# RATIONAL MAPS WITH INTEGER MULTIPLIERS

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Abstract. Let $O_K$ be the ring of integers of an imaginary quadratic field. Recently, Zhuchao Ji and Junyi Xie proved that rational maps whose multipliers at all periodic points belong to $O_K$ are power maps, Chebyshev maps or Lattès maps. Their proof relies on a non-archimedean result by Benedetto, Ingram, Jones and Levy. In this note, we show that one may avoid using this non-archimedean result by considering a differential equation instead.

## Introduction

Let $d \geq 2$ be an integer. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d$, where $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. Consider a sequence $(z_n)_{n\geq 0}$ satisfying $z_0 \in \widehat{\mathbb{C}}$ and $z_{n+1} = f(z_n)$ for $n \geq 0$. The point $z_0$ is a *periodic point of period* $k$ if $z_k = z_0$ for some minimal $k \geq 1$. In that case, the *multiplier of* $f$ *at* $z_0$ is the eigenvalue of $\mathrm{D}_{z_0} f^{\circ k} : \mathrm{T}_{z_0}\widehat{\mathbb{C}} \to \mathrm{T}_{z_0}\widehat{\mathbb{C}}$.

The rational map $f$ is

- a *power map* if it is conjugate to $z \mapsto z^{\pm d}$;
- a *Chebyshev map* if it is conjugate to $\pm T_d$ where $T_d$ is the unique polynomial of degree $d$ satisfying $T_d(z + z^{-1}) = z^d + z^{-d}$;
- a *Lattès map* if there exist a torus $\mathbb{T} = \mathbb{C}/\Lambda$, with $\Lambda \subset \mathbb{C}$ a lattice of rank 2, a holomorphic endomorphism $L : \mathbb{T} \to \mathbb{T}$ and a nonconstant holomorphic map $\Theta : \mathbb{T} \to \widehat{\mathbb{C}}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{L} & \mathbb{T} \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Theta} \\ \widehat{\mathbb{C}} & \xrightarrow[f]{} & \widehat{\mathbb{C}}. \end{array}$$

Power maps, Chebyshev maps and Lattès maps are called *finite quotients of affine maps* by Milnor [M] and *exceptional maps* by Ji and Xie [JX]. In this note, we will use the second terminology.

As observed by Milnor in [M], if $f$ is exceptional, the multipliers of $f$ at all periodic points are contained in a discrete subring of $\mathbb{C}$, thus in the ring of integers of some imaginary quadratic field. Milnor conjectured that the converse is true. In [H1] the third author proved the conjecture when $d = 2$ and in [JX] Ji and Xie proved the conjecture for all $d \geq 2$.

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**Theorem 1** (Ji-Xie)**.** *Assume that $O_K$ is the ring of integers of some imaginary quadratic field and $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$ whose multipliers all lie in $O_K$. Then $f$ is a power map, a Chebyshev map or a Lattès map.*

In this note, we present their proof with a minor modification for one of the arguments. More precisely, our main contribution is Proposition 1.

After writing this note, the third author [H2] proved the following stronger result.

**Theorem 2** (Huguin)**.** *Assume that $K$ is a number field and $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$ whose multipliers all lie in $K$. Then $f$ is a power map, a Chebyshev map or a Lattès map.*

## 1. Exceptional maps

Ritt [R] gave the following characterization of exceptional maps.

**Lemma 1** (Ritt)**.** *Assume that $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$, $\phi : \mathbb{C} \to \widehat{\mathbb{C}}$ is a nonconstant holomorphic map, $\alpha : \mathbb{C} \to \mathbb{C}$ is an affine map and $\tau : \mathbb{C} \to \mathbb{C}$ is a nontrivial translation such that*

- $\phi \circ \alpha = f \circ \phi$ *and*
- $\phi \circ \tau = \phi$.

*Then, $f$ is an exceptional map.*

The following generalization is essentially due to Ji and Xie (compare with [JX, Lemma 2.9]).

**Lemma 2.** *Assume that $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$, $\phi : \mathbb{C} \to \widehat{\mathbb{C}}$ is a nonconstant holomorphic map and $\alpha_1 : \mathbb{C} \to \mathbb{C}$ and $\alpha_2 : \mathbb{C} \to \mathbb{C}$ are affine maps such that*

- $\alpha_1$ *and* $\alpha_2$ *do not commute and*
- $\phi \circ \alpha_1 = f \circ \phi = \phi \circ \alpha_2$.

*Then, $f$ is an exceptional rational map.*

*Proof.* First, note that the affine map

$$\tau := \alpha_1 \alpha_2^{-1} \alpha_1 \alpha_2 \alpha_1^{-2} : \mathbb{C} \to \mathbb{C}$$

is a nontrivial translation. Indeed, the differentials of $\alpha_1$ and $\alpha_2$ are linear maps, thus commute. Therefore, we have $\mathrm{D}\tau = \mathrm{id} : \mathbb{C} \to \mathbb{C}$. In addition, $\tau \neq \mathrm{id}$ since otherwise

$$\begin{aligned} \alpha_1 \alpha_2^{-1} \alpha_1 \alpha_2 \alpha_1^{-2} = \mathrm{id} \quad &\Longrightarrow \quad \alpha_1 \alpha_2^{-1} \alpha_1 \alpha_2 = \alpha_1^2 \\ &\Longrightarrow \quad \alpha_2^{-1} \alpha_1 \alpha_2 = \alpha_1 \\ &\Longrightarrow \quad \alpha_1 \alpha_2 = \alpha_2 \alpha_1, \end{aligned}$$

contradicting the fact that $\alpha_1$ and $\alpha_2$ do not commute.

Second, observe that

$$\phi \circ \alpha_1 \alpha_2 = f \circ \phi \circ \alpha_2 = f^{\circ 2} \circ \phi = f \circ \phi \circ \alpha_1 = \phi \circ \alpha_1^2,$$

so that

$$\phi \circ \tau = \phi \circ \alpha_1 \alpha_2^{-1} \alpha_1 \alpha_2 \alpha_1^{-2} = \phi \circ \alpha_2 \alpha_2^{-1} \alpha_1 \alpha_2 \alpha_1^{-2} = \phi \circ \alpha_1^2 \alpha_1^{-2} = \phi.$$

The result then follows from Lemma 1. □

## 2. Escaping quadratic-like maps

An *escaping quadratic-like map* is a covering map $g : U \to V$ of degree 2 between open subsets of $\mathbb{C}$, with $V$ simply connected, and $U$ compactly contained in $V$. If $g : U \to V$ is such a map, then $U$ has two connected components $U_1$ and $U_2$, each of which is simply connected.

**Lemma 3.** *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$, there exist an integer $n \geq 1$ and open sets $U \Subset V \subset \mathbb{C}$, such that the restriction $f^{\circ n} : U \to V$ is an escaping quadratic-like map.*

We say that such a restriction of $f^{\circ n}$ is an *escaping quadratic-like map associated to $f$*.

*Proof.* Let $z_1 \in \mathbb{C}$ be a repelling periodic point for $f$ of period $r \geq 1$ which is not contained in the forward orbit of a critical point of $f$. Let $V_1 \subset \mathbb{C}$ be a simply connected neighborhood of $z_1$ such that the inverse branch $h_1$ of $f^{\circ r}$ fixing $z_1$ is defined on $V_1$ with $h_1(V_1) \Subset V_1$. Since $z_1$ is in the Julia set of $f$ and since the iterated preimages of $z_1$ are dense in the Julia set of $f$ which contains no isolated point, there exist $z_2 \in V_1 \smallsetminus \{z_1\}$ and an integer $s \geq 1$ such that $f^{\circ s}(z_2) = z_1$. Let $V \subset V_1$ be a simply connected neighborhood of $z_1$ such that

- $h_1(V) \Subset V$,
- the inverse branch $h_2$ of $f^{\circ s}$ sending $z_1$ to $z_2$ is defined on $V$ and
- $h_2(V) \Subset V_1 \setminus \{z_1\}$.

Let $m_1 \geq 1$ be sufficiently large so that $W_2 := h_1^{\circ m_1} \circ h_2(V) \Subset V$. Let $m_2 \geq 1$ be sufficiently large so that $W_1 := h_1^{\circ m_2}(V) \Subset V \smallsetminus W_2$. Set

$$k_1 := h_1^{\circ m_2} : V \to W_1 \quad \text{and} \quad k_2 := h_1^{\circ m_1} \circ h_2 : V \to W_2.$$

Note that $k_1$ is an inverse branch of $f^{\circ n_1}$ with $n_1 := m_2 r$ and $k_2$ is an inverse branch of $f^{\circ n_2}$ with $n_2 := m_1 r + s$. Set

$$n := n_1 n_2 = m_2 r(m_1 r + s) \quad \text{and} \quad U := k_1^{\circ n_2}(V) \cup k_2^{\circ n_1}(V).$$

Then, the restriction of $f^{\circ n}$ from $U$ to $V$ is an escaping quadratic-like map associated to $f$. ☐

## 3. Affine escaping quadratic-like maps

An escaping quadratic-like map $g : U \to V$ is *affine* if the restriction of $g$ to each connected component of $U$ coincides with the restriction of an affine map.

In addition, two escaping quadratic-like maps $g_1 : U_1 \to V_1$ and $g_2 : U_2 \to V_2$ are *conjugate* if there exists a holomorphic isomorphism $\phi : V_2 \to V_1$ such that the relation $\phi \circ g_2 = g_1 \circ \phi$ holds on $U_2$.

**Lemma 4.** *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. If an escaping quadratic-like map associated to $f$ is conjugate to an affine escaping quadratic-like map, then $f$ is an exceptional map.*

*Proof.* Assume that $f^{\circ n} : U \to V$ is an escaping quadratic-like map associated to $f$, that $g : U' \to V'$ is an affine escaping quadratic-like map and that $\phi : V' \to V$ conjugates $g : U' \to V'$ to $f^{\circ n} : U \to V$, i.e.,

$$\phi \circ g = f^{\circ n} \circ \phi.$$

By assumption, $U'$ has two connected components $U_1'$ and $U_2'$ and the restrictions of $g$ to $U_1'$ and $U_2'$ coincide with the restrictions of affine maps $\alpha_1 : \mathbb{C} \to \mathbb{C}$ and $\alpha_2 : \mathbb{C} \to \mathbb{C}$ to $U_1'$ and $U_2'$. The relations $\phi \circ \alpha_1 = f^{\circ n} \circ \phi$ and $\phi \circ \alpha_2 = f^{\circ n} \circ \phi$ hold on $U'$. Since the affine maps $\alpha_1 : \mathbb{C} \to \mathbb{C}$ and $\alpha_2 : \mathbb{C} \to \mathbb{C}$ are repelling, we may use any of those two relations to extend $\phi : V' \to V$ to a global meromorphic map $\hat{\phi} : \mathbb{C} \to \widehat{\mathbb{C}}$. We then have

$$\hat{\phi} \circ \alpha_1 = f^{\circ n} \circ \hat{\phi} = \hat{\phi} \circ \alpha_2 \quad \text{on } \mathbb{C}.$$

The affine maps $\alpha_1$ and $\alpha_2$ have distinct fixed points, respectively in $U_1'$ and $U_2'$. Thus, they do not commute. It follows from Lemma 2 that $f^{\circ n}$ is an exceptional map, and so, $f$ is an exceptional map (a rational map is exceptional if and only if its iterates are exceptional). □

## 4. The proof of Theorem 1

It follows from Lemma 3 and Lemma 4 that Theorem 1 is a consequence of the following result.

**Proposition 1.** *Let $O_K$ be the ring of integers of some quadratic imaginary field. If $g$ is an escaping quadratic-like map whose multipliers at all periodic points belong to $O_K$, then $g$ is conjugate to an affine escaping quadratic-like map.*

The proof will occupy the rest of the note. From now on, we assume that $g : U \to V$ is an escaping quadratic-like map whose multipliers at all periodic points belong to $O_K$. Let $U_1$ and $U_2$ be the two connected components of $U$. Set:

$$g_1 := g|_{U_1}, \quad g_2 := g|_{U_2}, \quad h_1 := g_1^{-1} : V \to U_1 \quad \text{and} \quad h_2 := g_2^{-1} : V \to U_2.$$

Let $p_1 \in U_1$ be the unique (repelling) fixed point of $g_1 : U_1 \to V$ and let $\lambda_1$ be its multiplier. Similarly, let $p_2 \in U_2$ be the unique (repelling) fixed point of $g_2 : U_2 \to V$ and let $\lambda_2$ be its multiplier.

The sequence of univalent maps $\psi_n : V \to \mathbb{C}$ defined by

$$\psi_n(z) := \frac{h_1^{\circ n}(z) - p_1}{h_1^{\circ n}(p_2) - p_1}$$

converges to a univalent map $\psi : V \to \mathbb{C}$ such that $\psi(p_1) = 0$, $\psi(p_2) = 1$ and $\psi \circ g_1 = \lambda_1 \times \psi$. Replacing $g$ by $\psi \circ g \circ \psi^{-1}$ if necessary, we may therefore assume that

$$p_1 = 0, \quad p_2 = 1 \quad \text{and} \quad g_1(z) = \lambda_1 z.$$

We need to prove that $g_2$ is an affine map.

### 4.1. A special sequence of periodic points.

In their proof, Ji and Xie consider a particular sequence of periodic points of $g$. This sequence may be defined as follows. For $n \geq 0$, let $z_n$ be the unique fixed point of the map $h_2 \circ h_1^{\circ n} : V \to U_2$. Then, $z_n$ is a periodic point of $g$ of period $n+1$. In particular the multiplier $\rho_n$ of $g$ at $z_n$ belongs to $O_K$. Note that as $n \to +\infty$, we have that

$$z_n \to \alpha := h_2(0)$$

so that

$$z_n = h_2\left(\frac{z_n}{\lambda_1^n}\right) = \alpha + \frac{\beta}{\lambda_1^n} + \mathrm{o}\left(\frac{1}{\lambda_1^n}\right) \quad \text{with} \quad \beta := \alpha h_2'(0).$$

Then,

$$\rho_n = \lambda_1^n g_2'(z_n) = \lambda_1^n a + b + \mathrm{o}(1) \quad \text{with} \quad a := g_2'(\alpha) \quad \text{and} \quad b := \beta g_2''(\alpha) = \alpha \frac{g''(\alpha)}{g'(\alpha)}.$$

**Lemma 5.** *We have that $\rho_n = a\lambda_1^n + b$ for $n$ large enough.*

*Proof.* Write $\rho_n = a\lambda_1^n + b + \varepsilon_n$ with $\varepsilon_n \to 0$ as $n \to +\infty$. We have that

$$\underbrace{\lambda_1 \rho_n - \rho_{n+1}}_{\in O_K} = (\lambda_1 - 1)b + \underbrace{\lambda_1 \varepsilon_n - \varepsilon_{n+1}}_{\underset{n\to+\infty}{\longrightarrow} 0}.$$

Thus, $(\lambda_1 - 1)b$ belongs to the closure of $O_K$, i.e., to $O_K$.

Since $O_K$ is discrete, we have that $\lambda_1 \rho_n - \rho_{n+1} = (\lambda_1 - 1)b$ for $n$ large enough, i.e., $\varepsilon_{n+1} = \lambda_1 \varepsilon_n$. Since $|\lambda_1| > 1$ and $\varepsilon_n \to 0$ as $n \to +\infty$, we have that $\varepsilon_n = 0$ for $n$ large enough. □

### 4.2. **A differential equation.**

**Lemma 6.** *The holomorphic map $g_2 : U_2 \to V$ satisfies the differential equation*

$$\text{(E)} \qquad \forall z \in U_2, \quad g_2'(z) = a + b\frac{g_2(z)}{z}.$$

*Proof.* Note that for $n$ large enough, we have that

$$\lambda_1^n g_2(z_n) = z_n \quad \text{and} \quad g_2'(z_n) = \frac{\rho_n}{\lambda_1^n} = a + \frac{b}{\lambda_1^n} = a + b\frac{g_2(z_n)}{z_n}.$$

Since the sequence $(z_n)_{n\geq 1}$ accumulates at $h_2(0) \in U_2$, the function $g_2$ satisfies the differential equation (E). □

**Remark.** Equation (E) is linear and so, may be easily solved. However, we shall not use the explicit form of the solutions.

**Lemma 7.** *We have that*

$$\frac{g_2'(\alpha)}{\lambda_2} = 1 + \frac{\nu}{1 - \lambda_2} \quad \textit{with} \quad \alpha := h_2(0) \quad \textit{and} \quad \nu := p_2 \frac{g_2''(p_2)}{g_2'(p_2)}.$$

*Proof.* Evaluating Equation (E) at $z = p_2 = g_2(p_2)$, we obtain

$$\lambda_2 = g_2'(p_2) = a + b.$$

In addition, differentiating Equation (E), we obtain

$$\forall z \in U_2, \quad g_2''(z) = b\left(\frac{g_2'(z)}{z} - \frac{g_2(z)}{z^2}\right).$$

Since $\lambda_2 = g_2'(p_2)$ and $a = g_2'(\alpha)$, we have that

$$\nu = p_2 \frac{g_2''(p_2)}{g_2'(p_2)} = \big(\lambda_2 - g_2'(\alpha)\big)\left(1 - \frac{1}{\lambda_2}\right).$$

This last equality may be rewritten in the required form. □

4.3. **Conclusion.** For each integer $k \geq 1$, consider the escaping quadratic-like map $g^k : U_1 \cup h_2^{\circ k}(V) \to V$ defined by

$$g^k(z) = \begin{cases} \lambda_1 z & \text{if } z \in U_1 \\ g_2^{\circ k}(z) & \text{if } z \in h_2^{\circ k}(V). \end{cases}$$

Then, all the periodic points of $g^k$ are periodic points of $g$ and their multipliers still belong to $O_K$. In addition, $g^k$ fixes $p_2$ with multiplier $\lambda_2^k$.

According to Lemma 7, we have that

$$\frac{(g_2^{\circ k})'(\alpha_k)}{\lambda_2^k} = 1 + \frac{\nu_k}{1-\lambda_2^k} \quad \text{with} \quad \alpha_k := h_2^{\circ k}(0) \quad \text{and} \quad \nu_k := p_2 \frac{(g_2^{\circ k})''(p_2)}{(g_2^{\circ k})'(p_2)}.$$

A rather elementary computation (in fact, it is the composition rule for nonlinearities) yields

$$\nu_k = \nu_1 + \lambda_2 \nu_1 + \lambda_2^2 \nu_1 + \cdots + \lambda_2^{k-1}\nu_1 = \frac{1-\lambda_2^k}{1-\lambda_2}\nu_1.$$

In particular,

$$\frac{\nu_k}{1-\lambda_2^k} = \frac{\nu_1}{1-\lambda_2}$$

does not depend on $k \geq 1$. As a consequence, for all $k \geq 1$,

$$\frac{g_2'(\alpha_{k+1})}{\lambda_2} \cdot \frac{(g_2^{\circ k})'(\alpha_k)}{\lambda_2^k} = \frac{(g_2^{\circ(k+1)})'(\alpha_{k+1})}{\lambda_2^{k+1}} = \frac{(g_2^{\circ k})'(\alpha_k)}{\lambda_2^k}.$$

Thus,

$$\forall k \geq 1, \quad g_2'(\alpha_{k+1}) = \lambda_2.$$

The sequence $(\alpha_k)_{k\geq 2}$ accumulates at $p_2 \in U_2$. So, $g_2'(z) = \lambda_2$ for all $z \in U_2$ and $g_2$ is an affine map as required. This completes the proof of Proposition 1.

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