The Julia-Wolff-Carathéodory theorem and its generalizations

Jasmin Raissy

Institut de Mathématiques de Toulouse; UMR5219, Université de Toulouse; CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France. E-mail: jraissy@math.univ-toulouse.fr

November 2014

This note is a short introduction to the Julia-Wolff-Carathéodory theorem, and its generalizations in several complex variables, up to very recent results for infinitesimal generators of semigroups.

1. The classical Julia-Wolff-Carathéodory theorem

One of the classical result in one-dimensional complex analysis is Fatou's theorem:

Theorem 1.1: (Fatou [Fa]) *Let $f\colon \Delta \to \Delta$ be a holomorphic self-map of the unit disk $\Delta \subset \mathbb{C}$. Then f admits non-tangential limit at almost every point of $\partial\Delta$.*

This result however does not give precise information about the behavior at a specific point σ of the boundary. Of course, to obtain a more precise statement in this case some hypotheses on f are needed. In fact, as it was found by Julia ([Ju1]) in 1920, the right hypothesis is to assume that $f(\zeta)$ approaches the boundary of Δ at least as fast as ζ, in a weak sense. More precisely, we have the classical *Julia's lemma:*

Theorem 1.2: (Julia [Ju1]) *Let $f\colon \Delta \to \Delta$ be a bounded holomorphic function such that*

$$\liminf_{\zeta\to\sigma} \frac{1-|f(\zeta)|}{1-|\zeta|} = \alpha < +\infty \tag{1.1}$$

for some $\sigma \in \partial\Delta$. Then f has non-tangential limit $\tau \in \partial\Delta$ at σ. Moreover, for all $\zeta \in \Delta$ one has

$$\frac{|\tau - f(\zeta)|^2}{1-|f(\zeta)|^2} \le \alpha \, \frac{|\sigma-\zeta|^2}{1-|\zeta|^2} \ . \tag{1.2}$$

The latter statement admits an interesting geometrical interpretation. The *horocycle* $E(\sigma, R)$ contained in Δ of *center* $\sigma \in \partial\Delta$ and *radius* $R > 0$ is the set

$$E(\sigma, R) = \left\{ \zeta \in \Delta \;\middle|\; \frac{|\sigma-\zeta|^2}{1-|\zeta|^2} < R \right\} \ .$$

Geometrically, $E(\sigma, R)$ is an euclidean disk of radius $R/(R+1)$ internally tangent to $\partial\Delta$ at σ. Therefore (1.2) becomes $f\big(E(\sigma,R)\big) \subseteq E(\tau, \alpha R)$ for all $R > 0$, and the existence of the non-tangential limit more or less follows from (1.2) and from the fact that horocycles touch the boundary in exactly one point.

2010 Mathematics Subject Classification: Primary 37L05; Secondary 32A40, 32H50, 20M20.

Keywords: infinitesimal generators, semigroups of holomorphic mappings, Julia-Wolff-Carathéodory theorem, boundary behaviour.

Partially supported by the FIRB2012 grant "Differential Geometry and Geometric Function Theory", and by the ANR project LAMBDA, ANR-13-BS01-0002.

A horocycle can be thought of as the limit of Poincaré disks of fixed euclidean radius and centers going to the boundary; so it makes sense to think of horocycles as Poincaré disks centered at the boundary, and of Julia's lemma as a Schwarz-Pick lemma at the boundary. This suggests that α might be considered as a sort of dilation coefficient: f expands horocycles by a ratio of α. If σ were an internal point and $E(\sigma, R)$ an infinitesimal euclidean disk actually centered at σ, one then would be tempted to say that α is (the absolute value of) the derivative of f at σ. This is exactly the content of the classical *Julia-Wolff-Carathéodory theorem:*

Theorem 1.3: (Julia-Wolff-Carathéodory) *Let $f\colon\Delta \to \Delta$ be a bounded holomorphic function such that*

$$\liminf_{\zeta\to\sigma} \frac{1-|f(\zeta)|}{1-|\zeta|} = \alpha < +\infty$$

for some $\sigma \in \partial\Delta$, and let $\tau \in \partial\Delta$ be the non-tangential limit of f at σ. Then both the incremental ratio $\big(\tau - f(\zeta)\big)/(\sigma - \zeta)$ and the derivative $f'(\zeta)$ have non-tangential limit $\alpha\bar{\sigma}\tau$ at σ.

So condition (1.1) forces the existence of the non-tangential limit of both f and its derivative at σ. This is the result of the work of several people: Julia [Ju2], Wolff [Wo], Carathéodory [C], Landau and Valiron [L-V], R. Nevanlinna [N] and others. We refer, for example, to [B] and [A1] for proofs, history and applications.

2. Generalizations to several variables

It was first remarked by Korányi and Stein ([Ko], [K-S], [St]) in extending Fatou's theorem to several complex variables, that the notion of non-tangential limit is not the right one to consider for domains in $\mathbb{C}^n$. In fact, it turns out that two notions are needed, and to introduce them it is useful to investigate the notion of non-tangential limit in the unit disk Δ.

The non-tangential limit can be defined in two equivalent ways. A function $f\colon\Delta \to \mathbb{C}$ is said to have *non-tangential limit* $L \in \mathbb{C}$ at $\sigma \in \partial\Delta$ if $f\big(\gamma(t)\big) \to L$ as $t \to 1^-$ for every curve $\gamma\colon[0,1) \to \Delta$ such that $\gamma(t)$ converges to σ non-tangentially as $t \to 1^-$. In $\mathbb{C}$, this is equivalent to having that $f(\zeta) \to L$ as $\zeta \to \sigma$ staying inside any *Stolz region* $K(\sigma, M)$ of *vertex* σ and *amplitude* $M > 1$, where

$$K(\sigma, M) = \left\{ \zeta \in \Delta \;\middle|\; \frac{|\sigma - \zeta|}{1-|\zeta|} < M \right\} ,$$

since Stolz regions are angle-shaped nearby the vertex σ, and the angle is going to π as $M \to +\infty$. These two approaches lead to different notions in several variables.

In the unit ball $B^n \subset \mathbb{C}^n$ the natural generalization of a Stolz region is the *Korányi region* $K(p, M)$ of *vertex* $p \in \partial B^n$ and *amplitude* $M > 1$ given by

$$K(p, M) = \left\{ z \in B^n \;\middle|\; \frac{|1 - \langle z, p\rangle|}{1-\|z\|} < M \right\} ,$$

where $\|\cdot\|$ denote the euclidean norm and $\langle\cdot\,,\cdot\rangle$ the canonical hermitian product. Then a function $f\colon B^n \to \mathbb{C}$ has *K-limit* (or *admissible limit*) $L \in \mathbb{C}$ at $p \in \partial B^n$, and we write

$$K\text{-}\lim_{z\to p} f(z)$$

if $f(z) \to L$ as $z \to p$ staying inside any Korányi region $K(\sigma, M)$. A Korányi region $K(p, M)$ approaches the boundary non-tangentially along the normal direction at p but tangentially along the complex tangential directions at p. Therefore, having K-limit is stronger than having non-tangential limit. However, as first noticed by Korányi and Stein, for holomorphic functions of several complex variables one is often able to prove the existence of K-limits. For instance, the best generalization of Julia's lemma to B^n is the following result (proved by Hervé [H] in terms of non-tangential limits and by Rudin [R] in general):

Theorem 2.1: (Rudin [R]) *Let $f\colon B^n \to B^m$ be a holomorphic map such that*

$$\liminf_{z\to p} \frac{1-\|f(z)\|}{1-\|z\|} = \alpha < +\infty ,$$

for some $p \in \partial B^n$. Then f admits K-limit $q \in \partial B^m$ at p, and furthermore for all $z \in B^n$ one has

$$\frac{|1-\langle f(z), q\rangle|^2}{1-\|f(z)\|^2} \le \alpha \frac{|1-\langle z, p\rangle|^2}{1-\|z\|^2} .$$

To define Korányi regions for more general domains in $\mathbb{C}^n$ than the unit ball, we need to briefly recall the definition of the Kobayashi distance (we refer, e.g., to [A1], [JP] and [Ko] for details and much more). We denote by k_Δ the Poincaré distance on the unit disk $\Delta \subset \mathbb{C}$. Given X a complex manifold, the *Lempert function* $\delta_X\colon X \times X \to \mathbb{R}^+$ of X is defined as

$$\delta_X(z, w) = \inf\{k_\Delta(\zeta, \eta) \mid \exists \phi\colon \Delta \to X \text{ holomorphic, with } \phi(\zeta) = z \text{ and } \phi(\eta) = w\}$$

for all z, $w \in X$. The *Kobayashi pseudodistance* $k_X\colon X\times X \to \mathbb{R}^+$ of X is then defined as the largest pseudodistance on X bounded above by δ_X. The manifold X is called *(Kobayashi) hyperbolic* if k_X is indeed a distance; X is called *complete hyperbolic* if k_X is a complete distance.

The main property of the Kobayashi (pseudo)distance is that it is contracted by holomorphic maps: if $f\colon X \to Y$ is a holomorphic map then

$$\forall z, w \in X \qquad\qquad k_Y\big(f(z), f(w)\big) \le k_X(z, w) .$$

In particular, the Kobayashi distance is invariant under biholomorphisms.

It is easy to see that the Kobayashi distance of the unit disk coincides with the Poincaré distance. Furthermore, the Kobayashi distance of the unit ball $B^n \subset \mathbb{C}^n$ coincides with the Bergman distance (see, e.g., [A1, Corollary 2.3.6]); and the Kobayashi distance of a bounded convex domain coincides with the Lempert function (see, e.g., [A1, Proposition 2.3.44]). Moreover, the Kobayashi distance of a bounded convex domain D is complete ([A1, Proposition 2.3.45]), and thus for each $p \in D$ we have that $k_D(p, z) \to +\infty$ if and only if z tends to the boundary ∂D.

Using the Kobayashi intrinsic distance we obtain the natural generalization to complete hyperbolic domains of Korányi regions of the balls.

Let $D \subset\subset \mathbb{C}^n$ be a complete hyperbolic domain and denote by k_D its Kobayashi distance. A *K-region* of *vertex* $x \in \partial D$, *amplitude* $M > 1$, and *pole* $z_0 \in D$ is the set

$$K_{D,z_0}(x, M) = \left\{ z \in D \mid \limsup_{w\to x} [k_D(z, w) - k_D(z_0, w)] + k_D(z_0, z) < \log M \right\}.$$

This definition clearly depends on the pole z_0. However, this dependence is not too relevant since changing the pole corresponds to shifting amplitudes. Moreover, it is elementary to check that in the unit ball K-regions coincide with Korányi regions, $K_{B^n,0}(x, M) = K(x, M)$. Therefore K-regions are a natural generalization of Korányi regions allowing us to generalize the notion of K-limit. A function $f\colon D \to \mathbb{C}^m$ has *K-limit* L at $x \in \partial D$ if $f(z) \to L$ as $z \to p$ staying inside any K-region of vertex x. The best generalization of Julia's lemma in this setting is then the following, due to Abate:

Theorem 2.2: (Abate [A2]) *Let $D \subset\subset \mathbb{C}^n$ be a complete hyperbolic domain and let $z_0 \in D$. Let $f\colon D \to \Delta$ be a holomorphic function and let $x \in \partial D$ be such that*

$$\liminf_{z\to x} [k_D(z_0, z) - k_\Delta(0, f(z))] < +\infty \ .$$

Then f admits K-limit $\tau \in \partial D$ at x.

In order to obtain a complete generalization of the Julia-Wolff-Carathéodory for B^n, Rudin discovered that he needed a different notion of limit, still stronger than non-tangential limit but weaker than K-limit. This notion is closely related to the other characterization of non-tangential limit in one variable we mentioned at the beginning of this section.

A crucial one-variable result relating limits along curves and non-tangential limits is *Lindelöf's theorem.* Given $\sigma \in \partial\Delta$, a *$\sigma$-curve* is a continuous curve $\gamma\colon [0, 1) \to \Delta$ such that $\gamma(t) \to \sigma$ as $t \to 1^-$. Then Lindelöf [Li] proved that if a bounded holomorphic function $f\colon \Delta \to \mathbb{C}$ admits limit $L \in \mathbb{C}$ along a given σ-curve then it admits limit L along all non-tangential σ-curves — and thus it has non-tangential limit L at σ.

In generalizing this result to several complex variables, Čirka [Č] realized that for a bounded holomorphic function the existence of the limit along a (suitable) p-curve (where $p \in \partial B^n$) implies not only the existence of the non-tangential limit, but also the existence of the limit along any curve belonging to a larger class of curves, including some tangential ones — but it does not in general imply the existence of the K-limit. To describe the version (due to Rudin [R]) of Čirka's result we shall state in this survey, let us introduce a bit of terminology.

Let $p \in \partial B^n$. As before, a *p-curve* is a continuous curve $\gamma\colon [0, 1) \to B^n$ such that $\gamma(t) \to p$ as $t \to 1^-$. A p-curve is *special* if

$$\lim_{t\to 1^-} \frac{\|\gamma(t) - \langle\gamma(t), p\rangle p\|^2}{1 - |\langle\gamma(t), p\rangle|^2} = 0 \ ; \tag{2.1}$$

and, given $M > 1$, it is *M-restricted* if

$$\frac{|1 - \langle\gamma(t), p\rangle|}{1 - |\langle\gamma(t), p\rangle|} < M$$

for all $t \in [0, 1)$. We also say that γ is *restricted* if it is M-restricted for some $M > 1$. In other words, γ is restricted if and only if $t \mapsto \langle\gamma(t), p\rangle$ goes to 1 non-tangentially in Δ.

It is not difficult to see that non-tangential curves are special and restricted; on the other hand, a special restricted curve approaches the boundary non-tangentially along the normal direction, but it can approach the boundary tangentially along complex tangential directions. Furthermore, a special M-restricted p-curve is eventually contained in any $K(p, M')$ with $M' > M$, and conversely a special p-curve eventually contained in $K(p, M)$ is M-restricted. However, $K(p, M)$ can contain p-curves that are restricted but not special: for these curves the limit in (2.1) might be a strictly positive number.

With these definitions in place, we shall say that a function $f\colon B^n \to \mathbb{C}$ has *restricted K-limit* (or *hypoadmissible limit*) $L \in \mathbb{C}$ at $p \in \partial B^n$, and we shall write

$$K'\text{-}\lim_{z\to p} f(z) = L \ ,$$

if $f\bigl(\gamma(t)\bigr) \to L$ as $t \to 1^-$ for any special restricted p-curve $\gamma\colon [0, 1) \to B^n$. It is clear that the existence of the K-limit implies the existence of the restricted K-limit, that in turns implies the existence of the non-tangential limit; but none of these implications can be reversed (see, e.g., [R] for examples in the ball).

Finally, we say that a function $f\colon B^n \to \mathbb{C}$ is *K-bounded* at $p \in \partial B^n$ if it is bounded in any Korányi region $K(p, M)$, where the bound can depend on $M > 1$. Then Rudin's version of Čirka's generalization of Lindelöf's theorem is the following:

Theorem 2.3: (Rudin [R]) *Let $f: B^n \to \mathbb{C}$ be a holomorphic function K-bounded at $p \in \partial B^n$. Assume there is a special restricted p-curve $\gamma^o: [0,1) \to B^n$ such that $f\big(\gamma^o(t)\big) \to L \in \mathbb{C}$ as $t \to 1^-$. Then f has restricted K-limit L at p.*

As before, it is possible to generalize this approach to a domain $D \subset \mathbb{C}^n$ different from the ball. A very precise and systematic presentation, providing clear proofs, details and examples, of various aspects of the problem of generalization of the classical Julia-Wolff-Carathéodory theorem to domains in several complex variables, and updated until 2004, can be found in [A6].

For the sake of simplicity we state here only the definitions needed to state Abate's version of Lindelöf's theorem in this setting. Given a point $x \in \partial D$, a *x-curve* is again a continuous curve $\gamma: [0,1) \to D$ so that $\lim_{t\to 1^-} \gamma(t) = x$. A *projection device* at $x \in \partial D$ is the data of: a neighbourhood U of x in $\mathbb{C}^n$, a holomorphic embedded disk $\varphi_x: \Delta \to D \cap U$, such that $\lim_{\zeta\to 1} \varphi_x(\zeta) = x$, a family $\mathcal{P}$ of x-curves in $D \cap U$, and a device associating to every x-curve $\gamma \in \mathcal{P}$ a 1-curve $\tilde{\gamma}_x$ in Δ, or equivalently a x-curve $\gamma_x = \varphi_x \circ \tilde{\gamma}_x$ in $\varphi_x(\Delta)$. If D is equipped with a projection device at $x \in \partial D$, then a curve $\gamma \in \mathcal{P}$ is *special* if $\lim_{t\to 1^-} k_{D\cap U}(\gamma(t), \gamma_x(t)) = 0$, and it is *restricted* if γ_x is a non-tangential 1-curve in Δ. A function $f: D \to C$ has *restricted K-limit* $L \in \mathbb{C}$ *at* x if $\lim_{t\to 1^-} f(\gamma(t)) = L$ for all special restricted x-curves. A projection device is *good* if: for any $M > 1$ there is a $M' > 1$ so that $\varphi_x(K(1,M)) \subset K_{D\cap U, z_0}(x, M')$, and for any special restricted x-curve γ there exists $M_1 = M_1(\gamma)$ such that $\lim_{t\to 1^-} k_{K_{D\cap U,z_0}(x,M_1)}(\gamma(t), \gamma_x(t)) = 0$. Good projection devices exist, and several examples can be found for example in [A6]. Finally, we say that a function $f: D \to \mathbb{C}$ is *K-bounded* at $p \in \partial B^n$ if it is bounded in any K-region $K_{D,z_0}(x, M)$, where the bound can depend on $M > 1$.

With these definitions we can state the generalization of Lindelöf principle given by Abate.

Theorem 2.4: (Abate [A6]) *Let $D \subset \mathbb{C}^n$ be a domain equipped with a good projection device at $x \in \partial D$. Let $f: D \to \Delta$ be a holomorphic function K-bounded at x. Assume there is a special restricted x-curve $\gamma^o: [0,1) \to D$ such that $f\big(\gamma^o(t)\big) \to L \in \mathbb{C}$ as $t \to 1^-$. Then f has restricted K-limit L at x.*

We can now deal with the generalization of the Julia-Wolff-Carathéodory theorem to several complex variables. With respect to the one-dimensional case there is an obvious difference: instead of only one derivative one has to deal with a whole (Jacobian) matrix of them, and there is no reason they should all behave in the same way. And indeed they do not, as shown in Rudin's version of the Julia-Wolff-Carathéodory theorem for the unit ball:

Theorem 2.5: (Rudin [R]) *Let $f: B^n \to B^m$ be a holomorphic map such that*

$$\liminf_{z\to p} \frac{1 - \|f(z)\|}{1 - \|z\|} = \alpha < +\infty \ ,$$

for some $p \in \partial B^n$. Then f admits K-limit $q \in \partial B^m$ at p. Furthermore, if we set $f_q(z) = \big\langle f(z), p\big\rangle q$ and denote by df_z the differential of f at z, we have:

(i) *the function $\big[1 - \big\langle f(z), q\big\rangle\big] / [1 - \langle z, p\rangle]$ is K-bounded and has restricted K-limit α at p;*
(ii) *the map $[f(z) - f_q(z)]/[1 - \langle z, p\rangle]^{1/2}$ is K-bounded and has restricted K-limit O at p;*
(iii) *the function $\big\langle df_z(p), q\big\rangle$ is K-bounded and has restricted K-limit α at p;*
(iv) *the map $[1 - \langle z, p\rangle]^{1/2} d(f - f_q)_z(p)$ is K-bounded and has restricted K-limit O at p;*
(v) *if v is any vector orthogonal to p, the function $\big\langle df_z(v), q\big\rangle / [1 - \langle z, p\rangle]^{1/2}$ is K-bounded and has restricted K-limit 0 at p;*
(vi) *if v is any vector orthogonal to p, the map $d(f - f_q)_z(v)$ is K-bounded at p.*

In the last twenty years this theorem (as well as Theorems 2.1 and 2.3) has been extended to domains much more general than the unit ball: for instance, strongly pseudoconvex domains

[A1, 2, 3], convex domains of finite type [AT], and polydisks [A5] and [AMY], (see also [A6] and references therein).

We end this section with the general version of the Julia-Wolff-Carathódory theorem obtained by Abate in [A6] for a complete hyperbolic domain D in $\mathbb{C}^n$. To formulate it, we need to introduce a couple more definitions. A projection device at $x \in \partial D$ is *geometrical* if there is a holomorphic function $\tilde{p}_x: D \cap U \to \Delta$ such that $\tilde{p}_x \circ \varphi_x = \mathrm{id}_\Delta$ and $\tilde{\gamma}_x = \tilde{p}_x \circ \gamma$ for all $\gamma \in \mathcal{P}$. A geometrical projection device at x is *bounded* if $d(z, \partial D)/|1-\tilde{p}_x(z)|$ is bounded in $D \cap U$, and $|1-\tilde{p}_x(z)|/d(z, \partial D)$ is K-bounded in $D \cap U$. The statement is then the following, where κ_D denotes the Kobayashi metric.

Theorem 2.6: (Abate [A6]) *Let $D \subset \mathbb{C}^n$ be a complete hyperbolic domain equipped with a bounded geometrical projection device at $x \in \partial D$. Let $f: D \to \Delta$ be a holomorphic function such that*

$$\liminf_{z \to x} [k_D(z_0, z) - k_\Delta(0, f(z))] = \frac{1}{2} \log \beta < +\infty .$$

Then for every $v \in \mathbb{C}^n$ and every $s \geq 0$ such that $d(z, \partial D)^s \kappa_D(z; v)$ is K-bounded at x the function

$$d(z, \partial D)^{s-1} \frac{\partial f}{\partial v} \tag{2.2}$$

is K-bounded at x. Moreover, if $s > \inf\{s \geq 0 \mid d(z, \partial D)^s \kappa_D(z; v)$ is K-bounded at $x\}$, then (2.2) has vanishing K-limit at x.

Depending on more specific properties of the projection device, it is indeed possible to deduce the existence of restricted K-limits, see [A6, Section 5].

Further generalizations of Julia-Wolff-Carathéodory theorem have been obtained in infinite-dimensional Banach and Hilbert spaces, and we refer to [EHRS], [ELRS], [ERS], [F], [MM], [SW], [Wł1, 2, 3], [Z], and references therein.

3. Julia-Wolff-Carathéodory theorem for infinitesimal generators

We conclude this survey focusing on a different kind of generalization in several complex variables: infinitesimal generators of one-parameter semigroups of holomorphic self-maps of B^n.

We consider $\mathrm{Hol}(B^n, B^n)$, the space of holomorphic self-maps of B^n, endowed with the usual compact-open topology. A *one-parameter semigroup* of holomorphic self-maps of B^n is a continuous semigroup homomorphism $\Phi: \mathbb{R}^+ \to \mathrm{Hol}(B^n, B^n)$. In other words, writing φ_t instead of $\Phi(t)$, we have $\varphi_0 = \mathrm{id}_{B^n}$, the map $t \mapsto \varphi_t$ is continuous, and the semigroup property $\varphi_t \circ \varphi_s = \varphi_{t+s}$ holds. An introduction to the theory of one-parameter semigroups of holomorphic maps can be found in [A1], [RS2] or [S].

One-parameter semigroups can be seen as the flow of a vector field (see, e.g., [A4]). Given a one-parameter semigroup Φ, it is possible to prove that there exists a holomorphic map $G: B^n \to \mathbb{C}^n$, the *infinitesimal generator* of the semigroup, such that

$$\frac{\partial \Phi}{\partial t} = G \circ \Phi . \tag{3.1}$$

It should be kept in mind, when reading the literature on this subject, that in some papers (e.g., in [ERS] and [RS1]) there is a change of sign with respect to our definition, due to the fact that the infinitesimal generator is defined there as the solution of the equation

$$\frac{\partial \Phi}{\partial t} + G \circ \Phi = O .$$

A Julia's lemma for infinitesimal generators was proved by Elin, Reich and Shoikhet in [ERS] in 2008, assuming that the radial limit of the generator at a point $p \in \partial B^n$ vanishes:

Theorem 3.1: ([ERS, Theorem p. 403]) *Let $G: B^n \to \mathbb{C}^n$ be the infinitesimal generator on B^n of a one-parameter semigroup $\Phi = \{\varphi_t\}$, and let $p \in \partial B^n$ be such that*

$$\lim_{t\to 1^-} G(tp) = O \; . \tag{3.2}$$

Then the following assertions are equivalent:

(I) $\alpha = \liminf_{t\to 1^-} \operatorname{Re} \frac{\langle G(tp),p\rangle}{t-1} < +\infty$;

(II) $\beta = 2\sup_{z\in B^n} \operatorname{Re}\left[\frac{\langle G(z),z\rangle}{1-\|z\|^2} - \frac{\langle G(z),p\rangle}{1-\langle z,p\rangle}\right] < +\infty$;

(III) *there exists $\gamma \in \mathbb{R}$ such that for all $z \in B^n$ we have* $\frac{|1-\langle \varphi_t(z),p\rangle|^2}{1-\|\varphi_t(z)\|^2} \le e^{\gamma t}\frac{|1-\langle z,p\rangle|^2}{1-\|z\|^2}$.

Furthermore, if any of these assertions holds then $\alpha = \beta = \inf\gamma$, and we have

$$\lim_{t\to 1^-} \frac{\langle G(tp),p\rangle}{t-1} = \beta \; . \tag{3.3}$$

If (3.2) and any (whence all) of the equivalent conditions (I)–(III) holds, $p \in \partial B^n$ is called a *boundary regular null point* of G with *dilation* $\beta \in \mathbb{R}$.

This result suggested that a Julia-Wolff-Carathéodory theorem could hold for infinitesimal generators along the line of Rudin's Theorem 2.5. A first partial generalization has been achieved by Bracci and Shoikhet in [BS]. In collaboration with Abate, in [AR] we have been able to give a full generalization of Julia-Wolff-Carathéodory theorem for infinitesimal generators, proving the following result.

Theorem 3.2: ([AR]) *Let $G: B^n \to \mathbb{C}^n$ be an infinitesimal generator on B^n of a one-parameter semigroup, and let $p \in \partial B^n$. Assume that*

$$\frac{\langle G(z),p\rangle}{\langle z,p\rangle - 1} \qquad \text{is } K\text{-bounded at } p \tag{3.4}$$

and

$$\frac{G(z) - \langle G(z),p\rangle p}{(\langle z,p\rangle - 1)^{\gamma}} \qquad \text{is } K\text{-bounded at } p \text{ for some } 0 < \gamma < 1/2. \tag{3.5}$$

Then $p \in \partial B^n$ is a boundary regular null point for G. Furthermore, if β is the dilation of G at p then:

(i) *the function $\langle G(z),p\rangle/(\langle z,p\rangle - 1)$ (is K-bounded and) has restricted K-limit β at p;*

(ii) *if v is a vector orthogonal to p, the function $\langle G(z),v\rangle/(\langle z,p\rangle - 1)^{\gamma}$ is K-bounded and has restricted K-limit 0 at p;*

(iii) *the function $\langle dG_z(p),p\rangle$ is K-bounded and has restricted K-limit β at p;*

(iv) *if v is a vector orthogonal to p, the function $(\langle z,p\rangle - 1)^{1-\gamma}\langle dG_z(p),v\rangle$ is K-bounded and has restricted K-limit 0 at p;*

(v) *if v is a vector orthogonal to p, the function $\langle dG_z(v),p\rangle \,/\, (\langle z,p\rangle - 1)^{\gamma}$ is K-bounded and has restricted K-limit 0 at p.*

(vi) *if v_1 and v_2 are vectors orthogonal to p the function $(\langle z,p\rangle - 1)^{1/2-\gamma}\langle dG_z(v_1),v_2\rangle$ is K-bounded at p.*

Sketch of Proof of Theorem 3.2. Statement (i) follows immediately from our hypotheses, thanks to Theorems 2.3 and 3.1. Statement (iii) follows by standard arguments, and (iv) follows from (ii), again by standard arguments.

The main point is the proof of part (ii). By Theorem 2.3, it suffices to compute the limit along a special restricted curve. We use the curve

$$\sigma(t) = tp + e^{-i\theta}\varepsilon(1-t)^{1-\gamma}v$$

which is always restricted, and *it is special if and only if* $\gamma < 1/2$. We then plug (i) and this curve into Theorem 3.1.(II), and we then let $\varepsilon \to 0^+$, using θ to get rid of the real part.

Statement (v) follows from (i), (ii) and by Theorem 3.1 using somewhat delicate arguments involving a curve of the form

$$\gamma(t) = \bigl(t + ic(1-t)\bigr)p + \eta(t)v\ ,$$

where $1 - t < |\eta(t)|^2 < 1 - |t + ic(1-t)|^2$, and the argument of $\eta(t)$ is chosen suitably. $\square$

A first difference with respect to Theorem 2.5 is that we have to assume (3.4) and (3.5) as separate hypotheses, whereas they appear as part of Theorem 2.5.(i) and (ii). Indeed, when dealing with holomorphic maps, conditions (3.4) and (3.5) are a consequence of the equivalent of condition (I) in Theorem 3.1, but in that setting the proof relies in the fact that there we have holomorphic *self-maps* of the ball. In our context, (3.5) is *not* a consequence of Theorem 3.1.(I), as shown in [AR, Example 1.2]; and it also seems that (3.4) is stronger than Theorem 3.1.(I).

A second difference is the exponent $\gamma < 1/2$. Bracci and Shoikhet proved Theorem 3.2 with $\gamma = 1/2$ but they couldn't prove the statements about restricted K-limits in cases (ii), (iv) and (v). This is due to an obstruction, which is not just a technical problem, but an inevitable feature of the theory. As mentioned in the sketch of the proof, in showing the existence of restricted K-limits, the curves one would like to use for obtaining the exponent 1/2 in the statements are *restricted but not special*, in the sense that the limit in (2.1) is a strictly positive (though finite) number. Actually the exponent 1/2 might not be the right one to consider in the setting of infinitesimal generators, as shown in [AR, Example 1.2].

An exact analogue of Theorem 2.5 with $\gamma = 1/2$ can be recovered assuming a slightly stronger hypothesis on the infinitesimal generator. Under assumptions (3.4) and (3.5) we have

$$\frac{\langle G\bigl(\sigma(t)\bigr), p\rangle}{\langle \sigma(t), p\rangle - 1} = \beta + o(1) \tag{3.6}$$

as $t \to 1^-$ for any special restricted p-curve $\sigma\colon [0,1) \to B^n$. Following ideas introduced in [ESY], [EKRS] and [EJ] in the context of the unit disk, p is said to be a *Hölder boundary null point* if there is $\alpha > 0$ such that

$$\frac{\langle G\bigl(\sigma(t)\bigr), p\rangle}{\langle \sigma(t), p\rangle - 1} = \beta + o\bigl((1-t)^\alpha\bigr) \tag{3.7}$$

for any special restricted p-curve $\sigma\colon [0,1) \to B^n$ such that $\langle \sigma(t), p\rangle \equiv t$. Using this notion we obtain the following result.

Theorem 3.3: ([AR]) *Let $G\colon B^n \to \mathbb{C}^n$ be the infinitesimal generator on B^n of a one-parameter semigroup, and let $p \in \partial B^n$. Assume that*

$$\frac{\langle G(z), p\rangle}{\langle z, p\rangle - 1} \quad \text{and} \quad \frac{G(z) - \langle G(z), p\rangle p}{(\langle z, p\rangle - 1)^{1/2}}$$

are K-bounded at p, and that p is a Hölder boundary null point. Then the statement of Theorem 3.2 holds with $\gamma = 1/2$.

Examples of infinitesimal generators with a Hölder boundary null point and satisfying the hypotheses of Theorem 3.3 are provided in [AR].

In a forthcoming paper in collaboration with Abate, we will deal with the generalization of Theorem 3.2 to strongly convex domains in $\mathbb{C}^n$.

References

[A1] M. Abate: **Iteration theory of holomorphic maps on taut manifolds.** Mediterranean Press, Rende, 1989.

[A2] M. Abate: *The Lindelöf principle and the angular derivative in strongly convex domains*. J. Analyse Math. **54** (1990), 189–228.

[A3] M. Abate: *Angular derivatives in strongly pseudoconvex domains*. Proc. Symp. Pure Math. **52,** Part 2 (1991), 23–40.

[A4] M. Abate: *The infinitesimal generators of semigroups of holomorphic maps*. Ann. Mat. Pura Appl. **161** (1992), 167–180.

[A5] M. Abate: *The Julia-Wolff-Carathéodory theorem in polydisks*. J. Analyse Math. **74** (1998), 275–306.

[A6] M. Abate: *Angular derivatives in several complex variables*. In **Real methods in complex and CR geometry,** Eds. D. Zaitsev, G. Zampieri, Lect. Notes in Math. 1848, Springer, Berlin, 2004, pp. 1–47.

[AR] M. Abate, J. Raissy: *A Julia-Wolff-Carathéodory theorem for infinitesimal generators in the unit ball*. to appear in Trans. AMS, (2014), 1–17.

[AT] M. Abate, R. Tauraso: *The Lindelöf principle and angular derivatives in convex domains of finite type*. J. Austr. Math. Soc. **73** (2002), 221–250.

[AMY] J. Agler, J.E. McCarthy, N.J. Young: *A Carathéodory theorem for the bidisk via Hilbert space methods*. Math. Ann. **352** (2012), 581–624.

[BCD] F. Bracci, M.D. Contreras, S. Díaz-Madrigal: *Pluripotential theory, semigroups and boundary behavior of infinitesimal generators in strongly convex domains*. J. Eur. Math. Soc. **12** (2010), 23–53.

[BS] F. Bracci, D. Shoikhet: *Boundary behavior of infinitesimal generators in the unit ball*. Trans. Amer. Math. Soc. **366** (2014), 1119–1140.

[B] R.B. Burckel: **An introduction to classical complex analysis.** Academic Press, New York,1979.

[C] C. Carathéodory: *Über die Winkelderivierten von beschränkten analytischen Funktionen*. Sitzungsber. Preuss. Akad. Wiss. Berlin (1929), 39–54.

[Č] E.M. Čirka: *The Lindelöf and Fatou theorems in $\mathbb{C}^n$*. Math. USSR-Sb. **21** (1973), 619–641.

[EHRS] M. Elin, L.A. Harris, S. Reich, D. Shoikhet: *Evolution equations and geometric function theory in J^*-algebras*. J. Nonlinear Conv. Anal. **3** (2002), 81–121.

[EKRS] M. Elin, D. Khavinson, S. Reich, D. Shoikhet: *Linearization models for parabolic dynamical systems via Abel's functional equation*. Ann. Acad. Sci. Fen. **35** (2010), 1–34.

[ELRS] M. Elin, M. Levenshtein, S. Reich, D. Shoikhet: *Some inequalities for the horosphere function and hyperbolically nonexpansive mappings on the Hilbert ball*. J. Math. Sci. (N.Y.) **201** (2014), no. 5, 595–613.

[EJ] M. Elin, F. Jacobzon: *Parabolic type semigroups: asymptotics and order of contact*. Preprint, arXiv:1309.4002, 2013.

[ERS] M. Elin, S. Reich, D. Shoikhet: *A Julia-Carathéodory theorem for hyperbolically monotone mappings in the Hilbert ball*. Israel J. Math. **164** (2008), 397–411.

[ES] M. Elin, D. Shoikhet: **Linearization models for complex dynamical systems. Topics in univalent functions, functional equations and semigroup theory.** Operator Theory: Advances and Applications, 208. Linear Operators and Linear Systems Birkhuser Verlag, Basel, 2010. xii+265 pp. ISBN: 978-3-0346-0508-3.

[ESY] M. Elin, D. Shoikhet, F. Yacobzon: *Linearization models for parabolic type semigroups*. J. Nonlinear Convex Anal. **9** (2008), 205–214.

[F] K. Fan: *The angular derivative of an operator-valued analytic function*. Pacific J. Math. **121** (1986), 67–72.

[Fa] P. Fatou: *Séries trigonométriques et séries de Taylor*. Acta Math. **30** (1906), 335–400.

[H] M. Hervé: *Quelques propriétés des applications analytiques d'une boule à m dimensions dans elle-même*. J. Math. Pures Appl. **42** (1963), 117–147.

[JP] M. Jarnicki, P. Pflug: **Invariant distances and metrics in complex analysis.** Walter de Gruyter & co., Berlin, 1993.

[Ju1] G. Julia: *Mémoire sur l'itération des fonctions rationnelles*. J. Math. Pures Appl. **1** (1918), 47–245.

[Ju2] G. Julia: *Extension nouvelle d'un lemme de Schwarz*. Acta Math. **42** (1920), 349–355.

[Ko] A. Korányi: *Harmonic functions on hermitian hyperbolic spaces*. Trans. Amer. Math. Soc. **135** (1969), 507–516.

[K-S] A. Korányi, E.M. Stein: *Fatou's theorem for generalized half-planes*. Ann. Scuola Norm. Sup. Pisa **22** (1968), 107–112.

[L-V] E. Landau, G. Valiron: *A deduction from Schwarz's lemma*. J. London Math. Soc. **4** (1929), 162–163.

[L] L. Lempert: *La métrique de Kobayashi et la représentation des domaines sur la boule*. Bull. Soc. Math. France **109** (1981), 427–474.

[Li] E. Lindelöf: *Sur un principe générale de l'analyse et ses applications à la theorie de la représentation conforme*. Acta Soc. Sci. Fennicae **46** (1915), 1–35.

[MM] M. Mackey, P. Mellon: *Angular derivatives on bounded symmetric domains*. Israel J. Math. **138** (2003), 291–315.

[N] R. Nevanlinna: *Remarques sur le lemme de Schwarz*. C.R. Acad. Sci. Paris **188** (1929), 1027–1029.

[RS1] S. Reich, D. Shoikhet: *Semigroups and generators on convex domains with the hyperbolic metric*. Atti Acc. Naz. Lincei Cl. Sc. Fis. Mat. Nat. Rend. Lincei **8** (1997), 231–250.

[RS2] S. Reich, D. Shoikhet: **Nonlinear semigroups, fixed points, and geometry of domains in Banach spaces.** Imperial College Press, London, 2005.

[R] W. Rudin: **Function theory in the unit ball of $\mathbb{C}^n$.** Springer, Berlin, 1980.

[S] D. Shoikhet: **Semigroups in geometrical function theory.** Kluwer Academic Publishers, Dordrecht, 2001.

[St] E.M. Stein: **The boundary behavior of holomorphic functions of several complex variables.** Princeton University Press, Princeton, 1972.

[SW] A. Szałowska, K. Włodarczyk: *Angular derivatives of holomorphic maps in infinite dimensions*. J. Math. Anal. Appl. **204** (1996), 1–28.

[W] S. Wachs: *Sur quelques propriétés des transformations pseudo-conformes avec un point frontière invariant*. Bull. Soc. Math. Fr. **68** (1940), 177–198.

[Wł1] K. Włodarczyk: *The Julia-Carathéodory theorem for distance decreasing maps on infinite-dimensional hyperbolic spaces*. Atti Accad. Naz. Lincei **4** (1993), 171–179.

[Wł2] K. Włodarczyk: *Angular limits and derivatives for holomorphic maps of infinite dimensional bounded homogeneous domains*. Atti Accad. Naz. Lincei **5** (1994), 43–53.

[Wł3] K. Włodarczyk: *The existence of angular derivatives of holomorphic maps of Siegel domains in a generalization of C^*-algebras*. Atti Accad. Naz. Lincei **5** (1994), 309–328.

[Wo] J. Wolff: *Sur une généralisation d'un théorème de Schwarz*. C.R. Acad. Sci. Paris **183** (1926), 500–502.

[Z] J.M. Zhu: *Angular derivatives of holomorphic maps in Hilbert spaces*. J. Math. (Wuhan) **19** (1999), 304–308.