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A local approach to holomorphic dynamics in higher dimension

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Contents

Présentation générale

Ce mémoire d'habilitation rend compte d'une large partie des recherches que j'ai effectuées après ma thèse en *dynamique discrète*, c'est-à-dire l'étude des itérés d'un endomorphisme $f\colon X \to X$, dans la catégorie *holomorphe*. J'ai organisé la lecture de mes travaux en trois parties correspondant à des thématiques distinctes et pouvant être lues indépendamment. Chacune présente un point de vue sur un sujet que j'ai exploré à travers plusieurs travaux. À la fin de ce texte je présenterai des commentaires et des questions ouvertes pour des recherches futures.

La première partie est consacrée à la dynamique dynamique holomorphe discrète, d'un point de vue local et à plusieurs variables. Plus précisément j'expliquerai ma contribution à la description de la dynamique locale des germes de biholomorphismes de $\mathbb{C}^n$ qui fixent l'origine, en distinguant le cas linárisable du cas non linéarisable.

Tout d'abord je présenterai brièvement l'état de l'art en dimension 1, puis je me concentrerai sur la dimension supérieure, où les *résonances* jouent un rôle important dans le problème de linéarisation. Je rappellerai donc le cadre de la normalisation formelle de Poincaré-Dulac et présenterai le travail [209] sur la linéarisation holomorphe en présence de résonances et le travail [210] sur la linéarisation holomorphe simultanée en présence de résonances.

Dans le deuxième chapitre, je me concentrerai sur la dynamique locale des germes tangents à l'identité en dimension supérieure. Je rappellerai les principaux résultats obtenus dans ce contexte, avant de me concentrer sur la dernière contribution que nous avons obtenue dans [174] pour les germes de biholomorphismes de $\mathbb{C}^2$ ayant une courbe formelle invariante. Nous avons travaillé sous l'hypothèse naturelle que la restriction du difféomorphisme à la courbe formelle invariante soit, ou bien hyperbolique attractive, ou bien rationnellement neutre et non périodique. J'expliquerai les idées principales de notre preuve de l'existence d'un nombre fini de variétés stables, qui sont soit des domaines ouverts soit des courbes paraboliques, contenant et consistant en toutes les orbites convergentes asymptotiquement à la courbe invariante formelle.

Le dernier chapitre de cette partie est consacré à la dynamique locale des germes résonants non linéarisables. L'intérêt pour la dynamique locale de tels germes s'est accru ces dernières années. Je présenterai les résultats obtenus dans [63] et dans [211] lorsque les résonances sont engendrées par un nombre fini de multi-indices. Dans un tel contexte, de manière générale, toute forme normale de germe de Poincaré-Dulac préserve un feuilletage singulier et y agit comme un germe tangent à l'identité. L'idée clé est donc d'utiliser les résultats sur la dynamique locale des germes tangents à l'identité afin d'obtenir des informations sur la dynamique locale des germes résonants considérés. Il s'avère que les résonances peuvent donner lieu à un comportement parabolique a priori inattendu, par exemple dans des situations de type elliptique.

Dans la deuxième partie je me concentrerai sur l'utilisation de techniques locales en dynamique holomorphe globale.

Je donnerai un compte rendu actualisé des résultats récents sur les composantes de Fatou pour les produits semi-directs polynomiaux de $\mathbb{C}^2$. L'*ensemble de Fatou* d'un endomorphisme holomorphe d'une variété complexe X est le sous-ensemble ouvert de X constitué par tous les points ayant un voisinage ouvert sur lequel les itérés de l'endomorphisme forment une famille normale. Une *composante de Fatou* est une composante connexe de l'ensemble Fatou. En dimension 1, le cas le plus important est celui où X est la sphère de Riemann. Dans ce cas, le Théorème du domaine non-errant de Sullivan, nous assure qu'il n'y a pas de composante de Fatou errante, c'est-à-dire que toutes les composantes de Fatou sont (pré-)périodiques. En dimension supérieure, la situation est plus compliquée et le cas particulier des produits semi-directs polynomiaux bidimensionnels s'est avéré être très riche et accessible avec des techniques locales. Je présenterai les résultats connus, ainsi que les étapes clés de la construction que nous avons obtenue dans [33] d'un produit semi-direct polynômial de $\mathbb{C}^2$ ayant une composante de Fatou errante. Je présenterai également les idées principales dans la description que nous avons obtenue dans [197] de la dynamique près d'une fibre invariante elliptique.

La seconde moitié de cette partie est consacrée aux composantes de Fatou pour les automorphismes holomorphes de $\mathbb{C}^k$. Je mentionnerai les résultats récents sur la classification des composantes périodiques. J'expliquerai ensuite la construction que nous avons obtenue dans [64] de la première famille d'automorphismes holomorphes de $\mathbb{C}^k$ ayants une composante de Fatou invariante, non récurrente, et qui *n'est pas* biholomorphe à $\mathbb{C}^k$. Une telle construction est basée sur les résultats locaux obtenus dans [63] et a comme corollaire l'existence d'un plongement de $\mathbb{C} \times (\mathbb{C}^*)^{k-1}$ en tant que domaine Runge dans $\mathbb{C}^k$. Ceci était une question ouverte de longue date, résolue donc positivement par notre construction.

La troisième partie a une saveur légèrement différente car elle est consacrée à la dynamique holomorphe sur les domaines convexes bornés de $\mathbb{C}^n$.

Je vais d'abord donner une courte introduction au Théorème de Wolff-Denjoy et ses généralisations à plusieurs variables complexes. En particulier, je me concentrerai sur la démonstration du Théorème de Wolff-Denjoy pour les domaines strictement convexes (qui ne sont pas nécessairement à bord lisse) que nous avons obtenue dans [17]. Avec des techniques similaires, nous sommes également capables de prouver un théorème de Wolff-Denjoy pour les domaines faiblement convexes, là encore sans aucune hypothèse de régularité pour la frontière du domaine.

Je me concentrerai ensuite sur l'étude des *orbites inverses* pour les endomorphismes holomorphes non inversibles des domaines bornés strictement convexes de $\mathbb{C}^n$ avec bord de classe C^2. Je commencerai par rappeler les résultats précédents dans le disque unité et dans la boule unité de $\mathbb{C}^n$, puis je me concentrerai sur la généralisation obtenue en [16]. Nous avons prouvé que les orbites inverses avec étape limitée par rapport à la distance de Kobayashi pour un endomorphisme hyperbolique ou fortement elliptique doivent nécessairement converger vers un point fixe isolé répulsif sur la frontière du domaine. En révisant le papier [16] pour la rédaction de ce manuscrit, j'ai trouvé une lacune dans une des démonstrations, que j'ai pu corriger comme je le montrerai dans ce chapitre.

Le dernier chapitre de cette partie sera ensuite consacré au Théorème de Julia-Wolff-Carathéodory sur les dérivées angulaires et à ses généralisations à plusieurs variables complexes. Je donnerai une brève introduction et je me concentrerai ensuite sur les résultats que nous avons obtenus dans [18] pour les générateurs infinitésimaux des semigroupes à un paramètre d'endomorphismes holomorphes de la boule unité de $\mathbb{C}^n$.

Certaines des questions que j'ai développées depuis ma thèse ne seront pas abordées dans ce mémoire, ou bien seulement indirectement, afin de maintenir une certaine homogénéité dans mon propos. Ainsi, le travail [26] ne sera que mentionné que brièvement dans les Sections 2.1 et 2.2, et de même pour le travail [15] mentionné dans la Section 1.2. D'autres travaux ne seront pas évoqués dans le texte, en particulier :

- le travail [87] sur une introduction au formalisme du calcul moulien d'Écalle, où nous avons révisé les résultats concernant la forme *trimmed* des germes de biholomorphismes de $\mathbb{C}^n$ à l'origine, et nous avons discuté une approche à travers le calcul moulien pour les formes normales de Poincaré-Dulac ;

- le travail [20] sur l'étude des propriétés des operateurs de Toeplitz associés à une mesure de Borel positive finie sur un domaine borné fortement pseudoconvexe $D \Subset \mathbb{C}^n$;
- le travail [19] sur une caractérisation à travers des produits de fonctions dans les espaces de Bergman à poids des mesures dites (λ, γ)-*skew Carleson*, avec $\lambda > 0$ et $\gamma > 1 - \frac{1}{n+1}$, sur un domaine borné fortement pseudoconvexe $D \Subset \mathbb{C}^n$ avec bord lisse.

Mes articles sont disponibles sur internet à l'adresse
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General presentation

In this dissertation, I will account for a large part of the research I did after my Ph.D. on *discrete dynamics*, that is the study of the iterates of a map $f\colon X \to X$ from a space to itself, in the *holomorphic* category. This report is composed of three main parts which correspond to distinct thematics and can be read independently. Each one presents a point of view on a topic I explored through several works. At the end of the manuscript I will present comments and open questions for future research.

Part I is devoted to local holomorphic discrete dynamics in several variables. More precisely I will explain my contribution to the description of the local dynamics of the iteration of germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin with respect to the dichotomy between linearizable and non linearizable germs.

I will first briefly present the state of the art in dimension 1, and then focus on the higher dimensional case, where an important rôle in the linearization problem is played by the so-called *resonances*. I will therefore recall the setting of Poincaré-Dulac formal normalization and present the works [209] on holomorphic linearization in presence of resonances and [210] on simultaneous holomorphic linearization in presence of resonances.

In the second chapter I will focus on the local dynamics of tangent to the identity germs in higher dimension. I will recall the main results in this setting, before focusing on the latest contribution that we obtained in [174], for germs of biholomorphisms in $\mathbb{C}^2$ with an invariant formal curve. We worked under the natural hypothesis that the restriction of the diffeomorphism to the formal invariant curve is either hyperbolic attracting or rationally neutral non-periodic. I will explain the main ideas in our proof of the existence of finitely many stable manifolds, that are either open domains or parabolic curves, consisting of and containing all converging orbits asymptotic to the formal invariant curve.

The final chapter of this part is devoted to the local dynamics of non-linearizable resonant germs. Interest in the local dynamics of such germs has grown in the last years. I will present the results we obtained in [63] and in [211] when the resonances are finitely generated. In such a setting, generically, any Poincaré-Dulac normal form of the germ preserves a singular foliation and acts on it as a tangent to the identity germ. The key idea is therefore to use results on the local dynamics of tangent to the identity germs to obtain information on the local dynamics of the considered resonant germs. It turns out that resonances can give rise to an a priori unexpected parabolic behaviour, for example in elliptic situations.

In Part II, I will focus on the use of local techniques in global holomorphic discrete dynamics in several variables.

I will give an updated account of the recent results on Fatou components for polynomial skew-products in $\mathbb{C}^2$. The *Fatou set* of a holomorphic endomorphism of a complex manifold X is the open subset of X consisting of all points having an open neighbourhood on which the iterates of the endomorphism form a normal family. A *Fatou component* is a connected component of the Fatou set. In dimension one, the most important case is when X is the Riemann sphere. In such case, Sullivan's non-wandering domains Theorem states that there are no wandering Fatou components, that is all Fatou components are preperiodic. In higher dimension, the situation is more complicated and the special case of two-dimensional polynomial skew-products turns out to be very rich and tractable using local techniques. I will present the known results, and the key steps in the construction we obtained in [33] of a polynomial skew-product having a wandering Fatou component. I will also present the main ideas in the description that we obtained in [197] of the dynamics near an elliptic invariant fiber.

The second half of this part is devoted to Fatou components for holomorphic automorphisms of $\mathbb{C}^k$. I will mention the recent results on the classification of periodic Fatou components for holomorphic automorphisms of $\mathbb{C}^k$. I will then explain the construction that we obtained [64] of the first family of holomorphic automorphisms of $\mathbb{C}^k$ with an invariant, non-recurrent, attracting Fatou component *not* biholomorphic to $\mathbb{C}^k$. Such construction is based on the local results we obtained in [63] and gives, as a corollary, the existence of an embedding of $\mathbb{C}\times(\mathbb{C}^*)^{k-1}$ as a Runge domain in $\mathbb{C}^k$. This was a long standing open question, positively settled by our construction.

Part III is of a slightly different flavour since it is devoted to holomorphic dynamics on bounded convex domains of $\mathbb{C}^n$.

I will first provide a short introduction to Wolff-Denjoy theorem, and its generalizations in several complex variables. In particular I will focus on the short proof of Wolff-Denjoy theorem for (not necessarily smooth) strictly convex domains that we obtained in [17]. With similar techniques we are also able to prove a Wolff-Denjoy theorem for weakly convex domains, again without any smoothness assumption on the boundary.

I will then focus on the study of *backward orbits* for non-invertible holomorphic self-maps of bounded strongly convex domains in $\mathbb{C}^n$ with C^2 boundary. I will start by recalling the previous results obtained in the unit disk and in the unit ball of $\mathbb{C}^n$ and then focus on the generalization we obtained in [16]. We proved that a backward orbit with bounded Kobayashi step for a hyperbolic or strongly elliptic holomorphic self-map of a bounded strongly convex C^2 domain in $\mathbb{C}^n$ necessarily converges to a repelling boundary fixed point. While checking our paper [16] for the writing of this manuscript, I found a gap in one of the proofs, that I have been able to fix, as I will show in this chapter.

The final chapter of this part will be then devoted to the Julia-Wolff-Carathéodory theorem on angular derivatives, and its generalizations in several complex variables. I will give a short introduction and I will then focus on the results for infinitesimal generators of one-parameter semigroups of holomorphic self-maps of the unit ball in $\mathbb{C}^n$ that we obtained in [18].

Some of the questions I have developed since my PhD will not be addressed in this text, or only indirectly, to maintain some homogeneity in the discussion. This is the case of the work [26] mentioned in Sections 2.1 and 2.2, and the work [15] mentioned in Section 1.2. Some other works will not be referred to in the text, in particular:

- the work [87] on a self-contained introduction to the mould formalism of Écalle, where we reviewed results about the trimmed form of local biholomorphisms of $\mathbb{C}^n$, and we discussed a mould approach to Poincaré-Dulac normal forms;
- the work [20] on the study of the mapping properties of Toeplitz operators associated to a finite positive Borel measure on a bounded strongly pseudoconvex domain $D \Subset \mathbb{C}^n$;
- the work [19] on a characterization through products of functions in weighted Bergman spaces of (λ,γ)-skew Carleson measures, with $\lambda > 0$ and $\gamma > 1 - \frac{1}{n+1}$, on a bounded strongly pseudoconvex domain D in $\mathbb{C}^n$ with smooth boundary.

My articles are available on the web at
`http://www.math.univ-toulouse.fr/~jraissy/papers/papers.html`

List of presented publications

Here is the list of the publications presented in this document.

[AR1] M. Abate, J. Raissy: *Backward iteration in strongly convex domains*, Adv. in Math., **228**, Issue 5, (2011), pp. 2837–2854.

[AR3] M. Abate, J. Raissy: *Wolff-Denjoy theorems in non-smooth convex domains*, Ann. Mat. Pura ed Appl. (4), Springer-Verlag, **193**, no. **5**, (2014), 1503–1518.

[AR4] M. Abate, J. Raissy: *A Julia-Wolff-Carathéodory theorem for infinitesimal generators in the unit ball*, Trans. Amer. Math. Soc. 368 (2016), no. 8, 5415–5431.

[ArR] M. Arizzi, J. Raissy: *On Écalle-Hakim's theorems in holomorphic dynamics*, in **Frontiers in Complex Dynamics: In celebration of John Milnor's 80th. birthday**, A. Bonifant, M. Lyubich, S. Sutherland editors, Princeton University Press, (2014), pp. 387–449.

[ABDPR] M. Astorg, X. Buff, R. Dujardin, H. Peters, J. Raissy: *A two-dimensional polynomial mapping with a wandering Fatou component*, Ann. of Math. (2) 184 (2016), no. 1, 263–313.

[BRZ] F. Bracci, J. Raissy, D. Zaitsev: *Dynamics of multi-resonant biholomorphisms*, Int. Math. Res. Notices, **23** (2013), no. 20, pp. 4772–4797.

[BRS] F. Bracci, J. Raissy, B. Stensønes: *Automorphisms of $\mathbb{C}^k$ with an invariant non-recurrent attracting Fatou component biholomorphic to $\mathbb{C} \times (\mathbb{C}^*)^{k-1}$*, Preprint 2017, 29 pages, arXiv:1703.08423.

[LRRS] L. López-Hernanz, J. Raissy, J. Ribón, F. Sanz-Sánchez: *Stable manifolds of two-dimensional biholomorphisms asymptotic to formal curves*, Preprint 2017, 30 pages, arXiv:1710.03728.

[PR] H. Peters, J. Raissy: *Fatou components of elliptic polynomial skew products*, Ergodic Theory and Dynamical Systems, (2017) https://doi.org/10.1017/etds.2017.112

[R1] J. Raissy: *Brjuno conditions for linearization in presence of resonances*, in **"Asymptotics in Dynamics, Geometry and PDE's; Generalized Borel Summation" vol. I**, O. Costin, F. Fauvet, F. Menous e D. Sauzin editori, "CRM series", Pisa, Edizioni Della Normale 2011, pp. 201–218.

[R2] J. Raissy: *Holomorphic linearization of commuting germs of holomorphic maps*, J. Geom. Anal. **23** (2013), no. 4, pp. 1993–2019.

[RV] J. Raissy, L. Vivas: *Dynamics of two-resonant biholomorphisms*, Math. Res. Lett., **20**, no. 4, (2013), 757–771.

Part I

Resonances, Normal Forms and Local Discrete Holomorphic Dynamics

Chapter 1

State of the art

In this chapter we give a brief account of the main results on the classification of normal forms and on the local dynamics of local holomorphic dynamical systems near an isolated fixed point. We refer to [78, 182] for details for the proofs in dimension 1 and to [10] for a more complete exposition of the state of the art.

Recall that, given $f\colon X \to X$ and $g\colon Y \to Y$ two dynamical systems, we say that f and g are *semi-conjugated* if there is a map $\varphi\colon X \to Y$ such that $\varphi \circ f = g \circ \varphi$. Semi-conjugacies map the sequences of iterates of f to the sequences of iterates of g. When φ is invertible, the equality $\varphi \circ f = g \circ \varphi$ can be rewritten as $f = \varphi^{-1} \circ g \circ \varphi$, and f and g are called *conjugated*. It is clear that the dynamics is invariant under conjugacy. When studying dynamical systems, we therefore constantly try to construct conjugacies (or at least semi-conjugacies) to simpler dynamical systems, usually called *in normal form*.

In the following we will denote by $\mathrm{End}(\mathbb{C}, 0)$ the set of germs of holomorphic functions of $\mathbb{C}$ fixing the origin, and by $\mathrm{End}(\mathbb{C}^n, O)$ the set of germs of holomorphic endomorphisms of $\mathbb{C}^n$ fixing the origin $O \in \mathbb{C}^n$.

1.1 Normal forms and local dynamics in dimension 1

In this section, we are interested in describing the *local dynamics*, that is the *asymptotic behaviour of the iterates* of a holomorphic function f defined *in a neighbourhood of a point p fixed by f*. Up to choosing local coordinates centered at the fixed point, we can locally write f as a power series without constant term converging in a neighbourhood of the origin:

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots .$$

Definition 1.1. *The number $\lambda = f'(0)$ is called the* multiplier of f at the fixed point.

The multiplier at a fixed point is invariant under analytic conjugacy.

A first useful fact is the following.

Proposition 1.2. *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic germ of function fixing the origin with multiplier $\lambda \in \mathbb{C}^*$. If λ is not a root of unity, then f is formally linearizable, that is there exists a unique formal power series φ of the form $\varphi(z) = z + \sum_{n\geq 2} b_n z^2$ such that $\varphi \circ L_\lambda = f \circ \varphi$ where $L_\lambda(z) = \lambda z$.*

Proof. It suffices to recursively find the coefficients b_k by solving the equation given by $\varphi(\lambda z) = f(\varphi(z))$ for the terms of degree k. In fact, if we assume to have determined the coefficients $b_2, \dots, b_{k-1}$, then the terms of degree k in $\varphi(\lambda z) = f(\varphi(z))$ give

$$\lambda^k b_k = \lambda b_k + P_k(a_2, \dots, a_k, b_2, \dots, b_{k-1}),$$

where P_k is a polynomial independent from λ. Since $\lambda^k \neq \lambda$ by hypothesis, we find the unique solution:

$$b_k = \frac{P_k(a_2, \dots, a_k, b_2, \dots, b_{k-1})}{\lambda^k - \lambda}. \tag{1.1}$$

□

The best linear approximation of f is λz, and we shall recall in this section how the local dynamics of f is strongly influenced by the value of λ. For this reason we introduce the following definition:

Definition 1.3. *Let $\lambda \in \mathbb{C}$ be the multiplier of $f \in \mathrm{End}(\mathbb{C}, 0)$. Then*

- *if $\lambda = 0$ we say that the fixed point 0 is* superattracting*,*
- *if $0 < |\lambda| < 1$ we say that the fixed point 0 is* attracting*,*
- *if $|\lambda| > 1$ we say that the fixed point 0 is* repelling*,*
- *if $|\lambda| \notin \{0, 1\}$ we say that the fixed point 0 is* hyperbolic*,*
- *if $\lambda \in \mathbb{S}^1$ is a root of unity, we say that the fixed point 0 is* parabolic *or* rationally indifferent*,*
- *if $\lambda \in \mathbb{S}^1$ is not a root of unity, we say that the fixed point 0 is* elliptic *or* irrationally indifferent*.*

We shall explain in the next subsections what is known on the normal forms and the local dynamics in the various cases.

1.1.1 Holomorphically normalizable germs

Hyperbolic case

We start with the hyperbolic case. A first observation is that if 0 is an attracting fixed point for $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ, then it is a repelling fixed point for the inverse map $f^{-1} \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ^{-1}.

It is also not difficult to find holomorphic and topological normal forms for one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point, as the following result shows.

Theorem 1.4 (Kœnigs, [161])**.** *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a one-dimensional holomorphic local dynamical system with a hyperbolic fixed point at the origin, and let $\lambda \in \mathbb{C}^* \setminus \mathbb{S}^1$ be its multiplier. Then:*

(i) *f is holomorphically (and hence formally) locally conjugated to its linear part $L_\lambda(z) = \lambda z$. The conjugacy φ is uniquely determined by the condition $\varphi'(0) = 1$.*

(ii) *Two such holomorphic local dynamical systems are holomorphically conjugated if and only if they have the same multiplier.*

(iii) *f is topologically locally conjugated to the map $g_<(z) = z/2$ if $|\lambda| < 1$, and to the map $g_>(z) = 2z$ if $|\lambda| > 1$.*

Idea of the proof. As we already remarked, it suffices to prove the statement in the attracting case, since if $|\lambda| > 1$ it will suffice to apply the same argument to f^{-1}.

(i) The sequence $\{\varphi_k\}$ defined by $\varphi_k = f^{\circ k}/\lambda^k$ converges locally uniformly in a neighbourhood of the origin to a holomorphic limit φ fixing the origin such that $\varphi'(0) = 1$ and $\varphi \circ f = \lambda\varphi$.

If ψ is another local holomorphic function such that $\psi'(0) = 1$ and $\psi^{-1} \circ L_\lambda \circ \psi = f$, it follows that $\psi \circ \varphi^{-1}(\lambda z) = \lambda\psi \circ \varphi^{-1}(z)$ and comparing the expansion in power series of both sides we find $\psi \circ \varphi^{-1} \equiv \mathrm{Id}$, that is $\psi \equiv \varphi$, and we are done.

(ii) Follows from the invariance of the multiplier under holomorphic local conjugacy.

(iii) Since $|\lambda| < 1$ it is easy to build a topological conjugacy between L_λ and $g_<$ on a disk $\mathbb{D}_\varepsilon$ of radius $\varepsilon > 0$ centered at the origin. First we choose a homeomorphism χ between the annulus $\{|\lambda|\varepsilon \leq |z| \leq \varepsilon\}$ and the annulus $\{\varepsilon/2 \leq |z| \leq \varepsilon\}$ which is the identity on the outer circle and which is given by $\chi(z) = z/(2\lambda)$ on the inner circle. Now we extend χ by induction to a homeomorphism between the annuli $\{|\lambda|^k\varepsilon \leq |z| \leq |\lambda|^{k-1}\varepsilon\}$ and $\{\varepsilon/2^k \leq |z| \leq \varepsilon/2^{k-1}\}$ by prescribing $\chi(\lambda z) = \frac{1}{2}\chi(z)$. We finally obtain a homeomorphism χ of $\mathbb{D}_\varepsilon$ with itself, such that $g = \chi^{-1} \circ g_< \circ \chi$, by putting $\chi(0) = 0$. □

Superattracting case

The superattracting case can be treated similarly to the hyperbolic case. If the origin 0 is an isolated superattracting fixed point for $f \in \mathrm{End}(\mathbb{C}, 0)$, we can write

$$f(z) = a_r z^r + a_{r+1} z^{r+1} + \cdots$$

with $a_r \neq 0$.

Definition 1.5. *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ and let 0 be a superattracting point for f. The* order *(or* local degree*) of the superattracting point is the minimal number $r \geq 2$ such that the coefficient of z^r in the power series expansion of f is non-zero.*

Theorem 1.6 (Böttcher, [50])**.** *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a one-dimensional holomorphic local dynamical system with a superattracting fixed point at the origin, and let $r \geq 2$ be its order. Then:*

(i) *f is holomorphically (and hence formally) locally conjugated to the map $g(z) = z^r$, and the conjugacy is unique up to multiplication by a $(r-1)$-root of unity,*

(ii) *two such holomorphic local dynamical systems are holomorphically (or topologically) conjugated if and only if they have the same order.*

Idea of the proof. (i) Up to linearly conjugating f we may assume that $a_r = 1$. Let $u\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be the germ defined by

$$u(z) := \log\left(\frac{f(z)}{z^r}\right)$$

where $\log\colon (\mathbb{C}, 1) \to (\mathbb{C}, 0)$ is the inverse branch of $\exp\colon (\mathbb{C}, 0) \to (\mathbb{C}, 1)$ vanishing at 1. For $R > 0$ sufficiently large, let H_R be the left half-plane $\{\mathrm{Re}(Z) < -R\}$ and $F\colon H_R \to H_R$ be the map defined by

$$F(w) = rw + u \circ \exp(w) \quad \text{so that} \quad \exp \circ F = f \circ \exp\,.$$

For $k \geq 0$, set

$$\Phi_k(w) := \frac{F^{\circ k}(w)}{r^k}.$$

The sequence $\{\Phi_k\}$ converges uniformly on H_R to the holomorphic function $\Phi\colon H_R \to \mathbb{C}$ so that $\Phi \circ F = r\Phi$.

It then suffices to take $\varphi\colon \mathbb{D}(0, \mathrm{e}^{-R}) \to \mathbb{C}$ the map such that $\exp\big(\Phi(w)\big) = \phi(z)$. Then

$$\big[\phi(z)\big]^r = \exp\big(r\Phi(w)\big) = \exp \circ \Phi \circ F(w) = \phi \circ \exp\big(F(w)\big) = \phi \circ f(z).$$

The uniqueness up to multiplication by $(r-1)$-roots of unity is easily deduced by comparing the power series expansions at the origin.

(ii) Since the order is the number of preimages of points close to the origin, z^r and z^s are locally topologically conjugated if and only if $r = s$, and hence we have (ii). □

Parabolic case

A holomorphic local dynamical system with a parabolic fixed point is never locally conjugated to its linear part, not even topologically, unless it is of finite order.

Proposition 1.7. *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system with multiplier λ, and assume that $\lambda = e^{2i\pi p/q}$ is a (primitive) root of the unity of order q. Then f is holomorphically (or topologically or formally) locally conjugated to $L_\lambda(z) = \lambda z$ if and only if $f^q \equiv \mathrm{Id}$.*

Proof. If $\varphi^{-1} \circ f \circ \varphi(z) = e^{2\pi i p/q} z$, then $\varphi^{-1} \circ f^q \circ \varphi = \mathrm{Id}$, and hence $f^q = \mathrm{Id}$.

Conversely, assume that $f^q \equiv \mathrm{Id}$ and set

$$\varphi(z) = \frac{1}{q} \sum_{j=0}^{q-1} \frac{f^j(z)}{\lambda^j}.$$

Then it is easy to check that $\varphi'(0) = 1$ and $\varphi \circ f(z) = \lambda\varphi(z)$, and so f is holomorphically (and topologically and formally) locally conjugated to λz. □

We will recall in Subsection 1.1.2 what is known on the dynamics of holomorphic local dynamical system with a parabolic fixed point that are not of finite order.

Elliptic case

In the elliptic case, that is

$$f(z) = e^{2\pi i\theta} z + a_2 z^2 + \cdots , \tag{1.2}$$

with $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the local dynamics depends mostly on numerical properties of θ. The main question here is whether such a local dynamical system is holomorphically conjugated to its linear part. Let us introduce a bit of terminology.

Definition 1.8. *If a holomorphic dynamical system of the form* (1.2) *is* holomorphically linearizable*, that is if it is holomorphically locally conjugated to its linear part, the irrational rotation $z \mapsto e^{2\pi i\theta} z$, we shall say that* 0 *is a* Siegel point *for f. Otherwise, we shall say that it is a* Cremer point *for f.*

It turns out that for a full measure subset B of $\theta \in [0,1] \setminus \mathbb{Q}$ all holomorphic local dynamical systems of the form (1.2) are holomorphically linearizable. Conversely, the complement $[0,1] \setminus B$ is a G_δ-dense set, and for all $\theta \in [0,1] \setminus B$ the quadratic polynomial $z \mapsto z^2 + e^{2\pi i\theta} z$ is not holomorphically linearizable. This is the gist of the results due to Cremer, Siegel, Brjuno and Yoccoz we shall describe in this section.

The first important fact in this setting is that it is possible to give a topological characterization of holomorphically linearizable local dynamical systems.

Definition 1.9. *Let $f\colon (\mathbb{C},0)\to(\mathbb{C},0)$ be a holomorphic map such that $f'(0)=\lambda$ with $|\lambda|=1$. The dynamics of f is* stable near 0 *if there exist $R>r>0$ such that for all $n\geq 0$, the iterates $f^{\circ n}|_{\mathbb{D}_r}$ are defined with $f^{\circ n}(\mathbb{D}_r)\subset\mathbb{D}_R$, where $\mathbb{D}_r$, resp. $\mathbb{D}_R$, denotes the disk of radius r, resp. R, and center at the origin.*

Theorem 1.10. *Let $f\in\operatorname{End}(\mathbb{C},0)$ be a holomorphic local dynamical system with multiplier $\lambda\in\mathbb{S}^1$ at the origin. Then f is holomorphically linearizable if and only if it is topologically linearizable if and only if the dynamics of f is stable near 0.*

Proof. If f is holomorphically linearizable, then it is topologically linearizable, and the dynamics is stable since the preimages by ϕ of Euclidean disks centered at 0 are invariant by f.

If f is topologically linearizable (and $|\lambda|=1$) then the dynamics is stable.

If the dynamics of f in a neighborhood of 0 is stable, consider the averages $\phi_n\colon\mathbb{D}_r\to\mathbb{D}_R$ defined by

$$\phi_n{:=}\frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{\lambda^j}f^{\circ j}.$$

We see that $\phi'_n(0)=1$, and

$$\phi_n\circ f=\frac{\lambda}{n}\sum_{j=0}^{n-1}\frac{1}{\lambda^{j+1}}f^{\circ(j+1)}=\lambda\phi_n+\frac{\lambda}{n}\left(\frac{1}{\lambda^n}f^{\circ n}-\mathrm{Id}\right).$$

Since the functions ϕ_n take their values in $\mathbb{D}_R$, they form a normal family by Montel's Theorem, and we can extract a converging subsequence ϕ_{n_k}. We then have that

$$\frac{\lambda}{n_k}\left(\frac{1}{\lambda^{n_k}}f^{\circ n_k}-\mathrm{Id}\right)\underset{n_k\to+\infty}{\longrightarrow}0.$$

The limit $\phi{:=}\lim\phi_{n_k}$ is holomorphic non constant since $\phi'_{n_k}(0)=1$, and linearizes f since $\phi\circ f(z)=\lambda\phi(z)$ as soon as z and $f(z)$ belong to $\mathbb{D}_r$. □

Since λ is not a root of unity, Proposition 1.2 ensures us that in this case f is always *formally* linerarizable. The formal power series linearizing f is not converging if its coefficients grow too fast. Thus (1.1) links the radius of convergence of the formal linearization φ to the behavior of $\lambda^k-\lambda$: if the latter becomes too small, the series defining φ does not converge. This is known in this context as the *small denominators problem.*

It is then natural to introduce the following quantity:

$$\omega_\lambda(m)=\min_{2\leq k\leq m}|\lambda^k-\lambda|\,,$$

for $\lambda\in\mathbb{S}^1$ and $m\geq 2$. Clearly, λ is a root of unity if and only if $\omega_\lambda(m)=0$ for all m greater than or equal to some $m_0\geq 2$; furthermore,

$$\lim_{m\to+\infty}\omega_\lambda(m)=0$$

for all $\lambda\in\mathbb{S}^1$.

The first one to actually prove the existence of non-linearizable elliptic holomorphic local dynamical systems has been Cremer, in [85]. His more general result is the following.

Theorem 1.11 (Cremer, [86]). *Let $\lambda \in \mathbb{S}^1$ be such that*

$$\limsup_{m \to +\infty} \frac{1}{m} \log \frac{1}{\omega_\lambda(m)} = +\infty \; . \tag{1.3}$$

Then there exists $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ which is not holomorphically linearizable. Furthermore, the set of $\lambda \in \mathbb{S}^1$ satisfying (1.3) *contains a G_δ-dense set.*

On the other hand, Siegel in [243] gave a condition on the multiplier ensuring holomorphic linearizability.

Theorem 1.12 (Siegel, [243]). *Let $\lambda \in \mathbb{S}^1$ be such that there exists $\beta > 1$ and $\gamma > 0$ so that for all $m \geq 2$*

$$\frac{1}{\omega_\lambda(m)} \leq \gamma \, m^\beta . \tag{1.4}$$

Then all $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ are holomorphically linearizable. Furthermore, the set of $\lambda \in \mathbb{S}^1$ satisfying (1.4) *for some $\beta > 1$ and $\gamma > 0$ is of full Lebesgue measure in $\mathbb{S}^1$.*

Notice that when $\theta \in [0, 1) \setminus \mathbb{Q}$ is algebraic then $\lambda = e^{2\pi i \theta}$ satisfies (1.4) for some $\beta > 1$ and $\gamma > 0$. However, the set of $\lambda \in \mathbb{S}^1$ satisfying (1.4) is much larger, being of full measure.

It is also interesting to notice that for generic (in a topological sense) $\lambda \in \mathbb{S}^1$ there is a non-linearizable holomorphic local dynamical system with multiplier λ, while for almost all (in a measure-theoretic sense) $\lambda \in \mathbb{S}^1$ every holomorphic local dynamical system with multiplier λ is holomorphically linearizable.

Theorem 1.12 suggests the existence of a number-theoretical condition on λ ensuring that the origin is a Siegel point for any holomorphic local dynamical system of multiplier λ. And indeed this is the content of the *Brjuno-Yoccoz theorem.*

Theorem 1.13 (Brjuno, [65, 66, 67], Yoccoz, [267, 268]). *Let $\lambda \in \mathbb{S}^1$. Then the following statements are equivalent:*

(i) *the origin is a Siegel point for the quadratic polynomial $f_\lambda(z) = \lambda z + z^2$;*

(ii) *the origin is a Siegel point for all $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ;*

(iii) *the number λ satisfies* Brjuno's condition

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} \log \frac{1}{\omega_\lambda(2^{k+1})} < +\infty \; . \tag{1.5}$$

Brjuno, using majorant series as in Siegel's proof of Theorem 1.12 (see also [141] and references therein) has proved that condition (iii) implies condition (ii). Yoccoz, using a more geometric approach based on conformal and quasi-conformal geometry, has proved that (i) is equivalent to (ii), and that (ii) implies (iii), that is that if λ does not satisfy (1.5) then the origin is a Cremer point for some $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ — and hence it is a Cremer point for the quadratic polynomial $f_\lambda(z)$. See also [195] for related results.

Let us recall here that the Brjuno condition (1.5) is usually expressed in a different way. Writing $\lambda = e^{2\pi i \theta}$, and letting $\{p_k/q_k\}$ be the sequence of rational numbers converging to θ given by the expansion in continued fractions, (1.5) is indeed equivalent to

$$\sum_{k=0}^{+\infty} \frac{1}{q_k} \log q_{k+1} < +\infty \; ,$$

while (1.4) is equivalent to

$$q_{n+1} = O(q_n^\beta) \ ,$$

and (1.3) is equivalent to

$$\limsup_{k\to+\infty} \frac{1}{q_k} \log q_{k+1} = +\infty \ .$$

We refer to [159] for a tractation on continued fractions, and to [141, 268, 182, 75, 76, 78] and references therein for further details on condition (1.5).

Remark 1.14. *A clear obstruction to the holomorphic linearization of an elliptic germ* $f \in \mathrm{End}(\mathbb{C}, 0)$ *with multiplier* $\lambda = e^{2\pi i\theta} \in \mathbb{S}^1$ *is the existence of* small cycles, *that is of periodic orbits contained in any neighbourhood of the origin. Pérez-Marco [188], using Yoccoz's techniques, has shown that when the series*

$$\sum_{k=0}^{+\infty} \frac{\log\log q_{k+1}}{q_k}$$

converges then every germ with multiplier λ *is either linearizable or has small cycles, and that when the series diverges there exists such germs with a Cremer point but without small cycles.*

The complete proof of Theorem 1.13 is beyond the scope of this chapter and we refer to the original papers and to [78].

1.1.2 Non-holomorphically linearizable germs

We have briefly seen in the previous subsection that there are non-holomorphically linearizable, and we are interested in describing the local dynamics in these cases. We will separately deal with the parabolic case and the elliptic non-holomorphically linearizable case.

Parabolic case: the Leau-Fatou flower Theorem

Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system with a parabolic fixed point at the origin not of finite order. Then we can write

$$f(z) = e^{2i\pi p/q} z + a_{k+1} z^{k+1} + a_{k+2} z^{k+2} + \cdots \ , \tag{1.6}$$

with $a_{k+1} \neq 0$.

Definition 1.15. *The rational number* $p/q \in \mathbb{Q} \cap [0, 1)$ *is the* rotation number *of* f*, and the* multiplicity *of* f *at the fixed point is the minimal number* $k + 1 \geq 2$ *such that the coefficient of* z^{k+1} *in the power series expansion of* f *is non-zero. If* $p/q = 0$ *(that is, if the multiplier is 1), we shall say that* f *is* tangent to the identity.

Proposition 1.7 tells us that if f is tangent to the identity then it *cannot* be locally conjugated to the identity (unless it was the identity from the beginning, which is not a very interesting case dynamically speaking).

In order to understand the local dynamics in this case, let us first consider the case of the map $f(z) = z \cdot (1 + az^k)$. Note that the k half-lines where az^k belongs to $\mathbb{R}^+$ are mapped into themselves and that on those lines $\big|f(z)\big| > |z|$. It follows easily that when z belongs to such a half-line, $f^{\circ n}(z) \to \infty$ as $n \to +\infty$. Those k half-lines will be called *repelling directions* for f at 0. Similarly, the k half-lines where az^k belongs to $\mathbb{R}^-$ are mapped into themselves and on those lines when z is sufficiently close to 0, more precisely when $|az^k| < 1$, we have that $\big|f(z)\big| < |z|$.

It follows easily than when z belongs to such a half-line and is sufficiently close to 0, $f^{\circ n}(z) \to 0$ as $n \to +\infty$. Those k half-lines will be called *attracting directions* for f at 0.

The Leau-Fatou flower Theorem states that a similar description is still valid for $f(z) = z \cdot \big(1 + az^k + \mathrm{o}(z^k)\big)$. We have the following definition.

Definition 1.16. *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be tangent to the identity of multiplicity $k+1 \geq 2$. Then a unit vector $v \in \mathbb{S}^1$ is an* attracting *(resp.,* repelling*) direction for f at the origin if $a_{k+1}v^k$ is real and negative (resp., positive), where a_{k+1} is the coefficient of z^{k+1} in the power series expansion of f.*

Clearly, there are k equally spaced attracting directions, separated by k equally spaced repelling directions: if $a_{k+1} = |a_{k+1}|e^{i\alpha}$, then $v = e^{i\theta}$ is attracting (resp., repelling) if and only if

$$\theta = \frac{2\ell + 1}{k}\pi - \frac{\alpha}{k} \qquad \left(\text{resp., } \theta = \frac{2\ell}{k}\pi - \frac{\alpha}{k}\right).$$

Furthermore, a repelling (resp., attracting) direction for f is attracting (resp., repelling) for f^{-1}, which is defined in a neighbourhood of the origin.

Definition 1.17. *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be tangent to the identity, and let (U, f) be a representative of f. An* attracting petal *centered at an attracting direction v of f is a non-empty open simply connected f-invariant set $P \subseteq U \setminus \{0\}$ such that the orbit of a point tends to $0 \in \partial P$ tangent to v if and only if it is eventually contained in P. A* repelling petal *(centered at a repelling direction) is an attracting petal for the inverse of f.*

The Leau-Fatou flower Theorem gives us a complete description of the local dynamics in a punctured neighbourhood of the origin.

Theorem 1.18 (Leau, [169], Fatou, [113, 114, 115])**.** *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $k+1 \geq 2$ at the fixed point. Let $v_1^+, \ldots, v_k^+ \in \mathbb{S}^1$ be the k attracting directions of f at the origin, and $v_1^-, \ldots, v_k^- \in \mathbb{S}^1$ the k repelling directions. Then:*

(i) *for each attracting (resp., repelling) direction $v_j^\pm$ there exists an attracting (resp., repelling) petal $P_j^\pm$, so that the union of these $2k$ petals together with the origin forms a neighbourhood of the origin. Furthermore, the $2k$ petals are arranged cyclically so that two petals intersect if and only if the angle between their central directions is π/k.*

(ii) *For each petal P centered at one of the attracting directions, then there is a holomorphic function $\varphi\colon P \to \mathbb{C}$ such that $\varphi \circ f(z) = \varphi(z) + 1$ for all $z \in P$ and φ is a biholomorphism with an open subset of the complex plane containing a right half-plane, and so $f|_P$ is holomorphically conjugated to the translation $z \mapsto z + 1$.*

Camacho, using fundamental domains, has pushed this argument even further, obtaining a complete topological classification of one-dimensional holomorphic local dynamical systems tangent to the identity (see also [78, Theorem 1.7]).

Theorem 1.19 (Camacho, [80], Shcherbakov, [240])**.** *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $k+1$ at the fixed point. Then f is topologically locally conjugated to the map*

$$g(z) = z - z^{k+1}\,.$$

It is clear from the proof of Camacho [80] that such topological conjugacy is indeed C^∞ in a punctured neighbourhood of the origin. We refer to [80] and [53] for a proof, and to [152] for a more detailed proof. Jenkins in [152] also proved that if $f \in \mathrm{End}(\mathbb{C}, 0)$ is a holomorphic local dynamical system tangent to the identity with multiplicity 2, such that there exists a topological conjugacy (conjugating it with $g(z) = z - z^2$), which is indeed real-analytic in a punctured neighbourhood of the origin, with real-analytic inverse, then there exists a holomorphic conjugacy between f and g. Finally, Martinet and Ramis [181] proved that if a germ $f \in \mathrm{End}(\mathbb{C}, 0)$, tangent to the identity, is C^1-conjugated (in a full neighbourhood of the origin) to $g(z) = z - z^{k+1}$, then it is holomorphically or antiholomorphically conjugated to it.

The formal classification is simple too, though different (see, e.g., Milnor [182]):

Proposition 1.20. *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $k + 1$ at the fixed point. Then f is formally conjugated to the map*

$$g(z) = z - z^{k+1} + \beta z^{2k+1} ,$$

where β is a formal (and holomorphic) invariant given by

$$\beta = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - f(z)} , \tag{1.7}$$

where the integral is taken over a small positive loop γ around the origin.

On the other hand, the holomorphic classification of maps tangent to the identity is much more complicated, and, as shown by Écalle [97, 98] and Voronin [257], it depends on functional invariants. We refer to [10] and [149, 179, 180, 160, 78] and the original papers for details.

Finally, if $f \in \mathrm{End}(\mathbb{C}, 0)$ satisfies $\lambda = e^{2\pi i p/q}$, then $f^{\circ q}$ is tangent to the identity. Therefore we can apply the previous results to $f^{\circ q}$ and then infer informations about the dynamics of the original f (see for example [78, 182]).

Elliptic case

If 0 is a Siegel point for $f \in \mathrm{End}(\mathbb{C}, 0)$, the local dynamics of f is completely clear, and simple enough. On the other hand, if 0 is a Cremer point of f, then the local dynamics of f is very complicated and not yet completely understood. Pérez-Marco in [189, 191, 192, 193, 194] and Biswas in [46, 47] studied the topology and the dynamics of the stable set in this case. Some of their results can be briefly summarized as follows.

Theorem 1.21 (Pérez-Marco, [193, 194])**.** *Assume that 0 is a Cremer point for an elliptic holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}, 0)$. Then:*

(i) *There is a completely invariant stable set K_f which is compact, connected, full (i.e., $\mathbb{C} \setminus K_f$ is connected), it is not reduced to $\{0\}$, and it is not locally connected at any point distinct from the origin.*

(ii) *Any point of $K_f \setminus \{0\}$ is recurrent (that is, a limit point of its orbit).*

(iii) *There is an orbit in K_f which accumulates at the origin, but no non-trivial orbit converges to the origin.*

Theorem 1.22 (Biswas, [47])**.** *The rotation number and the conformal class of K_f are a complete set of holomorphic invariants for Cremer points. In other words, two elliptic non-linearizable holomorphic local dynamical systems f and g are holomorphically locally conjugated if and only if they have the same rotation number and there is a biholomorphism of a neighbourhood of K_f with a neighbourhood of K_g.*

Thus if $\lambda \in \mathbb{S}^1$ is not a root of unity and does not satisfy Brjuno's condition (1.5), we can find f_1, $f_2 \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ such that f_1 is holomorphically linearizable while f_2 is not. Then f_1 and f_2 are formally conjugated without being neither holomorphically nor topologically locally conjugated.

Yoccoz [268] has proved that if $\lambda \in \mathbb{S}^1$ is not a root of unity and does not satisfy Brjuno's condition (1.5) then there is an uncountable family of germs in $\mathrm{End}(\mathbb{C}, 0)$ with multiplier λ which are not holomorphically conjugated to each other nor holomorphically conjugated to any entire function.

See also [188, 190] for other results on the dynamics about a Cremer point, and [196] for relationships with holomorphic foliations in $\mathbb{C}^2$.

1.2 Formal normal forms in higher dimension

Let $F \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system at $O \in \mathbb{C}^n$, with $n \geq 2$. The *homogeneous expansion* of F is

$$F(z) = P_1(z) + P_2(z) + \cdots \in \mathbb{C}\{z_1, \ldots, z_n\}^n ,$$

where P_j is an n-uple of homogeneous polynomials of degree j. In particular, P_1 is the differential DF_O of f at the origin, and f is locally invertible if and only if P_1 is invertible.

The holomorphic and even the formal classification are not as well understood as in dimension topological one. As we saw in Section 1.2, one of the main questions in the study of local holomorphic dynamics is when f is holomorphically linearizable. The answer to this question depends on the set of eigenvalues of DF_O, usually called the *spectrum* of DF_O, and the main problem is caused by the so-called *resonances*. In the rest, we shall need the following notation.

Let $p \geq 2$. We denote by $\mathcal{H}^p$ the complex vector space of homogeneous polynomial endomorphisms of $\mathbb{C}^n$ of degree p, and we consider on it the standard basis $\mathcal{B}^p = \{z^Q e_j \mid |Q| = p, 1 \leq j \leq n\}$. We shall denote by $o(k)$ every holomorphic map of the form $\sum_{p \geq k+1} h_p$ with $h_p \in \mathcal{H}^p$.

Let us first see what happens when we conjugate F by a germ of biholomorphism of the form $\Psi_p := \mathrm{Id} + \widehat{\Psi}_p$ with $\widehat{\Psi}_p \in \mathcal{H}^p$ and $p \geq 2$.

Lemma 1.23. *Let $\Psi := \mathrm{Id} + \widehat{\Psi}$ be a germ of biholomorphism of $\mathbb{C}^n$ with $\widehat{\Psi} \in \mathcal{H}^q$, and let $F = \Lambda + S_{q-1} + H_q + o(q)$ be a germ of biholomorphism with $S_{q-1} \in \mathcal{H}^1 \oplus \cdots \oplus \mathcal{H}^{q-1}$ and $H_q \in \mathcal{H}^q$. Then*

$$\Psi^{-1} \circ F \circ \Psi = \Lambda + S_{q-1} + [H_q + \Lambda \circ \widehat{\Psi} - \widehat{\Psi} \circ \Lambda] + o(q).$$

Thus the germ $\psi_q := I + \widehat{\psi}_q$ conjugates $f = \Lambda + H_q + o(q)$ with $\Lambda + o(q)$ if and only if $\widehat{\psi}_q$ is a solution of the equation $H_q = \widehat{\psi} \circ \Lambda - \Lambda \circ \widehat{\psi}$. We have then to study the invertibility of the linear operators

$$M^r_\Lambda \colon \mathcal{H}^r \to \mathcal{H}^r$$

defined by

$$M^r_\Lambda(h) = h \circ \Lambda - \Lambda \circ h.$$

When $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ it is easy to answer this question. In fact, for each element $z^Q e_j$ of the basis $\mathcal{B}^r$ we have

$$M^r_\Lambda(z^Q e_j) = (\lambda^Q - \lambda_j) z^Q e_j, \tag{1.8}$$

hence $\ker(M^r_\Lambda) = \{z^Q e_j \mid \lambda^Q - \lambda_j = 0, |Q| \geq 2, 1 \leq j \leq n\}$.

We are then led to give the following definition.

Definition 1.24. *Let* $\lambda \in (\mathbb{C}^*)^n$ *and let* $j \in \{1, \dots, n\}$. *We say that a multi-index* $Q \in \mathbb{N}^n$, *with* $|Q| \geq 2$, *gives a* multiplicative resonance relation for λ relative to the j-th coordinate *if*

$$\lambda^Q := \lambda_1^{q_1} \cdots \lambda_n^{q_n} = \lambda_j .$$

We denote

$$\mathsf{Res}_j(\lambda) := \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \lambda^Q = \lambda_j\}.$$

The elements of $\mathsf{Res}(\lambda) := \bigcup_{j=1}^n \mathsf{Res}_j(\lambda)$ *are simply called* resonant *multi-indices.*

Notice that when $n = 1$ there is a resonance if and only if the multiplier is a root of unity, or zero. This is no more true for $n > 1$.

Resonances are the obstruction to formal linearization. Indeed, a computation completely analogous to the one yielding Proposition 1.2 shows that the coefficients of a formal linearization have in the denominators quantities of the form $\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j$, and we can easily construct germs that are not formally linearizable. We have the following classical general result (see [27, pp. 192–193] or [206] for a proof), that shows that in presence of resonances, even the formal classification is not that easy.

Let us assume, for simplicity, that DF_O is in Jordan form, that is

$$P_1(z) = (\lambda_1 z, \epsilon_2 z_1 + \lambda_2 z_2, \dots, \epsilon_n z_{n-1} + \lambda_n z_n) ,$$

with $\epsilon_1, \dots, \epsilon_{n-1} \in \{0, 1\}$.

Definition 1.25. *We shall say that a monomial* $z^Q := z_1^{q_1} \cdots z_n^{q_n}$ *in the* j*-th coordinate of* f *is* resonant with respect to $\lambda_1, \dots \lambda_n \in \mathbb{C}^*$ *(or simply* $(\lambda_1, \dots \lambda_n)$-resonant*) if* $|Q| = \sum_{j=1}^n q_j \geq 2$ *and* $\lambda^Q := \lambda_1^{q_1} \cdots \lambda_n^{q_n} = \lambda_j$.

Theorem 1.26 (Poincaré, [203], Dulac, [96])**.** *Let* $F \in \mathrm{End}(\mathbb{C}^n, O)$ *be a locally invertible holomorphic local dynamical system. Then it is formally conjugated to a* $G \in \mathbb{C}[\![z_1, \dots, z_n]\!]^n$ *without constant term and such that* DG_O *is in Jordan normal form, and* g *has only resonant monomials. Moreover, the resonant part of the formal change of coordinates* ψ *can be chosen arbitrarily, but once this is done,* ψ *and* g *are uniquely determined.*

Definition 1.27. *A formal power series* $G \in \mathbb{C}[\![z_1, \dots, z_n]\!]^n$ *without constant term, and with linear part* Λ *in Jordan normal form with eigenvalues* $\lambda_1, \dots, \lambda_n \in \mathbb{C}^*$, *is called* in Poincaré-Dulac normal form *if it contains only resonant monomials with respect to* $\lambda_1, \dots, \lambda_n$.

Definition 1.28. *Let* F *be a germ of biholomorphism of* $\mathbb{C}^n$ *fixing the origin. A formal transformation* G *in Poincaré-Dulac normal form that can be formally conjugated to* F *is called a* Poincaré-Dulac (formal) normal form of F.

We already remarked that *there are germs of biholomorphisms of* $\mathbb{C}^n$ *fixing the origin and not linearizable, even formally.*

The main problem with *Poincaré-Dulac normal forms* is that they *are not unique.* In particular, one may wonder whether it could be possible to have such a normal form including *finitely many* resonant monomials only (as happened, for instance, in Proposition 1.20). We shall see in Subsection 1.4.1 that this is indeed the case when F belongs to the Poincaré domain, that is when DF_O is invertible and O is either attracting or repelling.

Definition 1.29. *We say that* F, *a germ of biholomorphism of* $\mathbb{C}^n$ *fixing the origin, is* holomorphically normalizable *if there exists a local change of coordinates* $\varphi \in \mathrm{End}(\mathbb{C}^n, O)$, *tangent to the identity, conjugating* F *to one of its Poincaré-Dulac normal forms.*

Remark 1.30. *Let us mention here that, together with Abate, we described in [15] an alternative approach to a general renormalization procedure for formal self-maps, originally suggested by Chen-Della Dora and Wang-Zheng-Peng, giving* formal *normal forms simpler than the classical Poincaré-Dulac normal form. Such an approach becomes particularily interesting for tangent to the identity maps, where since every monomial is resonant, Poincaré-Dulac Theorem 1.26 gives no simplification.*

We have seen that in dimension one the multiplier (i.e., the derivative at the origin) plays a main rôle. When $n > 1$, a similar rôle is played by the eigenvalues of the differential. We shall use the following classification.

Definition 1.31. *Let $F \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system at $O \in \mathbb{C}^n$, with $n \geq 2$. Then*

- *if all eigenvalues of DF_O have modulus strictly less than 1, we say that the fixed point O is* attracting;
- *if all eigenvalues of DF_O have modulus strictly greater than 1, we say that the fixed point O is* repelling;
- *if all eigenvalues of DF_O have modulus different from 1, we say that the fixed point O is* hyperbolic *(notice that we allow the eigenvalue zero);*
- *if O is attracting or repelling, and DF_O is invertible, we say that F is in the* Poincaré domain;
- *if O is hyperbolic, DF_O is invertible, and F is not in the Poincaré domain (and thus not all eigenvalues of DF_O are inside or outside the unit disk) we say that F is in the* Siegel domain;
- *if all eigenvalues of DF_O are roots of unity, we say that the fixed point O is* parabolic; *in particular, if $DF_O = \mathrm{Id}$ we say that F is* tangent to the identity;
- *if all eigenvalues of DF_O have modulus 1 but none is a root of unity, we say that the fixed point O is* elliptic;
- *if $DF_O = O$, we say that the fixed point O is* superattracting.

Other cases are clearly possible, but for the aim of this chapter this list is enough.

For the rest of the chapter we shall only consider germs of biholomorphisms. In the following we will restrict our attention to invertible holomorphic local dynamical system. We refer for example to [10, Section 5], [77], [236] and references therein for normal forms and dynamics of non-invertible local holomorphic dynamical systems.

1.2.1 Formal linearization

A consequence of Poincaré-Dulac Theorem 1.26 is the following classical result on formal linearization.

Theorem 1.32. *Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ fixing the origin O with no resonances. Then F is formally linearizable.*

In the resonant case, one can still find formally linearizable germs, (see for example [207, 205]), so it natural to ask whether a formally linearizable germ can have several different Poincaré-Dulac formal normal forms. The answer to such a question has been given by Rüssmann (this statement is slightly different from the original one presented in [238] but perfectly equivalent).

Theorem 1.33 (Rüssmann, [238])**.** *Let F be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin. If F is formally linearizable, then the linear form is its unique Poincaré-Dulac normal form.*

We refer to [209] for a direct proof of this result using power series expansions.

1.3 Holomorphic linearization in higher dimension

Each non-resonant germ which is in the Poincaré domain can be holomorphically linearized. In fact, in [203], using majorant series, Poincaré proved the following stronger result.

Theorem 1.34 (Poincaré, [203])**.** *Let $F \in \mathrm{End}(\mathbb{C}^n, O)$ be a locally invertible holomorphic local dynamical system in the Poincaré domain. Then F is holomorphically linearizable if and only if it is formally linearizable. In particular, if there are no resonances then F is holomorphically linearizable.*

Even when there are no resonances, or more generally, when we know a priori that a given germ is formally linearizable, not so much is known about the convergence of the linearizations in the cases different from the Poincaré domain.

A first easy observation is that Theorem 1.10 naturally generalizes to higher dimension. Here, given $F\colon (\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$ a holomorphic map such that all the eigenvalues of $DF(0)$ have modulus one, the dynamics of F is *stable near O* if there exist $R > r > 0$ such that for all $k \geq 0$, the iterates $F^{\circ k}|_{\mathbb{B}_r}$ are defined with $F^{\circ k}|_{\mathbb{B}_r} \subset \mathbb{B}_R$, where $\mathbb{B}_r$, resp. $\mathbb{B}_R$, denotes the ball of radius r, resp. R, and center at the origin.

Proposition 1.35. *Let $F \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system such that all the eigenvalues of $DF(0)$ have modulus one. Then F is holomorphically linearizable if and only if it is topologically linearizable if and only if the dynamics of F is stable near O.*

Proof. The proof is the same as the one of Theorem 1.10. Here it suffices to consider the averages $\Phi_N\colon \mathbb{B}_r \to \mathbb{B}_R$ defined by

$$\Phi_N := \frac{1}{N} \sum_{j=0}^{N-1} \Lambda^{-j} F^{\circ j},$$

where Λ is the diagonal matrix representing $DF(O)$ is local coordinates. □

Usually we have no information about stability near a fixed point. A first result about convergence is then the natural generalization of Siegel's Theorem 1.12.

Theorem 1.36. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$ be a non-resonant vector such that there exists $\beta > 1$ and $\gamma > 0$ so that for every $Q \in \mathbb{N}^n$, $|Q| \geq 2$*

$$\frac{1}{|\lambda^Q - \lambda_j|} \leq \gamma\, |Q|^\beta. \tag{1.9}$$

Then all $F \in \mathrm{End}(\mathbb{C}^n, O)$ such that DF_O is diagonalizable and has spectrum $\{\lambda_1, \ldots, \lambda_n\}$ are holomorphically linearizable.

As in one variable, (1.9) is a particular case of a more general condition, the multi-dimensional Brjuno condition.

When DF_O belongs to the Siegel domain, even without resonances, the formal linearization might diverge. To describe the known results, let us introduce the following definition.

Definition 1.37. *For $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $m \geq 2$ set*

$$\omega_{\lambda_1,\ldots,\lambda_n}(m) = \min\left\{ |\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j| \;\Big|\; k_1, \ldots, k_n \in \mathbb{N},\, 2 \leq \sum_{h=1}^{n} k_h \leq m,\, 1 \leq j \leq n \right\} . \quad (1.10)$$

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of DF_O, we shall write $\omega_f(m)$ for $\omega_{\lambda_1,\ldots,\lambda_n}(m)$.

It is clear that $\omega_f(m) \neq 0$ for all $m \geq 2$ if and only if there are no resonances. It is also not difficult to prove that if F belongs to the Siegel domain then

$$\lim_{m\to+\infty} \omega_f(m) = 0 ,$$

which is the reason why, even without resonances, the formal linearization might be diverging, exactly as in the one-dimensional case.

Definition 1.38. *Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. We say that λ* satisfies the Brjuno condition *if*

$$\sum_{\nu\geq 0} \frac{1}{2^\nu} \log \omega_{\lambda_1,\ldots,\lambda_n}(2^{\nu+1})^{-1} < \infty. \quad (1.11)$$

As far as we know, the best positive result in the non-resonant case is due to Brjuno [66, 67], using majorant series, and is a natural generalization of its one-dimensional counterpart:

Theorem 1.39 (Brjuno, [66, 67])**.** *Let F be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin, such that DF_O is diagonalizable. Assume moreover that the spectrum of DF_O has no resonances and it satisfies the Brjuno condition. Then F is holomorphically linearizable.*

It is also possible to generalize Theorem 1.11 proving that if $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ have no resonances and

$$\limsup_{m\to+\infty} \frac{1}{m} \log \frac{1}{\omega_{\lambda_1,\ldots,\lambda_n}(m)} = +\infty,$$

then there exists a germ of biholomorphism of $(\mathbb{C}^n, O)$, fixing the origin, with differential $DF_O = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$, and not holomorphically linearizable (see for example [206, Theorem 1.5.1]). Another result in the same spirit is the following.

Theorem 1.40 (Yoccoz, [268])**.** *Let $A \in GL(n, \mathbb{C})$ be an invertible matrix such that its eigenvalues have no resonances and such that its Jordan normal form contains a non-trivial block associated to an eigenvalue of modulus one. Then there exists $F \in \mathrm{End}(\mathbb{C}^n, O)$ with $DF_O = A$ which is not holomorphically linearizable.*

Remark 1.41. *Contrarily to the one-dimensional case, it is not yet known whether the Brjuno condition is necessary for the holomorphic linearizability of all holomorphic local dynamical systems with a given linear part belonging to the Siegel domain. However, it is easy to check that if $\lambda \in \mathbb{S}^1$ does not satisfy the one-dimensional Brjuno condition then any $F \in \mathrm{End}(\mathbb{C}^n, O)$ of the form*

$$f(z) = \big(\lambda z_1 + z_1^2, g(z)\big)$$

is not holomorphically linearizable: indeed, if $\varphi \in \mathrm{End}(\mathbb{C}^n, O)$ is a holomorphic linearization of F, then $\psi(\zeta) = \varphi(\zeta, O)$ is a holomorphic linearization of the quadratic polynomial $\lambda\zeta + \zeta^2$, contradicting Theorem 1.13.

Another approach to this kind of problems was given by Rüssmann in [237, 238]. Rüssmann introduced the following condition, that we shall call Rüssmann condition.

Definition 1.42. *Let $n \geq 2$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. We say that $\lambda = (\lambda_1, \dots, \lambda_n)$* satisfies the Rüssmann condition *if there exists a function $\Omega\colon \mathbb{N} \to \mathbb{R}$ such that:*

(i) $k \leq \Omega(k) \leq \Omega(k+1)$ *for all* $k \in \mathbb{N}$,

(ii) $\sum\limits_{k\geq 1} \frac{1}{k^2} \log \Omega(k) \leq +\infty$, *and*

(iii) $|\lambda^Q - \lambda_j| \geq \frac{1}{\Omega(|Q|)}$ *for all* $j = 1, \dots n$ *and for each multi-index* $Q \in \mathbb{N}$ *with* $|Q| \geq 2$ *not giving a resonance relative to* j.

Rüssmann then proved the following generalization of Brjuno Theorem 1.39 (the statement is slightly different from the original one presented in [238] but perfectly equivalent).

Theorem 1.43 (Rüssmann, [238])**.** *Let F be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that DF_O is diagonalizable. If F is formally linearizable and the spectrum of DF_O satisfies the Rüssmann condition, then it is holomorphically linearizable.*

We refer to the article [238] for the original proof of Rüssmann and to [209] where we recalled the main ideas of it. In [209] we gave the natural generalization of Brjuno's Theorem 1.39 in presence of resonances. The key remark is that, when there are no resonances, the function $\omega_f(m)$ defined in Definition 1.37 satisfies

$$|\lambda^Q - \lambda_j| \geq \omega_f(|Q|)$$

for each multi-index $Q \in \mathbb{N}$ with $|Q| \geq 2$.

This leads to the following natural definitions.

Definition 1.44. *Let $n \geq 2$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. For $m \geq 2$ set*

$$\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(m) = \min_{1\leq j\leq n} \min_{\substack{2\leq |K|\leq m \\ K\notin \mathsf{Res}_j(\lambda)}} |\lambda^K - \lambda_j|,$$

where $\mathsf{Res}_j(\lambda)$ is the set of multi-indices $K \in \mathbb{N}^n$, with $|K| \geq 2$, giving a resonance relation for $\lambda = (\lambda_1, \dots, \lambda_n)$ relative to $1 \leq j \leq n$, i.e., $\lambda^K - \lambda_j = 0$. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of DF_O, we shall write $\widetilde{\omega}_f(m)$ for $\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(m)$.

Definition 1.45. *Let $n \geq 2$ and let $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$. We say that λ* satisfies the reduced Brjuno condition *if*

$$\sum_{\nu\geq 0} \frac{1}{2^\nu} \log \widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(2^{\nu+1})^{-1} < \infty.$$

We have the following relation between the Rüssmann and the reduced Brjuno condition.

Lemma 1.46. *Let $n \geq 2$ and let $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$. If λ satisfies Rüssmann condition, then it also satisfies the reduced Brjuno condition.*

We proved in [210, Theorem 4.1] that the Rüssmann condition is indeed equivalent to the reduced Brjuno condition in the multi-dimensional case. Notice that Rüssmann was able to prove that in dimension 1, his condition is equivalent to Brjuno condition, but to do so he strongly used the one-dimensional characterization of these conditions via continued fraction.

In [209] we proved the following analog of Theorem 1.43.

Theorem 1.47 (R., [209]). *Let F be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that DF_O is diagonalizable. If F is formally linearizable and the spectrum of DF_O satisfies the reduced Brjuno condition, then F is holomorphically linearizable.*

Idea of the proof. The proof has two main parts. First we need explicit computation for the power series expansion of a suitable linearization, and then we prove its convergence via majorant series.

The key point for the first part is that, thanks to Theorem 1.33, when a germ is formally linearizable, then the linear form is its unique Poincaré-Dulac normal form. In fact as a consequence of such result, we obtain that *any formal normalization given by the Poincaré-Dulac procedure applied to a formally linerizable germ F is indeed a formal linearization of the germ.* In particular, we have *uniqueness of the Poincaré-Dulac normal form* (which is linear and hence holomorphic), but *not* of the formal linearizations. Hence a formally linearizable germ F is formally linearizable via a formal transformation $\varphi = \mathrm{Id} + \widehat{\varphi}$ containing only non-resonant monomials. In fact, thanks to the proof of Poincaré-Dulac Theorem 1.26, we can consider the formal normalization obtained with the Poincaré-Dulac procedure and imposing that the coefficient $\varphi_{Q,j}$ of the monomial z^Q in the j-th coordinate of φ vanishes for all Q and j such that $\lambda^Q = \lambda_j$; and this formal transformation φ conjugates F to its linear part.

The proof of the convergence then follows by carefully adapting Brjuno's proof of Theorem 1.39. □

It is natural to search for conditions for formal linearization in presence of resonances. A first result is due to Rong [220] when the spectrum of the differential at the origin of a given germ of biholomorphism fixing the origin contains 1 and λ_j's with $|\lambda_j| = 1$, but the λ_j's are not roots of unity, the germ F admits a curve of fixed points tangent to the generalized eigenspace of 1 and such that the other eigenvalues satisfy a Bryuno-type condition along the curve of fixed points. During the Ph.D. we generalized and specified Rong's result in [207].

1.3.1 Partial linearization

A way to generalize Brjuno's Theorem 1.39 is to look for partial linearization results, e.g., studying the linearization problem along submanifolds.

Pöschel [204] showed how to modify (1.10) and (1.11) to obtain partial linearization results along submanifolds. To do so, he uses a notion of partial Brjuno condition which is explained in the following definitions:

Definition 1.48. *Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. Fix $1 \leq s \leq n$ and let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_s)$. For any $m \geq 2$ put*

$$\omega_s(m) = \min_{2 \leq |K| \leq m} \min_{1 \leq j \leq n} |\underline{\lambda}^K - \lambda_j|,$$

where $K = (k_1, \ldots, k_s)$ and $\underline{\lambda}^K = \lambda_1^{k_1} \cdots \lambda_s^{k_s}$.

The following definition is a reformulation of Pöschel's definition of *admissible* eigenvalues as stated in [204].

Definition 1.49. *Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. Fix $1 \leq s \leq n$. We say that $\lambda = (\lambda_1, \ldots, \lambda_n)$* satisfies the partial Brjuno condition of order s *if*

$$\sum_{\nu \geq 0} \frac{1}{2^\nu} \log \omega_s(2^{\nu+1})^{-1} < \infty.$$

It is clear that $\omega_s(m) \neq 0$ for all $m \geq 2$ if and only if there are no resonant multi-indices Q of the form $Q = (q_1, \dots q_s, 0, \dots, 0)$.

Remark 1.50. *For $s = n$ the partial Brjuno condition of order s is nothing but the usual Brjuno condition introduced in [66, 67]. When $s < n$, the partial Brjuno condition of order s is indeed weaker than the Brjuno condition. Let us consider for example $n = 2$ and let $\lambda, \mu \in \mathbb{C}^*$ be distinct. To check whether the pair (λ, μ) satisfies the partial Brjuno condition of order 1, we have to consider only the terms $|\lambda^k - \lambda|$ and $|\lambda^k - \mu|$ for $k \geq 2$, whereas to check the full Brjuno condition we have to consider also the terms $|\mu^h - \lambda|$, $|\mu^h - \mu|$ for $h \geq 2$, and $|\lambda^k\mu^h - \lambda|$, $|\lambda^k\mu^h - \mu|$ for $k, h \geq 1$.*

Remark 1.51. *A n-tuple $\lambda = (\lambda_1, \dots, \lambda_s, 1, \dots, 1) \in (\mathbb{C}^*)^n$ satisfies the partial Brjuno condition of order s if and only if $(\lambda_1, \dots, \lambda_s)$ satisfies the Brjuno condition.*

We assume that the differential DF_O is diagonalizable. Then, possibly after a linear change of coordinates, we can write

$$F(z) = \Lambda z + \widehat{F}(z),$$

where $\Lambda = \mathrm{Diag}(\lambda_1, \dots, \lambda_n)$, and $\widehat{F}$ vanishes up to first order at $O \in \mathbb{C}^n$.

The linear map $z \mapsto \Lambda z$ has a very simple structure. For instance, for any subset $\lambda_1, \dots, \lambda_s$ of eigenvalues with $1 \leq s \leq n$, the direct sum of the corresponding eigenspaces obviously is an invariant manifold on which this map acts linearly with these eigenvalues.

We have the following result of Pöschel [204].

Theorem 1.52 (Pöschel, [204])**.** *Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ fixing the origin O. If there exists a positive integer $1 \leq s \leq n$ such that the eigenvalues $\lambda_1, \dots, \lambda_n$ of DF_O satisfy the partial Brjuno condition of order s, then there exists locally a complex analytic F-invariant manifold M of dimension s, tangent to the eigenspace of $\lambda_1, \dots, \lambda_s$ at the origin, on which the mapping is holomorphically linearizable.*

1.3.2 Simultaneous linearization

Another generalization of the linearization problem is to ask when $h \geq 2$ germs of biholomorphisms $F_1, \dots, F_h$ of $\mathbb{C}^n$ at the same fixed point, which we may place at the origin, are *simultaneously holomorphically linearizable*, i.e., there exists a local holomorphic change of coordinates conjugating F_k to its linear part for each $k = 1, \dots, h$.

In dimension 1, this problem has been thoroughly studied, also for commuting systems of analytic or smooth circle diffeomorphisms, that are indeed deeply related to commuting systems of germs of holomorphic functions, as explained in [194]. The question about the smoothness of a simultaneous linearization of such a system, raised by Arnold, was brilliantly answered by Herman [140], and extended by Yoccoz [266] (see also [269]). In [184], Moser raised the problem of smooth linearization of commuting circle diffeomorphisms in connection with the holonomy group of certain foliations of codimension 1, and, using the rapidly convergent Nash-Moser iteration scheme, he proved that if the rotation numbers of the diffeomorphisms satisfy a simultaneous Diophantine condition and if the diffeomorphisms are in some C^∞-neighborhood of the corresponding rotations then they are C^∞-linearizable, that is, C^∞-conjugated to rotations. We refer to [111] and references therein for a clear exposition of the one-dimensional problem and for the best results in such a context. Furthermore, the problem for commuting germs of holomorphic functions in dimension one has been studied by DeLatte [88], and by Biswas [48], under Brjuno-type conditions generalizing Moser's simultaneous Diophantine condition.

In dimension $n \geq 2$ much less is known in the formal and holomorphic settings. Gramchev and Yoshino [131] have proved a simultaneous holomorphic linearization result for pairwise commuting germs without simultaneous resonances, with diagonalizable linear parts, and under a simultaneous Diophantine condition (further studied by Yoshino in [270]) and a few more technical assumptions. In [89], DeLatte and Gramchev investigated on holomorphic linearization of germs with linear parts having Jordan blocks, leaving as an open problem the study of simultaneous formal and holomorphic linearization of commuting germs with non-diagonalizable linear parts.

Therefore, there are at least three natural questions arising in this setting:

(Q1) *Is it possible to say anything on the shape a (formal) simultaneous linearization can have?*

(Q2) *Are there any conditions on the eigenvalues of the linear parts of $h \geq 2$ germs of simultaneously formally linearizable biholomorphisms ensuring simultaneous holomorphic linearizability?*

(Q3) *Under which conditions on the eigenvalues of the linear parts of $h \geq 2$ pairwise commuting germs of biholomorphisms can one assert the existence of a simultaneous holomorphic linearization of the given germs? In particular, is there a Brjuno-type condition sufficient for convergence?*

Note that the third question is a natural generalization to dimension $n \geq 2$ of the question raised by Moser [184] in the one-dimensional case (see also the introduction of [111]).

In [210] we gave complete answers to these three questions without making any assumption on the resonances. Before stating our answer to the first question, we need the following definition.

Definition 1.53. *Let $M_1, \ldots, M_h$ be $h \geq 2$ complex $n \times n$ matrices. We say that $M_1, \ldots, M_h$ are* almost simultaneously Jordanizable*, it there exists a linear change of coordinates A such that $A^{-1}M_1A, \ldots, A^{-1}M_hA$ are* almost in simultaneous Jordan normal form*, i.e., for $k = 1, \ldots, h$ we have*

$$A^{-1}M_kA = \begin{pmatrix} \lambda_{k,1} & & & \\ \varepsilon_{k,1} & \lambda_{k,2} & & \\ & \ddots & \ddots & \\ & & \varepsilon_{k,n-1} & \lambda_{k,n} \end{pmatrix}, \quad \varepsilon_{k,j} \neq 0 \Longrightarrow \lambda_{k,j} = \lambda_{k,j+1}. \tag{1.12}$$

We say that $M_1, \ldots, M_h$ are simultaneously Jordanizable *if there exists a linear change of coordinates A such that we have (1.12) with $\varepsilon_{k,j} \in \{0, \varepsilon\}$.*

Notice that two commuting matrices are not necessarily almost simultaneously Jordanizable, and that two almost simultaneously Jordanizable matrices do not necessarily commute. However, the almost simultaneously Jordanizable hypothesis remains less restrictive than the simultaneously diagonalizable assumption usual in this context.

The following result gives an answer to (Q1).

Theorem 1.54 (R. [210])**.** *Let $F_1, \ldots, F_h$ be $h \geq 2$ formally linearizable germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin and with almost simultaneously Jordanizable linear parts. If $F_1, \ldots, F_h$ are simultaneously formally linearizable, then they are simultaneously formally linearizable via a linearization φ such that the coefficient $\varphi_{Q,j}$ of the monomial z^Q in the j-th coordinate of φ vanishes for each Q and j so that $Q \in \cap_{k=1}^h \mathsf{Res}_j(\Lambda_k)$, and such a linearization is unique.*

We also have a condition ensuring formal simultaneous linearizability.

Theorem 1.55 (R. [210]). *Let $F_1, \dots, F_h$ be $h \geq 2$ formally linearizable germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin and with almost simultaneously Jordanizable linear parts. If $F_1, \dots, F_h$ all commute pairwise, then they are simultaneously formally linearizable.*

To state our result on simultaneous holomorphic linearizability we introduced the following Brjuno-type condition.

Definition 1.56. *Let $n \geq 2$ and let $\Lambda_1 = (\lambda_{1,1}, \dots, \lambda_{1,n}), \dots, \Lambda_h = (\lambda_{h,1}, \dots, \lambda_{h,n})$ be $h \geq 2$ n-tuples of complex, not necessarily distinct, non-zero numbers. We say that $\Lambda_1, \dots, \Lambda_h$* satisfy the simultaneous Brjuno condition *if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu \geq 0}$ with $p_0 = 1$ such that*

$$\sum_{\nu \geq 0} \frac{1}{p_\nu} \log \frac{1}{\omega_{\Lambda_1,\dots,\Lambda_h}(p_{\nu+1})} < +\infty,$$

where for any $m \geq 2$ we set

$$\omega_{\Lambda_1,\dots,\Lambda_h}(m) = \min_{\substack{2 \leq |Q| \leq m \\ Q \notin \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)}} \varepsilon_Q,$$

with

$$\varepsilon_Q = \min_{1 \leq j \leq n} \max_{1 \leq k \leq h} |\Lambda_k^Q - \lambda_{k,j}|.$$

If $\Lambda_1, \dots, \Lambda_h$ are the sets of eigenvalues of the linear parts of $F_1, \dots, F_h$, we shall say that $F_1, \dots, F_h$ satisfy the simultaneous Brjuno condition.

Our holomorphic linearization result answering (Q2) is then the following.

Theorem 1.57 (R. [210]). *Let $F_1, \dots, F_h$ be $h \geq 2$ simultaneously formally linearizable germs of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that their linear parts $\Lambda_1, \dots, \Lambda_h$ are simultaneously diagonalizable. If $F_1, \dots, F_h$ satisfy the simultaneous Brjuno condition, then $F_1, \dots F_h$ are holomorphically simultaneously linearizable.*

Notice that the previous result can also be seen as a consequence of [247, Theorem 2.1].

Using Theorem 1.57 we are also able to give a positive answer to the generalization (Q3) of Moser's question.

Theorem 1.58 (R. [210]). *Let $F_1, \dots, F_h$ be $h \geq 2$ formally linearizable germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin, with simultaneously diagonalizable linear parts, and satisfying the simultaneous Brjuno condition. Then $F_1, \dots, F_h$ are simultaneously holomorphically linearizable if and only if they all commute pairwise.*

1.4 Holomorphic normalization in higher dimension and local dynamics

As we already remarked in Section 1.2, there are germs $F \in \mathrm{End}(\mathbb{C}^n, O)$ that are not formally linearizable. However, since by Poincaré-Dulac Theorem 1.26 every germ $F \in \mathrm{End}(\mathbb{C}^n, O)$ can be formally normalized, it is natural to ask whether a germ is holomorphically normalizable. As we will recall in the following subsection, this is always the case in the attracting (resp. repelling) case, but the non-uniqueness of Poincaré-Dulac normal forms makes the problem of finding canonical formal normal forms when F belongs to the Siegel domain more difficult. Furthermore, even if Écalle in his monumental work [99] provides completes sets of invariants characterizing the conjugacy classes of germs in $\mathrm{End}(\mathbb{C}^n, O)$, those invariants are not so easily computable and it remains somehow difficult to use them in studying particular cases. During the Ph.D., we were able to give in [208] a geometric characterization of holomorphically normalizable germs and we will briefly recall it in Subsection 1.4.2.

1.4.1 Attracting/Repelling germs

If a germ $F \in \mathrm{End}(\mathbb{C}^n, O)$ is in the Poincaré domain, that is the origin is an attracting or a repelling fixed point, then the holomorphic classification is clear. Since, as in the one-dimensional case, if the origin is a repelling fixed point for F then it is an attracting fixed point for f^{-1}, it suffices to study the attracting case.

The attracting (resp. repelling) case was first studied by Poincaré [203]; Fatou [116] and Bieberbach [45] used this case to construct the first examples of proper open subsets of $\mathbb{C}^n$ (with $n \geq 2$) biholomorphic to the whole of $\mathbb{C}^n$, a phenomenon that cannot occur in one variable. A very clear exposition of this case was given, using a functional approach, by Rosay and Rudin in the appendix of [233]. Recently, Berteloot in [42] provided a very beautiful exposition of this functional approach to the problem, and we refer to [42] for the proofs of the results presented here (see also [206]).

The first observation is that, in the attracting (and hence in the repelling) case, there can be only finitely many resonances.

Lemma 1.59. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$. If $|\lambda_j| < 1$ for all $j \in \{1, \dots, n\}$, then we have $\mathrm{card}(\bigcup_{j=1}^n \mathsf{Res}_j(\lambda)) < +\infty$. If moreover $0 < |\lambda_1| \leq \cdots \leq |\lambda_n| < 1$, then $Q \in \mathsf{Res}_j(\lambda)$ only if it is of the form $Q = (0, \dots, 0, q_{j+1}, \dots, q_n)$, and*

$$|Q| \leq \left\lfloor \frac{\log|\lambda_1|}{\log|\lambda_n|} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the integer part.

Proof. Up to reordering the coordinates, we may assume that

$$0 < |\lambda_1| \leq \cdots \leq |\lambda_n| < 1.$$

Hence $|\lambda_1| \leq |\lambda_j| \leq |\lambda_j|^{q_1+\cdots+q_j} |\lambda_n|^{|Q|-(q_1+\cdots+q_j)} \leq |\lambda_j|^{q_1+\cdots+q_j}$, for any multi-index Q with $|Q| \geq 2$, and we have the thesis. □

Moreover Poincaré proved the following result.

Theorem 1.60 (Poincaré, [203])**.** *Let G be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin, with linear part Λ such that $a\|z\| \leq \|\Lambda z\| \leq A\|z\|$, where $0 < a \leq A < 1$, and let $F\colon B_r \to \mathbb{C}^n$ be a holomorphic map such that*

$$F = G + \sum_{m \geq k} H_m$$

where $H_m \in \mathcal{H}^m$. Then, if $k > \log(a)/\log(A)$, the sequence $\{G^{-p} \circ G^p\}_p$ converges to a germ of biholomorphism Φ such that $\Phi(O) = O$, $D\Phi_O = \mathrm{Id}$ and $\Phi^{-1} \circ F \circ \Phi = G$.

Then, since the proof of Poincaré-Dulac Theorem 1.26 implies that we can always holomorphically conjugate a germ F to a germ G in Poincaré-Dulac form up to any finite given order, we have the following result.

Theorem 1.61 (Poincaré, [203]; Dulac, [96]). *Let F be a germ of biholomorphism of $\mathbb{C}^n$ in the Poincaré domain. Then F is locally holomorphically conjugated to one of its Poincaré-Dulac normal forms. Moreover, if the spectrum of DF_O is non-resonant, then F is holomorphically linearizable.*

Reich [212] describes holomorphic normal forms when DF_O belongs to the Poincaré domain and there are resonances (see also [102]).

1.4.2 Torus actions and Holomorphic normalization in higher dimension

In [208], following the analogous idea of Zung in [273] for germs of holomorphic vector fields, we found that commuting with a linearizable germ gives us information on the germs conjugated to a given one, and also on the linearization. More precisely we proved the following results (see [208] for the precise definitions and the proofs).

Theorem 1.62 (R. [208]). *Let F be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Then F commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $1 \leq r \leq n$ with weight matrix $\Theta \in M_{n\times r}(\mathbb{Z})$ if and only there exists a local holomorphic change of coordinates conjugating F to a germ with linear part in Jordan normal form and containing only Θ-resonant monomials.*

Theorem 1.63 (R. [208]). *Let F be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Then F is holomorphically linearizable if and only if it commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $1 \leq r \leq n$ with weight matrix Θ having no resonances.*

We then found out in a clear and computable manner what kind of structure a torus action must have in order to obtain a Poincaré-Dulac holomorphic normalization from Theorem 1.62. In particular, to do so we need to link in a clever way the eigenvalues of DF_O to the weight matrix of the action. Zung dealt with this problem in the case of holomorphic vector fields (see [273]), introducing the key notion of *toric degree* of a vector field.

It is a common thinking that once something can be done with germs of vector fields, i.e., for continuous local dynamical systems, then it can be translated analogously for germs of biholomorphisms, i.e., for discrete local dynamical systems. This is not completely true. At the very least there are torsion phenomena to be considered, preventing a straightforward translation from additive resonances (see below for the definition) to multiplicative resonances, and giving rise to new behaviors. One of the difficulties we overcame was exactly to understand up to which point one can push the analogies between continuous and discrete dynamics in the normalization problem. Following Écalle [101], we used the following definition of torsion.

Definition 1.64. *Let $\lambda \in (\mathbb{C}^*)^n$. The* torsion *of λ is the natural integer τ such that*

$$\frac{1}{\tau}2\pi i\mathbb{Z} = (2\pi i\mathbb{Q}) \cap \left((2\pi i\mathbb{Z}) \bigoplus_{1\leq j\leq n} (\log(\lambda_j)\mathbb{Z}) \right).$$

To understand what kind of structure a torus action must have in the case of germs of biholomorphisms to obtain a result equivalent to the one obtained by Zung, we first needed a right notion of toric degree for germs of biholomorphisms, and to link it to the torsion we introduced above. The link and the structure we found are more complicated than what one would expect: torsion is not enough to measure the difference between germs of holomorphic vector fields and germs of biholomorphisms. We therefore needed a more detailed study.

Notice that given $\lambda \in (\mathbb{C}^*)^n$, there is a unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$, i.e., $\lambda_j = e^{2\pi i[\varphi_j]}$ for every $j = 1, \dots, n$. The right definition of toric degree for maps is then the following

Definition 1.65. *Let $[\varphi] = ([\varphi_1], \dots, [\varphi_n]) \in (\mathbb{C}/\mathbb{Z})^n$, where $[\,\cdot\,]\colon \mathbb{C}^n \to (\mathbb{C}/\mathbb{Z})^n$ denotes the standard projection. The* toric degree of $[\varphi]$ *is the minimum $r \in \mathbb{N}$ such that there exist $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ and $\theta^{(1)}, \dots, \theta^{(r)} \in \mathbb{Z}^n$ such that*

$$[\varphi] = \left[\sum_{k=1}^{r} \alpha_k \theta^{(k)}\right].$$

The r-tuple $\theta^{(1)}, \dots, \theta^{(r)}$ is called a r-tuple of toric vectors associated to $[\varphi]$*, and the numbers $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ are* toric coefficients *of the toric r-tuple.*

Definition 1.66. *Given $\theta \in \mathbb{C}^n$ and $j \in \{1, \dots, n\}$, we say that a multi-index $Q \in \mathbb{N}^n$, with $|Q| = \sum_{h=1}^{n} q_h \geq 2$, gives an* additive resonance relation for θ relative to the j-th coordinate *if*

$$\langle Q, \theta \rangle = \sum_{h=1}^{n} q_h \theta_h = \theta_j$$

and we put

$$\mathrm{Res}_j^+(\theta) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \langle Q, \theta \rangle = \theta_j\}.$$

Given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$, the set

$$\mathsf{Res}_j([\varphi]) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, [\langle Q, \varphi \rangle - \varphi_j] = [0]\}$$

of multiplicative resonances of $[\varphi]$ is well-defined and we have $\mathsf{Res}_j(\lambda) = \mathsf{Res}_j([\varphi])$, where $\lambda = e^{2\pi i[\varphi]}$.

We then found relations between the additive resonances of toric vectors associated to $[\varphi]$ and the multiplicative resonances of $[\varphi]$. One of the advantages of the approach we found is that we shall be able to easily compute the multiplicative resonances, passing through the additive resonances of r-tuples of toric vectors.

Given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ of toric degree $1 \leq r \leq n$, even when the r-tuple of toric vectors associated to $[\varphi]$ is not unique, we can always say whether the toric coefficients are rationally independent with 1 or not.

Definition 1.67. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$. We say that $[\varphi]$ is in the* torsion-free case*, or simply* $[\varphi]$ is torsion-free*, if its r-tuples of toric vectors have toric coefficients rationally independent with* 1.

As a first application of our methods, we proved the following characterization of the vectors $\lambda \in (\mathbb{C}^*)^n$ without torsion.

Theorem 1.68 (R. [208])**.** *Let $\lambda = e^{2\pi i[\varphi]} \in (\mathbb{C}^*)^n$. Then $[\varphi]$ is torsion-free if and only if the torsion of λ is* 1.

In the torsion case, we can always find a more useful toric r-tuple.

Definition 1.69. *Let $[\varphi] = ([\varphi_1], \dots, [\varphi_n]) \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$ in the torsion case. We say that a r-tuple $\eta^{(1)}, \dots, \eta^{(r)}$ of toric vectors associated to $[\varphi]$ with toric coefficients $\beta_1, \dots, \beta_r$ rationally dependent with* 1 *is* reduced *if $\beta_1 = 1/m$ with $m \in \mathbb{N} \setminus \{0, 1\}$ and $m, \eta_1^{(1)}, \dots, \eta_n^{(1)}$ coprime. In this case the toric vectors $\eta^{(2)}, \dots, \eta^{(r)}$ are called* reduced torsion-free toric vectors *associated to $[\varphi]$.*

We have explicit techniques to compute the toric degree and toric r-tuples (reduced in the torsion case) of $[\varphi]$. Furthermore, we also showed that, in the torsion case, the torsion of $e^{2\pi i[\varphi]}$ always divides m.

As expected, we were able to show that the torsion-free case behaves as the vector fields case, proving the following result (which works even when DF_O is not diagonalizable).

Theorem 1.70 (R. [208])**.** *Let F be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Assume that, denoting by $\lambda = \{\lambda_1, \dots, \lambda_n\}$ the spectrum of DF_O, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$ is of toric degree $1 \le r \le n$ and torsion-free. Then F admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with F and such that the columns of the weight matrix of the action are a r-tuple of toric vectors associated to $[\varphi]$.*

The torsion case is more delicate and difficult to deal. First of all, given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ with toric degree $1 \le r \le n$ and torsion $\tau \ge 2$, and a reduced toric r-tuple $\eta^{(1)}, \dots, \eta^{(r)}$, we always have

$$\bigcap_{k=2}^{r} \mathsf{Res}_j^+(\eta^{(k)}) \supseteq \mathsf{Res}_j([\varphi]) \supseteq \bigcap_{k=1}^{r} \mathsf{Res}_j^+(\eta^{(k)}).$$

This suggests a subdivision in several subcases, all realizable (we found examples for all of them) and, surprisingly, having very different behaviours. There are cases similar to the case of germs of vector fields (even when the torsion is non zero), and cases that are indeed different. In particular, considering iterates of F to reduce to the torsion-free case hides very interesting phenomena, and it does not allow to see that some torsion cases can be directly studied. Moreover, we have explicit and computable techniques to decide in which subcase a given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ belongs to.

Definition 1.71. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$ and in the torsion case. We say that $[\varphi]$ is in the* impure torsion case *if, for one (and hence any: see Lemma 7.6) reduced r-tuple $\eta^{(1)}, \dots, \eta^{(r)}$ of toric vectors associated to $[\varphi]$ we have*

$$\mathsf{Res}_j([\varphi]) = \bigcap_{k=2}^{r} \mathsf{Res}_j^+(\eta^{(k)}), \tag{1.13}$$

for all $j \in \{1, \dots, n\}$. Otherwise we say that $[\varphi]$ is in the pure torsion case*.*

The impure torsion case is the subcase behaving as the case of germs of vector fields, and in which, again, we do not need DF_O diagonalizable. In fact, we could prove the following result.

Theorem 1.72 (R. [208])**.** *Let F be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Assume that, denoting by $\lambda = \{\lambda_1, \dots, \lambda_n\}$ the spectrum of DF_O, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$ is of toric degree $1 \le r \le n$ and in the impure torsion case. Then it admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $r-1$ commuting with F, and such that the columns of the weight matrix of the action are reduced torsion-free toric vectors associated to $[\varphi]$.*

The next subcase is the following.

Definition 1.73. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$ and in the pure torsion case. We say that $[\varphi]$* can be simplified *if it admits a reduced r-tuple of toric vectors $\eta^{(1)}, \dots, \eta^{(r)}$ such that*

$$\mathsf{Res}_j([\varphi]) = \bigcap_{k=1}^{r} \mathsf{Res}_j^+(\eta^{(k)}), \tag{1.14}$$

for all $j = 1, \dots, n$. The r-tuple $\eta^{(1)}, \dots, \eta^{(r)}$ is said a simple reduced *r-tuple associated to $[\varphi]$.*

Condition (1.14) depends on the chosen toric r-tuple. However, we have techniques to decide whether $[\varphi]$ can be simplified or not. The case in which $[\varphi]$ can be simplified is similar to the case of germs of vector fields, but we have a distinction between the case of DF_O diagonalizable and DF_O not diagonalizable, as we see in the following result.

Theorem 1.74 (R. [208]). *Let F be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Assume that, denoting by $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ the spectrum of DF_O, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$ is of toric degree $1 \le r \le n$ and in the pure torsion case and it can be simplified. Then*

(i) *if DF_O is diagonalizable, F admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with F and such that the columns of the weight matrix Θ of the action are a simple reduced r-tuple of toric vectors associated to $[\varphi]$;*

(ii) *if DF_O is not diagonalizable and there exists a simple reduced r-tuple of toric vectors associated to $[\varphi]$ such that its vectors are the columns of a matrix Θ compatible with DF_O, F admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with F and with weight matrix Θ.*

The case in which $[\varphi]$ cannot be simplified is the furthest from the case of germs of vectors fields, because we cannot reduce the multiplicative resonances to additive ones. In fact, in this case we only have a sufficient condition for holomorphic normalization.

Proposition 1.75 (R. [208]). *Let F be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Assume that, denoting by $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ the spectrum of DF_O, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$ is of toric degree $1 \le r \le n$ and in the pure torsion case and it cannot be simplified. If there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with F and such that the columns of the weight matrix of the action are a reduced r-tuple of toric vectors associated to $[\varphi]$, then F admits a holomorphic Poincaré-Dulac normalization.*

These results give a complete understanding of the relations between torus actions, holomorphic Poincaré-Dulac normalizations, and torsion phenomena. In [208] we also gave an example of techniques to construct torus actions.

1.5 Local dynamics in higher dimension

The description of the local dynamics of a holomorphic local dynamical system near an isolated fixed point in higher dimension cannot yet be given in a systematic way as we presented it in dimension 1.

As in dimension one, there is a natural dichotomy between the holomorphically linearizable case, where the dynamics is easy to understand, and the non linearizable case. However, the *non linearizable* case splits here into several different cases: a germ can be formally but not holomorphically linearizable, resonant not formally linearizable but holomorphically normalizable, resonant and not holomorphically normalizable. Moreover, unless for germs in the Poincaré domain (that is attracting or repelling), we do not have yet a complete description of the local holomorphic dynamics of holomorphically normalizable germs.

This could be one of the reasons why, until very recently, there has not been a systematic study of local dynamics in higher dimension using normal forms. We will present in Chapter

3 the recent results obtained using normal forms for non-formally linearizable resonant germs, when the resonances are finitely generated. These germs are called *multi-resonant*.

Another reason could be given by the fact that, as we pointed out in Remark 1.30, Poincaré-Dulac normal forms give no simplification in the tangent to the identity case. The local dynamics of tangent to the identity germs in higher dimension has been extensively studied since the eighties and we will present in the next chapter the description achieved so far, detailing our contribution to it.

We end this section and this chapter by citing the results we know on local dynamics in higher dimension in particular cases, without giving the precise statements.

We start by recalling that a local holomorphic invertible dynamical system with a hyperbolic fixed point is always topologically linearizable [132, 133, 137] and the *Stable Manifold Theorem* (see Wu [265] for a proof in the holomorphic category) gives us a description of the local dynamics near a hyperbolic fixed point.

The local dynamics of semi-attractive dynamical systems has been studied by Fatou [117], Nishimura [186], Ueda [251, 252], Hakim [134], Rivi [217, 218], Rong [226, 232]. Their results more or less say that the eigenvalue 1 yields the existence of a manifold M of a suitable dimension where orbits converge to the origin, which lies in the boundary of M, while the eigenvalues with modulus less than one ensure, roughly speaking, that the orbits of points in the normal bundle of M close enough to M are attracted to it.

The local dynamics of parabolic-elliptic dynamical systems has been studied by Bracci and Molino [56], Rong [221, 223], and Bracci and Rong [59]. Interesting very recent results generalizing Pérez-Marco's results in dimension 2 for *semi-indifferent* germs, that is with eigenvalues λ, μ with $|\lambda| = 1$ and $|\mu| < 1$ are due to Firsova, Lyubich, Radu and Tanase in [118] and Lyubich, Radu and Tanase in [177].

Chapter 2

Local dynamics of tangent to the identity germs

In this chapter we will deal with the local dynamics of tangent to the identity germs in higher dimension. that is $F \in \mathrm{End}(\mathbb{C}^n, O)$ whose local expansion as sum of homogeneous polynomials is

$$F(z) = z + P_{\nu+1}(z) + \cdots , \tag{2.1}$$

where $P_{\nu+1}$ is the first non-zero term in the homogeneous expansion of F, and $\nu + 1 \geq 2$ is called the *multiplicity* or *order* of F .

We will start by briefly recalling the main results in this setting, before focusing on our latest contribution and its applications to this topic. We refer to the interesting and complete survey [11] and references therein for a more detailed exposition of the state of the art.

In the following we will denote by $\mathrm{End}_1(\mathbb{C}^n, O)$ the set of germs of holomorphic endomorphisms of $\mathbb{C}^n$ fixing the origin $O \in \mathbb{C}^n$ and tangent to the identity, and by $\mathsf{Diff}(\mathbb{C}^n, O)$ the set of germs of holomorphic diffeomorphisms of $\mathbb{C}^n$ fixing the origin $O \in \mathbb{C}^n$.

2.1 Leau-Fatou flowers in higher dimension: petals

The natural question arising here is whether it is possible to find a multi-dimensional version of the Leau-Fatou flower theorem, giving a complete characterization of the local dynamics in a full neighbourhood of the origin.

A first special situation is when F admits a non-trivial one-dimensional F-invariant curve passing through the origin, that is an injective holomorphic map $\psi\colon \Delta \to \mathbb{C}^n$, where $\Delta \subset \mathbb{C}$ is a neighbourhood of the origin, such that $\psi(0) = O$, $\psi'(0) \neq O$ and $F\big(\psi(\Delta)\big) \subseteq \psi(\Delta)$ with $F|_{\psi(\Delta)} \not\equiv \mathrm{Id}$. In this case in fact we can apply Leau-Fatou flower theorem to $F|_{\psi(\Delta)}$ obtaining a one-dimensional Fatou flower for F inside the invariant curve. In particular, if $\widetilde{z} \in \psi(\Delta)$ belongs to an attractive petal, we have $F^{\circ k}(\widetilde{z}) \to O$ and $[F^{\circ k}(\widetilde{z})] \to [\psi'(0)]$, where $[\cdot]\colon \mathbb{C}^n \setminus \{O\} \to \mathbb{P}^{n-1}(\mathbb{C})$ is the canonical projection. It turns out that $[\psi'(0)]$ cannot be any direction in $\mathbb{P}^{n-1}(\mathbb{C})$ as proven by Hakim (see also [26]).

Proposition 2.1 (Hakim, [135]). *Let $F(z) = z + P_{\nu+1}(z) + \cdots \in \mathrm{End}_1(\mathbb{C}^n, O)$ be tangent to the identity of order $\nu + 1 \geq 2$. Assume there is $\widetilde{z}$ in a neighbourhood of the origing such that $F^{\circ k}(\widetilde{z}) \to O$ and $[F^{\circ k}(\widetilde{z})] \to [v] \in \mathbb{P}^{n-1}(\mathbb{C})$. Then $P_{\nu+1}(v) = \lambda v$ for some $\lambda \in \mathbb{C}$.*

This leads to the following definition.

Definition 2.2. *Let $P\colon \mathbb{C}^n \to \mathbb{C}^n$ be a homogeneous polynomial map. A direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ is* characteristic *for P if $P(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. Furthermore, we shall say that $[v]$ is* degenerate *if $P(v) = O$, and* non-degenerate *otherwise.*

From now on, given $F \in \mathrm{End}_1(\mathbb{C}^n, O)$ tangent to the identity of order $\nu + 1 \geq 2$, every notion/object/concept introduced for its leading term $P_{\nu+1}$ will be introduced also for F; for instance, a (degenerate/non-degenerate) characteristic direction for $P_{\nu+1}$ will also be a (degenerate/non-degenerate) characteristic direction for F.

Characteristic directions always exist, and it is not difficult to show (see for example [23]) that a *generic* F has exactly $(\nu^n - 1)/(\nu - 1)$ characteristic directions, counted with respect to a suitable multiplicity, but it can also happen that for special germs all directions are characteristic.

Definition 2.3. *We shall say that P is* dicritical *if all directions are characteristic;* non-dicritical *otherwise.*

The local dynamics of dicritical tangent to the identity germs has been investigated by Brochero-Martínez in [68, 69]. In the following we will only consider *non-dicritical* tangent to the identity germs.

Remark 2.4. *Note that if $F \in \mathrm{End}_1(\mathbb{C}^n, O)$ is given by* (2.1)*, then the local expansion of $F^{-1} \in \mathrm{End}_1(\mathbb{C}^n, O)$ is given by*

$$F^{-1}(z) = z - P_{\nu+1}(z) + \cdots .$$

In particular, F and F^{-1} have the same (degenerate/non-degenerate) characteristic directions.

If we have an F-invariant one-dimensional curve ψ through the origin then $[\psi'(0)]$ must be a characteristic direction, and if $\psi\colon \Delta \to \mathbb{C}^n$ is a one-dimensional curve with $\psi(0) = O$ and $\psi'(0) \neq O$ such that $F|_{\psi(\Delta)} \equiv \mathrm{Id}$, it is easy to see that $[\psi'(0)]$ must be a degenerate characteristic direction for F. However, in general there are non-degenerate characteristic directions which are not tangent to any F-invariant curve passing through the origin.

Example 2.5 (Hakim, [135])**.** *The germ $F \in \mathrm{End}(\mathbb{C}^2, O)$ given by*

$$F(z, w) = \left(\frac{z}{1+z}, w + z^2 \right) ,$$

is tangent to the identity of order 1, and $P_2(z, w) = (-z^2, z^2)$. In particular, F has a degenerate characteristic direction $[0 : 1]$ and a non-degenerate characteristic direction $[v] = [1 : -1]$. The degenerate characteristic direction is tangent to the curve $\{z = 0\}$, which is pointwise fixed by F. No F-invariant curve can be tangent to $[v]$.

Ribón has given in [216] examples of germs having no holomorphic invariant curves at all. For instance, this is the case for germs of the form $F(z, w) = (z + w^2, w + z^2 + \lambda z^5)$ for all $\lambda \in \mathbb{C}$ outside a polar Borel set.

The first result that we would like to cite here states that we do always have a Fatou flower tangent to a non-degenerate characteristic direction, even when there are no invariant complex curves containing the origin in their relative interior. To state it, we need to give a definition of *petal* in this multidimensional setting.

Definition 2.6. *A* parabolic curve *for $F \in \mathrm{End}_1(\mathbb{C}^n, O)$ tangent to the identity is an injective holomorphic map $\varphi\colon D \to \mathbb{C}^n \setminus \{O\}$ satisfying the following properties:*

(i) *D is a simply connected domain in $\mathbb{C}$ with $0 \in \partial D$;*

(ii) *φ is continuous at the origin, and $\varphi(0) = O$;*

(iii) *$\varphi(D)$ is F-invariant, and $(F|_{\varphi(D)})^{\circ k} \to O$ uniformly on compact subsets as $k \to +\infty$.*

Furthermore, if $[\varphi(\zeta)] \to [v]$ in $\mathbb{P}^{n-1}(\mathbb{C})$ as $\zeta \to 0$ in D, we shall say that the parabolic curve φ is tangent *to the direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$. Finally, a* Fatou flower *with ν* petals *tangent to a direction $[v]$ is a holomorphic map $\Phi\colon D_{\nu,\delta} \to \mathbb{C}$, where*

$$D_{\nu,\delta} = \{z \in \mathbb{C} \mid |z^\nu - \delta| < \delta\}$$

is such that Φ restricted to any connected component of $D_{\nu,\delta}$ is a parabolic curve tangent to $[v]$, a petal *of the Fatou flower. If ν is the order of F then we shall talk of a Fatou flower for F without mentioning the number of petals.*

Écalle, using his resurgence theory, and Hakim, using more classical methods, have proved the following result (see also [259]), of which a proof in full generality can be found in [26].

Theorem 2.7 (Écalle [99], Hakim [135, 136])**.** *Let $F \in \mathrm{End}(\mathbb{C}^n, O)$ be tangent to the identity, and $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ a non-degenerate characteristic direction for F. Then there exists (at least) one Fatou flower tangent to $[v]$. Furthermore, for every petal $\varphi\colon \Delta \to \mathbb{C}^n$ of the Fatou flower there exists a* Fatou coordinate*, that is an injective holomorphic map $\chi\colon \varphi(\Delta) \to \mathbb{C}$ such that $\chi\big(f(z)\big) = \chi(z) + 1$ for all $z \in \varphi(\Delta)$.*

A characteristic direction is a *complex* direction, not a real one; so it should not be confused with the attracting/repelling directions of Leau-Fatou Flower Theorem 1.18. All petals of a Fatou flower are tangent to the same characteristic direction, but each petal is tangent to a different real direction inside the same complex (characteristic) direction. In particular, Fatou flowers of F and F^{-1} are tangent to the same characteristic directions but the corresponding petals are tangent to different real directions, as in Leau-Fatou Flower Theorem 1.18.

In particular for the germ of Example 2.5 there exist parabolic curves tangent to $[1 : -1]$ even though there is no invariant curve passing through the origin tangent to that direction.

Theorem 2.7 applies to germs tangent to the identity having non-degenerate characteristic directions. However, it is not difficult to find examples of germs having only degenerate characteristic directions. In dimension 2 it is possible to obtain Fatou flowers also in this case.

Theorem 2.8 (Abate [8])**.** *Every germ $F \in \mathrm{End}(\mathbb{C}^2, O)$ tangent to the identity, with the origin as an isolated fixed point, admits at least one Fatou flower tangent to some direction.*

Abate proved in [8] a more general version of this result and we refer to [11] for its statement and a sketch of the proof.

Other interesting results in parabolic curves for tangent to the identity germs can be found in [23, 14, 51, 61, 183, 217, 224, 225, 228].

We end this section by mentioning that Abate and Tovena studied in [24] the dynamics of time-1 maps of n-dimensional homogeneous vector fields, reducing it to a study of singular holomorphic foliations in Riemann surfaces of $\mathbb{P}^{n-1}(\mathbb{C})$ and of geodesics for meromorphic connections on Riemann surfaces. Since a singular holomorphic foliation in $\mathbb{P}^1(\mathbb{C})$ is completely determined by its finite set of singular points, when $n = 2$ the problem reduces to the study of geodesics for a meromorphic connection on $\mathbb{P}^1(\mathbb{C})$.

They introduced a refined classification for characteristic directions in dimension 2. Given $\nu \geq 1$ and $P_{\nu+1} = (Q_1(z,w), Q_2(z,w))$ a homogeneous polynomial map of $\mathbb{C}^2$ of degree $\nu + 1$, we clearly have that $[1 : u_0]$ is a characteristic direction when $Q_2(1, u_0) = u_0 Q_1(1, u_0)$. Equivalently if we define $g_2(u) = Q_2(1, u) - uQ_1(1, u)$, then $[1 : u_0]$ is a characteristic direction if $g_2(u_0) = 0$. Define $g_1(u) = Q_1(1, u)$.

Let $\mu_j(u_0) \in \mathbb{N}$ be the order of vanishing of g_j at u_0. Then we say that the direction given by $[1 : u_0]$ is:

- an *apparent* characteristic direction if $\mu_2(u_0) < \mu_1(u_0) + 1$,
- a *Fuchsian* characteristic direction if $\mu_2(u_0) = \mu_1(u_0) + 1$,
- an *irregular* characteristic direction if $\mu_2(u_0) > \mu_1(u_0) + 1$.

As we shall recall in Section 2.2.1, it is possible to associate holomorphic invariants to characteristic directions. Here, we have the *index of a characteristic direction* $i_{[1:u_0]}$, which is defined as the residue $\mathrm{Res}_{u=u_0} \frac{g_1(u)}{g_2(u)}$.

Abate and Tovena gave a complete description of the dynamics for a substantial class of examples in $\mathbb{C}^2$ (we refer to [11, Section 6] for the definitions). For instance, they were able to completely describe the dynamics of most vector fields of the form

$$H(z, w) = \big(\rho z^2 + (1 + \tau)zw\big)\frac{\partial}{\partial z} + \big((1 + \rho)zw + \tau w^2\big)\frac{\partial}{\partial w}\ .$$

Proposition 2.9 (Abate, Tovena [24]). *Let $H \in \mathcal{X}_2^{\nu+1}$ be a homogeneous vector field on $\mathbb{C}^2$ of degree $\nu + 1 \geq 2$. Assume that H is non-dicritical and all its characteristic directions are Fuchsian of multiplicity* 1*. Assume moreover that for no set of characteristic directions the real part of the sum of the induced residues belongs to the interval $(-3/2, -1/2)$. Let $\gamma\colon [0, \varepsilon_0) \to \mathbb{C}^2$ be a maximal integral curve of H. Then:*

(a) *if $\gamma(0)$ belongs to a characteristic leaf L_{v_0}, then the image of γ is contained in L_{v_0} and moreover, either $\gamma(t) \to O$ (and this happens for a Zariski open dense set of initial conditions in L_{v_0}), or $\|\gamma(t)\| \to +\infty$;*

(b) *if $\gamma(0)$ does not belong to a characteristic leaf then either*

   (i) *γ converges to the origin tangentially to a characteristic direction $[v_0]$ whose index has positive real part, or*

   (ii) *$\|\gamma(t)\| \to +\infty$ tangentially to a characteristic direction $[v_0]$ whose index has negative real part.*

Furthermore, case (i) *happens for a Zariski open set of initial conditions.*

The conditions in Proposition 2.9 imply that there must be at least one index with positive real part.

2.1.1 The formal infinitesimal generator

A different approach to the study of parabolic curves in $\mathbb{C}^2$ has been suggested by Brochero-Martínez, Cano and López-Hernanz [70], and further developed by Câmara and Scárdua [81], López-Hernanz and Sánchez [173], López-Hernanz, R., Ribón and Sánchez [174], and López-Hernanz and Rosas [175]. It consists in using the formal infinitesimal generator of a germ tangent to the identity. To describe this approach, we need to introduce several definitions.

Definition 2.10. *We shall denote by* $\widehat{\mathcal{O}}_n = \mathbb{C}[\![z_1, \ldots, z_n]\!]$ *the space of formal power series in* n *variables. The* order $\mathrm{ord}(\widehat{\ell})$ *of* $\widehat{\ell} \in \widehat{\mathcal{O}}_n$ *is the lowest degree of a non-vanishing term in the Taylor expansion of* $\widehat{\ell}$. *A* formal map *is a* n*-tuple of formal power series in* n *variables; the space of formal maps will be denoted by* $\widehat{\mathcal{O}_n}^n$. *We shall denote by* $\widehat{\mathrm{End}}(\mathbb{C}^n, O)$ *the set of formal maps with vanishing constant term; by* $\widehat{\mathrm{End}}_1(\mathbb{C}^n, O)$ *the subset of formal maps tangent to the identity, and by* $\widehat{\mathrm{End}}_\nu(\mathbb{C}^n, O)$ *the subset of formal maps tangent to the identity of order at least* $\nu \geq 1$.

Definition 2.11. *We shall denote by* $\mathcal{X}_n$ *the space of germs at the origin of holomorphic vector fields in* $\mathbb{C}^n$. *A* formal vector field *is an expression of the form* $\widehat{X} = \widehat{X}_1 \frac{\partial}{\partial z_1} + \cdots + \widehat{X}_n \frac{\partial}{\partial z_n}$ *where* $\widehat{X}_1, \ldots, \widehat{X}_n \in \widehat{\mathcal{O}}_n$ *are the* components *of* $\widehat{X}$. *The space of* formal vector fields *will be denoted by* $\widehat{\mathcal{X}}_n$. *The* order $\mathrm{ord}(\widehat{X})$ *of* $\widehat{X} \in \widehat{\mathcal{X}}_n$ *is the minimum among the orders of its components. We put* $\widehat{\mathcal{X}}_n^k = \{\widehat{X} \in \widehat{\mathcal{X}}_n \mid \mathrm{ord}(\widehat{X}) \geq k\}$. *If* $\widehat{X} \in \widehat{\mathcal{X}}_n^k$, *the* principal part *of* $\widehat{X}$ *will be the unique polynomial homogeneous vector field* H_k *of degree exactly* k *such that* $\widehat{X} - H_k \in \widehat{\mathcal{X}}_n^{k+1}$. *A* characteristic direction *for* $\widehat{X}$ *is an invariant line for* H_k.

Remark 2.12. *There is a clear bijection between* $\widehat{\mathcal{X}}_n$ *and* $\widehat{\mathcal{O}_n}^n$ *obtained by associating to a formal vector field the* n*-tuple of its components; so we shall sometimes identify formal vector fields and formal maps without comments. In particular, this bijection preserves characteristic directions.*

If $X \in \mathcal{X}_n$ is a germ of holomorphic vector field vanishing at the origin (that is, of order at least 1), the associated time-1 map F_X will be a well-defined germ in $\mathrm{End}(\mathbb{C}^n, O)$, that can be recovered as follows (see, e.g., [70]):

$$F_X = \sum_{k \geq 0} \frac{1}{k!} X^{(k)}(\mathrm{Id}) , \tag{2.2}$$

where $X^{(k)}$ is the k-th iteration of X thought of as derivation of $\mathrm{End}(\mathbb{C}^n, O)$. Now, not every germ in $\mathrm{End}(\mathbb{C}^n, O)$ can be obtained as a time-1 map of a convergent vector field (see, e.g., [150, Theorem 21.31]). However, it turns out that the right-hand side of (2.2) is well-defined as a formal map for all $X \in \widehat{\mathcal{X}}_n^1$.

Definition 2.13. *The* exponential map $\exp\colon \widehat{\mathcal{X}}_n^1 \to \widehat{\mathrm{End}}(\mathbb{C}^n, O)$ *is defined by the right-hand side of* (2.2).

When $k \geq 2$, if $\widehat{X} \in \widehat{\mathcal{X}}_n^k$ has principal part H_k then it is easy to check that

$$\exp(\widehat{X}) = \mathrm{Id} + H_k + \text{h.o.t.} . \tag{2.3}$$

In particular, the exponential of a formal vector field of order k is a formal map tangent to the identity of order $k-1$. Takens (see, e.g., [150, Theorem 3.17]) has shown that on the formal level the exponential map is bijective:

Proposition 2.14. *The exponential map* $\exp\colon \widehat{\mathcal{X}}_n^{\nu+1} \to \widehat{\mathrm{End}}_\nu(\mathbb{C}^n, O)$ *is bijective for all* $\nu \geq 1$.

Definition 2.15. *If* $\widehat{F} \in \widehat{\mathrm{End}}_\nu(\mathbb{C}^n, O)$, *the unique formal vector field* $\widehat{X} \in \widehat{\mathcal{X}}_n^{\nu+1}$ *such that* $\exp(\widehat{X}) = \widehat{F}$ *is the* formal infinitesimal generator *of* $\widehat{F}$.

The idea is then to read properties of a holomorphic germ tangent to the identity from properties of its formal infinitesimal generator, using Theorem 2.7 as a bridge for going back from the formal side to the holomorphic side.

Let $\pi\colon (\widetilde{\mathbb{C}^2}, E) \to (\mathbb{C}^2, O)$ be the blow-up of the origin. If $\widehat{X} \in \widehat{\mathcal{X}}_2^2$ is a formal vector field and $[v] \in E$ is a characteristic direction of (the principal part of) $\widehat{X}$, then we can find a formal vector field $\widehat{X}_{[v]} \in \widehat{\mathcal{X}}_2^2$ such that $d\pi(\widehat{X}_{[v]}) = \widehat{X} \circ \pi$. This lifting is compatible with the exponential in the following sense.

Proposition 2.16 (Brochero-Martínez, Cano, López-Hernanz, [70]). *Let $F \in \mathrm{End}_1(\mathbb{C}^2, O)$ be tangent to the identity with formal infinitesimal generator $\widehat{X} \in \widehat{\mathcal{X}}_2^2$, and let $\widetilde{F} \in \mathrm{End}(\widetilde{\mathbb{C}^2}, E)$ be the lifting of F. Let $[v] \in E$ be a characteristic direction of F, and denote by $\widetilde{F}_{[v]}$ the germ of $\widetilde{F}$ at $[v]$. Then $\widetilde{F}_{[v]} = \exp(\widehat{X}_{[v]})$.*

In particular, Brochero-Martínez, Cano and López-Hernanz's proof of Theorems 2.8 go as follows: let $\widehat{X} \in \widehat{\mathcal{X}}_2^2$ be the formal infinitesimal generator of $F \in \mathrm{End}_1(\mathbb{C}^2, O)$ with an isolated fixed point (so that $\widehat{X}$ has an isolated singular point at the origin). Then the formal version of Camacho-Sad's theorem shows that we can find a finite composition $\pi\colon (M, E) \to (\mathbb{C}^2, O)$ of blow-ups at singular points and a smooth point $p \in E$ such that the lifting $\widehat{X}_p$ of $\widehat{X}$, in suitable coordinates centered at p adapted to E (in the sense that E is given by the equation $\{z = 0\}$ near p), has the expression

$$\widehat{X}_p(z, w) = z^m \left(\big(\lambda_1 z + O(z^2)\big)\frac{\partial}{\partial z} + \big(\lambda_2 w + O(z)\big)\frac{\partial}{\partial w} \right)$$

with $\lambda_1 \neq 0$, $\lambda_2/\lambda_1 \notin \mathbb{Q}^+$ and $m \geq \mathrm{ord}(\widehat{X}) - 1$. Then $\exp(\widehat{X}_p)$ has the form

$$\exp(\widehat{X}_p)(z, w) = \big(z + \lambda_1 z^{m+1} + O(z^{m+2}), w + \lambda_2 z^m w + O(z^{m+1})\big) ,$$

which has a non-degenerate characteristic direction transversal to E — and hence a Fatou flower outside the exceptional divisor. By Proposition 2.16, $\exp(\widehat{X}_p)$ is the blow-up of $\exp(\widehat{X}) = F$; therefore projecting this Fatou flower down by π we obtain a Fatou flower for F.

In [81] and [173] this approach has been pushed further showing how to relate formal separatrices and parabolic curves.

Definition 2.17. *A* formal curve *$\widehat{C}$ in $(\mathbb{C}^2, 0)$ is a reduced principal ideal of $\widehat{\mathcal{O}}_2$. Any generator of the ideal is an* equation *of the curve; the equation is defined up to an unit in $\widehat{\mathcal{O}}_2$. The* tangent cone *of a formal curve $\widehat{C}$ is the set of zeros of the homogeneous part of least degree of any equation of $\widehat{C}$; the* tangent directions *to $\widehat{C}$ are the points in $\mathbb{P}^1(\mathbb{C})$ determined by the tangent cone.*

It is known that a formal curve $\widehat{C}$ is irreducible if and only if it has a unique tangent direction.

Definition 2.18. *Let $\widehat{X} \in \widehat{\mathcal{X}}_2^2$. A* singular formal curve *for $\widehat{X}$ is a formal curve $\widehat{C} = (\widehat{\ell})$ such that $\widehat{X} = \widehat{\ell}\widehat{X}_1$ for some $\widehat{X} \in \widehat{\mathcal{X}}_2^1$. A* formal separatrix *of $\widehat{X}$ is a formal curve $\widehat{C} = (\widehat{\ell})$ such that $\widehat{X}(\widehat{\ell}) \in (\widehat{\ell})$. Clearly singular formal curves are formal separatrices.*

The corresponding notions for germs tangent to the identity are:

Definition 2.19. *Let $F \in \mathrm{End}_1(\mathbb{C}^2, O)$. A formal curve $\widehat{C} = (\widehat{\ell})$ is a* formal separatrix *for F if $\widehat{\ell} \circ f \in (\widehat{\ell})$. In particular, this means that F acts by composition on $\widehat{\mathcal{O}}_2/(\widehat{\ell})$; if the action is the identity, we say that $\widehat{C}$ is* completely fixed *by F. Notice that $\widehat{C}$ is completely fixed by F if and only if we can write $F = \mathrm{Id} + \widehat{\ell}\widehat{g}$ for some $\widehat{g} \in \widehat{\mathcal{O}}_2^2$.*

Proposition 2.20 (Camara, Scardua, [81]). *Let $\widehat{X} \in \widehat{\mathcal{X}}_2^2$ be the formal infinitesimal generator of $F \in \mathrm{End}_1(\mathbb{C}^2, O)$. Then:*

(i) *a formal curve is a formal separatrix for F if and only if it is a formal separatrix for $\widehat{X}$;*

(ii) *a formal curve is completely fixed for F if and only if it is a singular formal curve for $\widehat{X}$;*

(iii) *a completely fixed curve for F always has a convergent equation;*

(iv) *the tangent directions to a formal separatrix are characteristic directions for F, and the tangent directions to a completely fixed curve are degenerate characteristic directions for F.*

Let $\widehat{C} = (\widehat{\ell})$ be a formal curve, and $[v] \in \mathbb{P}^1(\mathbb{C})$ a tangent direction to $\widehat{C}$. If $\pi \colon (\widetilde{\mathbb{C}^2}, E) \to (\mathbb{C}^2, O)$ is the blow-up of the origin, we can find a formal curve $\pi^*\widehat{C}_{[v]} = (\widehat{\ell}_{[v]})$ at $[v]$ such that $\widehat{\ell}_{[v]} = \widehat{\ell} \circ \pi$; the tangent directions to $\pi^*\widehat{C}_{[v]}$ are higher order tangent directions of $\widehat{C}$. This construction can be iterated, and it gives a way of lifting formal curves along a finite sequence of blow-ups. Using this idea, and a generalization of Hakim's technique, López-Hernanz and Sánchez have been able to prove the following result.

Theorem 2.21 (López-Hernanz and Sánchez, [173])**.** *Let $F \in \mathrm{End}_1(\mathbb{C}^2, O)$ be a germ tangent to the identity admitting a formal separatrix $\widehat{C}$ not completely fixed. Then F or F^{-1} (or both) admit a parabolic curve tangent to (a tangent direction of) $\widehat{C}$.*

Together with López-Hernanz, Ribón and Sánchez, we generalized this result in a more general setting as we will explain in details in Section 2.3.

2.2 Leau-Fatou flowers in higher dimension: Parabolic manifolds

Parabolic curves are one-dimensional objects in an n-dimensional space; it is natural to wonder about the existence of higher dimensional invariant subsets. A sufficient condition for their existence has been given by Hakim; to state it we need to introduce another definition.

Definition 2.22. *Let $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ be a non-degenerate characteristic direction for a homogeneous map $P \colon \mathbb{C}^n \to \mathbb{C}^n$ of degree $\nu + 1 \geq 2$; in particular, $[v]$ is a fixed point for the meromorphic self-map $[P]$ of $\mathbb{P}^{n-1}(\mathbb{C})$ induced by P. The* directors *of P in $[v]$ are the eigenvalues $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$ of the linear operator*

$$\frac{1}{\nu}\left(d[P]_{[v]} - \mathrm{Id}\right) : T_{[v]}\mathbb{P}^{n-1}(\mathbb{C}) \to T_{[v]}\mathbb{P}^{n-1}(\mathbb{C}) .$$

As usual, if $F \in \mathrm{End}_1(\mathbb{C}^n, O)$ is of the form (2.1), then the directors *of F in a non-degenerate characteristic direction $[v]$ are the directors of $P_{\nu+1}$ in $[v]$. If all the directors of $[v]$ have strictly positive real parts, we call $[v]$ a* fully attractive *non-degenerate characteristic direction of h.*

Note that if $[v]$ is a non-degenerate characteristic direction of a homogeneous map P of degree $\nu + 1 \geq 2$, and $P(v) = \lambda v$. Then, up to replacing v by $(-\nu\lambda)^{-1/\nu} v$, we can assume

$$P(v) = -\frac{1}{\nu} v. \tag{2.4}$$

Definition 2.23. *A representative v of $[v]$ such that (2.4) is satisfied is called* normalized.

Note that a normalized representative is uniquely determined up to multiplication by ν-th roots of unity.

Remark 2.24. *Note that with this definition, in dimension 1 each tangent to the identity germ has exactly one non-degenerate characteristic direction which is clearly fully attractive.*

We showed in [26] that Definition 2.22 is equivalent to the original definition used by Hakim. Furthermore, in dimension 2 if $[v] = [1 : 0]$ is a non-degenerate characteristic direction of $P = (P_1, P_2)$ we have $P_1(1,0) \neq 0$, $P_2(1,0) = 0$ and the director can be easily computed:

$$\frac{1}{\nu} \frac{d}{d\zeta} \frac{P_2(1,\zeta) - \zeta P_1(1,\zeta)}{P_1(1,\zeta)}\bigg|_{\zeta=0} = \frac{1}{\nu}\left[\frac{\frac{\partial P_2}{\partial z_2}(1,0)}{P_1(1,0)} - 1\right].$$

Remark 2.25. *Recalling Remark 2.4 one sees that a germ $F \in \mathrm{End}_1(\mathbb{C}^n, O)$ tangent to the identity and its inverse F^{-1} have the same directors at their non-degenerate characteristic directions.*

Definition 2.26. *A* parabolic manifold *for a germ $F \in \mathrm{End}_1(\mathbb{C}^n, O)$ tangent to the identity is an F-invariant complex submanifold $M \subset \mathbb{C}^n \setminus \{O\}$ with $O \in \partial M$ such that $F^{\circ k}(z) \to O$ for all $z \in M$. A* parabolic domain *is a parabolic manifold of dimension n. We shall say that M is* centered *at the characteristic direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ if furthermore $[F^{\circ k}(z)] \to [v]$ for all $z \in M$.*

Hakim proved (see also [26] for the details of the proof) the following result.

Theorem 2.27 (Hakim [136])**.** *Let $F \in \mathrm{End}_1(\mathbb{C}^n, O)$ be tangent to the identity of order $\nu+1 \geq 2$. Let $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ be a non-degenerate characteristic direction, with directors $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{C}$. Furthermore, assume that $\mathrm{Re}\alpha_1, \dots, \mathrm{Re}\alpha_d > 0$ and $\mathrm{Re}\alpha_{d+1}, \dots, \mathrm{Re}\alpha_{n-1} \leq 0$ for a suitable $d \geq 0$. Then:*

(i) *There exist (at least) ν parabolic $(d+1)$-manifolds $M_1, \dots, M_\nu$ of $\mathbb{C}^n$ centered at $[v]$;*

(ii) *$F|_{M_j}$ is holomorphically conjugated to the translation $\tau(w_0, w_1, \dots, w_d) = (w_0+1, w_1, \dots, w_d)$ defined on a suitable right half-space in $\mathbb{C}^{d+1}$.*

2.2.1 Parabolic domains

Theorem 2.27 yields conditions ensuring the existence of parabolic domains attached to a non-degenerate characteristic direction. In fact, if all the directors of $[v]$ have positive real part, there is at least one parabolic domain.

Theorem 2.28 (Hakim [136])**.** *Let $F \in \mathrm{End}_1(\mathbb{C}^n, O)$ be tangent to the identity of order $\nu+1 \geq 2$. Let $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ be a fully attractive non-degenerate characteristic direction for F. Then there exist ν parabolic domains such that $F^{\circ j}(z) \neq 0$ for all j and $\lim_{j\to\infty}[F^{\circ j}(z)] = [v]$ for all fixed z in one such a parabolic domain. Moreover, if v is a normalized representative of $[v]$, then the parabolic domains can be chosen of the form*

$$M^i_{R,C} = \{(x,y) \in \mathbb{C} \times \mathbb{C}^{n-1} : x \in \Pi^i_R,\ \|y\| < C|x|\}, \tag{2.5}$$

where Π^i_R, $i = 1, \dots, k_0$, are the connected components of the set $\Delta_R = \{x \in \mathbb{C} : |x^{k_0} - \frac{1}{2R}| < \frac{1}{2R}\}$, and $R > 0$ is sufficiently large.

However, the condition given by Theorem 2.27 is not necessary for the existence of parabolic domains as shown for instance in [217] and [24], and in [229] where Rong gives conditions ensuring the existence of a parabolic domain when some directors have strictly positive real part and the others are equal to zero. Moreover, Lapan [166] has proved that if $n = 2$ and F has a unique characteristic direction $[v]$ which is non degenerate then there exists a parabolic domain centered at $[v]$ even though the director is necessarily 0.

In dimension 2, Vivas has found conditions ensuring the existence of a parabolic domain attached to Fuchsian and irregular degenerate characteristic directions, and Rong has found in [230] conditions ensuring the existence of a parabolic domain attached to apparent degenerate characteristic directions. Very recently, Lapan [167] has extended Rong's approach to cover more types of degenerate characteristic directions.

More precisely, Vivas has proved the following result:

Theorem 2.29 (Vivas [254])**.** *Let $F \in \mathrm{End}_1(\mathbb{C}^2, O)$ be tangent to the identity of order $\nu+1 \geq 2$, with O non-dicritical. Let $[v] \in \mathbb{P}^1(\mathbb{C})$ be a characteristic direction, and $\widetilde{f}$ the blow-up of F. Denote by $\mu+1 \geq 2$ the multiplicity, by $\tau \geq 0$ the transversal multiplicity, by $\iota \in \mathbb{C}$ the index, and by $\nu_o \geq 1$ the pure order of $\widetilde{f}$ at $[v]$. Assume that either*

(a) $[v]$ *is Fuchsian (thus necessarily $\tau \geq 1$ because $[v]$ is degenerate) and*

$$\mathrm{Re}(\iota) + \tau > 0\,, \qquad \left|\iota + \frac{\tau}{2} - \frac{\nu\mu}{2}\right| > \frac{\tau}{2} + \frac{\nu\mu}{2}\,,$$

or

(b) $[v]$ *is Fuchsian, $\nu_o = 1$ and*

$$\left|\iota - \frac{\mu\nu}{2}\right| < \frac{\mu\nu}{2}\,,$$

or

(c) $[v]$ *is Fuchsian, $\nu_o > 1$ and*

$$\mathrm{Re}\iota + \tau > 0\,, \qquad \left|\iota - \frac{(\nu+1)\tau}{2}\right| > \frac{(\nu+1)\tau}{2}\,,$$

or

(d) $[v]$ *is irregular.*

Then there is a parabolic domain centered at $[v]$.

2.3 Formal invariant curves

In collaboration with López-Hernanz, Ribón and Sanz-Sánchez we investigated in [174] the local dynamics of germs of holomorphic diffeomorphisms of $\mathbb{C}^2$ having a formal invariant curve. We will explain here the result we obtained and its consequences in the tangent to the identity case.

Let $F \in \mathsf{Diff}(\mathbb{C}^n, O)$ be a germ of a holomorphic diffeomorphism. Recall that a *stable set* of F is a subset $B \subset V$ of an open neighbourhood V of 0 where F is defined, which is invariant, i.e. $F(B) \subset B$, and such that the orbit of each point of B converges to 0. If B is an analytic, locally closed submanifold of V then we say that B is a *stable manifold* of F (in V).

As we briefly recalled in Chapter 1, for one-dimensional local diffeomorphisms the existence of stable manifolds depends mainly on the multiplier $\lambda = F'(0) \in \mathbb{C}$. More precisely, F has non-trivial stable manifolds when F is *(hyperbolic) attracting* ($|\lambda| < 1$), in which case a whole neighbourhood of $0 \in \mathbb{C}$ is a stable manifold, or *parabolic* (λ is a root of unity) and non-periodic, in which case the attracting petals of Leau-Fatou Flower Theorem 1.18 are stable manifolds. In the remaining cases, *(hyperbolic) repelling* ($|\lambda| > 1$), periodic or *irrationally neutral* ($|\lambda| = 1$ and λ is not a root of unity), the origin itself is the only stable manifold of F in any neighbourhood.

In [174] we studied the case of a planar diffeomorphism $F \in \mathsf{Diff}(\mathbb{C}^2, O)$ and we look for stable manifolds consisting of orbits which are asymptotic to a given invariant formal curve Γ. Moreover, we described a family of such stable manifolds whose union *captures* any orbit asymptotic to Γ. Following the terminology adopted by Ueda in [251], we constructed a *base of the set of orbits asymptotic to* Γ which is a union of stable manifolds. Our assumptions in order to guarantee the existence of such stable manifolds are just the necessary conditions inherited from the one-dimensional dynamics induced by F on Γ and we required no further hypotheses on the linear part at the origin DF_O.

We will now describe our main result in more precise terms. We will discuss at the end of this chapter its relations with some of the results mentioned earlier.

Recall that a formal curve Γ at $O \in \mathbb{C}^2$ is a reduced principal ideal of $\mathbb{C}[\![x, y]\!]$. It is called *irreducible* if Γ is a prime ideal. We say that Γ is *invariant* by F, or F-invariant, if $\Gamma \circ F = \Gamma$. If Γ is irreducible and F-invariant then we can consider the *restriction* $F|_\Gamma$, which is a formal diffeomorphism in one variable (see Section 2).

A formal irreducible curve Γ_0 is *m-periodic* if $\Gamma_0 \circ F^{\circ m} = \Gamma_0$ and m is the minimum positive integer holding such property. In that case, the formal curve

$$\Gamma = \bigcap_{j=0}^{m-1} \Gamma_0 \circ F^{\circ j}$$

is F-invariant. Note that if Γ_0 defines an analytic curve $V(\Gamma_0)$ then $V(\Gamma) = \cup_{j=0}^{m-1} F^{\circ j}(V(\Gamma_0))$. Thus $V(\Gamma)$ is the minimal F-invariant curve containing $V(\Gamma_0)$. Equivalently, Γ is the maximal F-invariant ideal contained in Γ_0, this conclusion being valid also in the formal setting. We say that Γ is the *invariant curve associated to* Γ_0. In this case, the irreducible components of Γ are the m-periodic curves $\Gamma_j := \Gamma_0 \circ F^{\circ j}$ for $j = 0, ..., m-1$.

Given a m-periodic curve Γ_0 of F, a non-trivial orbit O of F is said to be *asymptotic* to the associated invariant curve Γ if it converges to the origin and, for any finite composition of blow-ups of points $\sigma : M \to \mathbb{C}^2$, the ω-limit of the lifted sequence $\sigma^{-1}(O)$ is contained in the finite set determined by the components of Γ in the exceptional divisor $\sigma^{-1}(O)$ (see [174, Section 2] for details).

Our main result can be stated as follows.

Theorem 2.30 (López-Hernanz, R., Ribón, Sanz-Sánchez [174]). *Consider $F \in \mathsf{Diff}(\mathbb{C}^2, O)$ and let Γ_0 be a formal m-periodic curve of F whose associated invariant curve is denoted by Γ. Assume that the restriction $F^{\circ m}|_{\Gamma_0}$ is either attracting or parabolic and non-periodic. Then, in any sufficiently small open neighbourhood V of 0, there exists a non-empty finite family of pairwise disjoint stable manifolds $S_1, ..., S_r \subset V$ of F of pure positive dimension and with finitely many connected components such that the orbit of every point in S_j is asymptotic to Γ and such that any orbit of F asymptotic to Γ is eventually contained in $S_1 \cup \cdots \cup S_r$.*

It is worth mentioning that a diffeomorphism $F \in \mathsf{Diff}(\mathbb{C}^2, O)$ always has a formal periodic curve by a result of Ribón [216], although as we mentioned before they may be all divergent and non-invariant.

Roughly speaking, Theorem 2.30 can be interpreted by saying that the condition ensuring the existence of stable manifolds in dimension 1 also provides (applied to $F|_\Gamma$) stable manifolds of orbits asymptotic to Γ. Although these hypotheses are not necessary in general, if they are not satisfied then one can find simple examples where no orbit asymptotic to Γ exists. In the case where $F|_\Gamma$ is hyperbolic, being attracting is a necessary condition for having orbits asymptotic to Γ. In the case where $F|_\Gamma$ is periodic (and hence parabolic), since the set of fixed points of a diffeomorphism is an analytic set, either F is itself periodic or Γ is convergent. In the first case, there are no non-trivial orbits converging to the origin; in the second case, there are examples with no asymptotic orbits (for instance $F(x,y) = (-x, 2y)$ and $\Gamma = (y)$) and examples with asymptotic orbits (for instance $F = \mathrm{Exp}(y(x^2\partial/\partial x + y\partial/\partial y))$ and $\Gamma = (y)$). In the case where $F|_\Gamma$ is irrationally neutral, although we can also find simple linear examples with no asymptotic orbits, we still do not know whether there are examples with asymptotic orbits.

2.3.1 Idea of the proof of Theorem 2.30

It suffices to prove Theorem 2.30 in the case of an irreducible formal invariant curve, that is $m = 1$. In fact, if Γ_0 is m-periodic we can apply the theorem to $F^{\circ m}$ and the $F^{\circ m}$-invariant irreducible curve Γ_0. Let $\mathcal{F}_0 = \{S_1, ..., S_r\}$ be a family of stable manifolds of $F^{\circ m}$ obtained for a domain V in which every $F^{\circ j}$, for $j = 1, ..., m-1$, is defined and injective, and set $\mathcal{F} = \{\cup_{j=0}^{m-1} F^{\circ j}(S_1), \ldots, \cup_{j=0}^{m-1} F^{\circ j}(S_r)\}$. Then $\mathcal{F}$ is a family with the required properties of Theorem 2.30 for F and the invariant curve Γ. Notice that, since each component of Γ is invariant under $F^{\circ m}$, the points determined by Γ in the exceptional divisor after blow-ups are fixed points for the corresponding transform of $F^{\circ m}$. Therefore, an orbit $\mathcal{O} = \{F^{\circ n}(p)\}_{n\geq 0}$ of F is asymptotic to Γ if and only if each one of the m orbits $\mathcal{O}_j = \{F^{nm+j}(p)\}_{n\geq 0}$ of $F^{\circ m}$ for $j = 0, ..., m-1$ is asymptotic to one and only one of the components of Γ. Hence, the orbit under $F^{\circ m}$ of a point in $F^j(S_i)$ is asymptotic to $\Gamma_j = F^j(\Gamma_0)$ for any $j = 0, ..., m-1$ and any $i = 1, ..., r$ and thus $F^j(S_i) \cap F^k(S_l) = \emptyset$ whenever $i \neq l$ and $j, k \in \{0, \ldots, m-1\}$.

After this first reduction, the proof of Theorem 2.30 is divided into the two situations for $F|_\Gamma$, namely hyperbolic attracting or parabolic, since the arguments and the structure of the stable manifolds S_j are notably different in both cases.

Hyperbolic attracting case

The result in the case where $F|_\Gamma$ is hyperbolic attracting is a consequence of the classical Stable Manifold and Hartman-Grobman Theorems for diffeomorphisms. We show that Γ is an analytic curve eventually containing any orbit of F which is asymptotic to Γ. Indeed the hyperbolic case can be characterized in terms of the family of stable manifolds $\mathcal{F} = \{S_1, \ldots, S_r\}$ provided by Theorem 2.30 in the following way: $F|_\Gamma$ is hyperbolic if and only if $\overline{S}_j$ is a germ of analytic curve at 0 for some $1 \leq j \leq r$ and in this case $\mathcal{F} = \{\Gamma \setminus \{0\}\}$. We also prove that Γ is either non-singular or a cusp $y^p = x^q$ in some coordinates and that, in this last case, F is analytically linearizable.

The precise statement is the following.

Theorem 2.31 ([174, Theorem 2.5])**.** *Let $F \in \mathsf{Diff}(\mathbb{C}^2, O)$ and let Γ be an invariant formal curve of F. Assume that Γ is hyperbolic attracting. Then Γ is a germ of an analytic curve at the origin such that a (sufficiently small) representative of it is a stable manifold of F and contains the germ of any orbit of F asymptotic to Γ.*

Parabolic case

The case where $F|_\Gamma$ is parabolic is more involved. As usual, up to considering an iterate of F, we may assume that $F|_\Gamma$ is a *parabolic* formal diffeomorphism, i.e. $(F|_\Gamma)'(0) = 1$.

The first step is to show that, after finitely many blow-ups along Γ, we can consider analytic coordinates (x, y) at the origin such that Γ is non-singular and tangent to the x-axis and F is of the form

$$\begin{aligned} x \circ F(x,y) &= x - x^{k+p+1} + O(x^{2k+2p+1}) \\ y \circ F(x,y) &= \mu(y + x^k a(x) y + O(x^{k+p+1}y, x^{k+p+2})) \end{aligned} \tag{2.6}$$

where $k \geq 1$, $p \geq 0$ and $a(x)$ is a polynomial of degree at most p with $a(0) \neq 0$. Notice that $k+p+1$ is the *order of contact with the identity of the restriction* $F|_\Gamma$ and hence depends only on F and Γ.

Let $A(x) = A_0 + A_1 x + \cdots + A_p x^p$ be the polynomial defined by the formula

$$\log\mu + x^k \left(A_0 + A_1 x + \cdots + A_p x^p\right) = J_{k+p}\left(\log\left(\mu\left(1 + x^k a(x)\right)\right)\right),$$

where J_m denotes the truncation of a series up to degree m. As we mentioned in 2.1.1, the idea behind this definition is that the jets of order $k+p+1$ of F and of the exponential of the vector field

$$Z = -x^{k+p+1}\frac{\partial}{\partial x} + (\log\mu + x^k A(x))y\frac{\partial}{\partial y}$$

coincide, and the dynamics of F and $\mathrm{Exp}(Z)$ are somewhat related.

The behaviour of the orbits of the toy model $\mathrm{Exp}(Z)$ converging to the origin and asymptotic to the invariant curve $y = 0$, which plays the rôle of Γ, can be briefly described as follows. Given such an orbit $\mathcal{O} = \{(x_n, y_n)\}$, the sequence $\{x_n\}$ is an orbit of the one-dimensional parabolic diffeomorphism $x \mapsto \mathrm{Exp}(-x^{k+p+1}\frac{\partial}{\partial x})$ and hence it converges to $0 \in \mathbb{C}$ along a well-defined real limit direction, necessarily one of the $k+p$ half-lines $\xi\mathbb{R}^+$ with $\xi^{k+p} = 1$, called the *attracting directions* (they correspond to the central directions of the attracting petals in Leau-Fatou Flower Theorem). On the other hand, Z has a first integral $H(x, y) = yh(x)$, where

$$h(x) = \exp\left(\int \frac{\log\mu + x^k A(x)}{x^{k+p+1}}dx\right),$$

and the behaviour of the orbits of $\mathrm{Exp}(Z)$, since they are contained in fibers of H, depends on the asymptotics of H in a neighbourhood of the corresponding attracting direction ℓ.

Definition 2.32. *An attracting direction $\ell = \xi\mathbb{R}^+$ is a* node direction *for (F, Γ) if*

$$\left(\log|\mu|, \mathrm{Re}\left(\xi^k A_0\right), ..., \mathrm{Re}\left(\xi^{k+p-1}A_{p-1}\right)\right) < 0$$

in the lexicographic order; otherwise, it is a saddle direction. *In the case $|\mu| = 1$, we define the* first asymptotic significant order *of ℓ as p, if $\mathrm{Re}(\xi^{k+j}A_j) = 0$ for all $0 \leq j \leq p-1$, or as the first index $0 \leq r_\ell \leq p-1$ such that $\mathrm{Re}(\xi^{k+r_\ell}A_{r_\ell}) \neq 0$, otherwise.*

Therefore, up to making a linear change of variables so that $\ell = \mathbb{R}^+$, we have that ℓ is a *node* direction if $(\log|\mu|, \mathrm{Re}(A_0), ..., \mathrm{Re}(A_{p-1})) < 0$ in the lexicographic order, and otherwise, ℓ is a *saddle* direction.

In the simplest case where $|\mu| \neq 1$, that is F is *semi-hyperbolic*, then ℓ is a saddle or a node direction if $|\mu| > 1$ or $|\mu| < 1$, respectively. There exists a sector $\Omega \subset \mathbb{C}$ bisected by ℓ in which either $h(x)$ or $1/h(x)$ is a flat function depending on whether ℓ is a saddle or a node direction, respectively. Thus, the fibers of H in $\Omega \times \mathbb{C}$ behave correspondingly as a saddle (only $y = 0$ is bounded) or a node (any fiber is bounded and asymptotic to $y = 0$). In the general case, we can show a similar description for the fibers of H in $\Omega \times \mathbb{C}$, where Ω is a domain of $\mathbb{C}$ containing ℓ which is not necessarily a sector. Moreover, $\Omega \times \mathbb{C}$ eventually contains any orbit $\{(x_n, y_n)\}$ of $\mathrm{Exp}(Z)$ such that $\{x_n\}$ has ℓ as a limit direction. We obtain that $\Omega \times \mathbb{C}$ (respectively $\Omega \times \{0\}$) is a stable manifold of $\mathrm{Exp}(Z)$ when ℓ is a node direction (respectively saddle direction) composed of orbits asymptotic to the curve $y = 0$. The family of these stable manifolds satisfies the conclusions of Theorem 2.30.

For a general diffeomorphism F written in the reduced form (2.6), we obtain a similar description of the orbits asymptotic to Γ. In fact, we construct a family $\{S_\ell\}$ of stable manifolds of F, where ℓ varies in the set of attracting directions $\ell = \xi\mathbb{R}^+$, with $\xi^{k+p} = 1$, satisfying the assertion of Theorem 2.30. In particular, for the case of a saddle direction we obtain that S_ℓ is a parabolic curve, whereas in the case of a node direction we obtain that S_ℓ is a simply connected open set.

More precisely we obtain the following result, whose strategy of the proof is analogous to the one in [173], and inspired by the techniques used by Hakim in [135].

Theorem 2.33 (López-Hernanz, R., Ribón, Sanz-Sánchez [174, Theorem 2.5]). *Consider $F \in \mathsf{Diff}(\mathbb{C}^2, O)$ and let Γ be an invariant formal curve of F. Assume that Γ is parabolic and that the restricted diffeomorphism $F|_\Gamma$ is not periodic. Then, for any sufficiently small neighborhood V of the origin, there exists a non empty finite family of mutually disjoint stable manifolds $\{S_1, ..., S_r\}$ in V of pure positive dimension satisfying:*

(i) Every orbit in the union $S = \bigcup_{j=1}^r S_j$ is asymptotic to Γ.

(ii) S contains the germ of any orbit of F asymptotic to Γ.

(iii) If n is the order of the inner eigenvalue λ_Γ as a root of unity, then each S_j is a finite union of n connected and simply connected mutually disjoint stable manifolds $S_{j1}, \dots, S_{jn}$ of the iterated diffeomorphism $F^{\circ n}$ (i.e. either parabolic curves or open stable sets of $F^{\circ n}$). In fact, $S_{ji} = F(S_{j,i-1})$ for $i = 2, ..., n$ and for any j.

Moreover, if $\dim(S_j) = 1$ then S_j is asymptotic to Γ. If $\dim(S_j) = 2$, one can also choose S_j to be asymptotic to Γ.

2.3.2 Consequences of Theorem 2.30

A first consequence of our results in [174] is the following generalization of the results in [56] and [173].

Theorem 2.34 (López-Hernanz, R., Ribón, Sanz-Sánchez [174, Theorem 2]). *Let $F \in \mathsf{Diff}(\mathbb{C}^2, O)$. Let Γ be an irreducible formal invariant curve of F such that $F|_\Gamma$ is parabolic, with $F|_\Gamma \neq \mathrm{Id}$, and assume that $\mathrm{Spec}(DF_O) = \{1, \mu\}$, with $|\mu| \geq 1$. Then there exists a parabolic curve for F, which is asymptotic to Γ.*

This result gives a positive answer to the question stated in [33] concerning the existence of parabolic curves for the polynomial maps used in our construction of wandering domain.

We can also compare our approach to find stable manifolds with some special situations for the diffeomorphism F already treated in the literature.

Semi-hyperbolic attracting case: $|\mu| < 1$. In this case, every attracting direction is a node direction. We obtain $r = k + p$ open stable manifolds whose union forms a base for the set of orbits of F asymptotic to Γ. This case is the one considered by Ueda in [251], and our unified point of view recovers his result (observe that in the semi-hyperbolic case, the Poincaré-Dulac normal form $\widetilde{F}$ of F has a unique formal invariant curve $\widetilde{\Gamma}$ such that the restriction $\widetilde{F}|_{\widetilde{\Gamma}}$ is parabolic and hence so does F).

Semi-hyperbolic repelling case: $|\mu| > 1$. In this case, every attracting direction is a saddle direction and we obtain $r = k + p$ parabolic curves, defined as graphs of holomorphic functions over open sectors in the x-variable, whose union is a base of the set of orbits asymptotic to Γ. This case is also treated by Ueda in [252] and we again recover his conclusion.

Briot-Bouquet case: $\mathrm{Spec}(DF_O) = \{1\}$ and $p = 0$. In this case, every attracting direction is a saddle direction. We obtain, as in Écalle [99] and Hakim [135], that there exist k parabolic curves of F whose union is a base of convergent orbits asymptotic to Γ (notice that the tangent direction of Γ in this case is a characteristic direction of F). This result was used by Abate [8] (see also [70]) to show that every tangent to the identity diffeomorphism with isolated fixed point has a parabolic curve.

Parabolic-Elliptic case: $\mathrm{Spec}(DF_O) = \{1, \mu\}$**, with** $|\mu| = 1$**,** μ **not a root of unity and** $p = 0$**.** In this case, every attracting direction is a saddle direction, and Bracci and Molino [56] proved the existence of k parabolic curves of F. Since in this case there exists a formal invariant curve Γ such that $F|_\Gamma$ is parabolic, using the Poincaré-Dulac normal form, our approach recovers their result and generalizes it to the case $p > 0$.

Parabolic López and Sanz case: $\mathrm{Spec}(DF_O) = \{1\}$ **and** $\mathrm{Re}(A_0) > 0$**.** This is a particular case of a saddle direction, and López and Sanz proved in [173] the existence of a parabolic curve of F asymptotic to Γ. Following the same arguments (which are in turn a modification of Hakim's proof in [135]) we recovered that result and generalize it for an arbitrary saddle direction.

Parabolic Rong case: $\mathrm{Spec}(DF_O) = \{1\}$ **and** $\mathrm{Re}(A_0) < 0$**.** This is a particular case of a node direction, and Rong proved in [230] the existence of an open stable manifold. Note that, since $A_0 \neq 0$, applying Briot-Bouquet's theorem to the infinitesimal generator of F we conclude that there always exists a formal invariant curve Γ such that $F|_\Gamma$ is parabolic. Hence, our approach recovers Rong's result and generalizes it for an arbitrary node direction.

Stable manifolds for non-dicritical tangent to the identity germs in $\mathbb{C}^2$.

Theorem 2.30 is the key result in the following result obtained by López-Hernanz and Rosas in [175] which gives a positive answer to the question whether every characteristic direction for a non-dicritical tangent to the identity germs in $\mathbb{C}^2$ has some stable dynamics associated to it.

Theorem 2.35 (López-Hernanz, Rosas [175])**.** *Let $F \in \mathsf{Diff}(\mathbb{C}^2, O)$ be a tangent to the identity diffeomorphism of order $k + 1$, and let $[v]$ be a characteristic direction of F. Then at least one of the following possibilities holds:*

1. *There exists an analytic curve pointwise fixed by F and tangent to $[v]$.*
2. *There exist at least k invariant sets $S_1, \dots, S_k$, where each S_j is either a parabolic curve tangent to $[v]$ or a parabolic domain centered at $[v]$ and such that all the orbits in $S_1 \cup \cdots \cup S_k$ are mutually asymptotic. Moreover, at least one of the invariant sets S_j is a parabolic curve.*
3. *There exist at least k parabolic domains $S_1, \dots, S_k$ centered at $[v]$, where each S_j is foliated by parabolic curves and such that all the orbits in $S_1 \cup \cdots \cup S_k$ are mutually asymptotic.*

In particular, if F has an isolated fixed point then for any characteristic direction $[v]$ there is a parabolic curve tangent to $[v]$.

Chapter 3

Local dynamics of resonant germs

The study of the local dynamics of non-linearizable resonant germs is less developed than the one of tangent to the identity germs. The first contribution to such study has been given by Bracci and Zaitsev in [62] where they investigated the case of germs with one-dimensional sets of resonances. We completed such study and continued it in the case of resonances having finitely many generators in collaboration with Bracci and Zaitsev in [63] and in the case of germs with two-dimensional sets of resonances in collaboration with Vivas in [211].

Given a non-linearizable resonant germs F, we assume that its resonances are finitely generated by $m \geq 1$ multi-indices $P^1, \ldots, P^m \in \mathbb{N}^n$. Then, generically, any Poincaré-Dulac normal form of F preserves the singular foliation $\cup_{c \in \mathbb{C}^m} \{(z^{P^1}, \ldots, z^{P^m}) = (c_1, \ldots, c_m)\}$ and acts on it as a tangent to the identity germ. The key idea is therefore to use results on the local dynamics of tangent to the identity germs to obtain information on the local dynamics of the considered resonant germs. It turns out that resonances can give rise to an a priori unexpected parabolic behaviour, for example in elliptic situations.

In this chapter we will give an account of the results we obtained in this setting.

3.1 Multi-resonant germs

Definition 3.1. *Let F be in $\mathsf{Diff}(\mathbb{C}^n, O)$, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the differential $DF_)$. Let $m \geq 1$. We say that F is* m-resonant with respect to the first r eigenvalues $\lambda_1, \ldots, \lambda_r$ *$(1 \leq r \leq n)$ if there exist m multi-indices $P^1, \ldots, P^m \in \mathbb{N}^r \times \{0\}^{n-r}$ linearly independent over $\mathbb{Q}$, so that the resonant multi-indices L with respect to the j-th coordinate with $1 \leq j \leq r$ are precisely of the form*

$$L = e_j + k_1 P^1 + \cdots + k_m P^m \tag{3.1}$$

with $k_1, \ldots, k_m \in \mathbb{N}$ and $k_1 + \cdots + k_m \geq 1$ and where $e_j \in \mathbb{N}^n$ is the unit vector with 1 at the j-th place and 0 otherwise. The vectors $P^1, \ldots, P^m$ are called generators over $\mathbb{N}$ *of the resonances of F in the first r coordinates.*

We call F multi-resonant with respect to the first r eigenvalues *if it is m-resonant with respect to these eigenvalues for some $1 \leq m \leq r$.*

Example 3.2. *Assume that the differential of $F \in \mathsf{Diff}(\mathbb{C}^4, 0)$ has eigenvalues $\lambda_1, \ldots, \lambda_4$ such that $\lambda_1^3 = 1$ but $\lambda_1 \neq 1$, $\lambda_2 = -1$, and $\lambda_3^{-1}\lambda_4^2 = 1$. Then F is two-resonant with respect to λ_1, λ_2 with generators $P^1 = (3, 0, 0, 0)$, $P^2 = (0, 2, 0, 0)$. On the other hand, F is not multi-resonant with respect to all eigenvalues because it has the resonance $\lambda_3 = \lambda_4^2$ which is not of the form* (3.1).

We proved in [63] that if F is m-resonant with respect to $\lambda_1, \ldots, \lambda_r$, then $\lambda_j \neq \lambda_s$ for $1 \leq j \leq r$ and $1 \leq s \leq n$ with $j \neq s$.

Moreover, the set of generators is unique up to reordering (with respect to the lexicographic order) and if $P^1, \ldots, P^m$ are generators over $\mathbb{N}$ of (all, that is with $r = n$) the resonances of F, each multi-index $P \in \mathbb{N}^n$ such that $\lambda^P = 1$ is of the form

$$P = k_1 P^1 + \cdots + k_m P^m$$

with $k_1, \ldots, k_m \in \mathbb{N}$.

Definition 3.3. *The generators $P^1, \ldots, P^m$ are called* ordered *if $P^1 < \cdots < P^m$.*

Let $F \in \mathsf{Diff}(\mathbb{C}^n, O)$ be m-resonant (with respect to the first r eigenvalues), and let $P^1, \ldots, P^m$ be the ordered generators over $\mathbb{N}$ of its resonances. Thanks to Poincaré-Dulac Theorem 1.26, we can find a tangent to the identity (possibly) formal change of coordinates of $(\mathbb{C}^n, O)$ conjugating F to a germ in a (possibly formal) *Poincaré-Dulac normal form*, *i.e.*, of the form

$$\widetilde{F}(z) = Dz + \sum_{s=1}^{r} \sum_{\substack{|K\mathbf{P}| \geq 2 \\ K \in \mathbb{N}^m}} a_{K,s} z^{K\mathbf{P}} z_s e_s + \sum_{s=r+1}^{n} R_s(z) e_s, \tag{3.2}$$

where $D = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$, we denote by $K\mathbf{P} = \sum_{h=1}^{m} k_h P^h$, and $R_s(z) = O(\|z\|^2)$ for $s = r+1, \ldots, n$.

Moreover, it follows from the proof of the Poincaré-Dulac theorem that, given any $l \geq 2$, there exists a polynomial (hence holomorphic) change of coordinates tangent to the identity in $(\mathbb{C}^n, O)$ conjugating F to a Poincaré-Dulac normal form up to order l.

As we already mentioned, Poincaré-Dulac normal forms are not unique because they depend on the choice of the resonant part of the normalization. However, they are all conjugate to each other, which makes the following definition well-posed.

Definition 3.4. *Let $F \in \mathsf{Diff}(\mathbb{C}^n, O)$ be m-resonant with respect to $\{\lambda_1, \ldots, \lambda_r\}$, and let $P^1, \ldots, P^m$ be the ordered generators over $\mathbb{N}$ of its resonances. Let $\widetilde{F}$ be a Poincaré-Dulac normal form for F given by* (3.2). *The* weighted order *of F is the minimal $k_0 = |K| \in \mathbb{N} \setminus \{0\}$ such that the coefficient $a_{K,s}$ of $\widetilde{F}$ is non-zero for some $1 \leq s \leq r$.*

The weighted order of F is $+\infty$ if and only if F is formally linearizable in the first r coordinates.

Now, consider $F \in \mathsf{Diff}(\mathbb{C}^n, O)$ m-resonant, and let $P^1, \ldots, P^m$ be the ordered generators over $\mathbb{N}$ of its resonances. Write $P^j = (p_1^j, \ldots, p_r^j, 0, \ldots, 0)$, for $j = 1, \ldots, m$. Let $k_0 < \infty$ be the weighted order of F. Let $\widetilde{F}$ be a Poincaré-Dulac normal form for F given by (3.2). Then we set

$$G(z) = Dz + \sum_{s=1}^{r} \sum_{\substack{|K| = k_0 \\ K \in \mathbb{N}^m}} a_{K,s} z^{K\mathbf{P}} z_s e_s.$$

Consider the map $\pi\colon (\mathbb{C}^n, O) \to (\mathbb{C}^m, 0)$ defined by $\pi(z_1, \ldots, z_n) := (z^{P^1}, \ldots, z^{P^m}) = (u_1, \ldots, u_m)$. Therefore we can write

$$G(z) = Dz + \sum_{s=1}^{r} \sum_{\substack{|K| = k_0 \\ K \in \mathbb{N}^m}} a_{K,s} u^K z_s e_s,$$

and G induces a unique map $\Phi\colon (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ satisfying $\Phi \circ \pi = \pi \circ G$, which is tangent to the identity of order greater than or equal to $k_0 + 1$, and is of the form

$$\Phi(u) = u + H_{k_0+1}(u) + O(\|u\|^{k_0+2}),$$

where

$$H_{k_0+1}(u) = \begin{pmatrix} u_1 \displaystyle\sum_{|K|=k_0} \left(p_1^1 \frac{a_{K,1}}{\lambda_1} + \cdots + p_r^1 \frac{a_{K,r}}{\lambda_r} \right) u^K \\ \vdots \\ u_m \displaystyle\sum_{|K|=k_0} \left(p_1^m \frac{a_{K,1}}{\lambda_1} + \cdots + p_r^m \frac{a_{K,r}}{\lambda_r} \right) u^K \end{pmatrix}. \tag{3.3}$$

Therefore, roughly speaking, any Poincaré-Dulac normal form of F preserves the singular foliation $\cup_{c\in\mathbb{C}^m}\{(z^{P^1}, \ldots, z^{P^m}) = (c_1, \ldots, c_m)\}$.

Definition 3.5. *We call $u \mapsto u + H_{k_0+1}(u)$ a* parabolic shadow *of F.*

The key idea now is that we can infer on the local dynamics of F by analyzing the induced parabolic action of G on such foliation, since H_{k_0+1} remains unchanged under (holomorphic or formal) changes of coordinates preserving the Poincaré-Dulac normal forms of F. If $m = 1$, then thanks to the Leau-Fatou Flower Theorem 1.18 we have a complete description of the dynamics of the parabolic shadow f of F in a neighbourhood of the origin. If $m \geq 2$ we do not have a complete description in a neighbourhood of the origin but we can still use Hakim's Theorem 2.28 on the existence of parabolic domains.

Definition 3.6. *Let $F \in \mathsf{Diff}(\mathbb{C}^n, O)$ be m-resonant with respect to $\lambda_1, \ldots, \lambda_r$, with $P^1, \ldots, P^m$ being the ordered generators over $\mathbb{N}$ of the resonances, and of weighted order $k_0 < +\infty$. Let f be a parabolic shadow of F. We say that F is (f, v)-*attracting-non-degenerate *if $v \in \mathbb{C}^m$ is a normalized representative of a fully attractive non-degenerate characteristic direction for f.*

*If F is (f, v)-attracting-non-degenerate and $f(u) = u + H_{k_0+1}(u)$ with H_{k_0+1} as in (3.3), we say that F is (f, v)-*parabolically attracting *with respect to $\{\lambda_1, \ldots, \lambda_r\}$ if*

$$\mathrm{Re}\left(\sum_{\substack{|K|=k_0 \\ K\in\mathbb{N}^m}} \frac{a_{K,j}}{\lambda_j} v^K \right) < 0 \quad j = 1, \ldots, r. \tag{3.4}$$

We say that F is attracting-non-degenerate *(resp.* parabolically attracting*) if F is (f, v)-attracting-non-degenerate (resp. (f, v)-parabolically attracting) with respect to some parabolic shadow f and some normalized representative $v \in \mathbb{C}^m$ of a fully attractive non-degenerate characteristic direction for f.*

In [63] we were then able to prove the following general result.

Theorem 3.7 (Bracci, R., Zaitsev, [63])**.** *Let $F \in \mathsf{Diff}(\mathbb{C}^n, O)$ be m-resonant with respect to the eigenvalues $\{\lambda_1, \ldots \lambda_r\}$ and of weighted order $k_0 < +\infty$. Assume that $|\lambda_j| = 1$ for $j = 1, \ldots, r$ and $|\lambda_j| < 1$ for $j = r + 1, \ldots, n$. If F is parabolically attracting, then there exist (at least) k_0 disjoint basins of attraction having 0 at the boundary.*

Moreover, for each basin of attraction B of F there exists a Fatou coordinate $\mu\colon B \to \mathbb{C}$, that is a holomorphic function $\mu\colon B \to \mathbb{C}$ semi-conjugating F to a translation, $\mu \circ F(z) = \mu(z) + 1$.

Idea of the proof. Let $P^1, \dots, P^m$ be the ordered generators over $\mathbb{N}$ of the resonances of F. For any fixed $l \geq 2$, up to biholomorphic conjugacy, we can assume that $F(z) = (F_1(z), \dots, F_n(z))$ is of the form

$$F_j(z) = \lambda_j z_j \left(1 + \sum_{\substack{k_0 \leq |K| \leq k_l \\ K \in \mathbb{N}^m}} \frac{a_{K,j}}{\lambda_j} z^{K\mathbf{P}}\right) + O\left(\|z\|^{l+1}\right), \quad j = 1, \dots, r,$$
$$F_j(z) = \lambda_j z_j + O(\|z\|^2), \qquad j = r+1, \dots, n,$$

where

$$k_l := \max\{|K| : |K\mathbf{P}| \leq l\},$$

with $K\mathbf{P} := \sum_{h=1}^m k_h P^h$.

We consider the map $\pi\colon (\mathbb{C}^n, O) \to (\mathbb{C}^m, 0)$ defined by $\pi(z) = u := (z^{P^1}, \dots, z^{P^m})$. Then we can write

$$F_j(z) = G_j(u,z) + O\left(\|z\|^{l+1}\right), \quad G_j(u,z) := \lambda_j z_j \left(1 + \sum_{\substack{k_0 \leq |K| \leq k_l \\ K \in \mathbb{N}^m}} \frac{a_{K,j}}{\lambda_j} u^K\right), \quad j = 1, \dots, r.$$

The composition $\phi := \pi \circ F$ can be written as

$$\phi(z) = \Phi(u,z) := \overline{\Phi}(u) + g(z), \quad \overline{\Phi}(u) = u + H_{k_0+1}(u) + h(u),$$

where $\Phi\colon \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}^m$, $\overline{\Phi}$ is induced by G via $\pi \circ G = \overline{\Phi} \circ \pi$, the homogeneous polynomial $H_{k_0+1}(u)$ has the form (3.3), and where $h(u) = O(\|u\|^{k_0+2})$ and $g(z) = O(\|z\|^{l+1})$.

Since F is attracting-non-degenerate by hypothesis, its parabolic shadow $u \mapsto u + H_{k_0+1}(u)$ has a fully attractive non-degenerate characteristic direction $[v]$, such that $v \in \mathbb{C}^m$ is a normalized representative, *i.e.*, v satisfies (2.4) and the real parts of the directors of $[v]$ are all positive. In particular, we can apply Theorem 2.28 to $\overline{\Phi}(u)$. Then there exist k_0 disjoint parabolic domains $M^i_{R,C}$, $i = 1, \dots, k_0$, for $\overline{\Phi}$ at 0 in which every point is attracted to the origin along a trajectory tangent to $[v]$. We can use linear coordinates $(x,y) \in \mathbb{C} \times \mathbb{C}^{m-1}$ where v has the form $v = (1, 0, \dots, 0)$.

We therefore construct k_0 basins of attraction $\widetilde{B}^i_{R,C} \subset \mathbb{C}^n$, $i = 1, \dots, k_0$ for F in such a way that each $\widetilde{B}^i_{R,C}$ is projected into $M^i_{R,C}$ via π. The parabolic domains $M^i_{R,C}$'s are given by (2.5), and we can assume that the component Π^1_R is chosen centered at the direction 1. We first construct a basin of attraction based on $M^1_{R,C}$. We consider the sector

$$S_R(\varepsilon) := \{x \in \Delta_R : |\mathsf{Arg}(x)| < \varepsilon\} \subset \Pi^1_R,$$

for some $\varepsilon > 0$ small to be chosen later, and we let

$$B^1_{R,C}(\varepsilon) = \{(x,y) \in \mathbb{C} \times \mathbb{C}^{m-1} : x \in S_R(\varepsilon),\ \|y\| < C|x|\}.$$

Then for $\beta > 0$ we set

$$\widetilde{B} := \{z \in \mathbb{C}^n : |z_j| < |x|^\beta \text{ for } j = 1, \dots, n,\ u = \pi(z) \in B^1_{R,C}(\varepsilon),\ u = (x,y) \in \mathbb{C} \times \mathbb{C}^{m-1}\},$$

Taking $\beta > 0$ sufficiently small it is easy to check that $\widetilde{B}$ is an open non-empty set of $\mathbb{C}^n$ and $0 \in \partial \widetilde{B}$.

Next, we prove that $\widetilde{B}$ is F-invariant. We first prove that we can choose $\beta > 0$ small enough so that for any $l > 1$ such that $\beta l > k_0 + 1$ we have that if $z \in \widetilde{B}$ then $\pi(F(z)) = \Phi(u,z) \in B^1_{R,C}(\varepsilon)$.

Then, given $z \in \widetilde{B}$, we have to estimate $|F_j(z)|$ for $j = 1, \dots, n$. The estimates for the components F_j for $j = r+1, \dots, n$ follow easily since $|\lambda_j| < 1$ for $j = r+1, \dots, n$. For the other components we need more detailed and precise estimates that are satisfied thanks to condition (3.4).

Finally, setting inductively $u^{(l)} = (x^{(l)}, y^{(l)}) := \pi(F^{\circ(l-1)}(z))$, and denoting by $\pi_j \colon \mathbb{C}^n \to \mathbb{C}$ the projection $\pi_j(z) = z_j$, we obtain

$$\left|\pi_j \circ F^{\circ l}(z)\right| \leq \left|x^{(l)}\right|^{\beta}$$

for all $z \in \widetilde{B}$. Moreover, we have $\lim_{l\to+\infty} x^{(l)} = 0$, implying that $F^{\circ l}(z) \to 0$ as $l \to +\infty$. This proves that $\widetilde{B}$ is a basin of attraction of F at 0.

The construction of the Fatou coordinate μ follows as in the one-dimensional case (see [63, Proposition 4.3] for details).

The same argument can be repeated for each of the parabolic domains $M^i_{R,C}$ of $\overline{\Phi}$, and since those are disjoint, we obtain at least k_0 disjoint basins of attraction, concluding the proof. □

When F is m-resonant with respect to *all* the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ and $|\lambda_j| = 1$ for all $j = 1, \dots, n$, we showed that if F is attracting-non-degenerate or parabolically attracting then so is F^{-1}. This allowed us to show the existence of repelling basins for F in [63, Proposition 5.1, Corollary 5.2].

It is natural to ask what happens when condition (3.4) is satisfied only partially, that is for $j = 1, \dots, s$ with $1 \leq s < r$. In this case we obtain the existence of parabolic manifolds.

Corollary 3.8 (R., Vivas [211, Corollary 1]). *Let $G \in \mathsf{Diff}(\mathbb{C}^n, O)$ be m-resonant with respect to all eigenvalues $\{\lambda_1, \dots, \lambda_n\}$, with $|\lambda_j| = 1$ not a root of unity, for $j = 1, \dots, n$, and holomorphically normalizable. Assume that G is attracting-nondegenerate. If G is partially parabolically-attracting of order $1 \leq s < n$, that is condition (3.4) is satisfied only for $j = 1, \dots, s$, and the ordered generators over $\mathbb{N}$ of the resonances satisfy $P^1, \dots, P^m \in \mathbb{N}^s \times \{0\}^{n-s}$, then there exists a parabolic manifold of dimension s for G at O.*

3.1.1 Two-resonant germs

In this section we specify to the case of two-resonant germ. As we have seen in Chapter 2, we have more results on the existence of parabolic domains for tangent to the identity germs in dimension 2. In particular, it could seem possible to use all the results obtained by Vivas in Theorem 2.29 [254], together with the parabolically -attracting condition of [63], to obtain basins of attraction for two-resonant germs whose parabolic shadow has a degenerate or an irregular non-degenerate characteristic direction.

We proved in [211] that this is not possible for the degenerate characteristic directions. More precisely, we proved that a map whose parabolic shadow has a degenerate characteristic direction cannot be also parabolically-attracting.

Proposition 3.9 (R., Vivas [211]). *Let $F \in \mathsf{Diff}(\mathbb{C}^n, O)$ be m-resonant with respect to the first $r \leq n$ eigenvalues, and let f be a parabolic shadow of F. If $v \in \mathbb{C}^m$ is any representative of a degenerate characteristic direction $[v]$ for f, then F cannot be (f, v)-parabolically-attracting.*

We also obtained examples of germs with parabolic shadows having a basin along a degenerate characteristic direction but with no basins of attraction.

On the other hand, following the same strategy as in Theorem 3.7 we could prove the existence of basins when the parabolic shadow has an irregular non-degenerate characteristic direction and the map F is parabolically-attracting.

Theorem 3.10 (R., Vivas [211]). *Let $F \in \mathsf{Diff}(\mathbb{C}^n, O)$ be two-resonant with respect to the eigenvalues $\{\lambda_1, \dots \lambda_r\}$. Assume that $|\lambda_j| = 1$ for $j = 1, \dots, r$ and $|\lambda_j| < 1$ for $j = r+1, \dots, n$. If F is irregular-nondegenerate and parabolically-attracting, then there exists a basin of attraction having O at the boundary.*

This result shows the non-necessity of the hypothesis of *fully-attracting* for the characteristic direction of the parabolic shadow.

3.2 One-resonant germs

We end this chapter with the description of the results obtained by Bracci and Zaitsev in [62] and in collaboration with Bracci and Zaitsev in [63] on the dynamics of one-resonant germs. As one can expect, the complete description of the local dynamics in a neighbourhood of the origin for a tangent to the identity germ of holomorphic function in $(\mathbb{C}, 0)$ allows to have a better picture for one-resonant non-linearizable germs in higher dimension.

In this setting Definition 3.1 can be stated as follows.

Definition 3.11. *For $F \in \mathsf{Diff}(\mathbb{C}^n, O)$, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the differential DF_O. We say that F is* one-resonant with respect to the first r eigenvalues $\{\lambda_1, \dots, \lambda_r\}$ *$(1 \le r \le n)$ if there exists a fixed multi-index $P = (P_1, \dots, P_r, 0, \dots, 0) \neq 0 \in \mathbb{N}^n$ such that the resonant multi-indices with respect to the j-th coordinate with $j \in \{1, \dots, m\}$ are of the form $(j, Pk + e_j)$, where $e_j \in \mathbb{N}^n$ is the unit vector with 1 at the j-th place and 0 otherwise and where $k \ge 1 \in \mathbb{N}$ is arbitrary. The multi-index P is called the* index of resonance*. If F is one-resonant with respect to $\{\lambda_1, \dots, \lambda_n\}$ (i.e. $m = n$) we simply say that F is* one-resonant.

In particular, it follows that the relation $\lambda_1^{P_1} \cdots \lambda_r^{P_r} = 1$ holds and generates all other relations $\lambda_1^{Q_1} \cdots \lambda_r^{Q_r} = 1$ with $Q_s \ge 0$ for all s.

It also follows directly from the definition that, if F is one-resonant with respect to $\{\lambda_1, \dots, \lambda_r\}$, then for any $j \in \{1, \dots, r\}$ with $P_j \neq 0$, the eigenvalue λ_j differs from any other eigenvalue λ_s, $s \in \{1, \dots, n\} \setminus \{j\}$. Indeed, otherwise one would have resonances $(j, Q + e_j)$ where Q is obtained from P by replacing P_j and P_s with 0 and $P_j + P_s$ respectively.

Formal Poincaré-Dulac normal forms of a one-resonant germ F are of the form $G = (G_1, \dots, G_n)$ such that

$$G_j(z) = \lambda_j z_j + a_j z^{k_0 P} z_j + R_j(z), \quad j = 1, \dots, r, \tag{3.5}$$

where either $a = (a_1, \dots, a_r) \neq 0$ and $R_j(z)$ contains only resonant monomials $a_{j,s} z^{sP} z_j$ with $s > k_0$ or $a_j = 0$ and $R_j \equiv 0$ for all $j = 1, \dots, r$. Note that the second case occurs precisely when F is *formally linearizable* in the first r variables.

The integer k_0 in (3.5) is the weighted order defined in Definition 3.4, and here it will be simply called the *order* of F with respect to $\lambda_1, \dots, \lambda_r$.

Recall that the vector $a = (a_1, \dots, a_m)$ is invariant up to multiplication by a scalar, and setting

$$\Lambda(F) := \sum_{j=1}^{m} \frac{a_j}{\lambda_j} P_j.$$

We say that F is *non-degenerate* if $\Lambda(F) \neq 0$.

This non-degeneracy condition is invariant under conjugacies preserving the Poincaré-Dulac normal form. If F is one-resonant with respect to $\{\lambda_1\}$, then λ_1 is a root of unity. Moreover, in this case F is non-degenerate if and only if it is not formally linearizable in the first component.

As we already recalled in Section 3.1, any Poincaré-Dulac normal form of a non-linearizable one-resonant germ F preserves the singular foliation $\mathcal{F} = \cup_{c\in\mathbb{C}}\{z^P = c\}$ and acts on it as a tangent to the identity function. Similarly to dimension 1, Bracci and Zaitsev proved the following *normal form* for non-degenerate partially one-resonant diffeomorphisms.

Theorem 3.12 (Bracci, Zaitsev [62, Theorem 3.6]). *Let $F \in \mathsf{Diff}(\mathbb{C}^n, O)$ be one-resonant and non-degenerate with respect to $\lambda_1, \dots, \lambda_r$ with index of resonance P. Then there exist $k_0 \in \mathbb{N}$ and numbers $\mu, a_1, \dots, a_r \in \mathbb{C}$ such that F is formally conjugated to the map $\widehat{F}(z) = (\widehat{F}_1(z), \dots, \widehat{F}_n(z))$, where*

$$\widehat{F}_j(z) = \lambda_j z_j + a_j z^{k_0 P} z_j + \mu \frac{P_j}{\overline{\lambda_j}} z^{2k_0 P} z_j, \quad j = 1, \dots, r, \tag{3.6}$$

and the components $\widehat{F}_j(z)$ for $j = r+1, \dots, n$, contain only resonant monomials.

They also gave a description of the dynamics of Poincaré-Dulac normal forms on the foliation $\mathcal{F}$ in [62, Section 4].

In this setting, Definition 3.6 is given in the following easier form. Given $F \in \mathsf{Diff}(\mathbb{C}^n, O)$ one-resonant and non-degenerate with respect to $\{\lambda_1, \dots, \lambda_r\}$, we can set

$$\mathfrak{p}(F) := -\frac{|\Lambda(F)|}{\Lambda(F)}.$$

If $k_0 \in \mathbb{N}$ is the order of F with respect to $\lambda_1, \dots, \lambda_r$ and we choose coordinates such that (3.5) holds, we say that F is *parabolically attracting* with respect to $\{\lambda_1, \dots, \lambda_r\}$ if

$$|\lambda_j| = 1, \quad \mathrm{Re}\left(\frac{a_j}{\lambda_j}\mathfrak{p}(F)\right) < 0, \quad j = 1, \dots, r. \tag{3.7}$$

The statement of Theorem 3.7 then becomes:

Theorem 3.13 (Bracci, Zaitsev [62], Bracci, R., Zaitsev [63]). *Let $F \in \mathsf{Diff}(\mathbb{C}^n, O)$ be one-resonant with respect to the eigenvalues $\{\lambda_1, \dots \lambda_r\}$ and of order $k_0 < +\infty$. Assume that $|\lambda_j| < 1$ for $j = r+1, \dots, n$. If F is parabolically attracting, then there exist (at least) k_0 disjoint basins of attraction having 0 at the boundary.*

Moreover, for each basin of attraction B of F there exists a Fatou coordinate $\mu\colon B \to \mathbb{C}$, that is a holomorphic function $\mu\colon B \to \mathbb{C}$ semi-conjugating F to a translation, $\mu \circ F(z) = \mu(z) + 1$.

The proof is the same as in the multi-resonant case, except that here it is based on the Leau-Fatou Flower Theorem.

Leau-Fatou flower theorem for one-resonant Poincaré-Dulac normal form

We also obtained a full generalization of the Leau-Fatou Flower theorem in the fully one-resonant case for holomorphically normalizable germs to a Poincaré-Dulac normal form.

Theorem 3.14 (Bracci, R., Zaitsev [63, Theorem 5.3]). *Let $F \in \mathsf{Diff}(\mathbb{C}^n, O)$ be one-resonant with respect to all eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ with generator $P \in \mathbb{N}^n$. Assume that F is holomorphically conjugated to one of its Poincaré-Dulac normal forms. Suppose that $|\lambda_j| = 1$, $j = 1, \dots, n$ and F is parabolically attracting. Then for each $j \in \{1, \dots, n\}$ such that $P_j \neq 0$ there exists a germ M_j of a complex manifold tangent to $\{z_j = 0\}$ at 0 such that $F(M_j) \subset M_j$ and $F|_{M_j}$ is holomorphically linearizable. Moreover, there exists an open neighbourhood W of 0 such that $W \setminus \bigcup_j M_j$ is the union of attracting and repelling basins of F.*

We can also deduce the existence of parabolic manifolds when condition (3.4) is satisfied only partially, that is for $j = 1, \ldots, s$ with $1 \leq s < n$.

Proposition 3.15 (R., Vivas [211, Proposition 3]). *Let $F \in \mathsf{Diff}(\mathbb{C}^n, O)$ be one-resonant with respect to all eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, with $|\lambda_j| = 1$, but not a root of unity, for $j = 1, \ldots, n$, and non-degenerate. Assume that F is holomorphically normalizable germ, such that:*

$$\begin{cases} \mathrm{Re}\left(\dfrac{a_j}{\lambda_j}\dfrac{1}{\Lambda(G)}\right) > 0 & \textit{for } \; j = 1, \ldots, s, \\ \mathrm{Re}\left(\dfrac{a_h}{\lambda_h}\dfrac{1}{\Lambda(G)}\right) < 0 & \textit{for } \; h = s+1, \ldots, n, \end{cases}$$

for some $1 \leq s < n$, and let $P \in \mathbb{N}^n$ be the generator of the resonances of $\{\lambda_1, \ldots, \lambda_n\}$. Then:

1. *if $P \notin \mathbb{N}^s \times \{0\}^{n-s}$, then the unique point in a neighbourhood of the origin with orbit under F converging to O is the origin itself;*

2. *if $P \in \mathbb{N}^s \times \{0\}^{n-s}$, then there exists a parabolic manifold of dimension s for F at O.*

Part II

Local Techniques in Global Discrete Holomorphic Dynamics

Chapter 4

Bulging and Wandering Fatou components of polynomial skew-products in dimension 2

In this chapter we give an updated account of the recent results on Fatou components for polynomial skew-products in complex dimension two in a neighbourhood of a periodic fiber, dividing our discussion according to the different possible kinds of periodic fibers.

4.1 Preliminaries

Consider the discrete holomorphic dynamical system given by a complex manifold X and the iteration of a holomorphic endomorphism $F\colon X \to X$. In the investigation of the global behaviour of such a system it is natural to introduce the *Fatou set of* F, that is the largest open set $\mathcal{F}(F)$ where the family of iterates $\{F^{\circ n}\}_{n\in\mathbb{N}}$ of F is normal. A connected component of the Fatou set is called a *Fatou component.*

In complex dimension one, Fatou components of rational maps of degree at least 2 on the Riemann sphere are well understood. In fact, from one hand, Fatou conjectured the following classification for invariant Fatou components, which was partially proven by Fatou himself and completed by several authors (Julia, Leau, Siegel, Herman, Yoccoz...).

Theorem 4.1 (Fatou's Classification of invariant Fatou components)**.** *Let $f\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational map of degree $d \geq 2$ on the Riemann sphere. Let Ω be an invariant Fatou component of f. Then Ω is either:*

(i) *the basin of an attracting fixed point p, i.e. $|f'(p)| < 1$,*

(ii) *the parabolic basin of a parabolic fixed point p, i.e. $|f'(p)|$ is a root of unity, and in this case $f'(p) = 1$,*

(iii) *a rotation domain, that can be a Siegel disk of an elliptic fixed point p, i.e. $f'(p) = e^{2\pi i\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$, or a Herman ring.*

We recall that a connected open set U is called:

- a *basin of an attracting fixed point* p if it contains a fixed point p such that $|f'(p)| < 1$ and the sequence of iterates $\{f^{\circ k}\}$ converges uniformly to p on every compact subset of U;

- a *parabolic basin* if there is a fixed point $p \in \partial U$ with $|f'(p)| = 1$, and the sequence of iterates $\{f^{\circ k}\}$ converges uniformly to p on every compact subset of U;
- a *Siegel disk* if it is simply connected, and there exists a holomorphic isomorphism $h\colon U \to \mathbb{D}$ such that $h \circ f \circ h^{-1}(z) = e^{2\pi i\theta} z$, with $\theta \in \mathbb{R} \setminus \mathbb{Q}$;
- a *Herman ring* if it is doubly connected, and there exists a radius $r > 0$ and a holomorphic isomorphism $h\colon U \to A_r := \{z \in \mathbb{C} \mid r < |z| < 1\}$, such that $h \circ f \circ h^{-1}(z) = e^{2\pi i\theta} z$, with $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

On the other hand, Sullivan proved his celebrated *non-wandering domains* theorem.

Theorem 4.2 (Sullivan [248])**.** *Let $f\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational map of degree $d \geq 2$. Then every Fatou component of f is (pre-)periodic.*

Therefore, up to considering iterates of f we can describe all its Fatou components. The proof of Sullivan's non-wandering Theorem strongly relies on the Ahlfors-Bers mesurable mapping Theorem for quasi-conformal functions and we refer to the original paper of Sullivan [248] for it. The recent notes [74] by Buff present a proof due to Adam Epstein based on a density result of Bers for quadratic differentials. Such results are strongly one-dimensional and do not have an analogue in higher dimension, making impossible to mimic Sullivan's proof there. Besides this observation, little was known about this problem so far.

Moreover, in complex dimension two, the understanding of Fatou components is far less complete. A considerable progress in the classification of periodic Fatou components has been achieved thanks to Bedford and Smillie [37] [38] [39], Fornæss and Sibony [125], Lyubich and Peters [176] and Ueda [250].

The question of the existence of wandering (i.e., not pre-periodic) Fatou components in higher dimension was put forward by several authors since the 1990's (see e.g. [127]).

A first natural class of maps to consider are *direct product* polynomial endomorphisms of $\mathbb{C}^2$, that is maps $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ of the form

$$F(z, w) = (f(z), g(w)),$$

where f and g are complex polynomials in one variable. This allows us to recover the generalizations of one-dimensional dynamical behaviours in dimension two, for instance higher dimensional transcendental mappings with wandering domains can be constructed from one-dimensional examples by taking direct products, but without giving us a complete understanding of all possible behaviours of polynomial endomorphisms in $\mathbb{C}^2$, as direct products are a very particular class. An example of a transcendental *biholomorphism* of $\mathbb{C}^2$ with a wandering Fatou component oscillating to infinity was constructed by Fornæss and Sibony in [126]. Nonetheless, until recently very little was known about the existence of wandering Fatou components for holomorphic endomorphisms of $\mathbb{P}^2(\mathbb{C})$ or for polynomial endomorphisms of $\mathbb{C}^2$.

A more interesting class to consider is given by polynomial *skew-products* in $\mathbb{C}^2$, namely polynomial maps $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ of the form

$$F(z, w) = (f(z, w), g(w)), \tag{4.1}$$

where g is a complex polynomial in one variable and f is a complex polynomial in two variables. Since they leave invariant the fibration $\{w = \text{const.}\}$, skew-products allow us to *build on one-dimensional dynamics* and to get a first flavour of the richness of the higher dimension setting we are working in. This idea has been used by several authors to construct maps with particular dynamical properties. Dujardin, for example, used in [94] specific skew-products to construct a non-laminar Green current. Boc-Thaler, Fornæss and Peters constructed in [49] a map having a Fatou component with a punctured limit set. Last but not least, as we will explain in Section 4.3, skew-products are one of the key ingredients in the construction we recently obtained in [33] in collaboration with Astorg, Buff, Dujardin, Peters of holomorphic endomorphisms of $\mathbb{P}^2(\mathbb{C})$ having a wandering Fatou component.

The investigation of the holomorphic dynamics of polynomial skew-products was started by Heinemann [138] and then continued by Jonsson [153]. The topology of Fatou components of skew-products has been studied by Roeder in [219].

Given a Fatou component Ω of a polynomial skew-product F in $\mathbb{C}^2$, its projection on the second coordinate $\Omega_2 = \pi_2(\Omega)$ is a Fatou components for g and hence thanks to Sullivan's non-wandering Theorem 4.2, up to considering an iterate of F, it has to fall into one of the three cases given by Theorem 4.1, and moreover, since we are considering polynomials, Herman rings cannot occur. Therefore, since (pre-)periodic points for g correspond to (pre-)periodic fibers for F, up to considering an iterate of F, we can restrict ourselves to study what happens in neighbourhoods of invariant fibers of the form $\{w = c\}$. One-dimensional theory also describes the dynamics on the invariant fiber, which is given by the action of the one-dimensional polynomial $f(z, c) := f_c(z)$, and hence the Fatou components of f_c will be again all pre-periodic and, up to consider an iterate, we can assume that they are either attracting basins, or parabolic basins or Siegel disks. This structure leads us to two immediate questions.

(Q1) Do all Fatou components of f_c bulge to two-dimensional Fatou components of F?

(Q2) Is it possible to have wandering Fatou components for F in a neighbourhood of an invariant fiber?

In the following we shall call an invariant fiber $\{w = c\}$ *attracting, parabolic* or *elliptic* according to whether c is an attracting, parabolic or elliptic fixed point for g. A *bulging* Fatou component will be a Fatou component Ω of F such that $\Omega \cap \{w = c\}$ is a one-dimensional Fatou component of f_c on the invariant fiber $\{w = c\}$. With a slight abuse of terminology we shall say that a Fatou component Ω_c of f_c on the invariant fiber $\{w = c\}$ *is bulging* if there exists a bulging Fatou component Ω of F so that $\Omega_c = \Omega \cap \{w = c\}$.

The purpose of this chapter is to provide an updated account of the results related to these questions. We shall divide our discussion according to the different possible kinds of invariant fibers.

4.2 Attracting invariant fiber

Let us consider a polynomial skew-product $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ of degree $d \geq 2$

$$F(z, w) = (f(z, w), g(w)),$$

with an *attracting invariant fiber*. We can assume without loss of generality that the invariant fiber is $\{w = 0\}$. Therefore we have $g(0) = 0$ and $|g'(0)| < 1$. In this case it is a well-known one-dimensional result (see for exemple [83] or [182]) that there exists an *attracting basin*, containing the origin, of points whose iterates converge to the origin. The rate of convergence to the fixed point depends on whether $g'(0) = 0$, in which case the fixed point is called *superattracting*, or $g'(0) \neq 0$, in which case the fixed point is called *attracting* or *geometrically attracting.*

4.2.1 Superattracting case

This setting was studied by Lilov in [171] who was able to answer both questions stated in the introduction. He first proved the following result giving a positive answer to Question 1.

Theorem 4.3 (Lilov [171])**.** *Let $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial skew-product of the form* (4.1) *of degree $d \geq 2$. Let $\{w = c\}$ be a superattracting invariant fiber for F. Then all one-dimensional Fatou components of f_c bulge to Fatou components of F.*

Idea of the proof. We can assume $c = 0$ without loss of generality. Thanks to Theorem 4.1 and Theorem 4.2, all Fatou components of the restriction $f_0(z) = f(z, 0)$ of $f(z, w)$ to the invariant fiber are (pre-)periodic and are either attracting basins, or parabolic basins or Siegel domains. The strategy of the proof is to prove separately for each of these cases that the corresponding component is contained in a two-dimensional Fatou component of F. The bulging of one-dimensional Fatou components of attracting periodic points of $f_0(z)$ is well-known and follows for instance from the results of Rosay and Rudin [233]. For the remaining cases, by [171, Theorem 3.17] there exists a strong stable manifold through all point in the one-dimensional Fatou components of parabolic or elliptic periodic points of $f_0(z)$, and so the corresponding bulging Fatou components simply consist of the union of such manifolds. □

Then Lilov proved the following result implying the non-existence of wandering Fatou components in a neighbourhood of a superattracting invariant fiber.

Theorem 4.4 (Lilov [171])**.** *Let $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial skew-product of the form* (4.1) *of degree $d \geq 2$. Let $\{w = c\}$ be a superattracting invariant fiber for F and let $\mathcal{B}$ be the immediate basin of the superattracting fixed point c. Let $w_0 \in \mathcal{B}$ and let D_{w_0} be a one-dimensional open disk lying in the fiber over w_0 ($\mathbb{C} \times \{w_0\}$). Then the forward orbit of D must intersect one of the bulging Fatou components of f_c.*

The proof relies on the repeated use of [171, Lemma 3.2.4] to the orbit of a disk lying in a fiber over a point in the attracting basin, in order to obtain estimates from below for the radii of the images. Thanks to [171, Proposition 3.2.8], by studying the geometry of the bulging Fatou components, it is also possible to obtain an upper bound on the largest possible disk lying in a fiber over a point in the attracting basin that can lie in the complement of a bulging Fatou component, depending on the distance to the invariant fiber. The conclusion then follows combining these two estimates.

All bulging Fatou components are (pre-)periodic, therefore all Fatou components for F in a neighbourhood of a superattracting invariant fiber are (pre-)periodic, and then the non-existence of wandering Fatou components in a neighbourhood of a superattracting invariant fiber follows immediately.

Corollary 4.5 (Lilov [171])**.** *Let $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial skew-product of the form* (4.1) *of degree $d \geq 2$. Let $\{w = c\}$ be a superattracting invariant fiber for F and let $\mathcal{B}$ be the immediate basin of the superattracting fixed point c. Then there are no wandering Fatou components in $\mathcal{B} \times \mathbb{C}$.*

4.2.2 Geometrically attracting case

The geometrically attracting case was first partially addressed by Lilov in [171] even if not stated explicitly. In fact, the proof of Theorem 4.3 can be readily adjusted to this case obtaining the following statement answering Question 1.

Theorem 4.6 (Lilov [171]). *Let $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial skew-product of the form (4.1) of degree $d \geq 2$. Let $\{w = c\}$ be an attracting invariant fiber for F, that is $|g'(c)| < 1$. Then all one-dimensional Fatou components of f_c bulge to Fatou components of F.*

On the other hand, the proof of Theorem 4.4 cannot be generalized to this setting, which is indeed more complicated than the superattracting case. In fact, Theorem 4.4 does not hold in general, as showed by Peters and Vivas with the following result.

Theorem 4.7 (Peters, Vivas [199]). *Let $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial skew-product of the form*

$$F(z, w) = (p(z) + q(w), \lambda w), \tag{4.2}$$

with $0 < |\lambda| < 1$ and p and q complex polynomials. Then there exists a triple (λ, p, q) and a holomorphic disk $D \subset \{w = w_0\}$ whose forward orbit accumulates at a point $(z_0, 0)$, where z_0 is a repelling fixed point in the Julia set of f_0.

As a consequence, the forward orbits of D cannot intersect the bulging Fatou components of f_0. The family $\{F|_D^n\}_{n\in\mathbb{N}}$ is normal on the disks D, and so these are Fatou disks. However such disks are completely contained in the Julia set of F, which is the complement in $\mathbb{C}^2$ of the Fatou set (see [199, Theorem 6.1]).

The geometrically attracting case has been further investigated by Peters and Smit in [198]. They focused their investigation on polynomial skew-products such that the action on the invariant attracting fiber is *subhyperbolic*, that is the polynomial does not have parabolic periodic points and all critical points lying on the Julia set are pre-periodic. They proved the following result.

Proposition 4.8 (Peters, Smit [198]). *Let $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial skew-product of the form (4.1). Assume that the origin is an attracting, not superattracting, fixed point for g with corresponding basin B_g, and the polynomial $f_0(z) := f(z, 0)$ is subhyperbolic. Then there exists a set $E \subset \mathbb{C}$ of full mesure, such that for every $w_0 \in E$ the forward orbit of every disk in the fiber $\{w = w_0\}$ must intersect a bulging Fatou component of f_0.*

Idea of the proof. Notice that it suffices to prove the proposition in a neighbourhood of the attracting fiber $\{w = 0\}$. Therefore, up to considering a smaller neighbourhood, we can assume without loss of generality that $g(w) = \lambda w$, and

$$f(z, w) = a_0(w) + a_1(w)z + \cdots + a_d(w)z^d$$

where $a_0(w), \ldots, a_d(w)$ are holomorphic functions in w. The subhyperbolicity of the polynomial f_0 implies that its Fatou set is the union of finitely many attracting basins, and the orbits of the critical points contained in the Fatou set converge to one of these attracting cycles. The proof can be divided into 5 main steps.

Step 1. Fix $R > 0$ large enough so that for all z such that $|z| > R$ we have $|f_0(z)| > 2|z|$ and set

$$W_0 = \{|z| > R\} \cup \bigcup_{y \in \mathrm{Att}(f_0)} W_y$$

where $\mathrm{Att}(f_0)$ is the set of all attracting periodic points of f_0, and for each $y \in \mathrm{Att}(f_0)$ the set W_y is an open neighbourhood of the orbit of y such that $\overline{f_0(W_y)} \subset W_y$. Fix a neighbourhood U of the post-critical set of f_0. Then by [198, Proposition 15], there exists a set $E \subset \mathbb{C}$ of full mesure in a neighbourhood of the origin such that for all $w_0 \in E$ there exists a constant $C = C(w_0, U)$ such that for all $n \in \mathbb{N}$ we have

$$\operatorname{Card}\left\{z\colon \frac{\partial F_1^{\circ n}}{\partial z}(z, w_0) = 0 \text{ and } F_1^{\circ n}(z, w_0) \notin W_0 \times U\right\} \leq C\sqrt{n}, \tag{4.3}$$

where $F_1^{\circ n}$ is the first component of the n-th iterate of F.

Step 2. Assume by contradiction that a fiber $\{w = w_0\}$, with $w_0 \in E$, contains a disk D whose forward orbit avoids the bulging Fatou components of f_0. Then the restriction of $F^{\circ n}$ to D is bounded and hence a normal family. Therefore, up to shrinking D there exists a subsequence F^{n_j} such that $F^{n_j}|_D$ converges, uniformly on compact subsets of D, to a point ζ in the Julia set of f_0. Moreover, there exists $\varepsilon > 0$ so that $F^{\circ n}(D) \cap (W_0 \times D(0, \varepsilon))$ is empty for all $n \in \mathbb{N}$.

Step 3. Each critical point x contained in the Julia set is eventually mapped into a repelling periodic point, and up to considering an iterate of F we may assume that it is eventually mapped into a repelling fixed point with multiplier μ, with $|\mu| > 1$. The main tool to control the orbits of the critical points of F is obtained using a linearization map of the unstable manifold of the repelling fixed point, given by a map $\Phi\colon \mathbb{C} \to \mathbb{C}$ satisfying $\Phi(\mu t) = f_0^k \circ \Phi(t)$ for some $k \in \mathbb{N}$. Thanks to [198, Proposition 10], there exist $\widetilde{C} > 1$ and $0 < \gamma < 1$ so that

$$\operatorname{Area}(F^{\circ n}(D)) \leq \widetilde{C}\gamma^n. \tag{4.4}$$

Step 4. We may assume that ζ does not lie in the post-critical set, and we may choose U and $r > 0$ such that $D(\zeta, r) \cap (U \cup W_0) = \emptyset$. Let $j_1 \in \mathbb{N}$ be such that $F_1^{n_j}(D) \subseteq D(\zeta, \frac{r}{2})$ for all $j \geq j_1$, and consider O_j the connected component of $(F^{n_j})^{-1}(D(\zeta, r) \times \{\lambda^{n_j} w_0\})$ containing D. Then $D \subseteq O_j \subseteq D(0, R) \times \{w_0\}$, and we can study the proper holomorphic function $F_1^{n_j}\colon O_j \to D(\zeta, r)$. Thanks to (4.3), such a map has at most $d_j = C\sqrt{n_j}$ critical points.

Step 5. It is possible (see [198, Proposition 28]) to find a uniform constant $C_1 > 0$ so that if $f\colon \mathbb{D} \to \mathbb{D}$ is a proper holomorphic function of degree d, the set $R \subset \mathbb{D}$ has Poincaré area equal to A, and $d \cdot A^{1/2d} < 8$, then the Poincaré area of $f^{-1}(R)$ is at most $C_1 d^3 A^{1/d}$. Then, setting $R_j = F_1^{n_j}(D)$ and denoting by A_j its Poincaré area $\operatorname{Area}_{D(\zeta,r)}(R_j)$ with respect to $D(\zeta, r)$, for $j \geq j_1$, we have $R_j \subseteq D(\zeta, r)$, and we can estimate A_j applying (4.4). Therefore there exists $j_2 \geq j_1$ such that $d_j A_j^{1/2d_j} < 1/8$ for all $j \geq j_2$. This implies

$$\operatorname{Area}_{D(0,R)}(D) \leq \operatorname{Area}_{O_j}(D) \leq C_2 d_j^3 A_j^{1/d_j} \leq M n_j^{3/2} \gamma^{n_j^{3/2}}$$

where $M > 0$. The contradiction follows from the fact that the last expression will converge to zero as j increases towards infinity. □

Thanks to the fact that in particular E is dense, Peters and Smit are able to give a negative answer to Question 2 when the action on the invariant fiber is subhyperbolic. They also obtain as a corollary that the only Fatou components of F are the bulging ones, since the topological degree of F equals the one of f_0, implying that the only Fatou components that can be mapped onto the bulging Fatou components of f_0 are exactly those bulging Fatou components.

Theorem 4.9 (Peters-Smit, [198]). *Let $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial skew-product of the form (4.1). Assume that the origin is an attracting fixed point for g with corresponding basin B_g, and the polynomial $f_0(z) := f(z, 0)$ is subhyperbolic. Then F has no wandering Fatou component over B_g.*

Very recently, Ji was able in [154] to generalize Lilov's Theorem 4.4 to polynomial skew-products with an invariant geometrically attracting fiber under the hypothesis that the multiplier of the invariant fiber is small. More precisely he proved the following result.

Theorem 4.10 (Ji, [154])**.** *Let $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial skew-product of the form* (4.1) *of degree $d \geq 2$. Let $\{w = c\}$ be an attracting invariant fiber for F and let $\mathcal{B}_c$ be the basin of the attracting fixed point c. Then there exists $\lambda_0(c, f) > 0$ depending only on f and c such that if $|g'(c)| < \lambda_0$, then there are no wandering Fatou components in $\mathcal{B}_c \times \mathbb{C}$.*

The proof of this result follows Lilov's strategy. The main difficulty is due to the breaking down of Lilov's argument in the geometrically attracting case as we pointed out at the beginning of this section. Ji is able to overcome such difficulty by adapting a one-dimensional result due to Denker, Przytycki and Urbanski in [91] in this case. Such result is used to obtain estimates of the size of bulging Fatou components and of the size of forward images of wandering Fatou disks.

4.3 Parabolic invariant fiber and wandering domains

A first contribution to the investigation of this case is due to Vivas, who proved a parametrization result [256, Theorem 3.1] for the unstable manifolds for *special* parabolic skew-product of $\mathbb{C}^2$. Vivas used this parametrization as the main tool to prove the analogue of Theorem 4.7 for special parabolic skew-product. However, this construction does not allow to construct a wandering Fatou component in a neighbourhood of the parabolic invariant fiber.

In [33], together with Astorg, Buff, Dujardin, Peters, we proved the existence of polynomial skew-products of $\mathbb{C}^2$, extending to holomorphic endomorphisms of $\mathbb{P}^2(\mathbb{C})$, having a wandering Fatou component. The key tool consists in using parabolic implosion techniques on polynomial skew-products, and this idea was initially suggested by Lyubich. The main strategy is to combine slow convergence to an invariant parabolic fiber and parabolic transition in the fiber direction, to produce orbits shadowing those of the so-called Lavaurs map.

Theorem 4.11 (Astorg, Buff, Dujardin, Peters, R. [33])**.** *There exists a holomorphic endomorphism $F\colon \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$, induced by a polynomial skew-product mapping $F\colon \mathbb{C}^2 \to \mathbb{C}^2$, having a wandering Fatou component. More precisely, let $f\colon \mathbb{C} \to \mathbb{C}$ and $g\colon \mathbb{C} \to \mathbb{C}$ be polynomials of the form*

$$f(z) = z + z^2 + \mathrm{O}(z^3) \quad \text{and} \quad g(w) = w - w^2 + \mathrm{O}(w^3). \tag{4.5}$$

If the Lavaurs map $\mathcal{L}_f\colon \mathcal{B}_f \to \mathbb{C}$ has an attracting fixed point, then the skew-product $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$F(z, w) := \left(f(z) + \frac{\pi^2}{4} w, g(w)\right) \tag{4.6}$$

has a wandering Fatou component.

The orbits in these wandering Fatou components are bounded and the approach used in the proof is essentially local. Notice that if f and g have the same degree, F extends to a holomorphic endomorphism of $\mathbb{P}^2(\mathbb{C})$. Moreover we can obtain examples in arbitrary dimension $k \geq 2$ by simply considering products mappings of the form (F, Q), where Q has a fixed Fatou component.

To give the definition of Lavaurs map and the main ideas of the proof, we have to recall some facts on parabolic dynamics (more details can be found in [33, Appendix A]). Let f be a polynomial of the form

$$f(z) = z + z^2 + az^3 + \mathrm{O}(z^4) \text{ for some } a \in \mathbb{C}.$$

and denote by

$$\mathcal{B}_f := \left\{ z \in \mathbb{C} \ ; \ f^{\circ k}(z) \underset{n\to+\infty}{\overset{\neq}{\longrightarrow}} 0 \right\}$$

the parabolic basin of 0. It is a well-known one-dimensional result that there exists an *attracting Fatou coordinate* $\varphi_f \colon \mathcal{B}_f \to \mathbb{C}$ conjugating f to the translation T_1 by 1:

$$\varphi_f \circ f = T_1 \circ \varphi_f.$$

The Fatou coordinate can be normalized by requiring that

$$\varphi_f(z) = -\frac{1}{z} - (1-a)\log\left(-\frac{1}{z}\right) + \mathrm{o}(1) \quad \text{as} \quad \mathrm{Re}\left(-\frac{1}{z}\right) \to +\infty,$$

where the branch of log used in this normalization, as well as in the next one, is the one defined in $\mathbb{C} \setminus \mathbb{R}^-$ which vanishes at 1. There also exists a *repelling Fatou parameterization* $\psi_f \colon \mathbb{C} \to \mathbb{C}$ satisfying

$$\psi_f \circ T_1 = f \circ \psi_f,$$

which may be normalized by requiring that

$$-\frac{1}{\psi_f(Z)} = Z + (1-a)\log(-Z) + \mathrm{o}(1) \quad \text{as} \quad \mathrm{Re}(Z) \to -\infty.$$

The *Lavaurs map* $\mathcal{L}_f$ is then defined by

$$\mathcal{L}_f := \psi_f \circ \varphi_f \colon \mathcal{B}_f \to \mathbb{C},$$

and it clearly commutes with f since $\varphi_f \circ f = T_1 \circ \varphi_f$ and $\psi_f \circ T_1 = f \circ \psi_f$. This kind of functions appear in considering high iterates of small perturbations of f: this phenomenon is known as *parabolic implosion*, and plays a key rôle in our construction. A first introduction to this topic can be found in [92], and we also refer to [241] for a very detailed presentation. (Semi-)parabolic implosion was recently studied for dissipative polynomial automorphisms of $\mathbb{C}^2$ by Bedford, Smillie and Ueda in [40] (see also [95]) and their strategy was recently adapted by Bianchi in [44] for a perturbation of a class of holomorphic endomorphisms tangent to identity, establishing a two-dimensional Lavaurs theorem for such a class.

A first step in the proof of Theorem 4.11 is to find a parabolic polynomial f whose Lavaurs map $\mathcal{L}_f$ admits an attracting fixed point.

Proposition 4.12 (Astorg, Buff, Dujardin, Peters, R. [33, Proposition B]). *Let $f \colon \mathbb{C} \to \mathbb{C}$ be the cubic polynomial defined by*

$$f(z) = z + z^2 + az^3 \quad \textit{with} \quad a \in \mathbb{C}.$$

If $r > 0$ is sufficiently close to 0 and a belongs to the disk $D(1-r, r)$, then the Lavaurs map $\mathcal{L}_f \colon \mathcal{B}_f \to \mathbb{C}$ admits an attracting fixed point.

Idea of the proof. We consider

$$\mathcal{U}_f := \psi_f^{-1}(\mathcal{B}_f) \quad \text{and} \quad \mathcal{E}_f := \varphi_f \circ \psi_f \colon \mathcal{U}_f \to \mathbb{C}.$$

The open set $\mathcal{U}_f$ contains an upper half-plane and a lower half-plane, and it is invariant under T_1. Like the Lavaurs map, the map $\mathcal{E}_f$ commutes with T_1, therefore $\mathcal{E}_f - \mathrm{Id}$ is periodic of period 1 and admits a Fourier expansion in a upper half-plane:

$$\mathcal{E}_f(Z) = Z + \sum_{k\geq 0} c_k \mathrm{e}^{2\pi \mathrm{i} k Z}.$$

Using the expansion of φ_f and ψ_f near infinity, we obtain with an elementary computation:

$$\begin{aligned}\mathcal{E}_f(Z) = \varphi_f \circ \psi_f(Z) &= Z + (1-a)\log(-Z) + \mathrm{o}(1) \\ &\quad - (1-a)\log\big(Z + (1-a)\log(-Z) + \mathrm{o}(1)\big) + \mathrm{o}(1) \\ &= Z + (1-a)\log(Z) - \pi\mathrm{i}(1-a) - (1-a)\log(Z) + \mathrm{o}(1) \\ &= Z - \pi\mathrm{i}(1-a) + \mathrm{o}(1),\end{aligned}$$

and so $c_0 = -\pi\mathrm{i}(1-a)$.

Thanks to a more elaborate argument, based on the notion of finite type analytic map introduced by Adam Epstein, it is possible to prove that:

$$\mathcal{E}_f(Z) = Z - \pi\mathrm{i}(1-a) + c_1\mathrm{e}^{2\pi\mathrm{i}Z} + \mathrm{o}(\mathrm{e}^{2\pi\mathrm{i}Z}) \quad \text{with} \quad c_1 \neq 0.$$

It then follows that for $a \neq 1$ close to 1, $\mathcal{E}_f$ has a fixed point Z_f with multiplier ρ_f satisfying

$$c_1\mathrm{e}^{2\pi\mathrm{i}Z_f} \sim \pi\mathrm{i}(1-a) \quad \text{and} \quad \rho_f - 1 \sim 2\pi\mathrm{i}c_1\mathrm{e}^{2\pi\mathrm{i}Z_f} \sim -2\pi^2(1-a) \quad \text{as} \quad a \to 1.$$

Therefore, for $r > 0$ sufficiently close to 0 and $a \in D(1-r, 1)$, the multiplier ρ_f belongs to the unit disk and Z_f is an attracting fixed point of $\mathcal{E}_f$. This concludes the proof since $\psi_f : \mathcal{U}_f \to \mathcal{B}_f$ semi-conjugates $\mathcal{E}_f$ to $\mathcal{L}_f$, and so, the fact that Z_f is an attracting fixed point of $\mathcal{E}_f$ implies that the point $\psi_f(Z_f)$ is an attracting fixed point of $\mathcal{L}_f$. □

The key result in the proof of proof of Theorem 4.11 relies on a non-autonomous analog of Lavaurs estimates in the setting of skew-products.

Let $\mathcal{B}_f$ and $\mathcal{B}_g$ be the parabolic basins of 0 under iteration of respectively f and g. One of the key points is to choose $(z_0, w_0) \in \mathcal{B}_f \times \mathcal{B}_g$ so that the first coordinate of $F^{\circ m}(z_0, w_0)$ returns infinitely many times close to the attracting fixed point of $\mathcal{L}_f$. The proof is designed so that the return times are the integers n^2 for $n \geq n_0$. Therefore, we need to analyze the orbit segment between n^2 and $(n+1)^2$, which is of length $2n+1$.

Proposition 4.13 (Astorg, Buff, Dujardin, Peters, R. [33])**.** *As $n \to +\infty$, the sequence of maps*

$$\mathbb{C}^2 \ni (z, w) \mapsto F^{2n+1}\big(z, g^{n^2}(w)\big) \in \mathbb{C}^2$$

converges locally uniformly in $\mathcal{B}_f \times \mathcal{B}_g$ to the map

$$\mathcal{B}_f \times \mathcal{B}_g \ni (z, w) \mapsto \big(\mathcal{L}_f(z), 0\big) \in \mathbb{C} \times \{0\}.$$

Idea of the proof. Let $\mathcal{B}_g$ the parabolic basin of 0 under iteration of g. For all $w \in \mathcal{B}_g$, the orbit $g^m(w)$ converges to 0 like $1/m$. We want to analyze the behaviour of F starting at $\big(z, g^{\circ n^2}(w)\big)$ during $2n+1$ iterates. For large n, the first coordinate of F along this orbit segment is approximately

$$f(z) + \varepsilon^2 \quad \text{with} \quad \frac{\pi}{\varepsilon} \simeq 2n.$$

A rough statement of Lavaurs Theorem from parabolic implosion gives us that if $\frac{\pi}{\varepsilon} = 2n$, then for large n, the $(2n)^{\text{th}}$ iterate of $f(z) + \varepsilon^2$ is approximately equal to $\mathcal{L}_f(z)$ on $\mathcal{B}_f$.

Our setting is different since in our case ε keeps decreasing along the orbit. Indeed on the first coordinate we are taking the composition of $2n+1$ transformations of the form

$$f(z) + \varepsilon_k^2 \quad \text{with} \quad \frac{\pi}{\varepsilon_k} \simeq 2n + \frac{k}{n} \quad \text{and} \quad 1 \leq k \leq 2n+1.$$

The key step in the proof of the statement consists in a detailed analysis of this non-autonomous situation, proving that the decay of ε_k is counterbalanced by taking *exactly* one additional iterate of F. □

With this proposition in hand, the proof of the Theorem 4.11 is easily completed.

Proof of Theorem 4.11. Let ξ be an attracting fixed point of $\mathcal{L}_f$ and let $V \subset \mathcal{B}_f$ be a disk centered at ξ and such that $\mathcal{L}_f(V)$ is compactly contained in V. Therefore $\mathcal{L}_f^{\circ k}(V)$ converges to ξ as $k \to +\infty$. Let $W \Subset \mathcal{B}_g$ be an arbitrary disk.

Thanks to Proposition 4.13, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$,

$$\pi_1 \circ F^{\circ(2n+1)}(V \times g^{\circ n^2}(W)) \Subset V,$$

where $\pi_1 \colon \mathbb{C}^2 \to \mathbb{C}$ denotes the projection on the first coordinate, that is $\pi_1(z,w) := z$.

Let U be a connected component of the open set $F^{-n_0^2}\big(V \times g^{\circ n_0^2}(W)\big)$. Then for every integer $n \geq n_0$, we have

$$F^{\circ n^2}(U) \subseteq V \times g^{\circ n^2}(W). \tag{4.7}$$

In fact, this holds by assumption for $n = n_0$. Now if the inclusion is true for some $n \geq n_0$, then

$$\begin{aligned}\pi_1 \circ F^{\circ(n+1)^2}(U) &= \pi_1 \circ F^{\circ(2n+1)}\left(F^{\circ n^2}(U)\right)\\ &\subset \pi_1 \circ F^{\circ(2n+1)}\left(V \times g^{\circ n^2}(W)\right) \subset V,\end{aligned}$$

from which (4.7) follows. This yields that the sequence $\{F^{\circ n^2}\}_{n\geq 0}$ is uniformly bounded, and hence normal, on U. Moreover, any cluster value of this sequence of maps is constant and of the form $(z,0)$ for some $z \in V$, and $(z,0)$ is a limit value (associated to a subsequence $\{n_k\}$) if and only if $\big(\mathcal{L}_f(z), 0\big)$ is a limit value (associated to the subsequence $\{1+n_k\}$). Therefore the set of cluster limits is totally invariant under $\mathcal{L}_f \colon V \to V$, and so it must coincide with the attracting fixed point ξ of $\mathcal{L}_f$. Therefore the sequence $\{F^{\circ n^2}\}_{n\geq 0}$ converges locally uniformly to $(\xi, 0)$ on U.

The sequence $\{F^{\circ m}\}_{m\geq 0}$ is locally bounded on U if and only if there exists a subsequence $\{m_k\}$ such that $\{F^{\circ m_k}|_U\}_{k\geq 0}$ has the same property. In fact, since $\overline{W}$ is compact, there exists $R > 0$ such that if $|z| > R$, then for every $w \in W$, (z,w) escapes locally uniformly to infinity under iteration. The domain U is therefore contained in the Fatou set of F.

Let Ω be the component of the Fatou set $\mathcal{F}_F$ containing U. For any integer $j \geq 0$, the sequence of maps $\{F^{\circ n^2+j}\}_{n\in\mathbb{N}}$ converges locally uniformly to $F^{\circ j}(\xi,0) = (f^{\circ j}(\xi), 0)$ on U and hence on Ω. Therefore the sequence $\{F^{\circ n^2}\}_{n\in\mathbb{N}}$ converges locally uniformly to $(f^{\circ j}(\xi), 0)$ on $F^{\circ j}(\Omega)$. If i, j are nonnegative integers such that $F^{\circ i}(\Omega) = F^{\circ j}(\Omega)$, then $f^{\circ i}(\xi) = f^{\circ j}(\xi)$, and so $i = j$ because ξ cannot be pre-periodic under iteration of f, since it belongs to the parabolic basin $\mathcal{B}_f$. This proves that Ω is not (pre-)periodic under iteration of F, and so it is a wandering Fatou component for F. □

We end this section recalling some explicit examples satisfying the assumption of Theorem 4.11.

Example 4.14. *As a consequence of Proposition 4.12 we obtain that if* $f: \mathbb{C} \to \mathbb{C}$ *is the cubic polynomial* $f(z) = z + z^2 + az^3$*, and* g *is as in* (4.5)*, then the polynomial skew-product* F *defined in* (4.6) *admits a wandering Fatou component for* $r > 0$ *sufficiently small and* $a \in D(1-r, r)$*.*

It is also interesting to search for real polynomial mappings with wandering Fatou domains intersecting $\mathbb{R}^2$. We also have such examples.

Example 4.15 ([33, Proposition C])**.** *Let* $f: \mathbb{C} \to \mathbb{C}$ *be the degree* 4 *polynomial defined by*

$$f(z) := z + z^2 + bz^4 \text{ with } b \in \mathbb{R}.$$

There exist parameters $b \in (-8/27, 0)$ *such that for* g *as in* (4.5)*, the polynomial skew-product* F *defined in* (4.6) *has a wandering Fatou component intersecting* $\mathbb{R}^2$*.*

4.4 Elliptic invariant fiber

In collaboration with Peters, we investigated in [197] the case of invariant fibers at the center of a Siegel disk. More precisely, we considered a polynomial skew-product of the form (4.1) having an *elliptic invariant fiber*. As before, we can assume without loss of generality that the invariant fiber is $\{w = 0\}$ and so we have $g(0) = 0$ and $g'(0) = e^{2\pi i\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. We assume that the origin belongs to a Siegel disk for g and hence g is locally holomorphically linearizable near $w = 0$. Therefore, up to a local change of coordinates we may assume that F is of the form

$$F(z, w) = (f(z, w), \lambda \cdot w),$$

where $f_w(z) := f(z, w)$ is a polynomial in z with coefficients depending holomorphically on w. We assume that the degree of the polynomial f_w is constant near $\{w = 0\}$, and at least 2.

In this case we only have a partial answer to Question 1. In fact, while we already know that the attracting Fatou components of f_0 always bulge, the general situation appears to be more complicated as there might be resonance phenomena. In [197] we proved the following local result, implying that all parabolic Fatou components of a polynomial skew-product with an elliptic invariant fiber bulge if the rotation number satisfies the Brjuno condition (1.5).

Proposition 4.16 (Peters, R. [197, Proposition 2])**.** *Let* F *be a holomorphic skew-product of the form*

$$F(z, w) = (f_w(z), g(w)) \tag{4.8}$$

with $g(w) = \lambda w + O(w^2)$*,* $f_0(0) = 0$*, and* $f_0'(0) = 1$*. Assume* λ *is a Brjuno number. If* $f_0(w) \equiv w$*, then* F *is holomorphically linearizable. If* $f_0(z) = z + f_{0,k+1}z^{k+1} + O(z^{k+2})$ *with* $f_{0,k+1} \neq 0$ *for some* $k \geq 1$*, then for any* $h \geq 0$ *there exists a local holomorphic change of coordinates near the origin conjugating* F *to a map of the form* $\widetilde{F}(z, w) = (\widetilde{f}(z, w), \lambda w)$ *satisfying*

$$\widetilde{f}(z, w) = z + f_{0,k+1}z^{k+1} + \cdots + f_{0,k+h+1}z^{k+h+1} + \sum_{j \geq h} z^{k+j+2}\alpha_{k+j+2}(w), \tag{4.9}$$

where, for $j \geq h$*,* $\alpha_{k+j+2}(w)$ *is a holomorphic function in* w *such that* $\alpha_{k+j+2}(0) = f_{0,k+j+2}$*.*

The proof of the previous proposition makes use of a procedure inspired by the Poincaré-Dulac normalization process [27, Chapter 4] aiming to conjugate the given polynomial skew-product to a skew-product in a simpler form. At each step of the usual Poincaré-Dulac normalization procedure we can use a polynomial change of coordinates to eliminate all non-resonant monomials of a given degree. A similar idea is used here, and thanks to the skew-product structure of the germ and the Brjuno assumption we are able to eliminate all non-resonant terms of a given degree in the powers of z by means of changes of coordinates that are polynomial in z with holomorphic coefficients in w. It turns out that there are some differences in between the first degrees as it is pointed out in [197, Section 2].

The arithmetic Brjuno condition is strongly needed in the proof of Proposition 4.16 and it is natural to ask whether parabolic Fatou components of f_0 always bulge, or it is possible to characterize when they bulge. We do not have a complete answer to such question. However we can proove the following result.

Proposition 4.17 (Peters, R. [197, Proposition 10])**.** *Let $F(z,w) = (f_z(w), g(w))$ be a holomorphic skew-product with an elliptic linearizable invariant fiber. If F does not admit a holomorphic invariant curve on the invariant fiber, then the parabolic Fatou components of f_0 do not bulge.*

Moreover, we can construct examples showing that the Brjuno condition cannot be completely omitted. In fact by taking *Cremer* numbers, that is $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and satisfying (1.3), we can construct examples of polynomial skew-product not admitting a holomorphic invariant curve on the invariant fiber. An elementary example is the following polynomial skew-product

$$F(z,w) = (z + w + zw, \lambda w)$$

with λ a Cremer number (see [197, Example 8] for details). More generally, arguing as in Cremer's example [86] we can show (see [197, Proposition 9]) the existence of holomorphic skew-products without invariant holomorphic curves of the form $\{w = \varphi(z)\}$, having an elliptic invariant fiber that is not point-wise fixed.

Describing the general situation can also be complicated by resonance phenomena. For example, an invariant fiber at the center of a Siegel disk was used in [49] to construct a non-recurrent Fatou component with limit set isomorphic to a punctured disk, and in their construction the invariant fiber also contains a Siegel disk, but with opposite rotation number. Moreover, it might happen that Fatou components on the invariant fiber do not bulge. For example, we can consider the skew-product

$$F(z,w) = (\lambda z(1 + azw), \lambda^{-1}w),$$

with $a \in \mathbb{C}^*$, $\lambda = e^{2\pi i\theta}$ and $\theta \in \mathbb{R} \setminus \mathbb{Q}$. We have $F(z,0) = (\lambda z, 0)$, but the Siegel disk around the origin in $\{w = 0\}$ is not bulging, and in fact it follows from [62] and [63] that there exists a Fatou component of parabolic type having on its boundary the origin of $\mathbb{C}^2$, which is fixed by F.

We also have an answer to Question 2, under the assumption that the multiplier at the elliptic invariant fiber is Brjuno and all critical points of the polynomial acting on the invariant fiber lie in basins of attracting or parabolic cycles.

Theorem 4.18 (Peters, R. [197, Theorem 1])**.** *Let F be a polynomial skew-product of the form*

$$F(z,w) = (f_w(z), g(w)), \tag{4.10}$$

and let $\{w = c\}$ be an elliptic invariant fiber with multiplier λ. If λ is Brjuno and all critical points of the polynomial f_c lie in basins of attracting or parabolic cycles, then all Fatou components of f_c bulge, and there is a neighbourhood of the invariant fiber $\{w = c\}$ in which the only Fatou components of F are the bulging Fatou components of f_c. In particular there are no wandering Fatou components in this neighbourhood.

Idea of the proof. The proof relies on the fact that, in dimension 1, if every critical point of the polynomial either (1) lies in the basin of an attracting periodic cycle, (2) lies in the basin of a parabolic periodic cycle, or (3) lies in the Julia set and after finitely many iterates is mapped to a periodic point, then it is possible to construct a conformal metric μ, defined in a backward invariant neighbourhood of the Julia set minus the parabolic periodic orbits, so that f_c is expansive with respect to this metric. It then follows that there can be no wandering Fatou components.

The key step in the proof is that for fibers $\{w = w_0\}$ sufficiently close to the invariant fiber $\{w = c\}$ we can define conformal metrics μ_{w_0} that depend continuously on w_0, and so that F acts expansively with respect to this family of metrics (see [197, Section 3] for details). That is, for a point $(z_0, w_0) \in \mathbb{C}^2$ lying in the region where the metrics are defined, and for a non-zero tangent vector $\xi \in T_{z_0}(\mathbb{C}_{w_0})$ we have that

$$\mu_{w_1}(z_1, df_{w_0}\xi) > \mu_{w_0}(z_0, \xi),$$

where $(z_1, w_1) = F(z_0, w_0)$. Therefore, if (z_0, w_0) does not lie in one of the bulging Fatou components and its orbit (z_n, w_n) remains in the neighbourhood where the expanding metrics are defined, for any tangent vector $\xi \in T_{z_0}(\mathbb{C}_{w_0})$ we have that

$$\mu_{w_n} df_{z_{n-1}} \cdots df_{z_0}\xi \to \infty,$$

from which it follows that the family $\{F^{\circ n}\}$ cannot be normal on any neighbourhood of (z_0, w_0). This proves that there are no Fatou components but the bulging components near the invariant fiber $\{w = c\}$. □

Note that we only allow critical points of the type (1) and (2). Whether the same techniques could be used to deal with pre-periodic critical points lying on the Julia set is unclear to us. The difficulty is that the property that critical points are eventually mapped onto periodic cycles is not preserved in nearby fibers.

Chapter 5

Periodic Fatou components of holomorphic automorphisms of $\mathbb{C}^k$

In this chapter we give an updated account of the recent results on the classification of periodic Fatou components for holomorphic automorphisms of $\mathbb{C}^k$.

As we already mentioned in Chapter 4, the problem of describing the dynamics on the Fatou set, and the kinds of behaviour on its components, is now completely understood for rational endomorphisms of the Riemann sphere, thanks to Fatou's Classification of invariant Fatou components, Theorem 4.1, together with Sullivan's non-wandering domains Theorem 4.2. The situation in higher dimension is more complicated and the classification of periodic Fatou components is not yet completed even in dimension 2, as we will explain in the rest of this chapter. For simplicity, we will focus on the classification of invariant Fatou components.

5.1 Polynomial automorphisms

The classification of invariant Fatou components for polynomial automorphisms of $\mathbb{C}^k$ for any $k \geq 2$ is not yet complete as such group is big and complicated. For the rest of this section we will restrict ourselves to dimension 2, where the situation is better understood. In fact, interest in the dynamics of polynomial automorphisms of $\mathbb{C}^2$ and in particular of Hénon maps, rose at the end of the 1980's, also thanks to the fundamental work of Friedland and Milnor [119], who proved that every polynomial automorphisms of $\mathbb{C}^2$ is affinely conjugate to either an affine map, an elementary map, or a finite compositions of *generalized Hénon maps*, usually called *complex Hénon maps*.

Definition 5.1. *A* complex Hénon map *is a holomorphic map $H\colon \mathbb{C}^2 \to \mathbb{C}^2$ of the form*

$$H(z, w) = (p(z) - \delta w, z), \tag{5.1}$$

where p is a one-dimensional polynomial and $\delta \in \mathbb{C} \setminus \{0\}$.

It is easy to describe the dynamics of affine and elementary maps.

Complex Hénon maps have been extensively studied by Hubbard and Oberste-Vorth [146, 147, 148], Bedford-Smillie [37, 38], Fornæss-Sibony [123] and several other authors. A basic useful property of Hénon maps is the existence of the following filtration. For $R > 0$ large enough we set

$$W = \{(z,w) \in \mathbb{C}^2 \mid \max(|z|,|w|) \leq R\},$$
$$V_+ = \{(z,w) \in \mathbb{C}^2 \mid |z| \geq \max(|w|,R)\},$$
$$V_- = \{(z,w) \in \mathbb{C}^2 \mid |w| \geq \max(|z|,R)\}.$$

It is easy to check that $H(V_+) \subset V_+$, $H^{-1}(V_-) \subset V_-$, and the orbit of any point in V_+ converge to the attracting fixed point $[1:0:0]$ on the line at infinity. Therefore the escaping set

$$I_\infty = \bigcup_{n \in \mathbb{N}} H^{-n}(V_+) = \{(z,w) \in \mathbb{C}^2 \mid \|H(z,w)\| \to \infty\}$$

is a Fatou component, and the forward orbits for any other Fatou component are bounded.

First results on the dynamics on periodic Fatou components for Hénon maps have been obtained by Bedford and Smillie in 1991 [38], followed by Fornæss and Sibony in 1995 [125], with a later complement by Ueda [253]. Bedford and Smillie introduced the notion of recurrent Fatou component.

Definition 5.2. *An invariant Fatou component Ω for a holomorphic endomorphism F on a complex manifold X is called* recurrent *if there exists a point $p \in \Omega$ whose orbit accumulates at a point in Ω.*

Therefore, by normality, a Fatou component is recurrent exactly when it contains a recurrent orbit. For a non-recurrent Fatou component Ω all orbits in Ω converge to $\partial\Omega$. In dimension one, for rational endomorphisms of the Riemann sphere we therefore have that basins of attracting fixed points, Siegel disks and Herman rings are recurrent Fatou components, while the only non-recurrent components are basins of parabolic fixed points. Recurrent Fatou components for Hénon maps in dimension 2 have been studied by Bedford and Smillie [38, 39], Fornæss and Sibony [125] and Ueda [253]. Their results can be summarized as follows.

Theorem 5.3 ([38, 39, 125, 253])**.** *Let H be a Hénon map or a holomorphic endomorphism of $\mathbb{P}^2(\mathbb{C})$ with a recurrent invariant Fatou component Ω. Then one of the following holds:*

1. *Ω is an attracting basin of some fixed point in Ω, and Ω is biholomorphic to $\mathbb{C}^2$,*
2. *there exists a one-dimensional closed complex submanifold Σ of Ω and $H^{\circ n}(K) \to \Sigma$ for any compact set K in Ω. The Riemann surface Σ is biholomorphic to a disk or an annulus and $H|_\Sigma$ is conjugate to an irrational rotation,*
3. *Ω is a Siegel domain, that is there exists a sequence $\{n_j\}$ such that $\{H^{\circ n_j}\}$ converges uniformly on compact subsets of Ω to the identity.*

This result has been generalized to holomorphic endomorphisms of $\mathbb{P}^k(\mathbb{C})$, $k \geq 3$ by Fornæss and Rong in [121].

5.1.1 Non-recurrent invariant Fatou components

Non-recurrent invariant Fatou components for polynomial automorphisms in $\mathbb{C}^2$ have been first studied by Weickert [260] and Jupiter and Lilov [157]. One of the main difficulties consists in the fact that since for an invariant non-recurrent Fatou component Ω all orbits converge to the boundary of the component, by normality there exists a sequence $\{H^{\circ n_k}\}$ converging uniformly on compact subsets, to a limit map $h\colon \Omega \to \partial\Omega$. In general the map h is not unique, it depends on the sequence $\{n_k\}$, and a priori it is not even clear whether the limit set $h(\Omega)$ is always unique.

Recently, Lyubich and Peters were able to prove the following result giving a precise classification of non-recurrent invariant Fatou components under the assumption that $h(\Omega)$ is unique.

Theorem 5.4 (Lyubich, Peters [176, Theorem 6]). *Let H be a Hénon map and suppose that Ω is a non-recurrent invariant Fatou component. Then there exists a sequence $\{H^{\circ n_k}\}$ that converges uniformly on compact subsets of Ω to a fixed point $p \in \partial\Omega$. If the entire sequence $\{H^{\circ n}\}$ converges to p then the eigenvalues of DH_p are 1 and λ, with $|\lambda| < 1$. Moreover, Ω is biholomorphically equivalent to $\mathbb{C}^2$.*

They also obtained in [176, Theorem 7] that, for a polynomial endomorphism P of $\mathbb{P}^2(\mathbb{C})$ with a non-recurrent invariant Fatou component Ω, if the limit set is unique, then it either consists of one point, or it is an injectively immersed Riemann surface, conformally equivalent to either the unit disk, the punctured unit disk or an annulus, and P acts on it as an irrational rotation. Moreover, all cases were known to occur except for the punctured unit disk, whose existence has been proved by Boc-Thaler, Fornæss and Peters in [49] via a one-resonant *degenerate* polynomial endomorphism of $\mathbb{C}^2$.

Lyubich and Peters completed the classification of invariant Fatou components for *moderately dissipative* polynomial automorphisms of $\mathbb{C}^2$. More precisely they obtained the following result.

Theorem 5.5 (Lyubich, Peters [176, Theorem 1]). *Let $P\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a non-elementary polynomial automorphism of degree $d \geq 2$, and let $\delta = \det DP$ be its Jacobian. Assume that P is moderately dissipative, that is*

$$|\delta| < \frac{1}{d^2}.$$

Let Ω be an invariant non-recurrent Fatou component of P with bounded forward orbits. Then all the orbits in Ω converge to a parabolic point $\alpha \in \partial\Omega$ with multiplier 1.

Putting this result together with the previous ones, one obtains that for moderately dissipative polynomial automorphisms of $\mathbb{C}^2$ the description of invariant Fatou components is the same as in dimension 1.

Theorem 5.6 ([38, 39, 125, 253, 176]). *Let $P\colon \mathbb{C}^2 \to \mathbb{C}^2$ be non-elementary moderately dissipative polynomial automorphism of degree $d \geq 2$ and let Ω be an invariant Fatou component. Then one of the following holds:*

1. *Ω is an attracting basin of some fixed point p in Ω, and Ω is biholomorphic to $\mathbb{C}^2$,*
2. *there exists a properly embedded submanifold Σ of Ω, biholomorphic to a disk or an annulus, such that $H^{\circ n}(K) \to \Sigma$ for any compact set K in Ω and $H|_\Sigma$ is conjugate to an irrational rotation,*
3. *all orbits in Ω converge to a fixed point $p \in \partial\Omega$, the eigenvalues of DP_p are 1 and λ, with $|\lambda| < 1$, and Ω is biholomorphically equivalent to $\mathbb{C}^2$.*

Idea of the proof. Consider Γ_P the set of all limit functions $h\colon \Omega \to \overline{\Omega}$ for the family of the iterates of P. In the non-recurrent case the image of h is either a single point (rank 0 case) or a holomorphic curve (rank 1 case). The key point consists in analyzing the rank 1 case, that is when the image $h(\Omega)$ is an analytic curve lying in the boundary of Ω.

First, it is possible to show that the curve $h(\Omega)$ is non-singular. Then by analyzing the natural action of P on Γ_P it is possible to prove that Γ_P always contains a rank 0 map, and the image of such a map has to be a fixed point p. One then proves that if there exists a rank one limit map, then there also exists one whose image lies in the strong stable manifold of the fixed point p previously constructed. This last situation is ruled out by applying the classical Denjoy-Carleman-Ahlfors and Wiman Theorems, and this is the only step in the proof where the stronger assumption on the Jacobian of P is indeed required. Therefore, all orbits in Ω converge to a unique fixed point p. Finally, by adapting the classical Fatou's argument, it is possible to prove a Snail Lemma stating that the fixed point p is a semi-parabolic point with one multiplier equal to 1. □

5.2 Non-polynomial automorphisms

In this setting, less is known on the classification of invariant Fatou components, already for *attracting* Fatou components.

Definition 5.7. *Let F be a holomorphic endomorphism of $\mathbb{C}^k$, $k \geq 2$. An invariant Fatou component Ω for F is called* attracting *if there exists a point $p \in \overline{\Omega}$ with $\lim_{n\to\infty} F^{\circ n}(z) = p$ for all $z \in \Omega$.*

Note that, in particular, p is a fixed point for F, and Ω is recurrent if $p \in \Omega$, non-recurrent if $p \in \partial\Omega$.

It follows from Rosay-Rudin results in [233] and [200, Theorem 2.1] that

Theorem 5.8. *Let $k \geq 2$ and let $F\colon \mathbb{C}^k \to \mathbb{C}^k$ be a holomorphic automorphism. Every attracting recurrent Fatou component Ω of F is biholomorphic to $\mathbb{C}^k$.*

In fact it turns out that an attracting recurrent Fatou component is the global basin of attraction of F at p, which is an attracting fixed point (i.e. all eigenvalues of DF_p have modulus strictly less than 1).

The characterization of non-recurrent attracting Fatou components for non-polynomial automorphisms of $\mathbb{C}^k$ is more delicate, even when considering the two-dimensional case. It follows from the results obtained by Ueda in [250], that:

Theorem 5.9 (Ueda, [250])**.** *Let $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic automorphism and let Ω be a non-recurrent attracting Fatou component with fixed point $p \in \partial\Omega$. If the eigenvalues of DP_p are 1 and λ, with $|\lambda| < 1$, then Ω is biholomorphic to $\mathbb{C}^2$.*

Hakim [135] gave examples of automorphisms of $\mathbb{C}^k$ tangent to the identity at an isolated fixed point and such that the attracting set of the fixed point (which belongs to the boundary of such set) is biholomorphic to $\mathbb{C}^k$. Parabolic domains have indeed been often used to build Fatou-Bieberbach domains, that is proper subsets of $\mathbb{C}^n$ biholomorphic to $\mathbb{C}^n$; see, e.g., [255], [245] and references therein.

In the rest of the chapter we will explain the construction that we obtained with Bracci and Stensønes of the first family of holomorphic automorphisms of $\mathbb{C}^k$ with an invariant, non-recurrent, attracting Fatou component *not* biholomorphic to $\mathbb{C}^k$.

5.3 A non-recurrent component biholomorphic to $\mathbb{C}\times(\mathbb{C}^*)^{k-1}$

As we already recalled in the previous section, from the results obtained by Ueda in [251] and from [176, Theorem 6] by Lyubich and Peters, it follows that *every non-recurrent invariant attracting Fatou component Ω of a* <u>*polynomial*</u> *automorphism of $\mathbb{C}^2$ is biholomorphic to $\mathbb{C}^2$.*

Vivas and Stensønes in [245] produced examples of automorphisms of $\mathbb{C}^3$ having attracting non-recurrent Fatou component biholomorphic to $\mathbb{C}^2 \times \mathbb{C}^*$. However, the question whether there could exist a holomorphic automorphism of $\mathbb{C}^k$, with $k \geq 2$ and having an invariant, non-recurrent, attracting Fatou component biholomorphic to $\mathbb{C} \times (\mathbb{C}^*)^{k-1}$ remained open until recently.

In collaboration with Bracci and Stensønes, we gave in [64] a positive answer to such a question.

Theorem 5.10 (Bracci, R., Stensønes, [64]). *Let $k \geq 2$. There exist holomorphic automorphisms of $\mathbb{C}^k$ having an invariant, non-recurrent, attracting Fatou component biholomorphic to $\mathbb{C} \times (\mathbb{C}^*)^{k-1}$.*

Moreover as an immediate corollary of Theorem 5.10 and [251, Proposition 5.1], we obtain the following result.

Corollary 5.11 (Bracci, R., Stensønes, [64]). *Let $k \geq 2$. There exists a biholomorphic image of $\mathbb{C} \times (\mathbb{C}^*)^{k-1}$ in $\mathbb{C}^k$ which is Runge.*

The existence of an embedding of $\mathbb{C} \times \mathbb{C}^*$ as a Runge domain in $\mathbb{C}^2$ was a long standing open question, positively settled by our construction. After a preliminary version of [64] was circulating, Forstnerič and Wold constructed in [130] other examples of Runge embeddings of $\mathbb{C} \times \mathbb{C}^*$ in $\mathbb{C}^2$ (which do not arise from basins of attraction of automorphisms) using completely different techniques.

Notice that, thanks to the results obtained by Serre in [239] (see also [144, Theorem 2.7.11]), every Runge domain $D \subset \mathbb{C}^k$ satisfies $H^q(D) = 0$ for all $q \geq k$. Therefore the Fatou component in Theorem 5.10 has the highest possible admissible non-vanishing cohomological degree.

The proof of Theorem 5.10 is rather involved and in the next subsection we outline it in the case $k = 2$. We will explain in the last subsection the modifications needed to deal with the general case in any dimension.

5.3.1 Idea of the proof of Theorem 5.10 for $k = 2$

The strategy of the proof consists in three main steps. The first step consists in finding a holomorphic automorphism F of $\mathbb{C}^2$ with a not simply connected, completely F-invariant domain Ω having a fixed point in its boundary, that we might assume to be the origin, and such that all orbits of points in Ω converge to the fixed point. The second step consists in proving that such a domain is biholomorphic to $\mathbb{C} \times \mathbb{C}^*$. Finally we have to find a holomorphic automorphism F of $\mathbb{C}^2$ satisfying the first two steps and such that Ω is indeed one of its Fatou components.

The final step might seem unnecessary, but this is not the case, as this elementary example clearly shows: $\mathbb{C} \times \mathbb{C}^*$ (but also $\mathbb{C}^* \times \mathbb{C}^*$) is a completely invariant domain for the holomorphic automorphism $\mathbb{C}^2 \ni (z, w) \mapsto (\frac{z}{2}, \frac{w}{2}) \in \mathbb{C}^2$, but it is clearly not one of its Fatou component.

Step 1. The key point here is to start from a germ of biholomorphism of $(\mathbb{C}^2, O)$, with an isolated fixed point at the origin and a special dynamics near the origin:

$$F_N(z, w) = \left(\lambda z \left(1 - \frac{zw}{2}\right), \overline{\lambda} w \left(1 - \frac{zw}{2}\right)\right), \tag{5.2}$$

where $\lambda \in \mathbb{C}$, $|\lambda| = 1$, is not a root of unity.

This germ is a particular case of *one-resonant* germ and, as we already recalled in Chapter 3, the local dynamics of this kind of maps has been studied by Bracci with Zaitsev in [62], and by Bracci, R. and Zaitsev in [63]. As a consequence of Theorem 3.13, we have the following.

Proposition 5.12. *Let F_N be a germ of biholomorphism at $(0, 0)$ of the form (5.2). Let $\beta_0 \in (0, 1/2)$ and let $l \in \mathbb{N}$, $l \geq 4$ be such that $\beta_0(l + 1) \geq 4$. Then for every $\theta_0 \in (0, \pi/2)$ and for any germ of biholomorphism F at $(0, 0)$ of the form*

$$F(z, w) = F_N(z, w) + O(\|(z, w)\|^l)$$

there exists $R_0 > 0$ such that the (non-empty) open set

$$B = \{(z,w) \in \mathbb{C}^2 : zw \in S_{R_0}, |z| < |zw|^\beta, |w| < |zw|^\beta\},$$

with S_{R_0} a small sector in $\mathbb{C}$ with vertex at 0 around the positive real axis and radius R_0, is a uniform local basin of attraction for F, that is $F(B) \subseteq B$, and $\lim_{n\to\infty} F^{\circ n}(z,w) = (0,0)$ uniformly on B.

Setting $x = zw, y = w$ (which are coordinates on B) the domain B looks like $\{(x,y) \in \mathbb{C}\times\mathbb{C}^* : x \in S, |x|^{1-\beta} < |y| < |x|^\beta\}$, and hence B is doubly connected.

In order to obtain a holomorphic automorphism of $\mathbb{C}^2$ having the same property of F_N, it suffices to recall the results of Weickert in [259] and Forstnerič in [128], ensuring that for any large $l \in \mathbb{N}$ there exists an automorphism F of $\mathbb{C}^2$ such that

$$F(z,w) - F_N(z,w) = O(\|(z,w)\|^l). \tag{5.3}$$

Therefore it suffices to take one such holomorphic automorphism F of $\mathbb{C}^2$ and to consider

$$\Omega := \cup_{n\in\mathbb{N}} F^{-n}(B).$$

The domain Ω is connected but not simply connected, completely F-invariant, the origin belongs to the boundary of Ω and by construction all orbits in Ω converge to the origin.

We can also characterize the behaviour of the orbits in Ω (where for a point $(z,w) \in \mathbb{C}^2$, we denote $(z_n, w_n) := F^{\circ n}(z,w)$):

$$\Omega = \{(z,w) \in \mathbb{C}^2 \setminus \{(0,0)\} : \lim_{n\to\infty} \|(z_n,w_n)\| = 0, \quad |z_n| \sim |w_n|\},$$

and moreover, if $(z,w) \in \Omega$ then $|z_n| \sim |w_n| \sim \frac{1}{\sqrt{n}}$ (see [64, Theorem 5.2] for details).

Step 2. We already know that Ω is not simply connected, and so it cannot be biholomorphic to $\mathbb{C}^2$. We prove that Ω is biholomorphic to $\mathbb{C} \times \mathbb{C}^*$ by constructing a fibration from Ω to $\mathbb{C}$ in such a way that Ω is a line bundle minus the zero section over $\mathbb{C}$.

We first prove the existence of a univalent map Q on B which intertwines F on B with a simple overshear. The first component ψ of Q is morally the Fatou coordinate of the projection of F onto the zw-plane, as it was introduced in [63, Proposition 4.3] and satisfies

$$\psi \circ F = \psi + 1.$$

The second component σ is constructed [64, Proposition 3.4] as the local uniform limit on B of the sequence $\{\sigma_n\}$ defined by

$$\sigma_n(z,w) := \lambda^n \pi_2(F^{\circ n}(z,w)) \exp\left(\frac{1}{2}\sum_{j=0}^{n-1} \frac{1}{\psi(z,w)+j}\right),$$

where $\pi_2 \colon \mathbb{C}^2 \to \mathbb{C}$ is the projection on the second coordinate, $\pi_2(z,w) = w$, and satisfies the functional equation

$$\sigma \circ F = \overline{\lambda} e^{-\frac{1}{2\psi}} \sigma.$$

Next, using dynamics, we extend in [64, Section 4] such a map to a univalent map G defined on a domain $\Omega_0 \subset \Omega$, and we use it to prove that Ω is a line bundle minus the zero section over $\mathbb{C}$. Since all line bundles over $\mathbb{C}$ are globally holomorphically trivial, we obtain that Ω is biholomorphic to $\mathbb{C} \times \mathbb{C}^*$.

Step 3. To finish our construction and prove that Ω coincides with the Fatou component V containing it, we need to assume that the eigenvalue λ satisfies the Brjuno condition (1.5). In fact, this ensures (see for example [204]) the existence of two F-invariant analytic discs, tangent to the axes, where F acts as an irrational rotation. In particular, we can choose local coordinates at $(0,0)$, which we may assume to be defined on the unit ball $\mathbb{B}$ of $\mathbb{C}^2$ and $B \subset \mathbb{B}$, such that $\{z=0\}$ and $\{w=0\}$ are not contained in $V \cap \mathbb{B}$. Let $\mathbb{B}_* := \mathbb{B} \setminus \{zw=0\}$.

Assume by contradiction that $V \neq \Omega$. Then we can take $p_0 \in \Omega$, $q_0 \in V \setminus \Omega$, and Z a connected open set containing p_0 and q_0 and such that $\overline{Z} \subset V$. Moreover, since $\{F^{\circ n}\}$ converges uniformly to the origin on $\overline{Z}$, up to replacing F by one of its iterates, we can assume that the forward F-invariant set $W := \cup_{n\in\mathbb{N}} F^{\circ n}(Z)$ satisfies $W \subset \mathbb{B}_*$. By construction, for every $\delta > 0$ we can find $p \in Z \cap \Omega$ and $q \in Z \cap (V \setminus \Omega)$ such that $k_W(p,q) \le k_Z(p,q) < \delta$, where k_W is the Kobayashi (pseudo)distance of W. By the properties of the Kobayashi distance, for every $n \in \mathbb{N}$ we have

$$k_{\mathbb{B}_*}(F^{\circ n}(p), F^{\circ n}(q)) \le k_W(p,q) < \delta.$$

Also, if $(z_n, w_n) := F^{\circ n}(p)$, $(x_n, y_n) := F^{\circ n}(q)$, then

$$k_{\mathbb{D}^*}(z_n, x_n) < \delta, \quad k_{\mathbb{D}^*}(w_n, y_n) < \delta,$$

where $\mathbb{D}^*$ is the punctured unit disc, and therefore by the triangle inequality we have

$$k_{\mathbb{D}^*}(x_n, w_n) < 2\delta.$$

Since $q \notin \Omega$, $F^{\circ n}(q) \notin B$ for every $n \in \mathbb{N}$, and so we can prove (using [64, Lemma 2.5]) that, up to passing to a subsequence, we have $|x_n| \not\sim |y_n|$ and

$$|y_n| \le |x_n|^{\frac{1-\beta}{\beta}}.$$

The properties of the Kobayashi distance of $\mathbb{D}^*$ (see [64, Lemma5.6]) imply

$$k_{\mathbb{D}^*}(x_n, y_n) > \log \frac{1-\beta}{\beta} - \delta.$$

Therefore we have

$$\delta > k_{\mathbb{D}^*}(w_n, y_n) > k_{\mathbb{D}^*}(x_n, y_n) - k_{\mathbb{D}^*}(x_n, w_n) > \log \frac{1-\beta}{\beta} - 3\delta,$$

and so

$$4\delta \ge \log \frac{1-\beta}{\beta},$$

which leads to a contradiction since $\frac{1-\beta}{\beta} > 1$ is fixed and $\delta > 0$ is arbitrary.

□

5.3.2 Idea of the proof of Theorem 5.10 for $k \geq 3$

The strategy of the proof is the same as is dimension 2, with the following modifications.

In the general case, $k \geq 3$, we start with a germ of biholomorphism of $\mathbb{C}^k$ at the origin of the form

$$F_N(z_1, \ldots, z_k) = \left(\lambda_1 z_1 \left(1 - \frac{z_1 \cdots z_k}{k}\right), \ldots, \lambda_k z_k \left(1 - \frac{z_1 \cdots z_k}{k}\right)\right), \tag{5.4}$$

where:

(i) each $\lambda_j \in \mathbb{C}$, $|\lambda_j| = 1$, is not a root of unity for $j = 1, \dots, k$,

(ii) the k-tuple $(\lambda_1, \dots, \lambda_k)$ is *one-resonant with index of resonance* $(1, \dots, 1) \in \mathbb{N}^k$ in the sense of Definition 3.11,

(iii) the k-tuple $(\lambda_1, \dots, \lambda_k)$ is *admissible* in the sense of Pöschel (see [204]), that is we have

$$\sum_{n=0}^{+\infty} \frac{1}{2^n} \log \frac{1}{\omega_j(2^{n+1})} < +\infty \ , \ \text{for } j = 1, \dots, k$$

where $\omega_j(m) = \min_{2 \le h \le m} \min_{1 \le i \le k} |\lambda_j^h - \lambda_i|$ for any $m \ge 2$.

Thanks to a result of B. J. Weickert [259] and F. Forstnerič [128], for any large $l \in \mathbb{N}$ there exists an automorphism F of $\mathbb{C}^k$ such that

$$F(z_1, \dots, z_k) - F_N(z_1, \dots, z_k) = O(\|(z_1, \dots, z_k)\|^l). \tag{5.5}$$

Moreover, thanks to [62, Theorem 1.1], given $\beta \in (0, \frac{1}{k})$ and $l \in \mathbb{N}$, $l \ge 4$ such that $\beta(l+1) \ge 4$, there is $R > 0$ such that, denoting by S_R a small sector in $\mathbb{C}$ with vertex at 0 around the positive real axis and radius R, the set

$$B := \{(z_1, \dots, z_k) \in \mathbb{C}^k : u := z_1 \cdots z_k \in S_R, |z_j| < |u|^\beta \text{ for } j = 1, \dots, k\}$$

is non-empty, forward invariant under F, the origin is on the boundary of B and we have $\lim_{n\to\infty} F^{\circ n}(p) = 0$ for all $p \in B$, uniformly on compact sets. Arguing as in dimension 2, we obtain that for each $p \in B$, we have that $\lim_{n\to\infty} nu_n = 1$ and $|\pi_j(F^{\circ n}(p))| \sim n^{-1/k}$, for $j = 1, \dots, k$, where π_j is the projection on the j-th coordinate. Moreover, again thanks to [63, Proposition 4.3], there exists a local Fatou coordinate $\psi \colon B \to \mathbb{C}$ such that $\psi \circ F = \psi + 1$.

Here we need $k-1$ other local coordinates $\sigma_2, \dots, \sigma_k$. For $2 \le j \le k$, $\sigma_j \colon B \to \mathbb{C}$ is defined as the uniform limit on compact sets of the sequence $\{\sigma_{j,n}\}_n$ where

$$\sigma_{j,n}(z_1, \dots, z_k) := (\lambda_j \dots \lambda_k)^{-n} \Pi_j(F^{\circ n}(z_1, \dots, z_k)) \exp\left(\frac{k-j+1}{k} \sum_{m=0}^{n-1} \frac{1}{\psi(z_1, \dots, z_k) + m} \right),$$

and $\Pi_j \colon \mathbb{C}^k \to \mathbb{C}$ is defined as $\Pi_j(z_1, \dots, z_k) := z_j \cdots z_k$. The map σ_j satisfies the functional equation

$$\sigma_j \circ F = \lambda_j \cdots \lambda_k e^{-\frac{k-j+1}{k\psi}} \sigma_j.$$

Let $\Omega := \cup_{n\ge 0} F^{-n}(B)$. Arguing like in dimension 2, one can prove that $H^{k-1}(\Omega, \mathbb{C}) \ne 0$. Using the functional equation we can extend ψ to a map $g_1 \colon \Omega \to \mathbb{C}$. Moreover, set $H := g_1(B)$ and $\Omega_0 := g_1^{-1}(H)$. For $j = 2, \dots, k$, we can extend σ_j to Ω_0 by setting, for any $p \in \Omega_0$,

$$g_j(p) = (\lambda_j \cdots \lambda_k)^n \exp\left(-\frac{k-j+1}{k} \sum_{m=0}^{n-1} \frac{1}{g_1(p) + j} \right) \sigma_j(F^{\circ n}(p))$$

where $n \in \mathbb{N}$ is so that $F^{\circ n}(p) \in B$. As in dimension 2, the map $\Omega_0 \ni p \mapsto G(p) := (g_1(p), \dots, g_k(p)) \in H \times \mathbb{C}^{k-1}$ is univalent with image $H \times (\mathbb{C}^*)^{k-1}$. In fact, we can use coordinates

$$(u, y_2, \dots, y_k) := (z_1 \cdots z_k, z_2 \cdots z_k, \dots, z_k),$$

in B so that we have

$$B = \{u \in S(R, \theta), |u|^{1-k\beta} < |y_k| < |u|^\beta, \ |u|^{1-j\beta} < |y_j| < |u|^\beta |y_{j+1}| \text{ for } j = 2, \dots, k-1\}.$$

Following the proof of [64, Proposition 4.4], since, for $p \in \Omega_0$, $\lim_{n\to\infty} nu_n = 1$ and $|\Pi_j(F^{\circ n}(p))| \sim n^{-(k-j+1)/k}$ for $j = 2, \ldots, k$ we can show that for any $a \in H$ and $b_k \in \mathbb{C}^*$ there is a point $p \in \Omega_0$ such that $g_1(p) = a$ and $g_k(p) = b_k$. Now fix $a \in H$ and $b_k \in \mathbb{C}^*$. Using

$$|u|^{1-(k-1)\beta} < |y_{k-1}| < |u|^{\beta}|y_k|$$

we obtain that $\mathbb{C}^* \subseteq g_{k-1}(g_1^{-1}(a) \cap g_k^{-1}(b_k))$, and so on for every $j = 2, \ldots, k-2$. Therefore $G(\Omega_0) = H \times (\mathbb{C}^*)^{k-1}$, and as in Step 2 we prove that $g_1 \colon \Omega \to \mathbb{C}$ is a holomorphic fiber bundle map with fiber $(\mathbb{C}^*)^{k-1}$. Since the transition functions belong to $\mathrm{GL}_n(\mathbb{C})$, by [129, Corollary 8.3.3] we obtain that Ω is biholomorphic to $\mathbb{C} \times (\mathbb{C}^*)^{k-1}$.

To conclude, we assume that the k-tuple $(\lambda_1, \ldots, \lambda_k)$ is *admissible in the sense of Pöschel [204]* (see also Subsection 1.3.1). Therefore we can locally choose coordinates so that the Fatou component V containing Ω cannot intersect the coordinate axes in a small neighbourhood of the origin. Hence using the estimates for the Kobayashi distance as done in Step 3, we prove that $V = \Omega$.

□

Part III

Iteration in Convex Domains

Chapter 6

Wolff-Denjoy theorems in convex domains

In this chapter we provide a short introduction to Wolff-Denjoy theorem, and its generalizations in several complex variables, up to very recent results. A precise and systematic presentation can be found in [12] and references therein.

The Kobayashi distance is the main tool in describing the dynamics of holomorphic self-maps of bounded convex domains and we shall recall its definition and main properties in the next section.

6.1 The Kobayashi distance

Given X and Y two (finite dimensional) complex manifolds, in the following we shall denote by $\mathrm{Hol}(X, Y)$ the set of *all holomorphic maps from X to Y*, endowed with the compact-open topology so that it becomes a metrizable topological space. We shall denote by $\mathrm{Aut}(X) \subset \mathrm{Hol}(X, X)$ the set of *holomorphic automorphisms of X*, that is invertible holomorphic self-maps, of X. More generally, if X and Y are topological spaces we shall denote by $C^0(X, Y)$ the space of *continuous maps from X to Y*, again endowed with the compact-open topology.

We shall denote by $\mathbb{D} = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ the unit disk in the complex plane $\mathbb{C}$, by $\mathbb{B}^n = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$, where $\|\cdot\|$ is the Euclidean norm, the unit ball in the n-dimensional space $\mathbb{C}^n$, and by $\mathbb{D}^n \subset \mathbb{C}^n$ the unit polydisk in $\mathbb{C}^n$. Furthermore, $\langle \cdot , \cdot \rangle$ will denote the canonical Hermitian product on $\mathbb{C}^n$.

6.1.1 The Poincaré distance

The model for all invariant distances in complex analysis is the Poincaré distance on the unit disk of the complex plane.

Definition 6.1. *The* Poincaré *(or* hyperbolic*)* metric *on $\mathbb{D}$ is the Hermitian metric whose associated norm is given by*

$$\kappa_{\mathbb{D}}(\zeta; v) = \frac{1}{1 - |\zeta|^2} |v|$$

for all $\zeta \in \mathbb{D}$ and $v \in \mathbb{C} \simeq T_\zeta \mathbb{D}$.

The Poincaré metric is a complete Hermitian metric with constant Gaussian curvature -4.

Definition 6.2. *The* Poincaré *(or* hyperbolic*) distance $k_{\mathbb{D}}$ on $\mathbb{D}$ is the integrated form of the Poincaré metric. It is a complete distance, whose expression is*

$$k_{\mathbb{D}}(\zeta_1, \zeta_2) = \tfrac{1}{2} \log \frac{1 + \left|\frac{\zeta_1 - \zeta_2}{1 - \overline{\zeta_1}\zeta_2}\right|}{1 - \left|\frac{\zeta_1 - \zeta_2}{1 - \overline{\zeta_1}\zeta_2}\right|} .$$

In particular,

$$k_{\mathbb{D}}(0, \zeta) = \tfrac{1}{2} \log \frac{1 + |\zeta|}{1 - |\zeta|} .$$

It is useful to remark that the function

$$t \mapsto \tfrac{1}{2} \log \frac{1 + t}{1 - t}$$

is the inverse of the hyperbolic tangent $\tanh t = (e^t - e^{-t})/(e^t + e^{-t})$.

Besides being a metric with constant negative Gaussian curvature, the Poincaré metric strongly reflects the properties of the holomorphic self-maps of the unit disk. For instance, the isometries of the Poincaré metric coincide with the holomorphic or anti-holomorphic automorphisms of $\mathbb{D}$ (see, e.g., [2, Proposition 1.1.8]).

More importantly, the *Schwarz-Pick lemma* says that any holomorphic self-map of $\mathbb{D}$ is non-expansive for the Poincaré metric and distance (see, e.g., [2, Theorem 1.1.6]).

Theorem 6.3 (Schwarz-Pick lemma)**.** *Let $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ be a holomorphic self-map of $\mathbb{D}$. Then:*

(i)
$$\kappa_{\mathbb{D}}\big(f(\zeta); f'(\zeta)v\big) \leq \kappa_{\mathbb{D}}(\zeta; v) \tag{6.1}$$
for all $\zeta \in \mathbb{D}$ and $v \in \mathbb{C}$. Moreover, equality holds for some $\zeta \in \mathbb{D}$ and $v \in \mathbb{C}^*$ if and only if it holds for all $\zeta \in \mathbb{D}$ and all $v \in \mathbb{C}$, if and only if $f \in \mathrm{Aut}(\mathbb{D})$;

(ii)
$$k_{\mathbb{D}}\big(f(\zeta_1), f(\zeta_2)\big) \leq k_{\mathbb{D}}(\zeta_1, \zeta_2) \tag{6.2}$$
for all ζ_1, $\zeta_2 \in \mathbb{D}$. Moreover, equality holds for some $\zeta_1 \neq \zeta_2$ if and only if it holds for all ζ_1, $\zeta_2 \in \mathbb{D}$, if and only if $f \in \mathrm{Aut}(\mathbb{D})$.

Therefore *holomorphic self-maps of the unit disk are* 1*-Lipschitz, and hence equicontinuous, with respect to the Poincaré distance.*

As an immediate corollary, we can compute the group of automorphisms of $\mathbb{D}$, and thus the group of isometries of the Poincaré metric (see, e.g., [2, Proposition 1.1.2]):

Corollary 6.4. *The group* $\mathrm{Aut}(\mathbb{D})$ *of holomorphic automorphisms of $\mathbb{D}$ consists in all the functions $\gamma\colon \mathbb{D} \to \mathbb{D}$ of the form*

$$\gamma(\zeta) = e^{i\theta} \frac{\zeta - \zeta_0}{1 - \overline{\zeta_0}\zeta} \tag{6.3}$$

with $\theta \in \mathbb{R}$ and $\zeta_0 \in \mathbb{D}$. In particular, for every ζ_1, $\zeta_2 \in \mathbb{D}$ there exists $\gamma \in \mathrm{Aut}(\mathbb{D})$ such that $\gamma(\zeta_1) = 0$ and $\gamma(\zeta_2) \in [0, 1)$.

More generally, given $\zeta_1, \zeta_2 \in \mathbb{D}$ and $\eta \in [0, 1)$, it is not difficult to see that there is $\gamma \in \mathrm{Aut}(\mathbb{D})$ such that $\gamma(\zeta_1) = \eta$ and $\gamma(\zeta_2) \in [0, 1)$ with $\gamma(\zeta_2) \geq \eta$.

A consequence of (6.3) is that all automorphisms of $\mathbb{D}$ extends continuously to the boundary. It is classical to classify the elements of $\mathrm{Aut}(\mathbb{D})$ according to the number of fixed points in $\overline{\mathbb{D}}$.

Definition 6.5. *An automorphism* $\gamma \in \mathrm{Aut}(\mathbb{D}) \setminus \{\mathrm{Id}_{\mathbb{D}}\}$ *is called:*

(i) elliptic *if it has a unique fixed point in* $\mathbb{D}$*,*

(ii) parabolic *if it has a unique fixed point in* $\partial\mathbb{D}$*,*

(iii) hyperbolic *if it has exactly two fixed points in* $\partial\mathbb{D}$*.*

It is easy to check that these cases are mutually exclusive and exhaustive.

6.1.2 The Kobayashi distance on bounded convex domains

Given x, $y \in \mathbb{C}^n$, let

$$[x,y] = \{sx + (1-s)y \in \mathbb{C}^n \mid s \in [0,1]\} \qquad \text{and} \qquad (x,y) = \{sx + (1-s)y \in \mathbb{C}^n \mid s \in (0,1)\}$$

denote the *closed,* respectively *open, segment* connecting x and y. We recall that a set $D \subseteq \mathbb{C}^n$ is *convex* if $[x,y] \subseteq D$ for all x, $y \in D$; and *strictly convex* if $(x,y) \subseteq D$ for all x, $y \in \overline{D}$.

An easy but useful observation is that given $D \subset \mathbb{C}^n$ a convex domain, we have

(i) $(z,w) \subset D$ for all $z \in D$ and $w \in \partial D$,

(ii) if $x, y \in \partial D$ then either $(x,y) \subset \partial D$ or $(x,y) \subset D$.

This suggests the following definition

Definition 6.6. *Let* $D \subset \mathbb{C}^n$ *be a convex domain. Given* $x \in \partial D$*, we set*

$$\mathrm{ch}(x) = \{y \in \partial D \mid [x,y] \subset \partial D\} ;$$

we shall say that x *is a* strictly convex point *if* $\mathrm{ch}(x) = \{x\}$*. More generally, given* $F \subseteq \partial D$ *we set*

$$\mathrm{ch}(F) = \bigcup_{x\in F} \mathrm{ch}(x) .$$

A similar construction having a more holomorphic character is the following.

Definition 6.7. *Let* $D \subset \mathbb{C}^n$ *be a convex domain. A* complex supporting functional *at* $x \in \partial D$ *is a* $\mathbb{C}$*-linear map* $\sigma\colon \mathbb{C}^n \to \mathbb{C}$ *such that* $\mathrm{Re}\sigma(z) < \mathrm{Re}\sigma(x)$ *for all* $z \in D$*. A* complex supporting hyperplane *at* $x \in \partial D$ *is an affine complex hyperplane* $L \subset \mathbb{C}^n$ *of the form* $L = x + \ker\sigma$*, where* σ *is a complex supporting functional at* x *(the existence of complex supporting functionals and hyperplanes is guaranteed by the Hahn-Banach theorem). Given* $x \in \partial D$*, we shall denote by* $\mathrm{Ch}(x)$ *the intersection of* $\overline{D}$ *with of all complex supporting hyperplanes at* x*. Clearly,* $\mathrm{Ch}(x)$ *is a closed convex set containing* x*; in particular,* $\mathrm{Ch}(x) \subseteq \mathrm{ch}(x)$*. If* $\mathrm{Ch}(x) = \{x\}$ *we say that* x *is a* strictly $\mathbb{C}$-linearly convex point*; and we say that* D *is* strictly $\mathbb{C}$-linearly convex *if all points of* ∂D *are strictly* $\mathbb{C}$*-linearly convex. Finally, if* $F \subset \partial D$ *we set*

$$\mathrm{Ch}(F) = \bigcup_{x\in F} \mathrm{Ch}(x) \subseteq \mathrm{ch}(F) .$$

Note that if ∂D is of class C^1 then for each $x \in \partial D$ there exists a unique complex supporting hyperplane at x. Hence $\mathrm{Ch}(x)$ coincides with the intersection of the complex supporting hyperplane with ∂D, which is smaller than the flat region introduced in [2, p. 277] as the intersection of ∂D with the real supporting hyperplane. One should however keep in mind that non-smooth points can have more than one complex supporting hyperplanes, as it happens for instance in the polydisk.

The intrinsic complex geometry of convex domains can be described using the intrinsic Kobayashi distance. We refer to [2, 151, 162] for details and much more on the Kobayashi (pseudo)distance in complex manifolds; here we shall just recall what is needed for our aims.

If X is a complex manifold, the *Lempert function* $\delta_X \colon X \times X \to \mathbf{R}^+$ of X is

$$\delta_X(z,w) = \inf\{k_{\mathbb{D}}(\zeta,\eta) \mid \exists \phi \colon \mathbb{D} \to X \text{ holomorphic map with } \phi(\zeta) = z \text{ and } \phi(\eta) = w\}$$

for all z, $w \in X$. In general, the *Kobayashi pseudodistance* $k_X \colon X \times X \to \mathbf{R}^+$ of X is the largest pseudodistance on X bounded from above by δ_X. We say that X is *(Kobayashi) hyperbolic* if k_X is an actual distance. When $D \Subset \mathbb{C}^n$ is a bounded convex domain in $\mathbb{C}^n$, Lempert [170] has proved that δ_D is an actual distance, and thus it coincides with the Kobayashi distance k_D of D.

The main property of the *Kobayashi (pseudo)distance* is that it *is contracted by holomorphic maps*: if $f \colon X \to Y$ is a holomorphic map then

$$k_Y\big(f(z), f(w)\big) \le k_X(z,w)$$

for all z, $w \in X$. In particular, biholomorphisms are isometries, and holomorphic self-maps are *k_X-nonexpansive.*

The Kobayashi distance of convex domains has several interesting properties. For instance, it coincides with the Carathéodory distance, and it is a complete distance (see, e.g., [2] or [170]); in particular, k_D-bounded subsets of D are relatively compact in D.

Using the definition, it is easy to compute the Kobayashi pseudodistance of a few interesting manifolds (see, e.g., [2, Proposition 2.3.4, Corollaries 2.3.6, 2.3.7]):

Proposition 6.8. (i) *The Poincaré distance is the Kobayashi distance of the unit disk $\mathbb{D}$.*

(ii) *The Kobayashi distances of $\mathbb{C}^n$ and of the complex projective space $\mathbb{P}^n(\mathbb{C})$ vanish identically.*

(iii) *For every $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{D}^n$ we have*

$$k_{\mathbb{D}^n}(z,w) = \max_{j=1,\ldots,n}\{k_{\mathbb{D}}(z_j,w_j)\}\ .$$

(iv) *The Kobayashi distance of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ coincides with the classical Bergman distance; in particular, if $O \in \mathbb{C}^n$ is the origin and $z \in \mathbb{B}^n$ then*

$$k_{\mathbb{B}^n}(O,z) = \tfrac{1}{2}\log\frac{1+\|z\|}{1-\|z\|}\ .$$

The Kobayashi pseudodistance can be explicitly computed only in few cases. Besides the cases listed in Proposition 6.8, as far as we know there are formulas for some complex ellipsoids, bounded symmetric domains, the symmetrized bidisk and a few other scattered examples (see [12] for references).

6.2 Dynamics on bounded convex domains

Given a holomorphic self-map f of a bounded convex domains, we are again interested in studying the asymptotic behaviour of the sequence $\{f^{\circ k}\}$ of its iterates. Thanks to the fact that the Kobayashi distance of a bounded convex domain is contracted by holomorphic self-maps, the family of iterates here is always normal and we can therefore describe the asymptotic behaviour of all orbits.

The model theorem for this theory is the Wolff-Denjoy theorem (for a proof see, e.g., [2, Theorem 1.3.9]):

Theorem 6.9 (Wolff-Denjoy, [264, 90])**.** *Let* $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\mathrm{Id}_{\mathbb{D}}\}$ *be a holomorphic self-map of* $\mathbb{D}$ *different from the identity. Assume that* f *is not an elliptic automorphism, that is without fixed points in* $\mathbb{D}$*. Then the sequence of iterates of* f *converges, uniformly on compact subsets, to a constant map* $\tau \in \overline{\mathbb{D}}$*.*

When $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ has a fixed point $\zeta_0 \in \mathbb{D}$, the Wolff-Denjoy theorem is an easy consequence of the Schwarz-Pick lemma. Indeed if f is an automorphism the statement is clear; if it is not an automorphism, then f is a strict contraction of any Kobayashi ball centered at ζ_0, and thus the orbits must converge to the fixed point ζ_0. When f has no fixed points, this argument fails because there are no f-invariant Kobayashi balls. The key idea here is to replace Kobayashi balls by a sort of balls *centered* at points in the boundary, the *horocycles,* and to prove the existence of f-invariant horocycles, which completes the proof of the Wolff-Denjoy theorem.

This result leads to the following definition.

Definition 6.10. *Let* $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\mathrm{Id}_{\mathbb{D}}\}$ *be a holomorphic self-map of* $\mathbb{D}$ *different from the identity and not an elliptic automorphism. Then the point* $\tau \in \overline{\mathbb{D}}$ *whose existence is asserted by Theorem 6.9 is the* Wolff point *of* f*.*

We can even be slightly more precise, introducing a bit of terminology. Given $f\colon X \to X$ a self-map of a set X, recall that a *fixed point* of f is a point $x_0 \in X$ such that $f(x_0) = x_0$. We shall denote by $\mathrm{Fix}(f)$ the set of fixed points of f. More generally, $x_0 \in X$ is *periodic* of *period* $p \geq 1$ if $f^{\circ p}(x_0) = x_0$ and $f^{\circ j}(x_0) \neq x_0$ for all $j = 1, \ldots, p-1$. We say that f is *periodic* of *period* $p \geq 1$ if $f^{\circ p} = \mathrm{Id}_X$, that is if all points are periodic of period p.

Definition 6.11. *Let* $f\colon X \to X$ *be a continuous self-map of a topological space* X*. We shall say that a continuous map* $g\colon X \to X$ *is a* limit map *of* f *if there is a subsequence of iterates of* f *converging to* g *(uniformly on compact subsets). We shall denote by* $\Gamma(f) \subset C^0(X, X)$ *the set of limit maps of* f*. If* $\mathrm{Id}_X \in \Gamma(f)$ *we shall say that* f *is* pseudoperiodic*.*

Let $\gamma_\theta \in \mathrm{Aut}(\mathbb{D})$ be given by $\gamma_\theta(\zeta) = e^{2\pi i\theta}\zeta$. It is easy to check that γ_θ is periodic if $\theta \in \mathbb{Q}$, and it is pseudoperiodic (but not periodic) if $\theta \in \mathbb{R} \setminus \mathbb{Q}$. It is easy to check that any elliptic automorphism of $\mathbb{D}$ is biholomorphically conjugated to one of the automorphisms γ_θ. Therefore an elliptic automorphism of $\mathbb{D}$ is necessarily periodic or pseudoperiodic.

Wolff-Denjoy Theorem 6.9 gives us the following dichotomy. Given $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ different from the identity: either f has a fixed point $\tau \in \mathbb{D}$ or $\mathrm{Fix}(f) = \emptyset$ (in fact, by the Schwarz-Pick lemma and the structure of the automorphisms of $\mathbb{D}$, the only holomorphic self-map of $\mathbb{D}$ with at least two distinct fixed points is the identity). Therefore:

(a) if $\mathrm{Fix}(f) = \{\tau\}$, then either f is an elliptic automorphism—and hence it is periodic or pseudoperiodc—or the whole sequence of iterates converges to the constant function τ;

(b) if $\mathrm{Fix}(f) = \emptyset$ then there exists a unique point $\tau \in \partial\mathbb{D}$ such that the whole sequence of iterates converges to the constant function τ.

Hence there is a natural dichotomy between self-maps with fixed points and self-maps without fixed points.

Since the discovery of Wolff-Denjoy Theorem, a lot of work has been devoted to obtain similar statements in more general situations. In one complex variable, there are results in multiply connected domains, multiply and infinitely connected Riemann surfaces, and even in the settings of one-parameter semigroups and of random dynamical systems (see, e.g., [35, 139, 168]).

In the next section we will present a (suitable) generalization of the Wolff-Denjoy Theorem in $\mathbb{C}^n$.

6.3 Wolff-Denjoy theorem

In several complex variables, the first Wolff-Denjoy theorems are due to Hervé [142, 143]. In particular, in [143] he proved a statement identical to the one above for fixed points free self-maps of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$. Hervé's theorem has also been generalized in various ways to open unit balls of complex Hilbert and Banach spaces (see, e.g., [73, 244] and references therein).

Abate showed in [1] how to prove a Wolff-Denjoy theorem for holomorphic self-maps of smoothly bounded strongly convex domains in $\mathbb{C}^n$. The techniques introduced there have turned out to be quite effective in other contexts too and in particular they have led to Wolff-Denjoy theorems in smooth strongly pseudoconvex domains and smooth domains of finite type. The key idea is to use the Kobayashi distance to define a general notion of multidimensional analogue of the horocycles, the *horospheres*.

Definition 6.12. *Let $D \Subset \mathbb{C}^n$ be a bounded domain. The* small horosphere *of* center $x_0 \in \partial D$*,* radius $R > 0$ *and* pole $z_0 \in D$ *is the set*

$$E_{z_0}(x_0, R) = \left\{ z \in D \;\middle|\; \limsup_{w \to x_0} [k_D(z, w) - k_D(z_0, w)] < \tfrac{1}{2} \log R \right\} ;$$

the large horosphere *of* center $x_0 \in \partial D$*,* radius $R > 0$ *and* pole $z_0 \in D$ *is the set*

$$F_{z_0}(x_0, R) = \left\{ z \in D \;\middle|\; \liminf_{w \to x_0} [k_D(z, w) - k_D(z_0, w)] < \tfrac{1}{2} \log R \right\} .$$

The idea behind this definition is that a Kobayashi ball of center $w \in D$ and radius r is the set of $z \in D$ such that $k_D(z, w) < r$, but when we let w go to a point in the boundary $k_D(z, w)$ tends to infinity (at least when D is complete hyperbolic), and so we cannot use it to define all subsets of D. Therefore we renormalize $k_D(z, w)$ by subtracting the distance $k_D(z_0, w)$ from a reference point z_0, since by the triangular inequality the difference $k_D(z, w) - k_D(z_0, w)$ is bounded by $k_D(z_0, z)$, and thus we can consider the lim inf and the lim sup as w tends to $x_0 \in \partial D$ (in general the limit does not exist), and the sublevels provide some sort of balls centered at points in the boundary.

We list in the following lemma a few elementary properties of horospheres, which are an immediate consequence of the definition (see, e.g., [2, Lemmas 2.4.10 and 2.4.11]).

Lemma 6.13. *Let $D \Subset \mathbb{C}^n$ be a bounded domain of $\mathbb{C}^n$, and choose $z_0 \in D$ and $x \in \partial D$. Then:*

(i) *for every $R > 0$ we have $E_{z_0}(x, R) \subset F_{z_0}(x, R)$;*

(ii) *for every $0 < R_1 < R_2$ we have $E_{z_0}(x, R_1) \subset E_{z_0}(x, R_2)$ and $F_{z_0}(x, R_1) \subset F_{z_0}(x, R_2)$;*

(iii) *for every $R > 1$ we have $B_D(z_0, \frac{1}{2} \log R) \subset E_{z_0}(x, R)$;*

(iv) *for every $R < 1$ we have $F_{z_0}(x, R) \cap B_D(z_0, -\frac{1}{2} \log R) = \emptyset$;*

(v) $\bigcup_{R>0} E_{z_0}(x, R) = \bigcup_{R>0} F_{z_0}(x, R) = D$ *and* $\bigcap_{R>0} E_{z_0}(x, R) = \bigcap_{R>0} F_{z_0}(x, R) = \emptyset$*;*

(vi) *if* $\varphi \in \mathrm{Aut}(D) \cap C^0(\overline{D}, \overline{D})$*, then for every* $R > 0$

$$\varphi\big(E_{z_0}(x,R)\big) = E_{\varphi(z_0)}\big(\varphi(x),R\big) \qquad \text{and} \qquad \varphi\big(F_{z_0}(x,R)\big) = F_{\varphi(z_0)}\big(\varphi(x),R\big) \ ;$$

(vii) *if* $z_1 \in D$ *and we set*

$$\tfrac{1}{2}\log L = \limsup_{w\to x}\big[k_D(z_1,w) - k_D(z_0,w)\big] \ ,$$

then for every $R > 0$ *we have* $E_{z_1}(x,R) \subseteq E_{z_0}(x,LR)$ *and* $F_{z_1}(x,R) \subseteq F_{z_0}(x,LR)$*.*

It is also easy to check that the horospheres with pole at the origin in $\mathbb{B}^n$ (and thus in $\mathbb{D}$) coincide with the classical horospheres.

Lemma 6.14. *If* $x \in \partial\mathbb{B}^n$ *and* $R > 0$ *then*

$$E_O(x,R) = F_O(x,R) = \left\{ z \in \mathbb{B}^n \;\middle|\; \frac{|1 - \langle z, x\rangle|^2}{1 - \|z\|^2} < R \right\} \ .$$

Thus in $\mathbb{B}^n$ small and large horospheres coincide. Furthermore, the horospheres with pole at the origin are ellipsoids tangent to $\partial\mathbb{B}^n$ in x, because an easy computation yields

$$E_O(x,R) = \left\{ z \in \mathbb{B}^n \;\middle|\; \frac{\|P_x(z) - (1-r)x\|^2}{r^2} + \frac{\|z - P_x(z)\|^2}{r} < 1 \right\} ,$$

where $r = R/(1+R)$. In particular if $\tau \in \partial\mathbb{D}$ we have

$$E_0(\tau,R) = \big\{ \zeta \in \mathbb{D} \;\big|\; |\zeta - (1-r)\tau|^2 < r^2 \big\} \ ,$$

and so a horocycle is an Euclidean disk internally tangent to $\partial\mathbb{D}$ in τ.

Another domain where we can explicitly compute the horospheres is the polydisk; in this case large and small horospheres are actually different (see, e.g., [2, Proposition 2.4.12]):

Proposition 6.15. *Let* $x \in \partial\mathbb{D}^n$ *and* $R > 0$*. Then*

$$E_O(x,R) = \left\{ z \in \mathbb{D}^n \;\middle|\; \max_j \left\{ \frac{|x_j - z_j|^2}{1 - |z_j|^2} \;\middle|\; |x_j| = 1 \right\} < R \right\} ;$$

$$F_O(x,R) = \left\{ z \in \mathbb{D}^n \;\middle|\; \min_j \left\{ \frac{|x_j - z_j|^2}{1 - |z_j|^2} \;\middle|\; |x_j| = 1 \right\} < R \right\} .$$

The key in the proof of the classical Wolff-Denjoy theorem is Wolff's lemma.

Theorem 6.16 (Wolff's lemma, [264])**.** *Let* $f \in \mathrm{Hol}(\mathbb{D},\mathbb{D})$ *without fixed points. Then there exists a unique* $\tau \in \partial\mathbb{D}$ *such that*

$$f\big(E_0(\tau,R)\big) \subseteq E_0(\tau,R) \tag{6.4}$$

for all $R > 0$*.*

The proof of the Wolff-Denjoy theorem follows easily from Wolff's lemma. In fact, if $f \in \mathrm{Hol}(\mathbb{D},\mathbb{D})$ has no fixed points we know that the sequence of iterates is *compactly divergent*, which means that the image of any limit h of a converging subsequence is contained in $\partial\mathbb{D}$. By the maximum principle, the map h must be constant; and by Wolff's lemma this constant must be contained in $\overline{E_0(\tau,R)} \cap \partial\mathbb{D} = \{\tau\}$. So every converging subsequence of $\{f^{\circ k}\}$ must converge to the constant τ, which is equivalent to saying that the whole sequence of iterates converges to the constant map τ.

The proof of the Wolff-Denjoy theorem we just briefly described is therefore based on two ingredients: the existence of a f-invariant horocycle, and the fact that a horocycle touches the boundary in exactly one point. To generalize such argument to several variables we first need an analogous of Theorem 6.16 for our multidimensional horospheres, and then we need to know how the horospheres touch the boundary.

There exist several multidimensional versions of Wolff's lemma; we shall state in this chapter three of them (Theorems 6.18, 6.23 and 6.37). To state the first one we need a definition.

Definition 6.17. *Let $D \subset \mathbb{C}^n$ be a domain in $\mathbb{C}^n$. We say that D has* simple boundary *if every $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{C}^n)$ such that $\varphi(\mathbb{D}) \subseteq \overline{D}$ and $\varphi(\mathbb{D}) \cap \partial D \neq \emptyset$ is constant.*

Then Abate proved the following results.

Theorem 6.18 (Abate, [4])**.** *Let $D \Subset \mathbb{C}^n$ be a complete hyperbolic bounded domain with simple boundary, and take $f \in \mathrm{Hol}(D, D)$ with compactly divergent sequence of iterates. Fix $z_0 \in D$. Then there exists $x_0 \in \partial D$ such that*

$$f^{\circ k}\big(E_{z_0}(x_0, R)\big) \subseteq F_{z_0}(x_0, R)$$

for all $k \in \mathbb{N}$ and $R > 0$.

Theorem 6.19 (Abate, [1])**.** *Let $D \Subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then*

$$\overline{E_{z_0}(x_0, R)} \cap \partial D = \overline{F_{z_0}(x_0, R)} \cap \partial D = \{x_0\}$$

for every $z_0 \in D$, $x_0 \in \partial D$ and $R > 0$.

Such results enabled him to prove a Wolff-Denjoy theorem for strongly pseudoconvex domains.

Theorem 6.20 (Abate, [4])**.** *Let $D \Subset \mathbb{C}^n$ be a strongly pseudoconvex C^2 domain. Take $f \in \mathrm{Hol}(D, D)$ with compactly divergent sequence of iterates. Then $\{f^{\circ k}\}$ converges to a constant map $x_0 \in \partial D$.*

6.3.1 Strictly convex domains

The proof of Theorem 6.20 strongly depends on the fact that the boundary of the domain D is of class at least C^2. Recently, Budzyńska [71] (see also [72]) found a way to prove Theorem 6.20 in *strictly convex* domains without any assumption on the smoothness of the boundary; in this subsection we shall describe the simplified approach that we found in collaboration with Abate in [17].

Theorem 6.19 is replaced by the following result.

Proposition 6.21. *Let $D \subset \mathbb{C}^n$ be a hyperbolic convex domain, $z_0 \in D$, $R > 0$ and $x \in \partial D$. Then we have $[x, z] \subset \overline{F_{z_0}(x, R)}$ for all $z \in \overline{F_{z_0}(x, R)}$. Furthermore,*

$$x \in \bigcap_{R>0} \overline{F_{z_0}(x, R)} \subseteq \mathrm{ch}(x) \, . \tag{6.5}$$

In particular, if x is a strictly convex point then $\bigcap_{R>0} \overline{F_{z_0}(x, R)} = \{x\}$.

We can then prove the following Wolff-Denjoy theorem in strictly convex domains without any assumption on the regularity of the boundary.

Theorem 6.22 (Budzyńska, [71]; Abate, R., [17]). *Let $D \Subset \mathbb{C}^n$ be a bounded strictly convex domain, and take $f \in \mathrm{Hol}(D, D)$ without fixed points. Then the sequence of iterates $\{f^{\circ k}\}$ converges to a constant map $x \in \partial D$.*

Proof. Fix $z_0 \in D$, and let $x \in \partial D$ be given by Theorem 6.18, that can be applied because strictly convex domains are complete hyperbolic and have simple boundary. So, since D is bounded, it suffices to prove that every converging subsequence of $\{f^{\circ k}\}$ converges to the constant map x.

If $\{f^{\circ k_\nu}\}$ converges to a holomorphic map $h \in \mathrm{Hol}(D, \mathbb{C}^n)$. Clearly, $h(D) \subset \overline{D}$. Since the sequence of iterates is compactly divergent, we have $h(D) \subset \partial D$, and since D has simple boundary, it follows that $h \equiv y \in \partial D$. We thus have to prove that $y = x$.

Take $R > 0$, and choose $z \in E_{z_0}(x, R)$. Then Theorem 6.18 yields $y = h(z) \in \overline{F_{z_0}(x, R)} \cap \partial D$. Since this holds for all $R > 0$ we obtain $y \in \bigcap_{R>0} \overline{F_{z_0}(x, R)}$, and Proposition 6.21 yields the assertion. □

6.3.2 Weakly convex domains

The approach leading to Theorem 6.22 actually yields results for weakly convex domains too, even though in general we cannot expect the convergence to a constant map. In fact, if we consider $f \in \mathrm{Hol}(\mathbb{D}^2, \mathbb{D}^2)$ given by

$$f(z, w) = \left(\frac{z + 1/2}{1 + z/2}, w \right) ,$$

it is easy to check that the sequence of iterates of f converges to the non-constant map $h(z, w) = (1, w)$.

The first remark is that there is a version of Theorem 6.18 valid in all convex domains, without the requirement of simple boundary.

Theorem 6.23 (Abate, [1]). *Let $D \Subset \mathbb{C}^n$ be a bounded convex domain, and take a map $f \in \mathrm{Hol}(D, D)$ without fixed points. Then there exists $x \in \partial D$ such that*

$$f^{\circ k}\big(E_{z_0}(x, R)\big) \subset F_{z_0}(x, R)$$

for every $z_0 \in D$, $R > 0$ and $k \in \mathbb{N}$.

When D has C^2 boundary this is enough to obtain a Wolff-Denjoy theorem, thanks to the following result.

Proposition 6.24 (Abate, R., [17]). *Let $D \Subset \mathbb{C}^n$ be a bounded convex domain with C^2 boundary, and $x \in \partial D$. Then for every $z_0 \in D$ and $R > 0$ we have*

$$\overline{F_{z_0}(x, R)} \cap \partial D \subseteq \mathrm{Ch}(x) .$$

In particular, if x is a strictly $\mathbb{C}$-linearly convex point then $\overline{F_{z_0}(x, R)} \cap \partial D = \{x\}$.

To simplify subsequent statements, let us introduce a definition.

Definition 6.25. *Let $D \subset \mathbb{C}^n$ be a hyperbolic convex domain, and $f \in \mathrm{Hol}(D, D)$ without fixed points. The* target set *of f is defined as*

$$T(f) = \bigcup_h h(D) \subseteq \partial D ,$$

where the union is taken with respect to all the holomorphic maps $h \in \mathrm{Hol}(D, \mathbb{C}^n)$ obtained as limit of a subsequence of iterates of f. We have $T(f) \subseteq \partial D$ because the sequence of iterates $\{f^{\circ k}\}$ is compactly divergent.

As a consequence of Proposition 6.24 we obtain:

Corollary 6.26 (Abate, R., [17])**.** *Let $D \Subset \mathbb{C}^n$ be a C^2 bounded convex domain, and $f \in \mathrm{Hol}(D, D)$ without fixed points. Then there exists $x_0 \in \partial D$ such that*

$$T(f) \subseteq \mathrm{Ch}(x_0)\ .$$

In particular, if D is strictly $\mathbb{C}$-linearly convex then the sequence of iterates $\{f^{\circ k}\}$ converges to the constant map x_0.

Proof. Let $x_0 \in \partial D$ be given by Theorem 6.23, and fix $z_0 \in D$. Given $z \in D$, choose $R > 0$ such that $z \in E_{z_0}(x_0, R)$. If $h \in \mathrm{Hol}(D, \mathbb{C}^n)$ is the limit of a subsequence of iterates then Theorem 6.23 and Proposition 6.24 yield

$$h(z) \in \overline{Fz_0(x, R)} \cap \partial D \subset \mathrm{Ch}(x_0)\ ,$$

and we are done. □

Remark 6.27. *In [272], Zimmer has proved Corollary 6.26 for bounded convex domains with $C^{1,\alpha}$ boundary.*

We conjectured that the last statement of the previous Corollary should hold for strictly $\mathbb{C}$-linearly convex domains without smoothness assumptions on the boundary. Bracci and Gaussier recently introduced in [55] a prime end-type theory on complete Kobayashi hyperbolic manifolds using horosphere sequences, which allowed them to introduce a new notion of boundary, *the horosphere boundary*, and a topology on the manifold together with its horosphere boundary, *the horosphere topology*. In particular they obtained the following Wolff- Denjoy theorem, giving a partial positive answer to our question.

Proposition 6.28 (Bracci, Gaussier [55, Proposition 8.7])**.** *Let $D \subset \mathbb{C}^n$ be a bounded convex domain. Assume that either D is biholomorphic to a strongly convex domain with C^3 boundary or D is $\mathbb{C}$-strictly linearly convex and biholomorphic to a bounded strongly pseudoconvex domain with C^3 boundary. Let $f\colon D \to D$ be holomorphic without fixed points in D. Then there exists exists $p \in \partial D$ such that $T(f) = \{p\}$.*

They conjectured that in fact this holds for every bounded convex domain $D \subset \mathbb{C}^n$ whose boundary does not contain non-constant analytic discs.

Remark 6.29. *Bharali and Zimmer introduced and studied in [43] a class of domains, called* Goldilocks domains*, defined in terms of a lower bound on how fast the Kobayashi metric grows and an upper bound on how fast the Kobayashi distance grows as one approaches the boundary. Strongly pseudoconvex domains and weakly pseudoconvex domains of finite type always satisfy such condition. They proved that the Kobayashi metric on Goldilocks domains morally behaves as a negatively curved Riemannian metric and in particular, it satisfies a visibility condition in the sense of Eberlein and O'Neill. Such behaviour allows them to prove several results on boundary extension of maps and to establish Wolff-Denjoy theorems for a wide collection of domains.*

Dropping any smoothness or strict convexity condition on the boundary, a useful result is the following.

Lemma 6.30 (Abate, R., [17]). *Let $D \Subset \mathbb{C}^n$ be a convex domain. Then for every connected complex manifold X and every holomorphic map $h\colon X \to \mathbb{C}^n$ such that $h(X) \subset \overline{D}$ and $h(X) \cap \partial D \neq \emptyset$ we have*

$$h(X) \subseteq \bigcap_{z \in X} \mathrm{Ch}\big(h(z)\big) \subseteq \partial D\ .$$

We can then prove a weak Wolff-Denjoy theorem:

Proposition 6.31 (Abate, R., [17]). *Let $D \Subset \mathbb{C}^n$ be a bounded convex domain, and $f \in \mathrm{Hol}(D, D)$ without fixed points. Then there exists $x \in \partial D$ such that for any $z_0 \in D$ we have*

$$T(f) \subseteq \bigcap_{R>0} \mathrm{Ch}\big(\overline{F_{z_0}(x,R)} \cap \partial D\big)\ . \tag{6.6}$$

Proof. Let $x \in \partial D$ be given by Theorem 6.23. Choose $z_0 \in D$ and $R > 0$, and take $z \in E_{z_0}(x,R)$. Let $h \in \mathrm{Hol}(D, \mathbb{C}^n)$ be obtained as limit of a subsequence of iterates of f. Arguing as usual we know that $h(D) \subseteq \partial D$; therefore Theorem 6.23 yields $h(z) \in \overline{F_{z_0}(x,R)} \cap \partial D$. Then Lemma 6.30 yields

$$h(D) \subseteq \mathrm{Ch}\big(h(z)\big) \subseteq \mathrm{Ch}\big(\overline{F_{z_0}(x,R)} \cap \partial D\big)\ .$$

Since z_0 and R are arbitrary, we get the assertion. □

Using Lemma 6.13 it is easy to check that the intersection in (6.6) is independent of the choice of $z_0 \in D$.

Large horospheres can be too large. For instance, take $(\tau_1, \tau_2) \in \partial\mathbb{D} \times \partial\mathbb{D}$. Then Proposition 6.15 says that the horosphere of center (τ_1, τ_2) in the bidisk are given by

$$F_O\big((\tau_1,\tau_2), R\big) = E_0(\tau_1, R) \times \mathbb{D} \cup \mathbb{D} \times E_0(\tau_2, R)\ ,$$

where $E_0(\tau, R)$ is the horocycle of center $\tau \in \partial\mathbb{D}$ and radius $R > 0$ in the unit disk $\mathbb{D}$, and a not difficult computation shows that

$$\mathrm{Ch}\big(\overline{F_O\big((\tau_1,\tau_2),R\big)} \cap \partial\mathbb{D}^2\big) = \partial\mathbb{D}^2\ ,$$

making the statement of Proposition 6.31 irrelevant. So to obtain an effective statement we need to replace large horospheres with smaller sets.

Small horospheres might be too small; as shown by Frosini [120], there are holomorphic self-maps of the polydisk with no invariant small horospheres. We thus need another kind of horospheres, defined by Kapeluszny, Kuczumow and Reich [158], and studied in detail by Budzyńska [71]. To introduce them we begin with a definition:

Definition 6.32. *Let $D \Subset \mathbb{C}^n$ be a bounded domain, and $z_0 \in D$. A sequence $\mathbf{x} = \{x_\nu\} \subset D$ converging to $x \in \partial D$ is a* horosphere sequence *at x if the limit of $k_D(z, x_\nu) - k_D(z_0, x_\nu)$ as $\nu \to +\infty$ exists for all $z \in D$.*

It is easy to see that the notion of horosphere sequence does not depend on the point z_0. Moreover, it follows from the following topological lemma that horosphere sequences always exist.

Lemma 6.33 (Reich, [213]). *Let (X, d) be a separable metric space, and for each $\nu \in \mathbb{N}$ let $a_\nu \colon X \to \mathbb{R}$ be a 1-Lipschitz map, i.e., $|a_\nu(x) - a_\nu(y)| \leq d(x, y)$ for all x, $y \in X$. If for each $x \in X$ the sequence $\{a_\nu(x)\}$ is bounded, then there exists a subsequence $\{a_{\nu_j}\}$ of $\{a_\nu\}$ such that $\lim_{j\to\infty} a_{\nu_j}(x)$ exists for each $x \in X$.*

Then:

Proposition 6.34 (Budzyńska, Kuczumow, Reich [72]). *Let $D \Subset \mathbb{C}^n$ be a bounded convex domain, and $x \in \partial D$. Then every sequence $\{x_\nu\} \subset D$ converging to x contains a subsequence which is a horosphere sequence at x.*

We can then introduce a new kind of horospheres.

Definition 6.35. *Let $D \Subset \mathbb{C}^n$ be a bounded convex domain. Given $z_0 \in D$, let $\mathbf{x} = \{x_\nu\}$ be a horosphere sequence at $x \in \partial D$, and take $R > 0$. Then the* sequence horosphere $G_{z_0}(x, R, \mathbf{x})$ *is defined as*

$$G_{z_0}(x, R, \mathbf{x}) = \left\{z \in D \;\middle|\; \lim_{\nu\to+\infty}\left[k_D(z, x_\nu) - k_D(z_0, x_\nu)\right] < \tfrac{1}{2}\log R\right\} .$$

The basic properties of sequence horospheres are contained in the following result.

Proposition 6.36 ([158, 71, 72]). *Let $D \Subset \mathbb{C}^n$ be a bounded convex domain. Fix $z_0 \in D$, and let $\mathbf{x} = \{x_\nu\} \subset D$ be a horosphere sequence at $x \in \partial D$. Then:*

(i) $E_{z_0}(x, R) \subseteq G_{z_0}(x, R, \mathbf{x}) \subseteq F_{z_0}(x, R)$ *for all* $R > 0$*;*

(ii) $G_{z_0}(x, R, \mathbf{x})$ *is nonempty and convex for all* $R > 0$*;*

(iii) $\overline{G_{z_0}(x, R_1, \mathbf{x})} \cap D \subset G_{z_0}(x, R_2, \mathbf{x})$ *for all* $0 < R_1 < R_2$*;*

(iv) $B_D(z_0, \frac{1}{2}\log R) \subset G_{z_0}(x, R, \mathbf{x})$ *for all* $R > 1$*;*

(v) $B_D(z_0, -\frac{1}{2}\log R) \cap G_{z_0}(x, R, \mathbf{x}) = \emptyset$ *for all* $0 < R < 1$*;*

(vi) $\bigcup_{R>0} G_{z_0}(x, R, \mathbf{x}) = D$ *and* $\bigcap_{R>0} G_{z_0}(x, R, \mathbf{x}) = \emptyset$*.*

Note that if $\mathbf{x}$ is a horosphere sequence at $x \in \partial D$ then it is not difficult to check that the family $\{G_z(x, 1, \mathbf{x})\}_{z\in D}$ and the family $\{G_{z_0}(x, R, \mathbf{x})\}_{R>0}$ with $z_0 \in D$ given, coincide.

We then have the following version of Theorem 6.16.

Theorem 6.37 (Budzyńska, [71]; Abate, R., [17]). *Let $D \Subset \mathbb{C}^n$ be a convex domain, and let $f \in \mathrm{Hol}(D, D)$ without fixed points. Then there exist $x \in \partial D$ and a horosphere sequence $\mathbf{x}$ at x such that*

$$f\big(G_{z_0}(x, R, \mathbf{x})\big) \subseteq G_{z_0}(x, R, \mathbf{x})$$

for every $z_0 \in D$ and $R > 0$.

And we can prove the following Wolff-Denjoy theorem for (not necessarily strictly or smooth) convex domains.

Theorem 6.38 (Abate, R. [17]). *Let $D \Subset \mathbb{C}^n$ be a bounded convex domain, and $f \in \mathrm{Hol}(D, D)$ without fixed points. Then there exist $x \in \partial D$ and a horosphere sequence $\mathbf{x}$ at x such that for any $z_0 \in D$ we have*

$$T(f) \subseteq \bigcap_{z\in D} \mathrm{Ch}\big(\overline{G_z(x, 1, \mathbf{x})} \cap \partial D\big) = \bigcap_{R>0} \mathrm{Ch}\big(\overline{G_{z_0}(x, R, \mathbf{x})} \cap \partial D\big) .$$

Proof. The equality of the intersections is a consequence of the fact that the family $\{G_z(x, 1, \mathbf{x})\}_{z\in D}$ and the family $\{G_{z_0}(x, R, \mathbf{x})\}_{R>0}$ with $z_0 \in D$ given, coincide. Then the assertion follows from Theorem 6.37 and Lemma 6.30 as in the proof of Proposition 6.31. □

The Polydisk

In order to show that Theorem 6.38 is actually a better statement than Proposition 6.31 let us consider the case of the polydisk.

Sequence horospheres can be computed as follows.

Lemma 6.39. *Let $\mathbf{x} = \{x_\nu\} \subset \mathbb{D}^n$ be a horosphere sequence converging to $\xi = (\xi_1, \ldots, \xi_n) \in \partial\mathbb{D}^n$. Then for every $1 \le j \le n$ such that $|\xi_j| = 1$ the limit*

$$\alpha_j := \lim_{\nu\to+\infty} \min_h \left\{ \frac{1 - |(x_\nu)_h|^2}{1 - |(x_\nu)_j|^2} \right\} \le 1 \tag{6.7}$$

exists, and we have

$$G_O(\xi, R, \mathbf{x}) = \left\{ z \in \mathbb{D}^n \;\middle|\; \max_j \left\{ \alpha_j \frac{|\xi_j - z_j|^2}{1 - |z_j|^2} \;\middle|\; |\xi_j| = 1 \right\} < R \right\} = \prod_{j=1}^n E_j \ ,$$

where

$$E_j = \begin{cases} \mathbb{D} & \textit{if } |\xi_j| < 1, \\ E_0(\xi_j, R/\alpha_j) & \textit{if } |\xi_j| = 1. \end{cases}$$

An elementary computation shows that

$$\mathrm{Ch}(\xi) = \bigcap_{|\xi_j|=1} \{\eta \in \partial\mathbb{D}^n \mid \eta_j = \xi_j\}$$

for all $\xi \in \partial\mathbb{D}^n$. As a consequence,

$$\mathrm{Ch}\big(\overline{G_O(\xi, R, \mathbf{x})} \cap \partial\mathbb{D}^n\big) = \bigcup_{j=1}^n \overline{\mathbb{D}} \times \cdots \times C_j(\xi) \times \cdots \times \overline{\mathbb{D}} \ ,$$

where

$$C_j(\xi) = \begin{cases} \{\xi_j\} & \text{if } |\xi_j| = 1, \\ \partial\mathbb{D} & \text{if } |\xi_j| < 1. \end{cases}$$

Notice that the right-hand sides do not depend either on R or on the horosphere sequence $\mathbf{x}$, but only on ξ.

Therefore Theorem 6.38 in the polydisk takes the following form.

Corollary 6.40. *Let $f \in \mathrm{Hol}(\mathbb{D}^n, \mathbb{D}^n)$ be without fixed points. Then there exists $\xi \in \partial\mathbb{D}^n$ such that*

$$T(f) \subseteq \bigcup_{j=1}^n \overline{\mathbb{D}} \times \cdots \times C_j(\xi) \times \cdots \times \overline{\mathbb{D}} \ . \tag{6.8}$$

Roughly speaking, this is the best one can do, in the sense that while it might be true (for instance in the bidisk; see Theorem 6.41 below) that the image of a limit point of the sequence of iterates of f is always contained in just one of the sets appearing in the right-hand side of (6.8), it is impossible to determine a priori in which one it is contained on the basis of the point ξ only; it is necessary to know something more about the map f. Indeed, Hervé has proved the following result.

Theorem 6.41 (Hervé, [142])**.** *Let $F = (f, g) \colon \mathbb{D}^2 \to \mathbb{D}^2$ be a holomorphic self-map of the bidisk, and write $f_w = f(\cdot, w)$ and $g_z = g(z, \cdot)$. Assume that F has no fixed points in $\mathbb{D}^2$. Then one and only one of the following cases occurs:*

(i) *if* $g(z,w) \equiv w$ *(respectively,* $f(z,w) \equiv z$*) then the sequence of iterates of* F *converges uniformly on compact sets to* $h(z,w) = (\sigma, w)$*, where* σ *is the common Wolff point of the* f_w*'s (respectively, to* $h(z,w) = (z,\tau)$*, where* τ *is the common Wolff point of the* g_z*'s);*

(ii) *if* $\mathrm{Fix}(f_w) = \emptyset$ *for all* $w \in \mathbb{D}$ *and* $\mathrm{Fix}(g_z) = \{y(z)\} \subset \mathbb{D}$ *for all* $z \in \mathbb{D}$ *(respectively, if* $\mathrm{Fix}(f_w) = \{x(w)\}$ *and* $\mathrm{Fix}(g_z) = \emptyset$*) then* $T(F) \subseteq \{\sigma\} \times \overline{\mathbb{D}}$*, where* $\sigma \in \partial\mathbb{D}$ *is the common Wolff point of the* f_w*'s (respectively,* $T(F) \subseteq \overline{\mathbb{D}} \times \{\tau\}$*, where* τ *is the common Wolff point of the* g_z*'s);*

(iii) *if* $\mathrm{Fix}(f_w) = \emptyset$ *for all* $w \in \mathbb{D}$ *and* $\mathrm{Fix}(g_z) = \emptyset$ *for all* $z \in \mathbb{D}$ *then either* $T(F) \subseteq \{\sigma\} \times \overline{\mathbb{D}}$ *or* $T(F) \subseteq \overline{\mathbb{D}} \times \{\tau\}$*, where* $\sigma \in \partial\mathbb{D}$ *is the common Wolff point of the* f_w*'s, and* $\tau \in \partial\mathbb{D}$ *is the common Wolff point of the* g_z*;*

(iv) *if* $\mathrm{Fix}(f_w) = \{x(w)\} \subset \mathbb{D}$ *for all* $w \in \mathbb{D}$ *and* $\mathrm{Fix}(g_z) = \{y(z)\} \subset \mathbb{D}$ *for all* $z \in \mathbb{D}$ *then there are* σ, $\tau \in \partial D$ *such that the sequence of iterates converges to the constant map* (σ, τ)*.*

All four cases can occur: see [142] for the relevant examples.

Chapter 7

Backward iteration in strongly convex domains

The iteration theory of a non-invertible self-map $f\colon X \to X$ of a set X is usually devoted to the study of the behaviour of forward orbits of the system, as we have seen in the previous chapter.

It is a natural interesting question to investigate instead the behaviour of *backward orbits.*

Definition 7.1. *Let $f\colon X \to X$ be a self-map of a set X. A* backward orbit *(or* backward iteration sequence*) for f is a sequence $\{x_k\}_{k\in\mathbb{N}} \subset X$ so that $f(x_{k+1}) = x_k$ for all $k \in \mathbb{N}$.*

In collaboration with Abate, in [16] we studied such question in the context of holomorphic self-maps of bounded strongly convex domains as we shall explain in this chapter.

7.1 Backward iteration in $\mathbb{D}$ and in $\mathbb{B}^n$

Backward orbits for holomorphic self-maps of the unit disk $\mathbb{D} \subset \mathbb{C}$ have been first studied by Poggi-Corradini. In fact he introduced backward orbits in [201] to study intertwining linear models for holomorphic self-maps of the unit disk at boundary fixed points in the sense of non-tangential limits for which the boundary dilatation coefficient is strictly greater than 1, called *boundary regular repelling fixed points.* They were further studied by Bracci in [52] and finally a detailed study was performed by Poggi-Corradini in [202].

In general, if $\{z_k\}$ is a backward orbit, the sequence $d(z_k, z_{k+1})$ is increasing, where d is the Kobayashi distance. Therefore, in order to have convergence, one needs to impose an upper bound on the so-called *hyperbolic step*, that is

$$d(z_k, z_{k+1}) \leq a, \tag{7.1}$$

for all k and for a fixed $a < 1$.

This condition is nontrivial as there exist maps admitting backward orbits with unbounded Kobayashi steps, as shown by Poggi-Corradini.

Poggi-Corradini proved that, unless f is a non-Euclidean rotation of $\mathbb{D}$, a backward orbit satisfying (7.1) must converge to a point in the boundary of $\mathbb{D}$, which is a repelling or parabolic fixed point of the map f, in the sense of non-tangential limits.

Theorem 7.2 (Poggi-Corradini, [202])**.** *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map of $\mathbb{D}$ which is not an elliptic automorphism. Let $\{z_k\}$ be a backward orbit for f with bounded Kobayashi step $d_k = d(z_k, z_{k+1}) \uparrow a < 1$. Then:*

(i) *there is a point* $q \in \partial\mathbb{D}$ *such that* $z_k \to q$ *as* n *tends to infinity, and* q *is a fixed point for* f *with a well-defined multiplier* $f'(q) = \alpha < \infty$*;*

(ii) *if* $q \neq \tau$*, where* τ *is the Denjoy-Wolff point, then* $\alpha > 1$*, the sequence* z_k *tends to* q *along a non-tangential direction, and we call* q *a* boundary repelling fixed point*;*

(iii) *if* $q = \tau$*, then* f *is necessarily of parabolic type and* z_k *tends to* q *tangentially.*

Ostapyuk studied in [187] backward orbits for holomorphic self-maps of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ generalizing to this setting the results obtained by Poggi-Corradini.

She too considered backward orbits with bounded Kobayashi step, but this time using the Bergmann distance of the unit ball. In order to state her results, we need to recall the following classification for holomorphic self-maps of the unit ball. To avoid repetitions, we will give in the next section the precise definition of *boundary dilation* in the more general setting of strongly convex domains. Roughly speaking, the *boundary dilation* of a self-map f of the unit ball at a boundary fixed point is the derivative of the normal component of f along the normal direction to the boundary of the ball at the fixed point.

A self-map of the unit ball is therefore called:

- *hyperbolic* if it has no fixed points inside the ball and the Wolff-Denjoy point in the boundary has boundary dilation strictly less than one;
- *attracting-elliptic* if its sequence of iterates converges to a unique fixed point inside the ball.

Ostapyuk showed that a backward orbit with bounded Kobayashi step for a hyperbolic or attracting-elliptic holomorphic self-map of the ball necessarily converges, staying inside a Korányi region (see also Definition 7.12), to an a priori chosen repelling fixed point in the boundary of the ball and, conversely, that every boundary repelling fixed point, isolated in a suitable sense, is the limit of a backward iteration sequence with bounded Kobayashi step.

Theorem 7.3 (Ostapyuk, [187])**.** *Let* $f\colon \mathbb{B}^n \to \mathbb{B}^n$ *be a holomorphic self-map of* $\mathbb{B}^n$ *of hyperbolic or attracting-elliptic type with Wolff-Denjoy point* τ*. Let* $\{z_k\}$ *be a backward orbit for* f *with bounded Kobayashi step* $d_{\mathbb{B}^n}(z_k, z_{k+1}) \leq a < 1$*. Then:*

1. *there is a point* $q \in \partial\mathbb{B}^n$*,* $q \neq \tau$*, such that* $z_k \to q$ *as* n *tends to infinity,*
2. $\{z_k\}$ *stays in a Korányi region with vertex* q*,*
3. *for all* $R > 0$ *we have*
$$f\left(E_O(q, R)\right) \subseteq E_O(q, \alpha R),$$
with $\alpha \geq \frac{1}{c}$*, where* $c < 1$ *is a constant that depends on* f *and where* $E_O(x, R)$ *is the horosphere of center* $x \in \partial\mathbb{B}^n$*, radius* $R > 0$ *and pole at the origin* O*.*

Finally, using backward iteration sequences, Ostapyuk also studied seminormal forms for holomorphic self-maps of the unit ball near a boundary repelling fixed point under suitable assumptions (see [187, Theorems 1.15, 1.16]).

7.2 Backward iteration in strongly convex domains

In [16] we extended Poggi-Corradini's results to backward orbits in general bounded strongly convex domains in $\mathbb{C}^n$ with C^2 boundary. To do so, we systematically used the geometric properties of the Kobayashi distance of strongly convex domains. It is interesting to notice that the better geometric understanding given by this tool (and the impossibility of using the kind of explicit computations done in [187] for the ball) yields proofs that are both simpler and clearer than the previous ones, even for the ball and the unit disk.

While checking our paper [16] for this manuscript, we found a gap in one of the proofs, that we have been able to fix, as we shall show in the rest of this chapter. To precisely state our result and give the correct proof we need to fist recall some results on the dynamics of holomorphic self-maps of bounded strongly convex domains, giving a more precise description in this setting than the one that we provided in the previous chapter.

7.2.1 Dynamics of holomorphic self-maps of bounded strongly convex domains

We already recalled in the previous chapter the definition and the main properties of the Kobayashi distance. As we remarked, the main property of the Kobayashi (pseudo)distance is that it is contracted by holomorphic maps and in particular, the Kobayashi distance is invariant under biholomorphisms. In this chapter we will need some more properties of the Kobayashi distance.

Definition 7.4. *A* complex geodesic *in a hyperbolic manifold X is a holomorphic map $\varphi\colon \mathbb{D}\to X$ which is an isometry with respect to the Kobayashi distance of $\mathbb{D}$ and the Kobayashi distance of X.*

Lempert's theory (see [170] and [2, Chapter 2.6]) of complex geodesics in strongly convex domains is one of the main tools for the study of the geometric function theory of strongly convex domains. In particular, we have the following facts, summarizing the main results obtained by Lempert [170] and Royden-Wong [234] on complex geodesics in strongly convex domains.

Theorem 7.5 ([2, Theorem 2.6.19 and Corollary 2.6.30])**.** *Let $D \Subset \mathbb{C}^n$ be a bounded convex domain. Then for every pair of distinct points z, $w \in D$ there exists a complex geodesic $\varphi\colon \mathbb{D}\to D$ such that $\varphi(0) = z$ and $\varphi(r) = w$, where $0 < r < 1$ is such that $k_{\mathbb{D}}(0, r) = k_D(z, w)$; furthermore, if D is strongly convex then φ is unique. Moreover a holomorphic map $\varphi \in \mathrm{Hol}(\mathbb{D}, D)$ is a complex geodesic if and only if $k_D\big(\varphi(\zeta_1), \varphi(\zeta_2)\big) = k_{\mathbb{D}}(\zeta_1, \zeta_2)$ for a pair of distinct points ζ_1, $\zeta_2 \in \mathbb{D}$.*

Proposition 7.6 ([2, Proposition 2.6.22])**.** *Let $D \Subset \mathbb{C}^n$ be a bounded convex domain. Then every complex geodesic $\varphi \in \mathrm{Hol}(\mathbb{D}, D)$ admits a* left inverse, *that is a holomorphic map $\widetilde{p}_\varphi\colon D \to \mathbb{D}$ such that $\widetilde{p}_\varphi \circ \varphi = \mathrm{Id}_{\mathbb{D}}$. The map $p_\varphi = \varphi \circ \widetilde{p}_\varphi$ is then a holomorphic retraction of D onto the image of φ.*

Theorem 7.7 ([2, Theorem 2.6.29])**.** *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex with C^2 boundary. Then every complex geodesics φ extend continuously (actually, $\frac{1}{2}$-Hölder) to the boundary of $\mathbb{D}$, and the image of φ is transversal to ∂D.*

Theorem 7.8 ([2, Theorem 2.6.45])**.** *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex with C^2 boundary. Then for every $z \in D$ and $\tau \in \partial D$ there is a complex geodesic $\varphi \in \mathrm{Hol}(\mathbb{D}, D)$ with $\varphi(0) = z$ and $\varphi(1) = \tau$. Moreover for every pair of distinct points σ, $\tau \in \partial D$ there is a complex geodesic $\varphi \in \mathrm{Hol}(\mathbb{D}, D)$ such that $\varphi(-1) = \sigma$ and $\varphi(1) = \tau$.*

The statement of [2, Theorem 2.6.45] requires D with C^3 boundary, but the proof of the existence works assuming just C^2 smoothness.

Now let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary, and $f \in \mathrm{Hol}(D, D)$ a holomorphic self-map of D. As we saw in the previous chapter, if the set $\mathrm{Fix}(f)$ of fixed points of f in D is not empty, then the sequence $\{f^{\circ k}\}$ of iterates of f is relatively compact in $\mathrm{Hol}(D, D)$, and there exists a submanifold $D_0 \subseteq D$, the *limit manifold* of f, such that every limit of a subsequence of iterates is of the form $\gamma \circ \rho$, where $\rho\colon D \to D_0$ is a holomorphic retraction, and γ is a biholomorphism of D_0; furthermore, $f|_{D_0}$ is a biholomorphism of D_0, and $\mathrm{Fix}(f) \subseteq D_0$ (see [1] or [2, Theorem 2.1.29]).

Definition 7.9. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. A holomorphic map $f \in \mathrm{Hol}(D, D)$ is*

- elliptic *if* $\mathrm{Fix}(f) \neq \emptyset$,
- strongly elliptic *if its limit manifold reduces to a single point, the* Wolff point *of the strongly elliptic map.*

We say that a point $p \in \mathrm{Fix}(f)$ is attracting *if all the eigenvalues of df_p have modulus strictly less than 1.*

We also have an equivalent characterization of strongly elliptic maps.

Lemma 7.10 ([16, Lemma 1.1])**.** *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary, and $f \in \mathrm{Hol}(D, D)$. Then the following facts are equivalent:*

(i) f is strongly elliptic;

(ii) the sequence of iterates of f converges to a point $p \in D$;

(iii) f has an attracting fixed point $p \in D$;

(iv) there exists $p \in \mathrm{Fix}(f)$ such that $k_D\big(p, f(z)\big) < k_D(p, z)$ for all $z \in D \setminus \{p\}$.

As pointed out in the previous chapter, in the study of the dynamics of self-maps without fixed points, a crucial rôle is played by the horospheres defined in Definition 6.12, a generalization of the classical notion of horocycle.

It is a non-trivial fact (see [2, Theorem 2.6.47] or [57]) that for a bounded strongly convex domain with C^2 boundary $D \Subset \mathbb{C}^n$ the limit

$$\lim_{w \to \tau} [k_D(z, w) - k_D(p, w)]$$

exists for every $\tau \in \partial D$ and $p \in D$ and we can therefore define $h_{\tau,p}\colon D \to \mathbb{R}^+$ as

$$\tfrac{1}{2} \log h_{\tau,p}(z) = \lim_{w \to \tau} [k_D(z, w) - k_D(p, w)] \,.$$

Then, with this notation, the *horosphere* of *center* $\tau \in \partial D$, *radius* $R > 0$ and *pole* $p \in D$ is the set

$$E_p(\tau, R) = \{z \in D \mid h_{\tau,p}(z) < R\} \,.$$

We shall need the following fact.

Lemma 7.11 ([2, Lemma 2.7.16])**.** *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary, and $\varphi \in \mathrm{Hol}(\mathbb{D}, D)$ a complex geodesic. Put $p = \varphi(0)$ and $\tau = \varphi(1)$. Then*

$$\tilde{p}_\varphi\big(E_p(\tau, R)\big) = E_0^{\mathbb{D}}(1, R)$$

for any $R > 0$, where $E_0^{\mathbb{D}}(1, R)$ is the horosphere of center 1, pole 0 and radius R in $\mathbb{D}$.

An easy remark is that changing the pole amounts to multiplying the radius by a fixed constant, that is given $\tau \in \partial D$ we have

$$h_{\tau,q} = \frac{1}{h_{\tau,p}(q)} h_{\tau,p}$$

for all p, $q \in D$, and in particular for all $R > 0$ we have

$$E_q(\tau, R) = E_p\big(\tau, h_{\tau,p}(q)R\big) .$$

We can introduce *K-regions* in a similar way.

Definition 7.12. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. The* K-region $K_p(\tau, M)$ of center $\tau \in \partial D$, amplitude $M > 0$ and pole $p \in D$ *is the set*

$$K_p(\tau, M) = \{z \in D \mid \tfrac{1}{2} \log h_{\tau,p}(z) + k_D(p, z) < \log M\} .$$

It is well-known (see [2, 4]) that the K-regions with pole at the origin in the unit disk coincide with the classical Stolz regions, and that the K-regions with pole at the origin in the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ coincide with the usual Korányi approach regions.

Remark 7.13. *In strongly convex domains K-regions are comparable to* Stein admissible approach regions $A(\tau, M)$ of vertex $\tau \in \partial D$ and aperture $M > 1$*:*

$$A(\tau, M) = \{z \in D \mid \|z - \tau\|^2 < Md(z, \partial D), |\langle z - \tau, n_\tau\rangle| < Md(z, \partial D)\} , \tag{7.2}$$

where n_τ is the outer unit normal vector to ∂D at τ. Here comparable *means that for every $\tau \in \partial D$ there exists a neighbourhood $U \subset \mathbb{C}^n$ of τ such that for any $M > 1$ and $p \in D$ there are M_1, $M_2 > 1$ such that*

$$A(\tau, M_1) \cap U \subseteq K_p(\tau, M) \cap U \subseteq A(\tau, M_2) \cap U ;$$

see, e.g., [2, Propositions 2.7.4, 2.7.6 and p. 380]. Moreover, changing the pole does not change much the K-regions, because for each p, $q \in D$ there is $L > 0$ such that

$$K_p(\tau, M/L) \subseteq K_q(\tau, M) \subseteq K_p(\tau, ML) \tag{7.3}$$

for every $M > 0$ (see [2, Lemma 2.7.2]).

Definition 7.14. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Given $\tau \in \partial D$, we shall say that a function $F\colon D \to \mathbb{C}^n$ has* K-limit $\ell \in \mathbb{C}^n$ at τ *if $F(z) \to \ell$ as $z \to \tau$ inside any K-region centered at τ.*

Notice that the choice of the pole is immaterial because of (7.3). Since K-regions in strongly convex domains are comparable to Stein admissible regions, the notion of K-limit is equivalent to Stein admissible limit, and thus it is the right generalization to several variables of the one-dimensional notion of non-tangential limit (in particular, the existence of a K-limit always implies the existence of a non-tangential limit). Finally, the intersection of a horosphere (or K-region) of center $\tau \in \partial D$ and pole $p \in D$ with the image of a complex geodesic φ with $\varphi(0) = p$ and $\varphi(1) = \tau$ is the image via φ of the horosphere (or K-region) of center 1 and pole 0 in the unit disk ([2, Proposition 2.7.8 and Lemma 2.7.16]).

The good generalization of the one-variable notion of angular derivative is given by the *dilation coefficient* (see [2, Section 1.2.1 and Theorem 2.7.14])

Definition 7.15. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary, $f \in \mathrm{Hol}(D, D)$, and let $\sigma \in \partial D$. The* dilation coefficient $\beta_{\sigma,p} \in (0, +\infty]$ of f at $\sigma \in \partial D$ with pole $p \in D$ *is given by*

$$\tfrac{1}{2} \log \beta_{\sigma,p} = \liminf_{z \to \sigma} \big[k_D(p, z) - k_D\big(p, f(z)\big) \big] \ .$$

Furthermore, $\sigma \in \partial D$ is called a boundary fixed point *of f if f has K-limit σ at σ.*

Since

$$k_D(p, z) - k_D\big(p, f(z)\big) \geq k_D\big(f(p), f(z)\big) - k_D\big(p, f(z)\big) \geq -k_D\big(p, f(p)\big) \ ,$$

the dilation coefficient cannot be zero. We also recall the following useful formulas for computing the dilation coefficient obtained in [2, Lemma 2.7.22]:

$$\begin{aligned} \tfrac{1}{2} \log \beta_{\sigma,p} &= \lim_{t \to 1} \big[k_D\big(p, \varphi(t)\big) - k_D\big(p, f\big(\varphi(t)\big)\big) \big] \\ &= \lim_{t \to 1} \big[k_D\big(p, \varphi(t)\big) - k_D\big(p, p_\varphi \circ f\big(\varphi(t)\big)\big) \big] \ , \end{aligned} \tag{7.4}$$

where $\varphi \in \mathrm{Hol}(\mathbb{D}, D)$ is a complex geodesic with $\varphi(0) = p$ and $\varphi(1) = \sigma$, and $p_\varphi = \varphi \circ \widetilde{p}_\varphi$ is the holomorphic retraction associated to φ.

When σ is a boundary fixed point then the dilation coefficient does not depend on the pole (see for example [16, Lemma 1.3]) and we shall then simply denote by β_σ the dilation coefficient at a boundary fixed point.

Definition 7.16. *Let $\sigma \in \partial D$ be a boundary fixed point for a self-map $f \in \mathrm{Hol}(D, D)$ of a bounded strongly convex domain with C^2 boundary $D \Subset \mathbb{C}^n$. We shall say that σ is*

- attracting *if* $0 < \beta_\sigma < 1$,
- parabolic *if* $\beta_\sigma = 1$,
- repelling *if* $\beta_\sigma > 1$.

We can now quote the general version of Julia's lemma proved by Abate (see [2, Theorem 2.4.16 and Proposition 2.7.15]) that we shall need in this chapter.

Proposition 7.17 (Abate, [2])**.** *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary, and $f \in \mathrm{Hol}(D, D)$. Let $\sigma \in \partial D$ and $p \in D$ be such that the dilation coefficient $\beta_{\sigma,p}$ is finite. Then there exists a unique $\tau \in \partial D$ such that for all $R > 0$ we have*

$$f\big(E_p(\sigma, R)\big) \subseteq E_p\big(\tau, \beta_{\sigma,p} R\big) \ ,$$

and f has K-limit τ at σ. Moreover, if there is a sequence $\{w_\nu\} \subset D$ converging to $\sigma \in \partial D$ so that $\{f(w_\nu)\}$ converges to $\tau_1 \in \partial D$ then $\tau = \tau_1$.

Finally, we recall the several variable version of the Wolff-Denjoy theorem given in [A1] (see also the previous chapter and [A2, Theorems 2.4.19 and 2.4.23]).

Theorem 7.18 (Abate, [1])**.** *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary, and $f \in \mathrm{Hol}(D, D)$ without fixed points. Then there exists a unique $\tau \in \partial D$ such that the sequence of iterates of f converges to τ.*

Definition 7.19. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary, and $f \in \mathrm{Hol}(D, D)$ without fixed points. The point $\tau \in \partial D$ introduced in the previous theorem is the* Wolff point *of f.*

The dilation coefficient can also be used to characterize the Wolff point of $f \in \mathrm{Hol}(D,D)$ without fixed points in D defined above.

Proposition 7.20 (Abate, R. [16, Proposition 1.6]). *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary, and $f \in \mathrm{Hol}(D,D)$ without fixed points in D. Then the following assertions are equivalent for a point $\tau \in \partial D$:*

(i) *τ is a boundary fixed point with $0 < \beta_\tau \leq 1$;*

(ii) *$f\big(E_p(\tau,R)\big) \subseteq E_p(\tau,R)$ for all $R > 0$ and any (and hence all) $p \in D$;*

(iii) *τ is the Wolff point of f.*

Definition 7.21. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary, and $f \in \mathrm{Hol}(D,D)$ without fixed points and with Wolff point $\tau \in \partial D$. We shall say that f is* hyperbolic *if $0 < \beta_\tau < 1$ and* parabolic *if $\beta_\tau = 1$.*

Before turning our attention to backward orbits, a final remark is needed. Since the Kobayashi distance k_D is weakly contracted by holomorphic maps, forward orbits always have *bounded Kobayashi step,* that is the Kobayashi distance $k_D\big(f^{\circ k+1}(z), f^{\circ k}(z)\big)$ between two consecutive elements of the orbit is bounded by a constant independent of n (but depending on the orbit). We can then give the following definition.

Definition 7.22. *Let X be a Kobayashi hyperbolic manifold. We say that a sequence $\{z_k\} \subset X$ has* bounded Kobayashi step *if*

$$a = \sup_k k_X(z_{k+1}, z_k) < +\infty .$$

The number a is the Kobayashi step *of the sequence.*

Summing up, if $f \in \mathrm{Hol}(D,D)$ is strongly elliptic, hyperbolic or parabolic, then all forward orbits have bounded Kobayashi step and converge to the Wolff point $\tau \in \overline{D}$ (for the sake of uniformity, we are calling *Wolff point* the unique fixed point of a strongly elliptic map too), which is a (possibly boundary) fixed point.

7.2.2 Backward iteration in strongly convex domains

The main result in [16] states that backward orbits with bounded Kobayashi step for a hyperbolic or strongly elliptic self-map of a bounded strongly convex domain in $\mathbb{C}^n$ with C^2 boundary always converge to a boundary fixed point, where a *boundary fixed point* is a point $\sigma \in \partial D$ such that f has K-limit σ at σ. Moreover if such boundary fixed point σ does not coincide with the Wolff point, then it is *repelling*, that is the boundary dilation β_σ of f at σ is larger than 1. More precisely, we prove the following result.

Theorem 7.23 (Abate, R. [16, Theorem 0.1]). *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D,D)$ be either hyperbolic or strongly elliptic, with Wolff point $\tau \in \overline{D}$. Let $\{z_k\} \subset D$ be a backward orbit for f with bounded Kobayashi step. Then:*

(i) *the sequence $\{z_k\}$ converges to a boundary fixed point $\sigma \in \partial D$;*

(ii) *if $\sigma \neq \tau$ then σ is repelling;*

(iii) *$\sigma \neq \tau$ if and only if $\{z_k\}$ goes to σ inside a K-region, that is, there exists $M > 0$ so that $z_k \in K_p(\sigma, M)$ eventually, where p is any point in D.*

Remark 7.24. *If f is strongly elliptic then clearly $\sigma \neq \tau$. We conjecture that $\sigma \neq \tau$ in the hyperbolic case too.*

Remark 7.25. *The original statement of [16, Theorem 0.1] included also the parabolic case. We were not yet able to completely fix the proof in the parabolic case; thus the behavior of backward orbits for a parabolic self-map is still unknown, even (as far as we know) in the unit ball of $\mathbb{C}^n$ (see [187]).*

Proof. We begin with a first general lemma, saying that if a backward orbit with bounded Kobayashi step converges to a boundary point, then this point necessarily is a boundary fixed point.

Lemma 7.26 ([16, Lemma 2.3])**.** *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary, and let $f \in \mathrm{Hol}(D, D)$. Let $\{z_k\} \subset D$ be a backward orbit for f with bounded Kobayashi step $a = \frac{1}{2}\log\alpha$ converging to $\sigma \in \partial D$. Then σ is a boundary fixed point of f and $\beta_\sigma \leq \alpha$.*

Proof. Fix $p \in D$. First of all we have

$$\begin{aligned}\tfrac{1}{2}\log\beta_{\sigma,p} = \liminf_{w\to\sigma}\big[k_D(w,p) - k_D\big(f(w),p\big)\big] &\leq \liminf_{k\to+\infty}[k_D(z_{k+1},p) - k_D\big(z_k,p\big)] \\ &\leq \liminf_{k\to+\infty} k_D(z_{k+1}, z_k) \\ &\leq a = \tfrac{1}{2}\log\alpha\ .\end{aligned} \tag{7.5}$$

Since $z_k \to \sigma$ and $f(z_k) = z_{k-1} \to \sigma$ as $k \to +\infty$, Proposition 7.17 yields the assertion. □

The rest of the proof is divided into two cases according to whether f is hyperbolic or strongly elliptic.

Hyperbolic case. In this case, we first prove that any backward orbit has to accumulate to the boundary of the domain D.

Lemma 7.27 ([16, Lemma 2.1])**.** *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $\{z_k\} \subset D$ be a backward orbit for hyperbolic or parabolic self-map $f \in \mathrm{Hol}(D, D)$. Then $z_k \to \partial D$ as $k \to +\infty$.*

Proof. Assume, by contradiction, that the sequence does not converge to ∂D. Then there exists a subsequence $\{z_{k_n}\}$ converging to $w_0 \in D$, that is, such that

$$k_D(w_0, z_{k_n}) \to 0 \quad \text{as } k_n \to +\infty\ .$$

Therefore

$$k_D\big(f^{\circ k_n}(w_0), f^{\circ k_n}(z_{k_n})\big) \leq k_D(w_0, z_{k_n}) \to 0 \quad \text{as } k_n \to +\infty\ .$$

But, on the other hand, $f^{\circ k_n}(z_{k_n}) = z_0$ for all k_n; moreover, $f^{\circ k_n}(w_0) \to \tau$ as $k_n \to +\infty$, where $\tau \in \partial D$ is the Wolff point of f, and so

$$\lim_{k_n\to\infty} k_D\big(f^{\circ k_n}(w_0), f^{\circ k_n}(z_{k_n})\big) = +\infty\ ,$$

because k_D is complete, giving us a contradiction. □

In order to prove the convergence of the whole sequence towards a point $\sigma \in \partial D$, which is a boundary fixed point, we first need the following estimate.

Lemma 7.28 ([16, Lemma 2.6]). *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary, and fix $p \in D$. Let $f \in \mathrm{Hol}(D, D)$ be hyperbolic or parabolic with Wolff point $\tau \in \partial D$ and dilation coefficient $0 < \beta_\tau \leq 1$. Let $\{z_k\} \subset D$ be a backward orbit for f. Then for every $k \in \mathbb{N}$ we have*

$$h_{\tau,p}(z_k) \geq \left(\frac{1}{\beta_\tau}\right)^k h_{\tau,p}(z_0) \, .$$

Proof. Put $t_k = h_{\tau,p}(z_k)$. By definition, $z_k \in \partial E_p(\tau, t_k)$. By Proposition 7.17, if $z_{k+1} \in E_p(\tau, R)$ then $z_k \in E_p(\tau, \beta_\tau R)$. Since $z_k \notin E_p(\tau, t_k)$, we have that $z_{k+1} \notin E_p(\tau, \beta_\tau^{-1} t_k)$, that is

$$t_{k+1} \geq \frac{1}{\beta_\tau} t_k \, , \tag{7.6}$$

and the assertion follows by induction. □

This estimate allows us to prove part (i) in the hyperbolic case.

Lemma 7.29 ([16, Remark 2.1]). *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D, D)$ be hyperbolic with Wolff point $\tau \in \partial D$ and let $\{z_k\} \subset D$ be a backward orbit for f with bounded Kobayashi step $a > 0$. Then $\{z_k\}$ converges to a boundary fixed point $\sigma \in \partial D$.*

Proof. First of all, that [21, Lemma 2.4 and Remark 3] yield a constant $C_1 > 0$ such that

$$\|z_k - z_{k+1}\|^2 + |\langle z_k - z_{k+1}, z_k\rangle| \leq \frac{C_1^2}{1 - \hat{a}^2} d(z_k, \partial D), \tag{7.7}$$

and so

$$\|z_k - z_{k+1}\| \leq \frac{C_1}{\sqrt{1 - \hat{a}^2}} \sqrt{d(z_k, \partial D)} \leq \frac{C_1}{1 - \hat{a}} \sqrt{d(z_k, \partial D)} \, , \tag{7.8}$$

where $\hat{a} = \tanh a \in (0, 1)$. On the other hand, given $p \in D$ the triangular inequality and the upper estimate [2, Theorem 2.3.51] on the boundary behaviour of the Kobayashi distance yield a constant $C_2 > 0$ such that

$$\tfrac{1}{2} \log h_{\tau,p}(z_k) \leq k_D(p, z_k) \leq C_2 - \tfrac{1}{2} \log d(z_k, \partial D) \, ,$$

that is

$$d(z_k, \partial D) \leq \frac{e^{2C_2}}{h_{\tau,p}(z_k)} \, , \tag{7.9}$$

and thus

$$\|z_k - z_{k+1}\| \leq \frac{C}{1 - \hat{a}} \sqrt{\frac{1}{h_{\tau,p}(z_k)}} \, , \tag{7.10}$$

for a suitable $C > 0$. Therefore using (7.6) we obtain that for every $k, m \geq 0$ we have

$$\begin{aligned} \|z_k - z_{k+m}\| &\leq \sum_{j=k}^{k+m-1} \|z_j - z_{j+1}\| \leq \frac{C}{1 - \hat{a}} \frac{1}{\sqrt{h_{\tau,p}(z_k)}} \sum_{j=0}^{m-1} \beta_\tau^{j/2} \\ &\leq \frac{C}{1 - \hat{a}} \frac{1}{1 - \beta_\tau^{1/2}} \frac{1}{\sqrt{h_{\tau,p}(z_k)}} \, , \end{aligned} \tag{7.11}$$

Since $h_{p.\tau}(z_k) \to +\infty$ as $k \to +\infty$ by Lemma 7.28 it follows that $\{z_k\}$ is a Cauchy sequence in $\mathbb{C}^n$, converging to a point σ, necessarily belonging to ∂D by Lemma 7.27. The proof is then completed thanks to Lemma 7.26. □

The following lemma allows us to control the dilation coefficient at the limit of a backward orbit, giving in particular part (ii) of Theorem 7.23 in the hyperbolic case.

Lemma 7.30 ([16, Lemma 2.4])**.** *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D, D)$ be hyperbolic or parabolic with Wolff point $\tau \in \partial D$ and dilation coefficient $0 < \beta_\tau \leq 1$. Let $\sigma \in \partial D \setminus \{\tau\}$ be a boundary fixed point with finite dilation coefficient β_σ. Then*

$$\beta_\sigma \geq \frac{1}{\beta_\tau} \geq 1 \, .$$

In particular, if f is hyperbolic then σ is repelling.

Proof. Let $\varphi\colon \overline{\mathbb{D}} \to \overline{D}$ be a complex geodesic such that $\varphi(-1) = \sigma$ and $\varphi(1) = \tau$, and set $p = \varphi(0)$. Proposition 7.17 yields

$$p \in \overline{E_p(\sigma, 1)} \qquad \Longrightarrow \qquad f(p) \in \overline{E_p(\sigma, \beta_\sigma)}$$

and

$$p \in \overline{E_p(\tau, 1)} \qquad \Longrightarrow \qquad f(p) \in \overline{E_p(\tau, \beta_\tau)} \, .$$

Hence $\overline{E_p(\sigma, \beta_\sigma)} \cap \overline{E_p(\tau, \beta_\tau)} \neq \emptyset$.

Let $\widetilde{p}_\varphi\colon D \to \mathbb{D}$ be the left-inverse of φ. Then using Lemma 7.11 we get

$$\emptyset \neq \widetilde{p}_\varphi \left(\overline{E_p(\sigma, \beta_\sigma)} \cap \overline{E_p(\tau, \beta_\tau)} \right) \subseteq \widetilde{p}_\varphi(\overline{E_p(\sigma, \beta_\sigma)}) \cap \widetilde{p}_\varphi(\overline{E_p(\tau, \beta_\tau)}) = \overline{E_0^{\mathbb{D}}(-1, \beta_\sigma)} \cap \overline{E_0^{\mathbb{D}}(1, \beta_\tau)} \, .$$

Now, $E_0^{\mathbb{D}}(1, \beta_\tau)$ is an Euclidean disk of radius $\beta_\tau/(\beta_\tau + 1)$ tangent to $\partial\mathbb{D}$ in 1, and $E_0^{\mathbb{D}}(-1, \beta_\sigma)$ is an Euclidean disk of radius $\beta_\sigma/(\beta_\sigma + 1)$ tangent to $\partial\mathbb{D}$ in -1. So these disks intersect if and only if

$$1 - \frac{2\beta_\tau}{\beta_\tau + 1} \leq -1 + \frac{2\beta_\sigma}{\beta_\sigma + 1} \, ,$$

which is equivalent to $\beta_\sigma \beta_\tau \geq 1$, as claimed. □

We can now prove the first half of Theorem 7.23.(iii) for the hyperbolic case.

Lemma 7.31. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D, D)$ be hyperbolic with Wolff point $\tau \in \partial D$ and dilation coefficient $0 < \beta_\tau < 1$, and let $\{z_k\} \subset D$ be a backward orbit with bounded Kobayashi step $a = \frac{1}{2} \log \alpha$ converging to $\sigma \in \partial D \setminus \{\tau\}$. Then for every $p \in D$ there exists $M > 0$ such that $z_k \in K_p(\sigma, M)$ eventually.*

Proof. Fix $p \in D$. By Remark 7.13 it suffices to prove that there exists $M > 1$ such that $\{z_k\}$ converges to σ inside an admissible approach region $A(\sigma, M)$.

Set again $t_k := h_{\tau,p}(z_k)$. Thanks to (7.6), we have

$$\frac{1}{t_{k+m}} \leq \beta_\tau^k \frac{1}{t_k}$$

for all k, $m \geq 0$. Moreover, thanks to [2, Corollary 2.3.55], since $\sigma \neq \tau$, there exists $\varepsilon > 0$ and $K > 0$ such that for any $w \in D \cap B(\tau, \varepsilon)$ and k such that $z_k \in D \cap B(\sigma, \varepsilon)$ we have

$$k_D(z_k, w) \geq -\tfrac{1}{2} \log d(z_k, \partial D) - \tfrac{1}{2} \log d(w, \partial D) + K.$$

On the other hand, [2, Theorem 2.3.51] yields $c_1 \in \mathbb{R}$ such that

$$k_D(w, p) \leq c_1 - \tfrac{1}{2} \log d(w, \partial D)$$

for any $w \in D$. So for $w \in D \cap B(\tau, \varepsilon)$ and k sufficiently large we have

$$k_D(z_k, w) - k_D(w, p) \geq -\tfrac{1}{2}\log d(z_k, \partial D) - \tfrac{1}{2}\log d(w, \partial D) + \tfrac{1}{2}\log d(w, \partial D) - c_1 + K,$$

which implies

$$t_k = h_{\tau,p}(z_k) = \lim_{w \to \tau}[k_D(z_k, w) - k_D(p, w)] \geq -\tfrac{1}{2}\log d(z_k, \partial D) + K - c_1,$$

that is

$$\frac{1}{t_k} \leq \widetilde{C}_1 d(z_k, \partial D), \tag{7.12}$$

for some $\widetilde{C}_1 > 0$.

Therefore, thanks to (7.11), for all $m \geq 0$ and k large enough we have

$$\begin{aligned} \|z_k - z_{k+m}\| &\leq \frac{C}{1-\hat{a}} \frac{1}{\sqrt{t_k}} \sum_{j=0}^{\infty} \beta_\tau^{j/2} \leq \frac{C}{1-\hat{a}} \frac{1}{1-\beta_\tau^{1/2}} \frac{1}{\sqrt{t_k}} \\ &\leq \frac{C\widetilde{C}_1}{1-\hat{a}} \frac{1}{1-\beta_\tau^{1/2}} \sqrt{d(z_k, \partial D)} \end{aligned} \tag{7.13}$$

for some $C > 0$, and letting m tend to infinity we obtain that for k sufficiently large, there is $M_1 > 1$ such that

$$\|z_k - \sigma\| \leq M_1 \sqrt{d(z_k, \partial D)}. \tag{7.14}$$

On the other hand, up to translating the domain, without loss of generality we can assume that D contains the origin. In particular, D being bounded and strongly convex we can replace n_σ by σ in the definition of $A(\sigma, M)$. Therefore, to conclude the proof it suffices to prove that there exists $M_2 > 1$ such that

$$|\langle z_k - \sigma, \sigma \rangle| \leq M_2 d(z_k, \partial D)$$

for k large enough. Now

$$|\langle z_j - z_{j+1}, z_j - \sigma \rangle| \leq \|z_j - z_{j+1}\| \|z_j - \sigma\|,$$

and so, thanks to (7.7) and (7.14), for k large enough and $m \geq 0$ we have

$$\begin{aligned} |\langle z_k - z_{k+m}, \sigma \rangle| &\leq \sum_{j=k}^{k+m-1} |\langle z_j - z_{j+1}, \sigma \rangle| \\ &\leq \sum_{j=k}^{k+m-1} \Big(|\langle z_j - z_{j+1}, z_j - \sigma \rangle| + |\langle z_j - z_{j+1}, z_j \rangle| \Big) \\ &\leq \sum_{j=k}^{k+m-1} \Big(\|z_j - z_{j+1}\| \|z_j - \sigma\| + \frac{C_1^2}{1-\hat{a}^2} d(z_j, \partial D) \Big) \\ &\leq \sum_{j=k}^{k+m-1} \Big(\frac{M_1 C_1}{1-\hat{a}} d(z_j, \partial D) + \frac{C_1^2}{1-\hat{a}^2} d(z_j, \partial D) \Big) \\ &\leq C' \sum_{j=k}^{k+m-1} d(z_j, \partial D), \end{aligned} \tag{7.15}$$

for some $C' > 0$. Arguing as in (7.11) and (7.13), using (7.9) and (7.12) we obtain

$$|\langle z_k - z_{k+m}, \sigma\rangle| \leq M_2 d(z_k, \partial D)$$

for $m \geq 0$, k large enough and for some $M_2 > 1$. Letting m tend to infinity we finally have

$$|\langle z_k - \sigma, \sigma\rangle| \leq M_2 d(z_k, \partial D).$$

It then suffices to take $M = \max\{M_1, M_2\}$ to conclude the proof. □

The following lemma completes the proof of Theorem 7.23.(iii):

Lemma 7.32. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D, D)$ be hyperbolic with Wolff point $\tau \in \partial D$ and dilation coefficient $0 < \beta_\tau < 1$, and let $\{z_k\} \subset D$ be a backward orbit with bounded Kobayashi step $a = \frac{1}{2}\log\alpha$ converging to $\sigma \in \partial D \backslash \{\tau\}$ inside a K-region. Then $\sigma \neq \tau$.*

Proof. Assume, by contradiction, that $\sigma = \tau$. Fix $p \in D$, and let $M > 1$ be such that $z_k \in K_p(\tau, M)$. Given $\varepsilon > 0$, [2, Lemma 2.7.1] yields $r > 0$ such that if $k_D(z_k, p) \geq r$ then $z_k \in E_p(\tau, \varepsilon)$, that is $h_{\tau,p}(z_k) < \varepsilon$. Since $k_D(z_k, p) \to +\infty$, it follows that $h_{\tau,p}(z_k) \to 0$ as $k \to +\infty$. But Lemma 7.28 implies that $h_{\tau,p}(z_k) \to +\infty$, contradicting our hypotheses. □

Strongly elliptic case. Also in this case, we start by proving by contradiction that any backward orbit has to accumulate to the boundary of the domain D.

Lemma 7.33. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D, D)$ be strongly elliptic with Wolff point $p \in D$, and let $\{z_k\} \subset D$ be a backward orbit with bounded Kobayashi step $a = \frac{1}{2}\log\alpha$. Then $z_k \to \partial D$ as $k \to +\infty$.*

Proof. Define $\ell_k > 0$ by setting $\frac{1}{2}\log\ell_k = k_D(z_k, p)$. Since f is strongly elliptic, we have

$$k_D(z_k, p) < k_D(z_{k+1}, p)\ ,$$

and thus the sequence $\{\ell_k\}$ is strictly increasing. Assume, by contradiction, that it has a finite limit ℓ_∞. This means that every limit point z_∞ of the sequence $\{z_k\}$ satisfies $k_D(z_\infty, p) = \frac{1}{2}\log\ell_\infty$. But $f(z_\infty)$ is a limit point of the sequence $\{f(z_k)\} = \{z_{k-1}\}$ and thus we again have $k_D\big(f(z_\infty), p\big) = \frac{1}{2}\log\ell_\infty$, which is impossible by Lemma 7.10 because f is strongly elliptic. Therefore $\ell_\infty = +\infty$, which means that $z_k \to \partial D$. □

This allows us to prove the following key result.

Lemma 7.34. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D, D)$ be strongly elliptic with Wolff point $p \in D$. Let $\{z_k\} \subset D$ be a backward orbit with bounded Kobayashi step. Then there exists a constant $0 < c < 1$ such that*

$$k_D(z_k, p) - k_D(z_{k+1}, p) \leq \tfrac{1}{2}\log c < 0$$

for all $k \in \mathbb{N}$.

Proof. Assume, by contradiction, that for every $0 < c < 1$ there is $k(c) \in \mathbb{N}$ such that

$$k_D(z_{k(c)}, p) - k_D(z_{k(c)+1}, p) > \tfrac{1}{2}\log c\ ,$$

that is

$$k_D(z_{k(c)+1}, p) - k_D\big(f(z_{k(c)+1}, p\big) < -\tfrac{1}{2}\log c\ .$$

Consider the sequences $\{z_{k(1-\frac{1}{j})+1}\}$ and $\{z_{k(1-\frac{1}{j})} = f(z_{k(1-\frac{1}{j})+1})\}$. Thanks to Lemma 7.33, we know that both these sequences accumulate on ∂D; therefore, by extracting subsequences, we can find a subsequence $\{z_{k_j}\}$ such that $z_{k_j} \to \sigma_1 \in \partial D$, $f(z_{k_j}) \to \sigma_2 \in \partial D$ as $j \to +\infty$ and

$$\lim_{j\to+\infty} \big[k_D(z_{k_j}, p) - k_D\big(f(z_{k_j}), p\big)\big] \le 0 \, .$$

If $\sigma_1 \neq \sigma_2$, then [2, Corollary 2.3.55], together with the fact that $\{z_k\}$ has bounded Kobayashi step, lead to a contradiction since for k large enough there is $K \in \mathbb{R}$ such that

$$a \ge k_D\big(z_{k_j}, f(z_{k_j})\big) \ge -\tfrac{1}{2}\log d(z_{k_j}, \partial D) - \tfrac{1}{2}\log d\big(f(z_{k_j}), \partial D\big) + K$$

whereas the right-hand side tends to infinity. Therefore, $\sigma_1 = \sigma_2$ and we have

$$\liminf_{z\to\sigma_1} \big[k_D(z, p) - k_D\big(f(z), p\big)\big] \le 0 \, .$$

Then we can apply Proposition 7.17, obtaining that σ_1 is a boundary fixed point and that for any $R > 0$ we have $f\big(E_p(\sigma_1, R)\big) \subseteq E_p(\sigma_1, R)$. We can then choose $R < 1$ so that $p \notin \overline{E_p(\sigma_1, R)}$, and let $w \in \overline{E_p(\sigma_1, R)}$ be a point closest to p with respect to the Kobayashi distance. Since $f(w) \in \overline{E_p(\sigma_1, R)}$ this means that $k_D\big(f(w), p\big) \ge k_D(w, p)$, which is impossible because $w \neq p$ and f is strongly elliptic. □

This estimate allows us to prove that the whole backward orbit converges to a boundary fixed point $\sigma \in \partial D$, which is obviously different from the Wolff point $p \in D$.

Lemma 7.35 ([16, Remark 2.2])**.** *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D, D)$ be strongly elliptic with Wolff point $p \in D$, and let $\{z_k\} \subset D$ be a backward orbit with bounded Kobayashi step $a = \frac{1}{2}\log\alpha$. Then $\{z_k\}$ converges to a boundary fixed point $\sigma \in \partial D$ with $\beta_\sigma \le \alpha$.*

Proof. Without loss of generality, we can assume that $z_0 \neq p$. We consider $s_k > 0$ defined by setting $-\frac{1}{2}\log s_k = k_D(z_k, p)$. Taking the constant $0 < c < 1$ given by the Lemma 7.34, we therefore have

$$-\tfrac{1}{2}\log s_k + \tfrac{1}{2}\log s_{k+1} \le \tfrac{1}{2}\log c \, ,$$

that is

$$s_{k+1} \le c s_k \, . \tag{7.16}$$

Therefore $s_{k+m} \le c^m s_k$ for every $k, m \in \mathbb{N}$, and using again (7.7) and [2, Theorem 2.3.51] as in the proof of Lemma 7.29, for all $j \in \mathbb{N}$ we obtain

$$\|z_j - z_{j+1}\| \le \frac{C}{1-\hat{a}}\sqrt{s_j}$$

for a suitable $C > 0$, where $\hat{a} = \tanh a$. Arguing exactly as in (7.11) we then obtain that

$$\|z_k - z_{k+m}\| \le \frac{C}{1-\hat{a}}\frac{1}{1-c^{1/2}}\sqrt{s_k}, \tag{7.17}$$

for any $m \ge 0$ and k large enough. So $\{z_k\}$ is a Cauchy sequence in $\mathbb{C}^n$ converging to a point $\sigma \in \partial D$ by Lemma 7.34, and the assertion follows from Lemma 7.26. □

The following general result proves Theorem 7.23.(ii) in the strongly elliptic case.

Lemma 7.36. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D,D)$ be strongly elliptic with Wolff point $p \in D$. If $\sigma \in \partial D$ is a boundary fixed point then $\beta_\sigma > 1$.*

Proof. Since p is a fixed point of f, we already know that

$$\tfrac{1}{2}\log\beta_\sigma = \liminf_{z\to\sigma}\big[k_D(z,p) - k_D\big(f(z),p\big)\big] \geq 0 \,.$$

Assume, by contradiction, that $\beta_\sigma = 1$. Then Proposition 7.17 yields $f\big(E_p(\sigma,R)\big) \subseteq \overline{E_p(\sigma,R)}$ for any $R > 0$ because σ is a boundary fixed point. Choose $R < 1$ so that $p \notin \overline{E_p(\sigma,R)}$, and let $w \in \overline{E_p(\sigma_1,R)}$ be a point closest to p with respect to the Kobayashi distance. Since $f(w) \in \overline{E_p(\sigma_1,R)}$ this means that $k_D\big(f(w),p\big) \geq k_D(w,p)$, which is impossible because $w \neq p$ and f is strongly elliptic. □

We conclude by proving Theorem 7.23.(iii) in the strongly elliptic case.

Lemma 7.37. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary.Let $f \in \mathrm{Hol}(D,D)$ be strongly elliptic, with Wolff point $p \in D$. Let $\{z_k\} \subset D$ be a backward orbit for f with bounded Kobayashi step converging to $\sigma \in \partial D$. Then for every $q \in D$ there exists $M > 0$ such that $z_k \in K_q(\sigma, M)$ eventually.*

Proof. It suffices again to prove that there exists $M > 1$ such that $\{z_k\}$ converges to σ inside an admissible approach region $A(\sigma, M)$.

Without loss of generality, we can assume that $z_0 \neq p$. We consider again $s_k > 0$ defined by setting $-\frac{1}{2}\log s_k = k_D(z_k,p)$. Thanks to (7.16), there is a constant $0 < c < 1$ such that

$$s_{k+m} \leq c^m s_k \tag{7.18}$$

for all $k, m \geq 0$.

Now, [2, Theorem 2.3.51, Theorem 2.3.52] yield constants $\widetilde{C}_1, \widetilde{C}_2 > 0$ such that

$$\widetilde{C}_1 d(z_j,\partial D) \leq s_j \leq \widetilde{C}_2 d(z_j,\partial D) \tag{7.19}$$

for all $j \in \mathbb{N}$, and so plugging this in (7.17) we have

$$\|z_k - z_{k+m}\| \leq \frac{C}{1-\hat{a}}\frac{1}{1-c}\sqrt{s_k} \leq \frac{C}{1-\hat{a}}\frac{1}{1-c}\sqrt{\widetilde{C}_2}\sqrt{d(z_k,\partial D)}$$

for any $m \geq 0$ and k large enough, and letting m tend to infinity we obtain

$$\|z_k - \sigma\| \leq M_1\sqrt{d(z_k,\partial D)}, \tag{7.20}$$

for some $M_1 > 1$.

On the other hand, up to translating the domain, without loss of generality we can assume that D contains the origin. In particular, D being bounded and strongly convex we can replace n_σ by σ in the definition of $A(\sigma, M)$. Therefore, to conclude the proof it suffices to prove that there exists $M_2 > 1$ such that

$$|\langle z_k - \sigma, \sigma\rangle| \leq M_2 d(z_k,\partial D)$$

for k large enough. This follows by arguing as in the proof of Lemma 7.31 using s_k instead of t_k, thanks to (7.18) and (7.19). Then taking $M = \max\{M_1, M_2\}$ we conclude the proof. □

This concludes the proof of the Theorem 7.23 in both cases. □

Abate and Bracci used this result in [13] where they also adapted our proof to obtain a similar result about existence and convergence of backward orbits for holomorphic self-maps of *rotational elliptic* holomorphic self-maps.

7.2.3 Existence of backward orbits with bounded Kobayashi step

To show that our theorem is not empty we proved the existence of backward orbits with bounded Kobayashi step. Given $D \Subset \mathbb{C}^n$ a bounded strongly convex domain with C^2 boundary and $f \in \mathrm{Hol}(D, D)$, a boundary fixed point $\sigma \in \partial D$ with dilation coefficient β_σ is said *isolated* if there is a neighbourhood $U \subset \mathbb{C}^n$ of σ in $\mathbb{C}^n$ such that $U \cap \partial D$ contains no other boundary fixed point of f with dilation coefficient at most β_σ.

In [16, Section 3] we constructed backward orbits with bounded Kobayashi step converging to isolated boundary fixed points. More precisely we proved the following result.

Theorem 7.38 (Abate, R. [16, Theorem 3.3])**.** *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D, D)$ be either hyperbolic, parabolic, or strongly elliptic, with Wolff point $\tau \in \overline{D}$. Let $\sigma \in \partial D \setminus \{\tau\}$ be an isolated repelling boundary fixed point for f with dilation coefficient $\beta_\sigma > 1$. Then there is a backward orbit with Kobayashi step bounded by $\frac{1}{2}\log\beta_\sigma$ converging to σ.*

The construction is done by slightly adapting an argument due to Poggi-Corradini (see [201, 187]) and we refer to [16, Section 3.] for the details.

Notice that in general, contrary to the unit disk case, as an example of Ostapyuk shows, boundary regular repelling fixed points in the unit ball are not isolated. In the recent preprint [31], Arosio and Guerini proved that for $f\colon \mathbb{B}^n \to \mathbb{B}^n$ a holomorphic self-map of the unit ball in $\mathbb{C}^n$ if $\sigma \in \partial\mathbb{B}^n$ is a boundary repelling fixed point with dilation $\beta_\sigma > 1$, then there exists a backward orbit converging to σ with step $\frac{1}{2}\log\beta_\sigma$. Moreover, they proved that any two backward orbits converging at the same boundary repelling fixed point stay at finite distance, and deduced as a consequence the existence of a unique canonical pre-model in this case.

However the behaviour near non-isolated boundary regular repelling fixed points in general strongly convex bounded domain with C^2 boundary is still unknown.

Chapter 8

The Julia-Wolff-Carathéodory Theorem and its generalizations

This chapter is a short introduction to the Julia-Wolff-Carathéodory theorem, and its generalizations in several complex variables, up to very recent results for infinitesimal generators of semigroups. A very precise and systematic presentation, providing clear proofs, of various aspects of the problem of generalizing the classical Julia-Wolff-Carathéodory theorem to several complex variables can be found in [9].

8.1 The classical Julia-Wolff-Carathéodory theorem

One of the classical result in one-dimensional complex analysis is Fatou's theorem:

Theorem 8.1 (Fatou [112]). *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map of the unit disk $\mathbb{D} \subset \mathbb{C}$. Then f admits non-tangential limit at almost every point of $\partial\mathbb{D}$.*

This result however does not give any precise information about the behaviour at a specific point σ of the boundary. Of course, to obtain a more precise statement in this case some hypotheses on f are needed. In fact, as it was found by Julia ([155]) in 1920, the right hypothesis is to assume that $f(\zeta)$ approaches the boundary of $\mathbb{D}$ at least as fast as ζ, in a weak sense. More precisely, we have the classical *Julia's lemma:*

Theorem 8.2 (Julia [155]). *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a bounded holomorphic function such that*

$$\liminf_{\zeta\to\sigma} \frac{1-|f(\zeta)|}{1-|\zeta|} = \alpha < +\infty \tag{8.1}$$

for some $\sigma \in \partial\mathbb{D}$. Then f has non-tangential limit $\tau \in \partial\mathbb{D}$ at σ. Moreover, for all $\zeta \in \mathbb{D}$ one has

$$\frac{|\tau - f(\zeta)|^2}{1-|f(\zeta)|^2} \le \alpha \frac{|\sigma-\zeta|^2}{1-|\zeta|^2} \,. \tag{8.2}$$

The latter statement admits an interesting geometrical interpretation. Recall that the *horocycle* $E(\sigma, R)$ contained in $\mathbb{D}$ of *center* $\sigma \in \partial\mathbb{D}$ and *radius* $R > 0$ is the set

$$E(\sigma, R) = \left\{\zeta \in \mathbb{D} \;\middle|\; \frac{|\sigma-\zeta|^2}{1-|\zeta|^2} < R\right\} \,.$$

Geometrically, $E(\sigma, R)$ is an euclidean disk of radius $R/(R+1)$ internally tangent to $\partial\mathbb{D}$ at σ. Therefore (8.2) becomes $f\big(E(\sigma,R)\big) \subseteq E(\tau, \alpha R)$ for all $R > 0$, and the existence of the non-tangential limit more or less follows from (8.2) and from the fact that horocycles touch the boundary in exactly one point.

As we remarked in Chapter 6, a horocycle can be thought of as the limit of Poincaré disks of fixed euclidean radius and centers going to the boundary; so it makes sense to think of horocycles as Poincaré disks centered at the boundary, and of Julia's lemma as a Schwarz-Pick lemma at the boundary. This suggests that α might be considered as a sort of dilation coefficient: f expands horocycles by a ratio of α. If σ were an internal point and $E(\sigma, R)$ an infinitesimal euclidean disk actually centered at σ, one then would be tempted to say that α is (the absolute value of) the derivative of f at σ. This is exactly the content of the classical *Julia-Wolff-Carathéodory theorem:*

Theorem 8.3 (Julia-Wolff-Carathéodory)**.** Let $f\colon \mathbb{D} \to \mathbb{D}$ be a bounded holomorphic function such that

$$\liminf_{\zeta\to\sigma} \frac{1-|f(\zeta)|}{1-|\zeta|} = \alpha < +\infty$$

for some $\sigma \in \partial\mathbb{D}$, and let $\tau \in \partial\mathbb{D}$ be the non-tangential limit of f at σ. Then both the incremental ratio $\big(\tau - f(\zeta)\big)/(\sigma - \zeta)$ and the derivative $f'(\zeta)$ have non-tangential limit $\alpha\bar{\sigma}\tau$ at σ.

So condition (8.1) forces the existence of the non-tangential limit of both f and its derivative at σ. This is the result of the work of several people: Julia [156], Wolff [264], Carathéodory [82], Landau and Valiron [165], Nevanlinna [185] and others. We refer, for example, to [79] and [2] for proofs, history and applications.

8.2 Generalizations to several variables

It was first remarked by Korányi and Stein ([163, 164, 246]) in extending Fatou's theorem to several complex variables, that the notion of non-tangential limit is not the right one to consider for domains in $\mathbb{C}^n$. In fact, it turns out that two notions are needed, and to introduce them it is useful to investigate the notion of non-tangential limit in the unit disk $\mathbb{D}$.

The non-tangential limit can be defined in two equivalent ways. A function $f\colon \mathbb{D} \to \mathbb{C}$ is said to have *non-tangential limit* $L \in \mathbb{C}$ at $\sigma \in \partial\mathbb{D}$ if $f\big(\gamma(t)\big) \to L$ as $t \to 1^-$ for every curve $\gamma\colon [0,1) \to \mathbb{D}$ such that $\gamma(t)$ converges to σ non-tangentially as $t \to 1^-$. In $\mathbb{C}$, this is equivalent to having that $f(\zeta) \to L$ as $\zeta \to \sigma$ staying inside any *Stolz region* $K(\sigma, M)$ of *vertex* σ and *amplitude* $M > 1$, where

$$K(\sigma, M) = \left\{ \zeta \in \mathbb{D} \;\middle|\; \frac{|\sigma - \zeta|}{1-|\zeta|} < M \right\} ,$$

since Stolz regions are angle-shaped nearby the vertex σ, and the angle is going to π as $M \to +\infty$. These two approaches lead to different notions in several variables.

In the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ the natural generalization of a Stolz region is the *Korányi region* $K(p, M)$ of *vertex* $p \in \partial\mathbb{B}^n$ and *amplitude* $M > 1$ given by

$$K(p, M) = \left\{ z \in \mathbb{B}^n \;\middle|\; \frac{|1 - \langle z, p\rangle|}{1 - \|z\|} < M \right\} ,$$

where $\|\cdot\|$ denote the euclidean norm and $\langle\cdot\,,\cdot\rangle$ the canonical hermitian product. Then a function $f\colon \mathbb{B}^n \to \mathbb{C}$ has *K-limit* (or *admissible limit*) $L \in \mathbb{C}$ at $p \in \partial\mathbb{B}^n$, and we write

$$K\text{-}\lim_{z\to p} f(z)$$

if $f(z) \to L$ as $z \to p$ staying inside any Korányi region $K(\sigma, M)$. A Korányi region $K(p, M)$ approaches the boundary non-tangentially along the normal direction at p but tangentially along the complex tangential directions at p. Therefore, having K-limit is stronger than having non-tangential limit. However, as first noticed by Korányi and Stein, for holomorphic functions of several complex variables one is often able to prove the existence of K-limits. For instance, the best generalization of Julia's lemma to $\mathbb{B}^n$ is the following result (proved by Hervé [143] in terms of non-tangential limits and by Rudin [235] in general):

Theorem 8.4 (Rudin [235])**.** Let $f\colon \mathbb{B}^n \to \mathbb{B}^m$ be a holomorphic map such that

$$\liminf_{z\to p} \frac{1-\|f(z)\|}{1-\|z\|} = \alpha < +\infty \ ,$$

for some $p \in \partial\mathbb{B}^n$. Then f admits K-limit $q \in \partial\mathbb{B}^m$ at p, and furthermore for all $z \in \mathbb{B}^n$ one has

$$\frac{|1-\langle f(z), q\rangle|^2}{1-\|f(z)\|^2} \le \alpha \frac{|1-\langle z, p\rangle|^2}{1-\|z\|^2} \ .$$

We defined Korányi regions for more general domains in $\mathbb{C}^n$ than the unit ball, using the Kobayashi distance, in Definition 7.12. For the sake of completeness we recall here that given $D \Subset \mathbb{C}^n$ a complete hyperbolic domain and denoting by k_D its Kobayashi distance, a *K-region* of *vertex $x \in \partial D$, amplitude $M > 1$*, and *pole $z_0 \in D$* is the set

$$K_{D,z_0}(x, M) = \left\{ z \in D \mid \limsup_{w\to x} [k_D(z, w) - k_D(z_0, w)] + k_D(z_0, z) < \log M \right\} .$$

This definition clearly depends on the pole z_0. However, this dependence is not too relevant since changing the pole corresponds to shifting amplitudes. Since K-regions are a natural generalization of Korányi regions they allow us to generalize the notion of K-limit. A function $f\colon D \to \mathbb{C}^m$ has *K-limit* L at $x \in \partial D$ if $f(z) \to L$ as $z \to p$ staying inside any K-region of vertex x. The best generalization of Julia's lemma in this setting is then the following, due to Abate:

Theorem 8.5 (Abate [3])**.** Let $D \Subset \mathbb{C}^n$ be a complete hyperbolic domain and let $z_0 \in D$. Let $f\colon D \to \mathbb{D}$ be a holomorphic function and let $x \in \partial D$ be such that

$$\liminf_{z\to x} [k_D(z_0, z) - k_{\mathbb{D}}(0, f(z))] < +\infty \ .$$

Then f admits K-limit $\tau \in \partial D$ at x.

In order to obtain a complete generalization of the Julia-Wolff-Carathéodory for $\mathbb{B}^n$, Rudin introduced a different notion of limit, still stronger than non-tangential limit but weaker than K-limit. This notion is closely related to the other characterization of non-tangential limit in one variable we mentioned at the beginning of this section.

A crucial one-variable result relating limits along curves and non-tangential limits is *Lindelöf's theorem.* Given $\sigma \in \partial\mathbb{D}$, a *$\sigma$-curve* is a continuous curve $\gamma\colon [0, 1) \to \mathbb{D}$ such that $\gamma(t) \to \sigma$ as $t \to 1^-$. Then Lindelöf [172] proved that if a bounded holomorphic function $f\colon \mathbb{D} \to \mathbb{C}$ admits limit $L \in \mathbb{C}$ along a given σ-curve then it admits limit L along all non-tangential σ-curves — and thus it has non-tangential limit L at σ.

In generalizing this result to several complex variables, Čirka [84] realized that for a bounded holomorphic function the existence of the limit along a (suitable) p-curve (where $p \in \partial\mathbb{B}^n$) implies not only the existence of the non-tangential limit, but also the existence of the limit along any curve belonging to a larger class of curves, including some tangential ones — but it does not in general imply the existence of the K-limit. To describe the version (due to Rudin [235]) of Čirka's result we shall state here, let us introduce a bit of terminology.

Let $p \in \partial\mathbb{B}^n$. As before, a *p-curve* is a continuous curve $\gamma\colon [0,1) \to \mathbb{B}^n$ such that $\gamma(t) \to p$ as $t \to 1^-$. A p-curve is *special* if

$$\lim_{t\to 1^-} \frac{\|\gamma(t) - \langle\gamma(t),p\rangle p\|^2}{1 - |\langle\gamma(t),p\rangle|^2} = 0 . \tag{8.3}$$

Given $M > 1$, a special curve is *M-restricted* if

$$\frac{|1 - \langle\gamma(t),p\rangle|}{1 - |\langle\gamma(t),p\rangle|} < M$$

for all $t \in [0,1)$. We also say that γ is *restricted* if it is M-restricted for some $M > 1$. In other words, γ is restricted if and only if $t \mapsto \langle\gamma(t),p\rangle$ goes to 1 non-tangentially in $\mathbb{D}$.

It is not difficult to see that non-tangential curves are special and restricted; on the other hand, a special restricted curve approaches the boundary non-tangentially along the normal direction, but it can approach the boundary tangentially along complex tangential directions. Furthermore, a special M-restricted p-curve is eventually contained in any $K(p,M')$ with $M' > M$, and conversely a special p-curve eventually contained in $K(p,M)$ is M-restricted. However, $K(p,M)$ can contain p-curves that are restricted but not special: for these curves the limit in (8.3) might be a strictly positive number.

With these definitions in place, we shall say that a function $f\colon \mathbb{B}^n \to \mathbb{C}$ has *restricted K-limit* (or *hypoadmissible limit*) $L \in \mathbb{C}$ at $p \in \partial\mathbb{B}^n$, and we shall write

$$K'\text{-}\lim_{z\to p} f(z) = L ,$$

if $f\big(\gamma(t)\big) \to L$ as $t \to 1^-$ for any special restricted p-curve $\gamma\colon [0,1) \to \mathbb{B}^n$. It is clear that the existence of the K-limit implies the existence of the restricted K-limit, that in turns implies the existence of the non-tangential limit; but none of these implications can be reversed (see, e.g., [235] for examples in the ball).

Finally, we say that a function $f\colon \mathbb{B}^n \to \mathbb{C}$ is *K-bounded* at $p \in \partial\mathbb{B}^n$ if it is bounded in any Korányi region $K(p,M)$, where the bound can depend on $M > 1$. Then Rudin's version of Čirka's generalization of Lindelöf's theorem is the following:

Theorem 8.6 (Rudin [235])**.** Let $f\colon \mathbb{B}^n \to \mathbb{C}$ be a holomorphic function K-bounded at $p \in \partial\mathbb{B}^n$. Assume there is a special restricted p-curve $\gamma^o\colon [0,1) \to \mathbb{B}^n$ such that $f\big(\gamma^o(t)\big) \to L \in \mathbb{C}$ as $t \to 1^-$. Then f has restricted K-limit L at p.

As before, it is possible to generalize this approach to a domain $D \subset \mathbb{C}^n$ different from the ball. A very precise and systematic presentation, providing clear proofs, details and examples, of various aspects of the problem of generalization of the classical Julia-Wolff-Carathéodory theorem to domains in several complex variables can be found in [9].

For the sake of simplicity we state here only the definitions needed to state Abate's version of Lindelöf's theorem in this setting. Given a point $x \in \partial D$, a *x-curve* is again a continuous curve $\gamma\colon [0,1) \to D$ so that $\lim_{t\to 1^-}\gamma(t) = x$. A *projection device* at $x \in \partial D$ is the data of: a neighbourhood U of x in $\mathbb{C}^n$, a holomorphic embedded disk $\varphi_x\colon \mathbb{D} \to D\cap U$, such that $\lim_{\zeta\to 1}\varphi_x(\zeta) = x$, a family $\mathcal{P}$ of x-curves in $D \cap U$, and a device associating to every x-curve $\gamma \in \mathcal{P}$ a 1-curve $\widetilde{\gamma}_x$ in $\mathbb{D}$, or equivalently a x-curve $\gamma_x = \varphi_x \circ \widetilde{\gamma}_x$ in $\varphi_x(\mathbb{D})$. If D is equipped with a projection device at $x \in \partial D$, then a curve $\gamma \in \mathcal{P}$ is *special* if $\lim_{t\to 1^-} k_{D\cap U}(\gamma(t),\gamma_x(t)) = 0$, and it is *restricted* if γ_x is a non-tangential 1-curve in $\mathbb{D}$. A function $f\colon D \to C$ has *restricted K-limit*

$L \in \mathbb{C}$ *at* x if $\lim_{t\to 1^-} f(\gamma(t)) = L$ for all special restricted x-curves. A projection device is *good* if: for any $M > 1$ there is a $M' > 1$ so that $\varphi_x(K(1,M)) \subset K_{D\cap U,z_0}(x, M')$, and for any special restricted x-curve γ there exists $M_1 = M_1(\gamma)$ such that $\lim_{t\to 1^-} k_{K_{D\cap U,z_0}(x,M_1)}(\gamma(t), \gamma_x(t)) = 0$. Good projection devices exist, and several examples can be found for example in [9]. Finally, we say that a function $f\colon D \to \mathbb{C}$ is K-*bounded* at $p \in \partial\mathbb{B}^n$ if it is bounded in any K-region $K_{D,z_0}(x, M)$, where the bound can depend on $M > 1$.

With these definitions we can state the generalization of Lindelöf principle given by Abate.

Theorem 8.7 (Abate [9])**.** Let $D \subset \mathbb{C}^n$ be a domain equipped with a good projection device at $x \in \partial D$. Let $f\colon D \to \mathbb{D}$ be a holomorphic function K-bounded at x. Assume there is a special restricted x-curve $\gamma^o\colon [0,1) \to D$ such that $f\big(\gamma^o(t)\big) \to L \in \mathbb{C}$ as $t \to 1^-$. Then f has restricted K-limit L at x.

We can now deal with the generalization of the Julia-Wolff-Carathéodory theorem to several complex variables. With respect to the one-dimensional case there is an obvious difference: instead of only one derivative one has to deal with a whole (Jacobian) matrix of them, and there is no reason they should all behave in the same way. And indeed they do not, as shown in Rudin's version of the Julia-Wolff-Carathéodory theorem for the unit ball:

Theorem 8.8 (Rudin [235])**.** Let $f\colon \mathbb{B}^n \to \mathbb{B}^m$ be a holomorphic map such that

$$\liminf_{z\to p} \frac{1 - \|f(z)\|}{1 - \|z\|} = \alpha < +\infty\ ,$$

for some $p \in \partial\mathbb{B}^n$. Then f admits K-limit $q \in \partial\mathbb{B}^m$ at p. Furthermore, if we set $f_q(z) = \big\langle f(z), p\big\rangle q$ and denote by df_z the differential of f at z, we have:

(i) the function $\big[1 - \big\langle f(z), q\big\rangle\big]/[1 - \langle z, p\rangle]$ is K-bounded and has restricted K-limit α at p;

(ii) the map $[f(z) - f_q(z)/[1 - \langle z, p\rangle]^{1/2}$ is K-bounded and has restricted K-limit O at p;

(iii) the function $\big\langle df_z(p), q\big\rangle$ is K-bounded and has restricted K-limit α at p;

(iv) the map $[1 - \langle z, p\rangle]^{1/2} d(f - f_q)_z(p)$ is K-bounded and has restricted K-limit O at p;

(v) if v is any vector orthogonal to p, the function $\big\langle df_z(v), q\big\rangle\big/[1 - \langle z, p\rangle]^{1/2}$ is K-bounded and has restricted K-limit 0 at p;

(vi) if v is any vector orthogonal to p, the map $d(f - f_q)_z(v)$ is K-bounded at p.

In the last twenty years this theorem (as well as Theorems 8.4 and 8.6) has been extended to domains much more general than the unit ball: for instance, strongly pseudoconvex domains [2, 3, 5], convex domains of finite type [22], and polydisks [7] and [25], (see also [9] and references therein).

We end this section with the general version of the Julia-Wolff-Carathódory theorem obtained by Abate in [9] for a complete hyperbolic domain D in $\mathbb{C}^n$. To formulate it, we need to introduce a couple more definitions. A projection device at $x \in \partial D$ is *geometrical* if there is a holomorphic function $\widetilde{p}_x\colon D \cap U \to \mathbb{D}$ such that $\widetilde{p}_x \circ \varphi_x = \mathrm{Id}_{\mathbb{D}}$ and $\widetilde{\gamma}_x = \widetilde{p}_x \circ \gamma$ for all $\gamma \in \mathcal{P}$. A geometrical projection device at x is *bounded* if $d(z, \partial D)/|1 - \widetilde{p}_x(z)|$ is bounded in $D \cap U$, and $|1 - \widetilde{p}_x(z)|/d(z, \partial D)$ is K-bounded in $D \cap U$. The statement is then the following, where κ_D denotes the Kobayashi metric.

Theorem 8.9 (Abate [9]). Let $D \subset \mathbb{C}^n$ be a complete hyperbolic domain equipped with a bounded geometrical projection device at $x \in \partial D$. Let $f\colon D \to \mathbb{D}$ be a holomorphic function such that

$$\liminf_{z \to x} [k_D(z_0, z) - k_{\mathbb{D}}(0, f(z))] = \frac{1}{2} \log \beta < +\infty .$$

Then for every $v \in \mathbb{C}^n$ and every $s \geq 0$ such that $d(z, \partial D)^s \kappa_D(z; v)$ is K-bounded at x the function

$$d(z, \partial D)^{s-1} \frac{\partial f}{\partial v} \tag{8.4}$$

is K-bounded at x. Moreover, if $s > \inf\{s \geq 0 \mid d(z, \partial D)^s \kappa_D(z; v) \text{ is } K\text{-bounded at } x\}$, then (8.4) has vanishing K-limit at x.

Depending on more specific properties of the projection device, it is indeed possible to deduce the existence of restricted K-limits, see [9, Section 5].

Further generalizations of Julia-Wolff-Carathéodory theorem have been obtained in infinite-dimensional Banach and Hilbert spaces, and we refer to [103, 105, 107, 128, 178, 249, 261, 262, 263, 271] and references therein, and very recently also in the non-commutative setting [41].

8.3 Julia-Wolff-Carathéodory theorem for infinitesimal generators

We conclude this chapter focusing on a different kind of generalization in several complex variables: infinitesimal generators of one-parameter semigroups of holomorphic self-maps of $\mathbb{B}^n$.

We consider $\mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$, the space of holomorphic self-maps of $\mathbb{B}^n$, endowed with the usual compact-open topology. A *one-parameter semigroup* of holomorphic self-maps of $\mathbb{B}^n$ is a continuous semigroup homomorphism $\Phi\colon \mathbb{R}^+ \to \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$. In other words, writing φ_t instead of $\Phi(t)$, we have $\varphi_0 = \mathrm{Id}_{\mathbb{B}^n}$, the map $t \mapsto \varphi_t$ is continuous, and the semigroup property $\varphi_t \circ \varphi_s = \varphi_{t+s}$ holds. An introduction to the theory of one-parameter semigroups of holomorphic maps can be found in [2, 215, 242].

One-parameter semigroups can be seen as the flow of a vector field (see, e.g., [6]). Given a one-parameter semigroup Φ, it is possible to prove that there exists a holomorphic map $G\colon \mathbb{B}^n \to \mathbb{C}^n$, the *infinitesimal generator* of the semigroup, such that

$$\frac{\partial \Phi}{\partial t} = G \circ \Phi . \tag{8.5}$$

It should be kept in mind, when reading the literature on this subject, that in some papers (e.g., in [107, 214]) there is a change of sign with respect to our definition, due to the fact that the infinitesimal generator is defined there as the solution of the equation

$$\frac{\partial \Phi}{\partial t} + G \circ \Phi = O .$$

A Julia's lemma for infinitesimal generators was proved by Elin, Reich and Shoikhet in [107] in 2008, assuming that the radial limit of the generator at a point $p \in \partial \mathbb{B}^n$ vanishes:

Theorem 8.10 (Elin, Reich, Shoikhet [107, Theorem p. 403]). *Let $G\colon \mathbb{B}^n \to \mathbb{C}^n$ be the infinitesimal generator on $\mathbb{B}^n$ of a one-parameter semigroup $\Phi = \{\varphi_t\}$, and let $p \in \partial \mathbb{B}^n$ be such that*

$$\lim_{t \to 1^-} G(tp) = O . \tag{8.6}$$

Then the following assertions are equivalent:

(I) $\alpha = \liminf_{t\to 1^-} \operatorname{Re}\frac{\langle G(tp),p\rangle}{t-1} < +\infty$;

(II) $\beta = 2\sup_{z\in\mathbb{B}^n} \operatorname{Re}\left[\frac{\langle G(z),z\rangle}{1-\|z\|^2} - \frac{\langle G(z),p\rangle}{1-\langle z,p\rangle}\right] < +\infty$;

(III) *there exists* $\gamma \in \mathbb{R}$ *such that for all* $z \in \mathbb{B}^n$ *we have* $\frac{|1-\langle\varphi_t(z),p\rangle|^2}{1-\|\varphi_t(z)\|^2} \le e^{\gamma t}\frac{|1-\langle z,p\rangle|^2}{1-\|z\|^2}$.

Furthermore, if any of these assertions holds then $\alpha = \beta = \inf\gamma$, *and we have*

$$\lim_{t\to 1^-}\frac{\langle G(tp),p\rangle}{t-1} = \beta\ . \tag{8.7}$$

If (8.6) and any (and hence all) of the equivalent conditions (I)–(III) holds, $p \in \partial\mathbb{B}^n$ is called a *boundary regular null point* of G with *dilation* $\beta \in \mathbb{R}$.

This result suggested that a Julia-Wolff-Carathéodory theorem could hold for infinitesimal generators along the line of Rudin's Theorem 8.8. A first partial generalization has been achieved by Bracci and Shoikhet in [60]. In collaboration with Abate, in [18] we have been able to give a full generalization of Julia-Wolff-Carathéodory theorem for infinitesimal generators, proving the following result.

Theorem 8.11 (Abate, R. [18])**.** *Let* $G\colon \mathbb{B}^n \to \mathbb{C}^n$ *be an infinitesimal generator on* $\mathbb{B}^n$ *of a one-parameter semigroup, and let* $p \in \partial\mathbb{B}^n$. *Assume that*

$$\frac{\langle G(z),p\rangle}{\langle z,p\rangle - 1} \qquad \text{is } K\text{-bounded at } p \tag{8.8}$$

and

$$\frac{G(z) - \langle G(z),p\rangle p}{(\langle z,p\rangle - 1)^\gamma} \qquad \text{is } K\text{-bounded at } p \text{ for some } 0<\gamma<1/2. \tag{8.9}$$

Then $p \in \partial\mathbb{B}^n$ *is a boundary regular null point for* G. *Furthermore, if* β *is the dilation of* G *at* p *then:*

(i) *the function* $\langle G(z),p\rangle/(\langle z,p\rangle - 1)$ *(is* K*-bounded and) has restricted* K*-limit* β *at* p*;*

(ii) *if* v *is a vector orthogonal to* p, *the function* $\langle G(z),v\rangle/(\langle z,p\rangle - 1)^\gamma$ *is* K*-bounded and has restricted* K*-limit* 0 *at* p*;*

(iii) *the function* $\langle dG_z(p),p\rangle$ *is* K*-bounded and has restricted* K*-limit* β *at* p*;*

(iv) *if* v *is a vector orthogonal to* p, *the function* $(\langle z,p\rangle - 1)^{1-\gamma}\langle dG_z(p),v\rangle$ *is* K*-bounded and has restricted* K*-limit* 0 *at* p*;*

(v) *if* v *is a vector orthogonal to* p, *the function* $\langle dG_z(v),p\rangle \,/\, (\langle z,p\rangle - 1)^\gamma$ *is* K*-bounded and has restricted* K*-limit* 0 *at* p.

(vi) *if* v_1 *and* v_2 *are vectors orthogonal to* p *the function* $(\langle z,p\rangle - 1)^{1/2-\gamma}\langle dG_z(v_1),v_2\rangle$ *is* K*-bounded at* p.

Idea of the proof. Statement (i) follows immediately from our hypotheses, thanks to Theorems 8.6 and 8.10. Statement (iii) follows by standard arguments, and (iv) follows from (ii), again by standard arguments.

The main point is the proof of part (ii). By Theorem 8.6, it suffices to compute the limit along a special restricted curve. We use the curve

$$\sigma(t) = tp + e^{-i\theta}\varepsilon(1-t)^{1-\gamma}v$$

which is always restricted, and *it is special if and only if* $\gamma < 1/2$. We then plug (i) and this curve into Theorem 8.10.(II), and we then let $\varepsilon \to 0^+$, using θ to get rid of the real part.

Statement (v) follows from (i), (ii) and by Theorem 8.10 using somewhat delicate arguments involving a curve of the form

$$\gamma(t) = \big(t + ic(1-t)\big)p + \eta(t)v ,$$

where $1 - t < |\eta(t)|^2 < 1 - |t + ic(1-t)|^2$, and the argument of $\eta(t)$ is chosen suitably.

□

A first difference with respect to Theorem 8.8 is that we have to assume (8.8) and (8.9) as separate hypotheses, whereas they appear as part of Theorem 8.8.(i) and (ii). Indeed, when dealing with holomorphic maps, conditions (8.8) and (8.9) are a consequence of the equivalent of condition (I) in Theorem 8.10, but in that setting the proof relies in the fact that there we have holomorphic *self-maps* of the ball. In our context, (8.9) is *not* a consequence of Theorem 8.10.(I), as shown in [18, Example 1.2]; and it also seems that (8.8) is stronger than Theorem 8.10.(I).

A second difference is the exponent $\gamma < 1/2$. Bracci and Shoikhet proved Theorem 8.11 with $\gamma = 1/2$ but they couldn't prove the statements about restricted K-limits in cases (ii), (iv) and (v). This is due to an obstruction, which is not just a technical problem, but an inevitable feature of the theory. As mentioned in the sketch of the proof, in showing the existence of restricted K-limits, the curves one would like to use for obtaining the exponent 1/2 in the statements are *restricted but not special*, in the sense that the limit in (8.3) is a strictly positive (though finite) number. Actually the exponent 1/2 might not be the right one to consider in the setting of infinitesimal generators, as shown in [18, Example 1.2].

An exact analogue of Theorem 8.8 with $\gamma = 1/2$ can be recovered assuming a slightly stronger hypothesis on the infinitesimal generator. In fact, under the assumptions (8.8) and (8.9) we have

$$\frac{\big\langle G\big(\sigma(t)\big), p\big\rangle}{\langle \sigma(t), p\rangle - 1} = \beta + o(1) \tag{8.10}$$

as $t \to 1^-$ for any special restricted p-curve $\sigma\colon [0,1) \to \mathbb{B}^n$. Following ideas introduced in [109, 104, 106] in the context of the unit disk, we would like to have

$$\frac{\big\langle G\big(\sigma(t)\big), p\big\rangle}{\langle \sigma(t), p\rangle - 1} = \beta + o\big((1-t)^\alpha\big) \tag{8.11}$$

for some $\alpha > 0$ and any special restricted p-curve $\sigma\colon [0,1) \to \mathbb{B}^n$ such that $\langle \sigma(t), p\rangle \equiv t$. If there is $\alpha > 0$ such that (8.11) is satisfied, p is said to be a *Hölder boundary null point*. Using this notion we obtain the following result.

Theorem 8.12 (Abate, R. [18])**.** Let $G\colon \mathbb{B}^n \to \mathbb{C}^n$ be the infinitesimal generator on $\mathbb{B}^n$ of a one-parameter semigroup, and let $p \in \partial\mathbb{B}^n$. Assume that

$$\frac{\langle G(z), p\rangle}{\langle z, p\rangle - 1} \quad \text{and} \quad \frac{G(z) - \langle G(z), p\rangle p}{(\langle z, p\rangle - 1)^{1/2}}$$

are K-bounded at p, and that p is a Hölder boundary null point. Then the statement of Theorem 8.11 holds with $\gamma = 1/2$.

Examples of infinitesimal generators with a Hölder boundary null point and satisfying the hypotheses of Theorem 8.12 are provided in [18].

Perspectives

I will formulate here some questions related to the works presented in this manuscript.

Local Dynamics

Arithmetic conditions for holomorphic normalization

Holomorphic Linearization

Yoccoz proved the optimality of the Brjuno condition in dimension 1 for the class of quadratic polynomials, as we recalled in Theorem 1.13. In higher dimension it is still unknown whether the Brjuno condition is optimal for some classes of non-resonant polynomials. A first issue is to find the correct class to study. For instance, it is straightforward that if we consider direct products germs in two variables, that is $F\colon (\mathbb{C}^2, O) \to (\mathbb{C}^2, O)$ of the form $F(z,w) = (f_1(z), f_2(w))$, then we can directly apply Yoccoz's result separately to each component. Therefore we obtain that a quadratic polynomial map

$$(z, w) \mapsto (\lambda_1 z + z^2, \lambda_2 w + w^2), \tag{8.12}$$

with $|\lambda_j| = 1$ for $j = 1, 2$, is holomorphically linearizable if and only if each λ_j satisfies the one-dimensional Brjuno condition (1.5). This is always satisfied if (λ_1, λ_2) satisfies the reduced Brjuno condition given in Definition 1.45, thanks to the fact that $\omega_{\lambda_j}(m) \geq \widetilde{\omega}_{\lambda_1,\lambda_2}(m)$ for each $m \geq 2$. However the contrary is not true in general and, more importantly, it is not known whether there is any relation between the holomorphic linearizability of the quadratic polynomial (8.12) and that of a germ of biholomorphism of $(\mathbb{C}^2, O)$ with differential at the origin equal to $\mathrm{Diag}(\lambda_1, \lambda_2)$. It therefore seems more reasonable to start such investigation on a richer class, like that given by skew-products as defined in (4.1).

Holomorphic Normalization

In [66, 67], Brjuno was able to prove results ensuring the existence of a holomorphic normalization for resonant germs of holomorphic vector fields of $\mathbb{C}^n$ with a singular point at the origin, under a condition on the formal normal forms, called *condition A*, and an adapted arithmetical condition on the spectrum of the linear term of the vector field, analog to our *reduced Brjuno condition* defined in Definition 1.45. It is a natural interesting question to investigate the possibility of generalizing such results, or of finding analogous ones, to the case of germs of resonant biholomorphisms. One key fact to be kept in mind will be the differences between germs of vector fields and germs of biholomorphisms found in [208].

Another interesting and promising approach to the problem of holomorphic normalization is given by the use of Écalle's *arborification theory*. Recently, Fauvet, Menous and Sauzin revisited in [110] the linearization problem for non-resonant holomorphic diffeomorphisms in dimension

1 and, using a part of Écalle's arborification theory, they obtained explicit formulas, indexed by *forests*, for the linearizing function. This allowed them to recover Yoccoz's bound for the convergence radius of the linearizing function under the Brjuno condition. It seems to be possible to use the same method to obtain explicit formulas for the linearizing map of non-resonant biholomorphisms in higher dimension and to use them to find a bound on the convergence radius as well, again under the Brjuno condition. Together with Fauvet, we think that such techniques can be used to obtain explicit formulas for the normalizations of resonant germs of biholomorphisms and to study their convergence under the reduced Brjuno condition.

Tangent to the identity germs

A first natural question related to the results presented in Section 2.3 is whether they can be generalized in dimension 3 and higher, and in which way. In fact, in dimension $n \geq 3$ the existence of a parabolic curve for a tangent to the identity germ $F\colon (\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$ having a formal invariant curve Γ with $F|_\Gamma \not\equiv \mathrm{Id}$ might not always be possible. For instance, Abate and Tovena found in [23] examples of tangent to the identity germs in $\mathbb{C}^3$ without parabolic curves asymptotic to formal separatrices. It is still reasonable to expect that the examples in [23] are somehow exceptional and to investigate on the existence of *parabolic submanifolds* of dimension $s < n$ or parabolic domains. The further natural general case to consider is that of a germ of biholomorphism of $(\mathbb{C}^n, O)$ with $n \geq 3$ having a formal invariant curve Γ under the necessary hypotheses for the existence of orbits converging to the origin asymptotically to Γ.

A more general open problem is to completely describe the local dynamics of a tangent to the identity germ in dimension $n \geq 2$ in a punctured neighbourhood of the fixed point, possibly using characteristic directions, directors, indices and/or other invariants.

This could also shed some light on the other natural, still open, question in higher dimension concerning the topological classification of tangent to the identity germs. More precisely we would like to know whether an analog of Theorem 1.19 holds. A first class of germs to be considered could be that of non-dicritical tangent to the identity germ such that all characteristic directions are non-degenerate.

Resonant germs

The study of the local dynamics of resonant non-linearizable and non-tangent to the identity germs of biholomorphisms started only recently as we recalled in Chapter 3. The new phenomena that can occur in this setting, like the existence of elliptic germs having attracting basins with a parabolic-like dynamics, naturally lead to the question of classifying all possible behaviours that can occur in the resonant non-linearizable case.

A first class to study is that of *degenerate* multi-resonant germs of weighted order k_0. Such class splits into two subcases: either the parabolic shadow is a tangent to the identity germ of multiplicity $k_0 + h > k_0$ at the origin or it is the identity. In the first case, we need to understand the information that this gives on the local dynamics of germs in normal form, while in the second one the dynamics for normal forms can be readily described. In both cases the further question is whether we can infer on the local dynamics of general germs not in normal form.

The resonant setting where the resonances are not finitely generated, that is the germ is not multi-resonant, has not been attacked yet. A good starting point could be to investigate the local dynamics of germs in normal forms using the notions of *minimal* and *cominimal* elements introduced in [208].

Formally linearizable germs

Another class of germs of biholomorphisms whose local dynamics is still not understood is that of formally but not holomorphically linearizable germs. More precisely, also due to the lack of knowledge in the description of the dynamics of tangent to the identity germs, we do not know whether Pérez-Marco's Theorem 1.21 can be fully generalized in higher dimension, nor whether other new phenomena might occur.

A first generalization of Perez-Marco's *hedgehogs* has been recently given in $\mathbb{C}^2$ by Firsova, Lyubich, Radu and Tanase in [118] and Lyubich, Radu and Tanase in [177]. In their construction only one of the eigenvalues can have modulus equal to 1. A natural open question is whether a hedgehog always exists in this setting if all eigenvalues have modulus 1 but are not roots of unity.

Global Dynamics

Parabolic invariant fiber for skew-products in $\mathbb{C}^2$

In the presentation we gave in Chapter 4 of the dynamics for a polynomial skew-product $F(z, w) = (f(z, w), g(w))$ in $\mathbb{C}^2$ near an invariant fiber, we did not provide a full description for the case of a parabolic invariant fiber.

It is evident that in the case of a polynomial skew-product with a parabolic invariant fiber, Question 1 as formulated in Section 4.1 makes no sense. In fact, since the fixed point c for the polynomial $g(w)$ lies on the boundary of the Fatou component $\mathcal{B}_{g,c}$ for g, the Fatou components of the polynomial $f_c(z) := f(z, c)$ cannot be obtained as intersections of Fatou components of F with the parabolic invariant fiber $\{w = c\}$.

We can however weaken the definition of *bulging* by saying that a Fatou component Ω_{f_c} of the polynomial f_c *bulges* if there exists a Fatou component Ω for F such that $\overline{\Omega} \cap \{w = c\} = \Omega_{f_c}$, that is $\Omega_{f_c} \subset \partial\Omega$. Then the question whether all Fatou components of f_c bulge makes sense and we can start addressing it by using the results of Ueda in [251], for the attracting Fatou components, and the results we obtained with López-Hernanz, Ribón and Sanz-Sánchez in [174] for the Siegel and the parabolic components. Understanding the dynamics near a parabolic Fatou component for f_c might also give non-trivial examples of tangent to the identity germs where the local dynamics in a full neighbourhood of the origin is understood.

Coming back to the results we obtained in [33] on the existence of wandering domains, there still are several questions to consider: how prevalent is this behaviour in the parameter space? Is it a very special phenomenon, or is it typical? Can the orbit of the critical locus of the map somehow detect that there is a wandering domain? How many wandering orbits does such a map typically have? Can we determine if a given map admits a wandering domain? How does the wandering domain affect the global dynamics of the map? Are there other new phenomena, absent in dimension one, that are present when a map $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ has a wandering domain? Is it possible to find a Hénon map with a wandering domain?

Parabolic implosion in higher dimension

One essential ingredient of the construction of polynomial map with a wandering domain that we obtained in [33] involves adapting *parabolic implosion* to higher dimension in the setting of skew-products. Parabolic implosion is a phenomenon observed by Douady and Hubbard in their original study of quadratic polynomials and the Mandelbrot set, as we briefly recall here.

Let $f_c\colon \mathbb{C} \to \mathbb{C}$ be the quadratic family $f_c\colon z \mapsto z^2 + c$, parametrized by $c \in \mathbb{C}$. Let K_c be the *filled Julia set of* f_c, that is, the set of all points in $\mathbb{C}$ having bounded orbit under f_c. The set of all parameters $c \in \mathbb{C}$ such that K_c is connected is the *Mandelbrot set*, denoted by $\mathcal{M}$. There is a dense set of parameters in the boundary of $\mathcal{M}$ called *parabolic parameters*. These are values of c such that the polynomial f_c has a *parabolic cycle* and the cusp of $\mathcal{M}$, at $c = 1/4$, is one such parameter. Douady showed that K_c moves discontinuously, with respect to the Hausdorff metric, when c is a parabolic parameter, due to the phenomenon called of *parabolic implosion* [93]. In complex dimension one, there is an established *Lavaurs theory* about parabolic implosion.

In [40], Bedford, Smillie and Ueda established an analogous result for Hénon maps. Their work lay the foundations for *semi-parabolic implosion* in dimension two, that is, parabolic implosion with an eigenvalue of modulus strictly less than 1. In [44], Bianchi then adapted the Bedford, Smillie and Ueda strategies for a perturbation of a class of holomorphic endomorphisms tangent to identity, and he established a two-dimensional Lavaurs theorem for such a class, but the general case remains open. Understanding the situation in cases such as skew-products is a good starting point for studying the question of a general theory for parabolic implosion in dimension two.

Fatou components for automorphisms in $\mathbb{C}^k$

The results presented in Chapter 5, and in particular the results we obtained in [64], naturally lead to the question of classifying the kind of possible topological types for periodic Fatou components of holomorphic automorphisms of $\mathbb{C}^k$, for $k \geq 2$.

Hénon maps

Several precise questions have been formulated by Eric Bedford in [36] concerning Fatou components of *conservative* Hénon maps, that is maps of the form (5.1) with $|\delta| = 1$. We will mention here the questions that we think we can address using some of the results and techniques presented in this manuscript.

Given an invariant Fatou component Ω for a conservative Hénon map H, one can consider the set $\mathcal{G}$ of all limits of convergent subsequences of iterates of H. It can be shown that $\mathcal{G}$ is a compact Lie group, and Bedford and Smillie proved in [38] that the connected component $\mathcal{G}_0$ of the identity is isomorphic to a (real) torus of rank 1 or 2. Therefore Ω is invariant under a nontrivial torus of rotations, and we can call it a *rotation domain*. The rank of the torus is called the *rank of the rotation domain*.

Bedford asked whether a rotation domain always necessarily contains a fixed point. He then specified his question according to the rank of the rotation domain.

In the rank 1 case, he asked whether the (abstract) torus action on Ω is equivalent to a more familiar circle action. More precisely, given Ω a rank 1 rotation domain, can one find a (p,q)-*domain* $D \subset \mathbb{C}^2$, that is a connected open set $D \subset \mathbb{C}^2$ such that $(e^{ip\theta}z, e^{iq\theta}w) \in D$ whenever $(z,w) \in D$ and $\theta \in \mathbb{R}$, and a biholomorphism $\Phi\colon \Omega \to D$ conjugating the Hénon map H to a linear map $L = \mathrm{Diag}(\alpha^p, \alpha^q)$ with $\alpha \in \mathbb{S}^1$? Moreover, can the case $pq < 0$ occur?

In the rank 2 case, the results in [34] imply that the $\mathcal{G}$-action on Ω can be conjugated to the standard linear action on $\mathbb{C}^2$, that is there are a Reinhardt domain[1] $D \subset \mathbb{C}^2$, a linear map $L = \mathrm{Diag}(\alpha_1, \alpha_2)$, with $|\alpha_1| = |\alpha_2| = 1$ and a biholomorphisms $\Phi\colon \Omega \to D$ conjugating H to L. This statement leaves open the following question: on which Reinhardt domains can arise as rank 2 rotation domains? Since Ω is polynomially convex, the Reinhardt domain D is topologically equivalent to either a ball (in which case it contains a fixed point) or to the product of a disk

[1] We recall that $D \subset \mathbb{C}^2$ is a *Reinhardt domain* if $(e^{i\theta}z, e^{i\phi}w) \in D$ whenever $(z,w) \in D$ and $\theta, \phi \in \mathbb{R}$.

times an annulus (in which case it contains an invariant annulus inside one of the coordinate axes), and in this last case Ω is called *exotic*. We can therefore ask whether there are Hénon maps with *exotic* rotation domains, and whether Ω can be a *Siegel bidisk*, that is biholomorphic to the bidisk $\mathbb{D}^2$, or a *Siegel ball*, that is biholomorphic to the standard unit ball $\mathbb{B}^2$.

Post-critically finite maps

Among the questions that are well-understood in dimension 1 and whose analogues in higher dimension remain opens, we cite here the study of the global dynamics of *post-critically finite* endomorphisms of $\mathbb{P}^k(\mathbb{C})$.

Recall that Fatou and Julia established that the global dynamics of a rational map $f\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is governed by the forward orbits of the critical points of f. For example, if f has an attracting or a parabolic cycle, then a critical point is necessarily attracted to it. As another example, if f is a polynomial map, then the filled Julia set of f is connected if and only if it contains all (finite) critical points of f.

A rational map f of degree $d \geq 2$ has $2d-2$ critical points, counted with multiplicity. The *postcritical set of* f is the set

$$P_f := \bigcup_{n>0} f^{\circ n}(C_f) \quad \text{where} \quad C_f \text{ is the set of critical points of } f.$$

A rational function is then called *postcritically finite* (or PCF for short) if its postcritical set P_f is finite. If f is PCF, then every periodic cycle of f is either repelling, or superattracting (that is, the cycle contains a critical point). In other words, the dynamics of a PCF rational map is tame.

In higher dimension, a holomorphic endomorphism of $\mathbb{P}^k(\mathbb{C})$ is called *post-critically finite* if its post-critical set, that is the union of all forward images of its critical set, is algebraic. In [122], Fornæss and Sibony studied the dynamics of two specific examples of PCF endomorphisms of $\mathbb{P}^2(\mathbb{C})$. They then considered in [124] more general PCF endomorphisms of $\mathbb{P}^k(\mathbb{C})$ and studied the case of dimension $k = 2$. In particular, they notably proved that if the complement in $\mathbb{P}^2(\mathbb{C})$ of the postcritical set of a PCF endomorphism F is Kobayashi hyperbolic, then the only Fatou components of F are basins of superattracting cycles. Rong further generalized such result, proving in [222] that, still in dimension $k = 2$, the Fatou set of a PCF endomorphism is reduced to the union of basins of superattracting cycles. Another important more recent contribution is due to Astorg in [32]. However, it is still unknown whether it is possible to fully generalize this result for any $k \geq 2$. In a joint project with X. Buff and our Ph.D student Van Tu Le, we intend to prove that the eigenvalues of the differential of a PCF endomorphism of $\mathbb{P}^k(\mathbb{C})$ at a fixed point are either zero or of modulus strictly larger than 1. It is not difficult to show that the modulus of such eigenvalues are either zero or larger than or equal to 1. The non-trivial part is to exclude the case of modulus 1.

Dynamics in convex domains

Backward orbits and pre-models

Given a holomorphic self-map $f\colon \mathbb{B}^n \to \mathbb{B}^n$ of the unit ball of $\mathbb{C}^n$, backward orbits with bounded Kobayashi step, boundary repelling fixed points and pre-models are deeply related. We recall that a *pre-model* for f is a triple (Λ, h, φ), where Λ is a complex manifold called the *base space*, $h\colon \Lambda \to \mathbb{B}^n$ is a holomorphic mapping called the *intertwining mapping* and $\varphi\colon \Lambda \to \Lambda$ is an automorphism, such that the following diagram commutes:

$$\begin{array}{ccc} \Lambda & \xrightarrow{\varphi} & \Lambda \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{B}^n & \xrightarrow{f} & \mathbb{B}^n. \end{array}$$

A pre-model (Λ, h, φ) is called *associated* with the boundary repelling point ζ if for some (and hence for any) $x \in \Lambda$ we have $\lim_{n\to\infty} h(\varphi^{-n}(x)) = \zeta$.

Poggi-Corradini in [201] (see also the results obtained by Bracci in [52]) showed in the unit disk $\mathbb{D} \subset \mathbb{C}$ that given a boundary repelling fixed point ζ one can find a backward orbit with step equal to $\log \beta_\zeta$, where β_ζ is the dilation coefficient at ζ, converging to ζ. He then used such orbit to obtain an essentially unique pre-model $(\mathbb{D}, h, \tau)$ associated with ζ, where τ is a hyperbolic automorphism of the disk with dilation β_ζ at its repelling point.

This result was partially generalized by Ostapuyk [187] in the unit ball $\mathbb{B}^n$. She proved that given an *isolated* boundary repelling fixed point ζ one obtains with a similar method a pre-model $(\mathbb{D}, h, \tau)$ associated with ζ, where τ is a hyperbolic automorphism of the disk with dilation β_ζ at its repelling point. Such pre-model is one-dimensional and has no uniqueness property. It is then natural to ask, as in [187, Question 8], whether it is possible to describe the structure of the *stable subset* $\mathcal{S}(\zeta)$, that is the subset of starting points of backward orbits with bounded Kobayashi step converging to ζ, and whether one can find a *preferred* pre-model associated with ζ.

Recently Arosio gave in [28, 29] a partial answer to such questions, using the theory of canonical pre-models he developed together with Bracci in [30]. More precisely, he showed that every backward orbit (z_l) with bounded Kobayashi step converging to ζ gives rise in a natural way to a *canonical* pre-model $(\mathbb{B}^k, \ell, \tau)$ associated with ζ, where $1 \leq k \leq n$, and where τ is a hyperbolic automorphism of the ball $\mathbb{B}^k$ with dilation $\mu \geq \beta_\zeta$ at its repelling point. Moreover the canonical pre-model satisfies a universal property which can be roughly stated as follows: $(\mathbb{B}^k, \ell, \tau)$ is the "best possible" pre-model among all pre-models (Λ, h, φ) such that for some (and hence for any) $x \in \Lambda$ the backward orbit $h(\varphi^{-l}(x))$ stays at a finite Kobayashi distance from the given backward orbit (z_l).

Arosio and Guerini then proved in [31] that to every boundary repelling fixed point ζ is associated exactly one canonical pre-model which is the "best possible" among all pre-models associated with ζ and which has dilation β_ζ. Their construction is based on two key results. They first proved that for $f\colon \mathbb{B}^n \to \mathbb{B}^n$ a holomorphic self-map of the unit ball in $\mathbb{C}^n$, if $\sigma \in \partial\mathbb{B}^n$ is a boundary repelling fixed point with dilation $\beta_\sigma > 1$, then there exists a backward orbit converging to σ with step $\frac{1}{2}\log \beta_\sigma$. Then, they proved that any two backward orbits converging at the same boundary repelling fixed point stay at finite distance.

It is natural to ask whether it is possible to generalize the above mentioned results to bounded strongly convex domains with C^2 boundary.

Julia-Wolff-Carathéodory theorem for infinitesimal generators

In Section 8.3 we presented the generalization of Julia-Wolff-Carathéodory Theorem for infinitesimal generators of one-parameter semigroups of holomorphic self-maps of the unit ball in $\mathbb{C}^n$ obtained in [18]. In collaboration with Abate we think that it is possible to generalize our results to bounded strongly convex domains in $\mathbb{C}^n$ with some smoothness assumptions on the boundary. In order to obtain such generalization it should be possible to use pluripotential tools and in particular the pluricomplex Poisson kernel (see also [54, 58]) and Lempert's theory of complex geodesics in strongly convex domains.

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