(\mathbf{m}^{>i}))} + \frac{1}{l(\mathbf{m}^{>i})!} + \sum_{j=\max(d(\mathbf{m}^{>i})+1,2)}^{l(\mathbf{m}^{>i})} \frac{(-1)^{l(\mathbf{m}^{>j})^{\geq j}} [\mathrm{Dem}^\bullet]^{(\mathbf{m}^{>i})^{\geq j}}_{(\times l((\mathbf{m}^{>i})^{\geq j}))}}{l((\mathbf{m}^{>i})^{<j})! l((\mathbf{m}^{>i})^{\geq j})!} \right),
\end{aligned}$$

for $l(\mathbf{m}) > 1$.

Proof. — It follows obviously from (4.12), that $\mathrm{Sem}^{\mathbf{m}} = 1$ for every $\mathbf{m} \in \mathcal{A}(F)^*$ with $l(\mathbf{m}) = 0$ or $l(\mathbf{m}) = 1$.

Let us now consider $\mathbf{m} \in \mathcal{A}(F)^*$ with $l(\mathbf{m}) > 1$. In order to compute the mould $\mathrm{Sem}^\bullet$, we first compute $\mathrm{ExpI}^\bullet \cdot \mathrm{Exp}(-\mathrm{Dem}^\bullet)$. We have

$$\begin{aligned}
(\mathrm{ExpI}^\bullet \cdot \mathrm{Exp}(-\mathrm{Dem}^\bullet))^{\mathbf{n}} &= \sum_{\mathbf{n}^1\mathbf{n}^2=\mathbf{n}} (\mathrm{ExpI}^\bullet)^{\mathbf{n}^1} \mathrm{Exp}(-\mathrm{Dem}^\bullet)^{\mathbf{n}^2}, \\
&= \sum_{\mathbf{n}^1\mathbf{n}^2=\mathbf{n}} \left(1^{\mathbf{n}^1} + \frac{1}{l(\mathbf{n}^1)!} [\mathrm{I}^\bullet]^{\mathbf{n}^1}_{(\times l(\mathbf{n}^1))} \right) \left(1^{\mathbf{n}^2} + \frac{(-1)^{l(\mathbf{n}^2)}}{l(\mathbf{n}^2)!} [\mathrm{Dem}^\bullet]^{\mathbf{n}^2}_{(\times l(\mathbf{n}^2))} \right), \\
&= \sum_{\mathbf{n}^1\mathbf{n}^2=\mathbf{n}} \Bigg(1^{\mathbf{n}^1} 1^{\mathbf{n}^2} + 1^{\mathbf{n}^1} \frac{(-1)^{l(\mathbf{n}^2)}}{l(\mathbf{n}^2)!} [\mathrm{Dem}^\bullet]^{\mathbf{n}^2}_{(\times l(\mathbf{n}^2))} \\
&\qquad + 1^{\mathbf{n}^2} \frac{1}{l(\mathbf{n}^1)!} [\mathrm{I}^\bullet]^{\mathbf{n}^1}_{(\times l(\mathbf{n}^1))} + \frac{(-1)^{l(\mathbf{n}^2)}}{l(\mathbf{n}^1)! l(\mathbf{n}^2)!} [\mathrm{I}^\bullet]^{\mathbf{n}^1}_{(\times l(\mathbf{n}^1))} [\mathrm{Dem}^\bullet]^{\mathbf{n}^2}_{(\times l(\mathbf{n}^2))} \Bigg).
\end{aligned}$$

It is clear that $(\mathrm{ExpI}^\bullet \cdot \mathrm{Exp}(-\mathrm{Dem}^\bullet))^\emptyset = 1$. If $l(\mathbf{n}) \geq 1$ we have

$$\begin{aligned}
(\mathrm{ExpI}^\bullet \cdot \mathrm{Exp}(-\mathrm{Dem}^\bullet))^{\mathbf{n}} &= \frac{(-1)^{l(\mathbf{n})}}{l(\mathbf{n})!}\,[\mathrm{Dem}^\bullet]^{\mathbf{n}}_{(\times l(\mathbf{n}))} + \frac{1}{l(\mathbf{n})!}\,[\mathrm{I}^\bullet]^{\mathbf{n}}_{(\times l(\mathbf{n}))} \\
&\quad + \sum^{*}_{\mathbf{n}^1\mathbf{n}^2=\mathbf{n}} \left(\frac{(-1)^{l(\mathbf{n}^2)}}{l(\mathbf{n}^1)!\,l(\mathbf{n}^2)!}\,[\mathrm{Dem}^\bullet]^{\mathbf{n}^2}_{(\times l(\mathbf{n}^2))} \right) \\
&= \frac{(-1)^{l(\mathbf{n})}}{l(\mathbf{n})!}\,[\mathrm{Dem}^\bullet]^{\mathbf{n}}_{(\times l(\mathbf{n}))} + \frac{1}{l(\mathbf{n})!} + \sum_{j=\max(d(\mathbf{n})+1,2)}^{l(\mathbf{n})} \frac{(-1)^{l(\mathbf{n}^{\geq j})}\,[\mathrm{Dem}^\bullet]^{\mathbf{n}^{\geq j}}_{(\times l(\mathbf{n}^{\geq j}))}}{l(\mathbf{n}^{<j})!\,l(\mathbf{n}^{\geq j})!}.
\end{aligned}$$

Now we can compute $\mathrm{Sem}^\bullet$.

$$\begin{aligned}
\mathrm{Sem}^{\mathbf{n}} &= \left(\mathrm{e}^{\Delta}\left(\mathrm{Exp}(\mathrm{Dem}^\bullet)\right) \cdot \mathrm{Exp}(\mathrm{I}^\bullet) \cdot \mathrm{Exp}(-\mathrm{Dem}^\bullet)\right)^{\mathbf{n}} \\
&= \sum_{\mathbf{n}^1\mathbf{n}^2=\mathbf{n}} \left(\mathrm{e}^{\Delta}\mathrm{Exp}(\mathrm{Dem}^\bullet)\right)^{\mathbf{n}^1} \left(\frac{(-1)^{l(\mathbf{n}^2)}}{l(\mathbf{n}^2)!}\,[\mathrm{Dem}^\bullet]^{\mathbf{n}^2}_{(\times l(\mathbf{n}^2))} + \frac{1}{l(\mathbf{n}^2)!} \right. \\
&\qquad\qquad \left. + \sum_{j=\max(d(\mathbf{n}^2)+1,2)}^{l(\mathbf{n}^2)} \frac{(-1)^{l((\mathbf{n}^2)^{\geq j})}\,[\mathrm{Dem}^\bullet]^{(\mathbf{n}^2)^{\geq j}}_{(\times l((\mathbf{n}^2)^{\geq j}))}}{l((\mathbf{n}^2)^{<j})!\,l((\mathbf{n}^2)^{\geq j})!} \right), \\
&= \sum_{\mathbf{n}^1\mathbf{n}^2=\mathbf{n}} e^{-\langle \lambda, \|\mathbf{n}^1\|\rangle} \left(1^{\mathbf{n}^1} + \frac{1}{l(\mathbf{n}^1)!}\,[\mathrm{Dem}^\bullet]^{\mathbf{n}^1}_{(\times l(\mathbf{n}^1))} \right) \times \\
&\quad \left(\frac{(-1)^{l(\mathbf{n}^2)}}{l(\mathbf{n}^2)!}\,[\mathrm{Dem}^\bullet]^{\mathbf{n}^2}_{(\times l(\mathbf{n}^2))} + \frac{1}{l(\mathbf{n}^2)!} + \sum_{j=\max(d(\mathbf{n}^2)+1,2)}^{l(\mathbf{n}^2)} \frac{(-1)^{l((\mathbf{n}^2)^{\geq j})}\,[\mathrm{Dem}^\bullet]^{(\mathbf{n}^2)^{\geq j}}_{(\times l((\mathbf{n}^2)^{\geq j}))}}{l((\mathbf{n}^2)^{<j})!\,l((\mathbf{n}^2)^{\geq j})!} \right), \\
&= \frac{(-1)^{l(\mathbf{n})}}{l(\mathbf{n})!}\,[\mathrm{Dem}^\bullet]^{\mathbf{n}}_{(\times l(\mathbf{n}))} + \frac{1}{l(\mathbf{n})!} + \sum_{j=\max(d(\mathbf{n})+1,2)}^{l(\mathbf{n})} \frac{(-1)^{l(\mathbf{n}^{\geq j})}\,[\mathrm{Dem}^\bullet]^{\mathbf{n}^{\geq j}}_{(\times l(\mathbf{n}^{\geq j}))}}{l(\mathbf{n}^{<j})!\,l(\mathbf{n}^{\geq j})!} \\
&\quad + e^{-\langle \lambda, \|\mathbf{n}\|\rangle} \left(1^{\mathbf{n}} + \frac{1}{l(\mathbf{n})!}[\mathrm{Dem}^\bullet]^{\mathbf{n}}_{(\times l(\mathbf{n}))} \right) + \sum_{i=1}^{\min(q(\mathbf{n})l(\mathbf{n})-1)} \frac{e^{-\lambda.\|\mathbf{n}^{\leq i}\|}}{l(\mathbf{n}^{\leq i})!}\,[\mathrm{Dem}^\bullet]^{\mathbf{n}^{\leq i}}_{(\times l(\mathbf{n}^{\leq i}))} \times \\
&\quad \left(\frac{(-1)^{l(\mathbf{n}^{>i})}}{l(\mathbf{n}^{>i})!}\,[\mathrm{Dem}^\bullet]^{\mathbf{n}^{>i}}_{(\times l(\mathbf{n}^{>i}))} + \frac{1}{l(\mathbf{n}^{>i})!} + \sum_{j=\max(d(\mathbf{n}^{>i})+1,2)}^{l(\mathbf{n}^{>i})} \frac{(-1)^{l(\mathbf{n}^{>i})^{\geq j}}\,[\mathrm{Dem}^\bullet]^{(\mathbf{n}^{>i})^{\geq j}}_{(\times l((\mathbf{n}^{>i})^{\geq j}))}}{l((\mathbf{n}^{>i})^{<j})!\,l((\mathbf{n}^{>i})^{\geq j})!} \right).
\end{aligned}$$

This conclues the proof. □

4.2. The Trimmed form. — The Trimmed form is constructed by induction applying successively the previous simplification scheme to remove non-resonant terms of higher and higher degrees, and hence it will have non-trivial values only on resonant words. The mould formalism allows us to explicit some particular moulds underlying this construction as well as algorithmic and explicit formulae for some of them.

4.2.1. The Trimmed form up to order r. — We can use the simplification procedure previously defined inductively in order to cancel non-resonant terms of higher and higher degrees.

***Definition 9* (Trimmed form up to order r).** — *Given $r \in \mathbb{N}$, the Trimmed form up to order r is defined as $\mathrm{F}^r_{\mathrm{Sem}}$ obtained from F after r successive simplifications, i.e.,*

$$F = \mathrm{F}^0_{\mathrm{Sem}} \overset{\mathrm{Simp}^1}{\to} \mathrm{F}^1_{\mathrm{Sem}} \overset{\mathrm{Simp}^2}{\to} \cdots \overset{\mathrm{Simp}^r}{\to} \mathrm{F}^r_{\mathrm{Sem}}, \tag{4.16}$$

where Simp^i is the automorphism of simplification defined by

$$\mathrm{Simp}^i = \exp(\mathbf{V}_i), \tag{4.17}$$

with $\mathbf{V}_i$ the vector fields associated to the mould $\mathrm{Dem}^\bullet$ on the alphabet $\mathcal{A}(\mathrm{F}^{i-1}_{\mathrm{Sem}})$ associated to $\mathrm{F}^{i-1}_{\mathrm{Sem}}$.

Using Theorem 2, we deduce the following useful result:

Theorem 4. — *For all $r \in \mathbb{N}$, the Trimmed form up to order r denoted $\mathrm{F}^r_{\mathrm{Sem}}$ possesses a mould expansion, i.e., there exists moulds denoted by ${}_r\mathrm{Sem}^\bullet \in \mathcal{M}_{\mathbb{C}}(\mathcal{A}(F))$ and ${}_r\mathrm{sem}^\bullet \in \mathcal{M}_{\mathbb{C}}(A(F))$ such that*

$$\mathrm{F}^r_{\mathrm{sem}} = \mathrm{F}_{\mathrm{lin}} \left(\sum_\bullet {}_r\mathrm{Sem}^\bullet \mathrm{D}_\bullet \right) = \mathrm{F}_{\mathrm{lin}} \left(\sum_\bullet {}_r\mathrm{sem}^\bullet \mathrm{B}_\bullet \right). \tag{4.18}$$

Despite its moulds expansion, the Trimmed form up to order r is *not* a prenormal form since it can have non-resonant terms for words of length $l \geq r+1$.

4.2.2. The moulds ${}_r\mathrm{sem}^\bullet$ and ${}_r\mathrm{Sem}^\bullet$. — The mould ${}_r\mathrm{sem}^\bullet$ has a simple expression in function of $\mathrm{sem}^\bullet$.

Lemma 7. — *For all $r \in \mathbb{N}$, we have*

$${}_r\mathrm{sem}^\bullet = \underbrace{\mathrm{sem}^\bullet \circ \cdots \circ \mathrm{sem}^\bullet}_{r \text{ times}}. \tag{4.19}$$

Proof. — The simplification procedure can be written as follows:

$$\sum_\bullet \mathrm{I}^\bullet \mathrm{B}_\bullet \longmapsto \sum_\bullet \mathrm{sem}^\bullet \mathrm{B}_\bullet. \tag{4.20}$$

Iterating this mapping we go from step i to $i+1$

$$\sum_\bullet {}_i\mathrm{sem}^\bullet \mathrm{B}_\bullet = \sum_\bullet \mathrm{I}^\bullet{}_{i+1}\mathrm{B}_\bullet \longmapsto \sum_\bullet {}_{i+1}\mathrm{sem}^\bullet \mathrm{B}_\bullet = \sum_\bullet \mathrm{sem}^\bullet{}_{i+1}\mathrm{B}_\bullet, \tag{4.21}$$

where $\sum_\bullet \mathrm{I}^\bullet{}_{i+1}\mathrm{B}_\bullet$ denotes the homogeneous decomposition constructed on $\mathrm{F}^i_{\mathrm{sem}}$.

By definition of the composition for moulds we have

$$\sum_\bullet \mathrm{sem}^\bullet{}_{i+1}\mathrm{B}_\bullet = \sum_\bullet \left(\mathrm{sem}^\bullet \circ {}_i\mathrm{sem}^\bullet\right) \mathrm{B}_\bullet, \tag{4.22}$$

from which we deduce the recursive relation

$${}_{i+1}\mathrm{sem}^\bullet = \mathrm{sem}^\bullet \circ {}_i\mathrm{sem}^\bullet. \tag{4.23}$$

We conclude by induction on i. □

For the mould ${}_r\mathrm{Sem}^\bullet$ we have a more complicated formula:

Lemma 8. — *For all $r \in \mathbb{N}$, we have*

$$\mathrm{Log}[{}_r\mathrm{Sem}^\bullet_0] = \underbrace{\mathrm{Log}(\mathrm{Sem}^\bullet_0) \circ \cdots \circ \mathrm{Log}(\mathrm{Sem}^\bullet_0)}_{r \text{ times}}, \tag{4.24}$$

where we set $\mathrm{Sem}^\bullet_0 := \mathrm{Sem}^\bullet - 1^\bullet$.

The fact that we must take the Log of $\mathrm{Sem}^\bullet_0$ instead of $\mathrm{Sem}^\bullet_0$ is related to the fact that the alphabet of derivation ${}_{i+1}\mathrm{D}_\bullet$ constructed at step i from $\mathrm{F}^i_{\mathrm{sem}}$ is not related to $\sum_\bullet {}_i\mathrm{Sem}^\bullet_0\mathrm{D}_\bullet$ but to its logarithm.

Proof. — The simplification procedure can be written as follows:

$$\exp\left(\sum_\bullet \mathrm{I}^\bullet\mathrm{D}_\bullet\right) \longmapsto \sum_\bullet \mathrm{Sem}^\bullet_0\mathrm{D}_\bullet = \exp\left(\sum_\bullet \mathrm{Log}(\mathrm{Sem}^\bullet_0)\mathrm{D}_\bullet\right). \tag{4.25}$$

Iterating this mapping we go from step i to $i+1$

$$\begin{array}{c} \exp\left(\sum_\bullet \mathrm{Log}[{}_i\mathrm{Sem}^\bullet_0]\mathrm{D}_\bullet\right) = \exp\left(\sum_\bullet \mathrm{I}^\bullet{}_{i+1}\mathrm{D}_\bullet\right) \\ \downarrow \\ \exp\left(\sum_\bullet \mathrm{Log}[{}_{i+1}\mathrm{Sem}^\bullet_0]\mathrm{D}_\bullet\right) = \exp\left(\sum_\bullet \mathrm{Log}(\mathrm{Sem}^\bullet_0){}_{i+1}\mathrm{D}_\bullet\right), \end{array} \tag{4.26}$$

where $\sum_\bullet \mathrm{I}^\bullet{}_{i+1}\mathrm{D}_\bullet$ denotes the homogeneous decomposition constructed on $\sum_\bullet \mathrm{Log}[{}_i\mathrm{Sem}^\bullet_0]\mathrm{D}_\bullet$.

By definition of the composition of moulds, we deduce that

$$\mathrm{Log}[{}_{i+1}\mathrm{Sem}^\bullet_0] = \mathrm{Log}(\mathrm{Sem}^\bullet_0) \circ \mathrm{Log}[{}_i\mathrm{Sem}^\bullet_0]. \tag{4.27}$$

We conclude the proof by induction on i. □

4.2.3. The Trimmed form. —

Definition 10. — *The Trimmed form of F is the limit of the simplification procedure.*

Theorem 5. — *The Trimmed form is a continuous prenormal form given by*

$$\begin{aligned} \mathrm{F}_{\mathrm{Trem}} &= \mathrm{F}_{\mathrm{lin}}\left(\sum_{\mathbf{m}\in\mathcal{A}(F)^*}\mathrm{Trem}^{\mathbf{m}}D_{\mathbf{m}}\right), \\ &= \mathrm{F}_{\mathrm{lin}}\left(\sum_{\mathbf{n}\in A(F)^*}\mathrm{trem}^{\mathbf{n}}B_{\mathbf{n}}\right) \end{aligned} \tag{4.28}$$

with the moulds $\mathrm{Trem}^\bullet$ *and* $\mathrm{trem}^\bullet$ *defined by*

$$\begin{aligned} \mathrm{Trem}^\bullet - 1^\bullet := \mathrm{Trem}_0 = \mathrm{limstat}_{r\to\infty}\,[\mathrm{Sem}_0^\bullet]^{(\circ r)}, \\ \mathrm{trem}^\bullet - 1^\bullet := \mathrm{trem}_0 = \mathrm{limstat}_{r\to\infty}\,[\mathrm{sem}_0^\bullet]^{(\circ r)}, \end{aligned} \tag{4.29}$$

where $\mathrm{sem}_0^\bullet := \mathrm{sem}^\bullet - 1^\bullet$*, and* limstat *is the stationary limit (see* [**9**]*).*

The proof is a direct consequence of the simplification procedure.

Remark 1. — *Following (*[**8**] *§.7) we have divergence and resurgence of the simplification procedure. This is not the case when working directly with the diffeomorphism instead of its associated automorphism of substitution. However, this problem can be avoided (see* [**8**] *p.8).*

4.2.4. The mould $\mathrm{Trem}^\bullet$. — We can compute the mould $\mathrm{Trem}^\bullet$ using a simple remark. By definition, we have the following identities

$$\mathrm{Trem}_0^\bullet = \mathrm{Sem}_0^\bullet \circ \mathrm{Trem}_0^\bullet, \tag{4.30}$$

$$\mathrm{Trem}_0^\bullet = \mathrm{Trem}_0^\bullet \circ \mathrm{Sem}_0^\bullet. \tag{4.31}$$

Using the first equation and the definition of composition for moulds we obtain for all $\mathbf{m}\in\mathcal{A}(F)^*$

$$\mathrm{Trem}^{\mathbf{m}} = \mathrm{Sem}^{\|\mathbf{m}\|}\mathrm{Trem}^{\mathbf{m}} + \text{s.l.}, \tag{4.32}$$

where s.l. denotes terms which depend on $\mathrm{Trem}^\bullet$ for words with a length strictly shorter than $l(\mathbf{m})$.

By construction, the mould $\mathrm{Trem}^\bullet$ takes non-trivial values only on resonant words, *i.e.*, $\mathbf{m}\in\mathcal{A}(F)^*$ such that $\langle\|\mathbf{m}\|,\lambda\rangle\in 2\pi i\mathbb{Z}$. However, the mould $\mathrm{Sem}^\bullet$ is equal to 1 on resonant words of length 1. As a consequence, equation (4.32) cannot be used to compute the mould $\mathrm{Trem}^\bullet$ by induction on the length of words.

On the other hand, using equation (4.31), and the definition of composition for moulds, we obtain

$$\mathrm{Trem}^{\mathbf{m}} = \mathrm{Trem}^{\|\mathbf{m}\|}\mathrm{Sem}^{m_1}\dots\mathrm{Sem}^{m_r} + \mathrm{s.l} = \mathrm{Trem}^{\|\mathbf{m}\|} + \mathrm{s.l}, \tag{4.33}$$

and hence we can compute the mould $\mathrm{Trem}^{\bullet}$ by induction on the length of words.

4.3. About Écalle-Vallet results. — All our computations have been done in $\mathcal{D}_{\mathcal{A}(F)} := \{\mathrm{D}_{\mathbf{n}}\}_{\mathbf{n}\in\mathcal{A}(F)}$, that is for the mould $\mathrm{D}_{\bullet}$ with the alphabet $\mathcal{A}(F)$, whereas J. Écalle and B. Vallet used $\mathcal{B}_{A(F)} := \{\mathrm{B}_{\mathbf{n}}\}_{\mathbf{n}\in A(F)}$ to formulate their results in [**8**]. In order to compare our approach, we first give a simple formula connecting the two alphabets $\mathcal{D}_{\mathcal{A}(F)}$ and $\mathcal{B}_{A(F)}$. We then discuss some of the differences between the moulds $\mathrm{dem}^{\bullet}$, $\mathrm{sem}^{\bullet}$ and $\mathrm{trem}^{\bullet}$ with the moulds $\mathrm{Dem}^{\bullet}$, $\mathrm{Sem}^{\bullet}$, and $\mathrm{Trem}^{\bullet}$, showing that these moulds, except for the mould $\mathrm{dem}^{\bullet}$, can be expressed via closed formulae.

4.3.1. Relation between the alphabets $\mathcal{B}_{A(F)}$ and $\mathcal{D}_{\mathcal{A}(F)}$. — By definition, we have the identity

$$1 + \sum_{n\in A(F)} B_n = \exp\left(\sum_{m\in\mathcal{A}(F)} D_m\right). \tag{4.34}$$

Using the logarithm, we obtain

$$\log\left(1 + \sum_{n\in A(F)} B_n\right) = \sum_{m\in\mathcal{A}(F)} D_m. \tag{4.35}$$

As $\sum_{n\in A(F)} B_n = \sum\limits_{\mathbf{n}\in A^*(F)} \mathrm{I}^{\mathbf{n}} B_{\mathbf{n}}$, we have

$$\sum_{\mathbf{n}\in A^*(F)} (\mathrm{Log}\mathrm{I}^{\bullet})^{\mathbf{n}} B_{\mathbf{n}} = \sum_{m\in\mathcal{A}(F)} D_m. \tag{4.36}$$

We deduce the following relation between $\mathcal{D}_{\mathcal{A}(F)}$ and $\mathcal{B}_{A(F)}$:

Lemma 9. — *For all $D_m \in \mathcal{D}_{\mathcal{A}(F)}$, we have*

$$D_m = \sum_{\mathbf{n}\in A(F)^*,\ \|\mathbf{n}\|=m} (\mathrm{Log}\mathrm{I}^{\bullet})^{\mathbf{n}} B_{\mathbf{n}}. \tag{4.37}$$

The proof is based on the fact that a differential operator $B_{\mathbf{n}}$ is of order $\|\mathbf{n}\|$.

4.3.2. The mould dem$^\bullet$. — By definition, we have the identity

$$\sum_{\mathbf{n}\in A(F)^*} \mathrm{dem}^{\mathbf{n}} B_{\mathbf{n}} = \sum_{m\in\mathcal{A}(F)\setminus\mathcal{R}_{\mathcal{A}(F)}} \frac{D_m}{1-e^{\langle m,\lambda\rangle}}. \tag{4.38}$$

Using Lemma 9, we deduce:

Lemma 10. — *The mould* dem$^\bullet$ *of* $\mathcal{M}_{\mathbb{C}}(A(F))$ *is defined for all* $\mathbf{n}\in A(F)^*$ *by*

$$\mathrm{dem}^{\mathbf{n}} = \frac{(-1)^{l(\mathbf{n})+1}}{l(\mathbf{n})} \frac{1}{1-e^{\langle\|\mathbf{n}\|,\lambda\rangle}} \left[\mathrm{I}^\bullet\right]^{\mathbf{n}}_{(\times l(\mathbf{n}))} \mathbf{1}_{N(F)}(\mathbf{n}), \tag{4.39}$$

where $N(F) = \{\mathbf{n}\in A(F)^*, \langle\|\mathbf{n}\|,\lambda\rangle \notin 2\pi i\mathbb{Z}\}$ *is the set of non-resonant words of* $A(F)^*$ *and* $\mathbf{1}_J$ *is the indicatrix of the set* J, *i.e.,* $\mathbf{1}_J(x)$ *is equal to* 1 *if* $x\in J$, 0 *otherwise.*

This mould was defined by J. Écalle and B. Vallet (see [**8**], p.30).

Proof. — Equation (4.38) can be rewritten as

$$\sum_{\mathbf{n}\in A(F)^*} \mathrm{dem}^{\mathbf{n}} B_{\mathbf{n}} = \sum_{m\in\mathcal{A}(F)} \frac{D_m}{1-e^{\langle m,\lambda\rangle}} \mathbf{1}_{\{\langle m,\lambda\rangle\notin 2\pi i\mathbb{Z}\}}(m). \tag{4.40}$$

Using Lemma 9, we have

$$\begin{aligned}\sum_{m\in\mathcal{A}(F)} \frac{D_m}{1-e^{\langle m,\lambda\rangle}} \mathbf{1}_{\{\langle m,\lambda\rangle\notin 2\pi i\mathbb{Z}\}}(m) &= \sum_{m\in\mathcal{A}(F)} \sum_{\mathbf{n}\in A(F)^*,\ \|\mathbf{n}\|=m} \frac{(\mathrm{LogI}^\bullet)^{\mathbf{n}}}{1-e^{\langle m,\lambda\rangle}} \mathbf{1}_{\{\langle m,\lambda\rangle\notin 2\pi i\mathbb{Z}\}} B_{\mathbf{n}},\\ &= \sum_{\mathbf{n}\in A(F)^*} \frac{(\mathrm{LogI}^\bullet)^{\mathbf{n}}}{1-e^{\langle\|\mathbf{n}\|,\lambda\rangle}} \mathbf{1}_{N(F)} B_{\mathbf{n}},\end{aligned} \tag{4.41}$$

using the fact that

$$\bigcup_{m\in\mathcal{A}(F)} \{\mathbf{n}\in A(F)^*,\ \|\mathbf{n}\|=m\} = A(F)^*, \tag{4.42}$$

by assumption.

Using Lemma 3 for the mould $\mathrm{I}^\bullet$, we obtain for all $\mathbf{n}\in A(F)^*$

$$\mathrm{LogI}^{\mathbf{n}} = \frac{(-1)^{l(\mathbf{n})+1}}{l(\mathbf{n})} \left[\mathrm{I}^\bullet\right]^{\mathbf{n}}_{(\times l(\mathbf{n}))}. \tag{4.43}$$

Replacing $\mathrm{LogI}^\bullet$ by its expression in equation (4.41) we conclude the proof. □

5. The Poincaré-Dulac normal form

The Trimmed form is constructed using cancellation of non-resonant terms as the classical Poincaré-Dulac normal form. However, these two prenormal forms do not coincide in general. We introduce the universal mould associated to the Poincaré-Dulac normal form and the universal mould of the associated cancellation procedure. The difference between the two procedures lies in the treatment of the homogeneous components of the diffeomorphism. For a classical approach to the Poincaré-Dulac normal form we refer to ([**1**] §.B p.178).

5.1. Homogeneous components and the Trimmed form. — We keep the notations introduced in §.4.1. In order to discuss the cancellation of non-resonant terms, we must write our prepared form as follows:

$$\mathrm{Id} + \sum_{n\in A(F)} B_n = \exp \mathbf{D} = \exp\left(\sum_{m\in\mathcal{A}(F)} D_m\right) = \exp\left(\sum_{k\geq 1} \mathbf{D}_k\right), \tag{5.1}$$

where

$$\mathbf{D}_k = \sum_{n\in\mathcal{A}(F),\ |n|=k} D_m, \tag{5.2}$$

denotes the homogeneous component of degree k of the vector field $\mathbf{D}$.

For a given vector field $\mathbf{D}$ we introduce the following *degree of resonance*, denoted by K:

$$\mathrm{K} = \min_{k\geq 1} \{\mathrm{N}_k \neq \emptyset\}, \tag{5.3}$$

where N_k denotes the set of non-resonant letters $m \in \mathcal{A}(F)$ of degree k, *i.e.,*

$$\mathrm{N}_k = \{m \in \mathcal{A}(F) \mid |m| = k, \langle m, \lambda\rangle \in 2\pi i\mathbb{Z}\}. \tag{5.4}$$

So, if we write

$$\mathbf{D} = \sum_{1\leq k<K} \mathbf{D}_k + \mathbf{D}_K + \sum_{k>K} \mathbf{D}_k, \tag{5.5}$$

the first sum up to order $K-1$ is made of resonant terms. The first non-resonant terms belong to $\mathbf{D}_K$.

The field $\mathbf{V}$ introduced in §.4.1.2 cancels the non-resonant terms of degree K but it introduces several other terms in the homogeneous components of degree $>$ K which can be non-resonant. As a consequence, even if the field $\mathbf{V}$ is constructed in order to cancel *all* the non-resonant terms of the vector field $\mathbf{D}$, we have an effective cancellation only for the components of degree K.

As a consequence, the vector field $\mathbf{V}$ must be modified in order to cancel *only* non-resonant terms of degree K.

Definition 11. — *Let* $\mathbf{S}$ *be the vector field defined by the mould*

$$\mathrm{Den}^\bullet = \begin{cases} \dfrac{1}{1-e^{\langle m,\lambda\rangle}} & \text{for} \quad m \in \mathrm{N_K}(F), \\ 0 \quad \text{otherwise,} \end{cases} \tag{5.6}$$

We denote by $\mathrm{den}^\bullet$ *the associated mould on* $\mathcal{M}_{\mathbb{C}}(A(F))$, *i.e.,*

$$\mathbf{S} = \sum_\bullet \mathrm{Den}^\bullet \mathrm{D}_\bullet = \sum_\bullet \mathrm{den}^\bullet \mathrm{B}_\bullet. \tag{5.7}$$

We call Poincaré form of F the automorphism, denoted by $\mathrm{F_{Poin}}$, *obtained from F under the action of* $\exp \mathbf{S}$.

Arguing exactly as in the proof of Theorem 2, we then have the following result.

***Theorem 6* (Poincaré normalization procedure)**. — *Let* $\mathrm{F_{Poin}}$ *be the Poincaré form of F. Then we have*

$$\begin{aligned} \mathrm{F_{Poin}} &= \mathrm{F_{lin}} \left(\sum_{\mathbf{m}\in\mathcal{A}(F)^*} \mathrm{Poin}^{\mathbf{m}} D_{\mathbf{m}} \right), \\ &= \mathrm{F_{lin}} \left(\sum_{\mathbf{n}\in A(F)^*} \mathrm{poin}^{\mathbf{n}} B_{\mathbf{n}} \right) \end{aligned} \tag{5.8}$$

where the mould $\mathrm{Poin}^\bullet$ *is given by*

$$\mathrm{Poin}^\bullet = \mathrm{e}^{\Delta}\left(\mathrm{Exp}(\mathrm{Den}^\bullet)\right) \cdot \mathrm{Exp}(\mathrm{I}^\bullet) \cdot \mathrm{Exp}(-\mathrm{Den}^\bullet), \tag{5.9}$$

and the mould $\mathrm{poin}^\bullet$ *is given by*

$$\mathrm{poin}^\bullet = \mathrm{e}^{\Delta}\left(\mathrm{Exp}(\mathrm{den}^\bullet)\right) \cdot (1^\bullet + \mathrm{I}^\bullet) \cdot \mathrm{Exp}(-\mathrm{den}^\bullet). \tag{5.10}$$

5.2. The Poincaré normal form of order r. — We apply the Poincaré normalization procedure inductively in order to cancel non-resonant terms in homogeneous components of higher and higher degrees.

Definition 12 **(Poincaré normal form up to order r)**. — *Let $r \in \mathbb{N}$, the Poincaré normal form up to order r is defined as $\mathrm{F}^r_{\mathrm{Poin}}$ obtained from F after r successive simplifications, i.e.,*

$$F = \mathrm{F}^0_{\mathrm{Poin}} \overset{\mathrm{Simp}^1}{\to} \mathrm{F}^1_{\mathrm{Poin}} \overset{\mathrm{Simp}^2}{\to} \dots \overset{\mathrm{Simp}^r}{\to} \mathrm{F}^r_{\mathrm{Poin}}, \tag{5.11}$$

where Simp^i is the automorphism of simplification defined by

$$\mathrm{Simp}^i = \exp(\mathbf{S}_i), \tag{5.12}$$

with $\mathbf{S}_i$ the vector fields associated to the mould $\mathrm{Den}^\bullet$ on the alphabet $\mathcal{A}(\mathrm{F}^{i-1}_{\mathrm{Poin}})$ associated to $\mathrm{F}^{i-1}_{\mathrm{Poin}}$.

Using Theorem 6, we obtain:

Theorem 7. — *For all $r \in \mathbb{N}$, the Poincaré normal form up to order r denoted $\mathrm{F}^r_{\mathrm{Poin}}$ possesses a mould expansion, i.e., there exist moulds denoted by ${}_r\mathrm{Poin}^\bullet \in \mathcal{M}_{\mathbb{C}}(\mathcal{A}(F))$ and ${}_r\mathrm{poin}^\bullet \in \mathcal{M}_{\mathbb{C}}(A(F))$ such that*

$$\mathrm{F}^r_{\mathrm{Poin}} = \mathrm{F}_{\mathrm{lin}} \left(\sum_\bullet {}_r\mathrm{Poin}^\bullet \mathrm{D}_\bullet \right) = \mathrm{F}_{\mathrm{lin}} \left(\sum_\bullet {}_r\mathrm{poin}^\bullet \mathrm{B}_\bullet \right). \tag{5.13}$$

As for the moulds ${}_r\mathrm{sem}^\bullet$ and ${}_r\mathrm{Sem}^\bullet$, we have explicit inductive formulae to compute the moulds ${}_r\mathrm{poin}^\bullet$ and ${}_r\mathrm{Poin}^\bullet$ using only $\mathrm{poin}^\bullet$ and $\mathrm{Poin}^\bullet$.

5.3. The Poincaré-Dulac normal form.

— The mould formulation of the Poincaré-Dulac normal form is:

Definition 13. — *The Poincaré-Dulac normal form of F is the limit of the Poincaré normalization procedure.*

Theorem 8. — *The Poincaré-Dulac normal form is a continuous prenormal form given by*

$$\begin{aligned} \mathrm{F}_{\mathrm{Dulac}} &= \mathrm{F}_{\mathrm{lin}} \left(\sum_{\mathbf{m} \in \mathcal{A}(F)^*} \mathrm{Dulac}^{\mathbf{m}} D_{\mathbf{m}} \right), \\ &= \mathrm{F}_{\mathrm{lin}} \left(\sum_{\mathbf{n} \in A(F)^*} \mathrm{dulac}^{\mathbf{n}} B_{\mathbf{n}} \right) \end{aligned} \tag{5.14}$$

with the moulds $\mathrm{Dulac}^\bullet$ and $\mathrm{dulac}^\bullet$ defined by

$$\begin{aligned} \mathrm{Dulac}^\bullet - 1^\bullet &= \mathrm{limstat}_{r \to \infty} \left[\mathrm{Poin}^\bullet - 1^\bullet \right]^{(\circ r)}, \\ \mathrm{dulac}^\bullet - 1^\bullet &= \mathrm{limstat}_{r \to \infty} \left[\mathrm{poin}^\bullet - 1^\bullet \right]^{(\circ r)}, \end{aligned} \tag{5.15}$$

where limstat *is the stationary limit.*

The mould $\mathrm{Dulac}^\bullet$ (or $\mathrm{dulac}^\bullet$) is the *universal* part of the Poincaré-Dulac normal form as it does not depends on the exact values of the coefficients coming in the Taylor expansion of the diffeomorphism. It seems very difficult to characterize such kind of objects without using the mould formalism.

References

[1] V.I. Arnold, *Chapitres supplémentaires de la théorie des équations différentielles ordinaires*, Ed. Librairie du Globe, Paris (1996).

[2] A. Baider, Unique normal forms for vector fields and Hamiltonian, J. Diff. Eq. **78** (1989), 33–52.

[3] J. Cresson, *Mould calculus and normalization of vector fields and diffeomorphisms*, Lectures at the University of Pisa, Prépublications de l'I.H.É.S. (2006), 27 pp.

[4] J. Cresson, *Calcul Moulien*, Ann. Fac. Sci. Toulouse Math. **18** (2009), 307-395.

[5] J. Écalle, *Les fonctions résurgentes*, Publ. Math. d'Orsay [Vol. 1: 81–05, Vol. 2: 81–06, Vol. 3: 85–05] 1981, 1985.

[6] J. Écalle, Singularités non abordables par la géométrie, Ann. Inst. Fourier, **42**, 1992, 73–164.

[7] J. Écalle, D. Schlomiuk, The nilpotent and distinguished form of resonant vector fields or diffeomorphisms, Ann. Inst. Fourier **43** (1993) 1407–1483.

[8] J. Écalle, B. Vallet, Prenormalization, correction, and linearization of resonant vector fields or diffeomorphisms, Prepublication d'Orsay (1995), 101 pp.

[9] J. Écalle, B. Vallet, Correction and linearization of resonant vector fields and diffeomorphisms, Math. Z. **229** (1998) 249–318.

[10] P. W. Michor, *Topics in differential geometry.* Graduate Studies in Mathematics, **93**. American Mathematical Society, Providence, RI (2008).

Jacky CRESSON

Université de Pau et des Pays de l'Adour,
Laboratoire de Mathématiques appliquées de Pau, CNRS UMR 5142
jacky.cresson@univ-pau.fr

and

Institut de Mécanique Céleste et de Calcul des Éphémérides,
Observatoire de Paris, 77 avenue Denfert-Rochereau, 75014 Paris, France

Jasmin RAISSY
Università degli Studi di Milano Bicocca
Dipartimento di Matematica e Applicazioni
Via R. Cozzi, 53, 20125 Milano
jasmin.raissy@unimib.it

Jacky CRESSON
Jasmin RAISSY