Università di Pisa
Scuola di Dottorato Galileo Galilei
Dottorato in Matematica
xxii ciclo

Ph.D. Thesis

Geometrical methods in the normalization of germs of biholomorphisms

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– Academic Year 2009/2010 –

Contents

Introduction

In this thesis we shall discuss geometrical methods in the study of normal forms of germs of biholomorphisms of $\mathbb{C}^n$. Given a germ of biholomorphism f of $\mathbb{C}^n$ at a fixed point p, one would like to study the dynamics of f near the fixed point, i.e., for each point q in a (sufficiently) small neighbourhood of p, one would like to describe the asymptotic behavior of the sequence $\{f^k(q)\}_{k\geq 0}$ of the iterates of f at q, where f^k is the composition of f with itself k times. Since such a problem is invariant under translation, we can reduce ourselves to study germs of biholomorphisms of $(\mathbb{C}^n, O)$ fixing the origin.

Whereas many things are known in the one-dimensional case, for $n \geq 2$ such a study is far from being complete. Locally, f can be written as a convergent power series, that is, using the standard multi-index notation, we have

$$f(z) = \Lambda z + \sum_{\substack{Q\in\mathbb{N}^n \\ |Q|\geq 2}} f_Q z^Q,$$

where Λ is a $n \times n$ matrix with complex coefficients, $f_Q \in \mathbb{C}^n$, and, if $Q = (q_1, \dots, q_n)$, then $|Q| := \sum_{j=1}^n q_j$ and $z^Q := z_1^{q_1} \cdots z_n^{q_n}$. Up to a linear change of the coordinates, we can assume that Λ is in Jordan normal form, that is

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ \varepsilon_2 & \lambda_2 & & \\ & \ddots & \ddots & \\ & & \varepsilon_n & \lambda_n \end{pmatrix},$$

where the eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}^*$ are not necessarily distincts, and $\varepsilon_j \in \{0, \varepsilon\}$ can be non-zero only if $\lambda_{j-1} = \lambda_j$.

Since the dynamics does not change if we change coordinates, a natural idea is to look for a solution of a normalization problem: *given a germ of biholomorphism f of $\mathbb{C}^n$ fixing the origin and with linear part in Jordan normal form, does it exist a local change of coordinates φ of $\mathbb{C}^n$, fixing the origin, such that*

$$\varphi^{-1} \circ f \circ \varphi = \text{“simple form”?}$$

Moreover, one usually assumes $d\varphi_O = \mathrm{Id}$ because the linear part of f already is in (Jordan) normal form.

Of course, we have to specify what we mean by “simple form”. A natural choice for a “simple form” is the linear term of our given germ; so in this case we have to deal with the:

Linearization problem. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and with linear part Λ in Jordan normal form. Does it exist a local change of coordinates φ of $\mathbb{C}^n$, fixing the origin, with* $\mathrm{d}\varphi_O = \mathrm{Id}$, *such that*

$$\varphi^{-1} \circ f \circ \varphi = \Lambda?$$

A way to solve such a problem is to first look for a formal transformation φ solving

$$f \circ \varphi = \varphi \circ \Lambda,$$

and then to check whether φ is convergent.

The answer depends on the set of eigenvalues of Λ, usually called the *spectrum* of Λ. In fact it may happen that there exists a multi-index $Q = (q_1, \ldots, q_n) \in \mathbb{N}^n$, with $|Q| \geq 2$, such that

$$\lambda^Q - \lambda_j := \lambda_1^{q_1} \cdots \lambda_n^{q_n} - \lambda_j = 0 \tag{0.2}$$

for some $1 \leq j \leq n$; a relation of this kind is called a *multiplicative resonance* of f, and Q is called a *resonant multi-index*. A *resonant monomial* is a monomial z^Q in the j-th coordinate such that $\lambda^Q = \lambda_j$.

Resonances are the formal obstruction to linearization. Indeed, at formal level we have the following classical result:

Theorem. (Poincaré, 1893 [Po]; Dulac, 1904 [D]) *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O with linear part Λ in Jordan normal form. Then there exists a formal transformation φ of $\mathbb{C}^n$, without constant term and with linear part equal to the identity, conjugating f to a formal power series $g \in \mathbb{C}[\![z_1, \ldots, z_n]\!]^n$ without constant term, with linear term Λ and containing only resonant monomials.*

The formal series g is called a *Poincaré-Dulac normal form* of f. Hence the second natural choice for a "simple form" is a Poincaré-Dulac normal form; in this case we say that we have to deal with the:

Normalization problem. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and with linear part Λ in Jordan normal form. Does it exist a local change of coordinates φ of $\mathbb{C}^n$, fixing the origin, with* $\mathrm{d}\varphi_O = \mathrm{Id}$, *such that*

$$\varphi^{-1} \circ f \circ \varphi$$

is a Poincaré-Dulac normal form of f?

Even without resonances, the holomorphic linearization is not guaranteed. One has to study how the numbers $\lambda^Q - \lambda_j$ approach zero as $|Q| \to +\infty$; this is known as the *small divisors problem* in this context. Furthermore Poincaré-Dulac normal forms are not unique, and this makes particularly difficult the study of convergence.

In this thesis we shall use geometrical methods to study the linearization and the normalization problems, discussing both the formal level and convergence issues.

We first deal with the linearization problem in presence of resonances. In particular we find, under certain arithmetic conditions on the eigenvalues and some restrictions on the resonances, that a necessary and sufficient condition for holomorphic linearization in presence of resonances is the existence of a particular f-invariant complex manifold (see Chapter 2 for definitions and proofs):

Theorem 1. (Raissy, 2009 [R2]) *Let f be a germ of biholomorphism of $\mathbb{C}^n$ having the origin O as a quasi-Brjuno fixed point of order s. Then f is holomorphically linearizable if and only if it admits an osculating manifold M of codimension s such that $f|_M$ is holomorphically linearizable.*

Moreover such a result has as corollaries most of the known linearization results.

Secondly, we explore in our setting the consequences of the general heuristic principle saying that if a map f commutes with a map g, then some properties of g might be inherited by f.

For instance, one possible generalization of the linearization problem is to ask when a given set of $m \geq 2$ germs of biholomorphisms $f_1, \ldots, f_m$ of $\mathbb{C}^n$ at the same fixed point, which we may place at the origin, are *simultaneously holomorphically linearizable*, i.e., there exists a local holomorphic change of coordinates conjugating f_h to its linear part for each $h = 1, \ldots, m$.

We find that if $f_1, \ldots, f_m$ have diagonalizable linear part and are such that f_1 commutes with f_h for any $h = 2, \ldots, m$, under certain arithmetic conditions on the eigenvalues of $(\mathrm{d}f_1)_O$ and some restrictions on their resonances, $f_1, \ldots, f_m$ are simultaneously holomorphically linearizable if and only if there exists a particular complex manifold invariant under $f_1, \ldots, f_m$ (see Chapter 3 for definitions and proofs):

Theorem 2. (Raissy, 2009 [R3]) *Let $f_1, \ldots, f_m$ be $m \geq 2$ germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin. Assume that f_1 has the origin as a quasi-Brjuno fixed point of order s, with $1 \leq s \leq n$, and that it commutes with f_h for any $h = 2, \ldots, m$. Then $f_1, \ldots, f_m$ are simultaneously holomorphically linearizable if and only if there exists a germ of complex manifold M at O of codimension s, invariant under f_h for each $h = 1, \ldots, m$, which is a simultaneous osculating manifold for $f_1, \ldots, f_m$ and such that $f_1|_M, \ldots, f_m|_M$ are simultaneously holomorphically linearizable.*

Finally we study commutations with a particular kind of linearizable object: torus actions. We find out in a complete and computable manner what kind of structure a torus action must have in order to get a Poincaré-Dulac holomorphic normalization, studying the possible torsion phenomena. In particular, we link the eigenvalues of the linear part of our germ of biholomorphism to the weight matrix of the action. The link and the structure we find are more complicated than what one would expect; a detailed study is needed to completely understand the relations between torus actions, holomorphic Poincaré-Dulac normalizations, and torsion phenomena. An example of the results we get is (see Chapter 4 for definitions, proofs and other results):

Theorem 3. (Raissy, 2009 [R4]) *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Assume that, denoted by $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ the spectrum of the linear part Λ of f, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$ is of toric degree $1 \leq r \leq n$ and in the impure torsion case. Then f admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $r - 1$ commuting with f and such that the columns of the weight matrix of the action are reduced torsion-free toric vectors associated to $[\varphi]$.*

We end our work giving an example of techniques that can be used to construct torus actions.

The plan of the thesis is as follows.

In Chapter 1 we shall present a survey on local holomorphic discrete dynamics, focusing our attention on linearization and normalization problems. After fixing the setting and the notation, we shall first deal with the one-dimensional case, (mainly following Abate [A5]), and then with the multi-dimensional case. Among other things, we present a new proof of a linearization result in presence of resonances, originally proved by Rüssmann [Rü2] under a slightly different arithmetic hypothesis.

In Chapter 2 we shall prove our linearization result in presence of resonances (Theorem 1) for a germ f of biholomorphism of $\mathbb{C}^n$ fixing the origin O, and we shall also see that most of the classical linearization results can be obtained as corollaries of our result. The main results of this chapter are published in [R2].

In Chapter 3 we shall show how commuting with a linearizable germ gives us information on the germs that can be conjugated to a given one. We shall then deal with the simultaneous linearization problem, proving Theorem 2. Next we shall prove that commuting with a torus action yields the existence of a holomorphic linearization or normalization (not yet necessarily of Poincaré-Dulac type) for a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin. The main results of Section 3.2 and 3.3 are published in [R3], whereas the main results of Section 3.4 and 3.5 are published in [R4].

In Chapter 4 we shall describe in a complete and computable manner what kind of structure a torus action must have in order to infer a Poincaré-Dulac holomorphic normalization from the normalization theorem of Chapter 3. To do so, we shall link the eigenvalues of $\mathrm{d}f_O$ to the weight matrix of the action, and we shall introduce the new concepts of toric degree and toric vectors associated to the eigenvalues, needed to study the complicated torsion phenomena one has to deal with. We end the chapter giving an example of techniques that can be used to construct torus actions. The main results in this chapter are published in [R4].

Acknowledgments.

Let me end with the pleasant duty of acknowledgments. Several people have helped me in one way or another during the work culminating in this thesis. First of all, I wish to particularly thank M. Abate for his confidence, constant support and infinite patience during the past six years; the Scuola di Dottorato G. Galilei, its director, its council, and in particular the director of the Ph.D. program in Mathematics, F. Broglia, for the overall organization, and for financial and personal support; the Département de Mathématiques de la Faculté des Sciences d'Orsay, Université Paris-Sud 11, where part of this thesis was done, for its hospitality, and in particular J. Écalle for all his kindness and many useful conversations; the Mathematics Department of the University of Michigan for its warm hospitality; my "french family"; the Mathematics Department L. Tonelli of Pisa with all its residents, that by now are a second home and a second family for me; my family for supporting and enduring; and all the others helping me with their suggestions, unimaginable patience, and affection: Adriano, Alessandra, Ana, Andreea, Annalisa, Antonella, Berit, Carla, Caterina, Chiara, Claudio, Cristian, Daniele, David, Demdah, Elisa, Elizabeth, Enrica, Eric, Fabrizio, Federica, Feng, Filippo, Francesca, Francesco, François, Frédéric, Gennaro, Giorgio, Graziano, Ilaria, Isabelle, Isaia, Jean, Jeff, John Erik, José, Kingshook, Laura, Lazlo, Liz, Luigi Amedeo, Marco, Maria Antonietta, Marisa, Marko, Matteo, Mattias, Phil, Reza, Riccardo, Roberta, Sabrina, Sara, Shirin, Silvia, Tiziano, Valentina, Virginia, and all the others.

1

Linearization and normalization in local holomorphic dynamics

In this chapter we shall present a survey on local holomorphic discrete dynamics, focusing our attention on the linearization and on the normalization problems. After fixing the setting and the notation, we shall first deal with the one-dimensional case, (mainly following Abate [A5]), and then with the multi-dimensional case. Among other things, we present a new proof of a linearization result in presence of resonances, originally proved by Rüssmann [Rü2] under a slightly different arithmetic hypothesis.

1.1 Motivation

Let us introduce a few notations to formalize some of the concepts discussed in the introduction.

Definition 1.1.1. If M is a complex manifold, and $p \in M$, we shall denote by $\mathrm{End}(M, p)$ the set of germs about p of holomorphic self-maps of M fixing p. An element of $\mathrm{End}(M, p)$ will be called a *(discrete) holomorphic local dynamical system* at p. More generally, if S is a closed subset of M (e.g., a submanifold), we shall denote by $\mathrm{End}(M, S)$ the set of germs about S of holomorphic self-maps of M fixing S pointwise.

Remark 1.1.1. In this chapter we shall never have the occasion of discussing continuous holomorphic dynamical systems (i.e., holomorphic foliations). So from now on all dynamical systems will be discrete, except where explicitly noted otherwise.

Remark 1.1.2. If $M = \mathbb{C}^n$ and $p = O$, the set of germs $\mathrm{End}(\mathbb{C}^n, O)$ coincides with the space $\mathbb{C}_O\{z_1, \ldots, z_n\}^n$ of n-uples of converging power series fixing the origin (that is, without constant term), and thus it is naturally embedded into the space $\mathbb{C}_O[\![z_1, \ldots, z_n]\!]^n$ of n-uples of formal power series without constant terms. An element $\Phi \in \mathbb{C}_O[\![z_1, \ldots, z_n]\!]^n$ has an inverse (with respect to composition) still belonging to $\mathbb{C}_O[\![z_1, \ldots, z_n]\!]^n$ if and only if its linear part is a linear automorphism of $\mathbb{C}^n$.

An element $f \in \mathrm{End}(M, p)$ will be usually given by a representative (denoted by the same symbol) defined on a neighbourhood U of the fixed point p; for instance, U could be the domain of convergence of the power series defining f as element of $\mathbb{C}_O\{z_1, \ldots, z_n\}^n$.

If $z \in U$, it is not clear (and, in general, not true) whether $f(z)$ would still belong to U or not. Since we are interested in the dynamics of f, particular attention will be devoted to the set of points in U which remains in U under the action of f, that is to $U \cap f^{-1}(U)$. Indeed, if $q \in U \cap f^{-1}(U)$, we can define $f^2(q)$ by setting $f^2(q) = f\bigl(f(q)\bigr)$, because $f(q) \in U$. More generally, we shall use the following definitions:

Definition 1.1.2. Let $f \in \mathrm{End}(M,p)$. The *second iterate* $f^2 \in \mathrm{End}(M,p)$ of f is the germ represented by the map $f \circ f : U \cap f^{-1}(U) \to M$, where (U,f) is a representative of f (and it is clear that the germ f^2 does not depend on the chosen representative). If $k \in \mathbb{N}$, by induction we define the *k-th iterate* $f^k \in \mathrm{End}(M,p)$ as the germ represented by $f \circ f^{k-1} : U \cap \cdots \cap f^{k-1}(U) \to M$, where again (U,f) is a representative of f.

Remark 1.1.3. Clearly, if we think of $f \in \mathrm{End}(\mathbb{C}^n, O)$ as given by a n-uple of power series, then f^k is given by the k-th composition of those power series.

To introduce the next concept, we need the notion of germ of a set at $p \in M$.

Definition 1.1.3. Set $\mathcal{S} = \{(U,L) \mid U \subset M$ is an open neighbourhood of p, $L \subseteq U$ and $p \in \overline{L}\}$. On $\mathcal{S}$ we put the usual equivalence relation $(U_1, L_1) \sim (U_2, L_2)$ if there exists an open neighbourhood $W \subset U_1 \cap U_2$ of p such that $L_1 \cap W = L_2 \cap W$. A *germ of set at p* is an equivalence class $\mathbf{L} \in \mathcal{S}/\sim$. We shall say that a germ of set $\mathbf{L}_1$ at p is *contained* into another germ of set $\mathbf{L}_2$, and we shall write $\mathbf{L}_1 \subseteq \mathbf{L}_2$, if for any representative (U_1, L_1) of $\mathbf{L}_1$ and representative (U_2, L_2) of $\mathbf{L}_2$ there is an open neighbourhood $W \subset U_1 \cap U_2$ of p such that $L_1 \cap W \subseteq L_2 \cap W$.

Clearly, a germ $f \in \mathrm{End}(M,p)$ acts on germs of sets at p: if (U,f) is a representative of f and (V,L) is a representative of a germ $\mathbf{L}$ of set at p, then $f(\mathbf{L})$ is the germ of set represented by $\big(U \cap V, f(L \cap U) \cap U \cap V\big)$; analogously, if (W, f^{-1}) is a representative of f^{-1}, then $f^{-1}(\mathbf{L})$ is the germ of set represented by $\big(W \cap V, f^{-1}(L \cap W) \cap W \cap V\big)$. It is easy to verify that the germs $f(\mathbf{L})$ and $f^{-1}(\mathbf{L})$ do not depend on the representatives chosen, and thus we can introduce the following

Definition 1.1.4. Let $f \in \mathrm{End}(M,p)$. A germ of set $\mathbf{L}$ at p is *(forward) f-invariant* if $f(\mathbf{L}) \subseteq \mathbf{L}$. A germ of set $\mathbf{L}$ at p is *completely f-invariant* if $f^{-1}(\mathbf{L}) = \mathbf{L}$.

A bit more delicate is the definition of stable set of a germ:

Definition 1.1.5. Let (U,f) be a representative of a germ $f \in \mathrm{End}(M,p)$. Then the *stable set* of f (with respect to U) is the set

$$K_{(U,f)} = \bigcap_{k=0}^{\infty} f^{-k}(U).$$

If $z \in K_{(U,f)}$, the *(forward) orbit* of z is the set $O^+(z) = \{z, f(z), f^2(z), \dots\}$.

The problem with the notion of stable set in this generality is that even its germ at p might depend on the chosen representative. What may happen (and actually it happens) is the following: there might exist two representatives (U_1, f) and (U_2, f) of the same germ f, with $U_1 \subset U_2$, and points $z \in U_1$ as close as we want to p whose orbits escapes from U_1 but do not escape from U_2. If this happens, $K_{(U_1,f)}$ and $K_{(U_2,f)}$ do not agree in any neighbourhood of p, and thus they define two different germs of stable sets at p. As we shall see, this happens in the parabolic case.

Another definition of this kind that shall later be useful is the following:

Definition 1.1.6. Let $f \in \mathrm{End}(M,p)$. We shall say that p is *stable* for f if there is a representative (U,f) of f such that p is contained in the interior of $K_{(U,f)}$. In other words, there is an open neighbourhood $W \subset U$ of p such that $f^k(W) \subseteq U$ for all $k \in \mathbb{N}$.

We are now able to state the main problems of local holomorphic dynamics. Given a germ $f \in \mathrm{End}(M,p)$ we would like to:

(a) describe the invariant germs of sets at p, if any;

(b) decide whether p is stable for f, and, more generally, compute the stable set of sufficiently large representatives of f, and describe the orbit of any point in the stable set;

(c) describe the dynamics of f inside any invariant germ; and,

(d) describe the dynamics of f in a neighbourhood of p.

To deal with all these problems, the most efficient way is to replace f by a "dynamically equivalent" but simpler (e.g., linear) map g. In our context, "dynamically equivalent" means "locally conjugated"; and we have at least three kinds of conjugacy to consider.

It is clear what it means the composition of two germs in $\mathrm{End}(M,p)$, and thus for a germ to be invertible; more generally, it is clear what we mean by a germ of homeomorphism at p. Then

Definition 1.1.7. We shall say that two germs $f, g \in \mathrm{End}(M,p)$ are *holomorphically conjugate* [resp., *topologically conjugate*] if there exists an invertible germ $\varphi \in \mathrm{End}(M,p)$ [resp., a germ of homeomorphism φ at p] such that $g = \varphi \circ f \circ \varphi^{-1}$ as germs. In particular, $g^k = \varphi \circ f^k \circ \varphi^{-1}$ for all $k \in \mathbb{N}$, and thus the dynamics of f and g agree.

Remark 1.1.4. Using local coordinates centered at $p \in M$ it is easy to show that any holomorphic local dynamical system at p is holomorphically locally conjugated to a holomorphic local dynamical system at $O \in \mathbb{C}^n$, where $n = \dim M$.

Whenever we have an equivalence relation in a class of objects, there are classification problems. So a natural question in local holomorphic dynamics is to *find a (possibly small) class $\mathcal{F}$ of holomorphic local dynamical systems at $O \in \mathbb{C}^n$ such that every holomorphic local dynamical system f at a point in an n-dimensional complex manifold is holomorphically [resp., topologically] locally conjugated to a (possibly) unique element of $\mathcal{F}$, called the holomorphic [resp., topological] normal form of f.*

Unfortunately, the holomorphic classification is often too complicated to be practical; the family $\mathcal{F}$ of normal forms might be uncountable. A possible replacement is looking for invariants instead of normal forms, i.e., to *find a way to associate a (possibly small) class of (possibly computable) objects, called invariants, to any holomorphic local dynamical system f at $O \in \mathbb{C}^n$ so that two holomorphic local dynamical systems at O can be holomorphically conjugated only if they have the same invariants. The class of invariants is furthermore said complete if two holomorphic local dynamical systems at O are holomorphically conjugated if and only if they have the same invariants.*

Up to now all the questions we asked made sense for topological local dynamical systems; the next one instead makes sense only for holomorphic local dynamical systems.

Definition 1.1.8. We shall say that two germs f, $g \in \mathrm{End}(\mathbb{C}^n, O)$ are *formally conjugate* if there exists an invertible formal power series $\Phi \in \mathbb{C}_O[\![z_1,\dots,z_n]\!]^n$ such that $g = \Phi \circ f \circ \Phi^{-1}$ in $\mathbb{C}_O[\![z_1,\dots,z_n]\!]^n$.

It is clear that two holomorphically conjugated holomorphic local dynamical systems are both formally and topologically conjugated too. On the other hand, we shall see examples of holomorphic local dynamical systems that are topologically locally conjugated without being neither formally nor holomorphically locally conjugated, and examples of holomorphic local dynamical systems that are formally conjugated without being neither holomorphically nor topologically locally conjugated. So the last natural question in local holomorphic dynamics we shall deal with is to *find normal forms and invariants with respect to the relation of formal conjugacy for holomorphic local dynamical systems at $O \in \mathbb{C}^n$.*

In this chapter we shall present some of the main results known on these questions, starting with the one-dimensional situation.

1.2 One-dimensional case

Let us then start by discussing holomorphic local dynamical systems at $0 \in \mathbb{C}$. As remarked in the previous section, such a system is locally given by a converging power series f without constant term:

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots \in \mathbb{C}_0\{z\} \; .$$

Definition 1.2.1. Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be without constant term. The number $\lambda = f'(0)$ is called the *multiplier* of f.

The best linear approximation of f is λz, and we shall see that the local dynamics of f will be strongly influenced by the value of λ. For this reason we introduce the following definition:

Definition 1.2.2. Let $\lambda \in \mathbb{C}$ be the multiplier of $f \in \mathrm{End}(\mathbb{C}, 0)$. Then

- if $|\lambda| < 1$ we say that the fixed point 0 is *attracting;*
- if $\lambda = 0$ we say that the fixed point 0 is *superattracting;*
- if $|\lambda| > 1$ we say that the fixed point 0 is *repelling;*
- if $|\lambda| \neq 0, 1$ we say that the fixed point 0 is *hyperbolic;*
- if $\lambda \in \mathbb{S}^1$ is a root of unity, we say that the fixed point 0 is *parabolic* (or *rationally indifferent*);
- if $\lambda \in \mathbb{S}^1$ is not a root of unity, we say that the fixed point 0 is *elliptic* (or *irrationally indifferent*).

We shall explain in the next four subsections what is known on the dynamics in the various cases.

1.2.1 Hyperbolic case

The dynamics of one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point is pretty elementary; so we start with this case.

Assume first that 0 is attracting for the holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}, 0)$. Then we can write $f(z) = \lambda z + O(z^2)$, with $0 < |\lambda| < 1$; hence we can find a large constant $M > 0$, a small constant $\varepsilon > 0$ and $0 < \delta < 1$ such that if $|z| < \varepsilon$ then

$$|f(z)| \leq (|\lambda| + M\varepsilon)|z| \leq \delta |z| \; . \tag{1.2}$$

In particular, if Δ_ε denotes the disk of center 0 and radius ε, we have $f(\Delta_\varepsilon) \subset \Delta_\varepsilon$ for $\varepsilon > 0$ small enough, and the stable set of $f|_{\Delta_\varepsilon}$ is Δ_ε itself (in particular, a one-dimensional attracting fixed point is always stable). Furthermore,

$$|f^k(z)| \leq \delta^k |z| \to 0$$

as $k \to +\infty$, and thus every orbit starting in Δ_ε is attracted by the origin, which is the reason of the name "attracting" for such a fixed point.

Remark 1.2.1. Notice that if 0 is an attracting fixed point for $f \in \mathrm{End}(\mathbb{C}, 0)$ with non-zero multiplier, then it is a repelling fixed point for the inverse map $f^{-1} \in \mathrm{End}(\mathbb{C}, 0)$.

If instead 0 is a repelling fixed point, a similar argument (or the remark that 0 is attracting for f^{-1}) shows that for $\varepsilon > 0$ small enough the stable set of $f|_{\Delta_\varepsilon}$ reduces to the origin only: all (non-trivial) orbits escape.

It is also not difficult to find holomorphic and topological normal forms for one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point, as the following result shows, which can be considered as the beginning of the theory of holomorphic dynamical systems:

Theorem 1.2.2. (Kœnigs, 1884 [Kœ]) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a one-dimensional holomorphic local dynamical system with a hyperbolic fixed point at the origin, and let $\lambda \in \mathbb{C}^* \setminus \mathbb{S}^1$ be its multiplier. Then:*

(i) f is holomorphically (and hence formally) locally conjugated to its linear part $g(z) = \lambda z$. The conjugation φ is uniquely determined by the condition $\varphi'(0) = 1$.

(ii) Two such holomorphic local dynamical systems are holomorphically conjugated if and only if they have the same multiplier.

(iii) f is topologically locally conjugated to the map $g_<(z) = z/2$ if $|\lambda| < 1$, and to the map $g_>(z) = 2z$ if $|\lambda| > 1$.

Proof. Let us assume $0 < |\lambda| < 1$; if $|\lambda| > 1$ it will suffice to apply the same argument to f^{-1}.

(i) Choose $0 < \delta < 1$ such that $\delta^2 < |\lambda| < \delta$. Writing $f(z) = \lambda z + z^2 r(z)$ for a suitable holomorphic germ r, we can clearly find $\varepsilon > 0$ such that $|\lambda| + M\varepsilon < \delta$, where $M = \max_{z \in \overline{\Delta_\varepsilon}} |r(z)|$. Hence we have

$$|f(z) - \lambda z| \leq M|z|^2$$

and

$$|f^k(z)| \leq \delta^k |z|$$

for all $z \in \overline{\Delta_\varepsilon}$ and $k \in \mathbb{N}$.

Put $\varphi_k = f^k/\lambda^k$. Then the sequence $\{\varphi_k\}$ converges to a holomorphic map $\varphi \colon \Delta_\varepsilon \to \mathbb{C}$. In fact, we have

$$|\varphi_{k+1}(z) - \varphi_k(z)| = \frac{1}{|\lambda|^{k+1}} \left| f\big(f^k(z)\big) - \lambda f^k(z) \right| \leq \frac{M}{|\lambda|^{k+1}} |f^k(z)|^2 \leq \frac{M}{|\lambda|} \left(\frac{\delta^2}{|\lambda|} \right)^k |z|^2$$

for all $z \in \overline{\Delta_\varepsilon}$, and so the telescopic series $\sum_k (\varphi_{k+1} - \varphi_k)$ converges uniformly in Δ_ε to $\varphi - \varphi_0$.

Since $\varphi_k'(0) = 1$ for all $k \in \mathbb{N}$, we have $\varphi'(0) = 1$ and so, up to possibly shrinking ε, we can assume that φ is a biholomorphism with its image. Moreover, we have

$$\varphi\big(f(z)\big) = \lim_{k \to +\infty} \frac{f^k\big(f(z)\big)}{\lambda^k} = \lambda \lim_{k \to +\infty} \frac{f^{k+1}(z)}{\lambda^{k+1}} = \lambda \varphi(z) \;,$$

that is $f = \varphi^{-1} \circ g \circ \varphi$, as claimed.

If ψ is another local holomorphic function such that $\psi'(0) = 1$ and $\psi^{-1} \circ g \circ \psi = f$, it follows that $\psi \circ \varphi^{-1}(\lambda z) = \lambda \psi \circ \varphi^{-1}(z)$; comparing the expansion in power series of both sides we find $\psi \circ \varphi^{-1} \equiv \mathrm{Id}$, that is $\psi \equiv \varphi$, and we are done.

(ii) Since $f_1 = \varphi^{-1} \circ f_2 \circ \varphi$ implies $f_1'(0) = f_2'(0)$, the multiplier is invariant under holomorphic local conjugation, and so two one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point are holomorphically locally conjugated if and only if they have the same multiplier.

(iii) Since $|\lambda| < 1$ it is easy to build a topological conjugacy between g and $g_<$ on Δ_ε. First we choose a homeomorphism χ between the annulus $\{|\lambda|\varepsilon \leq |z| \leq \varepsilon\}$ and the annulus $\{\varepsilon/2 \leq |z| \leq \varepsilon\}$ which is the identity on the outer circle and which is given by $\chi(z) = z/(2\lambda)$

on the inner circle. Now we extend χ by induction to a homeomorphism between the annuli $\{|\lambda|^k\varepsilon \le |z| \le |\lambda|^{k-1}\varepsilon\}$ and $\{\varepsilon/2^k \le |z| \le \varepsilon/2^{k-1}\}$ by prescribing

$$\chi(\lambda z) = \tfrac{1}{2}\chi(z) \; .$$

We finally get a homeomorphism χ of Δ_ε with itself, such that $g = \chi^{-1} \circ g_< \circ \chi$, by putting $\chi(0) = 0$. □

Remark 1.2.3. Notice that $g_<(z) = \frac{1}{2}z$ and $g_>(z) = 2z$ cannot be topologically conjugated, because (for instance) for each representative $(U, g_<)$ of $g_<$ the stable set $K_{(U,g_<)}$ is open, whereas for each representative $(V, g_>)$ of $g_>$ the stable set $K_{(V,g_>)} = \{0\}$ is not.

Remark 1.2.4. The proof of this theorem is based on two techniques often used in dynamics to build conjugations. The first one is used in part (i). Suppose that we would like to prove that two invertible local dynamical systems f, $g \in \mathrm{End}(M,p)$ are conjugated. Set $\varphi_k = g^{-k} \circ f^k$, so that

$$\varphi_k \circ f = g^{-k} \circ f^{k+1} = g \circ \varphi_{k+1} \; .$$

Therefore if we can prove that $\{\varphi_k\}$ converges to an invertible map φ as $k \to +\infty$ we get $\varphi \circ f = g \circ \varphi$, and thus f and g are conjugated, as desired. This is exactly the way we proved Theorem 1.2.2.(i); and we shall see variations of this technique later on.

To describe the second technique we need a definition.

Definition 1.2.3. Let $f\colon X \to X$ be an open continuous self-map of a topological space X. A *fundamental domain* for f is an open subset $D \subset X$ such that

(i) $f^h(D) \cap f^k(D) = \varnothing$ for every $h \ne k \in \mathbb{N}$;

(ii) $\bigcup_{k\in\mathbb{N}} f^k(\overline{D}) = X$;

(iii) if z_1, $z_2 \in \overline{D}$ are so that $f^h(z_1) = f^k(z_2)$ for some $h > k \in \mathbb{N}$ then $h = k+1$ and $z_2 = f(z_1) \in \partial D$.

There are other possible definitions of a fundamental domain, but this will work for our aims.

Suppose that we would like to prove that two open continuous maps $f_1\colon X_1 \to X_1$ and $f_2\colon X_2 \to X_2$ are topologically conjugated. Assume we have fundamental domains $D_j \subset X_j$ for f_j (with $j = 1, 2$) and a homeomorphism $\chi\colon \overline{D_1} \to \overline{D_2}$ such that

$$\chi \circ f_1 = f_2 \circ \chi \tag{1.3}$$

on $\overline{D_1} \cap f_1^{-1}(\overline{D_1})$. Then we can extend χ to a homeomorphism $\tilde{\chi}\colon X_1 \to X_2$ conjugating f_1 and f_2 by setting

$$\forall z \in X_1 \qquad \tilde{\chi}(z) = f_2^k\big(\chi(w)\big) \; , \tag{1.4}$$

where $k = k(z) \in \mathbb{N}$ and $w = w(z) \in \overline{D}$ are chosen so that $f_1^k(w) = z$. The definition of fundamental domain and (1.3) imply that $\tilde{\chi}$ is well-defined. Clearly $\tilde{\chi} \circ f_1 = f_2 \circ \tilde{\chi}$; and using the openness of f_1 and f_2 it is easy to check that $\tilde{\chi}$ is a homeomorphism. This is the technique we used in the proof of Theorem 1.2.2.(iii); and we shall use it again later.

Thus the dynamics in the one-dimensional hyperbolic case is completely clear.

1.2.2 Superattracting case

The superattracting case can be treated similarly to the hyperbolic case. If the origin 0 is a superattracting fixed point for $f \in \mathrm{End}(\mathbb{C}, 0)$, we can write

$$f(z) = a_r z^r + a_{r+1} z^{r+1} + \cdots$$

with $a_r \neq 0$.

Definition 1.2.4. Let $f \in \mathrm{End}(\mathbb{C}, 0)$ and let 0 be a superattracting point for f. The *order* (or *local degree*) of the superattracting point is the minimal number $r \geq 2$ such that the coefficient of z^r in the power series expansion of f is non-zero.

Similarly to the attracting case, we can find a large constant $M > 1$ such that, for $\varepsilon > 0$ small enough we have

$$|f(z)| \leq M|z|^r$$

for $z \in \Delta_\varepsilon := \{|z| < \varepsilon\}$, hence the stable set of $f|_{\Delta_\varepsilon}$ still is all of Δ_ε, and the orbits converge (faster than in the attracting case) to the origin. Furthermore, we can prove the following

Theorem 1.2.5. (Böttcher, 1904 [Bö]) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a one-dimensional holomorphic local dynamical system with a superattracting fixed point at the origin, and let $r \geq 2$ be its order. Then:*

(i) f is holomorphically (and hence formally) locally conjugated to the map $g(z) = z^r$, and the conjugation is unique up to multiplication by an $(r-1)$-root of unity;

(ii) two such holomorphic local dynamical systems are holomorphically (or topologically) conjugated if and only if they have the same order.

Proof. First of all, up to a linear conjugation $z \mapsto \mu z$ with $\mu^{r-1} = a_r$ we can assume $a_r = 1$.

Now write $f(z) = z^r h_1(z)$ for a suitable holomorphic germ h_1 with $h_1(0) = 1$. By induction, it is easy to see that we can write $f^k(z) = z^{r^k} h_k(z)$ for a suitable holomorphic germ h_k with $h_k(0) = 1$. Furthermore, the equalities $f \circ f^{k-1} = f^k = f^{k-1} \circ f$ yield

$$h_{k-1}(z)^r h_1\big(f^{k-1}(z)\big) = h_k(z) = h_1(z)^{r^{k-1}} h_{k-1}\big(f(z)\big) \ . \tag{1.5}$$

Choose $0 < \delta < 1$. Then we can find $0 < \varepsilon < 1$ such that $M\varepsilon < \delta$, where $M = \max_{z \in \overline{\Delta_\varepsilon}} |h_1(z)|$; we can also assume that $h_1(z) \neq 0$ for all $z \in \overline{\Delta_\varepsilon}$. Since

$$\forall z \in \overline{\Delta_\varepsilon} \qquad\qquad |f(z)| \leq M|z|^r < \delta |z|^{r-1} \ ,$$

we have $f(\Delta_\varepsilon) \subset \Delta_\varepsilon$, as anticipated before.

We also remark that (1.5) implies that each h_k is well-defined and never vanishing on $\overline{\Delta_\varepsilon}$. So for every $k \geq 1$ we can choose a unique ψ_k holomorphic in Δ_ε such that $\psi_k(z)^{r^k} = h_k(z)$ on Δ_ε and with $\psi_k(0) = 1$.

Set $\varphi_k(z) = z\psi_k(z)$, so that $\varphi_k'(0) = 1$ and $\varphi_k(z)^{r^k} = f^k(z)$ on Δ_ε; in particular, formally we have $\varphi_k = g^{-k} \circ f^k$. We claim that the sequence $\{\varphi_k\}$ converges to a holomorphic function φ on Δ_ε. Indeed, we have

$$\begin{aligned}
\left| \frac{\varphi_{k+1}(z)}{\varphi_k(z)} \right| &= \left| \frac{\psi_{k+1}(z)^{r^{k+1}}}{\psi_k(z)^{r^{k+1}}} \right|^{1/r^{k+1}} = \left| \frac{h_{k+1}(z)}{h_k(z)^r} \right|^{1/r^{k+1}} = \left| h_1\big(f^k(z)\big) \right|^{1/r^{k+1}} \\
&= \left| 1 + O\big(|f^k(z)|\big) \right|^{1/r^{k+1}} = 1 + \frac{1}{r^{k+1}} O\big(|f^k(z)|\big) = 1 + O\left(\frac{1}{r^{k+1}} \right) \ ,
\end{aligned}$$

and so the telescopic product $\prod_k(\varphi_{k+1}/\varphi_k)$ converges to φ/φ_1 uniformly in Δ_ε.

Since $\varphi_k'(0) = 1$ for all $k \in \mathbb{N}$, we have $\varphi'(0) = 1$ and so, up to possibly shrinking ε, we can assume that φ is a biholomorphism with its image. Moreover, we have

$$\varphi_k\big(f(z)\big)^{r^k} = f(z)^{r^k}\psi_k\big(f(z)\big)^{r^k} = z^{r^k}h_1(z)^{r^k}h_k\big(f(z)\big) = z^{r^{k+1}}h_{k+1}(z) = \big[\varphi_{k+1}(z)^r\big]^{r^k},$$

and thus $\varphi_k \circ f = [\varphi_{k+1}]^r$. Passing to the limit we get $f = \varphi^{-1} \circ g \circ \varphi$, as claimed.

If ψ is another germ of biholomorphism of $(\mathbb{C}, 0)$ conjugating f with g, then we must have $\psi \circ \varphi^{-1}(z^r) = \psi \circ \varphi^{-1}(z)^r$ for all z in a neighbourhood of the origin; comparing the power series expansions at the origin we get $\psi \circ \varphi^{-1}(z) = az$ with $a^{r-1} = 1$, and hence $\psi(z) = a\varphi(z)$, as claimed.

Finally, since the order is the number of preimages of points close to the origin, z^r and z^s are locally topologically conjugated if and only if $r = s$, and hence we have (ii). □

Therefore also the one-dimensional local dynamics about a superattracting fixed point is completely clear; in the next subsection we shall discuss what happens about a parabolic fixed point.

1.2.3 Parabolic case

Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a (non-linear) holomorphic local dynamical system with a parabolic fixed point at the origin. Then we can write

$$f(z) = e^{2i\pi p/q}z + a_{r+1}z^{r+1} + a_{r+2}z^{r+2} + \cdots, \tag{1.6}$$

with $a_{r+1} \neq 0$.

Definition 1.2.5. Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be non-linear and with multiplier $\lambda = e^{2i\pi p/q}$. The rational number $p/q \in \mathbb{Q} \cap [0, 1)$ is the *rotation number* of f, and the *multiplicity* of f at the fixed point is the minimal number $r+1 \geq 2$ such that the coefficient of z^{r+1} in the power series expansion of f is non-zero. If $p/q = 0$ (that is, if the multiplier is 1), we shall say that f is *tangent to the identity.*

The first observation is that such a dynamical system is never locally conjugated to its linear part, not even topologically, unless it is of finite order:

Proposition 1.2.6. *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system with multiplier λ, and assume that $\lambda = e^{2i\pi p/q}$ is a (primitive) root of the unity of order q. Then f is holomorphically (or topologically or formally) locally conjugated to $g(z) = \lambda z$ if and only if $f^q \equiv \mathrm{Id}$.*

Proof. If $\varphi^{-1} \circ f \circ \varphi(z) = e^{2\pi i p/q}z$ then $\varphi^{-1} \circ f^q \circ \varphi = \mathrm{Id}$, and hence $f^q = \mathrm{Id}$.

Conversely, assume that $f^q \equiv \mathrm{Id}$ and set

$$\varphi(z) = \frac{1}{q}\sum_{j=0}^{q-1}\frac{f^j(z)}{\lambda^j}.$$

Then it is easy to check that $\varphi'(0) = 1$ and $\varphi \circ f(z) = \lambda\varphi(z)$, and so f is holomorphically (and topologically and formally) locally conjugated to λz. □

In particular, if f is tangent to the identity then it *cannot* be locally conjugated to the identity (unless it was the identity from the beginning, which is not a very interesting case dynamically speaking). More precisely, the stable set of such an f is never a neighbourhood of the origin. To understand why, let us first consider a map of the form

$$f(z) = z(1 + az^r)$$

for some $a \neq 0$. Let $v \in \mathbb{S}^1 \subset \mathbb{C}$ be such that av^r is real and positive. Then for any $c > 0$ we have

$$f(cv) = c(1 + c^r av^r)v \in \mathbb{R}^+ v;$$

moreover, $|f(cv)| > |cv|$. In other words, the half-line $\mathbb{R}^+ v$ is f-invariant and repelled from the origin, that is $K_{(U,f)} \cap \mathbb{R}^+ v = \varnothing$ for any representative (U, f) of f. Conversely, if av^r is real and negative then the segment $[0, |a|^{-1/r}]v$ is f-invariant and attracted by the origin. So $K_{(U,f)}$ neither is a neighbourhood of the origin nor reduces to $\{0\}$.

This example suggests the following definition:

Definition 1.2.6. Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be tangent to the identity of multiplicity $r+1 \geq 2$. Then a unit vector $v \in \mathbb{S}^1$ is an *attracting* [resp., *repelling*] *direction* for f at the origin if $a_{r+1}v^r$ is real and negative [resp., positive], where a_{r+1} is the coefficient of z^{r+1} in the power series expansion of f.

Clearly, there are r equally spaced attracting directions, separated by r equally spaced repelling directions: if $a_{r+1} = |a_{r+1}|e^{i\alpha}$, then $v = e^{i\theta}$ is attracting [resp., repelling] if and only if

$$\theta = \frac{2k+1}{r}\pi - \frac{\alpha}{r} \qquad \left[\text{resp., } \theta = \frac{2k}{r}\pi - \frac{\alpha}{r}\right].$$

Furthermore, a repelling [resp., attracting] direction for f is attracting [resp., repelling] for f^{-1}, which is defined in a neighbourhood of the origin.

Let (U, f) be a representative of f. To every attracting direction is associated a connected component of $K_{(U,f)} \setminus \{0\}$.

Definition 1.2.7. Let $v \in \mathbb{S}^1$ be an attracting direction for an $f \in \mathrm{End}(\mathbb{C}, 0)$ tangent to the identity, and let (U, f) be a representative of f. The *basin* centered at v is the set of points $z \in K_{(U,f)} \setminus \{0\}$ such that $f^k(z) \to 0$ and $f^k(z)/|f^k(z)| \to v$ (notice that, up to shrinking the domain of f, we can assume $f(z) \neq 0$ for all $z \in K_{(U,f)} \setminus \{0\}$). If z belongs to the basin centered at v, we shall say that the orbit of z *tends to* 0 *tangent to* v.

A slightly more specialized (but more useful) object is the following:

Definition 1.2.8. Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be tangent to the identity, and let (U, f) be a representative of f. An *attracting petal* centered at an attracting direction v of f is an open simply connected f-invariant set $P \subseteq K_{(U,f)} \setminus \{0\}$ such that a point $z \in K_{(U,f)} \setminus \{0\}$ belongs to the basin centered at v if and only if its orbit intersects P. In other words, the orbit of a point tends to 0 tangent to v if and only if it is eventually contained in P. A *repelling petal* (centered at a repelling direction) is an attracting petal for the inverse of f.

It turns out that the basins centered at the attracting directions are exactly the connected components of $K_{(U,f)} \setminus \{0\}$, as shown in the *Leau-Fatou flower theorem:*

Theorem 1.2.7. (Leau, 1897 [L]; Fatou, 1919-20 [F1–3]) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $r+1 \geq 2$ at the fixed point. Let $v_1^+, \ldots, v_r^+ \in \mathbb{S}^1$ be the r attracting directions of f at the origin, and $v_1^-, \ldots, v_r^- \in \mathbb{S}^1$ the r repelling directions. Then* *(i) for each attracting [resp., repelling] direction $v_j^{\pm}$ there exists an attracting [resp., repelling] petal $P_j^{\pm}$, so that the union of these $2r$ petals together with the origin forms a neighbourhood of the origin. Furthermore, the $2r$ petals are arranged ciclically so that two petals intersect if and only if the angle between their central directions is π/r.*

(ii) For any representative (U, f) of f, the stable set $K_{(U,f)} \setminus \{0\}$ is the (disjoint) union of the basins centered at the r attracting directions.

(iii) If B is a basin centered at one of the attracting directions, then there is a holomorphic function $\varphi\colon B \to \mathbb{C}$ such that $\varphi \circ f(z) = \varphi(z) + 1$ for all $z \in B$. Furthermore, if P is the corresponding petal constructed in part (i)*, then $\varphi|_P$ is a biholomorphism with an open subset of the complex plane containing a right half-plane, and so $f|_P$ is holomorphically conjugated to the translation $z \mapsto z + 1$.*

Proof. Up to a linear conjugation, we can assume that $a_{r+1} = -1$, so that the attracting directions are the r-th roots of unity. For any $\delta > 0$, the set $\{z \in \mathbb{C} \mid |z^r - \delta| < \delta\}$ has exactly r connected components, each one symmetric with respect to a different r-th root of unity; it will turn out that, for δ small enough, these connected components are attracting petals of f, even though to get a pointed neighbourhood of the origin we shall need larger petals.

For $j = 1, \ldots, r$ let $\Sigma_j \subset \mathbb{C}^*$ denote the sector centered about the attractive direction v_j^+ and bounded by two consecutive repelling directions, that is

$$\Sigma_j = \left\{ z \in \mathbb{C}^* \;\middle|\; \frac{2j-3}{r}\,\pi < \arg(z) < \frac{2j-1}{r}\,\pi \right\} .$$

Notice that each Σ_j contains a unique connected component $P_{j,\delta}$ of $\{z \in \mathbb{C} \mid |z^r - \delta| < \delta\}$; moreover, $P_{j,\delta}$ is tangent at the origin to the sector centered about v_j of amplitude π/r.

The main technical trick in this proof consists in transfering the setting to a neighbourhood of infinity in the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. Let $\psi\colon \mathbb{C}^* \to \mathbb{C}^*$ be given by

$$\psi(z) = \frac{1}{rz^r} ;$$

it is a biholomorphism between Σ_j and $\mathbb{C}^* \setminus \mathbb{R}^-$, with inverse $\psi^{-1}(w) = (rw)^{-1/r}$, choosing suitably the r-th root. Furthermore, $\psi(P_{j,\delta})$ is the right half-plane $H_\delta = \{w \in \mathbb{C} \mid \mathrm{Re}(w) > 1/(2r\delta)\}$.

When $|w|$ is so large that $\psi^{-1}(w)$ belongs to the domain of definition of f, the composition $F = \psi \circ f \circ \psi^{-1}$ makes sense, and we have

$$F(w) = w + 1 + O(w^{-1/r}) . \tag{1.7}$$

Thus to study the dynamics of f in a neighbourhood of the origin in Σ_j it suffices to study the dynamics of F in a neighbourhood of infinity.

The first observation is that when $\mathrm{Re}(w)$ is large enough then

$$\mathrm{Re}(F(w)) > \mathrm{Re}(w) + \frac{1}{2} ;$$

this implies that for δ small enough H_δ is F-invariant (and thus $P_{j,\delta}$ is f-invariant). Furthermore, by induction one has

$$\forall w \in H_\delta \qquad \mathrm{Re}(F^k(w)) > \mathrm{Re}(w) + \frac{k}{2} , \tag{1.8}$$

which implies that $F^k(w) \to \infty$ in H_δ (and $f^k(z) \to 0$ in $P_{j,\delta}$) as $k \to \infty$.

Now we claim that the argument of $w_k = F^k(w)$ tends to zero. Indeed, (1.7) and (1.8) yield

$$\frac{w_k}{k} = \frac{w}{k} + 1 + \frac{1}{k}\sum_{l=0}^{k-1} O(w_l^{-1/r}) \ ;$$

so Cesaro's theorem on the averages of a converging sequence implies

$$\frac{w_k}{k} \to 1 \ , \tag{1.9}$$

and thus $\arg(w_k) \to 0$ as $k \to \infty$. Going back to $P_{j,\delta}$, this implies that $f^k(z)/|f^k(z)| \to v_j$ for every $z \in P_{j,\delta}$. Since furthermore $P_{j,\delta}$ is centered about v_j^+, every orbit converging to 0 tangent to v_j^+ must intersect $P_{j,\delta}$, and thus we have proved that $P_{j,\delta}$ is an attracting petal.

Arguing in the same way with f^{-1} we get repelling petals; unfortunately, the petals obtained so far are too small to form a full pointed neighbourhood of the origin. In fact, as remarked before, each $P_{j,\delta}$ is contained in a sector centered about v_j of amplitude π/r; therefore the repelling and attracting petals obtained in this way do not intersect but are tangent to each other. We need larger petals.

So our aim is to find an f-invariant subset P_j^+ of Σ_j containing $P_{j,\delta}$ and which is tangent at the origin to a sector centered about v_j^+ of amplitude strictly greater than π/r. To do so, first of all remark that there are R, $C > 0$ such that

$$|F(w) - w - 1| \le \frac{C}{|w|^{1/r}} \tag{1.10}$$

as soon as $|w| > R$. Choose $\varepsilon \in (0,1)$ and select $\delta > 0$ so that $4r\delta < R^{-1}$ and $\varepsilon > 2C(4r\delta)^{1/r}$. Then $|w| > 1/(4r\delta)$ implies

$$|F(w) - w - 1| < \varepsilon/2 \ .$$

Set $M_\varepsilon = (1+\varepsilon)/(2r\delta)$ and let

$$\widetilde{H}_\varepsilon = \{w \in \mathbb{C} \mid |\operatorname{Im}(w)| > -\varepsilon \operatorname{Re}(w) + M_\varepsilon\} \cup H_\delta \ .$$

If $w \in \widetilde{H}_\varepsilon$ we have $|w| > 1/(2r\delta)$ and hence

$$\operatorname{Re}(F(w)) > \operatorname{Re}(w) + 1 - \varepsilon/2 \qquad \text{and} \qquad |\operatorname{Im}(F(w)) - \operatorname{Im}(w)| < \varepsilon/2 \ ; \tag{1.11}$$

it is then easy to check that $F(\widetilde{H}_\varepsilon) \subset \widetilde{H}_\varepsilon$ and that every orbit starting in $\widetilde{H}_\varepsilon$ must eventually enter H_δ. Thus $P_j^+ = \psi^{-1}(\widetilde{H}_\varepsilon)$ is as required, and we have proved (i).

To prove (ii) we need a further property of $\widetilde{H}_\varepsilon$. If $w \in \widetilde{H}_\varepsilon$, arguing by induction on $k \ge 1$ using (1.11) we get

$$k\left(1 - \frac{\varepsilon}{2}\right) < \operatorname{Re}(F^k(w)) - \operatorname{Re}(w)$$

and

$$\frac{k\varepsilon(1-\varepsilon)}{2} < |\operatorname{Im}(F^k(w))| + \varepsilon \operatorname{Re}(F^k(w)) - \bigl(|\operatorname{Im}(w)| + \varepsilon \operatorname{Re}(w)\bigr) \ .$$

This implies that for every $w_0 \in \widetilde{H}_\varepsilon$ there exists a $k_0 \ge 1$ so that we cannot have $F^{k_0}(w) = w_0$ for any $w \in \widetilde{H}_\varepsilon$. Coming back to the z-plane, this says that any inverse orbit of f must eventually leave P_j^+. Thus every (forward) orbit of f must eventually leave any repelling petal. So if $z \in K_{(U,f)} \setminus \{O\}$, where the stable set is computed working in the neighborhood U of the origin given by the union of repelling and attracting petals (together with the origin), the orbit of z must eventually land in an attracting petal, and thus z belongs to a basin centered at one of the r attracting directions — and (ii) is proved.

To prove (iii), first of all we notice that we have

$$|F'(w) - 1| \leq \frac{2^{1+1/r}C}{|w|^{1+1/r}} \tag{1.12}$$

in $\widetilde{H}_\varepsilon$. Indeed, (1.10) says that if $|w| > 1/(2r\delta)$ then the function $w \mapsto F(w) - w - 1$ sends the disk of center w and radius $|w|/2$ into the disk of center the origin and radius $C/(|w|/2)^{1/r}$; inequality (1.12) then follows from the Cauchy estimates on the derivative.

Now choose $w_0 \in H_\delta$, and set $\widetilde{\varphi}_k(w) = F^k(w) - F^k(w_0)$. Given $w \in \widetilde{H}_\varepsilon$, as soon as $k \in \mathbb{N}$ is so large that $F^k(w) \in H_\delta$ we can apply Lagrange's theorem to the segment from $F^k(w_0)$ to $F^k(w)$ to get a $t_k \in [0,1]$ such that

$$\begin{aligned}\left|\frac{\widetilde{\varphi}_{k+1}(w)}{\widetilde{\varphi}_k(w)} - 1\right| &= \left|\frac{F\big(F^k(w)\big) - F^k\big(F^k(w_0)\big)}{F^k(w) - F^k(w_0)} - 1\right| = \big|F'\big(t_k F^k(w) + (1-t_k)F^k(w_0)\big) - 1\big| \\ &\leq \frac{2^{1+1/r}C}{\min\{|\operatorname{Re}(F^k(w))|, |\operatorname{Re}(F^k(w_0))|\}^{1+1/r}} \leq \frac{C'}{k^{1+1/r}}\ ,\end{aligned}$$

where we used (1.12) and (1.9), and the constant C' is uniform on compact subsets of $\widetilde{H}_\varepsilon$ (and it can be chosen uniform on H_δ).

As a consequence, the telescopic product $\prod_k \widetilde{\varphi}_{k+1}/\widetilde{\varphi}_k$ converges uniformly on compact subsets of $\widetilde{H}_\varepsilon$ (and uniformly on H_δ), and thus the sequence $\widetilde{\varphi}_k$ converges, uniformly on compact subsets, to a holomorphic function $\widetilde{\varphi}\colon \widetilde{H}_\varepsilon \to \mathbb{C}$. Since we have

$$\begin{aligned}\widetilde{\varphi}_k \circ F(w) &= F^{k+1}(w) - F^k(w_0) \\ &= \widetilde{\varphi}_{k+1}(w) + F\big(F^k(w_0)\big) - F^k(w_0) \\ &= \widetilde{\varphi}_{k+1}(w) + 1 + O\big(|F^k(w_0)|^{-1/r}\big)\ ,\end{aligned}$$

it follows that

$$\widetilde{\varphi} \circ F(w) = \widetilde{\varphi}(w) + 1$$

on $\widetilde{H}_\varepsilon$. In particular, $\widetilde{\varphi}$ is not constant; being the limit of injective functions, by Hurwitz's theorem it is injective.

We now prove that the image of $\widetilde{\varphi}$ contains a right half-plane. First of all, we claim that

$$\lim_{\substack{|w|\to+\infty \\ w \in H_\delta}} \frac{\widetilde{\varphi}(w)}{w} = 1\ . \tag{1.13}$$

Indeed, choose $\eta > 0$. Since the convergence of the telescopic product is uniform on H_δ, we can find $k_0 \in \mathbb{N}$ such that

$$\left|\frac{\widetilde{\varphi}(w) - \widetilde{\varphi}_{k_0}(w)}{w - w_0}\right| < \frac{\eta}{3}$$

on H_δ. Furthermore, we have

$$\left|\frac{\widetilde{\varphi}_{k_0}(w)}{w-w_0}-1\right| = \left|\frac{k_0+\sum_{j=0}^{k_0-1}O(|F^j(w)|^{-1/r})+w_0-F^{k_0}(w_0)}{w-w_0}\right| = O(|w|^{-1})$$

on H_δ; therefore we can find $R>0$ such that

$$\left|\frac{\widetilde{\varphi}(w)}{w-w_0}-1\right| < \frac{\eta}{3}$$

as soon as $|w|>R$ in H_δ. Finally, if R is large enough we also have

$$\left|\frac{\widetilde{\varphi}(w)}{w-w_0}-\frac{\widetilde{\varphi}(w)}{w}\right| = \left|\frac{\widetilde{\varphi}(w)}{w-w_0}\right|\left|\frac{w}{w_0}\right| < \frac{\eta}{3}\ ,$$

and (1.13) follows.

Equality (1.13) clearly implies that $(\widetilde{\varphi}(w)-\overline{w})/(w-\overline{w}) \to 1$ as $|w|\to+\infty$ in H_δ for any $\overline{w}\in\mathbb{C}$. But this means that if $\mathrm{Re}(\overline{w})$ is large enough then the difference between the variation of the argument of $\widetilde{\varphi}-\overline{w}$ along a suitably small closed circle around $\overline{w}$ and the variation of the argument of $w-\overline{w}$ along the same circle will be less than 2π — and thus it will be zero. Then the argument principle implies that $\widetilde{\varphi}-\overline{w}$ and $w-\overline{w}$ have the same number of zeroes inside that circle, and thus $\overline{w}\in\widetilde{\varphi}(H_\delta)$, as required.

So setting $\varphi=\widetilde{\varphi}\circ\psi$, we have defined a function φ with the required properties on P_j^+. To extend it to the whole basin B it suffices to put

$$\varphi(z)=\varphi\big(f^k(z)\big)-k\ , \tag{1.14}$$

where $k\in\mathbb{N}$ is the first integer such that $f^k(z)\in P_j^+$. □

Remark 1.2.8. It is possible to construct petals that cannot be contained in any sector strictly smaller than Σ_j. To do so we need an F-invariant subset $\widehat{H}_\varepsilon$ of $C^*\setminus\mathbb{R}^-$ containing $\widetilde{H}_\varepsilon$ and containing eventually every half-line issuing from the origin (but $\mathbb{R}^-$). For $M>>1$ and $C>0$ large enough, replace the straight lines bounding $\widetilde{H}_\varepsilon$ on the left of $\mathrm{Re}(w)=-M$ by the curves

$$|\mathrm{Im}(w)|=\begin{cases} C\log|\mathrm{Re}(w)| & \text{if } r=1,\\ C|\mathrm{Re}(w)|^{1-1/r} & \text{if } r>1.\end{cases}$$

Then it is not too difficult to check that the domain $\widehat{H}_\varepsilon$ so obtained is as desired (see [CG]).

So we have a complete description of the dynamics in the neighbourhood of the origin. Actually, Camacho, using fundamental domains, has pushed this argument even further, obtaining a complete topological classification of one-dimensional holomorphic local dynamical systems tangent to the identity (see also [BH, Theorem 1.7]):

Theorem 1.2.9. (Camacho, 1978 [C]; Shcherbakov, 1982 [S]) *Let $f\in\mathrm{End}(\mathbb{C},0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $r+1$ at the fixed point. Then f is topologically locally conjugated to the map*

$$g(z)=z-z^{r+1}\ .$$

Remark 1.2.10. It is clear from the proof of Camacho [C] that the topological conjugation he founds is indeed C^∞ in a punctured neighbourhood of the origin. We refer to [C] and [Br2] for a proof, and to [J1] for a more detailed proof. Jenkins in [J1] also proved that if $f \in \mathrm{End}(\mathbb{C}, 0)$ is a holomorphic local dynamical system tangent to the identity with multiplicity 2, such that there exists a topological conjugation (conjugating it with $g(z) = z - z^2$), which is indeed real-analitic in a punctured neighbourhood of the origin, with real-analytic inverse, then there exists a holomorphic conjugation between f and g. Finally, Martinet and Ramis [MR] have proved that if a germ $f \in \mathrm{End}(\mathbb{C}, 0)$, tangent to the identity, is C^1-conjugated (in a full neighbourhood of the origin) to $g(z) = z - z^{r+1}$, then it is holomorphically or antiholomorphically conjugated to it.

The formal classification is simple too, though different (see, e.g., Milnor [Mi]):

Proposition 1.2.11. *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $r + 1$ at the fixed point. Then f is formally conjugated to the map*

$$g(z) = z - z^{r+1} + \beta z^{2r+1} , \tag{1.15}$$

where β is a formal (and holomorphic) invariant given by

$$\beta = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - f(z)} , \tag{1.16}$$

where the integral is taken over a small positive loop γ around the origin.

Proof. An easy computation shows that if f is given by (1.15) then (1.16) holds. Let us now show that the integral in (1.16) is a holomorphic invariant. Let φ be a local biholomorphism fixing the origin, and set $F = \varphi^{-1} \circ f \circ \varphi$. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z - f(z)} = \frac{1}{2\pi i} \int_{\varphi^{-1}\circ\gamma} \frac{\varphi'(w)\, dw}{\varphi(w) - f\big(\varphi(w)\big)} = \frac{1}{2\pi i} \int_{\varphi^{-1}\circ\gamma} \frac{\varphi'(w)\, dw}{\varphi(w) - \varphi\big(F(w)\big)} .$$

Now, we can clearly find M, $M_1 > 0$ such that

$$\begin{aligned}\left| \frac{1}{w - F(w)} - \frac{\varphi'(w)}{\varphi(w) - \varphi\big(F(w)\big)} \right| &= \frac{1}{\big|\varphi(w) - \varphi\big(F(w)\big)\big|} \left| \frac{\varphi(w) - \varphi\big(F(w)\big)}{w - F(w)} - \varphi'(w) \right| \\ &\le M \frac{|w - F(w)|}{\big|\varphi(w) - \varphi\big(F(w)\big)\big|} \le M_1 ,\end{aligned}$$

in a neighbourhood of the origin, where the last inequality follows from the fact that $\varphi'(0) \neq 0$. This means that the two meromorphic functions $1/\big(w - F(w)\big)$ and $\varphi'(w)/\big(\varphi(w) - \varphi(\big(F(w)\big)\big)$ differ by a holomorphic function; so they have the same integral along any small loop surrounding the origin, and

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z - f(z)} = \frac{1}{2\pi i} \int_{\varphi^{-1}\circ\gamma} \frac{dw}{w - F(w)},$$

as claimed.

To prove that f is formally conjugated to g, let us first take a local formal change of coordinates φ of the form

$$\varphi(z) = z + \mu z^d + O_{d+1} \tag{1.17}$$

with $\mu \neq 0$, and where we are writing O_{d+1} instead of $O(z^{d+1})$. Hence $\varphi^{-1}(z) = z - \mu z^d + O_{d+1}$, $(\varphi^{-1})'(z) = 1 - d\mu z^{d-1} + O_d$ and $(\varphi^{-1})^{(j)} = O_{d-j}$ for all $j \geq 2$. Then using the Taylor expansion of φ^{-1} we get

$$\begin{aligned}
\varphi^{-1} \circ f \circ \varphi(z) &= \varphi^{-1}\left(\varphi(z) + \sum_{j \geq r+1} a_j \varphi(z)^j\right) \\
&= z + (\varphi^{-1})'\bigl(\varphi(z)\bigr) \sum_{j \geq r+1} a_j z^j (1 + \mu z^{d-1} + O_d)^j + O_{d+2r} \\
&= z + [1 - d\mu z^{d-1} + O_d] \sum_{j \geq r+1} a_j z^j (1 + j\mu z^{d-1} + O_d) + O_{d+2r} \\
&= z + a_{r+1} z^{r+1} + \cdots + a_{r+d-1} z^{r+d-1} \\
&\qquad\qquad + [a_{r+d} + (r+1-d)\mu a_{r+1}] z^{r+d} + O_{r+d+1}.
\end{aligned} \tag{1.18}$$

This means that for all $d \neq r+1$ we can use a polynomial change of coordinates of the form $\varphi(z) = z + \mu z^d$ to remove the term of degree $r+d$ from the Taylor expansion of f without changing the lower degree terms.

So to conjugate f to g it suffices to use a linear change of coordinates to get $a_{r+1} = -1$, and then apply a sequence of change of coordinates of the form $\varphi(z) = z + \mu z^d$ to kill all the terms in the Taylor expansion of f but the term of degree z^{2r+1}.

Finally, formula (1.18) also shows that two maps of the form (1.15) with different β cannot be formally conjugated, and we are done. □

Definition 1.2.9. The number β given by (1.16) is called the *index* of f at the fixed point. The *iterative residue* of f is then defined by

$$\mathrm{Resit}(f) = \frac{r+1}{2} - \beta\ .$$

The iterative residue has been introduced by Écalle [É1], and it behaves nicely under iteration; for instance, it is possible to prove (see [BH, Proposition 3.10]) that

$$\mathrm{Resit}(f^k) = \frac{1}{k}\,\mathrm{Resit}(f)\ .$$

The holomorphic classification of maps tangent to the identity is much more complicated: as shown by Écalle [É2–3] and Voronin [Vo] in 1981, it depends on functional invariants. We shall now try and roughly describe it; see [I2], [M1-2], [Ki], [BH] and the original papers for details.

Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be tangent to the identity with multiplicity $r+1$ at the fixed point; up to a linear change of coordinates we can assume that $a_{r+1} = -1$. Let $P_j^\pm$ be a set of petals as in Theorem 1.2.7.(i), ordered so that P_1^+ is centered on the positive real semiaxis, and the others are arranged cyclically counterclockwise. Denote by φ_j^+ [resp., φ_j^-] the biholomorphism conjugating $f|_{P_j^+}$ [resp., $f|_{P_j^-}$] to the shift $z \mapsto z+1$ in a right [resp., left] half-plane given by Theorem 1.2.7.(iii) — applied to f^{-1} for the repelling petals. If we moreover require that

$$\varphi_j^\pm(z) = \frac{1}{rz^r} \pm \mathrm{Resit}(f) \cdot \log z + o(1)\ , \tag{1.19}$$

then φ_j is uniquely determined.

Put now $U_j^+ = P_j^- \cap P_{j+1}^+$, $U_j^- = P_j^- \cap P_j^+$, and $S_j^\pm = \bigcup_{k\in\mathbb{Z}} U_j^\pm$. Using the dynamics as in (1.14) we can extend φ_j^- to $S_j^\pm$, and φ_j^+ to $S_{j-1}^+ \cup S_j^-$; put $V_j^\pm = \varphi_j^-(S_j^\pm)$, $W_j^- = \varphi_j^+(S_j^-)$ and $W_j^+ = \varphi_{j+1}^+(S_j^+)$. Then let $H_j^-\colon V_j^- \to W_J^-$ be the restriction of $\varphi_j^+ \circ (\varphi_j^-)^{-1}$ to V_j^-, and $H_j^+\colon V_j^+ \to W_j^+$ the restriction of $\varphi_{j+1}^+ \circ (\varphi_j^-)^{-1}$ to V_j^+.

It is not difficult to see that $V_j^\pm$ and $W_j^\pm$ are invariant under translation by 1, and that V_j^+ and W_j^+ contain an upper half-plane while V_j^- and W_j^- contain a lower half-plane. Moreover, we have $H_j^\pm(z+1) = H_j^\pm(z)+1$; therefore using the projection $\pi(z) = \exp(2\pi i z)$ we can induce holomorphic maps $h_j^\pm\colon \pi(V_j^\pm) \to \pi(W_j^\pm)$, where $\pi(V_j^+)$ and $\pi(W_j^+)$ are pointed neighbourhood of the origin, and $\pi(V_j^-)$ and $\pi(W_j^-)$ are pointed neighbourhood of $\infty \in \mathbb{P}^1(\mathbb{C})$.

It is possible to show that one obtains a holomorphic germ $h_j^+ \in \mathrm{End}(\mathbb{C}, 0)$ setting $h_j^+(0) = 0$, and that one obtains a holomorphic germ $h_j^- \in \mathrm{End}\big(\mathbb{P}^1(C), \infty\big)$ setting $h_j^-(\infty) = \infty$. Furthermore, denoting by λ_j^+ [resp., λ_j^-] the multiplier of h_j^+ at 0 [resp., of h_j^- at ∞], it turns out that

$$\prod_{j=1}^{r} (\lambda_j^+ \lambda_j^-) = \exp\big[4\pi^2 \mathrm{Resit}(f)\big] \ . \tag{1.20}$$

Now, if we replace f by a holomorphic local conjugate $\widetilde{f} = \psi^{-1} \circ f \circ \psi$, and denote by $\widetilde{h}_j^\pm$ the corresponding germs, it is not difficult to check that (up to a cyclic renumbering of the petals) there are constants α_j, $\beta_j \in \mathbb{C}^*$ such that

$$\widetilde{h}_j^-(z) = \alpha_j h_j^-\left(\frac{z}{\beta_j}\right) \qquad \text{and} \qquad \widetilde{h}_j^+(z) = \alpha_{j+1} h_j^+\left(\frac{z}{\beta_j}\right) \ . \tag{1.21}$$

This suggests the introduction of an equivalence relation on the set of $2r$-uples of holomorphic germs $(h_1^\pm, \dots, h_r^\pm)$.

Definition 1.2.10. Let M_r denote the set of $2r$-uples of holomorphic germs $\mathbf{h} = (h_1^\pm, \dots, h_r^\pm)$, with $h_j^+ \in \mathrm{End}(\mathbb{C}, 0)$, $h_j^- \in \mathrm{End}\big(\mathbb{P}^1(\mathbb{C}), \infty\big)$, and whose multipliers satisfy (1.20). We shall say that $\mathbf{h}$, $\widetilde{\mathbf{h}} \in M_r$ are *equivalent* if up to a cyclic permutation of the indices we have (1.21) for suitable α_j, $\beta_j \in \mathbb{C}^*$. We denote by $\mathcal{M}_r$ the set of all equivalence classes.

The procedure described above allows then to associate to any $f \in \mathrm{End}(\mathbb{C}, 0)$ tangent to the identity with multiplicity $r + 1$ an element $\mu_f \in \mathcal{M}_r$.

Definition 1.2.11. Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be tangent to the identity. The element $\mu_f \in \mathcal{M}_r$ given by this procedure is the *sectorial invariant* of f.

Then the holomorphic classification proved by Écalle and Voronin is

Theorem 1.2.12. (Écalle, 1981 [É2–3]; Voronin, 1981 [Vo]) *Let f, $g \in \mathrm{End}(\mathbb{C}, 0)$ be two holomorphic local dynamical systems tangent to the identity. Then f and g are holomorphically locally conjugated if and only if they have the same multiplicity, the same index and the same sectorial invariant. Furthermore, for any $r \geq 1$, $\beta \in \mathbb{C}$ and $\mu \in \mathcal{M}_r$ there exists $f \in \mathrm{End}(\mathbb{C}, 0)$ tangent to the identity with multiplicity $r + 1$, index β and sectorial invariant μ.*

Remark 1.2.13. In particular, holomorphic local dynamical systems tangent to the identity give examples of local dynamical systems that are topologically conjugated without being neither holomorphically nor formally conjugated, and of local dynamical systems that are formally conjugated without being holomorphically conjugated. Particular examples of germs in $\mathrm{End}(\mathbb{C},0)$ tangent to the identity that are formally conjugated without being holomorphically conjugated can be found in [Na] and [Tr].

We would also like to mention the following result of Ribón appeared in the appendix of [CGBM]. It is known (see [Br2]) that for any germ $f \in \mathrm{End}(\mathbb{C},0)$ there exists a unique *formal* (not necessarily holomorphic) vector field, called the *infinitesimal generator of* f, whose time-one flow coincides with f.

Theorem 1.2.14. (Ribón, 2008 [CGBM]) *Let* $f \in \mathrm{End}(\mathbb{C},0) \setminus \{\mathrm{Id}\}$ *be a germ tangent to the identity. If there exists a germ of real-analytic foliation* $\mathcal{F}$*, defined by a 1-form having an isolated singularity, such that* $f^*\mathcal{F} = \mathcal{F}$*, then the formal infinitesimal generator of* f *is a germ of holomorphic vector field singular at the origin. In particular,* f *is holomorphically conjugated to the germ* $g(z) = z - z^{r+1}$*, where* $r+1$ *is the multiplicity of* f*.*

Finally, if $f \in \mathrm{End}(\mathbb{C},0)$ satisfies $\lambda = e^{2\pi i p/q}$, then f^q is tangent to the identity. Therefore we can apply the previous results to f^q and then infer informations about the dynamics of the original f, because of the following

Lemma 1.2.15. *Let* f*,* $g \in \mathrm{End}(\mathbb{C},0)$ *be two holomorphic local dynamical systems with the same multiplier* $e^{2\pi i p/q} \in \mathbb{S}^1$*. Then* f *and* g *are holomorphically locally conjugated if and only if* f^q *and* g^q *are.*

Proof. One direction is obvious. For the converse, let φ be a germ conjugating f^q and g^q; in particular,

$$g^q = \varphi^{-1} \circ f^q \circ \varphi = (\varphi^{-1} \circ f \circ \varphi)^q .$$

So, up to replacing f by $\varphi^{-1} \circ f \circ \varphi$, we can assume that $f^q = g^q$. Put

$$\psi = \sum_{k=0}^{q-1} g^{q-k} \circ f^k = \sum_{k=1}^{q} g^{q-k} \circ f^k .$$

The germ ψ is a local biholomorphism, because $\psi'(0) = q \neq 0$, and it is easy to check that $\psi \circ f = g \circ \psi$. □

We list here a few results; see [Mi], [Ma], [C], [É2–3], [Vo] and [BH] for proofs and further details.

Proposition 1.2.16. *Let* $f \in \mathrm{End}(\mathbb{C},0)$ *be a holomorphic local dynamical system with multiplier* $\lambda \in \mathbb{S}^1$*, and assume that* λ *is a primitive root of the unity of order* q*. Assume that* $f^q \not\equiv \mathrm{Id}$*. Then there exist* $n \geq 1$ *and* $\alpha \in \mathbb{C}$ *such that* f *is formally conjugated to*

$$g(z) = \lambda z - z^{nq+1} + \alpha z^{2nq+1} .$$

Definition 1.2.12. The number n is the *parabolic multiplicity* of f, and $\alpha \in \mathbb{C}$ is the *index* of f; the *iterative residue* of f is then given by

$$\mathrm{Resit}(f) = \frac{nq+1}{2} - \alpha .$$

Proposition 1.2.17. (Camacho, 1978 [C]) *Let* $f \in \mathrm{End}(\mathbb{C},0)$ *be a holomorphic local dynamical system with multiplier* $\lambda \in \mathbb{S}^1$*, and assume that* λ *is a primitive root of the unity of order* q*. Assume that* $f^q \not\equiv \mathrm{Id}$*, and has parabolic multiplicity* $n \geq 1$*. Then* f *is topologically conjugated to*

$$g(z) = \lambda z - z^{nq+1} .$$

Theorem 1.2.18. (Leau, 1897 [L]; Fatou, 1919-20 [F1–3]) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system with multiplier $\lambda \in \mathbb{S}^1$, and assume that λ is a primitive root of the unity of order q. Assume that $f^q \not\equiv \mathrm{Id}$, and let $n \geq 1$ be the parabolic multiplicity of f. Then f^q has multiplicity $nq + 1$, and f acts on the attracting [resp., repelling] petals of f^q as a permutation composed by n disjoint cycles. Finally, $K_{(U,f)} = K_{(U,f^q)}$ for any representatives (U, f) and (U, f^q) of f and f^q.*

Furthermore, it is possible to define the sectorial invariant of such a holomorphic local dynamical system, composed by $2nq$ germs whose multipliers still satisfy (1.20), and the analogue of Theorem 1.2.12 holds.

1.2.4 Elliptic case

We are left with the elliptic case:

$$f(z) = e^{2\pi i\theta} z + a_2 z^2 + \cdots \in \mathbb{C}_0\{z\} , \tag{1.22}$$

with $\theta \notin \mathbb{Q}$. It turns out that the local dynamics depends mostly on numerical properties of θ. The main question here is whether such a local dynamical system is holomorphically conjugated to its linear part. Let us introduce a bit of terminology.

Definition 1.2.13. We shall say that a holomorphic dynamical system of the form (1.22) is *holomorphically linearizable* if it is holomorphically locally conjugated to its linear part, the irrational rotation $z \mapsto e^{2\pi i\theta} z$. In this case, we shall say that 0 is a *Siegel point* for f; otherwise, we shall say that it is a *Cremer point.*

It turns out that for a full measure subset B of $\theta \in [0, 1] \setminus \mathbb{Q}$ all holomorphic local dynamical systems of the form (1.22) are holomorphically linearizable. Conversely, the complement $[0, 1]\setminus B$ is a G_δ-dense set, and for all $\theta \in [0, 1]\setminus B$ the quadratic polynomial $z \mapsto z^2 + e^{2\pi i\theta} z$ is not holomorphically linearizable. This is the gist of the results due to Cremer, Siegel, Brjuno and Yoccoz we shall describe in this section.

The first worthwhile observation in this setting is that it is possible to give a topological characterization of holomorphically linearizable local dynamical systems. Recall that a point p is stable for $f \in \mathrm{End}(M, p)$ if it belongs to the interior of $K_{(U,f)}$, where (U, f) is a representative of f.

Proposition 1.2.19. *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system with multiplier $\lambda \in \mathbb{S}^1$. Then f is holomorphically linearizable if and only if it is topologically linearizable if and only if 0 is stable for f.*

Proof. If f is holomorphically linearizable it is topologically linearizable, and if it is topologically linearizable (and $|\lambda| = 1$) then it is stable. Assume that 0 is stable, and set

$$\varphi_k(z) = \frac{1}{k} \sum_{j=0}^{k-1} \frac{f^j(z)}{\lambda^j} ,$$

so that $\varphi_k'(0) = 1$ and

$$\varphi_k \circ f = \lambda \varphi_k + \frac{\lambda}{k} \left(\frac{f^k}{\lambda^k} - \mathrm{Id} \right) . \tag{1.23}$$

The stability of 0 implies that there are bounded open sets $V \subset U$ containing the origin such that $f^k(V) \subset U$ for all $k \in \mathbb{N}$. Since $|\lambda| = 1$, it follows that $\{\varphi_k\}$ is a uniformly bounded family on V, and hence, by Montel's theorem, it admits a converging subsequence. But (1.23) implies that a converging subsequence converges to a conjugation between f and the rotation $z \mapsto \lambda z$, and so f is holomorphically linearizable. □

The second important observation is that two elliptic holomorphic local dynamical systems with the same multiplier are always formally conjugated:

Proposition 1.2.20. *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system of multiplier $\lambda = e^{2\pi i\theta} \in \mathbb{S}^1$ with $\theta \notin \mathbb{Q}$. Then f is formally conjugated to its linear part, by a unique formal power series tangent to the identity.*

Proof. We shall prove that there is a unique formal power series of the form

$$h(z) = z + h_2 z^2 + \cdots \in \mathbb{C}_0[\![z]\!]$$

such that $h(\lambda z) = f\big(h(z)\big)$. Indeed we have

$$\begin{aligned} h(\lambda z) - f\big(h(z)\big) &= \sum_{j\geq 2}\left\{\big[(\lambda^j - \lambda)h_j - a_j\big]z^j - a_j \sum_{\ell=1}^{j} \binom{j}{\ell} z^{\ell+j}\left(\sum_{k\geq 2} h_k z^{k-2}\right)^{\ell}\right\} \\ &= \sum_{j\geq 2}\big[(\lambda^j - \lambda)h_j - a_j - P_j(h_2, \dots, h_{j-1})\big]z^j\ , \end{aligned} \tag{1.24}$$

where P_j is a polynomial in $j-2$ variables with coefficients depending on $a_2, \dots, a_{j-1}$. It follows that the coefficients of h are uniquely determined by induction using the formula

$$h_j = \frac{a_j + P_j(h_2, \dots, h_{j-1})}{\lambda^j - \lambda}\ . \tag{1.25}$$

In particular, h_j depends only on λ, $a_2, \dots, a_j$. □

Remark 1.2.21. The same proof shows that any holomorphic local dynamical system with multiplier $\lambda \neq 0$ and not a root of unity is formally conjugated to its linear part.

The formal power series linearizing f is not converging if its coefficients grow too fast. Thus (1.25) links the radius of convergence of h to the behavior of $\lambda^j - \lambda$: if the latter becomes too small, the series defining h does not converge. This is known as the *small denominators problem* in this context.

It is then natural to introduce the following quantity:

$$\omega_\lambda(m) = \min_{1\leq k\leq m} |\lambda^k - \lambda|\ ,$$

for $\lambda \in \mathbb{S}^1$ and $m \geq 1$. Clearly, λ is a root of unity if and only if $\omega_\lambda(m) = 0$ for all m greater or equal to some $m_0 \geq 1$; furthermore,

$$\lim_{m\to+\infty} \omega_\lambda(m) = 0$$

for all $\lambda \in \mathbb{S}^1$.

The first one to actually prove that there are non-linearizable elliptic holomorphic local dynamical systems has been Cremer, in 1927 [Cr1]. His more general result is the following:

Theorem 1.2.22. (Cremer, 1938 [Cr2]) *Let $\lambda \in \mathbb{S}^1$ be such that*

$$\limsup_{m\to+\infty} \frac{1}{m} \log \frac{1}{\omega_\lambda(m)} = +\infty \,. \tag{1.26}$$

Then there exists $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ which is not holomorphically linearizable. Furthermore, the set of $\lambda \in \mathbb{S}^1$ satisfying (1.26) contains a G_δ-dense set.

Proof. Choose inductively $a_j \in \{0, 1\}$ so that $|a_j + P_j| \geq 1/2$ for all $j \geq 2$, where P_j is as in (1.25). Then

$$f(z) = \lambda z + a_2 z^2 + \cdots \in \mathbb{C}_0\{z\} \,,$$

while (1.26) implies that the radius of convergence of the formal linearization h is 0, and thus f cannot be holomorphically linearizable, as required.

Finally, let $C(q_0) \subset \mathbb{S}^1$ denote the set of $\lambda = e^{2\pi i\theta} \in \mathbb{S}^1$ such that

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{2^{q!}} \tag{1.27}$$

for some $p/q \in \mathbb{Q}$ in lowest terms, with $q \geq q_0$. Then it is not difficult to check that each $C(q_0)$ is a dense open set in $\mathbb{S}^1$, and that all $\lambda \in \mathcal{C} = \bigcap_{q_0 \geq 1} C(q_0)$ satisfy (1.26). Indeed, if $\lambda = e^{2\pi i\theta} \in \mathcal{C}$ we can find $q \in \mathbb{N}$ arbitrarily large such that there is $p \in \mathbb{N}$ so that (1.27) holds. Now, it is easy to see that

$$|e^{2\pi i t} - 1| \leq 2\pi |t|$$

for all $t \in [-1/2, 1/2]$. Then, letting p_0 be the closest integer to $q\theta$, so that $|q\theta - p_0| \leq 1/2$, we have

$$|\lambda^q - 1| = |e^{2\pi i q\theta} - e^{2\pi i p_0}| = |e^{2\pi i(q\theta - p_0)} - 1| \leq 2\pi|q\theta - p_0| \leq 2\pi|q\theta - p| < \frac{2\pi}{2^{q!-1}}$$

for arbitrarily large q, and (1.26) follows. □

On the other hand, Siegel in 1942 gave a condition on the multiplier ensuring holomorphic linearizability:

Theorem 1.2.23. (Siegel, 1942 [Si]) *Let $\lambda \in \mathbb{S}^1$ be such that there exists $\beta > 1$ and $\gamma > 0$ so that*

$$\forall m \geq 2 \qquad \frac{1}{\omega_\lambda(m)} \leq \gamma\, m^\beta. \tag{1.28}$$

Then all $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ are holomorphically linearizable. Furthermore, the set of $\lambda \in \mathbb{S}^1$ satisfying (1.28) for some $\beta > 1$ and $\gamma > 0$ is of full Lebesgue measure in $\mathbb{S}^1$.

Remark 1.2.24. If $\theta \in [0, 1) \setminus \mathbb{Q}$ is algebraic then $\lambda = e^{2\pi i\theta}$ satisfies (1.28) for some $\beta > 1$ and $\gamma > 0$. However, the set of $\lambda \in \mathbb{S}^1$ satisfying (1.28) is much larger, being of full measure.

Remark 1.2.25. It is interesting to notice that for generic (in a topological sense) $\lambda \in \mathbb{S}^1$ there is a non-linearizable holomorphic local dynamical system with multiplier λ, while for almost all (in a measure-theoretic sense) $\lambda \in \mathbb{S}^1$ every holomorphic local dynamical system with multiplier λ is holomorphically linearizable.

Theorem 1.2.23 suggests the existence of a number-theoretical condition on λ ensuring that the origin is a Siegel point for any holomorphic local dynamical system of multiplier λ. And indeed this is the content of the celebrated *Brjuno-Yoccoz theorem:*

Theorem 1.2.26. (Brjuno, 1965 [Brj1–3], Yoccoz, 1988 [Y1–2]) *Let $\lambda \in \mathbb{S}^1$. Then the following statements are equivalent:*

(i) the origin is a Siegel point for the quadratic polynomial $f_\lambda(z) = \lambda z + z^2$;

(ii) the origin is a Siegel point for all $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ;

(iii) the number λ satisfies Brjuno's condition

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} \log \frac{1}{\omega_\lambda(2^{k+1})} < +\infty \,. \tag{1.29}$$

Brjuno, using majorant series as in Siegel's proof of Theorem 1.2.23 (see also [He] and references therein) has proved that condition (iii) implies condition (ii). Yoccoz, using a more geometric approach based on conformal and quasi-conformal geometry, has proved that (i) is equivalent to (ii), and that (ii) implies (iii), that is that if λ does not satisfy (1.29) then the origin is a Cremer point for some $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ — and hence it is a Cremer point for the quadratic polynomial $f_\lambda(z)$. See also [P9] for related results.

Remark 1.2.27. Condition (1.29) is usually expressed in a different way. Write $\lambda = e^{2\pi i\theta}$, and let $\{p_k/q_k\}$ be the sequence of rational numbers converging to θ given by the expansion in continued fractions. Then (1.29) is equivalent to

$$\sum_{k=0}^{+\infty} \frac{1}{q_k} \log q_{k+1} < +\infty \,,$$

while (1.28) is equivalent to

$$q_{n+1} = O(q_n^\beta) \,,$$

and (1.26) is equivalent to

$$\limsup_{k\to+\infty} \frac{1}{q_k} \log q_{k+1} = +\infty \,.$$

See [K] for a tractation on continued fractions, [He], [Y2], [Mi] and references therein for other details on condition (1.29).

Remark 1.2.28. A clear obstruction to the holomorphic linearization of an elliptic germ $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier $\lambda = e^{2\pi i\theta} \in \mathbb{S}^1$ is the existence of *small cycles,* that is of periodic orbits contained in any neighbourhood of the origin. Pérez-Marco [P1], using Yoccoz's techniques, has shown that when the series

$$\sum_{k=0}^{+\infty} \frac{\log\log q_{k+1}}{q_k}$$

converges then every germ with multiplier λ is either linearizable or has small cycles, and that when the series diverges there exists such germs with a Cremer point but without small cycles.

The complete proof of Theorem 1.2.26 is beyond the scope of this chapter. We shall limit ourselves to describe a proof (adapted from the original one of [Brj1–3]) of the implication (iii)$\Longrightarrow$(ii), to report two of the easiest results of [Y2], and to illustrate what is the connection between condition (1.29) and the radius of convergence of the formal linearizing map.

Let us begin with Brjuno's theorem:

Theorem 1.2.29. (Brjuno, 1965 [Brj1–3]) *Assume that $\lambda = e^{2\pi i\theta} \in \mathbb{S}^1$ satisfies the Brjuno's condition*

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} \log \frac{1}{\omega_\lambda(2^{k+1})} < +\infty \ . \tag{1.30}$$

Then the origin is a Siegel point for all $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ.

Proof. We already know, thanks to Proposition 1.2.20, that there exists a unique formal power series

$$h(z) = z + z \sum_{k \geq 1} h_k z^k$$

such that $h^{-1} \circ f \circ h(z) = \lambda z$; we shall prove that h is actually converging. To do so it suffices to show that

$$\sup_k \frac{1}{k} \log |h_k| < \infty \ . \tag{1.31}$$

Writing $f(z) = \lambda z + z\widehat{f}(z)$ and $h(z) = z(1 + \widehat{h}(z))$, from $h(\lambda z) = f(h(z))$ we obtain

$$\lambda z \left(1 + \widehat{h}(\lambda z)\right) = \lambda z \left(1 + \widehat{h}(z)\right) + z \left(1 + \widehat{h}(z)\right) \widehat{f} \left(z + z\widehat{h}(z)\right),$$

hence, dividing by λz both sides and simplifying, we get

$$\widehat{h}(\lambda z) - \widehat{h}(z) = \lambda^{-1} \left(1 + \widehat{h}(z)\right) \widehat{f} \left(z + z\widehat{h}(z)\right),$$

thus, for each $k \geq 2$, we have

$$\left(\lambda^k - 1\right) h_k = \lambda^{-1} \left\{ \left(1 + \widehat{h}(z)\right) \widehat{f} \left(z + z\widehat{h}(z)\right) \right\}_k , \tag{1.32}$$

where, given a power series $g(z)$, we denote by $\{g(z)\}_k$ the coefficient of z^k in its power series expansion.

It is known (see [SM] pp. 110–113) that there exist constants c_1 and c_2 such that the power series of the function $c_1 z(c_2 - z)^{-1}$ dominates the Maclaurin series of the holomorphic function $f(z)$. Let $\|k\theta\|$ be the distance of $k\theta$ from the nearest integer, i.e., $\|k\theta\| = \min_{h\in\mathbb{Z}} |k\theta - h|$. Since the ratio of the length of a chord to the length of the smaller of the two corresponding arcs is at least $2/\pi$, we obtain

$$\frac{|\lambda^k - 1|}{2\pi\|k\theta\|} \geq \frac{2}{\pi},$$

hence

$$4\|k\theta\| \leq |\lambda^k - 1|.$$

Therefore, from (1.32) we obtain the estimate

$$\|k\theta\| \, |h_k| \leq \frac{1}{4} \left\{ \frac{c_1 z\big(1 + M(\widehat{h}\,)\big)^2}{c_2 - z\big(1 + M(\widehat{h}\,)\big)} \right\}_k ,$$

where $M(\widehat{h}\,)$ is the power series $\sum_{k\geq 1} |h_k| z^k$.

Let us now consider the power series $g(z) = \sum_{k\geq 1} g_k z^k$ with positive real coefficients defined recursively by

$$\|k\theta\|\, g_k = \frac{1}{4}\left\{\frac{c_1 z(1+g(z))^2}{c_2 - z(1+g(z))}\right\}_k . \tag{1.33}$$

It is not difficult to see by induction that $|h_k| \leq g_k$ for each $k \geq 1$; hence, to prove (1.31), it suffices to prove that

$$\sup_k \frac{1}{k} \log g_k < \infty . \tag{1.34}$$

Multiplying the k-th equation (1.33) by z^k and summing over all k, we get

$$\sum_{k\geq 1} \|k\theta\|\, g_k z^k = \frac{1}{4}\frac{c_1 z(1+g(z))^2}{c_2 - z(1+g(z))},$$

hence

$$c_2 \sum_{k\geq 1} \|k\theta\|\, g_k z^k = \frac{1}{4} c_1 z(1+g(z))^2 + z(1+g(z)) \sum_{k\geq 1} \|k\theta\|\, g_k z^k.$$

Since $\|k\theta\| \leq 1/2$ for any $k \in \mathbb{Z}$, last equality yields

$$\begin{aligned}\sum_{k\geq 1} \|k\theta\|\, g_k z^k &\prec \frac{1}{2c_2} z(1+g(z))g(z) + \frac{1}{4c_2} c_1 z(1+g(z))^2\\ &\prec \frac{2+c_1}{4c_2} z(1+g(z))^2,\end{aligned}$$

where, given two power series $\phi(z)$ and $\psi(z)$, $\phi \prec \psi$ means that $\phi_k \leq \psi_k$ for each $k \geq 1$. Therefore, setting $c_3 = (2+c_1)/(4c_2)$, for each $k \geq 1$ we have

$$\|k\theta\|\, g_k \leq c_3 \sum_{\substack{k_1+k_2+1=k\\ k_1,k_2\geq 0}} g_{k_1} g_{k_2}. \tag{1.35}$$

Let us now define inductively

$$\alpha_k = \begin{cases} 1 & \text{if } k = 0,\\ c_3 \displaystyle\sum_{\substack{k_1+k_2+1=k\\ k_1,k_2\geq 0}} \alpha_{k_1}\alpha_{k_2} & \text{if } k \geq 2,\end{cases}$$

and

$$\delta_k = \begin{cases} 1 & \text{if } k = 0,\\ \|k\theta\|^{-1} \displaystyle\max_{\substack{k_1+k_2+1=k\\ k_1,k_2\geq 0}} \delta_{k_1}\delta_{k_2}, & \text{if } k \geq 2.\end{cases}$$

Then it is easy to check by induction that

$$g_k \leq \alpha_k \delta_k$$

for all k. Therefore, to establish (1.34) it suffices to prove analogous estimates for α_k and δ_k.

To estimate α_k, let $\alpha(t) = \sum_{k\geq 0} \alpha_k t^k$. We have

$$\alpha(t) - 1 = \sum_{k\geq 1} \alpha_k t^k = c_3 t(\alpha(t))^2.$$

This equation has a unique holomorphic solution vanishing at zero

$$\alpha(t) = \frac{1 - \sqrt{1 - 4c_3 t}}{2c_3 t},$$

defined for $|t|$ small enough. Hence,

$$\sup_k \frac{1}{k} \log \alpha_k < \infty,$$

as we wanted.

To estimate δ_k we have to take care of small denominators. Let $\{p_m/q_m\}$ be the sequence of rational numbers converging to θ given by the expansion in continued fractions. By the best approximation theorem (see [Ma] pp. 22–23) $\{\|q_m\theta\|\}$ is the subsequence of successive minima of $\{\|k\theta\|\}$ as k varies from 1 to ∞, i.e., $q_0 = 1$, $q_m < q_{m+1}$ and $\|q_m\theta\| > \|q_{m+1}\theta\|$; thus for each $k < q_{m+1}$ we have $\|k\theta\| \geq \|q_{m+1}\theta\|$. We define $\|q_{-1}\theta\| = 1$. If $k \neq 0$, then $\|k\theta\| < \|q_{-1}\theta\|/2 = 1/2$.

Let us now introduce the following function on the natural numbers:

$$M_m(k) = \begin{cases} 1, & \text{if } \|k\theta\| < \frac{1}{2}\|q_m\theta\|, \\ 0, & \text{if } \|k\theta\| \geq \frac{1}{2}\|q_m\theta\|. \end{cases} \tag{1.36}$$

for $k \geq 1$ and $m \geq -1$. We have the following lemma.

Lemma 1.2.30. *Let $M_m(k)$ be the function defined by (1.36). If $M_m(k) = 1$ then, for all $0 < l < q_{m+1}$, we have $M_m(k-l) = 0$.*

Proof. By definition, there exist $m_1, m_2 \in \mathbb{Z}$ such that

$$\|k\theta\| = |k\theta - m_1| \quad \text{and} \quad \|(k-l)\theta\| = |(k-l)\theta - m_2|.$$

Since we have

$$\begin{aligned} |k\theta - m_1| + |(k-l)\theta - m_2| &\geq |k\theta - m_1 - ((k-l)\theta - m_2)| \\ &= |l\theta + m_2 - m_1| \\ &\geq \|l\theta\|, \end{aligned}$$

we have

$$\|(k-l)\theta\| \geq \|l\theta\| - \|k\theta\|.$$

By assumption, $\|l\theta\| \geq \|q_m\theta\|$ and $-\|k\theta\| > -\|q_m\theta\|/2$, hence

$$\|(k-l)\theta\| > \|q_m\theta\| - \frac{\|q_m\theta\|}{2} = \frac{\|q_m\theta\|}{2},$$

and we are done. □

For each $k \geq 2$ we associate to δ_k a specific decomposition of the form

$$\delta_k = \|k\theta\|^{-1}\delta_{k_1}\delta_{k_2}\ , \tag{1.37}$$

with $k > k_1 \geq k_2$, and $k = 1 + k_1 + k_2$, and hence, by induction, a specific decomposition of the form

$$\delta_k = \|l_0\theta\|^{-1}\|l_1\theta\|^{-1}\cdots\|l_h\theta\|^{-1}\ , \tag{1.38}$$

where $l_0 = k$ and $k > l_1 \geq \cdots \geq l_h \geq 2$. For $m \geq 2$ let $N_m(k)$ be the number of factors $\|l\theta\|^{-1}$ in the expression (1.38) of δ_k satisfying

$$\|l\theta\| < \frac{1}{2}\,\|q_m\theta\|.$$

The next lemma contains the key estimate.

Lemma 1.2.31. *For all $m \geq 2$ we have*

$$N_m(k) \leq \begin{cases} 0, & \textit{if } 0 < k < q_{m+1}\ , \\ \dfrac{2k}{q_{m+1}} - 1, & \textit{if } k \geq q_{m+1}\ . \end{cases}$$

Proof. We argue by induction on k. Writing δ_k as in (1.37), it is clear that we have

$$0 \leq N_m(k) \leq M_m(k) + N_m(k_1) + N_m(k_1).$$

If $0 \leq l \leq k < q_{m+1}$ we have $\|l\theta\| \geq \|q_{m+1}\theta\|$, and hence $N_m(k) = 0$.

Assume now $k > q_{m+1}$, so that $2k/q_{m+1} - 1 \geq 1$. We have a few cases to consider.

Case 1: $M_m(k) = 0$. Then

$$N_m(k) = N_m(k_1) + N_m(k_2),$$

and applying the induction hypotheses to each term we get $N_m(k) \leq (2k/q_{m+1}) - 1$.

Case 2: $M_m(k) = 1$. Then

$$N_m(k) = 1 + N_m(k_1) + N_m(k_2),$$

and there are three subcases.

Case 2.1: $k_1 < q_{m+1}$. Then

$$N_m(k) = 1 \leq \frac{2k}{q_{m+1}} - 1,$$

and we are done.

Case 2.2: $k_1 \geq k_2 \geq q_{m+1}$. Then we have

$$N_m(k) \leq 1 + N(k_1) + N(k_2) \leq 1 + \frac{2k_1}{q_{m+1}} - 1 + \frac{2k_2}{q_{m+1}} - 1 \leq \frac{2k}{q_{m+1}} - 1.$$

Case 2.3: $k_1 \geq q_{m+1} > k_2$. Then

$$N_m(k) = 1 + N_m(k_1),$$

and we have two different subsubcases.

Case 2.3.1: $k_1 \leq k - q_{m+1}$. Then

$$N_m(k) \leq 1 + 2\,\frac{k - q_{m+1}}{q_{m+1}} - 1 < \frac{2k}{q_{m+1}} - 1\ ,$$

and we are done in this case too.

Case 2.3.2: $k_1 > k - q_{m+1}$. Then, by Lemma 1.2.30, $M_m(k_1) = 0$. Therefore case 1 applies to δ_{k_1} and we have

$$N_m(k) = 1 + N_m(k_3) + N_m(k_4),$$

with $k > k_1 > k_3 \geq k_4$ and $k_1 = k_3 + k_4 + 1$. We can repeat the argument for this decomposition, and we finish unless we run into case 2.3.2 again. However, this loop cannot happen more than $q_{m+1} - 1$ times, and we eventually have to land into a different case. This completes the induction and the proof. □

Let us go back to the proof of Theorem 1.2.29. We have to estimate

$$\frac{1}{k}\log\delta_k = \sum_{j=0}^{q} \frac{1}{k}\log\|l_j\theta\|^{-1}\ .$$

Hence, by Lemma 1.2.31, letting ν be defined by $q_{\nu+1} > k \geq q_\nu$, we have

$$\frac{1}{k}\log\delta_k \leq \sum_{m=-1}^{\nu} 2\frac{k}{q_{m+1}}\log\bigl(2\|q_{m+1}\theta\|\bigr)^{-1}\bigr) < 2k\sum_{m\geq 0}\frac{1}{q_m}\log\bigl(2\|q_m\theta\|\bigr)^{-1}\bigr).$$

Now, recalling that $q_m \geq 2^{\frac{m-1}{2}}$ (see [K] Theorem 12 p. 13) and $\|q_m\theta\| \geq 1/(2q_{m+1})$ (it is a consequence of [K] Theorem 13 p. 15 and the proprieties of the convergents [K] p. 4), we get

$$\begin{aligned}
\frac{1}{k}\log\delta_k &< 2k\sum_{m\geq 0}\frac{1}{q_m}\log\bigl(4q_{m+1}\bigr)\bigr) \\
&= 2k\left(\log 4\sum_{m\geq 0}\frac{1}{q_m} + \sum_{m\geq 0}\frac{1}{q_m}\log q_{m+1}\right) \\
&< 2k\left(\frac{2\log 4}{\sqrt{2}-1} + \sum_{m\geq 0}\frac{\log q_{m+1}}{q_m}\right),
\end{aligned}$$

and we are done, since the last series converges by assumption. □

The second result we would like to present is Yoccoz's beautiful proof of the fact that almost every quadratic polynomial f_λ is holomorphically linearizable:

Proposition 1.2.32. *The origin is a Siegel point of $f_\lambda(z) = \lambda z + z^2$ for almost every $\lambda \in \mathbb{S}^1$.*

Proof. (Yoccoz [Y2]) The idea is to study the radius of convergence of the inverse of the linearization of $f_\lambda(z) = \lambda z + z^2$ when $\lambda \in \Delta^*$. Theorem 1.2.2 says that there is a unique map φ_λ defined in some neighbourhood of the origin such that $\varphi'_\lambda(0) = 1$ and $\varphi_\lambda \circ f = \lambda\varphi_\lambda$. Let ρ_λ be the radius of convergence of φ_λ^{-1}; we want to prove that φ_λ is defined in a neighbourhood of the unique critical point $-\lambda/2$ of f_λ, and that $\rho_\lambda = |\varphi_\lambda(-\lambda/2)|$.

Let $\Omega_\lambda \subset\subset \mathbb{C}$ be the basin of attraction of the origin, that is the set of $z \in \mathbb{C}$ whose orbit converges to the origin. Notice that setting $\varphi_\lambda(z) = \lambda^{-k}\varphi_\lambda\bigl(f_\lambda(z)\bigr)$ we can extend φ_λ to the whole of Ω_λ. Moreover, since the image of φ_λ^{-1} is contained in Ω_λ, which is bounded, necessarily $\rho_\lambda < +\infty$. Let $U_\lambda = \varphi_\lambda^{-1}(\Delta_{\rho_\lambda})$. Since we have

$$(\varphi_\lambda' \circ f)f' = \lambda\varphi_\lambda' \tag{1.39}$$

and φ_λ is invertible in U_λ, the function f cannot have critical points in U_λ.

If $z = \varphi_\lambda^{-1}(w) \in U_\lambda$, we have $f(z) = \varphi_\lambda^{-1}(\lambda w) \in \varphi_\lambda^{-1}(\Delta_{|\lambda|\rho_\lambda}) \subset\subset U_\lambda$; therefore

$$f(\overline{U}_\lambda) \subseteq \overline{f(U_\lambda)} \subset\subset U_\lambda \subseteq \Omega_\lambda,$$

which implies that $\partial U \subset \Omega_\lambda$. So φ_λ is defined on ∂U_λ, and clearly $|\varphi_\lambda(z)| = \rho_\lambda$ for all $z \in \partial U_\lambda$.

If f had no critical points in ∂U_λ, (1.39) would imply that φ_λ has no critical points in ∂U_λ. But then φ_λ would be locally invertible in ∂U_λ, and thus φ_λ^{-1} would extend across $\partial\Delta_{\rho_\lambda}$, impossible. Therefore $-\lambda/2 \in \partial U_\lambda$, and $|\varphi_\lambda(-\lambda/2)| = \rho_\lambda$, as claimed.

(Up to here it was classic; let us now start Yoccoz's argument.) Put $\eta(\lambda) = \varphi_\lambda(-\lambda/2)$. From the proof of Theorem 1.2.2 one easily sees that φ_λ depends holomorphically on λ; so $\eta\colon \Delta^* \to \mathbb{C}$ is holomorphic. Furthermore, since $\Omega_\lambda \subseteq \Delta_2$, Schwarz's lemma applied to $\varphi_\lambda^{-1}\colon \Delta_{\rho_\lambda} \to \Delta_2$ yields

$$1 = |(\varphi_\lambda^{-1})'(0)| \le 2/\rho_\lambda,$$

that is $\rho_\lambda \le 2$. Thus η is bounded, and thus it extends holomorphically to the origin.

So $\eta\colon \Delta \to \Delta_2$ is a bounded holomorphic function not identically zero; Fatou's theorem on radial limits of bounded holomorphic functions then implies that

$$\rho(\lambda_0) := \limsup_{r\to 1^-} |\eta(r\lambda_0)| > 0$$

for almost every $\lambda_0 \in \mathbb{S}^1$. This means that we can find $0 < \rho_0 < \rho(\lambda_0)$ and a sequence $\{\lambda_j\} \subset \Delta$ such that $\lambda_j \to \lambda_0$ and $|\eta(\lambda_j)| > \rho_0$. This means that $\varphi_{\lambda_j}^{-1}$ is defined in Δ_{ρ_0} for all $j \ge 1$; up to a subsequence, we can assume that $\varphi_{\lambda_j}^{-1} \to \psi\colon \Delta_{\rho_0} \to \Delta_2$. But then we have $\psi'(0) = 1$ and

$$f_{\lambda_0}\bigl(\psi(z)\bigr) = \psi(\lambda_0 z)$$

in Δ_{ρ_0}, and thus the origin is a Siegel point for f_{λ_0}. □

The third result we would like to present is the implication (i) $\Longrightarrow$ (ii) in Theorem 1.2.26. The proof depends on the following result of Douady and Hubbard, obtained using the theory of quasiconformal maps:

Theorem 1.2.33. (Douady-Hubbard, 1985 [DH]) *Given $\lambda \in \mathbb{C}^*$, let $f_\lambda(z) = \lambda z + z^2$ be a quadratic polynomial. Then there exists a universal constant $C > 0$ such that for every holomorphic function $\psi\colon \Delta_{3|\lambda|/2} \to \mathbb{C}$ with $\psi(0) = \psi'(0) = 0$ and $|\psi(z)| \le C|\lambda|$ for all $z \in \Delta_{3|\lambda|/2}$ the function $f = f_\lambda + \psi$ is topologically conjugated to f_λ in $\Delta_{|\lambda|}$.*

Then

Theorem 1.2.34. (Yoccoz, 1995 [Y2]) *Let $\lambda \in \mathbb{S}^1$ be such that the origin is a Siegel point for $f_\lambda(z) = \lambda z + z^2$. Then the origin is a Siegel point for every $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ.*

Sketch of proof: Write

$$f(z) = \lambda z + a_2 z^2 + \sum_{k\ge 3} a_k z^k ,$$

and let

$$f^a(z) = \lambda z + az^2 + \sum_{k\geq 3} a_k z^k \ ,$$

so that $f = f^{a_2}$. If $|a|$ is large enough then the germ

$$g^a(z) = af^a(z/a) = \lambda z + z^2 + a\sum_{k\geq 3} a_k (z/a)^k = f_\lambda(z) + \psi^a(z)$$

is defined on $\Delta_{3/2}$ and $|\psi^a(z)| < C$ for all $z \in \Delta_{3/2}$, where C is the constant given by Theorem 1.2.33. It follows that g^a is topologically conjugated to f_λ. By assumption, f_λ is topologically linearizable; hence g^a is too. Proposition 1.2.19 then implies that g^a is holomorphically linearizable, and hence f^a is too. Furthermore, it is also possible to show (see, e.g., [BH, Lemma 2.3]) that if $|a|$ is large enough, say $|a| \geq R$, then the domain of linearization of g^a contains Δ_r, where $r > 0$ is such that Δ_{2r} is contained in the domain of linearization of f_λ.

So we have proven the assertion if $|a_2| \geq R$; assume then $|a_2| < R$. Since λ is not a root of unity, there exists (Proposition 1.2.20) a unique formal power series $\widehat{h}^a \in \mathbb{C}[\![z]\!]$ tangent to the identity such that $g^a \circ \widehat{h}^a(z) = \widehat{h}^a(\lambda z)$. If we write

$$\widehat{h}^a(z) = z + \sum_{k\geq 2} h_k(a) z^k$$

then we have

$$\sum_{k\geq 2} (\lambda^k - \lambda) h_k(a) z^k = \sum_{l\geq 2} a_l \left(\sum_{m\geq 1} h_m(a) z^m \right)^l \ ,$$

implying that $h_k(a)$ is a polynomial in a of degree $k-1$. In particular, by the maximum principle we have

$$|h_k(a_2)| \leq \max_{|a|=R} |h_k(a)| \tag{1.40}$$

for all $k \geq 2$. Now, by what we have seen, if $|a| = R$ then $\widehat{h}^a$ is convergent in a disk of radius $r(a) > 0$, and its image contains a disk of radius r. Applying Schwarz's lemma to $(\widehat{h}^a)^{-1}\colon \Delta_r \to \Delta_{r(a)}$ we get $r(a) \geq r$. But then

$$\limsup_{k\to+\infty} |h_k(a_2)|^{1/k} \leq \max_{|a|=R} \limsup_{k\to+\infty} |h_k(a)|^{1/k} = \frac{1}{r(a)} \leq \frac{1}{r} < +\infty \ ;$$

hence $\widehat{h}^{a_2}$ is convergent, and we are done. □

Finally, we would like to describe the connection between condition (1.29) and linearization. From the function theoretical side, given $\theta \in [0,1)$ set

$$r(\theta) = \inf\{r(f) \mid f \in \mathrm{End}(\mathbb{C},0) \text{ has multiplier } e^{2\pi i\theta} \text{ and it is defined and injective in } \Delta\},$$

where $r(f) \geq 0$ is the radius of convergence of the unique formal linearization of f tangent to the identity.

From the number theoretical side, given an irrational number $\theta \in [0,1)$ let $\{p_k/q_k\}$ be the sequence of rational numbers converging to θ given by the expansion in continued fractions, and put

$$\begin{aligned} \alpha_n &= -\tfrac{q_n\theta - p_n}{q_{n-1}\theta - p_{n-1}}, & \alpha_0 &= \theta, \\ \beta_n &= (-1)^n (q_n\theta - p_n), & \beta_{-1} &= 1. \end{aligned}$$

Definition 1.2.14. The *Brjuno function* $B\colon [0,1)\setminus\mathbb{Q} \to (0,+\infty]$ is defined by

$$B(\theta) = \sum_{n=0}^{\infty} \beta_{n-1} \log \frac{1}{\alpha_n}.$$

Then Theorem 1.2.26 is consequence of what we have seen and the following

Theorem 1.2.35. (Yoccoz, 1995 [Y2]) *(i) $B(\theta) < +\infty$ if and only if $\lambda = e^{2\pi i\theta}$ satisfies Brjuno's condition (1.29);*

(ii) there exists a universal constant $C > 0$ such that

$$|\log r(\theta) + B(\theta)| \leq C$$

for all $\theta \in [0,1)\setminus\mathbb{Q}$ such that $B(\theta) < +\infty$;

(iii) if $B(\theta) = +\infty$ then there exists a non-linearizable $f \in \mathrm{End}(\mathbb{C},0)$ with multiplier $e^{2\pi i\theta}$.

See [BC] for deep results regarding the Brjuno function.

If 0 is a Siegel point for $f \in \mathrm{End}(\mathbb{C},0)$, the local dynamics of f is completely clear, and simple enough. On the other hand, if 0 is a Cremer point of f, then the local dynamics of f is very complicated and not yet completely understood. Pérez-Marco (in [P2, 4–7]) and Biswas ([Bis1, 2]) have studied the topology and the dynamics of the stable set in this case. Some of their results are summarized in the following

Theorem 1.2.36. (Pérez-Marco, 1995 [P6, 7]) *Assume that 0 is a Cremer point for an elliptic holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C},0)$, and let (U,f) be a representative of f. Then:*

(i) The stable set $K_{(U,f)}$ is compact, connected, full (i.e., $\mathbb{C}\setminus K_{(U,f)}$ is connected), it is not reduced to $\{0\}$, and it is not locally connected at any point distinct from the origin.

(ii) Any point of $K_{(U,f)}\setminus\{0\}$ is recurrent (that is, a limit point of its orbit).

(iii) There is an orbit in $K_{(U,f)}$ which accumulates at the origin, but no non-trivial orbit converges to the origin.

Theorem 1.2.37. (Biswas, 2007 [Bis2]) *The rotation number and the conformal class of $K_{(U,f)}$ are a complete set of holomorphic invariants for Cremer points. In other words, two elliptic non-linearizable holomorphic local dynamical systems f and g are holomorphically locally conjugated if and only if they have the same rotation number and there is a biholomorphism of a neighbourhood of $K_{(U,f)}$ with a neighbourhood of $K_{(U,g)}$.*

Remark 1.2.38. So, if $\lambda \in \mathbb{S}^1$ is not a root of unity and does not satisfy Brjuno's condition (1.29), we can find f_1, $f_2 \in \mathrm{End}(\mathbb{C},0)$ with multiplier λ such that f_1 is holomorphically linearizable while f_2 is not. Then f_1 and f_2 are formally conjugated without being neither holomorphically nor topologically locally conjugated.

Remark 1.2.39. Yoccoz [Y2] has proved that if $\lambda \in \mathbb{S}^1$ is not a root of unity and does not satisfy Brjuno's condition (1.29) then there is an uncountable family of germs in $\mathrm{End}(\mathbb{C},O)$ with multiplier λ which are not holomorphically conjugated to each other nor holomorphically conjugated to any entire function.

See also [P1, 3] for other results on the dynamics about a Cremer point, and [PY] for relationships with holomorphic foliations in $\mathbb{C}^2$.

1.3 Multi-dimensional case

Now we start the discussion of local dynamics in several complex variables. In this setting the theory is much less complete than its one-variable counterpart.

Definition 1.3.1. Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system at $O \in \mathbb{C}^n$, with $n \geq 2$. The *homogeneous expansion* of f is

$$f(z) = P_1(z) + P_2(z) + \cdots \in \mathbb{C}_O\{z_1, \ldots, z_n\}^n ,$$

where P_j is an n-uple of homogeneous polynomials of degree j. In particular, P_1 is the differential $\mathrm{d}f_O$ of f at the origin, and f is locally invertible if and only if P_1 is invertible.

We have seen that in dimension one the multiplier (i.e., the derivative at the origin) plays a main rôle. When $n > 1$, a similar rôle is played by the eigenvalues of the differential. We shall use the following classification.

Definition 1.3.2. Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system at $O \in \mathbb{C}^n$, with $n \geq 2$. Then:

- if all eigenvalues of $\mathrm{d}f_O$ have modulus less than 1, we say that the fixed point O is *attracting;*
- if all eigenvalues of $\mathrm{d}f_O$ have modulus greater than 1, we say that the fixed point O is *repelling;*
- if all eigenvalues of $\mathrm{d}f_O$ have modulus different from 1, we say that the fixed point O is *hyperbolic* (notice that we allow the eigenvalue zero);
- if O is attracting or repelling, and $\mathrm{d}f_O$ is invertible, we say that f is in the *Poincaré domain;*
- if O is hyperbolic, $\mathrm{d}f_O$ is invertible, and f is not in the Poincaré domain (and thus not all eigenvalues of $\mathrm{d}f_O$ are inside or outside the unit disk) we say that f is in the *Siegel domain;*
- if all eigenvalues of $\mathrm{d}f_O$ are roots of unity, we say that the fixed point O is *parabolic;* in particular, if $\mathrm{d}f_O = \mathrm{Id}$ we say that f is *tangent to the identity;*
- if all eigenvalues of $\mathrm{d}f_O$ have modulus 1 but none is a root of unity, we say that the fixed point O is *elliptic;*
- if $\mathrm{d}f_O = O$, we say that the fixed point O is *superattracting.*

Other cases are clearly possible, but for the aim of this chapter this list is enough.

Remark 1.3.1. A natural way for approaching the multi-dimensional case is to study situations where one can use more or less directly the one-dimensional theory. For example, it is possible to study the so-called *semi-direct product of germs*, namely germs $f \in \mathrm{End}(\mathbb{C}^n, O)$ of the form

$$f(z_1, \ldots, z_n) = (f_1(z_1), f_2(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n)),$$

or the so-called *unfoldings*, i.e., germs $f \in \mathrm{End}(\mathbb{C}^n, O)$ of the form

$$f(z_1, \ldots, z_n) = (f_1(z_1, \ldots, z_n), z_2, \ldots, z_n).$$

We refer to [J2] for the study of a particular class of semi-direct products, and to [Ri1–2] for interesting results on unfoldings.

In the rest of the chapter we shall give a survey of results in the multi-dimensional case to better understand the contest of our contribution, that will be presented in the next chapters.

1.3.1 Parabolic case

A first natural question in the several complex variables parabolic case is whether a result like the Leau-Fatou flower theorem holds, and, if so, in which form. To present what is known on this subject in this section we shall restrict our attention to holomorphic local dynamical systems tangent to the identity; consequences on dynamical systems with a more general parabolic fixed point can be deduced taking a suitable iterate (but see also the end of this section for results valid when the differential at the fixed point is not diagonalizable).

So we are interested in the local dynamics of a holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}^n, O)$ of the form

$$f(z) = z + P_\nu(z) + P_{\nu+1}(z) + \cdots \in \mathbb{C}_O\{z_1, \ldots, z_n\}^n , \tag{1.41}$$

where P_ν is the first non-zero term in the homogeneous expansion of f.

Definition 1.3.3. If $f \in \mathrm{End}(\mathbb{C}^n, O)$ is of the form (1.41), the number $\nu \geq 2$ is the *order* of f.

The two main ingredients in the statement of the Leau-Fatou flower theorem were the attracting directions and the petals. Let us first describe a several variables analogue of attracting directions.

Definition 1.3.4. Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be tangent at the identity and of order ν. A *characteristic direction* for f is a non-zero vector $v \in \mathbb{C}^n \setminus \{O\}$ such that $P_\nu(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. If $P_\nu(v) = O$ (that is, $\lambda = 0$) we shall say that v is a *degenerate* characteristic direction; otherwise, (that is, if $\lambda \neq 0$) we shall say that v is *non-degenerate.* We shall say that f is *dicritical* if all directions are characteristic; *non-dicritical* otherwise.

Remark 1.3.2. It is easy to check that $f \in \mathrm{End}(\mathbb{C}^n, O)$ of the form (1.41) is dicritical if and only if $P_\nu \equiv \lambda\,\mathrm{Id}$, where $\lambda\colon \mathbb{C}^n \to \mathbb{C}$ is a homogeneous polynomial of degree $\nu - 1$. In particular, generic germs tangent to the identity are non-dicritical.

Remark 1.3.3. There is an equivalent definition of characteristic directions that shall be useful later on. The n-uple of ν-homogeneous polynomials P_ν induces a meromorphic self-map of $\mathbb{P}^{n-1}(\mathbb{C})$, still denoted by P_ν. Then, under the canonical projection $\mathbb{C}^n \setminus \{O\} \to \mathbb{P}^{n-1}(\mathbb{C})$ non-degenerate characteristic directions correspond exactly to fixed points of P_ν, and degenerate characteristic directions correspond exactly to indeterminacy points of P_ν. In generic cases, there is only a finite number of characteristic directions, and using Bezout's theorem it is easy to prove (see, e.g., [AT1]) that this number, counting according to a suitable multiplicity, is given by $(\nu^n - 1)/(\nu - 1)$.

Remark 1.3.4. The characteristic directions are *complex* directions; in particular, it is easy to check that f and f^{-1} have the same characteristic directions. Later on we shall see how to associate to (most) characteristic directions $\nu - 1$ petals, each one in some sense centered about a *real* attracting direction corresponding to the same complex characteristic direction.

The notion of characteristic directions has a dynamical origin.

Definition 1.3.5. We shall say that an orbit $\{f^k(z_0)\}$ converges to the origin *tangentially* to a direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ if $f^k(z_0) \to O$ in $\mathbb{C}^n$ and $[f^k(z_0)] \to [v]$ in $\mathbb{P}^{n-1}(\mathbb{C})$, where $[\cdot]\colon \mathbb{C}^n \setminus \{O\} \to \mathbb{P}^{n-1}(\mathbb{C})$ denotes the canonical projection.

Then we have the following result (see [Ha2] for a proof)

Proposition 1.3.5. *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic dynamical system tangent to the identity. If there exists an orbit of f converging to the origin tangentially to a direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$, then v is a characteristic direction of f.*

Remark 1.3.6. There are examples of germs $f \in \mathrm{End}(\mathbb{C}^2, O)$ tangent to the identity with orbits converging to the origin without being tangent to any direction: for instance

$$f(z, w) = \big(z + \alpha zw, w + \beta w^2 + o(w^2)\big)$$

with α, $\beta \in \mathbb{C}^*$, $\alpha \neq \beta$ and $\mathrm{Re}(\alpha/\beta) = 1$ (see [Riv1] and [AT3]).

The several variables analogue of a petal is given by the notion of parabolic curve.

Definition 1.3.6. A *parabolic curve* for $f \in \mathrm{End}(\mathbb{C}^n, O)$ tangent to the identity is an injective holomorphic map $\varphi\colon \Delta \to \mathbb{C}^n \setminus \{O\}$ satisfying the following properties:

(a) Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$;

(b) φ is continuous at the origin, and $\varphi(0) = O$;

(c) $\varphi(\Delta)$ is f-invariant, and $(f|_{\varphi(\Delta)})^k \to O$ uniformly on compact subsets as $k \to +\infty$.

Furthermore, if $[\varphi(\zeta)] \to [v]$ in $\mathbb{P}^{n-1}(\mathbb{C})$ as $\zeta \to 0$ in Δ, we shall say that the parabolic curve φ is *tangent* to the direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$.

Then the first main generalization of the Leau-Fatou flower theorem to several complex variables is due to Écalle and Hakim (see [A5] for a sketch of proof and also Weickert [W]):

Theorem 1.3.7. (Écalle, 1985 [É4]; Hakim, 1998 [Ha2]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system tangent to the identity of order $\nu \geq 2$. Then for any non-degenerate characteristic direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ there exist (at least) $\nu - 1$ parabolic curves for f tangent to $[v]$.*

Definition 1.3.7. A set of $\nu - 1$ parabolic curves obtained in this way is a *Fatou flower* for f tangent to $[v]$.

Remark 1.3.8. When there is a one-dimensional f-invariant complex submanifold passing through the origin tangent to a characteristic direction $[v]$, the previous theorem is just a consequence of the usual one-dimensional theory. But it turns out that in most cases such an f-invariant complex submanifold does not exist: see [Ha2] for a concrete example, and [É4] for a general discussion.

We can also have f-invariant complex submanifolds of dimension strictly greater than one attracted by the origin.

Definition 1.3.8. Given a holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}^n, O)$ tangent to the identity of order $\nu \geq 2$, and a non-degenerate characteristic direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$, the eigenvalues $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$ of the linear operator $\frac{1}{\nu-1}(\mathrm{d}(P_\nu)_{[v]} - \mathrm{Id})\colon T_{[v]}\mathbb{P}^{n-1}(\mathbb{C}) \to T_{[v]}\mathbb{P}^{n-1}(\mathbb{C})$ are the *directors* of $[v]$.

Then, using a more elaborate version of her proof of Theorem 1.3.7, Hakim has been able to prove the following:

Theorem 1.3.9. (Hakim, 1997 [Ha3]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system tangent to the identity of order $\nu \geq 2$. Let $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ be a non-degenerate characteristic direction, with directors $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$. Furthermore, assume that, for a suitable $d \geq 0$, we have $\mathrm{Re}(\alpha_1), \ldots, \mathrm{Re}(\alpha_d) > 0$ and $\mathrm{Re}(\alpha_{d+1}), \ldots, \mathrm{Re}(\alpha_{n-1}) \leq 0$. Then:*

(i) There exists an f-invariant $(d+1)$-dimensional complex submanifold M of $\mathbb{C}^n$, with the origin in its boundary, such that the orbit of every point of M converges to the origin tangentially to $[v]$;

(ii) $f|_M$ *is holomorphically conjugated to the translation*

$$\tau(w_0, w_1, \ldots, w_d) = (w_0 + 1, w_1, \ldots, w_d)$$

defined on a suitable right half-space in $\mathbb{C}^{d+1}$.

Remark 1.3.10. In particular, if all the directors of $[v]$ have positive real part, there is an open domain attracted by the origin. However, the condition given by Theorem 1.3.9 is not necessary for the existence of such an open domain; see Rivi [Riv1] for an easy example, Ushiki [Us] for a more elaborate example with an open domain attracted by the origin, and Vivas [V] for a recent example with a domain attracted by the origin centered in a degenerate characteristic direction.

In his monumental work [É4] Écalle has given a complete set of formal invariants for holomorphic local dynamical systems tangent to the identity with at least one non-degenerate characteristic direction. For instance, he has proved the following

Theorem 1.3.11. (Écalle, 1985 [É4]) *Let* $f \in \mathrm{End}(\mathbb{C}^n, O)$ *be a holomorphic local dynamical system tangent to the identity of order* $\nu \geq 2$. *Assume that*

(a) f *has exactly* $(\nu^n - 1)/(\nu - 1)$ *distinct non-degenerate characteristic directions and no degenerate characteristic directions;*

(b) the directors of any non-degenerate characteristic direction are irrational and mutually independent over $\mathbb{Z}$.

Choose a non-degenerate characteristic direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$, *and let* $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$ *be its directors. Then there exist a unique* $\rho \in \mathbb{C}$ *and unique (up to dilations) formal series* $R_1, \ldots, R_n \in \mathbb{C}[\![z_1, \ldots, z_n]\!]$, *where each* R_j *contains only monomial of total degree at least* $\nu + 1$ *and of partial degree in* z_j *at most* $\nu - 2$, *such that* f *is formally conjugated to the time-1 map of the formal vector field*

$$X = \frac{1}{(\nu - 1)(1 + \rho z_n^{\nu-1})} \left\{ [-z_n^\nu + R_n(z)] \frac{\partial}{\partial z_n} + \sum_{j=1}^{n-1} [-\alpha_j z_n^{\nu-1} z_j + R_j(z)] \frac{\partial}{\partial z_j} \right\}.$$

Other approaches to the formal classification, at least in dimension 2, are described in [BM] and in [AT2].

Using his theory of resurgence, and always assuming the existence of at least one non-degenerate characteristic direction, Écalle has also provided a set of holomorphic invariants for holomorphic local dynamical systems tangent to the identity, in terms of differential operators with formal power series as coefficients. Moreover, if the directors of all non-degenerate characteristic directions are irrational and satisfy a suitable diophantine condition, then these invariants become a complete set of invariants. See [É5] for a description of his results, and [É4] for the details.

It is natural to ask what happens when there are no non-degenerate characteristic directions, which is, for instance, the case for

$$\begin{cases} f_1(z) = z_1 + bz_1z_2 + z_2^2, \\ f_2(z) = z_2 - b^2z_1z_2 - bz_2^2 + z_1^3, \end{cases}$$

for any $b \in \mathbb{C}^*$, (and it is easy to build similar examples of any order). At present, the theory in this case is satisfactorily developed for $n = 2$ only. In particular, in [A2] is proved the following

Theorem 1.3.12. (Abate, 2001 [A2]) *Every holomorphic local dynamical system tangent to the identity $f \in \mathrm{End}(\mathbb{C}^2, O)$, with an isolated fixed point, admits at least one Fatou flower tangent to some direction.*

Remark 1.3.13. Bracci and Suwa have proved a version of Theorem 1.3.12 for $f \in \mathrm{End}(M, p)$ where M is a *singular* variety with not too bad a singularity at p; see [BrS] for details.

We refer to [A5] for the main ideas in the proof of Theorem 1.3.12, and to the original article [A2] for the whole proof.

Actually, Abate have proved a slightly more precise result, for which we need the following definitions.

Let $f \in \mathrm{End}(M, E)$, where M is a complex n-dimensional manifold and $E \subset M$ is a compact smooth complex hypersurface pointwise fixed by f, and take $p \in E$. Then for every $h \in \mathcal{O}_{M,p}$ (where $\mathcal{O}_M$ is the structure sheaf of M) the germ $h \circ f$ is well-defined, and we have $h \circ f - h \in \mathcal{I}_{E,p}$, where $\mathcal{I}_E$ is the ideal sheaf of E.

Definition 1.3.9. The *f-order of vanishing* at p of $h \in \mathcal{O}_{M,p}$ is

$$\nu_f(h; p) = \max\{\mu \in \mathbb{N} \mid h \circ f - h \in \mathcal{I}^{\mu}_{E,p}\} ,$$

and the *order of contact* ν_f of f with E is

$$\nu_f = \min\{\nu_f(h; p) \mid h \in \mathcal{O}_{M,p}\} .$$

In [ABT1] Abate, Bracci and Tovena proved that ν_f does not depend on p, and that

$$\nu_f = \min_{j=1,\dots,n} \nu_f(z^j; p) ,$$

where (U, z) is any local chart centered at $p \in E$ and $z = (z^1, \dots, z^n)$. In particular, if the local chart (U, z) is such that $E \cap U = \{z^1 = 0\}$ (and we shall say that the local chart is *adapted* to E) then setting $f^j = z^j \circ f$ we can write

$$f^j(z) = z^j + (z^1)^{\nu_f} g^j(z) , \tag{1.42}$$

where at least one among $g^1, \dots, g^n$ does not vanish identically on $U \cap E$.

Definition 1.3.10. A map $f \in \mathrm{End}(M, E)$ is *tangential* to E if

$$\min\bigl\{\nu_f(h; p) \mid h \in \mathcal{I}_{E,p}\bigr\} > \nu_f$$

for some (and hence any) point $p \in E$.

Choosing a local chart (U, z) adapted to E so that we can express the coordinates of f in the form (1.42), it turns out that f is tangential if and only if $g^1|_{U\cap E} \equiv 0$.

The g^j's in (1.42) depend in general on the chosen chart; however, in [ABT1] Abate, Bracci and Tovena proved that setting

$$\mathcal{X}_f = \sum_{j=1}^{n} g^j \frac{\partial}{\partial z^j} \otimes (dz^1)^{\otimes \nu_f} \tag{1.43}$$

then $\mathcal{X}_f|_{U\cap E}$ defines a *global* section X_f of the bundle $TM|_E \otimes (N_E^*)^{\otimes \nu_f}$, where N_E^* is the conormal bundle of E into M. The bundle $TM|_E \otimes (N_E^*)^{\otimes \nu_f}$ is canonically isomorphic to the bundle $\mathrm{Hom}(N_E^{\otimes\nu_f}, TM|_E)$. Therefore the section X_f induces a morphism still denoted by $X_f\colon N_E^{\otimes\nu_f} \to TM|_E$.

Definition 1.3.11. The morphism $X_f\colon N_E^{\otimes\nu_f} \to TM|_E$ just defined is the *canonical morphism* associated to $f \in \mathrm{End}(M, E)$.

Remark 1.3.14. It is easy to check that f is tangential if and only if the image of X_f is contained in TE.

Definition 1.3.12. Assume that $f \in \mathrm{End}(M, E)$ is tangential. We shall say that $p \in E$ is a *singular point for f* if X_f vanishes at p.

Definition 1.3.13. Let M be the blow-up of $\mathbb{C}^n$ at the origin, and f the lift of a non-dicritical holomorphic local dynamical system $f_o \in \mathrm{End}(\mathbb{C}^n, O)$ tangent to the identity. We shall say that $[v] \in \mathbb{P}^{n-1}(\mathbb{C}) = E$ is a *singular direction* of f_o if it is a singular point of f.

Let $f \in \mathrm{End}(M, E)$, where E is a smooth complex hypersurface in a complex manifold M, and assume that f is tangential; let E^o denote the complement in E of the singular points of f. For simplicity of exposition we shall assume $\dim M = 2$ and $\dim E = 1$; but this part of the argument works for any $n \geq 2$ (even when E has singularities, and it can also be adapted to non-tangential germs).

Since $\dim E = 1 = \mathrm{rk} N_E$, the restriction of the canonical morphism X_f to $N_{E^o}^{\otimes\nu_f}$ is an isomorphism between $N_{E^o}^{\otimes\nu_f}$ and TE^o. Then in [ABT1] Abate, Bracci and Tovena showed that it is possible to define a holomorphic connection ∇ on N_{E^o} by setting

$$\nabla_u(s) = \pi([\mathcal{X}_f(\tilde{u}), \tilde{s}]|_S) , \tag{1.44}$$

where: s is a local section of N_{E^o}; $u \in TE^o$; $\pi\colon TM|_{E^o} \to N_{E^o}$ is the canonical projection; $\tilde{s}$ is any local section of $TM|_{E^o}$ such that $\pi(\tilde{s}|_{S^o}) = s$; $\tilde{u}$ is any local section of $TM^{\otimes\nu_f}$ such that $X_f\big(\pi(\tilde{u}|_{E^o})\big) = u$; and $\mathcal{X}_f$ is locally given by (1.43). In a chart (U, z) adapted to E, a local generator of N_{E^o} is $\partial_1 = \pi(\partial/\partial z^1)$, a local generator of $N_{E^o}^{\otimes\nu_f}$ is $\partial_1^{\otimes\nu_f} = \partial_1 \otimes \cdots \otimes \partial_1$, and we have

$$X_f(\partial_1^{\otimes\nu_f}) = g^2|_{U\cap E} \frac{\partial}{\partial z^2} ;$$

therefore

$$\nabla_{\partial/\partial z^2} \partial_1 = -\left.\frac{1}{g^2}\frac{\partial g^1}{\partial z^1}\right|_{U\cap E} \partial_1 .$$

In particular, ∇ is a meromorphic connection on N_E, with poles in the singular points of f.

Definition 1.3.14. The *index* $\iota_p(f, E)$ of f along E at a point $p \in E$ is by definition the opposite of the residue at p of the connection ∇:

$$\iota_p(f, E) = -\mathrm{Res}_p(\nabla) .$$

In particular, $\iota_p(f, E) = 0$ if p is not a singular point of f.

Remark 1.3.15. If $[v]$ is a non-degenerate characteristic direction of a non-dicritical germ $f_o \in \mathrm{End}(\mathbb{C}^2, O)$ with non-zero director $\alpha \in \mathbb{C}^*$, then it is not difficult to check that

$$\iota_{[v]}(f, E) = \frac{1}{\alpha} ,$$

where f is the lift of f_o to the blow-up of the origin.

The precise statement is then the following:

Theorem 1.3.16. (Abate, 2001 [A2]) *Let E be a (not necessarily compact) Riemann surface inside a 2-dimensional complex manifold M, and take $f \in \mathrm{End}(M, E)$ tangential to E. Let $p \in E$ be a singular point of f such that $\iota_p(f, E) \notin \mathbb{Q}^+$. Then there exists a Fatou flower for f at p. In particular, if $f_o \in \mathrm{End}(\mathbb{C}^2, O)$ is a non-dicritical holomorphic local dynamical system tangent to the identity with an isolated fixed point at the origin, and $[v] \in \mathbb{P}^1(\mathbb{C})$ is a singular direction such that $\iota_{[v]}\big(f, \mathbb{P}^1(\mathbb{C})\big) \notin \mathbb{Q}^+$, where f is the lift of f_o to the blow-up of the origin, then f_o has a Fatou flower tangent to $[v]$.*

Remark 1.3.17. This latter statement has been generalized in two ways. Degli Innocenti [DI1] has proved that we can allow E to be singular at p (but irreducible; in the reducible case one has to impose conditions on the indices of f along all irreducible components of E passing through p). Molino [Mo], on the other hand, has proved that the statement still holds assuming only $\iota_p(f, E) \neq 0$, at least for f of order 2 (and E smooth at p); it is natural to conjecture that this should be true for f of any order.

The problem of the validity of something like Theorem 1.3.12 remains open in dimension $n \geq 3$; see [AT1] and [Ro2] for some partial results.

It is also widely open, even in dimension 2, the problem of describing the stable set of a holomorphic local dynamical system tangent to the identity, as well as the more general problem of the topological classification of such dynamical systems. Some results in the case of a dicritical singularity are presented in [BM]; for non-dicritical singularities a promising approach in dimension 2 is described in [AT3].

Indeed, in [AT3], Abate and Tovena get a complete description of the local dynamics in a full neighbourhood of the origin for a large class of holomorphic local dynamical systems tangent to the identity. Since results like Theorem 1.2.9 seem to suggest that generic holomorphic local dynamical systems tangent to the identity might be topologically conjugated to the time-1 map of a homogeneous vector field, their approach might eventually lead to a complete topological description of the dynamics for generic holomorphic local dynamical systems tangent to the identity in dimension 2.

We end this section with a couple of words on holomorphic local dynamical systems with a parabolic fixed point where the differential is not diagonalizable. Particular examples are studied in detail in [CD], [A4] and [GS]. In [A1] it is described a canonical procedure for lifting an $f \in \mathrm{End}(\mathbb{C}^n, O)$ whose differential at the origin is not diagonalizable to a map defined in a suitable iterated blow-up of the origin (obtained blowing-up not only points but more general submanifolds) with a canonical fixed point where the differential is diagonalizable. Using this procedure it is for instance possible to prove the following

Corollary 1.3.18. (Abate, 2001 [A2]) *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be a holomorphic local dynamical system with $\mathrm{d}f_O = J_2$, the canonical Jordan matrix associated to the eigenvalue* 1, *and assume that the origin is an isolated fixed point. Then f admits at least one parabolic curve tangent to* $[1 : 0]$ *at the origin.*

1.3.2 Hyperbolic case

Let us now assume that the origin is a hyperbolic fixed point for an $f \in \mathrm{End}(\mathbb{C}^n, O)$ not necessarily invertible. We then have a canonical splitting

$$\mathbb{C}^n = E^s \oplus E^u ,$$

where E^s [resp., E^u] is the direct sum of the generalized eigenspaces associated to the eigenvalues of $\mathrm{d}f_O$ with modulus less [resp., greater] than 1. Then the first main result in this subject is the famous *stable manifold theorem* (originally due to Perron [Pe] and Hadamard [H]; see [FHY, HK, HPS, Pes, Sh, AM] for proofs in the C^∞ category, Wu [Wu] for a proof in the holomorphic category, and [A3] for a proof in the non-invertible case):

Theorem 1.3.19. *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system with a hyperbolic fixed point at the origin, and let (U, f) be a representative of f. Then:*

(i) the stable set $K_{(U,f)}$ is an embedded complex submanifold of (a neighbourhood of the origin in) $\mathbb{C}^n$, tangent to E^s at the origin;

(ii) there is an embedded complex submanifold $W_{(U,f)}$ of (a neighbourhood of the origin in) $\mathbb{C}^n$, called the unstable set of f, tangent to E^u at the origin, such that $f|_{W_{(U,f)}}$ is invertible, $f^{-1}(W_{(U,f)}) \subseteq W_{(U,f)}$, and $z \in W_{(U,f)}$ if and only if there is a sequence $\{z_{-k}\}_{k\in\mathbb{N}}$ in the domain of f such that $z_0 = z$ and $f(z_{-k}) = z_{-k+1}$ for all $k \geq 1$. Furthermore, if f is invertible then $W_{(U,f)}$ is the stable set of f^{-1}.

The proof is too involved to be summarized here; it suffices to say that both $K_{(U,f)}$ and $W_{(U,f)}$ can be recovered, for instance, as fixed points of a suitable contracting operator in an infinite dimensional space (see the references quoted above for details).

Remark 1.3.20. If the origin is an attracting fixed point, then $E^s = \mathbb{C}^n$, and $K_{(U,f)}$ is an open neighbourhood of the origin, its *basin of attraction*. However, as we shall discuss below, this does not imply that f is holomorphically linearizable, even when it is invertible. Conversely, if the origin is a repelling fixed point, then $E^u = \mathbb{C}^n$, and $K_{(U,f)} = \{O\}$. Again, not all holomorphic local dynamical systems with a repelling fixed point are holomorphically linearizable.

If a point in the domain U of a holomorphic local dynamical system with a hyperbolic fixed point does not belong either to the stable set or to the unstable set, it escapes both in forward time (that is, its orbit escapes) and in backward time (that is, it is not the end point of an infinite orbit contained in U). In some sense, we can think of the stable and unstable sets (or, as they are usually called in this setting, stable and unstable *manifolds*) as skewed coordinate planes at the origin, and the orbits outside these coordinate planes follow some sort of hyperbolic path, entering and leaving any neighbourhood of the origin in finite time.

Actually, this idea of straightening stable and unstable manifolds can be brought to fruition (at least in the invertible case), and it yields one of the possible proofs (see [HK, Sh, A3] and references therein) of the *Grobman-Hartman theorem:*

Theorem 1.3.21. (Grobman, 1959 [Gr1–2]; Hartman, 1960 [Har]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a locally invertible holomorphic local dynamical system with a hyperbolic fixed point. Then f is topologically locally conjugated to its differential $\mathrm{d}f_O$.*

Thus, at least from a topological point of view, the local dynamics about an invertible hyperbolic fixed point is completely clear. This is definitely not the case if the local dynamical system is not invertible in a neighbourhood of the fixed point. For instance, already Hubbard and Papadopol [HP] noticed that a Böttcher-type theorem for superattracting points in several complex variables is just not true: there are holomorphic local dynamical systems with a superattracting fixed point which are not even topologically locally conjugated to the first non-vanishing term of their homogeneous expansion. Recently, Favre and Jonsson (see, e.g., [Fa] and [FJ1, 2]) have begun a very detailed study of superattracting fixed points in $\mathbb{C}^2$, study that might lead to their topological classification. We shall limit ourselves to quote one result.

Definition 1.3.15. Given $f \in \mathrm{End}(\mathbb{C}^2, O)$, we shall denote by $\mathrm{Crit}(f)$ the set of critical points of f. Put

$$\mathrm{Crit}^\infty(f) = \bigcup_{k\geq 0} f^{-k}\bigl(\mathrm{Crit}(f)\bigr) ;$$

we shall say that f is *rigid* if (as germ in the origin) $\mathrm{Crit}^\infty(f)$ is f-invariantm and it is either empty, a smooth curve, or the union of two smooth curves crossing transversally at the origin. Finally, we shall say that f is *dominant* if $\det(\mathrm{d}f) \not\equiv 0$.

Rigid germs have been classified by Favre [Fa], which is the reason why next theorem can be useful for classifying superattracting dynamical systems:

Theorem 1.3.22. (Favre-Jonsson, 2007 [FJ2]) *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be superattracting and dominant. Then there exist:*

(a) a 2-dimensional complex manifold M (obtained by blowing-up a finite number of points);

(b) a surjective holomorphic map $\pi\colon M \to \mathbb{C}^2$ such that the restriction

$$\pi|_{M\setminus E}\colon M \setminus E \to \mathbb{C}^2 \setminus \{O\}$$

is a biholomorphism, where $E = \pi^{-1}(O)$;

(c) a point $p \in E$; and

(d) a rigid holomorphic germ $\widetilde{f} \in \mathrm{End}(M, p)$

so that $\pi \circ \widetilde{f} = f \circ \pi$.

1.3.3 Resonances and Poincaré-Dulac normal forms

Coming back to local invertible dynamical systems, the holomorphic and even the formal classification are not as simple as the topological one. As we saw in Section 1.2, one of the main questions in the study of local holomorphic dynamics is when f is holomorphically linearizable. The answer to this question depends on the set of eigenvalues of $\mathrm{d}f_O$, usually called the *spectrum* of $\mathrm{d}f_O$, and the main problem is caused by resonances. In the rest, we shall need the following notation.

Definition 1.3.16. Let $p \geq 2$. We denote by $\mathcal{H}^p$ the complex vector space of homogeneous polynomial endomorphisms of $\mathbb{C}^n$ of degree p, and we consider on it the standard basis $\mathcal{B}^p = \{z^Q e_j \mid |Q| = p, 1 \leq j \leq n\}$. We shall denote by $o(k)$ every holomorphic map of the form $\sum_{p\geq k+1} h_p$ with $h_p \in \mathcal{H}^p$.

Let us first see what happens when we conjugate f by a germ of biholomorphism of the form $\psi_p := I + \widehat{\psi}_p$ with $\widehat{\psi}_p \in \mathcal{H}^p$ and $p \geq 2$.

Lemma 1.3.23. *Let $\psi := I + \widehat{\psi}$ be a germ of biholomorphism of $\mathbb{C}^n$ with $\widehat{\psi} \in \mathcal{H}^q$, and let $f = \Lambda + S_{q-1} + H_q + o(q)$ be a germ of biholomorphism with $S_{q-1} \in \mathcal{H}^1 \oplus \cdots \oplus \mathcal{H}^{q-1}$ and $H_q \in \mathcal{H}^q$. Then*

$$\psi^{-1} \circ f \circ \psi = \Lambda + S_{q-1} + [H_q + \Lambda \circ \widehat{\psi} - \widehat{\psi} \circ \Lambda] + o(q).$$

Proof. It is useful to note that, for any $h \in \mathcal{H}^s$ with $s \geq 2$ and for any holomorphic map l with $l(O) = O$, we have $h(l + o(r)) = h \circ l + o(r + s - 1)$. Hence, we have

$$\begin{aligned} f \circ \psi &= \Lambda + \Lambda \circ \widehat{\psi} + S_{q-1} \circ (I + \widehat{\psi}) + H_q \circ (I + \widehat{\psi}) + o(q) \\ &= \Lambda + \Lambda \circ \widehat{\psi} + S_{q-1} + H_q + o(q). \end{aligned}$$

Moreover, it is easy to verify that $\psi^{-1} = I - \widehat{\psi} + o(q)$, so we have

$$\begin{aligned}\psi^{-1} \circ f \circ \psi &= \Lambda + \Lambda \circ \widehat{\psi} + S_{q-1} + H_q + o(q) - \widehat{\psi} \circ (\Lambda + o(1)) + o(q) \\ &= \Lambda + \Lambda \circ \widehat{\psi} + S_{q-1} + H_q - \widehat{\psi} \circ \Lambda + o(q),\end{aligned}$$

and this concludes the proof. □

Thus the germ $\psi_q := I + \widehat{\psi}_q$ conjugates $f = \Lambda + H_q + o(q)$ with $\Lambda + o(q)$ if and only if $\widehat{\psi}_q$ is a solution of the equation $H_q = \widehat{\psi} \circ \Lambda - \Lambda \circ \widehat{\psi}$. We have then to study the invertibility of the linear operators

$$M^r_\Lambda \colon \mathcal{H}^r \to \mathcal{H}^r$$

defined by

$$M^r_\Lambda(h) = h \circ \Lambda - \Lambda \circ h.$$

When $\Lambda = \mathrm{Diag}(\lambda_1, \dots, \lambda_n)$ it is easy to answer this question. In fact, for each element $z^Q e_j$ of the basis $\mathcal{B}^r$ we have

$$M^r_\Lambda(z^Q e_j) = (\lambda^Q - \lambda_j) z^Q e_j, \tag{1.45}$$

hence $\ker(M^r_\Lambda) = \{z^Q e_j \mid \lambda^Q - \lambda_j = 0, |Q| \geq 2, 1 \leq j \leq n\}$.

We are then led to give the following definition.

Definition 1.3.17. Let $\lambda \in (\mathbb{C}^*)^n$ and let $j \in \{1, \dots, n\}$. We say that a multi-index $Q \in \mathbb{N}^n$, with $|Q| \geq 2$, gives a *multiplicative resonance relation for λ relative to the j-th coordinate* if

$$\lambda^Q := \lambda_1^{q_1} \cdots \lambda_n^{q_n} = \lambda_j$$

and we put

$$\mathrm{Res}_j(\lambda) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \lambda^Q = \lambda_j\}.$$

The elements of $\mathrm{Res}_j(\lambda)$ are simply called *resonant* multi-indices.

When $n = 1$ there is a resonance if and only if the multiplier is a root of unity, or zero; but if $n > 1$ resonances may occur in the hyperbolic case and in other cases too.

Resonances are the obstruction to formal linearization. Indeed, as we shall see in a minute, a computation completely analogous to the one yielding Proposition 1.2.20 shows that the coefficients of a formal linearization have in the denominators quantities of the form $\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j$. In particular, from the formal point of view, we have the following classical result:

Theorem 1.3.24. *Let f be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ fixing the origin O with no resonances. Then f is formally conjugated to its differential $\mathrm{d}f_O$.*

We shall see in a minute that Theorem 1.3.24 is a consequence of next Theorem 1.3.25 (see also [Ar] pp. 192–193 for another proof).

In presence of resonances, even the formal classification is not that easy. Let us assume, for simplicity, that $\mathrm{d}f_O$ is in Jordan form, that is

$$P_1(z) = (\lambda_1 z, \epsilon_2 z_1 + \lambda_2 z_2, \dots, \epsilon_n z_{n-1} + \lambda_n z_n)\ ,$$

with $\epsilon_1, \dots, \epsilon_{n-1} \in \{0, 1\}$.

Definition 1.3.18. We shall say that a monomial $z^Q := z_1^{q_1} \cdots z_n^{q_n}$ in the j-th coordinate of f is *resonant with respect to* $\lambda_1, \dots \lambda_n \in \mathbb{C}^*$ (or simply $(\lambda_1, \dots \lambda_n)$*-resonant*) if $|Q| = \sum_{j=1}^n q_j \geq 2$ and $\lambda^Q := \lambda_1^{q_1} \cdots \lambda_n^{q_n} = \lambda_j$.

Then Theorem 1.3.24 can be generalized to:

Theorem 1.3.25. (Poincaré, 1893 [Po]; Dulac, 1904 [D]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a locally invertible holomorphic local dynamical system. Then it is formally conjugated to a $g \in \mathbb{C}_O[\![z_1, \dots, z_n]\!]^n$ such that $\mathrm{d}g_O$ is in Jordan normal form, and g has only resonant monomials. Moreover, the resonant part of the formal change of coordinates ψ can be chosen arbitrarily, but once this is done, ψ and g are uniquely determined.*

Proof. There are many ways to prove this result (see [Ar, p. 194] or [IY, p. 53] for other proofs). We would like to prove that formal solutions ψ and g of

$$f \circ \psi = \psi \circ g \tag{1.46}$$

exist with the required properties.

Write $f(z) = \Lambda z + \widehat{f}(z)$, $\psi(w) = w + \widehat{\psi}(w)$ and $g(z) = \Lambda z + \widehat{g}(z)$. Then equation (1.46) is equivalent to

$$\Lambda \widehat{\psi} + \widehat{f} \circ \psi = \widehat{g} + \widehat{\psi} \circ g. \tag{1.47}$$

Up to a linear change of coordinates, we can suppose that the matrix Λ is in Jordan normal form with eigenvalues $\lambda_1, \dots, \lambda_n$. Using the standard multi-index notation, i.e., writing

$$\widehat{f} = \sum_{|R| \geq 2} f_R z^R, \quad f_R \in \mathbb{C}^n,$$

where $R = (r_1, \dots, r_n) \in \mathbb{N}^n$,

$$\widehat{\psi} = \sum_{|Q| \geq 2} \psi_Q w^Q, \quad \psi_Q \in \mathbb{C}^n,$$

where $Q = (q_1, \dots, q_n) \in \mathbb{N}^n$, and

$$\widehat{g} = \sum_{|P| \geq 2} g_P z^P, \quad g_P \in \mathbb{C}^n,$$

where $P = (p_1, \dots, p_n) \in \mathbb{N}^n$, the j-th component of equation (1.47) becomes

$$\begin{aligned} \lambda_j \sum_{|Q| \geq 2} \psi_{Q,j} w^Q + \varepsilon_j \sum_{|Q| \geq 2} \psi_{Q,j-1} w^Q + \sum_{|R| \geq 2} f_{R,j} \left(\prod_{k=1}^n \left(w_k + \sum_{|P| \geq 2} \psi_{P,k} w^P \right)^{r_k} \right) \\ = \sum_{|Q| \geq 2} g_{Q,j} w^Q + \sum_{|R| \geq 2} \psi_{R,j} \left(\prod_{k=1}^n \left(\lambda_k w_k + \varepsilon_k w_{k-1} + \sum_{|P| \geq 2} g_{P,k} w^P \right)^{r_k} \right), \end{aligned} \tag{1.48}$$

where $\varepsilon_k \in \{0, \varepsilon\}$ and ε_k can be non-zero only if $\lambda_k = \lambda_{k-1}$. We want to compute the coefficient of w^Q. In the left-hand side we have the components $\lambda_j \psi_{Q,j}$ and $\varepsilon_j \psi_{Q,j-1}$ and a term depending polynomially on f and ψ, in which contribute only the coefficients $\psi_{P,k}$ with $|P| < |Q|$, because $\widehat{f}$ is of second order. In fact, in the product $\left(w_k + \sum_{|P| \geq 2} \psi_{P,k} w^P \right)^{r_k}$, $\psi_{P,k}$ belongs to powers of order more than or equal to $|P|$; the series $\widehat{f}$ has indices $|R| \geq 2$, then $\psi_{P,k} w^P$ will be multiplied at least by another w_s. In the right-hand side we have $g_{Q,j}$ and, analogously to the left-hand side, there is a term depending polynomially on ψ and g, in which contribute only the coefficients $\psi_{R,j}$ with $|R| \leq |Q|$ and the coefficients $g_{P,k}$ with $|P| < |Q|$. Moreover $R \leq Q$ in the lexicographic order. Indeed, if $R > Q$ then the ε_k's will give a contribute in the product $\left(\lambda_k w_k + \varepsilon_k w_{k-1} + \sum_{|P|\geq 2} g_{P,k} w^P\right)^{r_k}$, so we obtain a multi-index coming after Q; more precisely, for $R = Q$ we have only $\lambda^Q \psi_{Q,j}$.

Then we have:

$$\begin{aligned}\lambda_j \psi_{Q,j} + \varepsilon_j \psi_{Q,j-1} + \mathrm{Pol.}(f_{R,j}, \psi_{P,k}\colon |P| < |Q|)& \\ = g_{Q,j} + \lambda^Q \psi_{Q,j} + \mathrm{Pol.}(\psi_{R,j}, g_{P,k}\colon |P| < |Q|, R < Q),&\end{aligned} \tag{1.49}$$

hence

$$\begin{aligned}&(\lambda^Q - \lambda_j)\psi_{Q,j} + g_{Q,j} \\ &= \varepsilon_j \psi_{Q,j-1} + \mathrm{Pol.}(f_{R,j}, \psi_{P,k}\colon |P| < |Q|) - \mathrm{Pol.}(\psi_{R,j}, g_{P,k}\colon |P| < |Q|, R < Q) \\ &= C_{Q,j},\end{aligned}$$

so we can recursively solve the conjugacy equation imposing:

$$\begin{array}{lll} g_{Q,j} = 0, & \psi_{Q,j} = (\lambda^Q - \lambda_j)^{-1} C_{Q,j}, & \text{if } \lambda^Q - \lambda_j \neq 0, \\ g_{Q,j} = C_{Q,j}, & \psi_{Q,j} \text{ whatever}, & \text{if } \lambda^Q - \lambda_j = 0, \end{array}$$

and we are done. □

Definition 1.3.19. A formal power series $g \in \mathbb{C}_O[\![z_1, \dots, z_n]\!]^n$ without constant term, and with linear part Λ in Jordan normal form with eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}^*$, is called *in Poincaré-Dulac normal form* if it contains only resonant monomials with respect to $\lambda_1, \dots, \lambda_n$.

Definition 1.3.20. Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin. A series g in Poincaré-Dulac normal form that can be formally conjugated to f is called a *Poincaré-Dulac (formal) normal form of* f.

The problem with Poincaré-Dulac normal forms is that they are not unique. In particular, one may wonder whether it could be possible to have such a normal form including *finitely many* resonant monomials only (as happened, for instance, in Proposition 1.2.11). We shall see in the next subsection that this is indeed the case when f belongs to the Poincaré domain, that is when $\mathrm{d}f_O$ is invertible and O is either attracting or repelling.

Definition 1.3.21. We shall say that f, a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin, is *holomorphically normalizable* if there exists a local change of coordinates $\varphi \in \mathrm{End}(\mathbb{C}^n, O)$, tangent to the identity, conjugating f to one of its Poincaré-Dulac normal forms.

Even if Poincaré-Dulac normal forms associated to a same germ f are not unique, we can say something on the shape of the formal conjugations between them. We have in fact the following result.

Proposition 1.3.26. *Let f and g be two germs of biholomorphism of $\mathbb{C}^n$ fixing the origin, with the same linear term Λ and in Poincaré-Dulac normal form. If there exists φ a formal transformation of $\mathbb{C}^n$, with no constant term and tangent to the identity, conjugating f and g, then φ contains only monomials that are resonant with respect to the eigenvalues of Λ.*

Proof. Since f and g are in Poincaré-Dulac normal form, Λ is in Jordan normal form. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of Λ. We shall prove that a formal solution $\varphi = I + \widehat{\varphi}$ of

$$f \circ \varphi = \varphi \circ g \tag{1.50}$$

contains only monomials that are resonant with respect to $\lambda_1, \ldots, \lambda_n$. Using the standard multi-index notation, for each $j \in \{1, \ldots, n\}$ we can write

$$f_j(z) = \lambda_j z_j + \varepsilon_j z_{j-1} + z_j f_j^{\mathrm{res}}(z) = \lambda_j z_j + \varepsilon_j z_{j-1} + z_j \sum_{\substack{Q \in N_j \\ \lambda^Q = 1}} f_{Q,j} z^Q,$$

$$g_j(z) = \lambda_j z_j + \varepsilon_j z_{j-1} + z_j g_j^{\mathrm{res}}(z) = \lambda_j z_j + \varepsilon_j z_{j-1} + z_j \sum_{\substack{Q \in N_j \\ \lambda^Q = 1}} g_{Q,j} z^Q,$$

and

$$\varphi_j(z) = z_j \left(1 + \varphi_j^{\mathrm{res}}(z) + \varphi_j^{\neq \mathrm{res}}(z)\right) = z_j + z_j \sum_{\substack{Q \in N_j \\ \lambda^Q = 1}} \varphi_{Q,j} z^Q + z_j \sum_{\substack{Q \in N_j \\ \lambda^Q \neq 1}} \varphi_{Q,j} z^Q,$$

where

$$N_j := \{Q \in \mathbb{Z}^n \mid |Q| \geq 1, q_j \geq -1, q_h \geq 0 \text{ for all } h \neq j\},$$

and $\varepsilon_j \in \{0, 1\}$ can be non-zero only if $\lambda_j = \lambda_{j-1}$. With these notations, the j-th coordinate of the left-hand side of (1.50) becomes

$$\begin{aligned} (f \circ \varphi)_j(z) &= \lambda_j \varphi_j(z) + \varepsilon_j \varphi_{j-1}(z) + \varphi_j(z) \sum_{\substack{Q \in N_j \\ \lambda^Q = 1}} f_{Q,j} \prod_{k=1}^n \varphi_k(z)^{q_k} \\ &= \lambda_j z_j \left(1 + \varphi_j^{\mathrm{res}}(z) + \varphi_j^{\neq \mathrm{res}}(z)\right) \\ &\quad + \varepsilon_j z_{j-1} \left(1 + \varphi_{j-1}^{\mathrm{res}}(z) + \varphi_{j-1}^{\neq \mathrm{res}}(z)\right) \\ &\quad + z_j \left(1 + \varphi_j^{\mathrm{res}}(z) + \varphi_j^{\neq \mathrm{res}}(z)\right) \sum_{\substack{Q \in N_j \\ \lambda^Q = \lambda_j}} f_{Q,j} z^Q \prod_{k=1}^n \left(1 + \varphi_k^{\mathrm{res}}(z) + \varphi_k^{\neq \mathrm{res}}(z)\right)^{q_k}, \end{aligned} \tag{1.51}$$

while the right-hand side of the j-th coordinate of (1.50) becomes

$$\begin{aligned} (\varphi \circ g)_j(z) &= g_j(z) + g_j(z) \sum_{\substack{Q \in N_j \\ \lambda^Q = 1}} \varphi_{Q,j} \prod_{k=1}^n g_k(z)^{q_k} + g_j(z) \sum_{\substack{Q \in N_j \\ \lambda^Q \neq 1}} \varphi_{Q,j} \prod_{k=1}^n g_k(z)^{q_k} \\ &= \lambda_j z_j + \varepsilon_j z_{j-1} + z_j g_j^{\mathrm{res}}(z) \\ &\quad + \left(\lambda_j z_j + \varepsilon_j z_{j-1} + z_j g_j^{\mathrm{res}}(z)\right) \sum_{\substack{Q \in N_j \\ \lambda^Q = 1}} \varphi_{Q,j} z^Q \prod_{k=1}^n \left(\lambda_k + \varepsilon_k \frac{z_{k-1}}{z_k} + g_k^{\mathrm{res}}(z)\right)^{q_k} \\ &\quad + \left(\lambda_j z_j + \varepsilon_j z_{j-1} + z_j g_j^{\mathrm{res}}(z)\right) \sum_{\substack{Q \in N_j \\ \lambda^Q \neq 1}} \varphi_{Q,j} z^Q \prod_{k=1}^n \left(\lambda_k + \varepsilon_k \frac{z_{k-1}}{z_k} + g_k^{\mathrm{res}}(z)\right)^{q_k}. \end{aligned} \tag{1.52}$$

Furthermore, notice that if P and Q are two multi-indices such that $\lambda^P = \lambda^Q = 1$, then we have $\lambda^{\alpha P + \beta Q} = 1$ for every $\alpha, \beta \in \mathbb{Z}$.

We want to prove that $\varphi_{Q,j} = 0$ for each multi-index $Q \in N_j$ so that $\lambda^Q \neq 1$. Let us assume by contradiction that this is not true, and let $\widetilde{Q}$ be the first (with respect to the lexicographic order) multi-index in $N := \bigcup_{j=1}^n N_j$ so that $\lambda^{\widetilde{Q}} \neq 1$ and $\varphi_{\widetilde{Q},j} \neq 0$. Let j be the minimal in $\{1,\ldots,n\}$ such that $\widetilde{Q} \in N_j$, and let us compute the coefficient of the monomial $z^{\widetilde{Q}+e_j}$ in (1.51) and (1.52). In (1.51) we only have $\lambda_j \varphi_{\widetilde{Q},j}$ because, since $f - \Lambda$ is of second order and resonant, other contributions could come only from coefficients $\psi_{P,k}$ with $|P| < |\widetilde{Q}|$ and $\lambda^P \neq 1$, but there are no such coefficients thanks to the minimality of $\widetilde{Q}$ and j. In (1.52) we can argue analogously, but we have also to take care of the monomials divisible by $\varepsilon_k^h (z_{k-1}/z_k)^h z^P$, with $\lambda^P = 1$; in this last case, if $\varepsilon_k \neq 0$, we obtain a multi-index $P - he_k + he_{k-1}$, and again $\lambda^{P-he_k+he_{k-1}} = 1$ because $\lambda_k = \lambda_{k+1}$. Then in (1.52) we only have $\lambda^{\widetilde{Q}+e_j}\varphi_{\widetilde{Q},j}$. Hence, we have

$$(\lambda^{\widetilde{Q}+e_j} - \lambda_j)\varphi_{\widetilde{Q},j} = 0$$

yielding

$$\varphi_{\widetilde{Q},j} = 0$$

because $\lambda^{\widetilde{Q}} \neq 1$ and $\lambda_j \neq 0$, contradicting the hypothesis. □

Remark 1.3.27. It is clear from the proof that Proposition 1.3.26 holds also in the formal category, i.e., for $f, g \in \mathbb{C}_O[\![z_1,\ldots,z_n]\!]$ formal power series in Poincaré-Dulac normal form.

It should be remarked that, in the hyperbolic case, the problem of formal linearization is equivalent to the problem of smooth linearization. This has been proved by Sternberg [St1–2] and Chaperon [Ch]:

Theorem 1.3.28. (Sternberg, 1957 [St1–2]; Chaperon, 1986 [Ch]) *Let f, $g \in \mathrm{End}(\mathbb{C}^n, O)$ be two holomorphic local dynamical systems, and assume that f is locally invertible and with a hyperbolic fixed point at the origin. Then f and g are formally conjugated if and only if they are smoothly locally conjugated. In particular, f is smoothly linearizable if and only if it is formally linearizable. Thus if there are no resonances then f is smoothly linearizable.*

1.3.4 Attracting/Repelling case

If a germ $f \in \mathrm{End}(\mathbb{C}^n, O)$ is in the Poincaré domain, that is the origin is an attracting or a repelling fixed point, then the holomorphic classification is clear. Since, as in the one-dimensional case, if the origin is a repelling fixed point for f then it is an attracting fixed point for f^{-1}, it suffices to study the attracting case.

The attracting [resp. repelling] case was first studied by Poincaré [Po]; Fatou [F4] and Bieberbach [Bi] used this case to construct the first examples of proper open subsets of $\mathbb{C}^n$ (with $n \geq 2$) biholomorphic to the whole of $\mathbb{C}^n$, a phenomenon that cannot occur in one variable. A very clear exposition of this case was given, using a functional approach, by Rosay and Rudin in the appendix of [RR]. Recently, Berteloot in [B] provided a very beautiful exposition of this functional approach to the problem, that we shall present in this subsection.

In the rest of this subsection we shall use the following notation. Let B_r be the euclidian ball of $\mathbb{C}^n$ centered at the origin and with radius r. If $f\colon B_r \to \mathbb{C}^n$ is a holomorphic map fixing the origin and $\|\cdot\|$ is a fixed norm on $\mathbb{C}^n$, we put $\|f\|_\rho = \sup_{z \in B_\rho} \|f(z)\|$ for any $\rho \leq r$. Recall that $\mathcal{H}^p$ is the complex vector space of homogeneous polynomial endomorphisms of $\mathbb{C}^n$ of degree p, we consider on it the standard basis $\mathcal{B}^p = \{z^Q e_j, |Q| = p, 1 \leq j \leq n\}$, and we denote by $o(k)$ every holomorphic map of the form $\sum_{p \geq k+1} h_p$ with $h_p \in \mathcal{H}^p$.

We shall need the following lemma.

Lemma 1.3.29. *Let $k \geq 1$ and let $h = \sum_{p\geq k} H_p$ be holomorphic on B_r. Then:*

(i) for any $p \geq k$ and for any $z \in B_r$ we have

$$\|H_p(z)\| \leq \left(\frac{\|z\|}{r}\right)^p \|h\|_r;$$

(ii) for any $\rho < r$ and for any $z \in B_\rho$ we have

$$\|h(z)\| \leq \frac{\|h(z)\|_r}{r^k}\left(1 - \frac{\rho}{r}\right)^{-1} \|z\|^k.$$

Proof. For all $0 < \|z\| \leq \rho < r$ and $\theta \in [0, 2\pi]$ we have

$$e^{-ip\theta} h\left(\rho \frac{z}{\|z\|} e^{i\theta}\right) = \sum_{q\geq k} \left(\frac{\rho}{\|z\|}\right)^q e^{i(q-p)\theta} H_q(z). \tag{1.53}$$

Integrating (1.53) over $[0, 2\pi]$, we get (i). Furthermore (ii) follows immediately from (i). ☐

Theorem 1.3.30. (Poincaré, 1893 [Po]) *Let g be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin, with linear part Λ such that $a\|z\| \leq \|\Lambda z\| \leq A\|z\|$, where $0 < a \leq A < 1$, and let $f\colon B_r \to \mathbb{C}^n$ be a holomorphic map such that*

$$f = g + \sum_{m\geq k} H_m$$

where $H_m \in \mathcal{H}^m$. Then, if $k > \log(a)/\log(A)$, the sequence $\{g^{-p} \circ f^p\}_p$ converges to a germ of biholomorphism φ such that $\varphi(O) = O$, $\mathrm{d}\varphi_O = \mathrm{Id}$ and $\varphi^{-1} \circ f \circ \varphi = g$.

Proof. Up to shrinking r, we may assume that $f(B_\rho) \subset B_\rho$ and $g(B_\rho) \subset B_\rho$ for any $\rho \leq r$. Let $0 < a' < a \leq A < A' < 1$ and $0 < r_0 < r$ be such that $(A')^k/a' < 1$ and

$$\forall z, w \in B_{r_0} \qquad \|g^{-1}(z) - g^{-1}(w)\| \leq \frac{1}{a'}\|z - w\|, \tag{1.54}$$

$$\forall z \in B_{r_0} \qquad \|g(z)\| \leq A'\|z\|. \tag{1.55}$$

Fix $\varepsilon \geq 0$ such that $\gamma := (A' + \varepsilon)^k/a' < 1$. Thanks to Lemma 1.3.29, there exists a constant $C_0 > 0$ such that $\|f(z) - g(z)\| \leq C_0\|z\|^k$ on B_{r_0}. Hence, by (1.54) and (1.55), it follows that $\|g^{-1} \circ f(z) - z\| \leq C_1\|z\|^k$ and $\|f(z)\| \leq (A' + C_0\|z\|^{k-1})\|z\|$ on B_{r_0}. Taking $r_1 < r_0$ small enough, we have

$$\forall z \in B_{r_1} \qquad \|g^{-1} \circ f(z) - z\| \leq C_1\|z\|^k, \tag{1.56}$$

$$\forall z \in B_{r_1} \qquad \|f(z)\| \leq (A' + \varepsilon)\|z\|. \tag{1.57}$$

Up to taking a smaller r_1, we may also assume that $r_1 \sum_{p\geq 0} \gamma^p < r_0$ and $C_1 r_1^{k-1} \leq 1$.

Now we shall prove inductively over p that for any $z \in B_{r_1}$ we have

(i)$_p$ $\|g^{-(p+1)} \circ f^{p+1}(z) - g^{-p} \circ f^p z\| \leq C_1\gamma^p\|z\|^k$, (ii)$_p$ $\|g^{-p} \circ f^{p+1}(z)\| \le (1+\gamma+\cdots+\gamma^p)r_1$.

The assertions (i)$_0$ and (ii)$_0$ are, respectively, (1.56) and (1.57). Let us now assume that (i)$_p$ and (ii)$_p$ hold. Applying (i)$_p$ to $f(z)$, and using (1.57), since B_{r_1} is stable under f, we get

$$\forall z \in B_{r_1} \qquad \|g^{-(p+1)} \circ f^{p+2}(z) - g^{-p} \circ f^{p+1} z\| \le C_1 \gamma^p (A' + \varepsilon)^{\varepsilon} \|z\|^k. \tag{1.58}$$

Since we have

$$\begin{aligned} C_1 \gamma^p (A'+\varepsilon)^k \|z\|^k &= a' C_1 \gamma^{p+1} \|z\|^k \\ &\le a'(C_1 r_1)^{k-1} \gamma^{p+1} r_1 \\ &\le \gamma^{p+1} r_1, \end{aligned}$$

by (1.58) and (ii)$_p$ we get (ii)$_{p+1}$. Now, from (ii)$_p$ and (ii)$_{p+1}$ we get that $g^{-p} \circ f^{p+1}(z)$ and $g^{-(p+1)} \circ f^{p+2}(z)$ are in B_{r_0}, thus (i)$_{p+1}$ follows from (1.54) and (1.58).

Thanks to (i)$_p$, the sequence $\{g^{-p} \circ f^p\}_p$ converges uniformly on B_{r_1}, and its limit φ verifies $\varphi(O) = O$ and $\mathrm{d}\varphi_O = \lim \mathrm{d}(g^{-p} \circ f^p)_O = \mathrm{Id}$. Since we have

$$(g^{-p} \circ f^p) \circ f = g \circ (g^{-(p+1)} \circ f^{p+1}),$$

passing at the limit in both sides we get $\varphi \circ f = g \circ \varphi$ and we are done. $\square$

We want to prove that every germ of biholomorphism in the Poincaré domain is holomorphically normalizable.

The first thing to notice is that, in the attracting (and hence in the repelling) case, there can be only finitely many resonances.

Lemma 1.3.31. *Let* $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$. *If* $|\lambda_j| < 1$ *for all* $j \in \{1, \ldots, n\}$, *then* $\mathrm{card}(\bigcup_{j=1}^n \mathrm{Res}_j(\lambda)) < +\infty$. *If moreover* $0 < |\lambda_1| \le \cdots \le |\lambda_n| < 1$, *then* $Q \in \mathrm{Res}_j(\lambda)$ *only if it is of the form* $Q = (0, \ldots, 0, q_{j+1}, \ldots, q_n)$, *and*

$$|Q| \le \left[\frac{\log |\lambda_1|}{\log |\lambda_n|} \right],$$

where $[\cdot]$ *denotes the integer part.*

Proof. Up to reordering the coordinates, we may assume that

$$0 < |\lambda_1| \le \cdots \le |\lambda_n| < 1.$$

Hence $|\lambda_1| \le |\lambda_j| \le |\lambda_j|^{q_1+\ldots+q_j} |\lambda_n|^{|Q|-(q_1+\ldots+q_j)} \le |\lambda_j|^{q_1+\ldots+q_j}$, for any multi-index Q with $|Q| \ge 2$, and we have the thesis. $\square$

Theorem 1.3.32. (Poincaré, 1893 [Po]; Dulac, 1904 [D]) *Let f be a germ of biholomorphism of $\mathbb{C}^n$ in the Poincaré domain. Then f is locally holomorphically conjugated to one of its Poincaré-Dulac normal forms. Moreover, if the spectrum of $\mathrm{d}f_O$ is non-resonant, then f is holomorphically linearizable.*

Proof. It suffices to prove the statement for f having the origin as an attracting fixed point.

Up to linear changes of the coordinates we may assume that the linear term Λ of f is in Jordan normal form. Denote by D the diagonal of Λ, i.e., $D := \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1, \ldots, \lambda_n$ is the spectrum of $\mathrm{d}f_O$.

Up to reordering the coordinates, we may assume $0 < |\lambda_1| \leq \ldots \leq |\lambda_n| < 1$. Denote by $c_0(D)$ the quantity $(\log|\lambda_1|)/(\log|\lambda_n|)$ and take k a positive integer such that $k > c_0(D)$.

Let us first assume that there are no resonances. We have to distinguish between two cases: Λ diagonal and Λ not diagonal.

Assume first that $\Lambda = D$. Since we have no resonances, the operators $M_D^r\colon \mathcal{H}^r \to \mathcal{H}^r$, defined by $M_D^r(h) = h \circ D - D \circ h$, are invertible for each $r \geq 2$, and Lemma 1.3.23 applied to f gives us a germ of biholomorphism ψ_2 tangent to the identity at O and such that $\psi_2^{-1} \circ f \circ \psi_2 = \Lambda + o(2)$. Applying recursively 1.3.23 to $\psi_2^{-1} \circ f \circ \psi_2$ for $q \leq k$ times, we get a germ of biholomorphism tangent to the identity $\varphi := \psi_2 \circ \cdots \circ \psi_k$ such that $\widetilde{f} := \varphi^{-1} \circ f \circ \varphi = \Lambda + o(k)$. Since $k > c_0(D)$, we can apply Theorem 1.3.30 to $\widetilde{f}$ and we get that $\widetilde{f}$ is holomorphically linearizable, and therefore also f is. (Here the crucial fact is that $c_0(D)$ only depends on $\Lambda = \mathrm{d}f_O = \mathrm{d}\widetilde{f}_O$).

Let us now consider the case in which $\Lambda \neq D$. We want to reduce ourselves to the previous case conjugating Λ to a matrix sufficiently close to D. Let us then consider the matrix

$$S_\varepsilon := \begin{pmatrix} \varepsilon & & \\ & \ddots & \\ & & \varepsilon^n \end{pmatrix}$$

with $\varepsilon > 0$. Then the germ $f_\varepsilon = S_\varepsilon^{-1} \circ f \circ S_\varepsilon = \Lambda_\varepsilon + H_{2,\varepsilon} + o(2)$, where $\Lambda_\varepsilon = S_\varepsilon^{-1} \circ \Lambda \circ S_\varepsilon$, is close to D for ε small enough, because Λ_ε coincides with the sum of D and a strictly upper triangular matrix with coefficients $t_{i,j} = o(\varepsilon^{j-i})$. If ε is small enough, the operators $M^2_{\Lambda_\varepsilon}, \ldots, M^k_{\Lambda_\varepsilon}$ are invertible and $k > c_0(\Lambda_\varepsilon)$. Hence we apply to f_ε the same procedure we used with f, and we get a germ of biholomorphism φ_ε tangent to the identity and such that $\varphi_\varepsilon^{-1} \circ f_\varepsilon \circ \varphi_\varepsilon = \Lambda_\varepsilon + o(k)$. Hence $\widetilde{f}_\varepsilon := (S_\varepsilon \circ \varphi_\varepsilon \circ S_\varepsilon^{-1})^{-1} \circ f \circ (S_\varepsilon \circ \varphi_\varepsilon \circ S_\varepsilon^{-1}) = \Lambda + o(k)$ and by Theorem 1.3.30 applied to $\widetilde{f}_\varepsilon$ we have the assertion.

Let us now pass to the case in which there are resonances, thus $\ker(M_D^r)$ is non trivial for some r between 2 and $n_0 = [(\log|\lambda_1|)/(\log|\lambda_n|)]$. For those values, $\ker(M_D^r)$ has a basis given by the resonant monomials $z_{j+1}^{p_{j+1}} \cdots z_n^{p_n} e_j$, and the sum of an element of $\bigoplus \ker(M_D^r)$ and Λ (which is upper triangular) defines a polynomial triangular automorphism of $\mathbb{C}^n$:

$$(\lambda_1 z_1 + P_1(z_2, \ldots, z_n), \lambda_2 z_2 + P_2(z_3, \ldots, z_n), \ldots, \lambda_{n-1} z_{n-1} + P_{n-1}(z_n), \lambda_n z_n).$$

Let us fix an integer $k > c_0(D) = [(\log|\lambda_1|)/(\log|\lambda_n|)]$. For any $r \geq 2$, let $\mathcal{H}^r = \ker(M_D^r) \oplus X^r$ be the decomposition of $\mathcal{H}^r$, where X^r is the sum of the eigenspaces of M_D^r distinct from the kernel. We denote by π^r the projection on $\ker(M_D^r)$ with respect to that decomposition, and by $\widehat{h}_r := \pi^r(h_r)$. Fix $k_0 \geq 2$, since $\pi^r + M_D^r$ is invertible for any $r \geq 2$, $\pi^r + M_\Lambda^r$ will be invertible for $2 \leq r \leq k_0$ provided that Λ is sufficiently close to D. Let us assume for a moment that such a condition is satisfied and that $k > c_0(\Lambda)$. We apply Lemma 1.3.23 to $f = \Lambda + H_2 + o(2)$ with $h = h_2$, where h_2 is the unique element of $\mathcal{H}^2$ such that $\widehat{h}_2 + M_\Lambda^2(h_2) = H_2$. This gives us a germ of biholomorphism ψ_2 tangent to the identity such that

$$\psi_2^{-1} \circ f \circ \psi_2 = \Lambda + \widehat{h}_2 + \widetilde{H}_3 + o(3).$$

We apply Lemma 1.3.23 to $\psi_2^{-1} \circ f \circ \psi_2$ with $S_2 = \widehat{h}_2$ and $h = h_3$ such that $\widehat{h}_3 + M_\Lambda^3(h_3) = \widetilde{H}_3$, obtaining a germ of biholomorphism ψ_3 tangent to the identity such that

$$\psi_3^{-1} \circ \psi_2^{-1} \circ f \circ \psi_2 \circ \psi_3 = \Lambda + \widehat{h}_2 + \widehat{h}_3 + \widetilde{H}_4 + o(4).$$

Continuing with this procedure, we construct a germ of biholomorphism ψ fixing the origin, tangent to the identity and such that

$$\psi^{-1} \circ f \circ \psi = \Lambda + \widehat{h}_2 + \cdots + \widehat{h}_{n_0} + o(k).$$

Since $g := \Lambda + \widehat{h}_2 + \cdots + \widehat{h}_{n_0}$ is a triangular automorphism of $\mathbb{C}^n$, we can now apply Theorem 1.3.30 to $\psi^{-1} \circ f \circ \psi$ and deduce that f is locally holomorphically conjugated to g via a germ of biholomorphism tangent to the identity at the origin.

If Λ is not close enough to D, we can replace f with $S_\varepsilon^{-1} \circ f \circ S_\varepsilon$, with $\varepsilon > 0$ small enough, as we did before. As before, $S_\varepsilon^{-1} \circ f \circ S_\varepsilon$ is locally holomorphically conjugated to a triangular automorphism of $\mathbb{C}^n$ with linear part $S_\varepsilon^{-1} \circ \Lambda \circ S_\varepsilon$. Hence f is locally holomorphically conjugated to a triangular automorphism of $\mathbb{C}^n$ with linear part Λ. □

Reich [Re2] describes holomorphic normal forms when $\mathrm{d}f_O$ belongs to the Poincaré domain and there are resonances (see also [ÉV]).

1.4 Holomorphic linearization

We saw in the previous subsection that each non-resonant germ which is in the Poincaré domain can be holomorphically linearized. Poincaré proved something more, in fact, in [Po], using majorant series, he proved the following

Theorem 1.4.1. (Poincaré, 1893 [Po]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a locally invertible holomorphic local dynamical system in the Poincaré domain. Then f is holomorphically linearizable if and only if it is formally linearizable. In particular, if there are no resonances then f is holomorphically linearizable.*

Proof. If f is holomorphically linearizable, obviously it is also formally linearizable. If f is formally linearizable then it is holomorphically conjugated to its linear term up to order k for any positive integer k, and hence by Theorem 1.3.30 it is holomorphically linearizable. □

Even when there are no resonances, or more generally, when we know a priori that a given germ is formally linearizable, not so much is known about the convergence of the linearizations in the cases different from the Poincaré domain. A first result in this sense is the natural generalization of Theorem 1.2.23:

Theorem 1.4.2. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$ be a non-resonant vector such that there exists $\beta > 1$ and $\gamma > 0$ so that*

$$\forall Q \in \mathbb{N}^n,\ |Q| \geq 2 \qquad \frac{1}{|\lambda^Q - \lambda_j|} \leq \gamma\, |Q|^\beta. \tag{1.59}$$

Then all $f \in \mathrm{End}(\mathbb{C}^n, 0)$ such that $\mathrm{d}f_O$ is diagonalizable and has spectrum $\{\lambda_1, \ldots, \lambda_n\}$ are holomorphically linearizable.

As in one variable, (1.59) is a particular case of a more general condition, the multi-dimensional Brjuno condition. In the next subsection we shall introduced this condition and we shall show how to use it to prove convergence (and thus, in particular, Theorem 1.4.2).

1.4.1 Brjuno's result

When $\mathrm{d}f_O$ belongs to the Siegel domain, even without resonances, the formal linearization might diverge. To describe the known results, let us introduce the following definition:

Definition 1.4.1. For $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $m \geq 2$ set

$$\omega_{\lambda_1,\ldots,\lambda_n}(m) = \min\left\{|\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j| \;\Big|\; k_1, \ldots, k_n \in \mathbb{N},\, 2 \leq \sum_{h=1}^{n} k_h \leq m,\, 1 \leq j \leq n\right\} . \tag{1.60}$$

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathrm{d}f_O$, we shall write $\omega_f(m)$ for $\omega_{\lambda_1,\ldots,\lambda_n}(m)$.

It is clear that $\omega_f(m) \neq 0$ for all $m \geq 2$ if and only if there are no resonances. It is also not difficult to prove that if f belongs to the Siegel domain then

$$\lim_{m\to+\infty} \omega_f(m) = 0 ,$$

which is the reason why, even without resonances, the formal linearization might be diverging, exactly as in the one-dimensional case.

Definition 1.4.2. Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. We say that λ *satisfies the Brjuno condition* if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu\geq 0}$ with $p_0 = 1$ such that

$$\sum_{\nu\geq 0} p_\nu^{-1} \log \omega_{\lambda_1,\ldots,\lambda_n}(p_{\nu+1})^{-1} < \infty. \tag{1.61}$$

Lemma 1.4.3. (Brjuno, 1971 [Brj3]) *Let $\omega\colon \mathbb{N} \to (0, +\infty)$ be a monotone non-increasing function. Then there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu\geq 0}$ with $p_0 = 1$ such that*

$$\sum_{\nu\geq 0} p_\nu^{-1} \log \omega(p_{\nu+1})^{-1} < \infty \tag{1.62}$$

if and only if

$$\sum_{\nu\geq 0} \frac{1}{2^\nu} \log \omega(2^{\nu+1})^{-1} < \infty. \tag{1.63}$$

Proof. We claim that for any increasing sequence $\{p_\nu\}_{\nu\geq 0}$, we have

$$\sum_{\nu\geq 0} \frac{1}{2^\nu} \log \omega(2^{\nu+1})^{-1} < 4 \sum_{\nu\geq 1} p_\nu^{-1} \log \omega(p_{\nu+1})^{-1}. \tag{1.64}$$

In fact, for every ν we can find k and l satisfying the inequalities

$$2^k < p_\nu \leq 2^{k+1} < \cdots < 2^{k+l} < p_{\nu+1} \leq 2^{k+l+1},$$

implying

$$\log \omega(2^{k+1})^{-1} < \cdots < \log \omega(2^{k+l})^{-1} < \log \omega(p_{\nu+1})^{-1}.$$

Hence we have

$$\begin{aligned}\sum_{j=k}^{k+l-1} \frac{1}{2^j} \log \omega(2^{j+1})^{-1} &< \frac{1}{2^k} \log \omega(p_{\nu+1})^{-1} \sum_{j\geq 0} \frac{1}{2^j} \\ &= 4\, \frac{1}{2^{k+1}} \log \omega(p_{\nu+1})^{-1} \\ &\leq 4\, \frac{1}{p_\nu} \log \omega(p_{\nu+1})^{-1}.\end{aligned}$$

Decomposing the series in the left-hand side of (1.64) into corresponding pieces, and applying to each of them the last estimate, we get the claim. This shows that if (1.62) holds for a strictly increasing sequence of integers $\{p_\nu\}_{\nu\geq 0}$ with $p_0 = 1$, then also (1.63) holds.

The other direction is clear. □

Remark 1.4.4. It is clear from the proof that we get the same assertion even if in (1.63) we replace 2 by an arbitrary natural number $a > 1$. We refer to [Brj3] pp. 222–224 for other relations between the Brjuno condition and other similar arithmetic conditions.

As far as I know, the best positive result in the non-resonant case is due to Brjuno [Brj2–3], and is a natural generalization of its one-dimensional counterpart:

Theorem 1.4.5. (Brjuno, 1971 [Brj2–3]) *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin, such that $\mathrm{d}f_O$ is diagonalizable. Assume moreover that the spectrum of $\mathrm{d}f_O$ has no resonances and it satisfies the Brjuno condition. Then f is holomorphically linearizable.*

We shall see in Subsection 1.5.1 that it is instead possible to generalize Theorem 1.2.22 proving that if $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ have no resonances and

$$\limsup_{m\to+\infty} \frac{1}{m} \log \frac{1}{\omega_{\lambda_1,\ldots,\lambda_n}(m)} = +\infty,$$

then there exists a germ of biholomorphism of $(\mathbb{C}^n, O)$, fixing the origin, with differential $\mathrm{d}f_O = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$, and not holomorphically linearizable.

Remark 1.4.6. It should be remarked that, contrarily to the one-dimensional case, it is not yet known whether the Brjuno condition is necessary for the holomorphic linearizability of all holomorphic local dynamical systems with a given linear part belonging to the Siegel domain. However, it is easy to check that if $\lambda \in \mathbb{S}^1$ does not satisfy the one-dimensional Brjuno condition then any $f \in \mathrm{End}(\mathbb{C}^n, O)$ of the form

$$f(z) = \big(\lambda z_1 + z_1^2, g(z)\big)$$

is not holomorphically linearizable: indeed, if $\varphi \in \mathrm{End}(\mathbb{C}^n, O)$ is a holomorphic linearization of f, then $\psi(\zeta) = \varphi(\zeta, O)$ is a holomorphic linearization of the quadratic polynomial $\lambda\zeta + \zeta^2$, against Theorem 1.2.26.

We shall see in the next subsections possible generalizations of Brjuno's Theorem 1.4.5. We would also like to mention here that in [DG] are discussed results in the spirit of Theorem 1.4.5 without assuming that the differential is diagonalizable.

1.4.2 Linearization under the reduced Brjuno condition

Another approach to this kind of problems was given by Rüssmann in [Rü1], an I.H.E.S. preprint which is no longer available, and it was finally published in [Rü2]. Rüssmann introduced the following condition, that we shall call Rüssmann condition.

Definition 1.4.3. Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. We say that $\lambda = (\lambda_1, \ldots, \lambda_n)$ *satisfies the Rüssmann condition* if there exists a function $\Omega\colon \mathbb{N} \to \mathbb{R}$ such that:

(i) $k \leq \Omega(k) \leq \Omega(k+1)$ for all $k \in \mathbb{N}$,

(ii) $\sum\limits_{k\geq 1} \frac{1}{k^2} \log \Omega(k) \leq +\infty$, and

(iii) $|\lambda^Q - \lambda_j| \geq \frac{1}{\Omega(|Q|)}$ for all $j = 1, \ldots n$ and for each multi-index $Q \in \mathbb{N}$ with $|Q| \geq 2$ not giving a resonance relative to j.

Rüssmann proves that, in dimension 1, his condition is equivalent to Brjuno condition (see Lemma 8.2 of [Rü2]), and he also proves the following result.

Lemma 1.4.7. (Rüssmann, 2002 [Rü2]) *Let $\Omega\colon \mathbb{N} \to (0, +\infty)$ be a monotone non decreasing function, and let $\{s_\nu\}$ be defined by $s_\nu := 2^{q+\nu}$, with $q \in \mathbb{N}$. Then*

$$\sum_{\nu \geq 0} \frac{1}{s_\nu} \log \Omega(s_{\nu+1}) \leq \sum_{k \geq 2^{q+1}} \frac{1}{k^2} \log \Omega(k).$$

Proof. For each a, b integers with $0 < a < b$ we have

$$\frac{1}{a} - \frac{1}{b} = \sum_{k=a}^{b-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \sum_{k=a}^{b-1} \frac{1}{k(k+1)} \leq \sum_{k=a}^{b-1} \frac{1}{k^2},$$

hence we have

$$\frac{1}{2^{p+1}} = \frac{1}{2^p} - \frac{1}{2^{p+1}} \leq \sum_{k=2^p}^{2^{p+1}-1} \frac{1}{k^2}$$

for any $p \geq 0$.

Since Ω is non decreasing, we obtain

$$\frac{1}{2^{p+1}} \log \Omega(2^p) \leq \sum_{k=2^p}^{2^{p+1}-1} \frac{1}{k^2} \log \Omega(k),$$

hence

$$\sum_{\nu \geq 0} \frac{1}{2^{q+\nu+1}} \log \Omega(2^{q+\nu+2}) \leq \sum_{\nu \geq 0} \sum_{k=2^{q+\nu+1}}^{2^{q+\nu+2}-1} \frac{1}{k^2} \log \Omega(k) = \sum_{k \geq 2^{q+1}} \frac{1}{k^2} \log \Omega(k),$$

and we are done. □

Rüssmann proved the following generalization of Brjuno's Theorem 1.4.5 (the statement is slightly different from the original one presented in [Rü2] but perfectly equivalent).

Theorem 1.4.8. (Rüssmann, 2002 [Rü2]) *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that $\mathrm{d}f_O$ is diagonalizable. If f is formally linearizable and the spectrum of $\mathrm{d}f_O$ satisfies the Rüssmann condition, then it is holomorphically linearizable.*

We refer to the article [Rü2] for the original proof of Rüssmann and we limit ourselves to briefly recall here the main ideas of it. To prove this result, Rüssmann first studies the process of Poincaré-Dulac formal normalization carrying on the functional iterative process we saw in Lemma 1.3.23, without assuming anything on the diagonalizability of $\mathrm{d}f_O$. He then proves that the set of Poincaré-Dulac formal normal forms of a formally linearizable germ of biholomorphism f with linear part Λ reduces to Λ. He constructs a formal iteration process for a zero of the operator $\mathcal{F}(\varphi) = f \circ \varphi - \varphi \circ \Lambda$, and then, *assuming* Λ *diagonal*, he gives estimates for each iteration step, proving that, under what we called the Rüssmann condition, the process converges to a holomorphic linearization.

We would also like to mention here the articles of Zehnder [Z1–3] where one can find the modified Newton method used by Rüssmann.

Notice that, when there are no resonances, the function $\omega_f(m)$ defined in 1.4.1 satisfies

$$|\lambda^Q - \lambda_j| \geq \omega_f(|Q|)$$

for each multi-index $Q \in \mathbb{N}$ with $|Q| \geq 2$.

Let us then define:

Definition 1.4.4. Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. For $m \geq 2$ set

$$\widetilde{\omega}_{\lambda_1,\ldots,\lambda_n}(m) = \min_{\substack{2 \leq |K| \leq m \\ K \notin \mathrm{Res}_j(\lambda)}} \min_{1 \leq j \leq n} |\lambda^K - \lambda_j|,$$

where $\mathrm{Res}_j(\lambda)$ is the set of multi-indices $K \in \mathbb{N}^n$, with $|K| \geq 2$, giving a resonance relation for $\lambda = (\lambda_1, \ldots, \lambda_n)$ relative to $1 \leq j \leq n$, i.e., $\lambda^K - \lambda_j = 0$. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathrm{d}f_O$, we shall write $\widetilde{\omega}_f(m)$ for $\widetilde{\omega}_{\lambda_1,\ldots,\lambda_n}(m)$.

Definition 1.4.5. Let $n \geq 2$ and let $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$. We say that λ *satisfies the reduced Brjuno condition* if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu \geq 0}$ with $p_0 = 1$ such that

$$\sum_{\nu \geq 0} p_\nu^{-1} \log \widetilde{\omega}_{\lambda_1,\ldots,\lambda_n}(p_{\nu+1})^{-1} < \infty.$$

We have the following relation between the Rüssmann and the reduced Brjuno condition.

Lemma 1.4.9. *Let $n \geq 2$ and let $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$. If λ satisfies Rüssmann condition, then it also satisfies the reduced Brjuno condition.*

Proof. The function $\widetilde{\omega}_{\lambda_1,\ldots\lambda_n}(m)$ defined in Definition 1.4.4 satisfies

$$\widetilde{\omega}_{\lambda_1,\ldots\lambda_n}(m)^{-1} \leq \widetilde{\omega}_{\lambda_1,\ldots\lambda_n}(m+1)^{-1}$$

for all $m \in \mathbb{N}$, and

$$|\lambda^Q - \lambda_j| \geq \widetilde{\omega}_{\lambda_1,\ldots\lambda_n}(|Q|)$$

for each $j = 1, \ldots n$ and each multi-index $Q \in \mathbb{N}$ with $|Q| \geq 2$ not giving a resonance relative to j. Furthermore, by its definition, it is clear that any other function $\Omega \colon \mathbb{N} \to \mathbb{R}$ such that $k \leq \Omega(k) \leq \Omega(k+1)$ for all $k \in \mathbb{N}$, and satisfying, for any $j = 1, \ldots n$,

$$|\lambda^Q - \lambda_j| \geq \frac{1}{\Omega(|Q|)}$$

for each multi-index $Q \in \mathbb{N}$ with $|Q| \geq 2$ not giving a resonance relative to j, is such that

$$\widetilde{\omega}_{\lambda_1,\ldots\lambda_n}(m)^{-1} \leq \Omega(m)$$

for all $m \in \mathbb{N}$. Hence

$$\sum_{\nu \geq 0} p_\nu^{-1} \log \widetilde{\omega}_{\lambda_1,\ldots,\lambda_n}(p_{\nu+1})^{-1} < \sum_{\nu \geq 0} p_\nu^{-1} \log \Omega(p_{\nu+1})$$

for any strictly increasing sequence of integers $\{p_\nu\}_{\nu \geq 0}$ with $p_0 = 1$. Since λ satisfies Rüssmann condition, thanks to Lemma 1.4.7, there exists a function Ω as above such that

$$\sum_{\nu \geq 0} \frac{1}{s_\nu} \log \Omega(s_{\nu+1}) < +\infty,$$

with $\{s_\nu\}$ be defined by $s_\nu := 2^{q+\nu}$, with $q \in \mathbb{N}$, and we are done. $\square$

We do not know whether the Rüssmann condition is equivalent to the reduced Brjuno condition in the multi-dimensional case. As we said, Rüssmann is able to prove that this is true in dimension one, but to do so he strongly uses the one-dimensional characterization of these conditions via continued fraction.

We shall give a direct proof of an analogue of Rüssmann Theorem under a slightly different (and more natural) assumption, using explicit computation with the power series expansion and then proving convergence via majorant series. To do so, we first prove that when a germ is formally linearizable, then the linear form is its unique Poincaré-Dulac normal form.

Proposition 1.4.10. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin. If f is formally linearizable, and f is formally conjugated to a formal Poincaré-Dulac normal form g, then g is linear.*

Proof. Let Λ be the linear term of f. Up to linear conjugacy, we may assume that Λ is in Jordan normal form. If the eigenvalues $\lambda_1, \ldots, \lambda_n$ of Λ have no resonances, then there is nothing to prove. Let us then assume that we have resonances, and let us assume by contradiction that $g \not\equiv \Lambda$. Since f is formally linearizable and it is also formally conjugated to g, also g is formally linearizable. Thanks to Proposition 1.3.26, any formal linearization ψ of g tangent to the identity contains only $(\lambda_1, \ldots, \lambda_n)$-resonant monomials; hence, writing $g = \Lambda + g^{\mathrm{res}}$ and $\psi = I + \psi^{\mathrm{res}}$, the conjugacy equation $g \circ \psi = \psi \circ \Lambda$ becomes

$$\begin{aligned} \Lambda + \Lambda\psi^{\mathrm{res}} + g^{\mathrm{res}} \circ (I + \psi^{\mathrm{res}}) &= (\Lambda + g^{\mathrm{res}}) \circ (I + \psi^{\mathrm{res}}) \\ &= (I + \psi^{\mathrm{res}}) \circ \Lambda \\ &= \Lambda + \psi^{\mathrm{res}} \circ \Lambda \\ &= \Lambda + \Lambda\psi^{\mathrm{res}}, \end{aligned}$$

because $\psi^{\mathrm{res}} \circ \Lambda = \Lambda\psi^{\mathrm{res}}$. Hence there must be

$$g^{\mathrm{res}} \circ \psi \equiv 0,$$

and composing on the right with ψ^{-1} we get $g^{\mathrm{res}} \equiv 0$, contradicting the hypotheses. $\square$

Remark 1.4.11. As a consequence of the previous result, we get that *any formal normalization given by the Poincaré-Dulac procedure applied to a formally linerizable germ f is indeed a formal linearization of the germ.* In particular, we have *uniqueness of the Poincaré-Dulac normal form (which is linear and hence holomorphic), but not of the formal linearizations.* Hence *a formally linearizable germ f is formally linearizable via a formal transformation $\varphi = \mathrm{Id} + \widehat{\varphi}$ containing only non-resonant monomials.* In fact, thanks to the proof of Poincaré-Dulac Theorem 1.3.25, we can consider the formal normalization obtained with the Poincaré-Dulac procedure and imposing $\varphi_{Q,j} = 0$ for all Q and j such that $\lambda^Q = \lambda_j$; and this formal transformation φ, by Proposition 1.4.10, conjugates f to its linear part.

Now we have all the ingredients needed to prove the following result.

Theorem 1.4.12. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that $\mathrm{d}f_O$ is diagonalizable. If f is formally linearizable and the spectrum of $\mathrm{d}f_O$ satisfies the reduced Brjuno condition, then f is holomorphically linearizable.*

Proof. Up to linear changes of the coordinates, we may assume that the linear term Λ of f is diagonal, i.e., $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$. From the conjugacy equation

$$f \circ \varphi = \varphi \circ \Lambda, \tag{1.65}$$

writing $f(z) = \Lambda z + \sum_{|L|\geq 2} f_L z^L$, and $\varphi(w) = w + \sum_{|Q|\geq 2} \varphi_Q w^Q$, where f_L and φ_Q belong to $\mathbb{C}^n$, we have that coefficients of φ have to verify

$$\sum_{|Q|\geq 2} A_Q \varphi_Q w^Q = \sum_{|L|\geq 2} f_L \left(\sum_{|M|\geq 1} \varphi_M w^M \right)^L, \tag{1.66}$$

where

$$A_Q = \lambda^Q I_n - \Lambda.$$

The matrices A_Q are not invertible only when $Q \in \bigcup_{j=1}^n \mathrm{Res}_j(\lambda)$, but, thanks Remark 1.4.11, we can set $\varphi_{Q,j} = 0$ for all $Q \in \mathrm{Res}_j(\lambda)$; hence we just have to consider $Q \notin \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$, and, to prove the convergence of the formal conjugation φ in a neighbourhood of the origin, it suffices to show that

$$\sup_Q \frac{1}{|Q|} \log \|\varphi_Q\| < \infty, \tag{1.67}$$

for $|Q| \geq 2$ and $Q \notin \cap_{j=1}^n \mathrm{Res}_j(\lambda)$.

Since f is holomorphic in a neighbourhood of the origin, there exists a positive number ρ such that $\|f_L\| \leq \rho^{|L|}$ for $|L| \geq 2$. The functional equation (1.65) remains valid under the linear change of coordinates $f(z) \mapsto \sigma f(z/\sigma)$, $\varphi(w) \mapsto \sigma\varphi(w/\sigma)$ with $\sigma = \max\{1, \rho^2\}$. Therefore we may assume that

$$\forall |L| \geq 2 \qquad\qquad \|f_L\| \leq 1.$$

It follows from (1.66) that for any multi-index $Q \in \mathbb{N}^n \setminus \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ with $|Q| \geq 2$ we have

$$\|\varphi_Q\| \leq \varepsilon_Q^{-1} \sum_{\substack{Q_1+\cdots+Q_\nu=Q \\ \nu\geq 2}} \|\varphi_{Q_1}\| \cdots \|\varphi_{Q_\nu}\|, \tag{1.68}$$

where

$$\varepsilon_Q = \min_{\substack{1\leq j\leq n \\ Q\notin \mathrm{Res}_j(\lambda)}} |\lambda^Q - \lambda_j|.$$

We can define, inductively, for $m \geq 2$

$$\alpha_m = \sum_{\substack{m_1+\cdots+m_\nu=m \\ \nu\geq 2}} \alpha_{m_1} \cdots \alpha_{m_\nu},$$

and

$$\delta_Q = \varepsilon_Q^{-1} \max_{\substack{Q_1+\cdots+Q_\nu=Q \\ \nu\geq 2}} \delta_{Q_1} \cdots \delta_{Q_\nu},$$

for $Q \in \mathbb{N}^n \setminus \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ with $|Q| \geq 2$, with $\alpha_1 = 1$ and $\delta_E = 1$, where E is any integer vector with $|E| = 1$. Then, by induction, we have that

$$\|\varphi_Q\| \leq \alpha_{|Q|}\delta_Q,$$

for every $Q \in \mathbb{N}^n \setminus \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ with $|Q| \geq 2$. Therefore, to establish (1.67) it suffices to prove analogous estimates for α_m and δ_Q.

It is easy to estimate α_m. Let $\alpha = \sum_{m\geq 1} \alpha_m t^m$. We have

$$\begin{aligned}\alpha - t &= \sum_{m\geq 2} \alpha_m t^m \\ &= \sum_{m\geq 2} \left(\sum_{h\geq 1} \alpha_h t^h\right)^m \\ &= \frac{\alpha^2}{1-\alpha}.\end{aligned}$$

This equation has a unique holomorphic solution vanishing at zero

$$\alpha = \frac{t+1}{4}\left(1 - \sqrt{1 - \frac{8t}{(1+t)^2}}\right),$$

defined for $|t|$ small enough. Hence,

$$\sup_m \frac{1}{m} \log \alpha_m < \infty,$$

as we want.

To estimate δ_Q we have to take care of small divisors. First of all, for each multi-index $Q \notin \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ with $|Q| \geq 2$ we can associate to δ_Q a decomposition of the form

$$\delta_Q = \varepsilon_{L_0}^{-1} \varepsilon_{L_1}^{-1} \cdots \varepsilon_{L_p}^{-1}, \tag{1.69}$$

where $L_0 = Q$, $|Q| > |L_1| \geq \cdots \geq |L_p| \geq 2$ and $L_j \notin \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ for all $j = 1, \ldots, p$ and $p \geq 1$. In fact, we choose a decomposition $Q = Q_1 + \cdots + Q_\nu$ such that the maximum in the expression of δ_Q is achieved; obviously, Q_j does not belong to $\bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ for all $j = 1, \ldots, \nu$. We can then express δ_Q in terms of $\varepsilon_{Q_j}^{-1}$ and $\delta_{Q'_j}$ with $|Q'_j| < |Q_j|$. Carrying on this process, we eventually arrive at a decomposition of the form (1.69). Furthermore, for each multi-index $Q \notin \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ with $|Q| \geq 2$, we can choose an index i_Q so that

$$\varepsilon_Q = |\lambda^Q - \lambda_{i_Q}|.$$

The rest of the proof follows closely [Brj2–3]. For the benefit of the reader, we report here the main steps.

For $m \geq 2$ and $1 \leq j \leq n$, we can define

$$N_m^j(Q)$$

to be the number of factors ε_L^{-1} in the expression (1.69) of δ_Q, satisfying

$$\varepsilon_L < \theta\, \widetilde{\omega}_f(m), \quad \text{and} \quad i_L = j,$$

where $\widetilde{\omega}_f(m)$ is defined in Definition 1.4.4, and in this notation can be expressed as

$$\widetilde{\omega}_f(m) = \min_{\substack{2\leq |Q| \leq m \\ Q \notin \cap_{j=1}^n \mathrm{Res}_j(\lambda)}} \varepsilon_Q,$$

and θ is the positive real number satisfying

$$4\theta = \min_{1\le h\le n} |\lambda_h| \le 1.$$

The last inequality can always be satisfied by replacing f by f^{-1} if necessary. Moreover we also have $\widetilde{\omega}_f(m) \le 2$.

Notice that $\widetilde{\omega}_f(m)$ is non-increasing with respect to m and under our assumptions $\widetilde{\omega}_f(m)$ tends to zero as m goes to infinity. Following [Brj2–3], we have the key estimate.

Lemma 1.4.13. *For $m \ge 2$, $1 \le j \le n$ and $Q \notin \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$,we have*

$$N_m^j(Q) \le \begin{cases} 0, & \text{if } |Q| \le m, \\ \dfrac{2|Q|}{m} - 1, & \text{if } |Q| > m. \end{cases}$$

Proof. The proof is done by induction on $|Q|$. Since we fix m and j throughout the proof, we write N instead of N_m^j.

For $|Q| \le m$,

$$\varepsilon_Q \ge \widetilde{\omega}_f(|Q|) \ge \widetilde{\omega}_f(m) > \theta\,\widetilde{\omega}_f(m),$$

hence $N(Q) = 0$.

Assume now that $|Q| > m$. Then $2|Q|/m - 1 \ge 1$. Write

$$\delta_Q = \varepsilon_Q^{-1}\delta_{Q_1}\cdots\delta_{Q_\nu}, \quad Q = Q_1 + \cdots + Q_\nu, \quad \nu \ge 2,$$

with $|Q| > |Q_1| \ge \cdots \ge |Q_\nu|$; note that $Q - Q_1$ does not belong to $\bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$, otherwise the other Q_h's would be in $\bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$. We have to consider the following different cases.

Case 1: $\varepsilon_Q \ge \theta\,\widetilde{\omega}_f(m)$ and i_Q arbitrary, or $\varepsilon_Q < \theta\,\widetilde{\omega}_f(m)$ and $i_Q \ne j$. Then

$$N(Q) = N(Q_1) + \cdots + N(Q_\nu),$$

and applying the induction hypotheses to each term we get $N(Q) \le (2|Q|/m) - 1$.

Case 2: $\varepsilon_Q < \theta\,\widetilde{\omega}_f(m)$ and $i_Q = j$. Then

$$N(Q) = 1 + N(Q_1) + \cdots + N(Q_\nu),$$

and there are three different subcases.

Case 2.1: $|Q_1| \le m$. Then

$$N(Q) = 1 < \frac{2|Q|}{m} - 1,$$

as we want.

Case 2.2: $|Q_1| \ge |Q_2| > m$. Then there is ν' such that $2 \le \nu' \le \nu$ and $|Q_{\nu'}| > m \ge |Q_{\nu'+1}|$, and we have

$$N(Q) = 1 + N(Q_1) + \cdots + N(Q_{\nu'}) \le 1 + \frac{2|Q|}{m} - \nu' \le \frac{2|Q|}{m} - 1.$$

Case 2.3: $|Q_1| > m \ge |Q_2|$. Then

$$N(Q) = 1 + N(Q_1),$$

and there are again three different subcases.

Case 2.3.1: $i_{Q_1} \neq j$. Then $N(Q_1) = 0$ and we are done.

Case 2.3.2: $|Q_1| \leq |Q| - m$ and $i_{Q_1} = j$. Then

$$N(Q) \leq 1 + 2\,\frac{|Q| - m}{m} - 1 < \frac{2|Q|}{m} - 1.$$

Case 2.3.3: $|Q_1| > |Q| - m$ and $i_{Q_1} = j$. The crucial remark is that $\varepsilon_{Q_1}^{-1}$ gives no contribute to $N(Q_1)$, as shown in the next lemma.

Lemma 1.4.14. *If $Q > Q_1$ with respect to the lexicographic order, Q, Q_1 and $Q - Q_1$ are not in $\bigcap_{j=1}^{n} \mathrm{Res}_j(\lambda)$, $i_Q = i_{Q_1} = j$ and*

$$\varepsilon_Q < \theta\,\widetilde{\omega}_f(m) \quad \text{and} \quad \varepsilon_{Q_1} < \theta\,\widetilde{\omega}_f(m),$$

then $|Q - Q_1| = |Q| - |Q_1| \geq m$.

Proof. Before we proceed with the proof, notice that the equality $|Q - Q_1| = |Q| - |Q_1|$ is obvious since $Q > Q_1$.

Since we are supposing $\varepsilon_{Q_1} = |\lambda^{Q_1} - \lambda_j| < \theta\,\widetilde{\omega}_f(m)$, we have

$$\begin{aligned} |\lambda^{Q_1}| &> |\lambda_j| - \theta\,\widetilde{\omega}_f(m) \\ &\geq 4\theta - 2\theta = 2\theta. \end{aligned}$$

Let us suppose by contradiction $|Q - Q_1| = |Q| - |Q_1| < m$. By assumption, it follows that

$$\begin{aligned} 2\theta\,\widetilde{\omega}_f(m) &> \varepsilon_Q + \varepsilon_{Q_1} \\ &= |\lambda^Q - \lambda_j| + |\lambda^{Q_1} - \lambda_j| \\ &\geq |\lambda^Q - \lambda^{Q_1}| \\ &\geq |\lambda^{Q_1}|\,|\lambda^{Q-Q_1} - 1| \\ &\geq 2\theta\,\widetilde{\omega}_f(|Q - Q_1| + 1) \\ &\geq 2\theta\,\widetilde{\omega}_f(m), \end{aligned}$$

which is impossible. □

Using Lemma 1.4.14, case 1 applies to δ_{Q_1} and we have

$$N(Q) = 1 + N(Q_{1_1}) + \cdots + N(Q_{1_{\nu_1}}),$$

where $|Q| > |Q_1| > |Q_{1_1}| \geq \cdots \geq |Q_{1_{\nu_1}}|$ and $Q_1 = Q_{1_1} + \cdots + Q_{1_{\nu_1}}$. We can do the analysis of case 2 again for this decomposition, and we finish unless we run into case 2.3.2 again. However, this loop cannot happen more than $m+1$ times and we have to finally run into a different case. This completes the induction and the proof of Lemma 1.4.13. □

Since the spectrum of $\mathrm{d}f_O$ satisfies the reduced Brjuno condition, there exists a strictly increasing sequence $\{p_\nu\}_{\nu\geq 0}$ of integers with $p_0 = 1$ and such that

$$\sum_{\nu\geq 0} p_\nu^{-1} \log \widetilde{\omega}_f(p_{\nu+1})^{-1} < \infty. \tag{1.70}$$

We have to estimate

$$\frac{1}{|Q|}\log\delta_Q=\sum_{j=0}^{p}\frac{1}{|Q|}\log\varepsilon_{L_j}^{-1},\quad Q\notin\bigcap_{j=1}^{n}\mathrm{Res}_j(\lambda).$$

By Lemma 1.4.13,

$$\begin{aligned}\operatorname{card}\left\{0\le j\le p:\theta\,\widetilde{\omega}_f(p_{\nu+1})\le\varepsilon_{L_j}<\theta\,\widetilde{\omega}_f(p_\nu)\right\}&\le N_{p_\nu}^{1}(Q)+\cdots N_{p_\nu}^{n}(Q)\\&\le\frac{2n|Q|}{p_\nu}\end{aligned}$$

for $\nu\ge 1$. It is also easy to see from the definition of δ_Q that the number of factors $\varepsilon_{L_j}^{-1}$ is bounded by $2|Q|-1$. In particular,

$$\operatorname{card}\left\{0\le j\le p:\theta\,\widetilde{\omega}_f(p_1)\le\varepsilon_{L_j}\right\}\le 2n|Q|=\frac{2n|Q|}{p_0}.$$

Then,

$$\begin{aligned}\frac{1}{|Q|}\log\delta_Q&\le 2n\sum_{\nu\ge0}p_\nu^{-1}\log(\theta^{-1}\,\widetilde{\omega}_f(p_{\nu+1})^{-1})\\&=2n\left(\sum_{\nu\ge0}p_\nu^{-1}\log\widetilde{\omega}_f(p_{\nu+1})^{-1}+\log(\theta^{-1})\sum_{\nu\ge0}p_\nu^{-1}\right).\end{aligned}\tag{1.71}$$

Since $\widetilde{\omega}_f(m)$ tends to zero monotonically as m goes to infinity, we can choose some $\overline{m}$ such that $1>\widetilde{\omega}_f(m)$ for all $m>\overline{m}$, and we get

$$\sum_{\nu\ge\nu_0}p_\nu^{-1}\le\frac{1}{\log\widetilde{\omega}_f(\overline{m})^{-1}}\sum_{\nu\ge\nu_0}p_\nu^{-1}\log\widetilde{\omega}_f(p_{\nu+1})^{-1},$$

where ν_0 verifies the inequalities $p_{\nu_0-1}\le\overline{m}<p_{\nu_0}$. Thus both series in parentheses in (1.71) converge thanks to (1.70). Therefore

$$\sup_Q\frac{1}{|Q|}\log\delta_Q<\infty$$

and this concludes the proof. □

As a corollary, when there are no resonances, we obtain Brjuno's Theorem 1.4.5.

Recently, again using majorant series, Rong [Ro1] proved the following result in the case in which the spectrum of the differential at the origin of a given germ of biholomorphism fixing the origin contains 1 and λ_j's with $|\lambda_j|=1$, but the λ_j's are not roots of unity.

Theorem 1.4.15. (Rong, 2008 [Ro1]) *Let f be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$, fixing the origin with $\mathrm{d}f_O=\mathrm{Diag}(\Lambda_s,I_r)$, where $\Lambda_s=\mathrm{Diag}(\lambda_1,\dots,\lambda_s)$ with $\lambda_j=e^{2\pi i\theta_j}$, $\theta_j\in\mathbb{R}\setminus\mathbb{Q}$. Assume that there is M a pointwise fixed complex manifold through O of codimension s. Choose local coordinates (x,y) centered in O such that $M=\{x=0\}$. For any $p\in M$, write $\mathrm{d}f_p=\begin{pmatrix}\Lambda_s(y)&O\\ \star&I_r\end{pmatrix}$. Assume that $\Lambda_s(y)\equiv\Lambda_s$ for all $p\in M$. If the λ_j's satisfy the Brjuno condition, then there exists a local holomorphic change of coordinates ψ such that*

$$f\circ\psi=\psi\circ\Lambda,$$

where Λ is the linear part of f.

We shall extend and generalize these results in Chapter 2.

1.5 Non linearizable germs

There are germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin and not linearizable, even formally. In fact let us consider again (1.48), let $\widetilde{Q}$ be the first resonant multi-index with respect to the lexicographic order and let j be the minimal in $\{1,\dots,n\}$ such that $\widetilde{Q} \in \mathrm{Res}_j(\lambda)$. Hence

$$(\lambda^{\widetilde{Q}} - \lambda_j)\psi_{\widetilde{Q},j} + g_{\widetilde{Q},j} = \mathrm{Pol.}(f_{R,j}, \psi_{P,k}\colon |P| < |Q|)$$

and, thanks to the minimality of $\widetilde{Q}$ and j, all the coefficients in the right-hand side are uniquely determined by f and Λ, hence if $\mathrm{Pol.}(f_{R,j}, \psi_{P,k}\colon |P| < |Q|) \neq 0$ (and it can well happen), then $g_{\widetilde{Q},j} \neq 0$ and f is not formally linearizable because we can never delete the term $f_{\widetilde{Q},j} z^{\widetilde{Q}}$ of f.

Even without resonances, the holomorphic linearization is not guaranteed. It is not difficult to construct examples of germs of biholomorphisms that are not holomorphically linearizable using the known results of the one-dimensional case, as in Remark 1.4.6.

However it is also possible to give other kinds of examples. We shall see in the next two subsection two families of examples of germs that are formally but not holomorphically linearizable.

1.5.1 Cremer-like example

It is not difficult to prove that if f belongs to the Siegel domain then

$$\lim_{m\to+\infty} \omega_f(m) = 0\ ,$$

which is the reason why, even without resonances, the formal linearization might be diverging, exactly as in the one-dimensional case. As far as I know, the best negative result in this setting is due to Brjuno [Brj2–3], and is a natural generalization of its one-dimensional counterpart, Theorem 1.2.22:

Theorem 1.5.1. *Let $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ be without resonances and such that*

$$\limsup_{m\to+\infty} \frac{1}{m} \log \frac{1}{\omega_{\lambda_1,\dots,\lambda_n}(m)} = +\infty\ .$$

Then there exists $f \in \mathrm{End}(\mathbb{C}^n, O)$, with $\mathrm{d}f_O = \mathrm{Diag}(\lambda_1,\dots,\lambda_n)$, not holomorphically linearizable.

Proof. Let $\Lambda = \mathrm{Diag}(\lambda_1,\dots,\lambda_n)$. We want to define $f = \Lambda + \widehat{f}$ recursively in such a way that its unique formal linearization $\varphi = \mathrm{Id} + \widehat{\varphi}$ is divergent. Using the conjugacy equation $f \circ \varphi = \varphi \circ \Lambda$ we know that for each multi-index Q with $|Q| \geq 2$ we have

$$(\lambda^Q I - \Lambda)\varphi_Q = f_Q + \mathrm{Polynomial}(f_P, \varphi_R \text{ with } P < Q, |R| < |Q|) \tag{1.72}$$

where we are using the same notation used in the proof of Theorem 1.4.12, and, since there are no resonances, $\lambda^Q I - \Lambda$ is invertible for every multi-index Q with $|Q| \geq 2$. Using (1.72) we inductively choose $f_Q \in \{0,1\}^n$ such that

$$\|f_Q + \mathrm{Polynomial}(f_P, \varphi_R \text{ with } P < Q, |R| < |Q|)\| \geq \frac{1}{2},$$

hence we have

$$\begin{aligned}\sup_{|Q|\geq 2}\frac{1}{|Q|}\log\|\varphi_Q\| &\geq \sup_{|Q|\geq 2}\frac{1}{|Q|}\log\frac{\|\lambda^Q I-\Lambda\|^{-1}}{2}\\ &\geq \sup_{|Q|\geq 2}\frac{1}{|Q|}\log\frac{1}{\omega_{\lambda_1,\ldots,\lambda_n}(|Q|)}\\ &= +\infty,\end{aligned}$$

and we are done. □

1.5.2 Yoccoz's example

As already remarked, it is not known whether the Brjuno condition is still necessary for holomorphic linearizability. However, another result in the spirit of Theorem 1.5.1 is the following:

Theorem 1.5.2. (Yoccoz, 1995 [Y2]) *Let $A \in GL(n,\mathbb{C})$ be an invertible matrix such that its eigenvalues have no resonances and such that its Jordan normal form contains a non-trivial block associated to an eigenvalue of modulus one. Then there exists $f \in \mathrm{End}(\mathbb{C}^n, O)$ with $\mathrm{d}f_O = A$ which is not holomorphically linearizable.*

To prove this result we need a couple of preliminary results. The first one is due to Yoccoz, and is the following:

Proposition 1.5.3. (Yoccoz, 1995 [Y2]) *Let $\mathfrak{m}$ be the unique maximal ideal of $\mathbb{C}\{z_1, z_2\}$. Let $\lambda \in \mathbb{S}^1$. Then for $t \in \mathbb{C}^*$ the operator $A_{\lambda,t}\colon \mathfrak{m}^2 \to \mathfrak{m}^2$ defined by*

$$A_{\lambda,t}(g(z_1,z_2)) = g(\lambda(z_1+tz_2),\lambda z_2) - \lambda g(z_1,z_2) \tag{1.73}$$

where $g \in \mathfrak{m}^2$, is not invertible.

Remark 1.5.4. It is clear that (1.73) defines also a formal operator $\widehat{A}_{\lambda,t}$ on the square of the unique maximal ideal $\widehat{\mathfrak{m}}$ of $\mathbb{C}[\![z_1,z_2]\!]$, which is invertible if and only if λ is not a root of unity. Moreover it is well known that $A_{\lambda,t}$ is invertible if $|\lambda| \neq 1$.

To prove Proposition 1.5.3 when λ is not a root of unity, Yoccoz first constructs explicitly the formal solution of (1.73) (which exists thanks to the previous remark) and then he proves that, for $t \neq 0$, it is divergent at the origin. We refer to [Y2] pp. 79–82 for the complete proof.

The second ingredient we need is due to Il'yashenko [I1]. He proved it in the setting of normalization of germs of vector fields, but it works with the same proof for normalization of germs of biholomorphisms. We report here the statement and the proof in our setting.

Definition 1.5.1. A vector $\lambda \in (\mathbb{C}^*)^n$ is called *quasi-resonant* if the series

$$\sum_{\substack{|Q|\geq 2\\ 1\leq j\leq n}} \frac{z^Q e_j}{\lambda^Q - \lambda_j}$$

diverges for every $z \neq 0$.

Theorem 1.5.5. (Il'yashenko, 1979 [I1]) *Let $\lambda \in (\mathbb{C}^*)^n$ be a quasi-resonant vector and let $\widehat{f}$ be a germ of holomorphic mapping of $\mathbb{C}^n$ fixing the origin, without linear term, and such that its coefficients can be estimated from below in modulus by some geometric progression. Then the unique linearization φ_t of $f_t = \Lambda + t\widehat{f}$, where $\Lambda = \mathrm{Diag}(\lambda_1,\ldots,\lambda_n)$, diverges for almost all $t \in \mathbb{C}$ with respect to the Lebesgue measure.*

Proof. It follows from the hypotheses that the solution of the cohomological equation

$$\widehat{\varphi} \circ \Lambda - \Lambda \widehat{\varphi} = \widehat{f} \tag{1.74}$$

diverges. In fact, using the multi-index notation for each component j, from

$$\widehat{f}_j(z) = \sum_{|Q| \geq 2} f_{Q,j} z^Q$$

we get $\widehat{\varphi}_j(z) = \sum_{|Q|\geq 2} \frac{f_{Q,j}}{\lambda^Q - \lambda_j} z^Q$, and the coefficients of this series are not majorized by any geometric progression.

We shall need the following preliminary results:

Lemma 1.5.6. (Il'yashenko, 1979 [I1]) *Let $\varphi_t = \mathrm{Id} + \widehat{\varphi}_t$ be a formal power series linearizing f_t. Then $(\widehat{\varphi}_t)_j = \sum_{|Q|\geq 2} \varphi_{Q,j}(t) z^Q$ and $\varphi_{Q,j}(t)$ is a polynomial in t of degree at most $|Q|$, for each component j.*

Proof. We argue by induction on $s = |Q|$. The series $\widehat{\varphi}_t$ satisfies

$$\widehat{\varphi}_t \circ \Lambda - \Lambda \widehat{\varphi}_t = t \widehat{f} \circ (\mathrm{Id} + \widehat{\varphi}_t),$$

hence we have

$$\{\widehat{\varphi}_t\}_2 \circ \Lambda - \Lambda \{\widehat{\varphi}_t\}_2 = t\{\widehat{f}\}_2,$$

where we denote by $\{\widehat{\varphi}_t\}_s$ and by $\{\widehat{f}\}_s$ respectively the homogeneous components of degree s in z of $\widehat{\varphi}_t$ and $\widehat{f}$; since Λ is linear, we obtain that $\deg_t(\{\varphi_{Q,j}(t)\}_2) = 1$. Assume now that $\deg_t(\{\varphi_{Q,j}(t)\}_r) < r$ for all $r < s$. We prove that $\deg_t(\{\widehat{f} \circ (\mathrm{Id} + \widehat{\varphi}_t)\}_s) \leq s-1$. We remark that the s-jet of $\widehat{f} \circ (\mathrm{Id} + \widehat{\varphi}_t)$ depends only on the $(s-1)$-jet of the series $\widehat{\varphi}_t$. A monomial of degree s in z in the power series expansion of $\widehat{f} \circ (\mathrm{Id} + \widehat{\varphi}_t)$ either contains a factor of the form $\varphi_{K,j}(t) z^K$, or else more than one such factor. In the first case, we have $|K| \leq s-1$ by the above remark, and using the induction hypothesis $\deg_t(\varphi_{K,j}(t)) \leq s-1$; in the second case, again using the induction hypothesis, the degree in t of the coefficient multiplying the monomial considered is at least 2 less than the degree of the monomial in z, i.e., it is again less than $s-1$. The assertion follows then from the equality

$$\{\widehat{\varphi}_t \circ \Lambda - \Lambda \widehat{\varphi}_t\}_s = t\{\widehat{f} \circ (\mathrm{Id} + \widehat{\varphi}_t)\}_s,$$

and we are done. □

The second ingredient we need is due to Nadirashvili, and we refer to the original article [N] for a proof:

Lemma 1.5.7. (Nadirashvili, 1976 [N]) *Let $p_s(t)$ be a polynomial of degree s, let E be a set of positive Lebesgue plane measure in a disk K_R of radius R, and assume that there exists $q > 0$ such that $|p_s|_E| < q^s$. Then there exist C depending only on $\mu = \mathrm{Leb}_2(E)$ and R such that $|p_s|_{K_R}| < (Cq)^s$.*

Now we have all we need to finish the proof of Theorem 1.5.5.

Assume that the series $\widehat{\varphi}_t$ converges for all t in a set $M \subset \mathbb{C}$ of positive measure. Then for each $t \in M$ the function $q(t) = \inf\{q \in \mathbb{R} \mid q^{|K|} > |\varphi_{K,j}(t)|\}$ is well-defined and finite. This function is measurable, since $\varphi_{K,j}(t)$ are polynomials in t. Then there exists q such that the set $E = \{t \in M \mid q(t) < q\}$ has positive measure. By Lemma 1.5.7, there exist C and R such that $|\varphi_{K,j}(t)| < (Cq)^{|K|}$ on a disk $K_R = \{|t| < R\}$. Hence, by the Cauchy estimates, $\widehat{\varphi}_t$ converges in the polydisk $\{|z_j| < (Cq)^{-1}, |t| < R\}$ as a power series in z and t; hence, the function $\psi = \left(\frac{\partial}{\partial t}\widehat{\varphi}_t\right)|_{t=0}$ is holomorphic in a neighbourhood of zero. We have

$$\frac{\partial}{\partial t}(\widehat{\varphi}_t \circ \Lambda - \Lambda\widehat{\varphi}_t) = \frac{\partial}{\partial t}(t\widehat{f} \circ (\mathrm{Id} + \widehat{\varphi}_t));$$

thus

$$\left(\frac{\partial\widehat{\varphi}_t}{\partial t} \circ \Lambda - \Lambda\frac{\partial\widehat{\varphi}_t}{\partial t}\right)\bigg|_{t=0} = \widehat{f} \circ (\mathrm{Id} + \widehat{\varphi}_t)\,|_{t=0} + t\frac{\partial}{\partial t}(\widehat{f} \circ (\mathrm{Id} + \widehat{\varphi}_t))\,|_{t=0}\ ,$$

and hence

$$\psi \circ \Lambda - \Lambda\psi = \widehat{f} \circ (\mathrm{Id} + \widehat{\varphi}_0), \tag{1.75}$$

where $\widehat{\varphi}_0 = \frac{\partial\widehat{\varphi}_t}{\partial t}|_{t=0}$, and thus it is holomorphic. But $\widehat{\varphi}_0$ has to solve $\widehat{\varphi}_0 \circ \Lambda - \Lambda\widehat{\varphi}_0 = 0$, hence $\widehat{\varphi}_0 \equiv 0$ and (1.75) becomes

$$\psi \circ \Lambda - \Lambda\psi = \widehat{f}.$$

This means that ψ is a convergent solution of (1.74) contradicting the fact that any solution of (1.74) diverges. □

With this two ingredient the proof of Theorem 1.5.2 is almost immediate.

Proof of Theorem 1.5.2. It suffices to prove it for $n = 2$ and

$$A = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}.$$

Let $B\colon \mathfrak{m}^2 \to \mathfrak{m}^2$ be the operator defined by

$$B(g) = g \circ A - A \circ g.$$

Writing, $g = (g_1, g_2)$, the second component of $B(g)$ is exactly equal to $A_{\lambda,1}(g_2)$; then, thanks to Proposition 1.5.3, B is not surjective. Applying Theorem 1.5.5, we get that if $h = (h_1, h_2) \in \mathfrak{m}^2$ does not belong to the image of B, then the germ of biholomorphism $f_t = A + th$ is not linearizable for almost all $t \in \mathbb{C}$, and we are done. □

1.5.3 Pérez-Marco's results

In [P8] and [P9] Pérez-Marco develops the idea, originally due to Il'yashenko [I1], of using deformations and potential theory to study holomorphic small divisor problems.

As we recalled in the previous subsection, in the framework of the theory of linearization of holomorphic systems of differential equations, Il'yashenko studied linear deformations of a system and used the polynomial dependence of the new formal linearization to study the divergence of the series. He also introduced the idea of using potential theory since he applied Lemma 1.5.7, due to Nadirashvili [N], to control the norm of a polynomial on an arbitrary bounded domain through its norm on a set of positive measure.

We saw in the previous subsection that a variation of the argument of Il'yashenko [I1] has been used by J.-C. Yoccoz [Y2] in order to show that the quadratic polynomial is the worst linearizable holomorphic germ. Also, Herman [He] used potential theory (in parameter space) for studying small divisor problems.

The general principle suggested by this kind of results could be stated as follows:

There is either total convergence for all parameter values or general divergence except for a very small exceptional set of parameter values.

The exceptional set E in Pérez-Marco's work [P9] is a pluripolar set: for each point of E there is a neighborhood U and a plurisubharmonic function u such that the points of $U \cap E$ belong to the $-\infty$ level set of u. Of course these sets have zero Lebesgue measure, but are indeed "much smaller" (there exist smooth arcs which are not pluripolar). In one dimension a pluripolar set has zero logarithmic capacity, and hence zero Hausdorff dimension.

Pérez-Marco proves the following result (we refer to the original article [P9] for a proof).

Theorem 1.5.8. (Pérez-Marco, 2001 [P9]) *Let $n, m \geq 1$ and $d \geq 0$. Let us consider a family $\{\varphi_I\}_{I \in \mathbb{N}^m}$ of germs of holomorphic maps $\varphi_I \colon (\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$ of order larger or equal to 2 (i.e., $\varphi_I(z) = O(\|z\|^2)$) indicized over multi-indices $I = (i_1, \dots, i_m) \in \mathbb{N}^m$ with $0 \leq |I| \leq d$. For $t = (t_1, \dots, t_m) \in \mathbb{C}^m$ we consider the holomorphic family of germs of biholomorphisms of $\mathbb{C}^n$*

$$f_t(z) = Az + \sum_{\substack{I \in \mathbb{N}^m \\ 0 \leq |I| \leq d}} t^I \varphi_I(z)$$

where $A \in \mathrm{GL}_n(\mathbb{C})$ is a fixed linear map, $A = \mathrm{d}f_O$, with non-resonant eigenvalues. Then all maps f_t, with $t \in \mathbb{C}^m$ are formally linearizable, and we have the following dichotomy: either

(i) The holomorphic family $(f_t)_{t \in \mathbb{C}^m}$ is holomorphically linearizable. Moreover, the radius $R(h_t)$ of convergence of the linearization h_t is bounded from below on compact sets, more precisely, for some $C_0 > 0$, and any $t \in \mathbb{C}^m$,

$$R(h_t) \geq \frac{C_0}{1 + \|t\|^d}; \textit{or}$$

(ii) f_t is not holomophically linearizable, except for an exceptional pluri-polar set $E \subset \mathbb{C}^m$ of values of t.

In [P8] Pérez-Marco proves a similar result in the setting of the problem of holomorphic linearization in the presence of resonances.

1.6 Partial linearization results

Another way to generalize Brjuno's Theorem 1.4.5 is to look for partial linearization results, e.g., studying the linearization problem along submanifolds.

1.6.1 Pöschel's result

Pöschel [Pö] shows how to modify (1.60) and (1.61) to get partial linearization results along submanifolds. To do so, he uses a notion of partial Brjuno condition which is explained in the following definitions:

Definition 1.6.1. Let $n \geq 2$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. Fix $1 \leq s \leq n$ and let $\underline{\lambda} = (\lambda_1, \dots, \lambda_s)$. For any $m \geq 2$ put

$$\omega_s(m) = \min_{2 \leq |K| \leq m} \min_{1 \leq j \leq n} |\underline{\lambda}^K - \lambda_j|,$$

where $\underline{\lambda}^K = \lambda_1^{k_1} \cdots \lambda_s^{k_s}$.

Definition 1.6.2. Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. Fix $1 \leq s \leq n$. We say that $\lambda = (\lambda_1, \ldots, \lambda_n)$ *satisfies the partial Brjuno condition of order* s if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu \geq 0}$ with $p_0 = 1$ such that

$$\sum_{\nu \geq 0} p_\nu^{-1} \log \omega_s(p_{\nu+1})^{-1} < \infty.$$

It is clear that $\omega_s(m) \neq 0$ for all $m \geq 2$ if and only if there are no resonant multi-indices Q of the form $Q = (q_1, \ldots q_s, 0, \ldots, 0)$.

Remark 1.6.1. For $s = n$ the partial Brjuno condition of order s is nothing but the usual Brjuno condition introduced in [Brj2–3] (see also [M] pp. 25–37 for the one-dimensional case). When $s < n$, the partial Brjuno condition of order s is indeed weaker than the Brjuno condition. Let us consider for example $n = 2$ and let $\lambda, \mu \in \mathbb{C}^*$ be distinct. To check whether the pair (λ, μ) satisfies the partial Brjuno condition of order 1, we have to consider only the terms $|\lambda^k - \lambda|$ and $|\lambda^k - \mu|$ for $k \geq 2$, whereas to check the full Brjuno condition we have to consider also the terms $|\mu^h - \lambda|$, $|\mu^h - \mu|$ for $h \geq 2$, and $|\lambda^k \mu^h - \lambda|$, $|\lambda^k \mu^h - \mu|$ for $k, h \geq 1$.

Remark 1.6.2. A n-tuple $\lambda = (\lambda_1, \ldots, \lambda_s, 1, \ldots, 1) \in (\mathbb{C}^*)^n$ satisfies the partial Brjuno condition of order s if and only if $(\lambda_1, \ldots, \lambda_s)$ satisfies the Brjuno condition.

We assume that the differential $\mathrm{d}f_O$ is diagonalizable. Then, possibly after a linear change of coordinates, we can write

$$f(z) = \Lambda z + \widehat{f}(z),$$

where $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$, and $\widehat{f}$ vanishes up to first order at $O \in \mathbb{C}^n$.

The linear map $z \mapsto \Lambda z$ has a very simple structure. For instance, for any subset $\lambda_1, \ldots, \lambda_s$ of eigenvalues with $1 \leq s \leq n$, the direct sum of the corresponding eigenspaces obviously is an invariant manifold on which this map acts linearly with these eigenvalues.

We have the following result of Pöschel [Pö] that generalizes the one of Brjuno [Brj2–3]:

Theorem 1.6.3. (Pöschel, 1986 [Pö]) *Let f be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ fixing the origin O. If there exists a positive integer $1 \leq s \leq n$ such that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathrm{d}f_O$ satisfy the partial Brjuno condition of order s, then there exists locally a complex analytic f-invariant manifold M of dimension s, tangent to the eigenspace of $\lambda_1, \ldots, \lambda_s$ at the origin, on which the mapping is holomorphically linearizable.*

The proof of this result is similar to the one of Theorem 1.4.12 and the reader can also find it in the original article [Pö]. In the next chapter we shall present a generalization of this result to get a complete linearization result also in presence of resonances.

1.6.2 Other results

We also have the following partial linearization result obtained by Nishimura in [Ni] which generalizes the one of Reich [Re2] (the statement is slightly different from the original one presented in [Ni] but perfectly equivalent):

Theorem 1.6.4. (Nishimura, 1983 [Ni]) *Let f be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$, fixing the origin O. Assume that Y is a complex manifold through O of codimension s pointwise fixed by f. In coordinates $z = (x, y)$ in which $Y = \{x = 0\}$ we can write f in the form*

$$\begin{aligned} x_i' &= \textstyle\sum_{k=1}^s C_{ik}(y) x_k + f_i^1(x, y) \quad && \text{for } i = 1, \ldots, s, \\ y_j' &= y_j + f_j^2(x, y) && \text{for } j = 1, \ldots, r, \end{aligned}$$

with $\mathrm{ord}_x(f_i^1) \geq 2$ *and* $\mathrm{ord}_x(f_j^2) \geq 1$. *If for each point* $p \in Y$ *the eigenvalues* $\{\lambda_1(p), \dots, \lambda_s(p)\}$ *of the matrix* $C(p) = \big(C_{jk}(p)\big)$ *have modulus less than* 1 *and have no resonances, then there exists a unique holomorphic change of coordinates* ψ*, defined in a neighbourhood of* Y*, tangent to the identity such that*

$$f \circ \psi = \psi \circ L, \tag{1.76}$$

where L *is the germ*

$$\begin{array}{ll} x_i' = \sum_{k=1}^{s} C_{ik}(y) x_k & \text{for } i = 1, \dots, s, \\ y_j' = y_j & \text{for } j = 1, \dots, r. \end{array}$$

We refer to the original article [Ni] for the proof, which, also in this case is done by proving that the unique formal solution ψ of (1.76) is indeed majored by a convergent series, and hence is convergent.

Another kind of partial linearization results, namely "linearization modulo an ideal", can be found in [Sto].

1.7 Holomorphic normalization

As we saw in Section 1.5, there are germs $f \in \mathrm{End}(\mathbb{C}^n, O)$ not holomorphically linearizable. However, by Poincaré-Dulac Theorem 1.3.25 every germ $f \in \mathrm{End}(\mathbb{C}^n, O)$ can be formally normalized; hence it is natural to ask whether a germ is holomorphically normalizable. We saw that in the attracting [resp. repelling] case this is always the case, but the problem with Poincaré-Dulac normal forms is that, as we saw, they are not unique, and the problem of finding canonical formal normal forms when f belongs to the Siegel domain is still open, except for very few cases (see [J2] and [Ri1] for example). Furthermore, even if Écalle in his monumental work [É4] provides completes sets of invariants characterizing the conjugacy classes of germs in $\mathrm{End}(\mathbb{C}^n, O)$, those invariants are not so easily computable and it remains somehow difficult to use them in studying particular cases.

A case that has received some attention is the so-called semi-attractive case

Definition 1.7.1. A holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}^n, O)$ is said *semi-attractive* if the eigenvalues of $\mathrm{d}f_O$ are either equal to 1 or with modulus strictly less than 1.

The dynamics of semi-attractive dynamical systems has been studied by Fatou [F5], Nishimura [Ni], Ueda [U1–2], Hakim [H1] and Rivi [Riv1–2]. Their results more or less say that the eigenvalue 1 yields the existence of a "parabolic manifold" M — in the sense of Theorem 1.3.9.(ii) — of a suitable dimension, while the eigenvalues with modulus less than one ensure, roughly speaking, that the orbits of points in the normal bundle of M close enough to M are attracted to it. For instance, Rivi proved the following

Theorem 1.7.1. (Rivi, 1999 [Riv1–2]) *Let* $f \in \mathrm{End}(\mathbb{C}^n, O)$ *be a holomorphic local dynamical system. Assume that* 1 *is an eigenvalue of (algebraic and geometric) multiplicity* $q \geq 1$ *of* $\mathrm{d}f_O$*, and that the other eigenvalues of* $\mathrm{d}f_O$ *have modulus less than* 1. *Then:*

(i) *We can choose local coordinates* $(z, w) \in \mathbb{C}^q \times \mathbb{C}^{n-q}$ *such that* f *expressed in these coordinates becomes*

$$\begin{cases} f_1(z, w) = A(w)z + P_{2,w}(z) + P_{3,w}(z) + \cdots, \\ f_2(z, w) = G(w) + B(z, w)z, \end{cases}$$

where: $A(w)$ *is a* $q \times q$ *matrix with entries holomorphic in* w *and* $A(O) = I_q$*; the* $P_{j,w}$ *are* q*-uples of homogeneous polynomials in* z *of degree* j *whose coefficients are holomorphic* *in w; G is a holomorphic self-map of $\mathbb{C}^{n-q}$ such that $G(O) = O$ and the eigenvalues of dG_O are the eigenvalues of $\mathrm{d}f_O$ with modulus strictly less than 1; and $B(z, w)$ is an $(n - q) \times q$ matrix of holomorphic functions vanishing at the origin. In particular, $f_1(z, O)$ is tangent to the identity.*

(ii) If $v \in \mathbb{C}^q \subset \mathbb{C}^m$ is a non-degenerate characteristic direction for $f_1(z, O)$, and the latter map has order ν, then there exist $\nu - 1$ disjoint f-invariant $(n - q + 1)$-dimensional complex submanifolds M_j of $\mathbb{C}^n$, with the origin in their boundary, such that the orbit of every point of M_j converges to the origin tangentially to $\mathbb{C}v \oplus E$, where $E \subset \mathbb{C}^n$ is the subspace generated by the generalized eigenspaces associated to the eigenvalues of $\mathrm{d}f_O$ with modulus less than one.

Rivi also has results in the spirit of Theorem 1.3.9, and results when the algebraic and geometric multiplicities of the eigenvalue 1 differ; see [Riv1, 2] for details.

Building on work done by Canille Martins [CM] in dimension 2, and using Theorem 1.2.9 and general results on normally hyperbolic dynamical systems due to Palis and Takens [PT], Di Giuseppe has obtained the topological classification when the eigenvalue 1 has multiplicity 1 and the other eigenvalues are not resonant:

Theorem 1.7.2. (Di Giuseppe, 2004 [Di]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system such that $\mathrm{d}f_O$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$, where λ_1 is a primitive q-root of unity, and $|\lambda_j| \neq 0, 1$ for $j = 2, \ldots, n$. Assume moreover that $\lambda_2^{r_2} \cdots \lambda_n^{r^n} \neq 1$ for all multi-indices $(r_2, \ldots, r_n) \in \mathbb{N}^{n-1}$ such that $r_2 + \cdots + r_n \geq 2$. Then f is topologically locally conjugated either to $\mathrm{d}f_O$ or to the map*

$$z \mapsto (\lambda_1 z_1 + z_1^{kq+1}, \lambda_2 z_2, \ldots, \lambda_n z_n)$$

for a suitable $k \in \mathbb{N}^*$.

We end this chapter by recalling results by Bracci and Molino, and by Rong. Assume that $f \in \mathrm{End}(\mathbb{C}^2, O)$ is a holomorphic local dynamical system such that the eigenvalues of $\mathrm{d}f_O$ are 1 and $e^{2\pi i\theta} \neq 1$. If $e^{2\pi i\theta}$ satisfies the Brjuno condition, Pöschel [Pö], in Theorem 1.6.3, proved the existence of a 1-dimensional f-invariant holomorphic disk containing the origin where f is conjugated to the irrational rotation of angle θ. On the other hand, Bracci and Molino give sufficient conditions (depending on f but not on $e^{2\pi i\theta}$, expressed in terms of two new holomorphic invariants, and satisfied by generic maps) for the existence of parabolic curves tangent to the eigenspace of the eigenvalue 1; see [BrM] for details, and [Ro3] for generalizations to $n \geq 3$.

2

Linearization in presence of resonances

The main purpose of this chapter is to prove a linearization result in presence of resonances for a germ f of biholomorphism of $\mathbb{C}^n$ fixing the origin O, with $\mathrm{d}f_O$ diagonalizable. We shall prove that, under certain arithmetic conditions on the eigenvalues of $\mathrm{d}f_O$ and some restrictions on the resonances, f is locally holomorphically linearizable if and only if there exists a particular f-invariant complex manifold, and we shall also see that most of the classical linearization results can be obtained as corollaries of our result. The main results of this chapter are published in [R2].

2.1 Definitions and notations

In this chapter we shall prove a linearization result in presence of resonances for a germ of biholomorphism $f \in \mathrm{End}(\mathbb{C}^n, O)$. To do that we shall need a restriction on the resonances and a property for the germ to first prove a formal linearization result, and then we shall need a condition on the eigenvalues of $\mathrm{d}f_O$ assuring convergence.

The restriction on the admitted resonances is the following:

Definition 2.1.1. Let $1 \leq s \leq n$. We say that $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r) \in (\mathbb{C}^*)^n$ *has only level s resonances* if there are only two kinds of resonances:

(*a*)
$$\boldsymbol{\lambda}^Q = \lambda_h \iff Q \in \widetilde{K}_1,$$

where
$$\widetilde{K}_1 = \left\{ Q \in \mathbb{N}^n : |Q| \geq 2, \sum_{p=1}^{s} q_p = 1 \ \text{ and } \ \mu_1^{q_{s+1}} \cdots \mu_r^{q_n} = 1 \right\},$$

and

(*b*)
$$\boldsymbol{\lambda}^Q = \mu_j \iff Q \in \widetilde{K}_2,$$

where
$$\widetilde{K}_2 = \{ Q \in \mathbb{N}^n : |Q| \geq 2, q_1 = \cdots = q_s = 0 \text{ and } \exists j \in \{1, \ldots, r\} \text{ s.t. } \mu_1^{q_{s+1}} \cdots \mu_r^{q_n} = \mu_j \}.$$

Example 2.1.1. When $s < n$, if $(\lambda_1, \ldots, \lambda_s)$ has no resonances, it is not difficult to verify that $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s, 1, \ldots, 1)$ has only level s resonances.

Remark 2.1.2. It is obvious that if the set $\widetilde{K}_2$ is empty (which implies that the set $\widetilde{K}_1$ is empty as well), there are no resonances. If $\widetilde{K}_1 \neq \varnothing$, having only level s resonances implies that the sets $\{\lambda_1, \ldots, \lambda_s\}$ and $\{\mu_1, \ldots, \mu_r\}$ are disjoint. If $\widetilde{K}_1 = \varnothing$ but $\widetilde{K}_2 \neq \varnothing$, then the sets $\{\lambda_1, \ldots, \lambda_s\}$ and $\{\mu_1, \ldots, \mu_r\}$ may intersect only in elements not involved in resonances, i.e., we can have $\lambda_l = \mu_m$ for some l and m only if for every multi-index $(q_{s+1}, \ldots, q_n)$, we have $\mu_1^{q_{s+1}} \cdots \mu_r^{q_n} \neq \mu_m$, and for any resonance $\mu_1^{q_{s+1}} \cdots \mu_r^{q_n} = \mu_j$ with $j \neq m$, we have $q_{s+m} = 0$.

Example 2.1.3. Let $\gamma \geq 1$ and let μ_3 be a $(\gamma+1)$-th primitive root of unity. Let μ_1, μ_2 be two complex numbers of modulus different from 1 and such that

$$\mu_1^{\alpha} \mu_2^{\beta} = \mu_3$$

with $\alpha, \beta \in \mathbb{N} \setminus \{0\}$. Then we have

$$\mu_1^{\alpha} \mu_2^{\beta} \mu_3^{\gamma} = 1.$$

We can choose μ_1, μ_2 such that the only resonant multi-indices for the triple (μ_1, μ_2, μ_3) are $(\alpha, \beta, 0)$, $(\alpha-1, \beta, \gamma)$ and $(\alpha, \beta-1, \gamma)$. Then, if we consider λ such that $(\lambda, \mu_1, \mu_2, \mu_3)$ has only level 1 resonances, the admitted resonances are the following:

$$\begin{aligned} \widetilde{K}_1 &= \{(1, \alpha, \beta, \gamma)\}, \\ \widetilde{K}_2 &= \{(0, \alpha, \beta, 0), (0, \alpha-1, \beta, \gamma), (0, \alpha, \beta-1, \gamma)\}. \end{aligned}$$

Example 2.1.4. Let us consider $(\mu_1, \mu_2, \mu_3, \mu_4) \in (\mathbb{C}^*)^4$ with only one resonance, for example $\mu_1^p \mu_2^q = \mu_3$ with $p, q \geq 1$, and such that $(\lambda, \mu_1, \mu_2, \mu_3, \mu_4)$ has only level 1 resonances with $\lambda = \mu_4$. Then

$$\begin{aligned} \widetilde{K}_1 &= \varnothing, \\ \widetilde{K}_2 &= \{(0, p, q, 0, 0)\}. \end{aligned}$$

Recall that if $n \geq 2$ and we take $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ not necessarily distinct, for any $m \geq 2$ we put

$$\widetilde{\omega}_{\lambda_1, \ldots, \lambda_n}(m) = \min_{\substack{2 \leq |Q| \leq m \\ Q \notin \mathrm{Res}_j(\lambda)}} \min_{1 \leq j \leq n} |\lambda^Q - \lambda_j|,$$

where $\mathrm{Res}_j(\lambda)$ is the set of multi-indices $Q \in \mathbb{N}^n$ giving a resonance relation for $\lambda = (\lambda_1, \ldots, \lambda_n)$ relative to $1 \leq j \leq n$, i.e., $\lambda^Q - \lambda_j = 0$. We also said that λ *satisfies the reduced Brjuno condition* if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu \geq 0}$ with $p_0 = 1$ such that

$$\sum_{\nu \geq 0} p_\nu^{-1} \log \widetilde{\omega}_{\lambda_1, \ldots, \lambda_n}(p_{\nu+1})^{-1} < \infty.$$

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathrm{d}f_O$, we write $\widetilde{\omega}_f(m)$ for $\widetilde{\omega}_{\lambda_1, \ldots, \lambda_n}(m)$.

We saw in Section 1.4.2 that the reduced Brjuno condition plays a main rôle in the study of convergence in the linearization problem; hence it is natural to introduce the following new case in the classification we presented in Definition 1.3.2.

Definition 2.1.2. Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O and let $s \in \mathbb{N}$, with $1 \leq s \leq n$. The origin O is called a *quasi-Brjuno fixed point of order* s if $\mathrm{d}f_O$ is diagonalizable and, denoting by $\boldsymbol{\lambda}$ the spectrum of $\mathrm{d}f_O$, we have:

(i) $\boldsymbol{\lambda}$ has only level s resonances;

(ii) $\boldsymbol{\lambda}$ satisfies the reduced Brjuno condition.

We say that the origin is a *quasi-Brjuno fixed point* if there exists $1 \le s \le n$ such that it is a quasi-Brjuno fixed point of order s.

Note that we are not assuming anything on the modulus of the eigenvalues of $\mathrm{d}f_O$ (and, in this sense, this case is transversal to the other ones). In fact, as we saw in the proof of Theorem 1.4.12, we need to estimate the divisors $|\boldsymbol{\lambda}^Q - \lambda_j|$ as Q varies in the non-resonant multi-indices and j varies in the coordinates.

Since we assume that the differential $\mathrm{d}f_O$ is diagonalizable, possibly after a linear change of coordinates, we can write

$$f(z) = \Lambda z + \widehat{f}(z),$$

where $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$, and $\widehat{f}$ vanishes up to first order at $O \in \mathbb{C}^n$.

As we remarked in Section 1.6.1, the linear map $z \mapsto \Lambda z$ has a very simple structure. For instance, for any subset $\lambda_1, \ldots, \lambda_s$ of eigenvalues with $1 \le s \le n$, the direct sum of the corresponding eigenspaces obviously is an invariant manifold on which this map acts linearly with these eigenvalues.

In this chapter we would like to extend Pöschel Theorem 1.6.3 to get a complete linearization in a neighbourhood of the origin. In fact, as we shall see in the next section, a germ with a quasi-Brjuno fixed point satisfies the hypotheses of Pöschel Theorem 1.6.3, and we shall see in Sections 2.4 and 2.5 how to get a complete holomorphic linearization, adding a particular invariant manifold that will be introduced in Section 2.3.

In the rest of the chapter we shall denote by $\|\cdot\|$ the norm $\|\cdot\|_\infty$; but we could also had used the norm $\|\cdot\|_2$ thanks to the equivalence of such norms. We shall also need the following notation: if $g\colon \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function with $g(O) = 0$ (or a formal power series without constant term), and $z = (x, y) \in \mathbb{C}^n$ with $x \in \mathbb{C}^s$ and $y \in \mathbb{C}^{n-s}$, we shall denote by $\mathrm{ord}_x(g)$ the maximum positive integer m such that g belongs to the ideal $(x_1, \cdots, x_s)^m$.

2.2 Quasi-Brjuno condition vs partial Brjuno condition

As announced in this section we shall explain the relations between the quasi-Brjuno condition and the partial Brjuno condition.

Recall that if $n \ge 2$, we take $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ not necessarily distinct, we fix $1 \le s \le n$, and, letting $\underline{\lambda} = (\lambda_1, \ldots, \lambda_s)$, for any $m \ge 2$ we put

$$\omega_s(m) = \min_{2 \le |K| \le m} \min_{1 \le j \le n} |\underline{\lambda}^K - \lambda_j|,$$

where $\underline{\lambda}^K = \lambda_1^{k_1} \cdots \lambda_s^{k_s}$. We then said that $\lambda = (\lambda_1, \ldots, \lambda_n)$ *satisfies the partial Brjuno condition of order* s if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu \ge 0}$ with $p_0 = 1$ such that

$$\sum_{\nu \ge 0} p_\nu^{-1} \log \omega_s(p_{\nu+1})^{-1} < \infty.$$

Notice that whereas it is always possible to introduce the reduced Brjuno condition, the partial Brjuno condition makes sense only when there are no resonant multi-indices $Q \in \mathbb{N}^n$, with $|Q| \ge 2$ and $q_{s+1} = \ldots = q_n = 0$. Anyway, when we have only level s resonance, we can deal with these two condition at the same time.

Remark 2.2.1. If λ has only level s resonances, then we have

$$\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(m) = \min_{2\leq|Q|\leq m} \min\left\{ \min_{\substack{1\leq j\leq n\\ q_1+\cdots+q_s\geq 2}} |\lambda^Q-\lambda_j|, \min_{\substack{1\leq j\leq n-s\\ q_1+\cdots+q_s=1}} |\lambda^Q-\lambda_{s+j}| \right\},$$

therefore

$$\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(m) = \min\left\{\omega_s(m), \min_{\substack{2\leq|Q|\leq m\\ (q_{s+1},\dots,q_n)\neq O}} \left\{ \min_{\substack{1\leq j\leq n\\ q_1+\cdots+q_s\geq 2}} |\lambda^Q-\lambda_j|, \min_{\substack{1\leq j\leq n-s\\ q_1+\cdots+q_s=1}} |\lambda^Q-\lambda_{s+j}| \right\}\right\},$$

so it is obvious that, since $\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(m)\leq\omega_s(m)$ for every $m\geq 2$, *the reduced Brjuno condition implies the partial Brjuno condition of order s*. A partial converse is the following

Lemma 2.2.2. *Let $n\geq 2$ and let $\lambda_1,\dots,\lambda_n\in\mathbb{C}^*$ be not necessarily distinct. Let $1\leq s\leq n$ be such that $\lambda=(\lambda_1,\dots,\lambda_n)$ has only level s resonances. Then, if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu\geq 0}$ with $p_0=1$ such that*

$$\sum_{\nu\geq 0} p_\nu^{-1}\log\omega_s(p_{\nu+1})^{-1}<\infty,$$

(i.e., λ satisfies the partial Brjuno condition of order s), and there exist $k\in\mathbb{N}$ and $\alpha\geq 1$ such that

$$p_\nu>k\Rightarrow\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(p_\nu-k)\geq\omega_s(p_\nu)^\alpha,$$

then λ satisfies the reduced Brjuno condition.

Proof. Let $q_0=p_0$ and $q_j=p_{\nu_0+j}-k$ for $j\geq 1$, where ν_0 is the minimum index such that $p_\nu>k$ for all $\nu\geq\nu_0$. Then we have

$$\begin{aligned}\sum_{\nu\geq 0} q_\nu^{-1}\log\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(q_{\nu+1})^{-1} &\leq \alpha\sum_{\nu\geq 0} q_\nu^{-1}\log\omega_s(q_{\nu+1}+k)^{-1}\\ &= \alpha p_0^{-1}\log\omega_s(p_{\nu_0+1})^{-1}+\alpha\sum_{\nu\geq\nu_0+2}\frac{p_\nu}{p_\nu-k}p_\nu^{-1}\log\omega_s(p_{\nu+1})^{-1}\\ &\leq 2\alpha\sum_{\nu\geq 0}p_\nu^{-1}\log\omega_s(p_{\nu+1})^{-1}\\ &<\infty,\end{aligned}$$

and we are done. □

Remark 2.2.3. Suppose that λ has only level s resonances. Recall that a sequence $\{a_m\}$ is said to be *Diophantine of exponent* $\tau>1$ if there exist γ, $\gamma'>0$ so that $\gamma' m^{-\beta}\geq a_m\geq\gamma m^{-\beta}$ (see also [Car], [G] and [St3]). Then if $\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(m)$ is Diophantine of exponent $\beta>1$, and if $\omega_s(m)$ is Diophantine of exponent $\varepsilon>1$, there always exist $\alpha\geq 1$ and $\delta>0$ for which

$$\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(m)\geq\gamma m^{-\beta}\geq\delta m^{-\varepsilon\alpha}\geq\omega_s(m)^\alpha,$$

and thus the hypothesis of Lemma 2.2.2 is satisfied with $k=0$.

More in general, if we have

$$\forall m\geq k+2 \qquad \widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(m-k)\geq\omega_s(m)^\alpha$$

for some $k\in\mathbb{N}$ and $\alpha\geq 1$, the hypothesis of Lemma 2.2.2 is obviously satisfied. For example if $\lambda_1,\dots,\lambda_s\in\mathbb{R}$ are positive and $\lambda_{s+1},\dots,\lambda_n\in\{-1,+1\}$ then it is easy to verify that

$$\forall m\geq 3 \qquad \widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(m-1)\geq\omega_s(m).$$

Furthermore, if $\lambda_{s+1}=\cdots=\lambda_n=1$ then $\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(m)=\omega_s(m)$, and so in this case the partial Brjuno condition of order s coincides with the reduced Brjuno condition.

2.3 Osculating manifolds

In this section we shall introduce a particular kind of invariant manifolds for germs of biholomorphisms that give us information on the properties of such germs, and that will be used in proving our linearization result.

Let f be a germ of biholomorphism of $\mathbb{C}^n$ at a point which we may assume without loss of generality to be the origin O, and let M be an f-invariant complex manifold through O of codimension s, with $1 \le s \le n$. In this situation, the differential $\mathrm{d}f$ acts on the normal bundle $N_M = T\mathbb{C}^n/TM$.

Definition 2.3.1. Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O, and let $1 \le s \le n$. We will say that f *admits an osculating manifold M of codimension s* if there is a germ of f-invariant complex manifold M at O of codimension s such that the normal bundle N_M of M admits a holomorphic flat $(1,0)$-connection that commutes with $\mathrm{d}f|_{N_M}$.

It is obvious that locally every holomorphic bundle admits a holomorphic flat $(1,0)$-connection (it suffices to take the trivial connection on a trivialization). Moreover, it is easy to prove the following result, which has exactly the same proof as in the smooth case (adopting for instance the argument in [BCS] pp. 272–274).

Proposition 2.3.1. *Let $\pi\colon E \to M$ be a holomorphic vector bundle on a complex manifold M and let ∇ be a holomorphic flat $(1,0)$-connection. Then there are a local holomorphic coordinate system about O and a local holomorphic frame of E in which all the connection coefficients Γ^i_{jk} are zero.*

In the particular case of the normal bundle we have the following useful result.

Lemma 2.3.2. *Let $M \subset \mathbb{C}^n$ be a complex manifold of codimension s, $1 \le s \le n$ and let N_M be its normal bundle. Fix $p \in M$. Take a local holomorphic frame in a neighbourhood of p. Then for every local holomorphic frame $\{V_1, \dots, V_s\}$ of N_M we can find local holomorphic coordinates (U, z) with $z = (x, y)$, adapted to M (i.e., $M \cap U = \{x = 0\}$) such that, on $U \cap M$,*

$$V_j = \pi\left(\frac{\partial}{\partial x_j}\right)$$

for every $j = 1, \dots, s$, where $\pi\colon T\mathbb{C}^n \to N_M$ is the canonical projection.

Proof. Let us choose local holomorphic coordinates $\tilde{z} = (\tilde{x}, \tilde{y})$ centered at p adapted to M. Then for every point $(0, \tilde{y}) \in M$ there exists a non-singular matrix $A(\tilde{y}) = (a_{ij}(\tilde{y}))$, depending holomorphically on $\tilde{y}$, such that

$$V_j(\tilde{y}) = \sum_{i=1}^{s} a_{ij}(\tilde{y})\pi\left(\frac{\partial}{\partial \tilde{x}_i}\right)\Bigg|_{(0,\tilde{y})}.$$

Therefore, using the coordinates

$$\begin{aligned} x_i &= \sum_{i=1}^{s} a_{ij}(\tilde{y})\tilde{x}_i \qquad \text{for } i = 1, \dots, s,\\ y_j &= \tilde{y}_j \qquad\qquad\qquad\ \text{for } j = 1, \dots, r, \end{aligned}$$

we obtain the assertion. $\square$

Definition 2.3.2. Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O, and let M be a germ of f-invariant complex manifold at O of codimension s, with $1 \leq s \leq n$. We say that a holomorphic flat $(1,0)$-connection ∇ of the normal bundle N_M of M is *f-invariant* if it commutes with $\mathrm{d}f|_{N_M}$.

Theorem 2.3.3. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O, let M be a germ of f-invariant complex manifold through O of codimension s, with $1 \leq s \leq n$, and let ∇ be a holomorphic flat $(1,0)$-connection of the normal bundle N_M. Then ∇ is f-invariant if and only if there exist local holomorphic coordinates $z=(x,y)$ about O adapted to M in which f has the form*

$$\begin{aligned} x_i' &= \lambda_i x_i + \varepsilon_i x_{i+1} + f_i^1(x,y) && \text{for } i = 1, \dots, s, \\ y_j' &= \mu_j y_j + \varepsilon_{s+j} y_{j+1} + f_j^2(x,y) && \text{for } j = 1, \dots, r = n-s, \end{aligned} \tag{2.2}$$

where $\varepsilon_i, \varepsilon_{s+j} \in \{0,1\}$, and

$$\mathrm{ord}_x(f_i^1) \geq 2,$$

for any $i = 1, \dots, s$.

Proof. If there exist local holomorphic coordinates $z=(x,y)$ about O adapted to M, in which f has the form (2.2) with $\mathrm{ord}_x(f_i^1) \geq 2$ for any $i = 1, \dots, s$, then it is obvious to verify that the trivial holomorphic flat $(1,0)$-connection is f-invariant.

Conversely, let ∇ be a holomorphic flat f-invariant $(1,0)$-connection of the normal bundle N_M. Thanks to Proposition 2.3.1 and to Lemma 2.3.2 we can find local holomorphic coordinates $z=(x,y)$ adapted to M, in which all the connection coefficients Γ^i_{jk} with respect to the local holomorphic frame $\{\pi(\frac{\partial}{\partial x_1}), \dots, \pi(\frac{\partial}{\partial x_s})\}$ of N_M are zero. We may assume without loss of generality, (up to linear changes of the coordinates we can assume that the linear part of f is in Jordan normal form), that in such coordinates f has the form

$$\begin{aligned} x_i' &= \lambda_i x_i + \varepsilon_i x_{i+1} + f_i^1(x,y) && \text{for } i = 1, \dots, s, \\ y_j' &= \mu_j y_j + \varepsilon_{s+j} y_{j+1} + f_j^2(x,y) && \text{for } j = 1, \dots, r, \end{aligned}$$

where $\varepsilon_i, \varepsilon_{s+j} \in \{0,1\}$. Moreover, since $M = \{x = 0\}$ is f-invariant, we have

$$\mathrm{ord}_x(f_i^1) \geq 1.$$

Thanks to the f-invariance of ∇ we have

$$\nabla_{\mathrm{d}f\frac{\partial}{\partial y_k}}\left(\mathrm{d}f|_{N_M}\pi\left(\frac{\partial}{\partial x_j}\right)\right) = \mathrm{d}f|_{N_M}\nabla_{\frac{\partial}{\partial y_k}}\pi\left(\frac{\partial}{\partial x_j}\right)$$

for any $j = 1, \dots, s$ and $k = 1, \dots, r$. Now the right-hand side vanishes, because in the chosen coordinates we have $\nabla_{\frac{\partial}{\partial y_k}}\pi\left(\frac{\partial}{\partial x_j}\right) = 0$. So, using Leibniz formula, we obtain

$$\begin{aligned} 0 &= \nabla_{\mathrm{d}f\frac{\partial}{\partial y_k}}\left(\mathrm{d}f\pi\left(\frac{\partial}{\partial x_j}\right)\right) \\ &= \nabla_{\mathrm{d}f\frac{\partial}{\partial y_k}}\left(\sum_{h=1}^{s}\left(\lambda_h\delta_{hj} + \varepsilon_h\delta_{h,j+1} + \frac{\partial f_h^1}{\partial x_j}(0,y)\right)\pi\left(\frac{\partial}{\partial x_h}\right)\right) \\ &= \sum_{h=1}^{s}\left(\lambda_h\delta_{hj} + \varepsilon_h\delta_{h,j+1} + \frac{\partial f_h^1}{\partial x_j}(0,y)\right)\nabla_{\mathrm{d}f\frac{\partial}{\partial y_k}}\pi\left(\frac{\partial}{\partial x_h}\right) + \sum_{h=1}^{s}\mathrm{d}f\frac{\partial}{\partial y_k}\left(\frac{\partial f_h^1}{\partial x_j}(0,y)\right)\pi\left(\frac{\partial}{\partial x_h}\right) \\ &= \sum_{h=1}^{s}\mathrm{d}f\frac{\partial}{\partial y_k}\left(\frac{\partial f_h^1}{\partial x_j}(0,y)\right)\pi\left(\frac{\partial}{\partial x_h}\right). \end{aligned} \tag{2.3}$$

Therefore we obtain

$$\mathrm{d}f \frac{\partial}{\partial y_k}\left(\frac{\partial f_h^1}{\partial x_j}(0,y)\right) = 0$$

for every $j, h = 1, \ldots, s$ and $k = 1, \ldots r$, and, since $\mathrm{d}f$ is invertible, this implies

$$\frac{\partial}{\partial y_k}\left(\frac{\partial f_h^1}{\partial x_j}(0,y)\right) = 0$$

for every $j, h = 1, \ldots, s$ and $k = 1, \ldots r$, that is

$$\mathrm{ord}_x(f_h^1) \geq 2$$

for every $h = 1, \ldots, s$, and this concludes the proof. □

As a corollay of the previous result we obtain the following characterization of osculating manifolds.

Corollary 2.3.4. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O, and let $1 \leq s \leq n$. Then f admits an osculating manifold M of codimension s such that $f|_M$ is holomorphically linearizable if and only if there exist local holomorphic coordinates $z = (x, y)$ about O adapted to M in which f has the form*

$$\begin{aligned} x_i' &= \lambda_i x_i + \varepsilon_i x_{i+1} + f_i^1(x,y) \qquad \text{for } i = 1, \ldots, s, \\ y_j' &= \mu_j y_j + \varepsilon_{s+j} y_{j+1} + f_j^2(x,y) \quad \text{for } j = 1, \ldots, r, \end{aligned} \tag{2.4}$$

where $\varepsilon_i, \varepsilon_{s+j} \in \{0, 1\}$, and

$$\begin{aligned} \mathrm{ord}_x(f_i^1) &\geq 2, \\ \mathrm{ord}_x(f_j^2) &\geq 1, \end{aligned} \tag{2.5}$$

for any $i = 1, \ldots, s$ and $j = 1, \ldots, r$.

Proof. One direction is clear. Conversely, thanks to Theorem 2.3.3, the fact that M is osculating, is equivalent to the existence of local holomorphic coordinates $z = (x, y)$ about O adapted to M, in which f has the form (2.4) with $\mathrm{ord}_x(f_i^1) \geq 2$ for any $i = 1, \ldots, s$.

Furthermore, $f|_M$ is linearizable; therefore there exists a local holomorphic change of coordinate, tangent to the identity, and of the form

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= \Phi(y), \end{aligned}$$

conjugating f to $\tilde{f}$ of the form (2.4) satisfying (2.5), as we wanted. □

Then we could say that, if we write f as in (2.2), the hypothesis of f-invariance is equivalent to $\mathrm{ord}_x(f_i^1) \geq 1$; $f|_M$ linearized is equivalent to $\mathrm{ord}_x(f_j^2) \geq 1$; osculating means that f_i^1 has no terms of order 1 in x, that is, $f_i^1 = \sum_{h,k} x_h x_k \theta_i^{hk}(x,y)$.

Notice that in Theorem 2.3.3 and in Corollary 2.3.4, up to linear changes of coordinates, we can always assume $\varepsilon_i, \varepsilon_j \in \{0, \varepsilon\}$ instead of $\varepsilon_i, \varepsilon_j \in \{0, 1\}$ for every $\varepsilon > 0$ small enough.

In the next section, we shall first prove a formal linearization result, and so we need the formal analogue of Definition 2.3.1. We define a *formal complex manifold M of codimension s* by means of an ideal of formal complex power series generated by s power series $g_1, \ldots, g_s$ such that their differentials at the origin $\mathrm{d}g_1, \dots \mathrm{d}g_s$ are linearly independent (see also [BER] and [BMR]). Denote by $\widehat{T\mathbb{C}^n}$ the formal tangent bundle of $\mathbb{C}^n$, that is the space of all formal vector fields with complex coefficients. Then the *formal tangent bundle* $\widehat{TM}$ *to* M is well-defined as the set of formal vector fields of $\widehat{T\mathbb{C}^n}$ vanishing on the ideal of formal power series generated by $g_1, \dots, g_s$. The *formal normal bundle* $\widehat{N_M}$ *of* M is then the quotient $\widehat{T\mathbb{C}^n}/\widehat{TM}$. A *formal connection* on the formal normal bundle is a formal map $\widehat{\nabla}\colon \widehat{TM} \times \widehat{N_M} \to \widehat{N_M}$ which satisfies the usual properties of a connection but in the formal category. Thus the following definitions make sense.

Definition 2.3.3. Let f be a formal invertible transformation of $\mathbb{C}^n$ without constant term, and let M be an f-invariant formal complex manifold of codimension s, with $1 \le s \le n$. We say that a formal flat $(1,0)$-connection $\widehat{\nabla}$ of the formal normal bundle $\widehat{N_M}$ of M is *f-invariant* if it commutes with $\mathrm{d}f|_{\widehat{N_M}}$.

Definition 2.3.4. Let $1 \le s \le n$, and let f be a formal invertible transformation of $\mathbb{C}^n$ without constant term. We will say that f *admits a formal osculating manifold* M *of codimension* s if there is an f-invariant formal complex manifold M of codimension s such that the formal normal bundle $\widehat{N_M}$ of M admits a formal flat f-invariant $(1,0)$-connection.

Then, for the formal normal bundle we can prove the formal analogue of Proposition 2.3.1 (using a formal solution of the parallel transport equation that can be easily computed) and Lemma 2.3.2. We then have the following results, whose proofs are the formal rewritings of the ones of Theorem 2.3.3 and Corollary 2.3.4, and hence we omit to report them here.

Theorem 2.3.5. *Let f be a formal invertible transformation of $\mathbb{C}^n$ without constant term, let M be an f-invariant formal complex manifold through O of codimension s, with $1 \le s \le n$, and let $\widehat{\nabla}$ be a formal flat $(1,0)$-connection of the formal normal bundle $\widehat{N_M}$. Then $\widehat{\nabla}$ is f-invariant if and only if there exist local formal coordinates $z = (x, y)$ about O adapted to M in which f has the form*

$$\begin{aligned} x_i' &= \lambda_i x_i + \varepsilon_i x_{i+1} + f_i^1(x,y) && \text{for } i = 1, \dots, s, \\ y_j' &= \mu_j y_j + \varepsilon_{s+j} y_{j+1} + f_j^2(x,y) && \text{for } j = 1, \dots, r, \end{aligned} \tag{2.6}$$

where $\varepsilon_i, \varepsilon_{s+j} \in \{0,1\}$, and

$$\operatorname{ord}_x(f_i^1) \ge 2,$$

for any $i = 1, \dots, s$.

Corollary 2.3.6. *Let $1 \le s \le n$, and let f be a formal invertible transformation of $\mathbb{C}^n$ without constant term. Then f admits a formal osculating manifold M of codimension s such that $f|_M$ is formally linearizable if and only if there exist local formal coordinates $z = (x, y)$ about O adapted to M in which f has the form*

$$\begin{aligned} x_i' &= \lambda_i x_i + \varepsilon_i x_{i+1} + f_i^1(x,y) && \text{for } i = 1, \dots, s, \\ y_j' &= \mu_j y_j + \varepsilon_{s+j} y_{j+1} + f_j^2(x,y) && \text{for } j = 1, \dots, r, \end{aligned} \tag{2.7}$$

where $\varepsilon_i, \varepsilon_{s+j} \in \{0,1\}$, and

$$\begin{aligned} &\operatorname{ord}_x(f_i^1) \ge 2, \\ &\operatorname{ord}_x(f_j^2) \ge 1, \end{aligned} \tag{2.8}$$

for any $i = 1, \dots, s$ and $j = 1, \dots, r$.

2.4 Formal linearization

As announced at the beginning of the chapter, we first prove a formal linearization result.

Theorem 2.4.1. *Let f be a formal invertible transformation of $\mathbb{C}^n$ without constant term such that $\mathrm{d}f_O$ is diagonalizable and the spectrum of $\mathrm{d}f_O$ has only level s resonances, $1 \leq s \leq n$. Then f is formally linearizable if and only if it admits an osculating formal manifold of codimension s such that $f|_M$ is formally linearizable.*

Proof. If f is formally linearizable the assertion is obvious.

Conversely, using Corollary 2.3.6, we can choose formal local coordinates

$$(x, y) = (x_1, \ldots, x_s, y_1, \ldots, y_r)$$

such that, writing $(x', y') = f(x, y)$, f is of the form

$$\begin{aligned} x'_i &= \lambda_i x_i + f_i^1(x, y) \quad \text{for } i = 1, \ldots, s, \\ y'_j &= \mu_j y_j + f_j^2(x, y) \quad \text{for } j = 1, \ldots, r, \end{aligned}$$

where

$$\begin{aligned} \mathrm{ord}_x(f_i^1) &\geq 2, \\ \mathrm{ord}_x(f_j^2) &\geq 1. \end{aligned}$$

Denote by Λ the diagonal matrix $\mathrm{Diag}(\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r)$. We would like to prove that a formal solution ψ of

$$f \circ \psi = \psi \circ \Lambda \tag{2.9}$$

exists of the form

$$\begin{aligned} x_i &= u_i + \psi_i^1(u, v) \quad \text{for } i = 1, \ldots, s, \\ y_j &= v_j + \psi_j^2(u, v) \quad \text{for } j = 1, \ldots, r, \end{aligned}$$

where $(u, v) = (u_1, \ldots, u_s, v_1, \ldots, v_r)$ and ψ_i^1 and ψ_j^2 are formal power series with

$$\begin{aligned} \mathrm{ord}_u(\psi_i^1) &\geq 2, \\ \mathrm{ord}_u(\psi_j^2) &\geq 1. \end{aligned}$$

Write $f(z) = \Lambda z + \widehat{f}(z)$ and $\psi(w) = w + \widehat{\psi}(w)$, where $z = (x, y)$ and $w = (u, v)$. Then equation (2.9) is equivalent to

$$\widehat{\psi} \circ \Lambda - \Lambda \widehat{\psi} = \widehat{f} \circ \psi. \tag{2.10}$$

To obtain a formal solution, we first write

$$\widehat{\psi}(w) = \sum_{|Q| \geq 2} \psi_Q w^Q, \quad \psi_Q \in \mathbb{C}^n,$$

where $Q = (q_1, \ldots, q_n)$, and

$$\widehat{f}(z) = \sum_{|L| \geq 2} f_L z^L, \quad f_L \in \mathbb{C}^n,$$

where $L = (l_1, \ldots, l_n)$. Denoting $\widetilde{\lambda} = (\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r) = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$, equation (2.10) becomes

$$\sum_{|Q|\geq 2} A_Q \psi_Q w^Q = \sum_{|L|\geq 2} f_L \left(\sum_{|M|\geq 1} \psi_M w^M \right)^L, \tag{2.11}$$

where

$$A_Q = \widetilde{\lambda}^Q I_n - \Lambda.$$

The matrices A_Q might not be invertible for some choice of Q due to the presence of resonances. We can write $A_Q = \mathrm{Diag}(A_Q^1, A_Q^2)$ and recall that having only level s resonances means that $\det(A_Q^1) = 0$ if and only if

$$Q \in \widetilde{K}_1,$$

and $\det(A_Q^2) = 0$ if and only if

$$Q \in \widetilde{K}_2.$$

Moreover, from the hypotheses of the Theorem we have that $f_L^1 = 0$ for L in $K_1 \cup K_2$ and $f_L^2 = 0$ for L in K_2, where

$$K_1 = \{L \in \mathbb{N}^n : |L| \geq 2, L = (0, \ldots, 0, l_i, 0, \ldots, 0, l_{s+1}, \ldots, l_n), l_i = 1 \text{ and } i \in \{1, \ldots, s\}\}$$

$$K_2 = \{L \in \mathbb{N}^n : |L| \geq 2, L = (0, \ldots, 0, l_{s+1}, \ldots, l_n)\}.$$

Notice that $\widetilde{K}_1 \subseteq K_1$ and $\widetilde{K}_2 \subseteq K_2$. For each j in $\{1, \ldots, s\}$, let us denote by K_1^j the set $\{L \in \mathbb{N}^n : |L| \geq 2, L = (0, \ldots, 0, l_j, 0, \ldots, 0, l_{s+1}, \ldots, l_n), l_j = 1\}$, so that $K_1 = \cup_{j=1}^s K_1^j$. We look for a solution of (2.9) with $\psi_Q^1 = 0$ for $Q \in K_1 \cup K_2$ and $\psi_Q^2 = 0$ for $Q \in K_2$.

To do so, let us write (2.11) in a more explicit way: for $i = 1, \ldots, s$

$$\sum_{\substack{|Q|\geq 2 \\ Q \notin K_1 \cup K_2}} (\widetilde{\lambda}^Q - \lambda_i) \psi_{Q,i}^1 w^Q = \sum_{\substack{|L|\geq 2 \\ L \notin K_1 \cup K_2}} f_{L,i}^1 \left(\sum_{|M|\geq 1} \psi_M w^M \right)^L, \tag{2.12}$$

and for $j = 1, \ldots, r$

$$\begin{aligned} &\sum_{p=1}^s \sum_{\substack{|Q|\geq 2 \\ Q \in K_1^p}} (\widetilde{\lambda}^Q - \mu_j) \psi_{Q,j}^2 w^Q + \sum_{\substack{|Q|\geq 2 \\ Q \notin K_1 \cup K_2}} (\widetilde{\lambda}^Q - \mu_j) \psi_{Q,j}^2 w^Q \\ &\qquad = \sum_{p=1}^s \sum_{\substack{|L|\geq 2 \\ L \in K_1^p}} f_{L,j}^2 \left(\sum_{|M|\geq 1} \psi_M w^M \right)^L + \sum_{\substack{|L|\geq 2 \\ L \notin K_1 \cup K_2}} f_{L,j}^2 \left(\sum_{|M|\geq 1} \psi_M w^M \right)^L. \end{aligned} \tag{2.13}$$

Now, it is obvious that there are no terms w^Q with $Q \in K_2$ in either side of (2.12) and of (2.13), and we can obtain terms w^Q with $Q \in K_1$ in (2.13) only from terms with $L \in K_1$. In fact, if $L \in K_1^h$ then

$$\begin{aligned} \left(\sum_{|M|\geq 1} \psi_M w^M \right)^L &= \left(u_h + \sum_{p,q} u_p u_q \theta_h^{pq}(u,v) \right) \left(\prod_{j=1}^r \left(v_j + \sum_p u_p \theta_j^p(u,v) \right)^{l_{s+j}} \right) \\ &= u_h v_1^{l_{s+1}} \cdots v_r^{l_n} + \sum_{p,q} u_p u_q \chi^{pq}(u,v) \\ &= w^L + \sum_{p,q} u_p u_q \chi^{pq}(u,v). \end{aligned}$$

Therefore for $j = 1, \dots, r$, we have

$$\sum_{p=1}^{s} \sum_{\substack{|Q|\geq 2\\ Q\in K_1^p}} (\widetilde{\lambda}^Q - \mu_j)\psi^2_{Q,j} w^Q = \sum_{p=1}^{s} \sum_{\substack{|L|\geq 2\\ L\in K_1^p}} f^2_{L,j} \left(\sum_{|M|\geq 1} \psi_M w^M \right)^L$$

$$= \sum_{p=1}^{s} \sum_{\substack{|L|\geq 2\\ L\in K_1^p}} f^2_{L,j} \left(w^L + \sum_{a,b} u_a u_b \chi^{ab}(u,v) \right)$$

from which we conclude that for $Q \in K_1^p$ and $j = 1, \dots, r$ we have

$$\psi^2_{Q,j} = f^2_{Q,j} (\widetilde{\lambda}^Q - \mu_j)^{-1}. \tag{2.14}$$

The remaining ψ_Q with $Q \notin K_1 \cup K_2$ are easily determined by recursion, as usual. $\square$

2.5 Holomorphic linearization

Now we can prove the main result of this chapter.

Theorem 2.5.1. *Let f be a germ of a biholomorphism of $\mathbb{C}^n$ having the origin O as a quasi-Brjuno fixed point of order s, with $1 \leq s \leq n$. Then f is holomorphically linearizable if and only if it admits an osculating manifold M of codimension s such that $f|_M$ is holomorphically linearizable.*

Proof. If f is linearizable the assertion is obvious.

Conversely, we already know, thanks to the previous result, that f is formally linearizable, (notice that, thanks to Corollary 2.3.4, the changes of coordinates needed before finding ψ are holomorphic because now M is a complex manifold). Since the spectrum of $\mathrm{d}f_O$ satisfies the reduced Brjuno condition, the thesis follows from our Theorem 1.4.12, but we report here the proof in this particular case for the sake of completeness.

To prove the convergence of the formal conjugation ψ in a neighbourhood of the origin it suffices to show that

$$\sup_Q \frac{1}{|Q|} \log \|\psi_Q\| < \infty. \tag{2.15}$$

Since f is holomorphic in a neighbourhood of the origin, there exists a positive number ρ such that $\|f_L\| \leq \rho^{|L|}$ for $|L| \geq 2$. The functional equation (2.9) remains valid under the linear change of coordinates $f(z) \mapsto \sigma f(z/\sigma)$, $\psi(w) \mapsto \sigma\psi(w/\sigma)$ with $\sigma = \max\{1, \rho^2\}$. Hence we may assume that

$$\forall |L| \geq 2 \qquad\qquad \|f_L\| \leq 1.$$

It follows from (2.11) and (2.14) that

$$\|\psi_Q\| \leq \begin{cases} \varepsilon_Q^{-1} \displaystyle\sum_{\substack{Q_1+\cdots+Q_\nu = Q\\ \nu\geq 2}} \|\psi_{Q_1}\| \cdots \|\psi_{Q_\nu}\|, & |Q| \geq 2, \quad Q \notin K_1 \cup K_2, \\ \varepsilon_Q^{-1}, & |Q| \geq 2, \quad Q \in K_1, \end{cases} \tag{2.16}$$

where

$$\varepsilon_Q = \begin{cases} \min_{1\le i\le n} |\widetilde{\lambda}^Q - \tilde{\lambda}_i|, & Q \notin K_1 \cup K_2, \\ \min_{1\le h\le r} |\widetilde{\lambda}^Q - \mu_h|, & Q \in K_1. \end{cases}$$

We can define, inductively, for $j \ge 2$

$$\alpha_j = \sum_{\substack{j_1+\cdots+j_\nu=j\\ \nu\ge 2}} \alpha_{j_1}\cdots\alpha_{j_\nu},$$

and for $|Q| \ge 2$

$$\delta_Q = \begin{cases} \varepsilon_Q^{-1} \max_{\substack{Q_1+\cdots+Q_\nu=Q\\ \nu\ge 2}} \delta_{Q_1}\cdots\delta_{Q_\nu}, & Q \notin K_1\cup K_2, \\ \varepsilon_Q^{-1}, & Q\in K_1, \\ 0, & Q \in K_2, \end{cases}$$

with $\alpha_1 = 1$ and $\delta_E = 1$, where E is any integer vector with $|E| = 1$. Then, by induction, we have that

$$\forall |Q|\ge 1 \qquad \|\psi_Q\| \le \alpha_{|Q|}\delta_Q.$$

Therefore, to establish (2.15), it suffices to prove analogous estimates for α_j and δ_Q.

It is easy to estimate α_j. Let $\alpha = \sum_{j\ge 1}\alpha_j t^j$. We have

$$\begin{aligned} \alpha - t &= \sum_{j\ge 2}\alpha_j t^j \\ &= \sum_{j\ge 2}\left(\sum_{h\ge 1}\alpha_h t^h\right)^j \\ &= \frac{\alpha^2}{1-\alpha}. \end{aligned}$$

This equation has a unique holomorphic solution vanishing at zero

$$\alpha = \frac{t+1}{4}\left(1-\sqrt{1-\frac{8t}{(1+t)^2}}\right),$$

defined for $|t|$ small enough. Hence,

$$\sup_j \frac{1}{j}\log\alpha_j < \infty,$$

as we want.

To estimate δ_Q we have to take care of small divisors. First of all, for each $Q \notin K_2$ with $|Q|\ge 2$ we can associate to δ_Q a decomposition of the form

$$\delta_Q = \varepsilon_{L_0}^{-1}\varepsilon_{L_1}^{-1}\cdots\varepsilon_{L_p}^{-1}, \tag{2.17}$$

where $L_0 = Q$, $|Q| > |L_1| \geq \cdots \geq |L_p| \geq 2$ and $L_j \notin K_2$ for all $j = 1, \ldots, p$ and $p \geq 1$. If $Q \in K_1$ it is obvious by the definition of δ_Q. If $Q \notin K_1 \cup K_2$, choose a decomposition $Q = Q_1 + \cdots + Q_\nu$ such that the maximum in the expression of δ_Q is achieved. Obviously, Q_j doesn't belong to K_2 for all $j = 1, \ldots, \nu$. We can then express δ_Q in terms of $\varepsilon_{Q_j}^{-1}$ and $\delta_{Q'_j}$ with $|Q'_j| < |Q_j|$. Carrying on this process, we eventually arrive at a decomposition of the form (2.17). Furthermore,

$$\varepsilon_Q = |\widetilde{\lambda}^Q - \tilde{\lambda}_{i_Q}|, \quad |Q| \geq 2,\ Q \notin K_2,$$

the index i_Q being chosen in some definite way (of course, if $Q \in K_1$ then $i_Q \in \{s+1, \ldots, n\}$).

We can define

$$N_m^j(Q), \quad m \geq 2, \quad j \in \{1, \ldots, n\},$$

to be the number of factors ε_L^{-1} in δ_Q, $(L = L_0, \ldots, L_q)$ satisfying

$$\varepsilon_L < \theta\, \widetilde{\omega}_f(m), \quad \text{and} \quad i_L = j,$$

where θ is the positive real number satisfying

$$4\theta = \min_{1 \leq h \leq n} |\tilde{\lambda}_h| \leq 1.$$

The last inequality can always be satisfied by replacing f by f^{-1} if necessary. Then we also have $\widetilde{\omega}_f(m) \leq 2$, and in this notation $\widetilde{\omega}_f(m)$ can be expressed as

$$\widetilde{\omega}_f(m) = \min_{\substack{2 \leq |Q| \leq m \\ Q \notin K_2}} \varepsilon_Q, \quad m \geq 2.$$

Notice that $\widetilde{\omega}_f(m)$ is non-increasing with respect to m and under our assumptions $\widetilde{\omega}_f(m)$ tends to zero as m goes to infinity.

Lemma 2.5.2. *For $m \geq 2$, $1 \leq j \leq n$ and $Q \notin K_2$,*

$$N_m^j(Q) \leq \begin{cases} 0, & |Q| \leq m, \\ \dfrac{2|Q|}{m} - 1, & |Q| > m. \end{cases}$$

Proof. The proof is done by induction. Since we fix m and j throughout the proof, we write N instead of N_m^j.

For $|Q| \leq m$,

$$\varepsilon_Q \geq \widetilde{\omega}_f(|Q|) \geq \widetilde{\omega}_f(m) > \theta\, \widetilde{\omega}_f(m),$$

hence $N(Q) = 0$.

Assume now that $|Q| > m$. Then $2|Q|/m - 1 \geq 1$. If $Q \in K_1$ then, by definition, $\delta_Q = \varepsilon_Q^{-1}$, so $N(Q)$ can only be equal to 0 or 1 and we are done.

Let us suppose $Q \notin K_1 \cup K_2$. Write

$$\delta_Q = \varepsilon_Q^{-1} \delta_{Q_1} \cdots \delta_{Q_\nu}, \quad Q = Q_1 + \cdots + Q_\nu, \quad \nu \geq 2,$$

with $|Q| > |Q_1| \geq \cdots \geq |Q_\nu|$, and consider the following different cases. Note that $Q - Q_1 \notin K_2$, otherwise the other Q_h's would be in K_2.

Case 1: $\varepsilon_Q \geq \theta\,\widetilde{\omega}_f(m)$ and i_Q arbitrary, or $\varepsilon_Q < \theta\,\widetilde{\omega}_f(m)$ and $i_Q \neq j$. Then

$$N(Q) = N(Q_1) + \cdots + N(Q_\nu),$$

and applying the induction hypotheses to each term we get $N(Q) \leq (2|Q|/m) - 1$.

Case 2: $\varepsilon_Q < \theta\,\widetilde{\omega}_f(m)$ and $i_Q = j$. Then

$$N(Q) = 1 + N(Q_1) + \cdots + N(Q_\nu),$$

and there are three different cases.

Case 2.1: $|Q_1| \leq m$. Then

$$N(Q) = 1 < \frac{2|Q|}{m} - 1,$$

as we want.

Case 2.2: $|Q_1| \geq |Q_2| > m$. Then there is ν' such that $2 \leq \nu' \leq \nu$ and $|Q_{\nu'}| > m \geq |Q_{\nu'+1}|$, and we have

$$N(Q) = 1 + N(Q_1) + \cdots + N(Q_{\nu'}) \leq 1 + \frac{2|Q|}{m} - \nu' \leq \frac{2|Q|}{m} - 1.$$

Case 2.3: $|Q_1| > m \geq |Q_2|$. Then

$$N(Q) = 1 + N(Q_1),$$

and there are three different cases.

Case 2.3.1: $i_{Q_1} \neq j$. Then $N(Q_1) = 0$ and we are done.

Case 2.3.2: $|Q_1| \leq |Q| - m$ and $i_{Q_1} = j$. Then

$$N(Q) \leq 1 + 2\,\frac{|Q| - m}{m} - 1 < \frac{2|Q|}{m} - 1.$$

Case 2.3.3: $|Q_1| > |Q| - m$ and $i_{Q_1} = j$. The crucial remark is that $\varepsilon_{Q_1}^{-1}$ gives no contribute to $N(Q_1)$, as shown in the next lemma.

Lemma 2.5.3. *If $Q > Q_1$ with respect to the lexicographic order, Q, Q_1 and $Q - Q_1$ are not in K_2, $i_Q = i_{Q_1} = j$ and*

$$\varepsilon_Q < \theta\,\widetilde{\omega}_f(m) \quad \text{and} \quad \varepsilon_{Q_1} < \theta\,\widetilde{\omega}_f(m),$$

then $|Q - Q_1| = |Q| - |Q_1| \geq m$.

Proof. Before we proceed with the proof, notice that the equality $|Q - Q_1| = |Q| - |Q_1|$ is obvious since $Q > Q_1$.

Since we are supposing $\varepsilon_{Q_1} = |\tilde{\lambda}^{Q_1} - \tilde{\lambda}_j| < \theta\,\widetilde{\omega}_f(m)$, we have

$$\begin{aligned} |\widetilde{\lambda}^{Q_1}| &> |\tilde{\lambda}_j| - \theta\,\widetilde{\omega}_f(m) \\ &\geq 4\theta - 2\theta = 2\theta. \end{aligned}$$

Let us suppose by contradiction $|Q - Q_1| = |Q| - |Q_1| < m$. By assumption, it follows that

$$\begin{aligned} 2\theta\,\widetilde{\omega}_f(m) &> \varepsilon_Q + \varepsilon_{Q_1} \\ &= |\widetilde{\lambda}^Q - \tilde{\lambda}_j| + |\widetilde{\lambda}^{Q_1} - \tilde{\lambda}_j| \\ &\geq |\widetilde{\lambda}^Q - \widetilde{\lambda}^{Q_1}| \\ &\geq |\widetilde{\lambda}^{Q_1}|\,|\widetilde{\lambda}^{Q-Q_1} - 1| \\ &\geq 2\theta\,\widetilde{\omega}_f(|Q - Q_1| + 1) \\ &\geq 2\theta\,\widetilde{\omega}_f(m), \end{aligned}$$

which is impossible. □

Using Lemma 2.5.3, case 1 applies to δ_{Q_1} and we have

$$N(Q) = 1 + N(Q_{1_1}) + \cdots + N(Q_{1_{\nu_1}}),$$

where $|Q| > |Q_1| > |Q_{1_1}| \geq \cdots \geq |Q_{1_{\nu_1}}|$ and $Q_1 = Q_{1_1} + \cdots + Q_{1_{\nu_1}}$. We can do the analysis of case 2 again for this decomposition, and we finish unless we run into case 2.3.3 again. However, this loop cannot happen more than $m+1$ times and we have to finally run into a different case. This completes the induction and the proof of Lemma 2.5.2. $\square$

Since the origin is a quasi-Brjuno fixed point of order s, there exists a strictly increasing sequence $\{p_\nu\}_{\nu\geq 0}$ of integers with $p_0 = 1$ and such that

$$\sum_{\nu\geq 0} p_\nu^{-1} \log \widetilde{\omega}_f(p_{\nu+1})^{-1} < \infty. \tag{2.18}$$

Since $\delta_Q = 0$ for $Q \in K_2$, we have to estimate only

$$\frac{1}{|Q|}\log \delta_Q = \sum_{j=0}^{p} \frac{1}{|Q|} \log \varepsilon_{L_j}^{-1}, \quad Q \notin K_2.$$

By Lemma 2.5.2,

$$\begin{aligned}\operatorname{card}\left\{0 \leq j \leq p : \theta\,\widetilde{\omega}_f(p_{\nu+1}) \leq \varepsilon_{L_j} < \theta\,\widetilde{\omega}_f(p_\nu)\right\} &\leq N_{p_\nu}^1(Q) + \cdots N_{p_\nu}^n(Q)\\ &\leq \frac{2n|Q|}{p_\nu}\end{aligned}$$

for $\nu \geq 1$. It is also easy to see from the definition of δ_Q that the number of factors $\varepsilon_{L_j}^{-1}$ is bounded by $2|Q|-1$. In particular,

$$\operatorname{card}\left\{0 \leq j \leq p : \theta\,\widetilde{\omega}_f(p_1) \leq \varepsilon_{l_j}\right\} \leq 2n|Q| = \frac{2n|Q|}{p_0}.$$

Then,

$$\begin{aligned}\frac{1}{|Q|}\log \delta_Q &\leq 2n \sum_{\nu\geq 0} p_\nu^{-1} \log(\theta^{-1}\,\widetilde{\omega}_f(p_{\nu+1})^{-1})\\ &= 2n\left(\sum_{\nu\geq 0} p_\nu^{-1} \log \widetilde{\omega}_f(p_{\nu+1})^{-1} + \log(\theta^{-1}) \sum_{\nu\geq 0} p_\nu^{-1}\right).\end{aligned} \tag{2.19}$$

Since $\widetilde{\omega}_f(m)$ tends to zero monotonically as m goes to infinity, we can choose some $\overline{m}$ such that $1 > \widetilde{\omega}_f(m)$ for all $m > \overline{m}$, and we get

$$\sum_{\nu\geq\nu_0} p_\nu^{-1} \leq \frac{1}{\log \widetilde{\omega}_f(\overline{m})^{-1}} \sum_{\nu\geq\nu_0} p_\nu^{-1} \log \widetilde{\omega}_f(p_{\nu+1})^{-1},$$

where ν_0 verifies the inequalities $p_{\nu_0 - 1} \leq \overline{m} < p_{\nu_0}$. Thus both series in parentheses in (2.19) converge thanks to (2.18). Therefore

$$\sup_Q \frac{1}{|Q|}\log \delta_Q < \infty$$

and this concludes the proof. $\square$

2.6 Necessity of the hypotheses

Roughly speaking, we have seen that having only level s resonances and the existence of the osculating manifold on which f is holomorphically linearizable take cares of the resonances in the μ_j's and give the formal linearization. Under these hypotheses the partial Brjuno condition of order s holds, so we have a partial holomorphic linearization given by Pöschel Theorem 1.6.3, and the reduced Brjuno condition glues the formal linearization and the partial holomorphic linearization so to get a global holomorphic linearization.

We shall now discuss the hypotheses of Theorem 2.5.1.

Remark 2.6.1. Notice that *the osculating hypothesis on the f-invariant manifold is necessary.* In fact, let us take a look at the following example in $\mathbb{C}^2$. Let f be given by

$$\begin{aligned} x' &= \lambda(1+y)x + x^2 \\ y' &= y \end{aligned}$$

with $(\lambda, 1)$ satisfying the Brjuno condition of order 1 (in particular λ is not a root of unity). This germ is not linearizable. In fact, let $g_y(x) = \lambda(1+y)x + x^2$, so we can write $f(x,y) = (g_y(x), y)$. A linearization for f is a germ of biholomorphism $\psi = (\psi_1, \psi_2)$ fixing the origin, tangent to the identity, and such that

$$\big(g_{\psi_2(x,y)}(\psi_1(x,y)), \psi_2(x,y)\big) = \big(\psi_1(\lambda x, y), \psi_2(\lambda x, y)\big).$$

This last equality implies $\psi_2 \equiv \psi_2(y)$ and $g_{\psi_2(y)}(\psi_1(x,y)) = \psi_1(\lambda x, y)$. Composing on the right with ψ_2^{-1} and setting $h_y(x) = \psi_1(x, \psi_2^{-1}(y))$, we have

$$g_y\big(h_y(x)\big) = h_y(\lambda x). \tag{2.20}$$

From (2.20) we deduce that $h_y(0) \in \mathrm{Fix}(g_y) = \{0, 1 - \lambda(1+y)\}$. Now, $h_0(0) = 0$; hence, by continuity $h_y(0) = 0$ for $|y|$ small enough, and so $g_y'(0)h_y'(0) = \lambda h_y'(0)$ for $|y|$ small enough. But $h_0'(0) = 1 \neq 0$; therefore $\lambda(1+y) = g_y'(0) = \lambda$ for $|y|$ small enough, which is impossible. Since f is not linearizable it cannot admit an osculating invariant manifold of codimension 1, even if, obviously, the manifold $\{x = 0\}$ is f-invariant, and f is linear there.

Remark 2.6.2. *The reduced Brjuno condition and the hypothesis f holomorphically linearizable on the osculating manifold are necessary.* Consider the following example in $\mathbb{C}^n$ for $n \geq 2$. Let f be a biholomorphism of $\mathbb{C}^n$, fixing the origin, given by

$$\begin{aligned} x_i' &= \lambda_i x_i + f_i^i(x,y) \quad \text{for } i = 1, \ldots, n-1, \\ y' &= \mu y + y^2, \end{aligned} \tag{2.21}$$

with $\mathrm{ord}_x(f_i^1) \geq 2$ for every $i = 1, \ldots, n-1$, $(\lambda_1, \ldots, \lambda_{n-1}, \mu)$ non resonant, and $\mu = e^{2\pi\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$ not a Brjuno number. Then $M = \{x = 0\}$ is an osculating manifold of codimension $n-1$, but $(\lambda_1, \ldots, \lambda_{n-1}, \mu)$ does not satisfy the reduced Brjuno condition (which, since we have no resonances, coincides with the usual Brjuno condition). Furthermore, thanks to Yoccoz's Theorem [Y2], $f|_M$ is not holomorphically linearizable. This germ is not holomorphically linearizable. In fact, assume by contradiction that ψ is a holomorphic linearization. Then $\widetilde{M} = \psi(M) = \{\psi_1^{-1}(\tilde{x}, \tilde{y}) = 0, \ldots, \psi_{n-1}^{-1}(\tilde{x}, \tilde{y}) = 0\}$ is an osculating manifold of codimension $n-1$ for $\tilde{f}(\tilde{x}, \tilde{y}) = \psi \circ f \circ \psi^{-1} \equiv \mathrm{Diag}(\lambda_1, \ldots, \lambda_{n-1}, \mu)(\tilde{x}, \tilde{y})$. Thanks to the implicit function Theorem there exist $n-1$ holomorphic functions $\chi_1(\tilde{y}), \dots, \chi_{n-1}(\tilde{y})$, such that $\tilde{M} = \{\tilde{x}_1 = \chi_1(\tilde{y}), \dots, \tilde{x}_{n-1} = \chi_{n-1}(\tilde{y})\}$. The $\tilde{f}$-invariance of $\widetilde{M}$ yields

$$\lambda_i \chi_i(\tilde{y}) = \chi(\mu\tilde{y}) \quad \text{for } i = 1, \dots, n-1,$$

and this is equivalent, writing $\chi_i(\tilde{y}) = \sum_{m\geq 1} \chi^i_m \tilde{y}^m$, to

$$\sum_{m\geq 1} \lambda_i \chi^i_m \tilde{y}^m = \sum_{m\geq 1} \chi^i_m \mu^m \tilde{y}^m,$$

which implies $\chi^i_m \equiv 0$ for every $i = 1, \dots, n-1$ and $m \geq 0$, because $(\lambda_1, \dots, \lambda_{n-1}, \mu)$ is not resonant. Then $\widetilde{M} = \{\tilde{x} = 0\}$ and, since $\tilde{f}|_{\widetilde{M}}$ is linear, we have a holomorphic linearization of $f|_M$, contradiction.

2.7 Final remarks

We can obtain many of the result recalled in Chapter 1 as corollaries of our Theorems. If there are no resonances Theorem 2.4.1 with $s = n$ yields Theorem 1.3.24. If there are no resonances and the origin is an attracting [resp., repelling] fixed point then Theorem 2.5.1 with $s = n$ yields Theorem 1.4.1 because the Brjuno condition is automatically satisfied.

Our result can be also compared with Theorem 1.6.4 obtained by Nishimura in [Ni]. The hypotheses of Nishimura are slightly different from ours, and, in fact, he does not prove a true linearization theorem. However, his result becomes a linearization result when $C(y)$ is a constant matrix, which is equivalent to requiring that Y is an osculating fixed manifold. In this situation our result can be seen as a generalization of Theorem 1.6.4 in the case of $\mathrm{d}f_O$ diagonalizable. In fact while he needs an osculating fixed manifold and a strong hypothesis on the modulus of the eigenvalues, we only need an osculating manifold on which our germ is holomorphically linearizable and the origin as a quasi-Brjuno fixed point of order s.

Also Theorem 1.4.15 obtained by Rong in [Ro1] can be seen as a particular case of Theorem 2.5.1. In fact, if we are in the hypotheses of Rong, our hypotheses are automatically verified: M is an osculating fixed manifold thanks to the hypothesis $\Lambda_s(y) \equiv \Lambda_s$ for all $p \in M$, and the hypotheses on the eigenvalues follow immediately from the fact that a n-tuple $\lambda = (\lambda_1, \dots, \lambda_s, 1, \dots, 1) \in (\mathbb{C}^*)^n$ satisfies the partial Brjuno condition of order s if and only if $(\lambda_1, \dots, \lambda_s)$ satisfies the Brjuno condition, and from Remark 2.2.3.

A similar topic is studied in [Sto]. However, his results are not comparable with ours, because his notion of "linearization modulo an ideal" is not suitable for producing a full linearization result except when there are no resonances at all, whereas in our result we explicitly admit some resonances.

What it is new in our result is that we are not assuming anything on the modulus of the eigenvalues, so we are really dealing with the mixed case. In fact we are allowing cases in which there are some eigenvalues with modulus greater than 1, some eigenvalues with modulus 1, and the remaining eigenvalues with modulus less than 1. Finally, our Theorem applies in cases not covered by the previous results, as shown by Remark 2.2.3.

An application to global holomorphic dynamics of a particular case of Theorem 2.5.1 is given by Bedford and Kim in [BK].

3

Commuting with a linearizable object

In this chapter we shall show how commuting with a linearizable germ gives us information on the germs that can be conjugated to a given one. We shall then deal with the simultaneous linearization problem, proving that, given $f_1, \ldots, f_m$, (with $m \geq 2$) germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin, with $(\mathrm{d}f_1)_O$ diagonalizable and such that f_1 commutes with f_h for any $h = 2, \ldots, m$, under certain arithmetic conditions on the eigenvalues of $(\mathrm{d}f_1)_O$ and some restrictions on their resonances, $f_1, \ldots, f_m$ are simultaneously holomorphically linearizable if and only if there exists a particular complex manifold invariant under $f_1, \ldots, f_m$. Then we shall see that if a germ of biholomorphism of $\mathbb{C}^n$, fixing the origin, commutes with a torus action, then we get the existence of a holomorphic linearization or of a holomorphic normalization of f.

The main results of Section 3.2 and 3.3 are published in [R3], whereas the main results of Section 3.4 and 3.5 are published in [R4].

3.1 Commuting with a linearizable germ

A general heuristic principle says that if a map f commutes with a map g, then some properties of g might be inherited by f. Here we shall explore this heuristic principle in our setting. Our first result is:

Theorem 3.1.1. *Let f and g be two commuting germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin, such that g is holomorphically linearizable and $\mathrm{d}g_O$ is diagonalizable. Let $\mu = (\mu_1, \ldots, \mu_n)$ be the spectrum of $\mathrm{d}g_O$. Then f is holomorphically conjugated to a germ containing only μ-resonant monomials.*

Proof. From the hypotheses there exists a germ of biholomorphism ψ of $\mathbb{C}^n$ fixing the origin and such that $\psi^{-1} \circ g \circ \psi = B$ is linear. Since $\mathrm{d}g_O = B$ and it is diagonalizable, there exists a linear map R such that $R^{-1}BR = \mathrm{Diag}(\mu_1, \ldots, \mu_n)$. Then, since $\tilde{f} := R^{-1}\psi^{-1} \circ f \circ \psi R$ commutes with $M := \mathrm{Diag}(\mu_1, \ldots, \mu_n)$ we get the thesis. In fact, writing

$$\tilde{f}_j(z) = \sum_{p=1}^{n} a_{jp} z_p + \sum_{|Q| \geq 2} \tilde{f}_{Q,j} z^Q$$

for each coordinate $j \in \{1, \ldots, n\}$, we have

$$\begin{aligned}\tilde{f}_j(\mu_1 z_1, \ldots, \mu_n z_n) &= \sum_{p=1}^{n} a_{jp}\mu_p z_p + \sum_{|Q|\geq 2} \tilde{f}_{Q,j}\mu^Q z^Q \\ &= \mu_j \left(\sum_{p=1}^{n} a_{jp} z_p + \sum_{|Q|\geq 2} \tilde{f}_{Q,j} z^Q \right) \\ &= \mu_j \tilde{f}_j(z_1, \ldots, z_n)\end{aligned}$$

if and only if

$$a_{jp}(\mu_p - \mu_j) = 0 \quad \text{and} \quad \tilde{f}_{Q,j}(\mu^Q - \mu_j) = 0,$$

which imply the thesis. □

We shall use the previous result to catch as more information as we can on the linearization and on the normalization problems.

One possible generalization of the linearization problem is to ask when a given set of $m \geq 2$ germs of biholomorphisms $f_1, \ldots, f_m$ of $\mathbb{C}^n$ at the same fixed point, which we may place at the origin, are *simultaneously holomorphically linearizable*, i.e., there exists a local holomorphic change of coordinates conjugating f_h to its linear part for each $h = 1, \ldots, m$.

In Chapter 2 we found, under certain arithmetic conditions on the eigenvalues and some restrictions on the resonances, a necessary and sufficient condition for holomorphic linearization. In the next two sections we shall use that result to find a necessary and sufficient condition for holomorphic simultaneous linearization.

A similar topic is studied in [Sto]. However, his results are not comparable with ours, because his notion of "linearization modulo an ideal" is not suitable for producing a full linearization result, except when there are no resonances at all, whereas in our result we explicitly admit some resonances.

We shall need some notations and definition that we introduded in the previous chapters; thus we recall them here for the benefit of the reader.

Let $1 \leq s \leq n$. We say that $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r) \in (\mathbb{C}^*)^n$ *has only level s resonances* if there are only two kinds of resonances:

$$(a) \qquad \boldsymbol{\lambda}^Q = \lambda_h \iff Q \in \widetilde{K}_1,$$

where

$$\widetilde{K}_1 = \left\{ Q \in \mathbb{N}^n \;\middle|\; |Q| \geq 2, \sum_{p=1}^{s} q_p = 1 \ \text{ and } \ \mu_1^{q_{s+1}} \cdots \mu_r^{q_n} = 1 \right\};$$

and

$$(b) \qquad \boldsymbol{\lambda}^Q = \mu_j \iff Q \in \widetilde{K}_2,$$

where

$$\widetilde{K}_2 = \left\{ Q \in \mathbb{N}^n \;\middle|\; |Q| \geq 2, q_1 = \cdots = q_s = 0 \text{ and } \exists j \in \{1, \ldots, r\} \text{ s.t. } \mu_1^{q_{s+1}} \cdots \mu_r^{q_n} = \mu_j \right\}.$$

Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. For any $m \geq 2$ put

$$\widetilde{\omega}(m) = \min_{\substack{2 \leq |Q| \leq m \\ Q \notin \mathrm{Res}_j(\lambda)}} \min_{1 \leq j \leq n} |\lambda^Q - \lambda_j|,$$

where $\mathrm{Res}_j(\lambda)$ is the set of multi-indices $Q \in \mathbb{N}^n$, with $|Q| \geq 2$, giving a resonance relation for $\lambda = (\lambda_1, \ldots, \lambda_n)$ relative to $1 \leq j \leq n$, i.e., such that $\lambda^Q - \lambda_j = 0$. We say that λ *satisfies the reduced Brjuno condition* if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu \geq 0}$ with $p_0 = 1$ such that

$$\sum_{\nu \geq 0} p_\nu^{-1} \log \widetilde{\omega}(p_{\nu+1})^{-1} < \infty.$$

Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O and let $s \in \mathbb{N}$, with $1 \leq s \leq n$. The origin O is called a *quasi-Brjuno fixed point of order* s if $\mathrm{d}f_O$ is diagonalizable and, denoting by $\boldsymbol{\lambda}$ the spectrum of $\mathrm{d}f_O$, we have:

(i) $\boldsymbol{\lambda}$ has only level s resonances;

(ii) $\boldsymbol{\lambda}$ satisfies the reduced Brjuno condition.

We say that f has the origin as a *quasi-Brjuno fixed point* if there exists $1 \leq s \leq n$ such that it is a quasi-Brjuno fixed point of order s.

In the previous chapter we saw that the osculating condition was necessary and sufficient to extend a holomorphic linearization from an invariant submanifold to a whole neighbourhood of the origin for a germ f_1 of biholomorphism with a quasi-Brjuno fixed point. Using that result, we shall first prove a simultaneous linearization result. Later on in this chapter we shall restrict ourselves to study commutations with a particular kind of linearizable object: torus actions.

We recall the following notation from the previous chapter: if $g\colon \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function with $g(O) = 0$, and $z = (x, y) \in \mathbb{C}^n$ with $x \in \mathbb{C}^s$ and $y \in \mathbb{C}^{n-s}$, we shall denote by $\mathrm{ord}_x(g)$ the maximum positive integer m such that g belongs to the ideal $\langle x_1, \cdots, x_s \rangle^m$. Furthermore, we shall say that the local coordinates $z = (x, y)$ are *adapted* to the complex submanifold M if in those coordinates M is given by $\{x = 0\}$.

3.2 Simultaneous osculating manifold

We introduced *osculating manifolds* in the previous chapter. A germ f of biholomorphism of $\mathbb{C}^n$ fixing the origin O admits an *osculating manifold M of codimension* $1 \leq s \leq n$ if there is a germ of f-invariant complex manifold M at O of codimension s such that the normal bundle N_M of M admits a holomorphic flat $(1,0)$-connection that commutes with $\mathrm{d}f|_{N_M}$. Next definition is the natural extension of this object to the case of simultaneous linearization.

Definition 3.2.1. Let $f_1, \ldots, f_m$ be m germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin, with $m \geq 2$, and let M be a germ of complex manifold at O of codimension $1 \leq s \leq n$, and f_h-invariant for each $h = 1, \ldots, m$. We say that M is a *simultaneous osculating manifold for $f_1, \ldots, f_m$* if there exists a holomorphic flat $(1,0)$-connection ∇ of the normal bundle N_M of M in $\mathbb{C}^n$ commuting with $\mathrm{d}f_h|_{N_M}$ for each $h = 1, \ldots, m$.

We shall need the following characterization of simultaneous osculating manifolds.

Proposition 3.2.1. *Let $f_1, \ldots, f_m$ be m germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin, with $m \geq 2$, and let M be a germ of complex manifold at O of codimension $1 \leq s \leq n$, and f_h-invariant for each $h = 1, \ldots, m$. Then M is a simultaneous osculating manifold for $f_1, \ldots, f_m$*

if and only if there exist local holomorphic coordinates $z = (x, y)$ about O adapted to M in which f_h has the form

$$\begin{aligned} x_i' &= \sum_{p=1}^{s} a_{i,p}^{(h)} x_p + \widehat{f}_i^{(h)}(x, y) && \text{for } i = 1, \dots, s, \\ y_j' &= f_j^{(h)}(x, y) && \text{for } j = 1, \dots, r = n - s, \end{aligned} \tag{3.2}$$

with

$$\operatorname{ord}_x(\widehat{f}_i^{(h)}) \geq 2,$$

for any $i = 1, \dots, s$ and $h = 1, \dots, m$.

Proof. If there exist local holomorphic coordinates $z = (x, y)$ about O adapted to M in which f_h has the form (3.2) with $\operatorname{ord}_x(\widehat{f}_i^{(h)}) \geq 2$ for any $i = 1, \dots, s$ and $h = 1, \dots, m$, then it is obvious to verify that the trivial holomorphic flat $(1,0)$-connection commutes with $\mathrm{d}f_h|_{N_M}$ for each $h = 1, \dots, m$.

Conversely, let ∇ be a holomorphic flat $(1,0)$-connection of the normal bundle N_M commuting with $\mathrm{d}f_h|_{N_M}$ for each $h = 1, \dots, m$. It suffices to choose local holomorphic coordinates $z = (x, y)$ adapted to M in which all the connection coefficients Γ^i_{jk} with respect to the local holomorphic frame $\{\pi(\frac{\partial}{\partial x_1}), \dots, \pi(\frac{\partial}{\partial x_s})\}$ of N_M are zero (see Proposition 2.3.1 and Lemma 2.3.2), and then the assertion follows immediately from the proof of Theorem 2.3.3. In fact, in such coordinates, since $M = \{x = 0\}$ is f_h-invariant, f_h has the form

$$\begin{aligned} x_i' &= \sum_{p=1}^{s} a_{i,p}^{(h)} x_p + \widehat{f}_i^{(h)}(x, y) \quad \text{for } i = 1, \dots, s, \\ y_j' &= f_j^{(h)}(x, y) \quad \text{for } j = 1, \dots, r = n - s, \end{aligned}$$

with

$$\operatorname{ord}_x(\widehat{f}_i^{(h)}) \geq 1.$$

Thanks to the hypotheses we have

$$\nabla_{\mathrm{d}f_h \frac{\partial}{\partial y_k}} \left(\mathrm{d}f_h|_{N_M} \pi \left(\frac{\partial}{\partial x_j} \right) \right) = \mathrm{d}f_h|_{N_M} \nabla_{\frac{\partial}{\partial y_k}} \pi \left(\frac{\partial}{\partial x_j} \right)$$

for any $j = 1, \dots, s$ and $h = 1, \dots, r$. Now the right-hand side vanishes, because in the chosen coordinates we have $\nabla_{\frac{\partial}{\partial y_k}} \pi \left(\frac{\partial}{\partial x_j} \right) = 0$. So, using Leibniz formula, we obtain

$$\begin{aligned} 0 &= \nabla_{\mathrm{d}f_h \frac{\partial}{\partial y_k}} \left(\mathrm{d}f_h|_{N_M} \pi \left(\frac{\partial}{\partial x_j} \right) \right) \\ &= \nabla_{\mathrm{d}f_h \frac{\partial}{\partial y_k}} \left(\sum_{i=1}^{s} \left(\sum_{p=1}^{s} a_{i,p}^{(h)} \delta_{pj} + \frac{\partial \widehat{f}_i^{(h)}}{\partial x_j}(0, y) \right) \pi \left(\frac{\partial}{\partial x_i} \right) \right) \\ &= \sum_{i=1}^{s} \left(\sum_{p=1}^{s} a_{i,p}^{(h)} \delta_{pj} + \frac{\partial \widehat{f}_i^{(h)}}{\partial x_j}(0, y) \right) \nabla_{\mathrm{d}f_h \frac{\partial}{\partial y_k}} \pi \left(\frac{\partial}{\partial x_i} \right) + \sum_{i=1}^{s} \mathrm{d}f_h \frac{\partial}{\partial y_k} \left(\frac{\partial \widehat{f}_i^{(h)}}{\partial x_j}(0, y) \right) \pi \left(\frac{\partial}{\partial x_i} \right) \\ &= \sum_{i=1}^{s} \mathrm{d}f_h \frac{\partial}{\partial y_k} \left(\frac{\partial \widehat{f}_i^{(h)}}{\partial x_j}(0, y) \right) \pi \left(\frac{\partial}{\partial x_i} \right). \end{aligned} \tag{3.3}$$

Therefore we obtain

$$\mathrm{d}f_h \frac{\partial}{\partial y_k}\left(\frac{\partial \widehat{f}_i^{(h)}}{\partial x_j}(0,y)\right) = 0$$

for every $j, i = 1, \ldots, s$ and $k = 1, \ldots r$, and, since $\mathrm{d}f_h$ is invertible, this implies

$$\frac{\partial}{\partial y_k}\left(\frac{\partial \widehat{f}_i^{(h)}}{\partial x_j}(0,y)\right) = 0$$

for every $j, i = 1, \ldots, s$ and $k = 1, \ldots r$, that is

$$\mathrm{ord}_x(\widehat{f}_i^{(h)}) \geq 2$$

for every $i = 1, \ldots, s$, and this concludes the proof. $\square$

Similarly to Corollary 2.3.4, we have the following result.

Corollary 3.2.2. *Let $f_1, \ldots, f_m$ be m germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin, with $m \geq 2$, and let M be a germ of complex manifold at O of codimension $1 \leq s \leq n$, and f_h-invariant for each $h = 1, \ldots, m$. Then M is a simultaneous osculating manifold for $f_1, \ldots, f_m$ such that $f_1|_M, \ldots, f_m|_M$ are simultaneously holomorphically linearizable if and only if there exist local holomorphic coordinates $z = (x, y)$ about O adapted to M in which f_h has the form*

$$\begin{aligned} x_i' &= \sum_{p=1}^{s} a_{i,p}^{(h)} x_h + \underset{i}{\widehat{f}^{(h)},1}\ (x,y) \quad \text{for } i = 1, \ldots, s, \\ y_j' &= f_j^{(h)\mathrm{lin}}(x,y) + \widehat{f}_j^{(h),2}(x,y) \quad \text{for } j = 1, \ldots, r = n - s, \end{aligned} \tag{3.4}$$

where $f_j^{(h)\mathrm{lin}}(x,y)$ is linear and

$$\begin{aligned} \mathrm{ord}_x(\widehat{f}_i^{(h),1}) &\geq 2, \\ \mathrm{ord}_x(\widehat{f}_j^{(h),2}) &\geq 1, \end{aligned} \tag{3.5}$$

for any $i = 1, \ldots, s$, $j = 1, \ldots, r$ and $h = 1, \ldots, m$.

Proof. One direction is clear.

Conversely, thanks to Proposition 3.2.1, the fact that M is a simultaneous osculating manifold for $f_1, \ldots, f_m$ is equivalent to the existence of local holomorphic coordinates $z = (x, y)$ about O adapted to M, in which f_h has the form (3.4) with $\mathrm{ord}_x(\widehat{f}_i^{(h),1}) \geq 2$ for any $i = 1, \ldots, s$ and $h = 1, \ldots, m$. Furthermore, $f_1|_M, \ldots, f_m|_M$ are simultaneously holomorphically linearizable; therefore there exists a local holomorphic change of coordinate, tangent to the identity, and of the form

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= \Phi(y), \end{aligned}$$

conjugating f_h to $\tilde{f}_h$ of the form (3.4) satisfying (3.5), for each $h = 1, \ldots, m$, as we wanted. $\square$

Remark 3.2.3. It is possible to give the formal analogous of Definition 3.2.1, and then to prove a formal analogous of Proposition 3.2.1 and Corollary 3.2.2, exactly as in the previous chapter.

3.3 Simultaneous linearization in presence of resonances

As announced we shall use Theorem 2.5.1 we proved in the previous chapter. We report here the statement for the sake of completeness.

Theorem 3.3.1. (Raissy, 2009 [R2]) *Let f be a germ of a biholomorphism of $\mathbb{C}^n$ having the origin O as a quasi-Brjuno fixed point of order s. Then f is holomorphically linearizable if and only if it admits an osculating manifold M of codimension s such that $f|_M$ is holomorphically linearizable.*

We can now state and prove our simultaneous linearization result.

Theorem 3.3.2. *Let $f_1, \ldots, f_m$ be $m \geq 2$ germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin. Assume that f_1 has the origin as a quasi-Brjuno fixed point of order s, with $1 \leq s \leq n$, and that it commutes with f_h for any $h = 2, \ldots, m$. Then $f_1, \ldots, f_m$ are simultaneously holomorphically linearizable if and only if there exists a germ of complex manifold M at O of codimension s, invariant under f_h for each $h = 1, \ldots, m$, which is a simultaneous osculating manifold for $f_1, \ldots, f_m$ and such that $f_1|_M, \ldots, f_m|_M$ are simultaneously holomorphically linearizable.*

Proof. Let M be a germ of complex manifold at O of codimension s, invariant under f_h for each $h = 1, \ldots, m$ which is a simultaneous osculating manifold for $f_1, \ldots, f_m$ and such that $f_1|_M, \ldots, f_m|_M$ are simultaneously holomorphically linearizable. Thanks to the hypotheses we can choose local holomorphic coordinates

$$(x, y) = (x_1, \ldots, x_s, y_1, \ldots, y_r)$$

such that f_1 is of the form

$$\begin{aligned} x'_i &= \lambda_{1,i} x_i + f_i^{(1),1}(x, y) \quad \text{for } i = 1, \ldots, s, \\ y'_j &= \mu_{1,j} y_j + f_j^{(1),2}(x, y) \quad \text{for } j = 1, \ldots, r = n - s, \end{aligned}$$

and, for $h = 2, \ldots, m$, each f_h is of the form

$$\begin{aligned} x'_i &= \sum_{p=1}^{s} a_{i,p}^{(h)} x_p + f_i^{(h),1}(x, y) \qquad \text{for } i = 1, \ldots, s, \\ y'_j &= f_j^{(h)\mathrm{lin}}(x, y) + f_j^{(h),2}(x, y) \quad \text{for } j = 1, \ldots, r = n - s, \end{aligned}$$

where $f_j^{(h)\mathrm{lin}}(x, y)$ is linear, and for each $k = 1, \ldots, m$

$$\begin{aligned} \mathrm{ord}_x(f_i^{(k),1}) &\geq 2, \\ \mathrm{ord}_x(f_j^{(k),2}) &\geq 1, \end{aligned}$$

that is

$$\begin{aligned} f_i^{(k),1}(x, y) &= \sum_{\substack{|Q| \geq 2 \\ |Q'| \geq 2}} f_{Q,i}^{(k),1} x^{Q'} y^{Q''} \quad \text{for } i = 1, \ldots, s, \\ f_j^{(k),2}(x, y) &= \sum_{\substack{|Q| \geq 2 \\ |Q'| \geq 1}} f_{Q,j}^{(k),2} x^{Q'} y^{Q''} \quad \text{for } j = 1, \ldots, r, \end{aligned}$$

where $Q = (Q', Q'') \in \mathbb{N}^s \times \mathbb{N}^r = \mathbb{N}^n$ and $|Q| = \sum_{p=1}^n q_p$.

Thanks to Theorem 3.3.1 and its proof, we know that f_1 is holomorphically linearizable via a linearization ψ of the form

$$\begin{aligned} x_i &= u_i + \psi_i^1(u,v) \quad \text{for } i = 1, \dots, s, \\ y_j &= v_j + \psi_j^2(u,v) \quad \text{for } j = 1, \dots, r, \end{aligned}$$

where $(u,v) = (u_1, \dots, u_s, v_1, \dots, v_r)$ and

$$\begin{aligned} \mathrm{ord}_u(\psi_i^1) &\geq 2, \\ \mathrm{ord}_u(\psi_j^2) &\geq 1, \end{aligned}$$

that is

$$\begin{aligned} \psi_i^1(u,v) &= \sum_{\substack{|Q|\geq 2 \\ |Q'|\geq 2}} \psi_{Q,i}^1 u^{Q'} v^{Q''} \quad \text{for } i = 1, \dots, s, \\ \psi_j^2(u,v) &= \sum_{\substack{|Q|\geq 2 \\ |Q'|\geq 1}} \psi_{Q,j}^2 u^{Q'} v^{Q''} \quad \text{for } j = 1, \dots, r. \end{aligned}$$

Since $\psi^{-1} \circ f_1 \circ \psi = \mathrm{Diag}(\lambda_{1,1}, \dots, \lambda_{1,s}, \mu_{1,1}, \dots, \mu_{1,r})$ commutes with $\tilde{f}_h = \psi^{-1} \circ f_h \circ \psi$ for each $h = 2, \dots, m$, and $(\lambda_{1,1}, \dots, \lambda_{1,s}, \mu_{1,1}, \dots, \mu_{1,r})$ has only level s resonances, it is immediate to verify that $\tilde{f}_h$ has the form

$$\begin{aligned} u_i' &= \sum_{p=1}^s a_{i,p}^{(h)} u_p + \sum_{\substack{1\leq l\leq n \\ \lambda_{1,l}=\lambda_{1,i}}} u_l \tilde{f}_{l,i}^{(h),1}(v) \quad \text{for } i = 1, \dots, s, \\ v_j' &= f_j^{(h)\mathrm{lin}}(u,v) + \tilde{f}_j^{(h),2}(v) \quad \text{for } j = 1, \dots, r. \end{aligned}$$

Moreover, since $f_h \circ \psi = \psi \circ \tilde{f}_h$, we have

$$\begin{aligned} &\sum_{p=1}^s a_{i,p}^{(h)} \sum_{\substack{|Q|\geq 2 \\ |Q'|\geq 2}} \psi_{Q,p}^1 u^{Q'} v^{Q''} + \sum_{\substack{|Q|\geq 2 \\ |Q'|\geq 2}} f_{Q,i}^{(h),1} (u + \psi^1(u,v))^{Q'} (v + \psi^2(u,v))^{Q''} \\ &\quad = \sum_{\substack{1\leq l\leq n \\ \lambda_{1,l}=\lambda_{1,i}}} u_l \tilde{f}_{l,i}^{(h),1}(v) \\ &\quad + \sum_{\substack{|Q|\geq 2 \\ |Q'|\geq 2}} \psi_{Q,i}^1 \left(\sum_{p=1}^s a_{1,p}^{(h)} u_p + \sum_{\substack{1\leq l\leq n \\ \lambda_{1,l}=\lambda_{1,1}}} u_l \tilde{f}_{l,1}^{(h),1}(v) \right)^{q_1} \cdots \left(\sum_{p=1}^s a_{s,p}^{(h)} u_p + \sum_{\substack{1\leq l\leq n \\ \lambda_{1,l}=\lambda_{1,s}}} u_l \tilde{f}_{l,s}^{(h),1}(v) \right)^{q_s} \\ &\qquad \times (f^{(h)\mathrm{lin}}(u,v) + \tilde{f}^{(h),2}(v))^{Q''} \end{aligned} \tag{3.6}$$

for $i = 1, \dots, s$, and

$$\begin{aligned}
&\sum_{q=1}^{r} b_{j,q}^{(h)} \sum_{\substack{|Q|\geq 2\\ |Q'|\geq 1}} \psi^2_{Q,q} u^{Q'} v^{Q''} + \sum_{p=1}^{s} c_{j,p}^{(h)} \sum_{\substack{|Q|\geq 2\\ |Q'|\geq 2}} \psi^1_{Q,p} u^{Q'} v^{Q''} \\
&\qquad\qquad + \sum_{\substack{|Q|\geq 2\\ |Q'|\geq 1}} f^{(h),2}_{Q,j} (u + \psi^1(u,v))^{Q'} (v + \psi^2(u,v))^{Q''} \\
&\quad = \tilde{f}^{(h),2}_j(v) \\
&\qquad + \sum_{\substack{|Q|\geq 2\\ |Q'|\geq 1}} \psi^2_{Q,j} \left(\sum_{p=1}^{s} a^{(h)}_{1,p} u_p + \sum_{\substack{1\leq l\leq n\\ \lambda_{1,l}=\lambda_{1,1}}} u_l \tilde{f}^{(h),1}_{l,1}(v) \right)^{q_1} \cdots \left(\sum_{p=1}^{s} a^{(h)}_{s,p} u_p + \sum_{\substack{1\leq l\leq n\\ \lambda_{1,l}=\lambda_{1,s}}} u_l \tilde{f}^{(h),1}_{l,s}(v) \right)^{q_s} \\
&\qquad\qquad\qquad \times (f^{(h)\mathrm{lin}}_j(u,v) + \tilde{f}^{(h),2}(v))^{Q''}
\end{aligned} \tag{3.7}$$

for $j = 1, \dots, r$.

Now, it is not difficult to verify that there are no terms of the form $u^{Q'} v^{Q''}$ with $|Q'| = 1$ in the left-hand side of (3.6), whereas in the right-hand side terms of this form are given only by the sum of the $u_l \tilde{f}^{(h),1}_{l,i}(v)$; therefore it must be

$$\tilde{f}^{(h),1}_{l,i}(v) \equiv 0,$$

for all pairs l, i. Similarly, there are no terms of the form $u^{Q'} v^{Q''}$ with $Q' = O$ in the left-hand side of (3.7), whereas, again, in the right-hand side terms of this form are given by $\tilde{f}^{(h),2}_j(v)$ only; so

$$\tilde{f}^{(h),2}_j(v) \equiv 0 \quad \text{for } j = 1, \dots, r.$$

This proves that $\tilde{f}_h$ is linear for every $h = 2, \dots, m$, that is ψ is a simultaneous holomorphic linearization for $f_1, \dots, f_m$.

The other direction is clear. In fact, if f_1 commutes with $f_2, \dots, f_m$ and $f_1, \dots, f_m$ are linear, then the eigenspace of f_1 relative to the eigenvalues $\mu_{1,1}, \dots, \mu_{1,r}$ is a simultaneous osculating manifold for $f_1, \dots, f_m$ (and $f_1|_M, \dots, f_m|_M$ are linear), where $(\lambda_{1,1}, \dots, \lambda_{1,s}, \mu_{1,1}, \dots, \mu_{1,r})$ is the spectrum of f_1. □

Corollary 3.3.3. *Let $f_1, \dots, f_m$ be $m \geq 2$ germs of commuting biholomorphisms of $\mathbb{C}^n$, fixing the origin. Assume that f_1 has the origin as a quasi-Brjuno fixed point of order s, with $1 \leq s \leq n$. Then $f_1, \dots, f_m$ are simultaneously holomorphically linearizable if and only if there exists a germ of complex manifold M at O of codimension s, invariant under f_h for each $h = 1, \dots, m$ which is a simultaneous osculating manifold for $f_1, \dots, f_m$ and such that $f_1|_M, \dots, f_m|_M$ are simultaneously holomorphically linearizable.*

As a final corollary, taking $s = n$ in Theorem 3.3.2, one gets

Corollary 3.3.4. *Let $f_1, \dots, f_m$ be $m \geq 2$ germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin. Assume that f_1 has the origin as a Brjuno fixed point, and that it commutes with f_h for any $h = 2, \dots, m$. Then $f_1, \dots, f_m$ are simultaneously holomorphically linearizable.*

3.4 Torus actions

As announced in Section 3.1, we shall now focus our attention on a particular object: torus actions. Holomorphic torus actions are holomorphically linearizable thanks to Bochner linearization theorem (which is usually proved in the real-analytic setting but also holds, with the same proof, in the holomorphic setting; see [R1]).

Theorem 3.4.1. (Bochner, 1945 [Bo]) *Let A be a local holomorphic action of a compact Lie group on (M,p), where M is a holomorphic manifold and $p \in M$. Then the action is holomorphically linearizable via a local holomorphic change of coordinates tangent to the identity.*

Let $A\colon \mathbb{T}^r \times M \to M$ be a torus action on a complex manifold M, with a fixed point $p_0 \in M$ (that is $A(x,p_0) = A_x(p_0) = p_0$ for all $x \in \mathbb{T}^r$). The differential $\mathrm{d}(A_x)_{p_0}\colon T_{p_0}M \to T_{p_0}M$ is then well-defined, and thus we have a linear torus action on $T_{p_0}M$. A linear torus action can be thought of as a Lie group homomorphism $A\colon \mathbb{T}^r \to \mathrm{Aut}(T_{p_0}M)$, that is as a representation of $\mathbb{T}^r$ on the vector space $V = T_{p_0}M$.

Characters and weights of $\mathbb{T}^r$ are well known. All characters of $\mathbb{T}^1 = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ are of the form

$$\chi_\theta(x) = \exp(2\pi i x\theta)$$

with $\theta \in \mathbb{Z}$; hence the character group of $\mathbb{T}^1$ is isomorphic to $\mathbb{Z}$. Since $\mathbb{T}^r = \mathbb{T}^1 \times \cdots \times \mathbb{T}^1$, the characters of $\mathbb{T}^r$ are obtained multiplying characters of $\mathbb{T}^1$, that is they are of the form

$$\chi_\theta(x) = \exp\left(2\pi i \sum_{k=1}^{r} x_k\theta^k\right),$$

with $\theta = (\theta^1, \ldots, \theta^r) \in (\mathbb{Z}^r)^*$, where the * denotes the dual. In particular, θ should be thought of as a row vector. The weights of $\mathbb{T}^r$ are then the differential of the characters computed at the identity element, and thus are given by

$$w_\theta(v) = 2\pi i \sum_{k=1}^{r} v_k\theta^k$$

with $\theta \in (\mathbb{Z}^r)^*$ and $v \in \mathbb{R}^r$. If we write $\theta_j = (\theta_j^1, \ldots, \theta_j^r) \in (\mathbb{Z}^r)^*$, then the matrix representation of the linear action A in the eigenvector basis is given by

$$A(x) = \mathrm{diag}\left(\chi_{\theta_j}(x)\right) = \mathrm{diag}\left(\exp\left(2\pi i \sum_{k=1}^{r} x_k\theta_j^k\right)\right).$$

We have then associated to our torus action a matrix $\Theta = (\theta_j^k) \in M_{n\times r}(\mathbb{Z})$, whose columns do not depend on the particular coordinates chosen to express the torus action, but can be uniquely (up to order) recovered by the action itself.

Definition 3.4.1. The matrix Θ just defined is called the *weight matrix* of the torus action.

Definition 3.4.2. Let $\theta \in \mathbb{C}^n$ and let $j \in \{1, \ldots, n\}$. We say that a multi-index $Q \in \mathbb{N}^n$, with $|Q| = \sum_{h=1}^{n} q_h \geq 2$, gives an *additive resonance relation for θ relative to the j-th coordinate* if

$$\langle Q, \theta\rangle = \sum_{h=1}^{n} q_h\theta_h = \theta_j$$

and we put

$$\mathrm{Res}_j^+(\theta) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \langle Q, \theta\rangle = \theta_j\}.$$

Recall that, given $\lambda \in (\mathbb{C}^*)^n$ and let $j \in \{1, \dots, n\}$, we say that a multi-index $Q \in \mathbb{N}^n$, with $|Q| \geq 2$, gives a *multiplicative resonance relation for λ relative to the j-th coordinate* if

$$\lambda^Q = \lambda_1^{q_1} \cdots \lambda_n^{q_n} = \lambda_j$$

and we put

$$\mathrm{Res}_j(\lambda) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \lambda^Q = \lambda_j\}.$$

Remark 3.4.2. Given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$, where $[\,\cdot\,]\colon \mathbb{C}^n \to (\mathbb{C}/\mathbb{Z})^n$ is the standard projection, the set

$$\{Q \in \mathbb{N}^n \mid |Q| \geq 2, \langle Q, \varphi\rangle - \varphi_j \in \mathbb{Z}\}.$$

does not depend on the specific representative $\varphi \in \mathbb{C}^n$ but only on the class $[\varphi]$, and so it is well defined the set $\mathrm{Res}_j([\varphi])$ as

$$\mathrm{Res}_j([\varphi]) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, [\langle Q, \varphi\rangle - \varphi_j] = [0]\}.$$

Remark 3.4.3. Notice that given $\lambda \in (\mathbb{C}^*)^n$, we can always find a unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$, i.e., $\lambda_j = e^{2\pi i[\varphi_j]}$ for every $j = 1, \dots, n$. Then $\mathrm{Res}_j(\lambda) = \mathrm{Res}_j([\varphi])$, thus justifying the definitions and the terminology.

3.5 Commuting with torus actions

In this section we shall describe the relations between the existence of torus actions with certain properties and the possibility of conjugating a given germ of biholomorphism to another of a particular form.

Definition 3.5.1. Let $\theta^{(1)}, \dots, \theta^{(r)} \in \mathbb{Z}^n$. We say that a monomial $z^Q e_j$, with $Q \in \mathbb{N}^n$, $|Q| \geq 1$ and $j \in \{1, \dots, n\}$, is Θ*-resonant*, where Θ is the $n \times r$ matrix whose columns are $\theta^{(1)}, \dots, \theta^{(r)}$, if

$$\langle Q, \theta^{(k)}\rangle = \theta_j^{(k)}$$

for every $k = 1, \dots, r$. In other words, $z_h e_j$ is Θ-resonant if $\theta_h^{(k)} = \theta_j^{(k)}$, for all $k = 1, \dots, r$, and $z^Q e_j$, with $|Q| \geq 2$ is Θ-resonant, if

$$Q \in \mathcal{R}_j(\Theta) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\theta^{(k)}). \tag{3.8}$$

We say that Θ has *no resonances* if $\mathcal{R}_j(\Theta) = \varnothing$ for every $j = 1, \dots, n$.

Definition 3.5.2. Let $\theta^{(1)}, \dots, \theta^{(r)} \in \mathbb{Z}^n$ and let T be a linear map of $\mathbb{C}^n$. We say that the matrix Θ, with columns $\theta^{(1)}, \dots, \theta^{(r)}$, is *compatible with* T if and only if we can write T in Jordan form with all monomials Θ-resonant. In other words, a matrix $T = (t_{ij})$ in Jordan form is compatible with Θ if and only if $\theta_j^{(k)} = \theta_{j+1}^{(k)}$ for all $k = 1, \dots, r$ when $t_{j,j+1} \neq 0$, that is in a Jordan block of dimension at least 2.

Theorem 3.5.1. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Then f commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $1 \leq r \leq n$*

with weight matrix $\Theta \in M_{n\times r}(\mathbb{Z})$ *if and only there exists a local holomorphic change of coordinates conjugating* f *to a germ with linear part in Jordan normal form and containing only* Θ*-resonant monomials.*

Proof. Let us suppose that the linear part of f is in Jordan normal form and f contains only Θ-resonant monomials. Then we claim that f commutes with the linear effective torus action

$$A\colon \mathbb{T}^r \times (\mathbb{C}^n, O) \to (\mathbb{C}^n, O),$$

defined by

$$A(x,z) = \mathrm{Diag}\left(e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}}\right) z.$$

In fact in these hypotheses the j-th coordinate of f is

$$\lambda_j z_j + \varepsilon_j z_{j-1} + \sum_{\substack{|Q|\geq 2\\ Q\in\mathcal{R}_j(\Theta)}} f_{Q,j} z^Q$$

where $\varepsilon_j \in \{0,1\}$ can be different from 0 only if $\lambda_j = \lambda_{j-1}$, the set $\mathcal{R}_j(\Theta)$ is defined in (3.8) and the assumption that $\varepsilon_j z_{j-1} e_j$ is Θ-resonant implies $\theta_{j-1}^{(k)} = \theta_j^{(k)}$ for $k = 1,\dots,r$ if $\varepsilon_j \neq 0$. Then for every $x \in \mathbb{T}^r$ we have

$$\begin{aligned}
f_j(A(x,z)) &= \lambda_j e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} z_j + \varepsilon_j e^{2\pi i \sum_{k=1}^r x_k \theta_{j-1}^{(k)}} z_{j-1} + \sum_{\substack{|Q|\geq 2\\ Q\in\mathcal{R}_j(\Theta)}} f_{Q,j} e^{2\pi i \sum_{k=1}^r x_k \langle Q,\theta^{(k)}\rangle} z^Q \\
&= \lambda_j e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} z_j + \varepsilon_j e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} z_{j-1} + \sum_{\substack{|Q|\geq 2\\ Q\in\mathcal{R}_j(\Theta)}} f_{Q,j} e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} z^Q \\
&= e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} \left(\lambda_j z_j + \varepsilon_j z_{j-1} + \sum_{\substack{|Q|\geq 2\\ Q\in\mathcal{R}_j(\Theta)}} f_{Q,j} z^Q \right) \\
&= e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} (f_j(z)) \\
&= A(x, f(z))_j.
\end{aligned}$$

Conversely, let us suppose that f commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $1 \leq r \leq n$ with weight matrix Θ. Then, by Bochner linearization theorem 3.4.1, there exists a tangent to the identity holomorphic change of variables ψ linearizing the torus action. Furthermore, up to a linear change of coordinates we can assume that in the new coordinates the action is given by

$$A(x,z) = \mathrm{Diag}\left(e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}}\right) z,$$

and that f (still commuting with the torus action) has linear part in Jordan normal form compatible with Θ, and thus its j-th coordinate is

$$\lambda_j z_j + \varepsilon_j z_{j-1} + \sum_{|Q|\geq 2} f_{Q,j} z^Q$$

where $\varepsilon_j \in \{0,1\}$ can be different from 0 only if $\lambda_{j-1} = \lambda_j$ and $\theta_{j-1} = \theta_j$. For every $x \in \mathbb{T}^r$, we have

$$f_j(A(x,z)) = \lambda_j e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} z_j + \varepsilon_j e^{2\pi i \sum_{k=1}^r x_k \theta_{j-1}^{(k)}} z_{j-1} + \sum_{|Q|\geq 2} f_{Q,j} e^{2\pi i \sum_{k=1}^r x_k \langle Q, \theta^{(k)}\rangle} z^Q,$$

and

$$A(x, f(z))_j = e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} \left(\lambda_j z_j + \varepsilon_j z_{j-1} + \sum_{|Q|\geq 2} f_{Q,j} z^Q \right).$$

Then $f_j(A(x,z)) = A(x, f(z))_j$ if and only if

$$f_{Q,j}\left(e^{2\pi i \sum_{k=1}^r x_k \langle Q, \theta^{(k)}\rangle} - e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} \right) = 0$$

for every $x \in \mathbb{T}^r$, $j = 1, \cdots, n$, $Q \in \mathbb{N}^n$ with $|Q| \geq 2$, i.e., $f_{Q,j}$ can be non-zero only when

$$\sum_{k=1}^r x_k (\langle Q, \theta^{(k)}\rangle - \theta_j^{(k)}) \in \mathbb{Z} \quad \forall x \in \mathbb{T}^r,$$

which is equivalent to

$$\langle Q, \theta^{(k)}\rangle - \theta_j^{(k)} = 0$$

for every $k = 1, \ldots, r$, meaning that f contains only Θ-resonant monomials. □

As a consequence of this result we have

Corollary 3.5.2. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Then f is holomorphically linearizable if and only if it commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $1 \leq r \leq n$ with weight matrix Θ having no resonances.*

Proof. If f is linear and in Jordan normal form, then it commutes with any linear action of $\mathbb{T}^1$ with compatible weight matrix Θ; so it suffices to choose Θ with $\mathcal{R}_1(\Theta) = \ldots = \mathcal{R}_n(\Theta) = \varnothing$.

Conversely, if f commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $1 \leq r \leq n$ with weight matrix Θ, then, by the previous result, Θ is compatible with the linear part of f and there exists a local holomorphic change of coordinates such that f is conjugated to a germ with the same linear part and containing only Θ-resonant monomials. But each $\mathcal{R}_j(\Theta)$ is empty; hence there are no Θ-resonant monomials of degree at least 2, and thus f is holomorphically linearizable. □

The last corollary shows that it is possible to characterize the holomorphic linearization problem using Theorem 3.5.1. It is then natural to also try and use it to prove holomorphic normalization results. To do that, we need to find a link between a weight matrix Θ, with its additive resonances, and the spectrum of $\mathrm{d}f_O$, with its multiplicative resonances. We shall see in the next chapter how to do that and the (surprising) answers we found.

4

Torus Actions vs Normalization

In this final chapter we shall find out in a complete and computable manner what kind of structure a torus action must have in order to give a Poincaré-Dulac holomorphic normalization for a germ f of biholomorphism of $\mathbb{C}^n$ fixing the origin. In particular, we shall link the eigenvalues of $\mathrm{d}f_O$ to the weight matrix of the action. The link and the structure we found are more complicated than what one would expect; a detailed study was needed to completely understand the relations between torus actions, holomorphic Poincaré-Dulac normalizations, and torsion phenomena. We end the chapter giving an example of techniques that can be used to construct torus actions.

The main results in this chapter are published in [R4].

4.1 Preliminaries

In this chapter we want to discuss and solve the following problem: to find out in a clear (and possibly computable) manner what kind of structure a torus action must have in order to get a Poincaré-Dulac holomorphic normalization from the results we proved in the previous chapter. In particular, to do so we need to link in a clever way the eigenvalues of $\mathrm{d}f_O$ to the weight matrix of the action.

Before we go on, let us recall here, for the sake of completeness, notations, definitions and results that we shall use.

We have associated to any torus action $A\colon \mathbb{T}^r \times (\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$ fixing the origin a matrix $\Theta = (\theta_j^k) \in M_{n\times r}(\mathbb{Z})$, called the *weight matrix* of the torus action, whose columns do not depend on the particular coordinates chosen to express the torus action, but can be uniquely (up to order) recovered by the action itself.

We said that a monomial $z^Q e_j$, with $Q \in \mathbb{N}^n$, $|Q| \geq 1$ and $j \in \{1, \ldots, n\}$, is Θ-*resonant* if

$$\langle Q, \theta^{(k)} \rangle = \theta_j^{(k)}$$

for every $k = 1, \ldots, r$, where $\theta^{(1)}, \ldots, \theta^{(r)} \in \mathbb{Z}^n$ are the columns of Θ. In other words, $z_h e_j$ is Θ-resonant if $\theta_h^{(k)} = \theta_j^{(k)}$, for all $k = 1, \ldots, r$, and $z^Q e_j$, with $|Q| \geq 2$ is Θ-resonant if

$$Q \in \mathcal{R}_j(\Theta) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\theta^{(k)}), \tag{4.2}$$

where for each $k = 1, \ldots, r$

$$\mathrm{Res}_j^+(\theta^{(k)}) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \langle Q, \theta^{(k)} \rangle = \theta_j^{(k)}\}$$

is the set of the multi-indices giving an additive resonance for $\theta^{(k)}$ relative to the j-th component.

Then we said that Θ has *no resonances* if $\mathcal{R}_j(\Theta) = \varnothing$ for every $j = 1, \dots, n$.

Moreover, Θ is *compatible with* a linear map T of $\mathbb{C}^n$ if and only if we can write T in Jordan form with all monomials Θ-resonant. In other words, a matrix $T = (t_{ij})$ in Jordan form is compatible with Θ if and only if $\theta_j^{(k)} = \theta_{j+1}^{(k)}$ for all $k = 1, \dots, r$ when $t_{j,j+1} \neq 0$, that is in a Jordan block of dimension at least 2.

As announced we shall use Theorem 3.5.1 we proved in the previous chapter. We report here the statement for the sake of completeness.

Theorem 4.1.1. (Raissy, 2009 [R4]) *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Then f commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $1 \leq r \leq n$ with weight matrix $\Theta \in M_{n\times r}(\mathbb{Z})$ if and only there exists a local holomorphic change of coordinates conjugating f to a germ with linear part in Jordan normal form and containing only Θ-resonant monomials.*

Remark 4.1.2. As we noticed in the previous chapter, given $\lambda \in (\mathbb{C}^*)^n$, we can always find a unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$, i.e., $\lambda_j = e^{2\pi i[\varphi_j]}$ for every $j = 1, \dots, n$. Then we have

$$\mathrm{Res}_j(\lambda) = \mathrm{Res}_j([\varphi]) := \{Q \in \mathbb{N}^n \mid |Q| \geq 2, [\langle Q, \varphi\rangle - \varphi_j] = [0]\},$$

and to find a link between torus actions and holomorphic Poicaré-Dulac normalization we need to find when it is possible to translate the multiplicative resonances of $[\varphi]$ into additive ones.

4.2 Toric degree

We want to study the relations between the resonances of the eigenvalues of the differential $\mathrm{d}f_O$ of a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin, and the weight matrices of torus actions to understand in which cases Theorem 4.1.1 gives us a Poincaré-Dulac holomorphic normalization. Thanks to Remark 4.1.2 we have to deal with vectors of $(\mathbb{C}/\mathbb{Z})^n$. A concept that turns out to be crucial for this study is that of *toric degree.*

Definition 4.2.1. Let $[\varphi] = ([\varphi_1], \dots, [\varphi_n]) \in (\mathbb{C}/\mathbb{Z})^n$. The *toric degree of* $[\varphi]$ is the minimum $r \in \mathbb{N}$ such that there exist $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ and $\theta^{(1)}, \dots, \theta^{(r)} \in \mathbb{Z}^n$ such that

$$[\varphi] = \left[\sum_{k=1}^{r} \alpha_k \theta^{(k)}\right]. \tag{4.3}$$

The r-tuple $\theta^{(1)}, \dots, \theta^{(r)}$ is called a *r-tuple of toric vectors associated to* $[\varphi]$, and the numbers $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ are *toric coefficients* of the toric r-tuple.

Remark 4.2.1. Note that the toric degree is necessarily at most n, since

$$[\varphi] = \left[\sum_{k=1}^{n} \varphi_k e_k\right].$$

We did not say *the* toric coefficients because we have the following result.

Lemma 4.2.2. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$ and let $\theta^{(1)}, \dots, \theta^{(r)}$ be a r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $\alpha_1, \dots, \alpha_r \in \mathbb{C}$. Then $\beta_1, \dots, \beta_r \in \mathbb{C}$ satisfy*

$$[\varphi] = \left[\sum_{k=1}^{r} \beta_k \theta^{(k)}\right]$$

if and only if

$$\Theta \begin{pmatrix} \alpha_1 - \beta_1 \\ \vdots \\ \alpha_r - \beta_r \end{pmatrix} \in \mathbb{Z}^n$$

where Θ is the $n \times r$ matrix whose columns are $\theta^{(1)}, \dots, \theta^{(r)}$.

Proof. We have

$$\left[\sum_{k=1}^{r} \alpha_k \theta^{(k)}\right] = \left[\sum_{k=1}^{r} \beta_k \theta^{(k)}\right]$$

if and only if

$$\sum_{k=1}^{r} \alpha_k \theta^{(k)} - \sum_{k=1}^{r} \beta_k \theta^{(k)} \in \mathbb{Z}^n,$$

that is

$$\Theta \begin{pmatrix} \alpha_1 - \beta_1 \\ \vdots \\ \alpha_r - \beta_r \end{pmatrix} \in \mathbb{Z}^n,$$

which is the assertion. □

Thanks to Remark 4.1.2 the following definition makes sense.

Definition 4.2.2. Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and denote by $\lambda = \{\lambda_1, \dots, \lambda_n\}$ the spectrum of $\mathrm{d}f_O$. The *toric degree of f* is the toric degree of the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i [\varphi]}$.

Toric r-tuples and toric coefficients have to satisfy certain arithmetic properties, as the following result shows.

Lemma 4.2.3. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$ and let $\theta^{(1)}, \dots, \theta^{(r)}$ be a r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $\alpha_1, \dots, \alpha_r \in \mathbb{C}$. Then:*

(i) $\alpha_1, \dots, \alpha_r$ is a set of rationally independent complex numbers;

(ii) every r-tuple of toric vectors associated to $[\varphi]$ is a set of $\mathbb{Q}$-linearly independent vectors.

Proof. (i) Let us suppose by contradiction that $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ are rationally dependent. Then there exists $(c_1, \dots, c_r) \in \mathbb{Z}^r \setminus \{O\}$ such that

$$c_1\alpha_1 + \cdots + c_r\alpha_r = 0.$$

Up to reordering we may assume $c_1 \neq 0$. Then

$$\alpha_1 = -\frac{1}{c_1}(c_2\alpha_2 + \cdots + c_r\alpha_r),$$

and hence

$$\begin{aligned} [\varphi] &= \left[\sum_{k=1}^{r} \alpha_k \theta^{(k)}\right] \\ &= \left[-\frac{1}{c_1}(c_2\alpha_2 + \cdots + c_r\alpha_r)\theta^{(1)} + \alpha_2\theta^{(2)} + \cdots + \alpha_r\theta^{(r)}\right] \\ &= \left[\frac{\alpha_2}{c_1}(c_1\theta^{(2)} - c_2\theta^{(1)}) + \cdots + \frac{\alpha_r}{c_1}(c_1\theta^{(r)} - c_r\theta^{(1)})\right], \end{aligned}$$

and this contradicts the definition of toric degree.

(ii) The proof is analogous to the previous one. □

Remark 4.2.4. Given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$, of toric degree $1 \le r \le n$, if $\theta^{(1)}, \dots, \theta^{(r)}$ is a r-tuple of toric vectors associated to $[\varphi]$, the $n \times r$ matrix Θ whose columns are $\theta^{(1)}, \dots, \theta^{(r)}$, has maximal rank r.

Remark 4.2.5. Note that, if $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ has toric degree $1 \le r \le n$, and $\theta^{(1)}, \dots, \theta^{(r)}$ is a r-tuple of toric vectors associated to $[\varphi]$, up to change the toric coefficients $\alpha_1, \dots, \alpha_r$, we can always assume $\theta_1^{(k)}, \dots, \theta_n^{(k)}$ coprime for each $1 \le k \le r$. In fact, if $d_k \in \mathbb{Z}$ is the greatest common divisor of $\theta_1^{(k)}, \dots, \theta_n^{(k)}$, then

$$[\varphi] = \left[\sum_{k=1}^{r} \alpha_k \theta^{(k)}\right] = \left[\sum_{k=1}^{r} d_k \alpha_k \widetilde{\theta}^{(k)}\right],$$

where

$$\widetilde{\theta}^{(k)} = \begin{pmatrix} \theta_1^{(k)}/d_k \\ \vdots \\ \theta_n^{(k)}/d_k \end{pmatrix}$$

for $k = 1, \dots, r$.

Remark 4.2.6. *Given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$, of toric degree $1 \le r \le n$, the r-tuple of toric vectors associated to $[\varphi]$ is not necessarily unique.* Let us consider, for example

$$[\varphi] = \begin{bmatrix} 3\sqrt{2} + 4i \\ 2\sqrt{2} + 6i \\ -\sqrt{2} + 2i \end{bmatrix}.$$

The toric degree cannot be 1, since it is immediate to verify that φ cannot be written as the product of a complex number times an integer vector. The toric degree is in fact 2, since we have

$$[\varphi] = \left[\sqrt{2}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + 2i \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\right].$$

However we can also write $[\varphi]$ as

$$[\varphi] = \left[\frac{-3\sqrt{2} + 16i}{6}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{3\sqrt{2} + 4i}{6}\begin{pmatrix} 6 \\ 5 \\ -1 \end{pmatrix}\right].$$

Note that, in both cases, the toric coefficients are rationally independent with 1.

Example 4.2.7. The vector of $(\mathbb{C}/\mathbb{Z})^2$

$$[\varphi] = \begin{bmatrix} (1 + 6\sqrt{2})/6 \\ (1 - 2\sqrt{2})/2 \end{bmatrix},$$

has toric degree 2, since we have

$$[\varphi] = \left[\frac{1 + 6\sqrt{2}}{6}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1 - 2\sqrt{2}}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right],$$

and it is not difficult to verify that it cannot have toric degree 1. We can also write $[\varphi]$ as

$$[\varphi] = \left[\frac{1}{6}\begin{pmatrix}1\\3\end{pmatrix} + \sqrt{2}\begin{pmatrix}1\\-1\end{pmatrix}\right].$$

Note that this time, in both cases, the toric coefficients are rationally dependent with 1.

We shall prove that, given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ of toric degree $1 \le r \le n$, even when the r-tuple of toric vectors associated to $[\varphi]$ is not unique, we can always say whether the toric coefficients are rationally independent with 1 or not, so this will be an intrinsic property of the vector. Before proving this, we shall need the following result that gives us a way to find a more useful toric r-tuple when the toric coefficients are rationally dependent with 1.

Remark 4.2.8. Note that $\alpha \in \mathbb{C}$ is rationally dependent with 1 if and only if it belongs to $\mathbb{Q}$.

Lemma 4.2.9. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$, and let $\theta^{(1)}, \dots, \theta^{(r)}$ be a r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ rationally dependent with 1. Then there exists a r-tuple of toric vectors $\eta^{(1)}, \dots, \eta^{(r)}$ associated to $[\varphi]$ with toric coefficients $\beta_1, \dots, \beta_r \in \mathbb{C}$ such that $\beta_1 = 1/m$ with $m \in \mathbb{N}\setminus\{0,1\}$ and $m, \eta_1^{(1)}, \dots, \eta_n^{(1)}$ coprime. Moreover $\beta_2, \dots, \beta_r$ are rationally independent with 1.*

Proof. If $r = 1$, then α is rationally dependent with 1 if and only if it belongs to $\mathbb{Q}$, i.e.,

$$[\varphi] = \left[\frac{p}{q}\theta\right]$$

where we may assume without loss of generality p and q coprime and $q, \theta_1, \dots, \theta_n$ coprime. Then

$$[\varphi] = \left[\frac{1}{q}\eta\right]$$

where $\eta = p \cdot \theta \in \mathbb{Z}^n$ and we are done.

Let us suppose now $r \ge 2$. Since $\alpha_1, \dots, \alpha_r$ are (rationally independent and) rationally dependent with 1, we can consider the minimum positive integer $m_0 \in \mathbb{N} \setminus \{0\}$ so that there exists $(m_1, \dots, m_r) \in \mathbb{Z}^r \setminus \{O\}$ such that

$$m_1\alpha_1 + \cdots + m_r\alpha_r = m_0.$$

Thanks to the minimality of m_0, we have that $m_1, \dots, m_r, m_0$ are coprime. Up to reordering we may assume $m_1 \neq 0$. Then

$$\begin{aligned}\alpha_1 &= \frac{m_0}{m_1} - \left(\frac{m_2}{m_1}\alpha_2 + \cdots + \frac{m_r}{m_1}\alpha_r\right)\\ &= \frac{m_0'}{m_1'} - \left(\frac{m_2}{m_1}\alpha_2 + \cdots + \frac{m_r}{m_1}\alpha_r\right),\end{aligned}$$

where $\frac{m_0}{m_1} = \frac{m_0'}{m_1'}$ with $(m_0', m_1') = 1$ and $m_1' \in \mathbb{N} \setminus \{0,1\}$. Let d be the greatest common divisor of m_1' and the components of $\theta^{(1)}$, and consider

$$\widetilde{\theta}^{(1)} = \frac{1}{d}\theta^{(1)}, \quad \widetilde{m}_1 = \frac{m_1'}{d}.$$

Hence

$$\begin{aligned}[\varphi] &= \left[\frac{m_0'}{m_1'}\theta^{(1)} + \sum_{k=2}^{r}\frac{\alpha_k}{m_1}(m_1\theta^{(k)} - m_k\theta^{(1)})\right]\\ &= \left[\frac{m_0'}{\widetilde{m}_1}\widetilde{\theta}^{(1)} + \sum_{k=2}^{r}\frac{\alpha_k}{m_1}(m_1\theta^{(k)} - m_k\theta^{(1)})\right]\\ &= \left[\frac{1}{\widetilde{m}_1}m_0'\widetilde{\theta}^{(1)} + \sum_{k=2}^{r}\frac{\alpha_k}{m_1}(m_1\theta^{(k)} - m_k\theta^{(1)})\right]\\ &= \left[\sum_{k=1}^{r}\beta_k\eta^{(k)}\right],\end{aligned}$$

where

$$\beta_1 = \frac{1}{\widetilde{m}_1}, \beta_2 = \frac{\alpha_2}{m_1}, \dots, \beta_r = \frac{\alpha_r}{m_1},$$

and

$$\eta^{(1)} = m_0'\widetilde{\theta}^{(1)},\ \eta^{(2)} = m_1\theta^{(2)} - m_2\theta^{(1)},\ \dots,\ \eta^{(r)} = m_1\theta^{(r)} - m_r\theta^{(1)}.$$

Notice that $\widetilde{m}_1$ is necessarily greater than 1, because otherwise the toric degree of $[\varphi]$ would be less than r.

Now, if $\beta_2, \dots, \beta_r$ were rationally dependent with 1, then we would have an integer vector $(k_2, \dots, k_r) \in \mathbb{Z}^{r-1} \setminus \{O\}$ such that

$$k_2\beta_2 + \cdots + k_r\beta_r = k \in \mathbb{Z} \setminus \{0\},$$

then

$$-k\,\widetilde{m}_1 \cdot \frac{1}{\widetilde{m}_1} + k_2\beta_2 + \cdots + k_r\beta_r = 0,$$

contradicting Lemma 4.2.3. This concludes the proof. $\square$

Definition 4.2.3. Let $[\varphi] = ([\varphi_1], \dots, [\varphi_n]) \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$. We say that a r-tuple $\eta^{(1)}, \dots, \eta^{(r)}$ of toric vectors associated to $[\varphi]$ with toric coefficients $\beta_1, \dots, \beta_r$ rationally dependent with 1 is *reduced* if $\beta_1 = 1/m$ with $m \in \mathbb{N} \setminus \{0, 1\}$ and $m, \eta_1^{(1)}, \dots, \eta_n^{(1)}$ coprime. In this case the toric vectors $\eta^{(2)}, \dots, \eta^{(r)}$ are called *reduced torsion-free toric vectors* associated to $[\varphi]$.

Now we can prove that the rational independence with 1 of the coefficients of toric r-tuples associated to a given vector $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ of toric degree $1 \le r \le n$ is an intrinsic property of $[\varphi]$.

Proposition 4.2.10. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$, and let $\theta^{(1)}, \dots, \theta^{(r)}$ be a r-tuple of toric vectors associated to $[\varphi]$, with toric coefficients $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ rationally independent with* 1*. Then every other r-tuple of toric vectors associated to $[\varphi]$ has toric coefficients rationally independent with* 1.

Proof. Let us assume by contradiction that there exists a r-tuple $\eta^{(1)}, \dots, \eta^{(r)}$ of toric vectors associated to $[\varphi]$ with toric coefficients $\beta_1, \dots, \beta_r$ rationally dependent with 1. Thanks to Lemma 4.2.9, we may assume without loss of generality $\beta_1 = 1/m$ with $m \in \mathbb{N} \setminus \{0, 1\}$

and $m, \eta_1^{(1)}, \ldots, \eta_n^{(1)}$ coprime. Let N be the matrix with columns $\eta^{(1)}, \ldots, \eta^{(r)}$, and let Θ be the matrix with columns $\theta^{(1)}, \ldots, \theta^{(r)}$. We have

$$[\varphi] = \left[N \cdot \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_r \end{pmatrix} \right] = \left[\Theta \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} \right],$$

that is, there exists an integer vector $\mathbf{k} \in \mathbb{Z}^n$ such that

$$N \cdot \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_r \end{pmatrix} = \Theta \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} + \mathbf{k}.$$

Since N has maximal rank r, the linear map $N: \mathbb{Q}^r \to \mathbb{Q}^n$ is injective and, for every $U \subseteq \mathbb{Q}^n$ such that $\mathbb{Q}^n = \mathrm{Im}(N) \oplus U$, there is a linear map $L_U: \mathbb{Q}^n \to \mathbb{Q}^r$ such that $\ker(L_U) = U$ and $L_U N = \mathrm{Id}$; hence there is a linear map $\widetilde{L}_U: \mathbb{Z}^n \to \mathbb{Z}^r$ that $\widetilde{L}_U N = h\,\mathrm{Id}$, with $h \in \mathbb{Z} \setminus \{0\}$. Then

$$h \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_r \end{pmatrix} = \widetilde{L}_U \Theta \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} + \widetilde{L}_U \mathbf{k}.$$

Moreover, we can choose U so that the first row of $\widetilde{L}_U \Theta$ is not identically zero. In fact, the first row of $\widetilde{L}_U \Theta$ is identically zero if and only if the first vector e_1 of the standard basis belongs to $\ker(\Theta^T \widetilde{L}_U^T)$, and hence it is orthogonal to $\mathrm{Im}(\widetilde{L}_U \Theta)$, because for any $u \in \mathbb{Q}^r$ we have

$$0 = \langle u, \Theta^T \widetilde{L}_U^T e_1 \rangle = \langle \Theta u, \widetilde{L}_U^T e_1 \rangle = \langle \widetilde{L}_U \Theta u, e_1 \rangle.$$

In particular $\mathrm{Im}(\Theta) \cap U \neq \{O\}$; otherwise $\widetilde{L}_U|_{\mathrm{Im}(\Theta)}$ would be injective, thus $\mathrm{Im}(\widetilde{L}_U \Theta) = \mathbb{Q}^r$, and e_1 could not be orthogonal to $\mathrm{Im}(\widetilde{L}_U \Theta)$. Now, it is a well-known fact of linear algebra that given two subspaces V, W of a vector space T having the same dimension there exists a subspace U such that $T = V \oplus U = W \oplus U$. Hence choosing U so that $\mathbb{Q}^n = \mathrm{Im}(N) \oplus U = \mathrm{Im}(\Theta) \oplus U$, we have $\mathrm{Im}(\Theta) \cap U = \{O\}$, and thus the first row of $\widetilde{L}_U \Theta$ is not identically zero.

Then

$$h \frac{1}{m} = (\widetilde{L}_U \Theta)_1 \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} + (\widetilde{L}_U \mathbf{k})_1$$

and this gives us a contradiction since $\alpha_1, \ldots, \alpha_r$ are rationally independent with 1 by assumption. □

We have then two cases to deal with: the rationally independent with 1 case, and the rationally dependent with 1 case.

Definition 4.2.4. Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$. We say that $[\varphi]$ is in the *torsion-free case*, or simply $[\varphi]$ *is torsion-free*, if its r-tuples of toric vectors have toric coefficients rationally independent with 1.

A notion of torsion-free germ of biholomorphism was firstly introduced by Écalle in [É6]. We shall show in the next section that our notion is equivalent to his; our approach however gives more information on the normalization problem.

We end this section with a couple of results showing how to compute the toric degree, starting with toric degree 1.

Proposition 4.2.11. *Let* $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$*. Then:*

(i) $[\varphi]$ *has toric degree* 1 *with rational toric coefficient if and only if it belongs to* $(\mathbb{Q}/\mathbb{Z})^n$*;*

(ii) $[\varphi]$ *has toric degree* 1 *with toric coefficient in* $\mathbb{C} \setminus \mathbb{Q}$ *if and only if* $[\varphi] \notin (\mathbb{Q}/\mathbb{Z})^n$*, and there exists* $\theta \in \mathbb{Z}^n \setminus \{O\}$*, with* $\theta_k = 0$ *if* $[\varphi_k] = [0]$*, such that there is* $j_0 \in \{1, \dots, n\}$ *so that*

(a) $[\varphi_{j_0}] \notin (\mathbb{Q}/\mathbb{Z})^n$ *and*
$$\theta_k[\varphi_{j_0}] - \theta_{j_0}[\varphi_k] = [0] \tag{4.4}$$
for any k *so that* $[\varphi_k] \neq [0]$*; and*

(b) for any representatives φ_k *of* $[\varphi_k]$*, the integer vector* $\varphi_{j_0}\theta - \theta_{j_0}\varphi$ *belongs to the subspace* $\mathrm{Span}_{\mathbb{Z}}\{\widetilde{\theta}, -\theta_{j_0}e_1, \dots, \widehat{-\theta_{j_0}e_{j_0}}, \dots, -\theta_{j_0}e_n\}$*, where* $\widetilde{\theta} = \theta - \theta_{j_0}e_{j_0}$*.*

Proof. (i) If $\alpha = p/q \in \mathbb{Q}$ then
$$[\varphi] = \left[\frac{p}{q}\theta\right],$$
hence $[\varphi] \in (\mathbb{Q}/\mathbb{Z})^n$.

Conversely, if $[\varphi_j] = [p_j/q_j]$ with $p_j/q_j \in \mathbb{Q}$ for $j = 1, \dots, n$, then, considering $q = q_1 \cdots q_n$ we get
$$[\varphi] = \begin{bmatrix} \frac{p_1 q_2 \cdots q_n}{q} \\ \vdots \\ \frac{p_n q_1 \cdots q_{n-1}}{q} \end{bmatrix} = \left[\frac{1}{q}\theta\right],$$
and we are done.

(ii) If
$$[\varphi] = \left[\alpha \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}\right],$$
with $\alpha \in \mathbb{C} \setminus \mathbb{Q}$ and $\theta \in \mathbb{Z}^n \setminus \{O\}$ then it is immediate to verify that $[\varphi] \notin (\mathbb{Q}/\mathbb{Z})^n$, and θ satisfies (a). By assumption, once we choose arbitrarily representatives φ_k of $[\varphi_k]$, we can write $\varphi_k = \alpha\theta_k + m_k$ for suitable $m_k \in \mathbb{Z}$. Then
$$\theta_k\varphi_j - \theta_j\varphi_k = \theta_k(\alpha\theta_j + m_j) - \theta_j(\alpha\theta_k + m_k) = \theta_k m_j - \theta_j m_k,$$
for any j and k, thus (b) is verified.

Conversely, let $\theta \in \mathbb{Z}^n \setminus \{O\}$ satisfy the hypotheses. By assumption $[\varphi] \notin (\mathbb{Q}/\mathbb{Z})^n$ and there is $j_0 \in \{1, \dots, n\}$ such that $[\varphi_{j_0}] \notin (\mathbb{Q}/\mathbb{Z})^n$ satisfies (a) and (b); for the sake of simplicity, we may assume, without loss of generality, $j_0 = 1$. Let us choose a representative φ of $[\varphi]$ and set
$$\theta_j\varphi_1 - \theta_1\varphi_j = k_j \in \mathbb{Z}$$
for $j = 2, \dots, n$. Condition (b) means that we can find $m_1, \dots, m_n \in \mathbb{Z}$ so that
$$\begin{pmatrix} \theta_2 & -\theta_1 & & \\ \vdots & & \ddots & \\ \theta_n & & & -\theta_1 \end{pmatrix} \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} k_2 \\ \vdots \\ k_n \end{pmatrix}, \tag{4.5}$$

that is

$$k_j = \theta_j m_1 - \theta_1 m_j.$$

Now we put

$$\alpha = \frac{\varphi_1 - m_1}{\theta_1} \notin \mathbb{Q}.$$

Then $[\varphi] = [\alpha\theta]$; indeed

$$\alpha\theta_j = \frac{\theta_j(\varphi_1 - m_1)}{\theta_1} = \frac{\theta_j\varphi_1 - k_j - \theta_1 m_j}{\theta_1} = \varphi_j - m_j.$$

□

Remark 4.2.12. Condition (b) of the previous Proposition is necessary. In fact, if we just assume that condition (a) holds, then it is always possible to solve (4.5) in $\mathbb{Q}$, but this does not imply that it is solvable in $\mathbb{Z}$. For example the vector

$$[\varphi] = \begin{bmatrix} (2i+1)/3 \\ i \\ (11+10i)/6 \end{bmatrix}$$

has toric degree 2, but if we consider

$$\theta = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

we get condition (a) for $j = 1$. Moreover, choosing $((2i+4)/3, i, (11+10i)/6)$ as representative of $[\varphi]$, we get

$$\begin{pmatrix} k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

and it is not difficult to verify that

$$\begin{pmatrix} 3 & -2 & 0 \\ 5 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

has no solution $(m_1, m_2, m_3) \in \mathbb{Z}^3$.

Example 4.2.13. The vector of $(\mathbb{C}/\mathbb{Z})^3$

$$[\varphi_1] = \begin{bmatrix} (\sqrt{2}+i)/6 \\ (\sqrt{2}+i)/3 \\ 5(\sqrt{2}+i)/6 \end{bmatrix}$$

has toric degree 1, since it can be written as

$$[\varphi_1] = \left[\frac{\sqrt{2}+i}{6} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \right].$$

In general, to compute the toric degree of a vector one starts from the trivial representation of Remark 4.2.1, and then uses (the proof of) Lemma 4.2.3 to obtain rationally independent toric coefficients and toric vectors. Then the toric degree is computed as follows (see also Proposition 4.3.5)

Proposition 4.2.14. *Let $\alpha_1, \ldots, \alpha_r$ be $1 \le r \le n$ rationally independent complex numbers and let $\theta^{(1)}, \ldots, \theta^{(r)} \in \mathbb{Z}^n$ be $\mathbb{Q}$-linearly independent integer vectors. Then:*

(i) if $\alpha_1, \ldots, \alpha_r$ are rationally independent with 1, then $[\varphi] = \left[\sum_{k=1}^r \alpha_k \theta^{(k)}\right]$ has toric degree r;

(ii) if $\alpha_1, \ldots, \alpha_r$ are rationally dependent with 1, then $[\varphi] = \left[\sum_{k=1}^r \alpha_k \theta^{(k)}\right]$ has toric degree $r-1$ or r.

Proof. (i) Let $\alpha_1, \ldots, \alpha_r$ be rationally independent with 1. The toric degree of $[\varphi]$ is not greater than r. Let us suppose by contradiction that $[\varphi]$ has toric degree $s < r$. Then there exist $\eta^{(1)}, \ldots, \eta^{(s)} \in \mathbb{Z}^n$ and $\beta_1, \ldots, \beta_s \in \mathbb{C}$ rationally independent such that

$$\left[\sum_{k=1}^r \alpha_k \theta^{(k)}\right] = \left[\sum_{k=1}^s \beta_k \eta^{(k)}\right].$$

Let N be the matrix with columns $\eta^{(1)}, \ldots, \eta^{(s)}$, and Θ the matrix with columns $\theta^{(1)}, \ldots, \theta^{(r)}$. We have

$$[\varphi] = \left[N \cdot \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_s \end{pmatrix}\right] = \left[\Theta \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix}\right],$$

that is there exists an integer vector $\mathbf{k} \in \mathbb{Z}^n$ such that

$$N \cdot \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_s \end{pmatrix} = \Theta \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} + \mathbf{k}.$$

Since Θ has maximal rank r, the linear map $\Theta\colon \mathbb{Q}^r \to \mathbb{Q}^n$ is injective and, for every $U \subseteq \mathbb{Q}^n$ such that $\mathbb{Q}^n = \mathrm{Im}(\Theta) \oplus U$, there is a linear map $L_U\colon \mathbb{Q}^n \to \mathbb{Q}^r$ such that $\ker(L_U) = U$ and $L_U\Theta = \mathrm{Id}$; hence there is a linear map $\widetilde{L}_U\colon \mathbb{Z}^n \to \mathbb{Z}^r$ such that $\widetilde{L}_U\Theta = h\,\mathrm{Id}$, with $h \in \mathbb{Z} \setminus \{0\}$. Then

$$\widetilde{L}_U N \cdot \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_s \end{pmatrix} - \widetilde{L}_U \mathbf{k} = h \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix}.$$

Now, $\dim(\ker(\widetilde{L}_U N)^T) \ge 1$. In particular, there exists $\xi \in \mathbb{Z}^r \setminus \{O\}$ such that $(\widetilde{L}_U N)^T \xi = O$, that is $\xi^T \widetilde{L}_U N = O$; therefore

$$\mathbb{Z} \ni -\xi^T \widetilde{L}_U \mathbf{k} = h \xi^T \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = h\langle \xi, \alpha \rangle,$$

which is an absurdum, because $\alpha_1, \ldots, \alpha_r, 1$ are rationally independent.

(ii) Now we have $\alpha_1, \ldots, \alpha_r$ rationally dependent with 1, and, arguing as in the proof of Lemma 4.2.9, we can suppose, without loss of generality, $\alpha_1 = 1/m$ and $\alpha_2, \ldots, \alpha_r$ rationally independent with 1. If m divides $\theta_1^{(1)}, \ldots, \theta_n^{(1)}$, then $[\varphi] = [\sum_{k=2}^r \alpha_k \theta^{(k)}]$ has toric degree $r-1$

thanks to (i). Otherwise, we may assume, without loss of generality, $m, \theta_1^{(1)}, \dots, \theta_n^{(1)}$ coprime. The toric degree of $[\varphi]$ is not greater than r. Let us suppose that $[\varphi]$ has toric degree $s < r$. Then there exist $\eta^{(1)}, \dots, \eta^{(s)} \in \mathbb{Z}^n$ and $\beta_1, \dots, \beta_s \in \mathbb{C}$ such that

$$\left[\sum_{k=1}^{r} \alpha_k \theta^{(k)}\right] = \left[\sum_{k=1}^{s} \beta_k \eta^{(k)}\right],$$

thus we have

$$[m\varphi] = \left[\sum_{k=2}^{r} \alpha_k \cdot m\theta^{(k)}\right] = \left[\sum_{k=1}^{s} \beta_k \cdot m\eta^{(k)}\right],$$

and, since $\alpha_2, \dots, \alpha_r$ are rationally independent with 1, by (i) we get $s = r - 1$. $\square$

Remark 4.2.15. Note that both cases in (ii) can occur. In fact, it is not difficult to verify that

$$[\varphi_1] = \begin{bmatrix} 1/2 \\ \sqrt{2} \\ i \end{bmatrix} = \left[\frac{1}{2}e_1 + \sqrt{2}e_2 + ie_3\right]$$

has toric degree 3. However, if we consider,

$$[\varphi_2] = \left[\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{\sqrt{2}-1}{2}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right],$$

then

$$[\varphi_2] = \begin{bmatrix} \sqrt{2}/2 \\ (\sqrt{2}+i)/2 \\ (-2+3\sqrt{2}+i)/2 \end{bmatrix} = \left[\frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right],$$

so the toric degree is 2. Proposition 4.3.5 will show how to distinguish between the two cases of Proposition 4.2.14.(ii).

4.3 Torsion

In [É6], Écalle introduced the following notion.

Definition 4.3.1. Let $\lambda \in (\mathbb{C}^*)^n$. The *torsion* of λ is the natural integer τ such that

$$\frac{1}{\tau}2\pi i\mathbb{Z} = (2\pi i\mathbb{Q}) \cap \left((2\pi i\mathbb{Z}) \bigoplus_{1\le j\le n} ((\log \lambda_j)\mathbb{Z})\right). \tag{4.6}$$

Translated in our notation, (4.6) becomes

$$\frac{1}{\tau}\mathbb{Z} = \mathbb{Q} \cap \left(\mathbb{Z} \bigoplus_{1\le j\le n} \varphi_j\mathbb{Z}\right),$$

where φ is a representative of the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = \exp(2\pi i[\varphi])$.

The torsion is well-defined, as the following result shows (and whose proof describes how to explicitly compute the torsion).

Proposition 4.3.1. *The torsion of a n-tuple $(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$ is a well-defined natural integer. Furthermore, writing $\lambda = e^{2\pi i[\varphi]}$, if $[\varphi]$ is torsion-free, then $\tau = 1$; otherwise τ divides the denominator of the first toric coefficient in a reduced representation of $[\varphi]$.*

Proof. Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be the unique vector such that $\lambda = \exp(2\pi i[\varphi])$, let $1 \le r \le n$ be its toric degree and let $\theta^{(1)}, \ldots, \theta^{(r)}$ be a r-tuple of toric vectors associated to $[\varphi]$ with coefficients $\alpha_1, \ldots, \alpha_r$.

Our aim is to determine the structure of the set

$$R = \mathbb{Q} \cap \left(\mathbb{Z} \bigoplus_{j=1}^{n} \varphi_j \mathbb{Z} \right),$$

that is of the set of rational numbers x that can be expressed in the form

$$\mathbb{Q} \ni x = m_0 + m_1\varphi_1 + \cdots + m_n\varphi_n$$

with $m_0, \ldots, m_n \in \mathbb{Z}$. Write, as usual,

$$\varphi_j = h_j + \sum_{k=1}^{r} \alpha_k \theta_j^{(k)},$$

with $h_j \in \mathbb{Z}$. Then

$$\begin{aligned} x &= (m_0 + m_1h_1 + \cdots + m_nh_n) + m_1 \sum_{k=1}^{r} \alpha_k\theta_1^{(k)} + \cdots + m_n \sum_{k=1}^{r} \alpha_k\theta_n^{(k)} \\ &= \widetilde{m} + \sum_{k=1}^{r} \alpha_k \langle M, \theta^{(k)} \rangle, \end{aligned}$$

where $\widetilde{m} \in \mathbb{Z}$ and $M \in \mathbb{Z}^n$ are generic. If $\alpha_1, \ldots, \alpha_r$ are rationally independent with 1, it follows that $x \in \mathbb{Q}$ if and only if $\langle M, \theta^{(1)} \rangle = \cdots = \langle M, \theta^{(r)} \rangle = 0$, and thus $R = \mathbb{Z}$ and $\tau = 1$.

If $\alpha_1, \ldots, \alpha_r$ are not rationally independent with 1, let us use instead the reduced representation, with $\beta_1 = 1/m$, the remaining coefficients $\beta_2, \ldots, \beta_r$ rationally independent with 1, and with $\eta^{(1)}, \ldots, \eta^{(r)}$ as toric vectors. We get

$$x = \widetilde{m} + \frac{1}{m}\langle M, \eta^{(1)} \rangle + \sum_{k=2}^{r} \beta_k \langle M, \eta^{(k)} \rangle.$$

Therefore $x \in \mathbb{Q}$ if and only if $\langle M, \eta^{(2)} \rangle = \cdots = \langle M, \eta^{(r)} \rangle = 0$, and moreover in that case

$$x = \widetilde{m} + \frac{1}{m}\langle M, \eta^{(1)} \rangle.$$

Now, the set

$$S = \{\langle M, \eta^{(1)} \rangle \mid M \in \mathbb{Z}^n, \langle M, \eta^{(2)} \rangle = \cdots = \langle M, \eta^{(r)} \rangle = 0\}$$

is an ideal of $\mathbb{Z}$; therefore $S = q\mathbb{Z}$ for some $q \in \mathbb{N}$. It follows that

$$R = \mathbb{Z} \oplus \frac{q}{m}\mathbb{Z} = \mathbb{Z} \oplus \frac{\widetilde{q}}{\widetilde{m}}\mathbb{Z} = \frac{1}{\widetilde{m}}\mathbb{Z},$$

where $\widetilde{q}$ and $\widetilde{m}$ are coprime, and $q/m = \widetilde{q}/\widetilde{m}$. Hence $\tau = \widetilde{m}$, and we are done. $\square$

Remark 4.3.2. Note that, in the previous proof, $S \neq \{O\}$, i.e., $q \neq 0$. Indeed, $S = \{O\}$ if and only if the kernel in $\mathbb{Z}^n$ of the linear form $(\eta^{(1)})^T$ contains the intersection of the kernels in $\mathbb{Z}^n$ of the linear forms $(\eta^{(2)})^T, \ldots, (\eta^{(r)})^T$. It is easy to see that this implies that the kernel in $\mathbb{Q}^n$ of the linear form $(\eta^{(1)})^T$ contains the intersection of the kernels in $\mathbb{Q}^n$ of the linear forms $(\eta^{(2)})^T, \ldots, (\eta^{(r)})^T$. But this implies that the linear form $(\eta^{(1)})^T$ is a $\mathbb{Q}$-linear combination of $(\eta^{(2)})^T, \ldots, (\eta^{(r)})^T$, and so $\eta^{(1)}, \ldots, \eta^{(r)}$ are $\mathbb{Q}$-linearly dependent, impossible.

The next result explains the terminology of Definition 4.2.4.

Theorem 4.3.3. *Let $\lambda = e^{2\pi i[\varphi]} \in (\mathbb{C}^*)^n$. Then $[\varphi]$ is torsion-free if and only if the torsion of λ is* 1.

Proof. If $[\varphi]$ is torsion-free, then the toric coefficients of a toric r-tuple associated to $[\varphi]$ are rationally independent with 1, and the torsion τ is 1, by Proposition 4.3.1.

Conversely, let $\eta^{(1)}, \ldots, \eta^{(r)}$ be a reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/m, \beta_2, \ldots, \beta_r$. Let us assume by contradiction that the torsion τ of $[\varphi]$ is 1. From the proof of Proposition 4.3.1 it is clear that we have $\tau = 1$ if and only if $\langle P, \eta^{(1)} \rangle \in m\mathbb{Z}$, for any $P \in \mathbb{Z}^n$ such that $\langle P, \eta^{(k)} \rangle = 0$ for $k = 2, \ldots, r$.

Since $\eta^{(1)}, \ldots, \eta^{(r)}$ are a toric r-tuple, we may assume, without loss of generality, that the matrix A of $M_{n\times n}(\mathbb{Z})$ with columns $\eta^{(2)}, \ldots, \eta^{(r)}, e_r, \ldots, e_n$ is invertible in $M_{n\times n}(\mathbb{Q})$. Denote by N' the matrix in $M_{(r-1)\times(r-1)}(\mathbb{Z})$

$$N' = \begin{pmatrix} \eta_1^{(2)} & \ldots & \eta_1^{(r)} \\ \vdots & & \vdots \\ \eta_{r-1}^{(2)} & \ldots & \eta_{r-1}^{(r)} \end{pmatrix},$$

and by N'' the matrix in $M_{(n-r+1)\times(r-1)}(\mathbb{Z})$

$$N'' = \begin{pmatrix} \eta_r^{(2)} & \ldots & \eta_r^{(r)} \\ \vdots & & \vdots \\ \eta_n^{(2)} & \ldots & \eta_n^{(r)} \end{pmatrix}.$$

Then

$$A = \begin{pmatrix} N' & O \\ N'' & I_{n-r+1} \end{pmatrix}$$

and $\det(A) = \det(N') \neq 0$.

We claim that, up to pass to another toric r-tuple $\widehat{\eta}^{(1)}, \eta^{(2)}, \ldots, \eta^{(r)}$, we may assume that $m = \det(N')$ and $\widehat{\eta}^{(1)} \in \{0\}^{r-1} \times \mathbb{Z}^{n-r+1}$. In fact, $\eta^{(k)} = A^{-1}e_{k-1}$ for $k = 2, \ldots, r$, with $A^{-1} \in M_{n\times n}(\mathbb{Q})$. Hence $P \in \mathbb{Z}^n$ is such that $\langle P, \eta^{(k)} \rangle = 0$ for $k = 2, \ldots, r$ if and only if $\langle A^T P, e_j \rangle = 0$ for $j = 1, \ldots, r-1$, that is $A^T P \in \{0\}^{r-1} \times \mathbb{Z}^{n-r+1}$. Now, we have

$$A^T P = \begin{pmatrix} N'^T & N''^T \\ O & I_{n-r+1} \end{pmatrix} \begin{pmatrix} P' \\ P'' \end{pmatrix} \in \{0\}^{r-1} \times \mathbb{Z}^{n-r+1}$$

if and only if

$$P = \begin{pmatrix} -(N'^T)^{-1} N''^T P'' \\ P'' \end{pmatrix} \quad \text{with} \quad P'' \in \mathbb{Z}^{n-r+1} \text{ and } (N'^T)^{-1} N''^T P'' \in \mathbb{Z}^{r-1},$$

that is

$$P'' \in \mathbb{Z}^{n-r+1} \text{ and } (N'^+)^T N''^T P'' \in \det(N')\mathbb{Z}^{r-1}$$

where $(N'^{+})^T \in M_{(r-1)\times(r-1)}(\mathbb{Z})$ and $(N'^{+})^T N' = \det(N') I_{r-1}$. In particular, since we are assuming

$$\langle P, \eta^{(k)} \rangle = 0 \text{ for } k = 2, \dots, r \implies \langle P, \eta^{(1)} \rangle \in m\mathbb{Z}, \tag{4.7}$$

we get

$$\langle A^T P, A^{-1}\eta^{(1)} \rangle = \left\langle \begin{pmatrix} O \\ P'' \end{pmatrix}, A^{-1}\eta^{(1)} \right\rangle \in m\mathbb{Z}$$

for any $P'' \in \det(N')\mathbb{Z}^{n-r+1}$. Then there exist $q_1, \dots, q_{r-1} \in \mathbb{Q}$ and $\widehat{\eta}^{(1)} \in \{0\}^{r-1} \times \mathbb{Z}^{n-r+1}$ such that

$$A^{-1}\eta^{(1)} = q_1 e_1 + \cdots + q_{r-1} e_{r-1} + \frac{m}{\det(N')}\widehat{\eta}^{(1)},$$

that is

$$\eta^{(1)} = q_1 \eta^{(2)} + \cdots + q_{r-1}\eta^{(r)} + \frac{m}{\det(N')}\widehat{\eta}^{(1)},$$

thus we get

$$\begin{aligned} [\varphi] &= \left[\frac{1}{m}\eta^{(1)} + \sum_{k=2}^{r} \beta_k \eta^{(k)} \right] \\ &= \left[\frac{1}{m}\frac{m}{\det(N')}\widehat{\eta}^{(1)} + \sum_{k=2}^{r} \left(\beta_k + \frac{q_{k-1}}{m} \right) \eta^{(k)} \right] \\ &= \left[\frac{1}{\det(N')}\widehat{\eta}^{(1)} + \sum_{k=2}^{r} \widetilde{\beta}_k \eta^{(k)} \right]. \end{aligned}$$

Note that $\widetilde{\beta}_2, \dots, \widetilde{\beta}_r$ are rationally independent with 1.

Now we can assume that (4.7) holds with $m = \det(N')$ and $\eta^{(1)} \in \{0\}^{r-1} \times \mathbb{Z}^{n-r+1}$. We claim that there exist $\gamma_2, \dots, \gamma_r \in \mathbb{C}^*$ such that $[\varphi] = [\sum_{k=2}^{r} \gamma_k \eta^{(k)}]$, i.e., $[\varphi]$ has toric degree $r-1$, contradicting the hypotheses. We can have $[\varphi] = [\sum_{k=2}^{r} \gamma_k \eta^{(k)}]$ with $\gamma_2, \dots, \gamma_r \in \mathbb{C}^*$, if there exists $\theta' \in \mathbb{Z}^{r-1}$ such that

$$\begin{pmatrix} \gamma_2 \\ \vdots \\ \gamma_r \end{pmatrix} = \begin{pmatrix} \beta_2 \\ \vdots \\ \beta_r \end{pmatrix} + N'^{-1}\theta',$$

and $\theta' \in \mathbb{Z}^{r-1}$ is a solution

$$N'' N'^{+} \begin{pmatrix} x_1 \\ \vdots \\ x_{r-1} \end{pmatrix} \equiv \begin{pmatrix} \eta_r^{(1)} \\ \vdots \\ \eta_n^{(1)} \end{pmatrix} \mod m\mathbb{Z}^{n-r+1}. \tag{4.8}$$

In fact, since $N'' N'^{-1} = (1/m) N'' N'^{+}$, this implies

$$\frac{1}{m}\eta^{(1)} = \frac{1}{m}\begin{pmatrix} O \\ \eta''^{(1)} \end{pmatrix} \equiv \begin{pmatrix} O \\ N'' N'^{-1}\theta' \end{pmatrix}, \tag{4.9}$$

modulo $\mathbb{Z}$, where $\eta''^{(1)} = (\eta_r^{(1)}, \dots, \eta_n^{(1)})$, hence

$$\begin{aligned}
[\varphi] &= \left[\frac{1}{m}\eta^{(1)} + N \begin{pmatrix} \beta_2 \\ \vdots \\ \beta_r \end{pmatrix} \right] \\
&= \left[\begin{pmatrix} O \\ N''N'^{-1}\theta' \end{pmatrix} + \begin{pmatrix} N' \\ N'' \end{pmatrix} \left(\begin{pmatrix} \gamma_2 \\ \vdots \\ \gamma_r \end{pmatrix} - N'^{-1}\theta' \right) \right] \\
&= \left[\begin{pmatrix} O \\ N''N'^{-1}\theta' \end{pmatrix} + \begin{pmatrix} N' \\ N'' \end{pmatrix} \begin{pmatrix} \gamma_2 \\ \vdots \\ \gamma_r \end{pmatrix} - \begin{pmatrix} \theta' \\ N''N'^{-1}\theta' \end{pmatrix} \right] \\
&= \left[N \begin{pmatrix} \gamma_2 \\ \vdots \\ \gamma_r \end{pmatrix} \right].
\end{aligned}$$

Now we prove that, if (4.7) holds with $m = \det(N')$ and $\eta^{(1)} \in \{0\}^{r-1} \times \mathbb{Z}^{n-r+1}$, then there exists a solution $\theta' \in \mathbb{Z}^{r-1}$ of (4.8). In fact, if $P'' \notin m\mathbb{Z}^{n-r+1}$ is a multi-index such that $P''^T N'' N'^{+} \in m\mathbb{Z}^{r-1}$, then by (4.7) we have $P''^T \eta''^{(1)} \in m\mathbb{Z}$, where we use the same notation of (4.9); thus, since up to reorder the indices we may assume that the last coordinate of P'' is not in $m\mathbb{Z}$, we can substitute $P''^T N'' N'^{+} x \equiv P''^T \eta''^{(1)}$ to the last equation of (4.8), and we have to solve a system with one equation less. We iterate this procedure for a set of generators of a complement of $m\mathbb{Z}^{n-r+1}$ in the lattice of P'' until, up to reordering, we get

$$B \begin{pmatrix} x_1 \\ \vdots \\ x_{r-1} \end{pmatrix} \equiv \begin{pmatrix} \eta_r^{(1)} \\ \vdots \\ \eta_{r+h-1}^{(1)} \end{pmatrix} \mod m\mathbb{Z}^h$$

where $1 \le h \le n - r + 1$, $B \in M_{h\times(r-1)}(\mathbb{Z})$ is the matrix of the first h rows of $N''N''^{+}$, and for any $R \notin m\mathbb{Z}^h$, we have $R^T B \notin m\mathbb{Z}^{r-1}$, that is B has maximal rank modulo m.

If $h = 1$, then we have

$$b_1 x_1 + \dots + b_{r-1} x_{r-1} \equiv \eta_1^{(1)} \mod m\mathbb{Z}. \tag{4.10}$$

If $b_1, \dots, b_{r-1}, m$ are coprime it is obvious that (4.10) is solvable. If the greatest common divisor of $b_1, \dots, b_{r-1}, m$ is $p > 1$, then $m = qp$ and $q(b_1, \dots, b_{r-1}) \in m\mathbb{Z}^{r-1}$, hence, by (4.7), we must have $\eta_1^{(1)} \in p\mathbb{Z}$ too, thus

$$\frac{b_1}{p} x_1 + \dots + \frac{b_{r-1}}{p} x_{r-1} \equiv \frac{\eta_1^{(1)}}{p} \mod \frac{m}{p}\mathbb{Z}$$

is solvable.

Let us now suppose $1 < h \le n - r + 1$. Since B has maximal rank modulo m, there exists B^+ in $M_{(r-1)\times h}(\mathbb{Z})$ such that $B^+ B \equiv dI_{r-1}$, modulo $m\mathbb{Z}$ where $d \ne m$. Thus we have

$$d \begin{pmatrix} x_1 \\ \vdots \\ x_{r-1} \end{pmatrix} \equiv B^+ \begin{pmatrix} \eta_r^{(1)} \\ \vdots \\ \eta_{r+h-1}^{(1)} \end{pmatrix} \mod m\mathbb{Z}^h.$$

If d and m are coprime, we are done. Otherwise, let p be greatest common divisor of d and m, and let $q = m/p$. Since $B^+B \equiv dI_{r-1}$ modulo $m\mathbb{Z}$, we have $qB^+B \equiv O$ modulo $m\mathbb{Z}$, thus, since we are assuming that for any $R \notin m\mathbb{Z}^h$, we have $R^TB \notin m\mathbb{Z}^{r-1}$, it has to be $qB^+ \equiv O$ modulo $m\mathbb{Z}$, that is $B^+ \equiv p\widetilde{B}$ modulo $m\mathbb{Z}$. Therefore we have

$$\frac{d}{p}\begin{pmatrix} x_1 \\ \vdots \\ x_{r-1} \end{pmatrix} \equiv \widetilde{B}\begin{pmatrix} \eta_r^{(1)} \\ \vdots \\ \eta_{r+h-1}^{(1)} \end{pmatrix} \quad \bmod \frac{m}{p}\mathbb{Z}^h,$$

which is solvable, as we wanted. □

Example 4.3.4. Let us consider the vector

$$[\varphi] = \left[\frac{1}{6}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \sqrt{2}\begin{pmatrix} 1 \\ -6 \end{pmatrix}\right] \in (\mathbb{C}/\mathbb{Z})^2,$$

of toric degree 2. We have

$$\langle P, \eta^{(2)} \rangle = p_1 - 6p_2 = 0$$

if and only if

$$P \in \begin{pmatrix} 6 \\ 1 \end{pmatrix}\mathbb{Z},$$

hence

$$\mathrm{Res}_1^+(\eta^{(2)}) = \{(6h+1, h) \mid h \in \mathbb{N} \setminus \{0\}\} \quad \text{and} \quad \mathrm{Res}_2^+(\eta^{(2)}) = \{(6h, h+1) \mid h \in \mathbb{N} \setminus \{0\}\},$$

and

$$\langle (6h, h), \eta^{(1)} \rangle \in 9\mathbb{Z},$$

that is

$$S = 9\mathbb{Z},$$

and the torsion is clearly 2. Moreover, we have

$$[\varphi] = \left[\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3\sqrt{2}-1}{3}\begin{pmatrix} 1 \\ -6 \end{pmatrix}\right] \in (\mathbb{C}/\mathbb{Z})^2.$$

Using the torsion τ of a vector, we obtain a complete criterion to compute the toric degree of a vector, as next result shows.

Proposition 4.3.5. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ and let τ be its torsion. If*

$$[\varphi] = \left[\frac{1}{m}\eta^{(1)} + \sum_{k=2}^{r} \beta_k \eta^{(k)}\right],$$

with $\eta^{(1)} \notin m\mathbb{Z}^n$, then $[\varphi]$ has toric degree r if and only if the torsion of $[\varphi]$ is $\tau > 1$, the coefficients $\beta_2, \ldots, \beta_r$ are rationally independent with 1, and the integer vectors $\eta^{(1)}, \ldots, \eta^{(r)}$ are $\mathbb{Q}$-linearly independent.

Proof. It follows from Lemma 4.2.9, Proposition 4.3.1 and from the proof of Theorem 4.3.3.

4.4 Poincaré-Dulac normal form in the torsion-free case

In the torsion-free case, it is not difficult to show that we can compute the resonances of $[\varphi]$, which are multiplicative, using the additive resonances of one of its associated r-tuples of toric vectors;

Lemma 4.4.1. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$ and torsion-free. Then for any r-tuple of toric vectors, $\theta^{(1)}, \dots, \theta^{(r)}$, associated to $[\varphi]$ we have*

$$\operatorname{Res}_j([\varphi]) = \bigcap_{k=1}^{r} \operatorname{Res}_j^+(\theta^{(k)})$$

for every $j = 1, \dots, n$.

Proof. We have

$$[\langle Q, \varphi \rangle - \varphi_j] = \left[\sum_{k=1}^{r} \alpha_k \left(\langle Q, \theta^{(k)} \rangle - \theta_j^{(k)} \right) \right] \tag{4.11}$$

and, since $\alpha_1, \dots, \alpha_r$ are rationally independent with 1, the right-hand side of (4.11) vanishes if and only if $\langle Q, \theta^{(k)} \rangle - \theta_j^{(k)} = 0$ for every $k = 1, \dots, r$. $\square$

Example 4.4.2. Let us consider the torsion-free vector

$$[\varphi] = \left[\sqrt{2} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + 2i \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right] \in (\mathbb{C}/\mathbb{Z})^3,$$

of toric degree 2. Then

$$\begin{cases} \langle P, \theta^{(1)} \rangle = 3p_1 + 2p_2 - p_3 = 0 \\ \langle P, \theta^{(2)} \rangle = 2p_1 + 3p_2 + p_3 = 0 \end{cases}$$

for some $P \in \mathbb{Z}^n$, if and only if

$$P \in \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \mathbb{Z}.$$

Hence in this case

$$\operatorname{Res}_1([\varphi]) = \operatorname{Res}_3([\varphi]) = \varnothing \quad \text{and} \quad \operatorname{Res}_2([\varphi]) = \{(1, 0, 1)\}.$$

Example 4.4.3. Let us consider the vector

$$[\varphi] = \left[\sqrt{2} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + 2i \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right] \in (\mathbb{C}/\mathbb{Z})^3.$$

Again, $[\varphi]$ has toric degree 2 and it is torsion-free. In this case, we have

$$\begin{cases} \langle P, \theta^{(1)} \rangle = 3p_1 + 2p_2 - p_3 = 0 \\ \langle P, \theta^{(2)} \rangle = 2p_1 - 3p_2 + p_3 = 0 \end{cases}$$

for some $P \in \mathbb{Z}^n$, if and only if

$$P \in \begin{pmatrix} 1 \\ 5 \\ 13 \end{pmatrix} \mathbb{Z}.$$

Hence

$$\begin{aligned} \mathrm{Res}_1([\varphi]) &= \{(q+1, 5q, 13q) \mid q \in \mathbb{N} \setminus \{0\}\} \\ \mathrm{Res}_2([\varphi]) &= \{(q, 5q+1, 13q) \mid q \in \mathbb{N} \setminus \{0\}\} \\ \mathrm{Res}_3([\varphi]) &= \{(q, 5q, 13q+1) \mid q \in \mathbb{N} \setminus \{0\}\}\,. \end{aligned}$$

We have the following immediate corollary of Lemma 4.4.1.

Corollary 4.4.4. *Let $\lambda \in (\mathbb{C}^*)^n$ and let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be such that $\lambda = e^{2\pi i[\varphi]}$. If $[\varphi]$ is torsion-free and has toric degree $1 \le r \le n$, then for every r-tuple $\theta^{(1)}, \dots, \theta^{(r)}$ of toric vectors associated to $[\varphi]$ we have*

$$\mathrm{Res}_j(\lambda) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\theta^{(k)})$$

for every $j = 1, \dots, n$.

Lemma 4.4.5. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$ and torsion-free. Then for any r-tuple of toric vectors, $\theta^{(1)}, \dots, \theta^{(r)}$, associated to $[\varphi]$ we have $\theta_j^{(k)} = \theta_h^{(k)}$ whenever $[\varphi_j] = [\varphi_h]$, for every $k = 1, \dots, r$.*

Proof. If $[\varphi_j] = [\varphi_h]$, then

$$\left[\alpha_1 \theta_j^{(1)} + \cdots + \alpha_r \theta_j^{(r)}\right] = \left[\alpha_1 \theta_h^{(1)} + \cdots + \alpha_r \theta_h^{(r)}\right];$$

hence there exists $m \in \mathbb{Z}$, such that

$$\alpha_1 \left(\theta_j^{(1)} - \theta_h^{(1)}\right) + \cdots + \alpha_r \left(\theta_j^{(r)} - \theta_h^{(r)}\right) = m,$$

and, since $\theta_j^{(k)} - \theta_h^{(k)} \in \mathbb{Z}$ for $k = 1, \dots, r$, the assertion follows from the rational independence with 1 of $\alpha_1, \dots, \alpha_r$. □

Definition 4.4.1. Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin. We say that f is *torsion-free* if, denoted by $\lambda = \{\lambda_1, \dots, \lambda_n\}$ the spectrum of $\mathrm{d}f_O$, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$ is torsion-free.

We have then the following complete description of Poincaré-Dulac holomorphic normalization in the torsion-free case.

Theorem 4.4.6. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O, of toric degree $1 \le r \le n$ and in the torsion-free case. Then f admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with f and such that the columns of the weight matrix of the action are a r-tuple of toric vectors associated to f.*

Proof. It follows from Theorem 4.1.1, Lemma 4.4.5 and Corollary 4.4.4. □

4.5 Poincaré-Dulac normal form in presence of torsion

The torsion case is more delicate and difficult to deal with. First, the next lemma yields a subdivision in more subcases, all realizable (we have examples for all of them) and, surprisingly, having very different behaviours; we have cases similar to the torsion-free case (even if we have torsion!), and cases that are indeed different. In particular, considering iterates of f to reduce to the torsion-free case hides very interesting phenomena, and it does not allow to see that some torsion cases can be directly studied.

Let us consider $[\varphi] \in \mathbb{C}/\mathbb{Z}$, of toric degree $1 \leq r \leq n$ and let $\theta^{(1)}, \dots, \theta^{(r)}$ be a r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $\alpha_1, \dots, \alpha_r$ rationally dependent with 1. We shall put

$$\mathcal{D}(\alpha_1, \dots, \alpha_r) = \{M \in \mathbb{Z}^r \mid m_1\alpha_1 + \cdots + m_r\alpha_r \in \mathbb{Z}\},$$

and

$$\mathrm{Adm}(\theta^{(1)}, \dots, \theta^{(r)}) = \bigcup_{j=1}^{n} \mathrm{Adm}_j(\theta^{(1)}, \dots, \theta^{(r)}),$$

where

$$\mathrm{Adm}_j(\theta^{(1)}, \dots, \theta^{(r)}) = \{M \in \mathbb{Z}^r \mid \exists Q \in \mathbb{N}^n, |Q| \geq 2 \,\text{s.t.}\, m_k = \langle Q - e_j, \theta^{(k)} \rangle \,\forall k = 1, \dots, r\} \cup \{O\},$$

for all $j \in \{1, \dots, n\}$.

Even if, in this case, it is not always true that we can compute the resonances of $[\varphi]$ as intersection of additive resonances, we can say many things on the resonant multi-indices using reduced r-tuples associated to $[\varphi]$.

Lemma 4.5.1. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$ and let $\eta^{(1)}, \dots, \eta^{(r)}$ be a reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/m, \beta_2, \dots, \beta_r$. Then*

(i) $\mathcal{D}(1/m, \beta_2, \dots, \beta_r) = \{(hm, 0, \dots, 0) \mid h \in \mathbb{Z}\} \subset \mathbb{Z}^r$;

(ii) we have

$$\mathcal{D}(1/m, \beta_2, \dots, \beta_r) \cap \mathrm{Adm}(\eta^{(1)}, \dots, \eta^{(r)}) \neq \{O\}$$

if and only there exist $Q \in \mathbb{N}^n$, with $|Q| \geq 2$ and $j \in \{1, \dots, n\}$ such that

$$\langle Q - e_j, \eta^{(1)} \rangle \in m\mathbb{Z} \setminus \{0\} \quad \text{and} \quad Q \in \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)});$$

(iii) we have

$$\mathrm{Res}_j([\varphi]) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \langle Q - e_j, \eta^{(1)} \rangle \in m\mathbb{Z}\} \cap \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}),$$

for any $j \in \{1, \dots, n\}$. In particular,

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) \supseteq \mathrm{Res}_j([\varphi]) \supseteq \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\eta^{(k)}), \tag{4.12}$$

for all $j \in \{1, \dots, n\}$.

(iv) $[\varphi_j] = [\varphi_h]$ *implies that* m *divides* $\eta_j^{(1)} - \eta_h^{(1)}$, *and that* $\eta_j^{(k)} = \eta_h^{(k)}$ *for any* $k = 2, \dots, r$.

Proof. (i) One inclusion is obvious. Conversely, let $M \in \mathcal{D}(1/m, \beta_2, \dots, \beta_r)$; then

$$m_1 \frac{1}{m} + m_2\beta_2 + \cdots + m_r\beta_r \in \mathbb{Z}.$$

Since $\beta_2, \dots, \beta_r$ are rationally independent with 1, this implies $m_2 = \cdots = m_r = 0$, thus we must have $m_1/m \in \mathbb{Z}$, and we are done.

(ii) It is immediate from the definitions of $\mathcal{D}(1/m, \beta_2, \dots, \beta_r)$ and $\mathrm{Adm}(\eta^{(1)}, \dots, \eta^{(r)})$ and from (i).

(iii) It is immediate from (ii) and from

$$[\langle Q, \varphi\rangle - \varphi_j] = \left[\frac{1}{m}\langle Q - e_j, \eta^{(1)}\rangle + \sum_{k=2}^{r} \beta_k \langle Q - e_j, \eta^{(k)}\rangle\right]. \tag{4.13}$$

(iv) If $[\varphi_j] = [\varphi_h]$, then

$$\left[\frac{1}{m}\eta_j^{(1)} + \beta_2\eta_j^{(2)} + \cdots + \beta_r\eta_j^{(r)}\right] = \left[\frac{1}{m}\eta_h^{(1)} + \beta_2\eta_h^{(2)} + \cdots + \beta_r\eta_h^{(r)}\right],$$

hence

$$\frac{1}{m}\left(\eta_j^{(1)} - \eta_h^{(1)}\right) + \beta_2\left(\eta_j^{(2)} - \eta_h^{(2)}\right) \cdots + \beta_r\left(\eta_j^{(r)} - \eta_h^{(r)}\right) \in \mathbb{Z},$$

and, since $\eta_j^{(k)} - \eta_h^{(k)} \in \mathbb{Z}$ for $k = 1, \dots, r$, the assertion follows as in (i). $\square$

Remark 4.5.2. Note that, given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ of toric degree $1 \le r \le n$, if $\eta^{(1)}, \dots, \eta^{(r)}$ is a reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/m, \beta_2, \dots, \beta_r$, and such that $[\varphi_j] = [\varphi_h]$ for some distinct coordinates j and h, but $\eta_j^{(1)} \neq \eta_h^{(1)}$, then, since m divides $\eta_j^{(1)} - \eta_h^{(1)}$, we have

$$\frac{1}{m}\eta_j^{(1)} = \frac{1}{m}\eta_h^{(1)} + \frac{1}{m}\left(\eta_j^{(1)} - \eta_h^{(1)}\right);$$

thus

$$[\varphi] = \left[\frac{1}{m}\widetilde{\eta}^{(1)} + \sum_{k=2}^{r}\beta_k\eta^{(k)}\right]$$

where, $\widetilde{\eta}_p^{(1)} = \eta_p^{(1)}$ for any $p \neq j, h$ and $\widetilde{\eta}_j^{(1)} = \widetilde{\eta}_h^{(1)}$, that is $\widetilde{\eta}^{(1)} = \eta^{(1)} - (\eta_j^{(1)} - \eta_h^{(1)})e_j$, obtaining a reduced r-tuple compatible with the structure of λ.

Even in the torsion case, toric r-tuples associated to a same vector $[\varphi]$ have to verify certain properties on the resonances, as next result shows.

Lemma 4.5.3. *Let* $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ *be of toric degree* $1 \le r \le n$ *and in the torsion case. Let* $\eta^{(1)}, \dots, \eta^{(r)}$ *be a reduced* r*-tuple of toric vectors associated to* $[\varphi]$ *with toric coefficients* $1/m, \beta_2, \dots, \beta_r$ *and let* $\xi^{(1)}, \dots, \xi^{(r)}$ *be a reduced* r*-tuple of toric vectors associated to* $[\varphi]$ *with toric coefficients* $1/\widetilde{m}, \gamma_2, \dots, \gamma_r$. *Then we have*

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\xi^{(k)}),$$

for all $j = 1, \dots, n$.

Proof. We have

$$[\varphi] = \left[\frac{1}{m}\eta^{(1)} + \sum_{k=2}^{r} \beta_k \eta^{(k)}\right] = \left[\frac{1}{\widetilde{m}}\xi^{(1)} + \sum_{k=2}^{r} \gamma_k \xi^{(k)}\right].$$

Then

$$[m\widetilde{m}\varphi] = \left[\sum_{k=2}^{r} m\widetilde{m}\beta_k \eta^{(k)}\right] = \left[\sum_{k=2}^{r} m\widetilde{m}\gamma_k \xi^{(k)}\right],$$

and, by Proposition 4.2.14, $[m\widetilde{m}\varphi]$ has toric degree $r-1$ and is torsion-free, because $\beta_2, \dots, \beta_r$ and $\gamma_2, \dots, \gamma_r$ are rationally independent with 1. Therefore, by Lemma 4.4.1, we have

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) = \mathrm{Res}_j([m\widetilde{m}\varphi]) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\xi^{(k)}),$$

for any $j = 1, \dots, n$, and we are done. $\square$

As Theorem 4.3.3 shows, it is not possible that $\langle P, \eta^{(1)} \rangle \in m\mathbb{Z}$ for any $P \in \mathbb{Z}^n$ such that $\langle P, \eta^{(k)} \rangle = 0$ for $k = 2, \dots, r$. However, it is possible that

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)})$$

for all $j \in \{1, \dots, n\}$, as next example shows.

Example 4.5.4. Let us consider the vector

$$[\varphi] = \left[\frac{1}{3}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + \sqrt{2}\begin{pmatrix}-12\\0\\0\\1\end{pmatrix} + \sqrt{3}\begin{pmatrix}0\\5\\2\\0\end{pmatrix}\right] \in (\mathbb{C}/\mathbb{Z})^4,$$

of toric degree 3. In this case $\mathcal{D}(1/3, \sqrt{2}, \sqrt{3}) = \{(3h, 0, 0) \mid h \in \mathbb{Z}\}$. We have

$$\langle P, \eta^{(2)} \rangle = -12p_1 + p_4 = 0$$

if and only if

$$P \in \begin{pmatrix}1\\0\\0\\12\end{pmatrix}\mathbb{Z} \oplus e_2\mathbb{Z} \oplus e_3\mathbb{Z},$$

and

$$\langle P, \eta^{(3)} \rangle = 5p_2 + 2p_3 = 0$$

if and only if

$$P \in \begin{pmatrix}0\\-2\\5\\0\end{pmatrix}\mathbb{Z} \oplus e_1\mathbb{Z} \oplus e_4\mathbb{Z}.$$

We have

$$\begin{aligned}
\mathrm{Res}_1^+(\eta^{(2)}) &= \{(q_1, q_2, q_3, 12(q_1-1)) \mid q_1, q_2, q_3 \in \mathbb{N}, 13q_1 + q_2 + q_3 \geq 14\}\\
\mathrm{Res}_2^+(\eta^{(2)}) &= \{(q_1, q_2, q_3, 12q_1) \mid q_1, q_2, q_3 \in \mathbb{N}, 13q_1 + q_2 + q_3 \geq 2\}\\
\mathrm{Res}_3^+(\eta^{(2)}) &= \mathrm{Res}_2^+(\eta^{(2)})\\
\mathrm{Res}_4^+(\eta^{(2)}) &= \{(q_1, q_2, q_3, 12q_1+1) \mid q_1, q_2, q_3 \in \mathbb{N}, 13q_1 + q_2 + q_3 \geq 1\},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Res}_1^+(\eta^{(3)}) &= \{(q_1, 0, 0, q_4) \mid q_1, q_4 \in \mathbb{N}, q_1 + q_4 \geq 2\}\\
\mathrm{Res}_2^+(\eta^{(3)}) &= \{(q_1, 1, 0, q_4) \mid q_1, q_4 \in \mathbb{N}, q_1 + q_4 \geq 1\}\\
\mathrm{Res}_3^+(\eta^{(3)}) &= \{(q_1, 0, 1, q_4) \mid q_1, q_4 \in \mathbb{N}, q_1 + q_4 \geq 1\}\\
\mathrm{Res}_4^+(\eta^{(3)}) &= \mathrm{Res}_1^+(\eta^{(3)}).
\end{aligned}$$

Moreover for each multi-index of the form $(p, 0, 0, 12p)$ with $p \geq 1$, we get

$$\left\langle \begin{pmatrix} p \\ 0 \\ 0 \\ 12p \end{pmatrix}, \eta^{(1)} \right\rangle = 12p \in 3\,\mathbb{Z}.$$

Then it is easy to verify that

$$\mathrm{Res}_j([\varphi]) = \mathrm{Res}_j^+(\eta^{(2)}) \cap \mathrm{Res}_j^+(\eta^{(3)}),$$

for $j = 1, \ldots, 4$.

Remark 4.5.5. This last example shows that, even in the torsion case, there are vectors $[\varphi]$ of $(\mathbb{C}/\mathbb{Z})^n$ such that, for any j, $\mathrm{Res}_j([\varphi])$ can be written as intersection of sets of additive resonances.

We have then the following definition.

Definition 4.5.1. Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$ and in the torsion case. We say that $[\varphi]$ is in the *impure torsion case* if, given $\eta^{(1)}, \ldots, \eta^{(r)}$ a reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/m, \beta_2, \ldots, \beta_r$, we have

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}), \tag{4.14}$$

for all $j \in \{1, \ldots, n\}$. Otherwise we say that $[\varphi]$ is in the *pure torsion case.*

The next result shows that the impure torsion case is well-defined, i.e., it does not depend on the chosen toric r-tuple.

Lemma 4.5.6. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$ and in the torsion case. Let $\eta^{(1)}, \ldots, \eta^{(r)}$ be a reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/m, \beta_2, \ldots, \beta_r$. If*

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}), \tag{4.15}$$

for all $j \in \{1, \ldots, n\}$, then (4.15) holds for any other reduced toric r-tuple associated to $[\varphi]$.

Proof. Let $\xi^{(1)}, \ldots, \xi^{(r)}$ be another reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/\widetilde{m}, \gamma_2, \ldots, \gamma_r$. Since $\eta^{(1)}, \ldots, \eta^{(r)}$ is in the impure torsion case, we have

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}),$$

but, thanks to Lemma 4.5.3, we have

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\xi^{(k)}),$$

for any $j = 1, \ldots, n$, that is $\xi^{(1)}, \ldots, \xi^{(r)}$ satisfy (4.15). □

Definition 4.5.2. Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin. We say that f is in the *impure torsion case* [resp., *in the pure torsion case*] if, denoting with $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ the spectrum of $\mathrm{d}f_O$, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$ is in the impure torsion case [resp., in the pure torsion case].

Theorem 4.5.7. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O of toric degree $1 \le r \le n$ and in the impure torsion case. Then it admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $r-1$ commuting with f, and such that the columns of the weight matrix of the action are reduced torsion-free toric vectors associated to f.*

Proof. It follows from Theorem 4.1.1, Lemma 4.5.1 (iv) and Lemma 4.5.6. □

The next examples show that, in case of pure torsion more cases are possible.

Example 4.5.8. Let us consider the vector

$$[\varphi] = \left[\frac{1}{6}\begin{pmatrix}1\\3\end{pmatrix} + \sqrt{2}\begin{pmatrix}1\\6\end{pmatrix}\right] \in (\mathbb{C}/\mathbb{Z})^2,$$

of toric degree 2. In this case $\mathcal{D}(1/6, \sqrt{2}) = \{(6h, 0) \mid h \in \mathbb{Z}\}$. We have

$$\langle P, \eta^{(2)} \rangle = p_1 + 6p_2 = 0$$

if and only if

$$P \in \begin{pmatrix}-6\\1\end{pmatrix}\mathbb{Z};$$

hence

$$\mathrm{Res}_1^+(\eta^{(2)}) = \varnothing \quad \text{and} \quad \mathrm{Res}_2^+(\eta^{(2)}) = \{(6, 0)\}.$$

Since

$$\langle (6, -1), \eta^{(1)} \rangle = 3 \notin 6\mathbb{Z},$$

we have

$$\mathcal{D}(1/6, \sqrt{2}) \cap \mathrm{Adm}_j(\eta^{(1)}, \eta^{(2)}) = \{O\}$$

for $j = 1, 2$, so we have

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\eta^{(k)}) = \varnothing,$$

for $j = 1, 2$. Moreover, it is evident that the torsion is 2.

Example 4.5.9. Let us consider the vector

$$[\varphi] = \left[\frac{1}{7}\begin{pmatrix}1\\3\end{pmatrix} + \sqrt{2}\begin{pmatrix}1\\-6\end{pmatrix}\right] \in (\mathbb{C}/\mathbb{Z})^2,$$

of toric degree 2. In this case $\mathcal{D}(1/7, \sqrt{2}) = \{(7h, 0) \mid h \in \mathbb{Z}\}$. We have

$$\mathrm{Res}_1^+(\eta^{(1)}) = \varnothing \quad \text{and} \quad \mathrm{Res}_2^+(\eta^{(1)}) = \{(3, 0)\},$$

and

$$\mathrm{Res}_1^+(\eta^{(2)}) = \{(6h+1, h) \mid h \in \mathbb{N} \setminus \{0\}\} \quad \text{and} \quad \mathrm{Res}_2^+(\eta^{(2)}) = \{(6h, h+1) \mid h \in \mathbb{N} \setminus \{0\}\};$$

then

$$\mathrm{Res}_1^+(\eta^{(1)}) \cap \mathrm{Res}_1^+(\eta^{(2)}) = \varnothing \quad \text{and} \quad \mathrm{Res}_2^+(\eta^{(1)}) \cap \mathrm{Res}_2^+(\eta^{(2)}) = \varnothing.$$

However, we have

$$\langle (6h, h), \eta^{(1)} \rangle \in 9\mathbb{Z};$$

hence we have

$$\begin{aligned}
&\mathrm{Res}_1^+(\eta^{(2)}) \supset \mathrm{Res}_1([\varphi]) = \{(42h+1, 7h) \mid h \in \mathbb{N} \setminus \{0\}\} \supset \mathrm{Res}_1^+(\eta^{(1)}) \cap \mathrm{Res}_1^+(\eta^{(2)})\\
&\mathrm{Res}_2^+(\eta^{(2)}) \supset \mathrm{Res}_2([\varphi]) = \{(42h, 7h+1) \mid h \in \mathbb{N} \setminus \{0\}\} \supset \mathrm{Res}_2^+(\eta^{(1)}) \cap \mathrm{Res}_2^+(\eta^{(2)}).
\end{aligned}$$

Moreover, it is not difficult to verify that the torsion is 7.

In the pure torsion case, one could ask whether, given a toric r-tuple $\eta^{(1)}, \dots, \eta^{(r)}$ associated to $[\varphi]$ such that

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) \supset \mathrm{Res}_j([\varphi]) \supset \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\eta^{(k)}), \tag{4.16}$$

for some $j \in \{1, \dots, n\}$, then this is true for any other toric r-tuple associated to $[\varphi]$. This is not always true, as next example shows.

Example 4.5.10. Let us consider the vector

$$[\varphi] = \left[\frac{1}{3}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \sqrt{2}\begin{pmatrix}1\\6\\0\\0\end{pmatrix} + \sqrt{3}\begin{pmatrix}0\\0\\-1\\5\end{pmatrix}\right] \in (\mathbb{C}/\mathbb{Z})^4,$$

of toric degree 3. In this case $\mathcal{D}(1/3, \sqrt{2}, \sqrt{3}) = \{(3h, 0, 0) \mid h \in \mathbb{Z}\}$. We have

$$\mathrm{Res}_j^+(\eta^{(1)}) = \varnothing,$$

for $j = 1, \dots, 4$,

$$\begin{aligned}
\mathrm{Res}_1^+(\eta^{(2)}) &= \{(1, 0, p, q) \mid p, q \in \mathbb{N}, p+q \geq 1\}\\
\mathrm{Res}_2^+(\eta^{(2)}) &= \{(6, 0, p, q) \mid p, q \in \mathbb{N}\} \cup \{(0, 1, p, q) \mid p, q \in \mathbb{N}, p+q \geq 1\}\\
\mathrm{Res}_3^+(\eta^{(2)}) &= \{(0, 0, p, q) \mid p, q \in \mathbb{N}, p+q \geq 2\}\\
\mathrm{Res}_4^+(\eta^{(2)}) &= \mathrm{Res}_3^+(\eta^{(2)}),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Res}_1^+(\eta^{(3)}) &= \{(h,k,5q,q) \mid h,k,q \in \mathbb{N}, h+k+6q \geq 2\} \\
\mathrm{Res}_2^+(\eta^{(3)}) &= \mathrm{Res}_1^+(\eta^{(3)}) \\
\mathrm{Res}_3^+(\eta^{(3)}) &= \{(h,k,5q+1,q) \mid h,k,q \in \mathbb{N}, h+k+6q \geq 1\} \\
\mathrm{Res}_4^+(\eta^{(3)}) &= \{(h,k,5(q-1),q) \mid h,k,q \in \mathbb{N}, h+k+6q \geq 7\}\,.
\end{aligned}$$

Then we have

$$\bigcap_{k=1}^{3} \mathrm{Res}_j^+(\eta^{(k)}) = \varnothing,$$

for $j = 1, \ldots, 4$, but it is not difficult to verify that

$$\mathrm{Res}_2([\varphi]) = \{(0,1,5q,q) \mid q \in \mathbb{N}^*\} \neq \varnothing \quad \text{and} \quad \mathrm{Res}_j([\varphi]) = \mathrm{Res}_j^+(\eta^{(2)}) \cap \mathrm{Res}_j^+(\eta^{(3)}) \;\; j = 1,3,4.$$

Then, since

$$\mathrm{Res}_2^+(\eta^{(2)}) \cap \mathrm{Res}_2^+(\eta^{(3)}) = \{(6,0,5q,q) \mid q \in \mathbb{N}\} \cup \{(0,1,5q,q) \mid q \in \mathbb{N}^*\} \neq \mathrm{Res}_2([\varphi]),$$

we are in the pure torsion case, but we cannot write all the resonances of $[\varphi]$ as intersection of the additive resonances of $\eta^{(1)}$, $\eta^{(2)}$ and $\eta^{(3)}$. However, we can write

$$[\varphi] = \left[\frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \\ -5 \end{pmatrix} + \sqrt{2} \begin{pmatrix} 1 \\ 6 \\ 0 \\ 0 \end{pmatrix} + \sqrt{3} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 5 \end{pmatrix} \right],$$

and it is not difficult to verify that, in this representation, we have

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=1}^{3} \mathrm{Res}_j^+(\xi^{(k)}),$$

for $j = 1, \ldots, 4$.

Example 4.5.11. If $[\varphi] \in (\mathbb{C}/\mathbb{Z})^2$ is given by Example 4.5.9, we saw that we can write it in the form

$$[\varphi] = \left[\frac{1}{\tau} \eta^{(1)} + \beta \eta^{(2)} \right]$$

so that

$$\mathrm{Res}_j^+(\eta^{(2)}) \supset \mathrm{Res}_j([\varphi]) \supset \mathrm{Res}_j^+(\eta^{(1)}) \cap \mathrm{Res}_j^+(\eta^{(2)}), \tag{4.17}$$

for all j. Furthermore, it is easy to check that $[\varphi]$ does not admit any reduced representation

$$[\varphi] = \left[\frac{1}{\tau q} \xi^{(1)} + \gamma \xi^{(2)} \right]$$

such that for all j we have

$$\mathrm{Res}_j([\varphi]) = \mathrm{Res}_j^+(\xi^{(1)}) \cap \mathrm{Res}_j^+(\xi^{(2)}). \tag{4.18}$$

We are then led to the following

Definition 4.5.3. Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$ and in the pure torsion case. We say that $[\varphi]$ *can be simplified* if it admits a reduced r-tuple of toric vectors $\eta^{(1)}, \dots, \eta^{(r)}$ such that

$$\operatorname{Res}_j([\varphi]) = \bigcap_{k=1}^{r} \operatorname{Res}_j^+(\eta^{(k)}), \tag{4.19}$$

for all $j = 1, \dots, n$. The r-tuple $\eta^{(1)}, \dots, \eta^{(r)}$ is said a *simple reduced* r-tuple associated to $[\varphi]$.

Definition 4.5.4. Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin in the pure torsion case. We say that f *can be simplified* if, denoting with $\lambda = \{\lambda_1, \dots, \lambda_n\}$ the spectrum of $\mathrm{d}f_O$, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$ can be simplified.

Theorem 4.5.12. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O of toric degree $1 \le r \le n$ and in the pure torsion case, such that it can be simplified. Then:*

(i) if $\mathrm{d}f_O$ is diagonalizable, f admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with f and such that the columns of the weight matrix Θ of the action are a simple reduced r-tuple of toric vectors associated to f;

(ii) if $\mathrm{d}f_O$ is not diagonalizable and there exists a simple reduced r-tuple of toric vectors associated to $[\varphi]$ such that its vectors are the columns of a matrix Θ compatible with $\mathrm{d}f_O$, f admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with f and with weight matrix Θ.

Proof. It follows from Theorem 4.1.1. □

Remark 4.5.13. Note that we cannot get rid of the compatibility hypothesis in the case of $\mathrm{d}f_O$ non diagonalizable, because if we change a simple reduced toric r-tuple as in Remark 4.5.2, it is not true that we obtain another simple reduced r-tuple. In fact, if $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ has toric degree $1 \le r \le n$, and $\eta^{(1)}, \dots, \eta^{(r)}$ is a simple reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/m, \beta_2, \dots, \beta_r$, but we have $[\varphi_j] = [\varphi_h]$ for some distinct coordinates j and h, and $\eta_j^{(1)} \neq \eta_h^{(1)}$, then for every $P \in \operatorname{Res}_l([\varphi])$, the equality

$$[\varphi] = \left[\frac{1}{m}\widetilde{\eta}^{(1)} + \sum_{k=2}^{r} \beta_k \eta^{(k)}\right]$$

with $\widetilde{\eta}^{(1)} = \eta^{(1)} - (\eta_j^{(1)} - \eta_h^{(1)})e_j$, only implies

$$\frac{\eta_j^{(1)} - \eta_h^{(1)}}{m}(\delta_{lh} - p_h) \in \mathbb{Z}$$

and there well can be resonant multi-indices with $p_h \neq 1$.

In case of pure torsion that cannot be simplified, we have the following results.

Proposition 4.5.14. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O of toric degree $1 \le r \le n$ and in the pure torsion case, such that it cannot be simplified. If there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with f and such that the columns of the weight matrix of the action are a reduced r-tuple of toric vectors associated to f, then f admits a holomorphic Poincaré-Dulac normalization.*

Proof. It follows from Theorem 4.1.1 and Lemma 4.5.1. □

Proposition 4.5.15. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O of toric degree $1 \le r \le n$ and in the pure torsion case, such that it cannot be simplified. If f admits a holomorphic Poincaré-Dulac normalization, then it commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $r-1$, such that the columns of the weight matrix of the action are reduced torsion-free toric vectors associated to f.*

Proof. It follows from Theorem 4.1.1 and Lemma 4.5.1. □

We end this section describing an algorithm to decide when a vector $[\varphi]$ can be simplified.

We want to know when, given $[\varphi]$ in the torsion case,

$$[\varphi] = \left[\frac{1}{\tau p}\eta^{(1)} + \sum_{k=2}^{r} \beta_k \eta^{(k)}\right]$$

of toric degree r, torsion $\tau \ge 2$, and such that there is $j \in \{1, \dots, n\}$ so that

$$\bigcap_{k=1}^{r} \mathrm{Res}_j^+(\eta^{(k)}) \subset \mathrm{Res}_j([\varphi]) \subset \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}), \tag{4.20}$$

there is another reduced representation

$$[\varphi] = \left[\frac{1}{\tau q}\xi^{(1)} + \sum_{k=2}^{r} \gamma_k \xi^{(k)}\right]$$

such that for any $j = 1, \dots, n$ we have

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\xi^{(k)}). \tag{4.21}$$

We know that there must be $H \in \mathbb{Z}^n \setminus \{O\}$ such that

$$\frac{1}{\tau p}\eta^{(1)} + \sum_{k=2}^{r} \beta_k \eta^{(k)} = \frac{1}{\tau q}\xi^{(1)} + \sum_{k=2}^{r} \gamma_k \xi^{(k)} + H.$$

Since

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\xi^{(k)}),$$

for any $j = 1, \dots, n$, we have that

$$\frac{1}{\tau p}\langle \eta^{(1)}, P - e_j\rangle = \frac{1}{\tau q}\langle \xi^{(1)}, P - e_j\rangle + \langle H, P - e_j\rangle$$

for any $P \in \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)})$. Now, if $\langle \xi^{(1)}, P - e_j\rangle = 0$ it must be $\langle \eta^{(1)}, P - e_j\rangle \in \tau p\mathbb{Z}$. On the contrary, if $\langle \eta^{(1)}, P - e_j\rangle \in \tau p\mathbb{Z}$, then we would like to find H such that $\langle \xi^{(1)}, P - e_j\rangle = 0$ that is, for any $j = 1, \dots, n$,

$$\frac{1}{\tau p}\langle \eta^{(1)}, P - e_j\rangle = \langle H, P - e_j\rangle$$

for any $P \in \bigcap_{k=2}^r \mathrm{Res}_j^+(\eta^{(k)})$ with $\langle \eta^{(1)}, P - e_j \rangle \in \tau p\mathbb{Z}$. In fact, if such a vector exists, then, setting $q = p$, $\xi^{(1)} = \eta^{(1)} - \tau p H$, $\gamma_k = \beta_k$ and $\eta^{(k)} = \xi^{(k)}$ for $k = 2, \ldots, r$, we get

$$[\varphi] = \left[\frac{1}{\tau p} \xi^{(1)} + \sum_{k=2}^r \gamma_k \xi^{(k)} \right],$$

and for any $P \in \mathrm{Res}_j([\varphi])$ we have $P \in \bigcap_{k=2}^r \mathrm{Res}_j^+(\xi^{(k)})$, and

$$\langle \xi^{(1)}, P - e_j \rangle = \langle \eta^{(1)}, P - e_j \rangle - \langle H, P - e_j \rangle = 0,$$

that is (4.21).

We then have to study the structure of the intersection of a submodule of $\mathbb{Z}^n$ with $\mathbb{N}^n$. It turns out that such a structure is the following. We thank Jean Écalle for suggesting the gist of the following argument.

Let $\mathcal{A} \subset \mathbb{Z}^n$ be a sub-module of $\mathbb{Z}^n$ where $n \in \mathbb{N}^*$, and let us denote by $\mathcal{A}^+$ the set $\mathcal{A} \cap \mathbb{N}^n$. For any vector $A = (a_1, \ldots, a_n) \in \mathcal{A}$, we denote by

$$\mathrm{red}(A) = \frac{1}{\alpha} A = \left(\frac{a_1}{\alpha}, \ldots, \frac{a_n}{\alpha} \right) \tag{4.22}$$

where α is the greatest common divisor of $a_1, \ldots, a_n$. The *support* of a vector $A \in \mathbb{Z}^n$ is the set

$$\mathrm{supp}(A) = \{ j \in \{1, \ldots, n\} \mid a_j \neq 0 \} \subseteq \{1, \ldots, n\}.$$

Using the support we can then define a partial order on $\mathcal{A}^+$ as follows: we say that $A \subseteq B$ if $\mathrm{supp}(A) \subset \mathrm{supp}(B)$, or the supports are equal and $A \leq B$ in the usual lexicographic order.

Definition 4.5.5. Let $\mathcal{A} \subset \mathbb{Z}^n$ be any sub-module of $\mathbb{Z}^n$, where $n \in \mathbb{N}^*$, and let $\mathcal{A}^+$ be the set $\mathcal{A} \cap \mathbb{N}^n$. For any $A, B \in \mathcal{A}^+$ we define

$$A/B = \mathrm{red}(qA - pB) \tag{4.23}$$

where

$$\frac{p}{q} = \min_{j \in \mathrm{supp}(B)} \left(\frac{a_j}{b_j} \right).$$

Obviously, if $\mathrm{supp}(B) \subseteq \mathrm{supp}(A)$, then $A/B \in \mathcal{A}^+$ and $A/B \subseteq A$.

Definition 4.5.6. Let $\mathcal{A} \subset \mathbb{Z}^n$ be any sub-module of $\mathbb{Z}^n$, where $n \in \mathbb{N}^*$, and let $\mathcal{A}^+$ be the set $\mathcal{A} \cap \mathbb{N}^n$. An element M of $\mathcal{A}^+$ is said *minimal* if it is minimal with respect to the partial order $\subseteq$. An element C of $\mathcal{A}^+$ is said *cominimal* if for any minimal element M of $\mathcal{A}^+$ we have $C - M \notin \mathcal{A}^+$.

Minimal elements have to satisfy certain properties.

Lemma 4.5.16. *Let $\mathcal{A} \subset \mathbb{Z}^n$ be any sub-module of $\mathbb{Z}^n$, where $n \in \mathbb{N}^*$, and let $\mathcal{A}^+$ be the set $\mathcal{A} \cap \mathbb{N}^n$. Two minimal elements of $\mathcal{A}^+$ have distinct supports.*

Proof. Let M and P be two distinct minimal elements of $\mathcal{A}^+$ and suppose by contradiction that $\mathrm{supp}(M) = \mathrm{supp}(P)$. Then $A = M/P$ and $B = P/M$ both have supports strictly contained in the ones of M and P contradicting their minimality with respect to $\subseteq$. $\square$

Corollary 4.5.17. *Let $\mathcal{A} \subset \mathbb{Z}^n$ be any sub-module of $\mathbb{Z}^n$, where $n \in \mathbb{N}^*$, and let $\mathcal{A}^+$ be the set $\mathcal{A} \cap \mathbb{N}^n$. Then $\mathcal{A}^+$ contains only a finite number of minimal elements.*

Minimal elements are a sort of generators of $\mathcal{A}^+$ in a sense that next result clarifies.

Lemma 4.5.18. *Let $\mathcal{A} \subset \mathbb{Z}^n$ be any sub-module of $\mathbb{Z}^n$, where $n \in \mathbb{N}^*$, and let $\mathcal{A}^+$ be the set $\mathcal{A} \cap \mathbb{N}^n$. Then every element A of $\mathcal{A}^+$ can be written in the form*

$$A = \frac{1}{\delta}(\alpha_1 M_1 + \cdots + \alpha_d M_d) \tag{4.24}$$

where $\alpha_1, \dots, \alpha_d \in \mathbb{N}$, $M_1, \dots, M_d$ are the minimal elements, and $\delta = \delta(\mathcal{A}^+) \in \mathbb{N}^*$ depends only on $\mathcal{A}^+$.

Proof. If A is non minimal, then there exists a minimal element $M_{j_1} \subseteq A$, and there exist $\gamma_1, \delta_1 \in \mathbb{Q}^+$ such that

$$A = \gamma_1 M_{j_1} + \delta_1 A_1,$$

where

$$A_1 = A/M_{j_1},$$

and $\operatorname{supp}(A_1) \subset \operatorname{supp}(A)$. If A_1 is not minimal, we can iterate this procedure getting

$$A_1 = \gamma_2 M_{j_2} + \delta_2 A_2,$$

with $\operatorname{supp}(A_2) \subset \operatorname{supp}(A_1) \subset \operatorname{supp}(A)$. The chain $\operatorname{supp}(A) \supset \operatorname{supp}(A_1) \supset \operatorname{supp}(A_3) \supset \cdots$ has to end because $\mathcal{A}^+ \subset \mathbb{N}^n$, then we eventually arrive to a decomposition of the form (4.24). Now $\delta = \delta(\mathcal{A}^+)$ cannot be greater than the least common multiple of all $|\det(\mathrm{M}^*)|$ where M^* varies in the square submatrices of order equal to the rank of the matrix having as columns all the minimal elements $M_1, \dots, M_d$ of $\mathcal{A}^+$. □

The cominimal elements are finite too.

Lemma 4.5.19. *Let $\mathcal{A} \subset \mathbb{Z}^n$ be any sub-module of $\mathbb{Z}^n$, where $n \in \mathbb{N}^*$, and let $\mathcal{A}^+$ be the set $\mathcal{A} \cap \mathbb{N}^n$. Then $\mathcal{A}^+$ contains only a finite number of cominimal elements.*

Proof. Let us assume by contradiction that there is an infinite sequence of distinct cominimal elements $\{C_j\}$. Thanks to Lemma 4.5.18, for each $j \geq 1$, we have

$$C_j = \frac{1}{\delta} \sum_{k=1}^{d} \gamma_{jk} M_k$$

where $\gamma_{jk} \in \mathbb{N}$. Then there is an infinite subsequence $\{C_{j'}\}$ such that all the corresponding $(\gamma_{j',1}, \dots, \gamma_{j',d})$ belong to a same class $(\gamma_1^*, \dots, \gamma_d^*)$ modulo $\delta\mathbb{Z}^d$. Hence there is an infinite subsequence $\{C_{j''}\}$ such that at least one component γ_{j'',k_0} diverges as j'' tends to infinity, and such that the other components $\gamma_{j'',k}$ with $k \neq k_0$ do not decrease. Then there exist at least two cominimal elements $C_{j_1} \leq C_{j_2}$ such that

$$C_{j_2} - C_{j_1} = \sum_{k=1}^{d} \tilde{\gamma}_k M_k$$

with

$$\tilde{\gamma}_k = \frac{1}{\delta}\left(\gamma_{j_2,k} - \gamma_{j_1,k}\right) \in \mathbb{N}$$

meaning that C_{j_2} is not cominimal against the assumption. □

For each element of $\mathcal{A}^+$, we want to find a decomposition with natural coefficients into linear combination of a finite number of elements of $\mathcal{A}^+$. This is possible using minimal and cominimal elements, as shown in next result.

Proposition 4.5.20. *Let $\mathcal{A} \subset \mathbb{Z}^n$ be any sub-module of $\mathbb{Z}^n$, where $n \in \mathbb{N}^*$, and let $\mathcal{A}^+$ be the set $\mathcal{A} \cap \mathbb{N}^n$. Then for any $A \in \mathcal{A}^+$ there exist $l_1, \ldots, l_d \in \mathbb{N}$ such that*

$$A = \sum_{j=1}^{d} l_j M_j \tag{4.25}$$

or

$$A = C_h + \sum_{j=1}^{d} l_j M_j \tag{4.26}$$

for some $h \in \{1, \ldots, e\}$, where $M_1, \ldots, M_d$ are the minimal elements of $\mathcal{A}^+$, and $C_1, \ldots, C_e$ are the cominimal elements of $\mathcal{A}^+$.

Proof. If A is non cominimal, there exists a minimal element $M_{j_1} \leq A$; thus if $A_1 = A - M_{j_1}$ is not cominimal, we iterate the procedure. The chain $A \geq A_1 \geq A_3 \geq \cdots$ has to end with a zero, i.e., we get a decomposition of the form (4.25), or with a cominimal element, i.e., we get a decomposition of the form (4.26). $\square$

Remark 4.5.21. Note that it can happen that the number of minimal elements of $\mathcal{A}^+$ is not equal to the maximum number of $\mathbb{Q}$-linearly independent elements of $\mathcal{A}^+$. In fact, if we consider the submodule $\mathcal{A}$ of $\mathbb{Z}^4$ orthogonal to $(1, -1, -1, 1)^T$, and $\mathcal{A}^+$, such a maximum is clearly 3, but we have four minimal elements

$$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix},$$

and we need all of them (and no cominimal) to ensure (4.25) and (4.26).

Returning to our problem, if now we consider

$$\mathcal{A} = \{Q \in \mathbb{Z}^n \mid \langle Q, \eta^{(k)} \rangle = 0, \text{ for } k = 2, \ldots, r\},$$

it is easy to verify that

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) = \mathcal{B}_0^+ \cup \mathcal{B}_j^+$$

where

$$\mathcal{B}_0^+ = \{P \in \mathbb{N}^n \mid P = Q + e_j, Q \in \mathcal{A}^+, |Q| \geq 1\}$$

and

$$\mathcal{B}_j^+ = \{P \in \mathbb{N}^n \mid P = Q + e_j, Q \in \mathcal{A}, q_h \geq 0, \text{ for } h \neq j, \ q_j = -1, |Q| \geq 1\}.$$

Notice that $Q \in \mathcal{B}_j^+$ if and only if we have

$$\langle \widehat{\eta}^{(k)}, \widehat{Q} \rangle = \eta_j^{(k)} \quad \text{for } k = 2, \ldots, r \tag{4.27}$$

where $\widehat{Q} = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n) \in \mathbb{N}^{n-1}$ and $\widehat{\eta}^{(k)} = (\eta_1^{(k)}, \ldots, \eta_{j-1}^{(k)}, \eta_{j+1}^{(k)}, \ldots, \eta_n^{(k)})$, i.e., $\widehat{Q}$ is a solution in $\mathbb{N}^{n-1}$ of the linear system with integer coefficients (4.27). Moreover, since $\mathcal{A}$ is a submodule of $\mathbb{Z}^n$, Proposition 4.5.20 applies to $\mathcal{A}^+$. Let $\mathfrak{M} = \{M_1, \ldots, M_d\}$ be the set of minimal elements of $\mathcal{A}^+$ and let $\mathfrak{C} = \{C_1, \ldots, C_e\}$ be the set of cominimal elements of $\mathcal{A}^+$ (recall that they all are different from O, hence their modulus is at least 1). We can thus consider the subsets $\{M_1', \ldots, M_s'\} \subset \mathfrak{M}$ and $\{C_1', \ldots, C_t'\} \subset \mathfrak{C}$ of the minimal and cominimal elements R of $\mathcal{A}^+$ such that $\langle \eta^{(1)}, R\rangle \in \tau p\mathbb{Z}$. Then $[\varphi]$ can be simplified if and only if there exists $H \in \mathbb{Z}^n$ such that

$$\langle H, M_h'\rangle = \frac{1}{\tau p}\langle \eta^{(1)}, M_h'\rangle,$$

for $1 \le h \le s$,

$$\langle H, C_l'\rangle = \frac{1}{\tau p}\langle \eta^{(1)}, C_l'\rangle,$$

for $1 \le l \le t$, and such that, for any $j = 1, \ldots, n$, we have

$$\langle \widehat{H}, \widehat{Q}\rangle - h_j = \frac{1}{\tau p}\left(\langle \widehat{\eta}^{(1)}, \widehat{Q}\rangle - \eta_j^{(1)}\right),$$

for every solution $\widehat{Q} \in \mathbb{N}^n$ of (4.27), with $|Q| \ge 1$, such that $\langle \widehat{Q}, \widehat{\eta}^{(1)}\rangle \in \tau p\, \eta_j^{(1)}\mathbb{Z}$.

4.6 Construction of torus actions

In this last section we shall see some conditions assuring the existence of the torus actions we need.

It is possible to introduce *formal Poincaré-Dulac normal forms* and the *normalization problem* for *germs of holomorphic vector field of* $(\mathbb{C}^n, O)$ *with a singular point at the origin*, i.e., for local continuous dynamical systems. We refer to [Ar] pp. 180–191 for a more detailed exposition, and shall restrict ourselves to recall here the main facts that we shall need in the following.

Let $X \in \mathfrak{X}_n$ be a germ of holomorphic vector field of $(\mathbb{C}^n, O)$ singular at the origin, in Poincaré-Dulac normal form, i.e.,

$$X = X^{\mathrm{dia}} + X^{\mathrm{nil}} + X^{\mathrm{res}}$$

where, denoting with ∂_j the partial derivative $\partial/\partial z_j$,

$$X^{\mathrm{dia}} = \sum_{j=1}^{n} \varphi_j z_j \partial_j,$$

X^{nil} is a linear nilpotent vector field singular at the origin such that

$$[X^{\mathrm{dia}}, X^{\mathrm{nil}}] = 0,$$

X^{res} is a holomorphic vector field singular at the origin with no linear part and such that

$$[X^{\mathrm{dia}}, X^{\mathrm{res}}] = 0.$$

In particular

$$[X^{\mathrm{dia}}, X^{\mathrm{nil}} + X^{\mathrm{res}}] = 0.$$

A germ of holomorphic vector field X of $(\mathbb{C}^n, O)$ singular at the origin is *in Poincaré-Dulac normal form up to order* k if

$$X = X^{\mathrm{dia}} + X^{\mathrm{nil}} + \widetilde{X},$$

where X^{dia} and X^{nil} are as above, and $\widetilde{X}$ is a holomorphic vector field singular at the origin with no linear part and such that

$$[X^{\mathrm{dia}}, \widetilde{X}] = O(\|z\|^k).$$

Recall that the flows of two commuting vector fields also commute (see [Le] Prop. 18.5). We have

$$\exp(X^{\mathrm{dia}}) = \mathrm{Diag}(e^{\varphi_1}, \ldots, e^{\varphi_n})z.$$

and, in general for a linear vector field $X^{\mathrm{lin}} = \sum_{j=1}^n \left(\sum_{h=1}^n a_{hj} z_h\right) \partial_j$, we have

$$\exp(X^{\mathrm{lin}}) = e^A z,$$

where A is the matrix (a_{hj}). If Y is a holomorphic vector field singular at the origin with no linear part, then we have

$$\exp(tY)z = \sum_{k\geq 0} \frac{t^k}{k!} Y^k(z). \tag{4.28}$$

In fact, defining $K_t(z) = z + tY(z)$, we get $K_0(z) = z$ and $\frac{\partial}{\partial t} K_t(z)|_{t=0} = Y(z)$, then we have $\exp(tY)z = \lim_{m\to\infty} (K_{1/m})^m$, (see [AMR] Theorem 4.1.26), that is (4.28). Moreover, if V, W are two commuting vector fields, we have

$$\exp(t(V+W)) = \left(\sum_{k\geq 0} \frac{(tV)^k}{k!}\right)\left(\sum_{k\geq 0} \frac{(tW)^k}{k!}\right) = \exp(tV)\exp(tW).$$

Then we have the following result.

Proposition 4.6.1. *Let X be a germ of holomorphic vector field of $(\mathbb{C}^n, O)$, singular at the origin, and in Poincaré-Dulac normal form. Then its flow is a germ of biholomorphism of $(\mathbb{C}^n, O)$ in Poincaré-Dulac normal form.*

Proof. The flow of $X^{\mathrm{nil}} + X^{\mathrm{res}}$ is unipotent, then the linear part of the flow of X is Az with A triangular matrix with diagonal $\mathrm{Diag}(e^{\varphi_1}, \ldots, e^{\varphi_n})$, and the flow of X has to commute with the flow of X^{dia}. □

In [Zu1], Zung found that to find a Poincaré-Dulac holomorphic normalization for a germ of holomorphic vector field is the same as to find (and linearize) a suitable torus action which preserves the vector field. To deal with this problem he introduced the notion of *toric degree* of a vector field. The following definition is a reformulation of Zung's original one, clearer and more suitable to our needs.

Definition 4.6.1. The *toric degree* of a germ of holomorphic vector field X of $(\mathbb{C}^n, O)$ singular at the origin is the minimum $r \in \mathbb{N}$ such that the diagonalized semi-simple part

$X^{\mathrm{dia}} = \sum_{j=1}^{n} \varphi_j z_j \partial_j$ of the linear term of X can be written as linear combination with complex coefficients of r diagonal vector fields with integer coefficients, i.e.,

$$X^{\mathrm{dia}} = \sum_{k=1}^{r} \alpha_k Z_k,$$

where $\alpha_1, \ldots, \alpha_r \in \mathbb{C}^*$ and $Z_k = \sum_{j=1}^{n} \rho_j^{(k)} z_j \partial_j$ with $\rho^{(k)} \in \mathbb{Z}^n$. The r-tuple $Z_1, \ldots, Z_k$ is called a *r-tuple of toric vector fields associated to* X, and the numbers $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$ are a r-tuple of *toric coefficients* of the toric r-tuple.

In particular, we have

$$\varphi = \sum_{k=1}^{r} \alpha_k \rho^{(k)},$$

and, similarly to the case of germs of biholomorphisms, toric r-tuples of vector fields and their toric coefficients have to satisfy certain arithmetic properties, as the following result shows.

Lemma 4.6.2. *Let $X^{\mathrm{dia}} = \sum_{j=1}^{n} \varphi_j z_j \partial_j$ be a germ of semi-simple linear holomorphic vector field of $(\mathbb{C}^n, O)$ singular at the origin, of toric degree r, and let $Z_1, \ldots, Z_r$ be a r-tuple of toric vector fields associated to X^{dia} with toric coefficients $\alpha_1, \ldots, \alpha_r$ and $Z_k = \sum_{j=1}^{n} \rho_j^{(k)} z_j \partial_j$. Then:*

(i) $\alpha_1, \ldots, \alpha_r$ *is a set of rationally independent complex numbers;*

(ii) $Z_1, \ldots, Z_r$ *is a set of $\mathbb{Q}$-linearly independent vectors;*

(iii) for every $j = 1, \ldots, n$ we have

$$\mathrm{Res}_j^+(\varphi) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\rho^{(k)});$$

(iv) we have $\rho_j^{(k)} = \rho_h^{(k)}$ whenever $\varphi_j = \varphi_h$, for every $k = 1, \ldots, r$.

Proof. (i) Let us suppose by contradiction that $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$ are rationally dependent. Then there exists $(c_1, \ldots, c_r) \in \mathbb{Z}^r \setminus \{O\}$ such that

$$c_1 \alpha_1 + \cdots + c_r \alpha_r = 0.$$

Up to reordering we may assume $c_1 \neq 0$. Then

$$\alpha_1 = -\frac{1}{c_1}(c_2 \alpha_2 + \cdots + c_r \alpha_r),$$

and hence

$$\begin{aligned} X^{\mathrm{dia}} &= \sum_{k=1}^{r} \alpha_k Z_k \\ &= -\frac{1}{c_1}(c_2\alpha_2 + \cdots + c_r\alpha_r) Z_1 + \alpha_2 Z_2 + \cdots + \alpha_r Z_r \\ &= \frac{\alpha_2}{c_1}(c_1 Z_2 - c_2 Z_1) + \cdots + \frac{\alpha_r}{c_1}(c_1 Z_r - c_r Z_1), \end{aligned}$$

and this contradicts the definition of toric degree.

(ii) The proof is analogous to the previous one.

(iii) Let Q be in $\mathbb{N}^n$ with $|Q| \geq 2$ and le $1 \leq j \leq n$. We have

$$\langle Q, \varphi\rangle - \varphi_j = \sum_{k=1}^{r} \alpha_k \left(\langle Q, \rho^{(k)}\rangle - \rho_j^{(k)}\right) \tag{4.29}$$

and, since $\alpha_1, \ldots, \alpha_r$ are rationally independent, the right-hand side of (4.29) vanishes if and only if $\langle Q, \rho^{(k)}\rangle - \rho_j^{(k)} = 0$ for every $k = 1, \ldots, r$.

(iv) If $\varphi_j = \varphi_h$, then

$$\alpha_1 \rho_j^{(1)} + \cdots + \alpha_r \rho_j^{(r)} = \alpha_1 \rho_h^{(1)} + \cdots + \alpha_r \rho_h^{(r)},$$

hence

$$\alpha_1 \left(\rho_j^{(1)} - \rho_h^{(1)}\right) + \cdots + \alpha_r \left(\rho_j^{(r)} - \rho_h^{(r)}\right) = 0,$$

and, since $\rho_j^{(k)} - \rho_h^{(k)} \in \mathbb{Z}$ for $k = 1, \ldots, r$, the assertion follows from the rational independence of $\alpha_1, \ldots, \alpha_r$. □

Lemma 4.6.3. *Let $X^{\mathrm{dia}} = \sum_{j=1}^n \varphi_j z_j \partial_j$ be a germ of semi-simple linear holomorphic vector field of $(\mathbb{C}^n, O)$ singular at the origin. Then X^{dia} has toric degree 1 if and only if, chosen a non-zero eigenvalue of its linear part, all the other eigenvalues are rational multiplies of it. In particular we have uniqueness of the toric vector field associated to X^{dia} up to multiplication by a non-zero integer.*

Proof. Up to reorderings we may assume $\varphi_1 \neq 0$. The vector field X^{dia} has toric degree 1 if and only if there exist a non zero complex number α and a diagonal vector field with integer coefficients $Y = \sum_{j=1}^n m_j z_j \partial_j$ such that $X^{\mathrm{dia}} = \alpha Y$, that is

$$\varphi_j = \alpha m_j \quad \forall j = 1, \ldots, n,$$

which is equivalent to

$$\frac{\varphi_j}{\varphi_1} = \frac{m_j}{m_1} \quad \forall j = 1, \ldots, n,$$

and, since $m_j/m_1 \in \mathbb{Q}$ for each $j = 1, \ldots, n$, this concludes the proof. □

We shall use the following definitions, that are a generalization of the one of [Zu1].

Definition 4.6.2. Let $1 \leq m \leq n$. A *set of m integrable vector fields* of $(\mathbb{C}^n, O)$ is a set $X_1, \ldots, X_m$ of germs of holomorphic vector fields of $(\mathbb{C}^n, O)$ singular at the origin, and such that:

(i) $X_1, \ldots, X_m$ commute pairwise and are linearly independent, i.e., $X_1 \wedge \cdots \wedge X_m \not\equiv 0$;

(ii) there exist $n - m$ germs of holomorphic functions $g_1, \ldots, g_{n-m}$ in $(\mathbb{C}^n, O)$ which are common first integrals of $X_1, \ldots, X_m$, i.e., $X_j(g_k) = 0$ for any j and k, and they are functionally independent almost everywhere, i.e., $\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_{n-m} \not\equiv 0$.

The *minimal* [resp. *maximal*] *order* of the set is the minimum [resp. maximum] of the orders of vanishing at the origin of the vector fields in the set; when all the vector fields have the same order of vanishing at the origin we simply call it the *order* of the set.

Definition 4.6.3. A germ of holomorphic vector field X of $(\mathbb{C}^n, O)$ singular at the origin is said *integrable* if there exists a positive integer $1 \le m \le n$ such that X belongs to a set of m integrable vector fields.

Theorem 4.6.4. (Zung, 2002 [Zu1]) *Let X be a germ of holomorphic vector field of $(\mathbb{C}^n, O)$ singular at the origin with non-nilpotent linear part which is integrable. Then X admits a holomorphic Poincaré-Dulac normalization.*

As a corollary of Proposition 4.6.1, we obtain

Corollary 4.6.5. *The flow of a germ of integrable holomorphic vector field of $(\mathbb{C}^n, O)$ singular at the origin and with non-nilpotent linear part admits a holomorphic Poincaré-Dulac normalization.*

Moreover we have the following result

Theorem 4.6.6. (Zung, 2002 [Zu1]) *Let $1 \le m \le n$. Every set of m integrable vector fields of order of vanishing at the origin 1 and with non-nilpotent linear parts admits a simultaneous holomorphic Poincaré-Dulac normalization.*

Thus we have the following corollary

Corollary 4.6.7. *Let $1 \le m \le n$. The flows of a set of m integrable vector fields of order of vanishing at the origin 1 with non-nilpotent linear parts admit a simultaneous holomorphic Poincaré-Dulac normalization.*

Remark 4.6.8. Theorem 4.6.6 means that we can conjugate $X_1, \dots, X_m$ to a m-tuple of vector fields containing only monomials belonging to the intersection of the additive resonances of the eigenvalues of the linear terms of $X_1, \dots, X_m$.

Now, we introduce an analogous for germs of biholomorphisms of the notion of integrability we described above.

Definition 4.6.4. A germ of biholomorphism f of $(\mathbb{C}^n, O)$ fixing the origin *commutes with a set of integrable vector fields* if there exists a positive integer $1 \le m \le n$, such that there exists a set of m germs of holomorphic integrable vector fields $X_1, \dots, X_m$ such that

$$\mathrm{d}f(X_j) = X_j \circ f$$

for each $j = 1, \dots, m$.

Remark 4.6.9. A germ of biholomorphism f of $(\mathbb{C}^n, O)$ commutes with a vector field X according to the previous definition if and only if it commutes with the flow generated by X.

In the following we shall need the following results of Zung [Zu1] that we report here with their proof, since we are using a different (but equivalent) definition for the toric degree. Next result also shows that the vector field case is similar to the torsion-free case but simpler.

Lemma 4.6.10. *Let $X = X^{\mathrm{dia}} + \widetilde{X}$ be a germ of holomorphic vector field of $(\mathbb{C}^n, O)$ singular at the origin, with $X^{\mathrm{dia}} = \sum_{j=1}^n \varphi_j z_j \partial_j$ of toric degree r, in Poincaré-Dulac normal form up to order $k \ge 1$ and let $Z_1, \dots, Z_r$ be a toric r-tuple of vector fields associated to X. Then*

(i) if Y is a germ of holomorphic vector field of $(\mathbb{C}^n, O)$ commuting with X, then we have $[Z_h, Y] = O(\|z\|^k)$ for each $h = 1, \dots, r$;

(ii) if g is a germ of holomorphic function of $(\mathbb{C}^n, O)$ such that $X(g) = 0$, then we have $Z_h(g) = O(\|z\|^k)$ for every $h = 1, \dots, r$.

Moreover, if X is in Poincaré-Dulac normal form, Y commutes with X and $X(g) = 0$, then Y commutes with each Z_h, and $Z_h(g) = 0$ for any $h = 1, \dots, r$.

Proof. (i) Let Y be a germ of holomorphic vector field of $(\mathbb{C}^n, O)$ commuting with X. Then we have

$$\pi_k\left([X,Y]\right) = 0, \tag{4.30}$$

where π_k is the projection on the space of k-jets of holomorphic vector field of $(\mathbb{C}^n, O)$ and it verifies

$$\pi_k\left([X,Y]\right) = [\pi_k(X), \pi_k(Y)]. \tag{4.31}$$

Since X is in Poincaré-Dulac normal form up to order k, the vector field $\pi_k(X^{\mathrm{dia}})$ is semi-simple; thus thanks to the uniqueness of the Jordan-Chevalley decomposition in finite-dimensional vector spaces, the semi-simple part of $\mathrm{ad}(\pi_k(X))$ coincides with $\mathrm{ad}(\pi_k(X^{\mathrm{dia}}))$, and hence $\pi_k(X^{\mathrm{dia}})$ coincides with the semi-simple part of the vector field $\pi_k(X)$. Furthermore, the semi-simple part if $\mathrm{ad}(\pi_k(X))$ is well-known to be a polynomial in $\mathrm{ad}(\pi_k(X))$, hence, by equations (4.30) and (4.31), we have

$$\begin{aligned} 0 &= [\pi_k(X)^{\mathrm{dia}}, \pi_k(Y)] \\ &= [\pi_k(X^{\mathrm{dia}}), \pi_k(Y)] \\ &= \pi_k([X^{\mathrm{dia}}, Y]), \end{aligned}$$

that is $[X^{\mathrm{dia}}, Y] = O(\|z\|^k)$, which means that all the monomials of Y of degree $l \leq k-1$ are resonant, and we get the thesis by Lemma 4.6.2.

Moreover, if X is in Poincaré-Dulac normal form and it commutes with Y, we have $[X^{\mathrm{dia}}, Y] = O(\|z\|^k)$ for every positive k, hence Y commutes with X^{dia} and therefore with each Z_h.

(ii) Each germ of holomorphic vector field of $(\mathbb{C}^n, O)$ singular at the origin is a derivation on the space $\mathbb{C}_O\{z_1, \ldots, z_n\}$ of germs holomorphic function of $(\mathbb{C}^n, O)$, hence it acts linearly on $\mathbb{C}_O\{z_1, \ldots, z_n\}$. Let $g \in \mathbb{C}_O\{z_1, \ldots, z_n\}$ be such that $X(g) = 0$. Then we have

$$\rho_k(X(g)) = 0,$$

where ρ_k is the projection of $\mathbb{C}_O\{z_1, \ldots, z_n\}$ over the space of k-jets J_n^k of germs of holomorphic functions of $(\mathbb{C}^n, O)$; as above we have

$$\rho_k(X(g)) = \pi_k(X)(\rho_k(g)).$$

In fact, if $X = \sum_{j,P} X_{j,P}\partial_j$, for any monomial $f_Q z^Q$ of g we have

$$\begin{aligned} \rho_k(X(z^Q)) &= \sum_{j=1}^n \sum_{|Q+P-e_j| \leq k} X_{j,P} g_Q q_j \, z^{Q+P-e_j} \\ &= \sum_{j=1}^n \sum_{\substack{|P|\leq k, |Q| \leq k \\ |Q+P-e_j| \leq k}} X_{j,P} g_Q q_j \, z^{Q+P-e_j}, \end{aligned} \tag{4.32}$$

where in the last equality we used that $|Q+P-e_j| \leq k$ yields $|P| \leq k$ e $|Q| \leq k$; moreover, if az^Q is a monomial with $|Q| \leq k$ and $a \in \mathbb{C}$, we have

$$\pi_k(X)(az^Q) = \sum_{j=1}^n \sum_{\substack{|P|\leq k, |Q| \leq k \\ |Q+P-e_j| \leq k}} X_{j,P} \, a \, q_j \, z^{Q+P-e_j}. \tag{4.33}$$

Since each monomial of $\rho_k(g)$ is of the form $g_Q z^Q$ with $|Q| \leq k$, by (4.32) and (4.33) we get $\rho_k(X(g)) = \pi_k(X)(\rho_k(g))$.

It follows that

$$\begin{aligned} 0 &= \pi_k(X)^{\mathrm{dia}}(\rho_k(g)) \\ &= \pi_k(X^{\mathrm{dia}})(\rho_k(g)) \\ &= \rho_k(X^{\mathrm{dia}}(g)), \end{aligned}$$

that is $X^{\mathrm{dia}}(g) = O(\|z\|^k)$, thus, for any $h = 1, \ldots, n$, we have $Z_h(g) = O(\|z\|^k)$.

If, moreover, X is in Poincaré-Dulac normal form and $X(g) = 0$, then for any positive k, we have $X^{\mathrm{dia}}(g) = O(\|z\|^k)$, implying $X^{\mathrm{dia}}(g) = 0$. Hence, writing $g = \sum_{|Q|\geq 0} g_Q z^Q$, we get

$$\begin{aligned} 0 &= X^{\mathrm{dia}}(g) \\ &= \sum_{h=1}^{r} \alpha_h Z_h(g) \\ &= \sum_{h=1}^{r} \alpha_h \left(\sum_{j=1}^{n} \rho_j^{(h)} z_j \partial_j(g) \right). \end{aligned} \tag{4.34}$$

Since, we have

$$\begin{aligned} \sum_{j=1}^{n} \rho_j^{(h)} z_j \partial_j(g) &= \sum_{j=1}^{n} \rho_j^{(h)} z_j \partial_j \left(\sum_{|Q|\geq 0} g_Q z^Q \right) \\ &= \sum_{j=1}^{n} \rho_j^{(h)} z_j \sum_{|Q|\geq 0} g_Q q_j \frac{z^Q}{z_j} \\ &= \sum_{|Q|\geq 0} g_Q z^Q \sum_{j=1}^{n} \rho_j^{(h)} q_j \\ &= M_h g, \end{aligned}$$

with $M_h \in \mathbb{Z}$, equation (4.34) becomes

$$0 = \left(\sum_{h=1}^{r} \alpha_h M_h \right) g,$$

and we get the thesis from the rational independence of $\alpha_1, \ldots, \alpha_r$. □

Lemma 4.6.11. (Zung, 2005 [Zu2]) *Let (ε_d) be a sequence of positive numbers converging to 0 and let S be a complex analytic subset of $\mathbb{C}^n$ containing the origin, of complex codimension $d \geq 1$. Then any bounded holomorphic function defined on $U = \bigcup_{d=1}^{\infty} U_d$, where U_d is the set $\{z \in \mathbb{C}^n : \|z\| < \varepsilon_d,\ d(z,S) > \|z\|^d\}$, admits a holomorphic extension to a whole neighbourhood of the origin in $\mathbb{C}^n$.*

Theorem 4.6.12. *Let f be a germ of biholomorphism of $(\mathbb{C}^n, O)$ fixing the origin. Let f commute with a set of n integrable holomorphic vector fields $X_1, \ldots, X_n$ of minimal order 1 such that X_j has non-nilpotent linear part for some $j \in \{1, \ldots, m\}$. Then f commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension equal to the toric degree r*

of X_j and such that the columns of the weight matrix of the action are a r-tuple of toric vectors associated to X_j.

Proof. Let $\ell \geq 1$ be the maximal order of the set $X_1, \dots X_n$. Up to reordering we may assume $j = 1$. Let us fix a holomorphic system of coordinates $z = (z_1, \dots, z_n)$ in a neighbourhood of the origin of $\mathbb{C}^n$ in which X_1 is in Poincaré-Dulac normal form up to order $D \in \mathbb{N}$, with $D \geq 2\ell$ sufficiently large. Let

$$Z^D = \sum_{j=1}^{n} i\rho_j z_j \partial_j$$

be a toric vector field associated to X_1. Since $[X_1, X_h] = 0$ for $h = 2, \dots, n$, from Lemma 4.6.10, we have

$$[Z^D, X_j] = O(\|z\|^D), \tag{4.35}$$

for $j = 1, \dots, n$. Thanks to the hypotheses, there exist $a_1^D, \dots, a_n^D \colon \mathbb{C}^n \to \mathbb{C}$ holomorphic functions such that

$$Z^D(z) = \sum_{j=1}^{n} a_j^D(z) X_j(z);$$

then (4.35) implies, since the maximal order of the set $X^1, \dots, X_n$ is ℓ, that

$$\left\| a_j^D(z) - a_j^D(0) \right\| = O(\|z\|^{D-2\ell+1}).$$

Now we normalize X_1 up to order $D+1$ via a holomorphic, tangent to the identity, change of coordinate $w = \varphi^{D+1}(z)$ in a neighbourhood of O (we can always do it, up to shrinking the neighbourhood, and $\varphi^{D+1} - \mathrm{Id}$ will be of order $D+1$). Setting

$$Z^{D+1}(w) = \sum_{j=1}^{n} i\rho_j w_j \frac{\partial}{\partial w_j},$$

as before, from Lemma 4.6.10, we have

$$[Z^{D+1}, X_j](w) = O(\|w\|^{D+1}), \tag{4.36}$$

for $j = 1, \dots, n$, and there exist $a_1^{D+1}, \dots, a_n^{D+1} \colon \mathbb{C}^n \to \mathbb{C}$ holomorphic functions such that

$$Z^{D+1}(w) = \sum_{j=1}^{n} a_j^{D+1}(w) X_j(w),$$

with

$$\left\| a_j^{D+1}(w) - a_j^{D+1}(0) \right\| = O(\|w\|^{D-2\ell+2}).$$

In the new coordinates, writing $\varphi^{D+1} = \mathrm{Id} + \widehat{\varphi}^{D+1}$, we have

$$\begin{aligned} Z^D(w) &= \sum_{j=1}^{n} \left(i\rho_j w_j + i\rho_j \widehat{\varphi}_j^{D+1}(w) \right) \sum_{k=1}^{n} \frac{\partial w_k}{\partial z_j} \frac{\partial}{\partial w_k} \\ &= \sum_{k=1}^{n} \left(i\rho_k w_k + \psi_k(w) \right) \frac{\partial}{\partial w_k} \end{aligned}$$

where $\psi_k(w) = O(\|w\|^{D+1})$. Moreover, the Z^D are defined in a uniform neighbourhood of the origin because they are obtained by polynomial changes of variables. We also have

$$Z^D(w) = \sum_{j=1}^{n} a_j^D(w) X_j(w),$$

with

$$\left\| a_j^D(w) - a_j^D(0) \right\| = O(\|w\|^{D-2l+1}). \tag{4.37}$$

Then, since Z^{D+1} coincides with Z^D up to order D,

$$\begin{aligned} \left\| Z^{D+1}(w) - Z^D(w) \right\| &= \left\| \sum_{j=1}^{n} (a_j^{D+1}(w) - a_j^D(w)) X_j(w) \right\| \\ &= \left\| \sum_{j=1}^{n} -\psi_k(w) \frac{\partial}{\partial w_k} \right\| \\ &= O(\|w\|^{D+1}) \end{aligned}$$

thus

$$\left\| a_j^{D+1}(w) - a_j^D(w) \right\| = O(\|w\|^{D-2\ell+1}) \tag{4.38}$$

for $j = 1, \ldots, n$. Then $a_j^D(0) = a_j(0)$ does not depend on D for any $j = 1, \ldots, n$. Set

$$Z = \sum_{j=1}^{n} a_j(0) X_j.$$

The holomorphic vector field Z is 2π-periodic because, from (4.37), it is arbitrarily close to a 2π-periodic vector field. Then we have

$$\begin{aligned} \mathrm{d}f(Z) &= \mathrm{d}f\left(\sum_{j=1}^{n} a_j(0) X_j \right) \\ &= \sum_{j=1}^{n} a_j(0) \mathrm{d}f(X_j) \\ &= \sum_{j=1}^{n} a_j(0) X_j \circ f \\ &= Z \circ f. \end{aligned}$$

It is evident that we can apply the same procedure to any toric vector field associated to X_1, hence we get r 2π-periodic germs of holomorphic vector fields, which are linearly independent, commute pairwise and with f, and such that their linear parts form a r-tuple of toric vectors associated to the linear semi-simple part of X_1, implying the thesis. $\square$

Theorem 4.6.13. *Let f be a germ of biholomorphism of $(\mathbb{C}^n, O)$ fixing the origin and commuting with a set of integrable holomorphic vector fields $X_1, \ldots, X_m$ such that X_j has non-nilpotent linear part for some $j \in \{1, \ldots, m\}$. Then f commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension equal to the toric degree r of X_j and such that the columns of the weight matrix of the action are a r-tuple of toric vectors associated to X_j.*

Proof. Up to reordering we may assume $j = 1$. We dealt with the case $n = m$ in Theorem 4.6.12. Let us now consider the case $1 \le m < n$.

Let us fix a holomorphic system of coordinates $z = (z_1, \ldots, z_n)$ in a neighbourhood of the origin of $\mathbb{C}^n$, a standard Hermitian metric in $\mathbb{C}^n$ and a positive sufficiently small number ε_0. Let S be the singular locus of the n-tuple of vector fields $X_1, \ldots, X_m$ and of the functions $g_1, \ldots, g_{n-m}$, i.e.,

$$\{z \in \mathbb{C}^n : \|z\| < \varepsilon_0,\ X_1 \wedge \cdots \wedge X_m(z) = 0\} \cup \{z \in \mathbb{C}^n : \|z\| < \varepsilon_0,\ \mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_{n-m}(z) = 0\}.$$

Thanks to the hypotheses, S is a complex analytic set of complex codimension at least 1; then it is possible to write it locally as the zero locus of a finite number of complex holomorphic functions, $S = \{h_1 = 0, \ldots, h_l = 0\}$, and, using Łojasiewicz inequalities (see [Ło] pp. 242–245), there exist a positive integer $N > 0$ and a positive constant $C > 0$ such that, for any z with $\|z\| < \varepsilon_0$ we have the following Łojasiewicz inequalities

$$\begin{aligned} \|X_1 \wedge \cdots \wedge X_m(z)\| &\ge C\, d(z, S)^N \\ \|\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_{n-m}(z)\| &\ge C\, d(z, S)^N, \end{aligned} \tag{4.39}$$

where the norms are the standard norms on the considered spaces and the distance is the Euclidean distance.

For each positive integer d and small positive number $\varepsilon(d)$ (which shall be chosen later in function of d with $\lim_{d\to\infty} \varepsilon(d) = 0$), let us define the following open subset of $\mathbb{C}^n$

$$U_{d,\varepsilon(d)} = \{z \in \mathbb{C}^n : \|z\| < \varepsilon(d),\ d(z, S) > \|z\|^d\}.$$

We will define a holomorphic vector field $\mathcal{Z}$ in $U_{d,\varepsilon(d)}$, periodic with period 2π, and in such a way that, for any two positive distinct integers d_1, d_2, the vector field $\mathcal{Z}$ defined in $U_{d_1,\varepsilon(d_1)}$ coincides, in the intersection $U_{d_1,\varepsilon(d_1)} \cap U_{d_2,\varepsilon(d_2)}$, with the one defined in $U_{d_2,\varepsilon(d_2)}$.

Up to holomorphic, tangent to the identity, changes of coordinates, we may assume X_1 to be in Poincaré-Dulac normal form up to order $D(d) \in \mathbb{N}$, with $D(d) = 4d\ell N + 2 \ge 2\ell$, where $\ell \ge 1$ is the maximal order of the set $X_1, \ldots, X_m$, (in particular $\lim_{d\to\infty} D(d) = +\infty$). Let

$$Z^d = \sum_{j=1}^n i\rho_j z_j \partial_j$$

be a toric vector field associated to X_1. Since $[X_1, X_h] = 0$ for $h = 2, \ldots, n$, from Lemma 4.6.10, we have

$$[Z^d, X_j] = O(\|z\|^{D(d)}),$$

for any $j = 1, \ldots, m$, and

$$Z^d(\mathbf{g})(z) = O(\|z\|^{D(d)})$$

where $\mathbf{g} = (g_1, \ldots, g_{n-m})$ is the $(n - m)$-tuple of common first integrals of $X_1, \ldots, X_m$.

Let y be an arbitrary point in $U_{d,\varepsilon(d)}$. Then, thanks to inequalities (4.39) and to the definition of $U_{d,\varepsilon(d)}$, we have

$$\begin{aligned} &\|X_1 \wedge \cdots \wedge X_m(y)\| \geq C\,\|y\|^{dN} \\ &\|\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_{n-m}(y)\| \geq C\,\|y\|^{dN}. \end{aligned} \tag{4.40}$$

Let us denote by $\Gamma^d(t,y) = \Gamma^d(t)$ the closed curve, $t \in [0, 2\pi]$, which is the orbit of the periodic vector field Z^d starting at y. Then we have $\Gamma^d(0) = y$ and, for $\varepsilon(d)$ small enough, we have $\frac{1}{2}\|y\| \leq \|\Gamma^d(t)\| \leq 2\|y\|$ for any t in $[0, 2\pi]$. Then, for any x in Γ^d we have

$$\begin{aligned} &\|X_1 \wedge \cdots \wedge X_m(x)\| > \frac{C}{2^{dN}}\,\|y\|^{dN} \\ &\|\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_{n-m}(x)\| > \frac{C}{2^{dN}}\,\|y\|^{dN}. \end{aligned} \tag{4.41}$$

Since Z^d commutes with $X_1, \ldots, X_m$ up to order $D(d)$ and $\mathbf{g}$ is a first integral of Z^d up to order $D(d)$, for $\varepsilon(d)$ small, we have the following inequalities

$$\begin{aligned} &\|\mathbf{g}(x) - \mathbf{g}(y)\| < \|y\|^{D_1(d)} \\ &\|[X_j, Z^d](x)\| < \|y\|^{D_1(d)} \quad \forall j = 1, \ldots, m \end{aligned} \tag{4.42}$$

for any x belonging to Γ^d_k, where $D_1(d) = dN + 3$, (which is larger than $dN + 2$ and verifies $D_1(d) < D(d) - 1 = 4dN + 1$ for every d). In fact

$$\|\mathbf{g}(x) - \mathbf{g}(y)\| \leq C_1\|Z^d(\mathbf{g})(y)\| \leq C_2\|y\|^{D(d)} < \|y\|^{D_1(d)}$$

and, for any $j = 1, \ldots, m$, we have

$$\|[X_j, Z^d](x)\| \leq C_3\|x\|^{D(d)} \leq 2^{D(d)}C_3\|y\|^{D(d)} < \|y\|^{D_1(d)}.$$

The inequalities (4.41) and (4.42) imply the following facts:

a) For any point y the regular part of the level set $L_y = \mathbf{g}^{-1}(\mathbf{g}(y))$ has complex dimension m, and its tangent space at each point is spanned by $X_1, \ldots, X_m$. Moreover, the regular part of L_y has an affine flat structure given by the vector fields $X_1, \ldots, X_m$, because they commute.

b) The curve Γ^d can be projected orthogonally on a smooth closed curve $\widehat{\Gamma}^d(t)$ lying on L_y and close to Γ^d in the C^1-topology: the distance between $\widehat{\Gamma}^d$ and Γ^d in the C^1-topology is bounded from above by $\|y\|^{D_2(d)}$, where $D_2(d) = dN + 1$.

c) We can write $\mathrm{d}\widehat{\Gamma}^d(t)/\mathrm{d}t$ in the form $\sum_{j=1}^m \mathrm{Re}(a^j(t)X_j(\widehat{\Gamma}^d(t)))$, and the holomorphic functions $a^j(t)$ are almost constant, in the sense that

$$\left\|a^j(t) - a^j_y(0)\right\| \leq \|y\|^{D_3(d)},$$

for $t \in [0, 2\pi]$, where $D_3(d)$ is positive, for example $D_3(d) = D_2(d) - 1 = dN$. This follows from the almost commutativity of $X_1, \ldots, X_m$ with Z^d and from the fact that, thanks to b), we have $\|\mathrm{d}\widehat{\Gamma}^d(t)/\mathrm{d}t - \mathrm{Re}(Z^d(\widehat{\Gamma}^d_k(t)))\| < \|y\|^{D_2(d)}$. In fact, since $X_1, \ldots, X_m$ commute, in a suitable system of coordinates $z_1, \ldots, z_n$ we may assume that each X_j coincides with ∂_j

for $j = 1, \ldots, m$. Writing Z^d in the form $\sum_{j=1}^n \zeta_j(z)\partial_j$ in these coordinates, since Z^d almost commutes with $X_1, \ldots, X_m$ and it is almost tangent to the level sets, the functions $\zeta_1, \ldots, \zeta_m$, are almost constant along the chosen orbit of Z^d, whereas $\zeta_{m+1}, \ldots, \zeta_n$ are almost zero. Projecting on the level set those functions remain almost constant.

d) Arguing analogously to what we did in the proof of Theorem 4.6.12, there exist complex numbers $a^1, \ldots, a^m$ such that $\|a^j - a^j_y(0)\| \leq \|y\|^{D_3(d)}$, and the time-$2\pi$ flow of the vector field $\sum_{j=1}^m a^j X_j$ in L_y fixes y. Then the real vector field $\mathrm{Re}(\sum_{j=1}^m a^j X_j)$ has a periodic orbit of period 2π passing through y, and this orbit is C^1-close to $\widehat{\Gamma}^d(t, y)$.

e) Thanks to the affine flat structure of L_y, the numbers $a^1, \ldots, a^m$ are well-defined, i.e., unique, and they do not depend, at least locally, on the choice of y in L_y. We can consider $a^1, \ldots, a^m$ as functions of y: $a^1(y), \ldots, a^m(y)$. These functions are holomorphic, due to the holomorphic implicit function theorem, constant on the connected components in $U_{d,\varepsilon(d)}$ of the level sets of $\mathbf{g}$, and they are uniformly bounded in $U_{d,\varepsilon(d)}$ by a constant, provided that $\varepsilon(d)$ is small enough.

Let us now define the vector field $\mathcal{Z}$ as follows

$$\mathcal{Z}(y) = \sum_{j=1}^m a^j(y) X_j(y).$$

Then $\mathcal{Z}$ is a holomorphic vector field in $U_{d,\varepsilon(d)}$ with the following properties:

(a) $\mathcal{Z}$ is uniformly bounded by a constant, and it is periodic with period 2π, at least in an open subset of $U_{d,\varepsilon(d)}$.

(b) If $\mathcal{Z}$ is a vector field defined as above for $U_{d,\varepsilon(d)}$, and $\mathcal{Z}'$ is another vector field defined as above but for $U_{d',\varepsilon(d')}$, with $d \neq d'$, then $\mathcal{Z}$ and $\mathcal{Z}'$ coincide in $U_{d,\varepsilon(d)} \cap U_{d',\varepsilon(d')}$. In fact, the vector field $\mathcal{Z}$ commutes with $\mathcal{Z}'$ on $U_{d,\varepsilon(d)} \cap U_{d',\varepsilon(d')}$ by construction, and $\mathcal{Z} - \mathcal{Z}'$ is tangent to the level sets of $\mathbf{g}$ in $U_{d,\varepsilon(d)} \cap U_{d',\varepsilon(d')}$ and it is a constant vector field with respect to the affine flat structure on each level set. Moreover $\mathcal{Z} - \mathcal{Z}'$ is periodic of period 2π on the considered intersection; but the coefficients of $\mathcal{Z} - \mathcal{Z}'$, when they are written as a linear combination of $X_1, \ldots, X_m$, are bounded from above by $\|y\|^{\min(D_3(d), D_3(d'))}$, therefore $\mathcal{Z} - \mathcal{Z}'$ is too small to be 2π-periodic unless it is zero. Thus $\mathcal{Z} = \mathcal{Z}'$ in $U_{d,\varepsilon(d)} \cap U_{d',\varepsilon(d')}$.

We have then defined a bounded holomorphic vector field $\mathcal{Z}$ on the open set

$$U = \bigcup_{d=1}^{\infty} U_{d,\varepsilon(d)},$$

which is constant on each L_y with respect to the affine flat structure. Moreover $\mathcal{Z}$ is 2π-periodic, and there exist $a_1, \ldots, a_m$ holomorphic functions constant on the connected components of each level set, such that

$$\mathcal{Z} = \sum_{j=1}^m a_j X_j$$

in U. Then

$$\begin{aligned}
\mathrm{d}f(\mathcal{Z}) &= \mathrm{d}f\left(\sum_{j=1}^{m} a_j X_j\right) \\
&= \sum_{j=1}^{m} (a_j \circ f)\, \mathrm{d}f(X_j) \\
&= \sum_{j=1}^{m} (a_j \circ f)\, (X_j \circ f) \\
&= \mathcal{Z} \circ f.
\end{aligned}$$

Applying Lemma 4.6.11, there exists a holomorphic vector field, defined in a whole neighbourhood of the origin, coinciding with $\mathcal{Z}$ on U.

It is evident that we can apply the same procedure to any toric vector field associated to X_1. Hence get r germs of holomorphic 2π-periodic vector fields $\mathcal{Z}_1, \ldots, \mathcal{Z}_r$ which are linearly independent, commute pairwise and with each X_j, and thus they generate a $\mathbb{T}^r$ action preserving $X_1, \ldots, X_m$. Moreover for each $k = 1, \ldots, r$, there exist $a_{1,k}, \ldots, a_{m,k}$ germs of holomorphic functions of $(\mathbb{C}^n, O)$, constant on the connected components of each level set $L_y = \mathbf{g}^{-1}(\mathbf{g}(y))$, where we denote by $\mathbf{g} = (g_1, \ldots, g_{n-m})$ the $(n-m)$-tuple of common first integrals of $X_1, \ldots, X_m$, such that

$$\mathcal{Z}_k = \sum_{j=1}^{m} a_{j,k} X_j,$$

for each $k = 1, \ldots, r$. Then, for each $k = 1, \ldots, r$, we have

$$\begin{aligned}
\mathrm{d}f(\mathcal{Z}_k) &= \mathrm{d}f\left(\sum_{j=1}^{m} a_{j,k} X_j\right) \\
&= \sum_{j=1}^{m} (a_{j,k} \circ f)\, \mathrm{d}f(X_j) \\
&= \sum_{j=1}^{m} (a_{j,k} \circ f)\, (X_j \circ f) \\
&= \mathcal{Z}_k \circ f.
\end{aligned}$$

Thus the torus action commutes with f as we wanted, and this concludes the proof. □

Corollary 4.6.14. *Let f be a germ of biholomorphism of $(\mathbb{C}^n, O)$ fixing the origin and commuting with a set $X_1, \ldots, X_m$ of m integrable holomorphic vector fields of order of vanishing at the origin 1, with non-nilpotent linear parts. Then f is holomorphically conjugated to a germ containing only monomials belonging to the intersection of the additive resonances of the eigenvalues of the linear terms of $X_1, \ldots, X_m$.*

Proof. It follows from the previous proof that, for each X_j, we can find r holomorphic periodic vector fields, such that their linear terms form a r-tuple of toric vector fields associated to X_j, which commute pairwise, are linearly independent, and they commute with f. Then the assertion follows from Corollary 4.6.7 and Theorem 4.1.1. □

Then we also have the following

Corollay 4.6.15. *Let f be a germ of biholomorphism of $(\mathbb{C}^n, O)$ fixing the origin and commuting with a set $X_1, \dots, X_m$ of m integrable holomorphic vector fields of order of vanishing at the origin 1, with non-nilpotent linear parts, and such that the intersection of the additive resonances of the eigenvalues of the linear terms of $X_1, \dots, X_m$ is equal or contained in the set of resonances of the spectrum of $\mathrm{d}f_O$. Then f admits a holomorphic Poincaré-Dulac normalization.*

References

[A1] M. Abate: *Diagonalization of non-diagonalizable discrete holomorphic dynamical systems*. Amer. J. Math. **122** (2000), 757–781.

[A2] M. Abate: *The residual index and the dynamics of holomorphic maps tangent to the identity*. Duke Math. J. **107** (2001), 173–207.

[A3] M. Abate: **An introduction to hyperbolic dynamical systems.** I.E.P.I. Pisa, 2001.

[A4] M. Abate: *Basins of attraction in quadratic dynamical systems with a Jordan fixed point*. Nonlinear Anal. **51** (2002), 271–282.

[A5] M. Abate: *Discrete holomorphic local dynamical systems*. to appear in **Holomorphic Dynamical Systems**, Eds. G. Gentili, J. Guenot, G. Patrizio, **Lectures notes in Math.**, Springer-Verlag, Berlin, 2009.

[AT1] M. Abate, F. Tovena: *Parabolic curves in* $\mathbb{C}^3$. Abstr. Appl. Anal. **2003** (2003), 275–294.

[AT2] M. Abate, F. Tovena: *Formal classification of holomorphic maps tangent to the identity*. Disc. Cont. Dyn. Sys. **Suppl.** (2005), 1–10.

[AT3] M. Abate, F. Tovena: *Poincaré-Bendixson theorems for meromorphic connections and homogeneous vector fields*. Preprint, Institut Mittag-Leffler, 2008.

[ABT1] M. Abate, F. Bracci, F. Tovena: *Index theorems for holomorphic self-maps*. Ann. of Math. **159** (2004), 819–864.

[ABT2] M. Abate, F. Bracci, F. Tovena: *Index theorems for holomorphic maps and foliations*. Indiana Univ. Math. J., 57 2008 2999–3048.

[AM] A. Abbondandolo, P. Majer: *On the global stable manifold*. Studia Math. **177** (2006), 113–131.

[AMR] R. Abraham, J. E. Marsden, T. Ratiu: **Manifolds, tensor analysis, and applications.** Second edition, Applied Mathematical Sciences, **75**, Springer-Verlag, New York, 1988.

[Ar] V.I. Arnold: **Geometrical methods in the theory of ordinary differential equations.** Springer-Verlag, Berlin, 1988.

[BCS] D. Bao, S.-S. Chern, Z. Shen: **An introduction to Riemann-Finsler geometry.** Graduate Texts in Mathematics **200**, Springer-Verlag, New York, 2000.

[BER] M. S. Baouendi, P. Ebenfelt, L. P. Rothschild: *Dynamics of the Segre varieties of a real submanifold in complex space*. J. Algebraic Geom. **12** (2003), no. 1 81–106.

[BMR] M. S. Baouendi, N. Mir, L. P. Rothschild: *Reflection ideals and mappings between generic submanifolds in complex space*. J. Geom. Anal. **12** (2002), no. 4 543–580.

[BK] E. Bedford, K. Kim: *Dynamics of Rational Surface Automorphisms: Rotation Domains*. Preprint, arXiv: 0907.3339v1, 2009.

[B] F. Berteloot: *Méthodes de changement d'échelles en analyse complexe*. Ann. Fac. Sci. Toulouse Math. **(6)** (2006), 15 no. 3 427–483.

[Bi] L. Bieberbach: *Beispiel zweier ganzer Funktionen zweier komplexer Variablen, welche eine schlichte volumtreue Abbildung des R*. S. B. Preuss Akad. Wiss. **14/15** (1933), 476–479.

[Bis1] K. Biswas: *Smooth combs inside hedgehogs*. Discrete Contin. Dyn. Syst. **12** (2005), 853–880.

[Bis2] K. Biswas: *Complete conjugacy invariants of nonlinearizable holomorphic dynamics*. Preprint, University of Pisa, 2007.

[Bo] S. Bochner: *Compact Groups of Differentiable Transformation*. Annals of Mathematics **(2)** (1945), **46**, **3** 372–381.

[Bö] L.E. Böttcher: *The principal laws of convergence of iterates and their application to analysis*. Izv. Kazan. Fiz.-Mat. Obshch. **14** (1904), 155–234.

[Br1] F. Bracci: *The dynamics of holomorphic maps near curves of fixed points*. Ann. Scuola Norm. Sup. Pisa **2** (2003), 493–520.

[Br2] F. Bracci: *Local dynamics of holomorphic diffeomorphisms*. Boll. UMI **7–B** (2004), 609–636.

[BrM] F. Bracci, L. Molino: *The dynamics near quasi-parabolic fixed points of holomorphic diffeomorphisms in* $\mathbb{C}^2$. Amer. J. Math. **126** (2004), 671–686.

[BrS] F. Bracci, T. Suwa: *Residues for singular pairs and dynamics of biholomorphic maps of singular surfaces*. Internat. J. Math. **15** (2004), 443–466.

[BrT] F. Bracci, F. Tovena: *Residual indices of holomorphic maps relative to singular curves of fixed points on surfaces*. Math. Z. **242** (2002), 481–490.

[BM] F.E. Brochero Martínez: *Groups of germs of analytic diffeomorphisms in* $(\mathbb{C}^2, O)$. J. Dynamic. Control Systems **9** (2003), 1–32.

[BCL] F.E. Brochero Martínez, F. Cano, L. López-Hernanz: *Parabolic curves for diffeomorphisms in* $\mathbb{C}^2$. Publ. Mat. **52** (2008), 189–194.

[Brj1] A.D. Brjuno: *Convergence of transformations of differential equations to normal forms*. Dokl. Akad. Nauk. USSR **165** (1965), 987–989.

[Brj2] A.D. Brjuno: *Analytical form of differential equations, I*. Trans. Moscow Math. Soc. **25** (1971), 131–288.

[Brj3] A.D. Brjuno: *Analytical form of differential equations, II*. Trans. Moscow Math. Soc. **26** (1972), 199–239.

[BC] X. Buff, A. Chéritat: *The Brjuno function continuously estimates the size of quadratic Siegel disks*. Ann. of Math. (2) **164** (2006), no. 1, 265–312.

[BH] X. Buff, J.H. Hubbard: **Dynamics in one complex variable.** To be published by Matrix Edition, Ithaca, NY.

[C] C. Camacho: *On the local structure of conformal mappings and holomorphic vector fields*. Astérisque **59-60** (1978), 83–94.

[CS] C. Camacho, P. Sad: *Invariant varieties through singularities of holomorphic vector fields*. Ann. of Math. **115** (1982), 579–595.

[CM] J.C. Canille Martins: *Holomorphic flows in* $(\mathbb{C}^3, O)$ *with resonances*. Trans. Am. Math. Soc. **329** (1992), 825–837.

[Ca] J. Cano: *Construction of invariant curves for singular holomorphic vector fields*. Proc. Am. Math. Soc. **125** (1997), 2649–2650.

[Car] T. Carletti: *Exponentially long time stability for non-linearizable analytic germs of* $(\mathbb{C}^n, 0)$. Annales de l'Institut Fourier (Grenoble) **54** (2004), no. 4 989–1004.

[CG] S. Carleson, F. Gamelin: **Complex dynamics.** Springer, Berlin, 1994.

[CGBM] D. Cerveau, D. Garba Belko, R. Meziani: *Techniques complexes d'étude d'E.D.O. (avec un appendice de J. Ribón)*. Publ. Mat. **52** (2008), no. 2 473–502.

[Ch] M. Chaperon: **Géométrie différentielle et singularités des systèmes dynamiques.** Astérisque **138–139,** 1986.

[CD] D. Coman, M. Dabija: *On the dynamics of some diffeomorphisms of* $\mathbb{C}^2$ *near parabolic fixed points*. Houston J. Math. **24** (1998), 85–96.

[Cr1] H. Cremer: *Zum Zentrumproblem*. Math. An.. **98** (1927), 151–163.

[Cr2] H. Cremer: *Über die Häufigkeit der Nichtzentren*. Math. Ann. **115** (1938), 573–580.

[DI1] F. Degli Innocenti: *Holomorphic dynamics near germs of singular curves*. Math. Z. **251** (2005), 943–958.

[DI2] F. Degli Innocenti: *On the relations between discrete and continuous dynamics in* $\mathbb{C}^2$. Ph.D. Thesis, Università di Pisa, 2007.

[DG] D. DeLatte, T. Gramchev: *Biholomorphic maps with linear parts having Jordan blocks: linearization and resonance type phenomena*. Math. Phys. El. J. **8** (2002), 1–27.

[Di] P. Di Giuseppe: *Topological classification of holomorphic, semi-hyperbolic germs in quasi-absence of resonances*. Math. Ann. **334** (2006), 609–625.

[DH] A. Douady, J.H. Hubbard: *On the dynamics of polynomial-like mappings*. Ann. Sc. Éc. Norm. Sup. **18** (1985), 287–343.

[D] H. Dulac: *Recherches sur les points singuliers des équationes différentielles*. J. Éc. Polytechnique **IX** (1904), 1–125.

[É1] J. Écalle: *Théorie itérative: introduction à la théorie des invariants holomorphes*. J. Math. Pures Appl. **54** (1975), 183–258.

[É2] J. Écalle: **Les fonctions résurgentes. Tome I: Les algèbres de fonctions résurgentes.** Publ. Math. Orsay **81-05,** Université de Paris-Sud, Orsay, 1981 `http://www.math.u-psud.fr/~biblio/numerisation/docs/E_ECALLE_81_05/pdf/E_ECALLE_81_05.pdf`.

[É3] J. Écalle: **Les fonctions résurgentes. Tome II: Les fonctions résurgentes appliquées à l'itération.** Publ. Math. Orsay **81-06,** Université de Paris-Sud, Orsay, 1981 `http://www.math.u-psud.fr/~biblio/numerisation/docs/E_ECALLE_81_06/pdf/E_ECALLE_81_06.pdf`.

[É4] J. Écalle: **Les fonctions résurgentes. Tome III: L'équation du pont et la classification analytique des objects locaux.** Publ. Math. Orsay **85-05,** Université de Paris-Sud, Orsay, 1985 `http://www.math.u-psud.fr/~biblio/numerisation/docs/E_ECALLE_85_05/pdf/E_ECALLE_85_05.pdf`.

[É5] J. Écalle: *Iteration and analytic classification of local diffeomorphisms of* $\mathbb{C}^\nu$. In **Iteration theory and its functional equations,** Lect. Notes in Math. **1163,** Springer-Verlag, Berlin, 1985, pp. 41–48.

[É6] J. Écalle: *Singularités non abordables par la géométrie*. Annales de l'Institut Fourier (Grenoble) **42** (1992), no. 1–2 73–164.

[ÉV] J. Écalle, B. Vallet: *Correction and linearization of resonant vector fields and diffeomorphisms*. Math. Z. **229** (1998), 249–318.

[F1] P. Fatou: *Sur les équations fonctionnelles, I*. Bull. Soc. Math. France **47** (1919), 161–271.

[F2] P. Fatou: *Sur les équations fonctionnelles, II*. Bull. Soc. Math. France **48** (1920), 33–94.

[F3] P. Fatou: *Sur les équations fonctionnelles, III*. Bull. Soc. Math. France **48** (1920), 208–314.

[F4] P. Fatou: *Sur certaines fonctions uniformes de deux variables*. C. R. Acad. Sci. Paris **175** (1922), 1030–1033..

[F5] P. Fatou: *Substitutions analytiques et équations fonctionnelles à deux variables*. Ann. Sc. Ec. Norm. Sup. **40** (1924), 67–142.

[Fa] C. Favre: *Classification of 2-dimensional contracting rigid germs and Kato surfaces. I*. J. Math. Pures Appl. **79** (2000), 475–514.

[FJ1] C. Favre, M. Jonsson: **The valuative tree.** Lect. Notes in Math. 1853, Springer Verlag, Berlin, 2004.

[FJ2] C. Favre, M. Jonsson: *Eigenvaluations*. Ann. Sci. École Norm. Sup. **40** (2007), 309–349.

[FHY] A. Fathi, M. Herman, J.-C. Yoccoz: *A proof of Pesin's stable manifold theorem.* In **Geometric Dynamics,** Lect Notes in Math. 1007, Springer Verlag, Berlin, 1983, pp. 177–216.

[GS] V. Gelfreich, D. Sauzin: *Borel summation and splitting of separatrices for the Hénon map*. Ann. Inst. Fourier Grenoble **51** (2001), 513–567.

[G] A. Gray: *A fixed point theorem for small divisors problems*. J. Diff. Eq. **18** (1975), 346–365.

[GH] P. Griffiths, J. Harris: **Principles of algebraic geometry.** Wyley, New York, 1978.

[Gr1] D.M. Grobman: *Homeomorphism of systems of differential equations*. Dokl. Akad. Nauk. USSR **128** (1959), 880–881.

[Gr2] D.M. Grobman: *Topological classification of neighbourhoods of a singularity in n-space*. Math. Sbornik **56** (1962), 77–94.

[H] J.S. Hadamard: *Sur l'itération et les solutions asymptotyques des équations différentielles*. Bull. Soc. Math. France **29** (1901), 224–228.

[Ha1] M. Hakim: *Attracting domains for semi-attractive transformations of* $\mathbb{C}^p$. Publ. Matem. **38** (1994), 479–499.

[Ha2] M. Hakim: *Analytic transformations of* $(\mathbb{C}^p, 0)$ *tangent to the identity*. Duke Math. J. **92** (1998), 403–428.

[Ha3] M. Hakim: *Transformations tangent to the identity. Stable pieces of manifolds*. Preprint, 1997.

[Har] P. Hartman: *A lemma in the theory of structural stability of differential equations*. Proc. Am. Math. Soc. **11** (1960), 610–620.

[HK] B. Hasselblatt, A. Katok: **Introduction to the modern theory of dynamical systems.** Cambridge Univ. Press, Cambridge, 1995.

[He] M. Herman: *Recent results and some open questions on Siegel's linearization theorem of germs of complex analytic diffeomorphisms of $\mathbb{C}^n$ near a fixed point*. In **Proc.** 8^{th} **Int. Cong. Math. Phys.,** World Scientific, Singapore, 1986, pp. 138–198.

[HPS] M. Hirsch, C.C. Pugh, M. Shub: **Invariant manifolds.** Lect. Notes Math. **583,** Springer-Verlag, Berlin, 1977.

[HP] J.H. Hubbard, P. Papadopol: *Superattractive fixed points in $\mathbb{C}^n$*. Indiana Univ. Math. J. **43** (1994), 321–365.

[I1] Yu.S. Il'yashenko: *Divergence of series reducing an analytic differential equation to linear normal form at a singular point*. Funct. Anal. Appl. **13** (1979), 227–229.

[I2] Yu.S. Il'yashenko: *Nonlinear Stokes phenomena*. In **Nonlinear Stokes phenomena,** Adv. in Soviet Math. **14,** Am. Math. Soc., Providence, 1993, pp. 1–55.

[IY] Y.S. Il'yashenko, S. Yakovenko: **Lectures on analytic differential equations.** Graduate Studies in Mathematics 86, American Mathematical Society, Providence, RI, 2008.

[J1] A. Jenkins: *Holomorphic germs and the problem of smooth conjugacy in a punctured neighborhood of the origin.*. Trans. Amer. Math. Soc. **360** (2008), no. 1, 331–346.

[J2] A. Jenkins: *Further reductions of Poincaré-Dulac normal forms in $\mathbb{C}^{n+1}$*. Proc. Am. Math. Soc. **136** (2008), 1671–1680.

[K] A. Ya. Khinchin: **Continued fractions.** Translated from the third (1961) Russian edition. Reprint of the 1964 translation. Dover Publications, Inc., Mineola, NY, 1997.

[Ki] T. Kimura: *On the iteration of analytic functions*. Funk. Eqvacioj **14** (1971), 197–238.

[Kœ] G. Kœnigs: *Recherches sur les integrals de certain equations fonctionelles*. Ann. Sci. Éc. Norm. Sup. **1** (1884), 1–41.

[L] L. Leau: *Étude sur les equations fonctionelles à une ou plusieurs variables*. Ann. Fac. Sci. Toulouse **11** (1897), E1–E110.

[Le] J.M. Lee: **Introduction to Smooth Manifolds.** Springer-Verlag, New York 2002.

[Ło] S. Łojasiewicz: **Introduction to Complex Analytic Geometry.** Birkhäuser-Verlag, Basel, 1991.

[M1] B. Malgrange: *Travaux d'Écalle et de Martinet-Ramis sur les systèmes dynamiques*. Astérisque **92-93** (1981/82), 59–73.

[M2] B. Malgrange: *Introduction aux travaux de J. Écalle*. Ens. Math. **31** (1985), 261–282.

[Ma] S. Marmi: **An introduction to small divisors problems.** I.E.P.I., Pisa, 2000.

[MR] J. Martinet, J.-P. Ramis: *Classification analytique des équations différentielles non linéaires résonnantes du premier ordre*. Ann. Scient. École Norm. Sup., série 4 **16** (1983), 571–621.

[MM] J.F. Mattei, R. Moussu: *Holonomie et intégrales premières*. Ann. Scient. Ec. Norm. Sup. **13** (1980), 469–523.

[Mi] J. Milnor: **Dynamics in one complex variable. Third edition.** Annals of Mathematics Studies, 160, Princeton University Press, Princeton, NJ, 2006.

[Mo] L. Molino: *The dynamics of maps tangent to the identity and with nonvanishing index*. Trans. Amer. Math. Soc. **361** (2009), 1597–1623.

[N] N. S. Nadirashvili: *A generalization of Hadamard's three circles theorem.* Moscow Univ. Math. Bull. **31** (1976), no. 3 30—32.

[Na] I. Nakai: *The classification of curvilinear angles in the complex plane and the groups of ± holomorphic diffeomorphisms*. Ann. Fac. Sci. Toulouse Math. **6** (1998), 7 no. 2 313—334.

[Ni] Y. Nishimura: *Automorphismes analytiques admettant des sousvariétés de point fixes attractives dans la direction transversale*. J. Math. Kyoto Univ. **23** (1983), 289–299.

[PT] J. Palis, F. Takens: *Topological equivalence of normally hyperbolic dynamical systems*. Topology **16** (1977), 335–345.

[P1] R. Pérez-Marco: *Sur les dynamiques holomorphes non linéarisables et une conjecture de V.I. Arnold*. Ann. Sci. École Norm. Sup. **26** (1993), 565–644.

[P2] R. Pérez-Marco: *Topology of Julia sets and hedgehogs*. Preprint, Université de Paris-Sud, 94-48, 1994.

[P3] R. Pérez-Marco: *Non-linearizable holomorphic dynamics having an uncountable number of symmetries*. Invent. Math. **199** (1995), 67–127.

[P4] R. Pérez-Marco: *Holomorphic germs of quadratic type*. Preprint, 1995.

[P5] R. Pérez-Marco: *Hedgehogs dynamics*. Preprint, 1995.

[P6] R. Pérez-Marco: *Sur une question de Dulac et Fatou.* C.R. Acad. Sci. Paris **321** (1995), 1045–1048.

[P7] R. Pérez-Marco: *Fixed points and circle maps*. Acta Math. **179** (1997), 243–294.

[P8] R. Pérez-Marco: *Linearization of holomorphic germs with resonant linear part*. Preprint, arXiv: math.DS/0009030, 2000.

[P9] R. Pérez-Marco: *Total convergence or general divergence in small divisors*. Comm. Math. Phys. **223** (2001), 451–464.

[PY] R. Pérez-Marco, J.-C. Yoccoz: *Germes de feuilletages holomorphesà holonomie prescrite*. Complex analytic methods in dynamical systems (Rio de Janeiro, 1992), Astérisque **222** (1994), 7, 345–371.

[Pe] O. Perron: *Über Stabilität und asymptotisches Verhalten der Integrale von Differentialgleichungssystemen.* Math. Z. **29** (1928), 129–160.

[Pes] Ja.B. Pesin: *Families of invariant manifolds corresponding to non-zero characteristic exponents*. Math. USSR Izv. **10** (1976), 1261–1305.

[Po] H. Poincaré: **Œuvres, Tome I.** Gauthier-Villars, Paris, 1928, pp. XXXVI–CXXIX.

[Pö] J. Pöschel: *On invariant manifolds of complex analytic mappings near fixed points*. Exp. Math. **4** (1986), 97–109.

[R1] J. Raissy: *Normalizzazione di campi vettoriali olomorfi*. Tesi di Laurea Specialistica, http://etd.adm.unipi.it/theses/available/etd-06022006-141206/, 2006.

[R2] J. Raissy: *Linearization of holomorphic germs with quasi-Brjuno fixed points*. Mathematische Zeitschrift Online First, (2009) http://www.springerlink.com/content/3853667627008057/fulltext.pdf.

[R3] J. Raissy: *Simultaneous linearization of holomorphic germs in presence of resonances*. Conform. Geom. Dyn. **13** (2009), 217–224.

[R4] J. Raissy: *Torus actions in the normalization problem*. Journal of Geometric Analysis Online First, (2009) http://springerlink.com/content/r20323122p74h06j/fulltext.pdf.

[Re1] L. Reich: *Das Typenproblem bei formal-biholomorphien Abbildungen mit anziehendem Fixpunkt*. Math. Ann. **179** (1969), 227–250.

[Re2] L. Reich: *Normalformen biholomorpher Abbildungen mit anziehendem Fixpunkt*. Math. Ann. **180** (1969), 233–255.

[Ri1] J. Ribón: *Formal classification of unfoldings of parabolic diffeomorphisms*. Ergodic Theory Dynam. Systems **28** (2008), no. 4, 1323–1365.

[Ri2] J. Ribón: *Modulus of analytic classification for unfoldings of resonant diffeomorphisms*. Mosc. Math. J. **8** (2008), no. 2, 319–395, 400.

[Riv1] M. Rivi: **Local behaviour of discrete dynamical systems.** Ph.D. Thesis, Università di Firenze, 1999.

[Riv2] M. Rivi: *Parabolic manifolds for semi-attractive holomorphic germs*. Mich. Math. J. **49** (2001), 211–241.

[Ro1] F. Rong: *Linearization of holomorphic germs with quasi-parabolic fixed points*. Ergodic Theory Dynam. Systems **28** (2008), 979–986.

[Ro2] F. Rong: *Robust parabolic curves in* $\mathbb{C}^m$ *(*$m \geq 3$*)*. To appear in Houston J. Math., 2008.

[Ro3] F. Rong: *Quasi-parabolic analytic transformations of* $\mathbb{C}^n$. J. Math. Anal. Appl. **343** (2008), 99–109.

[RR] J.-P. Rosay, W. Rudin: *Holomorphic maps from* $\mathbb{C}^n$ *to* $\mathbb{C}^n$. Trans. Amer. Math. Soc. **310** (1988), no. 1, 47–86.

[Rü1] H. Rüssmann: *On the convergence of power series transformations of analytic mappings near a fixed point into a normal form*. Preprint I.H.E.S. **M/77/178**, 1977.

[Rü2] H. Rüssmann: *Stability of elliptic fixed points of analytic area-preserving mappings under the Brjuno condition*. Ergodic Theory Dynam. Systems **22** (2002), 1551–1573.

[S] A.A. Shcherbakov: *Topological classification of germs of conformal mappings with identity linear part*. Moscow Univ. Math. Bull. **37** (1982), 60–65.

[Sh] M. Shub: **Global stability of dynamical systems.** Springer, Berlin, 1987.

[Si] C.L. Siegel: *Iteration of analytic functions*. Ann. of Math. **43** (1942), 607–612.

[SM] C.L. Siegel, J.K. Moser: **Lectures on celestial mechanics.** Translation by Charles I. Kalme. Die Grundlehren der mathematischen Wissenschaften, Band 187. Springer-Verlag, New York-Heidelberg, 1971.

[St1] S. Sternberg: *Local contractions and a theorem of Poincaré*. Amer. J. Math. **79** (1957), 809–824.

[St2] S. Sternberg: *The structure of local homomorphisms, II*. Amer. J. Math. **80** (1958), 623–631.

[St3] S. Sternberg: *Infinite Lie groups and the formal aspects of dynamical systems*. J. Math. Mech. **10** (1961), 451–474.

[Sto] L. Stolovitch, L.: *Family of intersecting totally real manifolds of* $(\mathbb{C}^n, 0)$ *and CR-singularities*. Preprint arXiv: math/0506052v2, 2005.

[T] M. Toma: *A short proof of a theorem of Camacho and Sad*. Enseign. Math. **45** (1999), 311–316.

[Tr] J.-M. Trépreau: *Discrimination analytique des difféomorphismes résonnants de* $(\mathbb{C},0)$ *et réflexion de Schwarz*. Astérisque **284** (2003), 271—319.

[U1] T. Ueda: *Local structure of analytic transformations of two complex variables, I*. J. Math. Kyoto Univ. **26** (1986), 233–261.

[U2] T. Ueda: *Local structure of analytic transformations of two complex variables, II*. J. Math. Kyoto Univ. **31** (1991), 695–711.

[Us] S. Ushiki: *Parabolic fixed points of two-dimensional complex dynamical systems*. Sūrikaisekikenkyūsho Kōkyūroku **959** (1996), 168–180.

[V] L. R. Vivas: *Fatou-Bieberbach domains as basins of attraction of automorphisms tangent to the identity*. Preprint, arXiv: 0907.2061v1, 2009.

[Vo] S.M. Voronin: *Analytic classification of germs of conformal maps* $(\mathbb{C},0) \to (\mathbb{C},0)$ *with identity linear part*. Func. Anal. Appl. **15** (1981), 1–17.

[W] B.J. Weickert: *Attracting basins for automorphisms of* $\mathbb{C}^2$. Invent. Math. **132** (1998), 581–605.

[Wu] H. Wu: *Complex stable manifolds of holomorphic diffeomorphisms*. Indiana Univ. Math. J. **42** (1993), 1349–1358.

[Y1] J.-C. Yoccoz: *Linéarisation des germes de difféomorphismes holomorphes de* $(\mathbb{C},0)$. C.R. Acad. Sci. Paris **306** (1988), 55–58.

[Y2] J.-C. Yoccoz: *Théorème de Siegel, nombres de Brjuno et polynômes quadratiques*. Astérisque **231** (1995), 3–88.

[Z1] E. Zehnder: *Generalized implicit function theorems with applications to some small divisor problems I*. Comm. Pure Appl. Math. **28** (1975), 91–140.

[Z2] E. Zehnder: *Generalized implicit function theorems with applications to some small divisor problems II*. Comm. Pure Appl. Math. **29** (1976), no. 1 49–111.

[Z3] E. Zehnder: *A simple proof of a generalization of a theorem by C. L. Siegel. Geometry and topology*. In **Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976,** Lecture Notes in Math., Vol. 597, Springer, Berlin, 1977, 855–866.

[Zu1] N. T. Zung: *Convergence versus integrability in Poincaré-Dulac normal form*. Math. Res. Lett. **9** (2002), 2–3 217–228.

[Zu2] N. T. Zung: *Convergence versus integrability in Birkhoff normal form*. Ann. of Math. (2) **161** (2005), no. 1 141–156.