Local dynamics of singular holomorphic foliations

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Introduction

Roughly speaking, a (regular) *holomorphic foliation* of rank r on a complex manifold M of dimension n is a partition of M in disjoint submanifolds of dimension r, with the same local structure of parallel r-dimensional planes in $\mathbb{C}^n$. A typical way (but not the unique one) to get a holomorphic foliation is via an integrable distribution of rank r, that is a complex subbundle F of rank r of the (holomorphic) tangent bundle TM satisfying the Frobenius condition $[F, F] \subseteq F$; the foliation is given by the integral manifolds of F. Given an integrable distribution F of rank r, we can find an open cover $\{U_\alpha\}$ of M such that F is generated over each U_α by r *linearly independent* holomorphic vector fields $X_1^\alpha, \ldots, X_r^\alpha$ such that

$$F_p = \mathrm{Span}\{X_1^\alpha(p), \ldots, X_r^\alpha(p)\}$$

for every $p \in U_\alpha$, and satisfying

$$[X_h^\alpha, X_k^\alpha] = \sum_{j=1}^{r} c_{hk,\alpha}^{j} X_j^\alpha \tag{1}$$

for suitable local holomorphic functions $c_{hk,\alpha}^j \in \mathcal{O}(U_\alpha)$. Furthermore, on $U_\alpha \cap U_\beta$ we can find an invertible matrix $\big((a_{\alpha\beta})_k^h\big) \in GL\big(r, \mathcal{O}(U_\alpha \cap U_\beta)\big)$ of holomorphic functions such that

$$X_k^\beta = \sum_{h=1^r} (a_{\alpha\beta})_k^h X_h^\alpha \tag{2}$$

for all $k = 1, \ldots, r$. If we drop the condition that $X_1^\alpha, \ldots, X_r^\alpha$ are *everywhere* linearly independent, we get a possible definition of *singular holomorphic foliation* of rank r: it is given by an open covering $\{U_\alpha\}$ of M and, for each α, a family $\{X_1^\alpha, \ldots, X_\alpha^r\}$ of holomorphic vector fields on U_α, satisfying (1) and (2) and linearly independent off a singular set Σ of codimension at least 2. For instance, a rank 1 singular holomorphic foliation is given by a family $\{(U_\alpha, X_\alpha)\}$, where $\{U_\alpha\}$ is an open cover of M, each X_α is a not identically zero vector field on U_α, and $X_\beta = a_{\alpha\beta} X_\alpha$ on $U_\alpha \cap U_\beta$ for a suitable never vanishing holomorphic function $a_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. The singular set is then the set of zeroes of the local vector fields X_α.

The local structure of a regular holomorphic foliation is quite easy to study: as mentioned above, a regular foliation locally is like parallel r-dimensional planes in $\mathbb{C}^n$. For this reason, when studying a regular holomorphic foliation one is mostly interested in global phenomena. On the other hand, even the *local* structure of a singular holomorphic foliation around the singular set Σ can be quite interesting. Off the singular set we have a regular holomorphic foliation, and thus a partition of $M \setminus \Sigma$ in r-dimensional complex submanifolds; but the way these submanifolds fit together nearby the singular set can be quite complicated (and intriguing). The study of the (local or global) structure of (singular or regular) holomorphic foliations is the aim of the subject known as (local or global) *continuous holomorphic dynamics*, which has been a very active field of research in the last forty years, particularly thanks to the Latino American and French schools. However, while it is relatively easy to find books on the theory of real foliations (see, e.g., [CC00], [CC03], [CLN85], [Ton88]), the results on holomorphic foliations are still mostly scattered in the research literature.

In the year 2007-08, I organized a reading course on *Local continuous holomorphic dynamics*, attended by Ph.D. students and post-docs of the Universities of Pisa and Firenze and of the Scuola Normale Superiore, where (relying on some notes written by Frank Leray a few years ago and on the original literature) they presented the basis of the local theory of singular holomorphic foliations, at least in dimension 2, starting from scratch and arriving up to Yoccoz-Perez-Marco's construction of foliations with prescribed holonomy. This booklet is the result of that course; we hope it will be useful as a starting reference for whoever would like to begin studying this beautiful subject.

As mentioned above, the content of this booklet is mostly limited to the case of rank 1 singular holomorphic foliations in 2-dimensional manifolds, for a couple of reasons: definitions and proofs are clearer in this setting than in the general case, and yet the main features of the subject are already evident here; and some important results are not known (or simply false) in higher rank or dimension. There is no claim of completeness here; but possibly after reading this booklet tackling important papers like [MR82], [MR83], [Str02] or [SŻa02] would be easier. Chapter 1 contains several equivalent definitions of regular and singular holomorphic foliations, and presents several basic concepts needed later. In particular, the fundamental notion of holonomy of a foliation is introduced here. Chapter 2 contains a proof of the Reduction of singularities Theorem, showing that (for rank 1 foliations in a complex surface) after a finite number of blow-ups one can reduce the local study of a singular holomorphic foliation to the study of elementary singularities, where an elementary singularity is the zero of a vector field with a not nilpotent linear part. Chapter 3 introduces the Poincaré-Dulac (formal and, when it exists, holomorphic) normal form of a singular vector field with an elementary singularity. Furthermore, it contains the holomorphic classification of foliations with an elementary singularity in the Poincaré domain (if α denotes the ratio of the two eigenvalues of the linear part of the vector field X with an elementary singularity at the origin, to be in the Poincaré domain is equivalent to having $\alpha \in \mathbb{C}^* \setminus \mathbb{R}^-$), and the description of the topology of the leaves in this case. Finally, Chapter 4 studies the elementary singularities in the Siegel domain (that is, with $\alpha < 0$), describing the formal classification and the topology of the leaves in this case, and proving Mattei-Moussu Theorem giving their holomorphic classification in terms of the holonomy, and Yoccoz-Perez-Marco Theorem constructing singular foliations with prescribed holonomy.

Each chapter is attributed to whoever originally presented the material in the reading course; but the actual writing of this booklet has been a collective affair, and the final version is both merit and responsability of all four of them. Matteo Ruggiero coordinated the merging of four separate drafts in an unitary ensemble, and I just did what you are going to do now: I read it all, and enjoyed the mathematics.

Pisa, February 17, 2009 *Marco Abate*

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Chapter 1

Holomorphic foliations and holonomy

Isaia Nisoli[1]

1.1 Regular foliations

Definition 1.1.1. Let M be a complex manifold of dimension m. A **(regular) holomorphic foliation** $\mathcal{F}$ of M of complex codimension k, or complex dimension $m-k$, is a maximal holomorphic atlas

$$\{(U_j, \phi_j) \mid \phi_j \colon U_j \to \phi_j(U_j) \subset \mathbb{C}^{m-k} \times \mathbb{C}^k\},$$

such that the transition maps

$$\phi_j \circ \phi_i^{-1} \colon \phi_i(U_j \cap U_i) \to \phi_j(U_j \cap U_i)$$

are of the form

$$(x, y) \mapsto \big(g_{ij}(x, y), h_{ij}(y)\big), \quad x \in U \subseteq \mathbb{C}^{m-k},\ y \in V \subseteq \mathbb{C}^k, \tag{1.1}$$

with g_{ij}, h_{ij} holomorphic maps.

We shall call (U_j, ϕ_j) a **chart of the foliation** $\mathcal{F}$. We shall often denote a chart of a foliation simply by U_j if we are interested only on the open set U_j and not on the map ϕ_j.

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Definition 1.1.2. Let $\mathcal{F}$ be a codimension k holomorphic foliation of the m-dimensional complex manifold M. Let (U, ϕ) be a chart of the foliation. A **plaque of the foliation** is a set of the form $\phi^{-1}(V \times \{c\})$ where V is an open set in $\mathbb{C}^{m-k}$ and c is in $\mathbb{C}^k$.

Let $\mathcal{F}$ be a foliation on a complex manifold M, let U and $\tilde{U}$ be two charts of $\mathcal{F}$, such that $U \cap \tilde{U}$ is not empty, and let P be a plaque in U and $\tilde{P}$ a plaque in $\tilde{U}$. Either these two plaques coincide on the intersection of U and $\tilde{U}$ or they have empty intersection. We define an equivalence relation saying that two points are equivalent under $\mathcal{F}$ if there exists a finite sequence $P_0, \ldots, P_n$ of plaques of $\mathcal{F}$ such that $p \in P_0$, $q \in P_n$ and $P_i \cap P_{i+1} \neq \emptyset$ for $i = 0, \ldots, n-1$.

Definition 1.1.3. Let $\mathcal{F}$ be a codimension k holomorphic foliation of the m-dimensional complex manifold M. The **leaves** of $\mathcal{F}$ are the equivalence classes of the points of M with respect to the equivalence relation above.

In this section we restrict ourselves to the case of complex dimension 2*, so when we write foliation we mean a holomorphic foliation of complex dimension* 1 *of* M*.*

Intuitively, a foliation is a partition of a manifold into submanifolds of constant dimension. This is a known phenomenon in geometry; let us introduce some standard results and definitions, and describe several examples.

Suppose that on an open set U we have a holomorphic submersion $f : U \to \mathbb{C}$. Then we have the following result, whose proof can be found in [FG02, Chapter IV, Section 1, Theorem 1.16].

Theorem 1.1.4 (**Holomorphic submersion theorem**)**.** *Let* M *be a complex manifold of complex dimension* 2*,* $U \subset M$ *a domain in* M*, and* $f : U \to \mathbb{C}$ *a holomorphic submersion. Then for every point* $p \in U$ *there exist an open neighborhood* U_p *of* p*, an open neighborhood* W *of* $f(p)$*, an open domain* $V \subset \mathbb{C}$ *and a holomorphic map* $g : U_p \to V$ *such that* $q \mapsto (g(q), f(q))$ *defines a biholomorphism from* U_p *to an open subset of* $V \times W$*.*

Example 1.1.5. Let U be an open set of a complex 2-manifold M, and $f : U \to \mathbb{C}$ a holomorphic submersion; then thanks to Theorem 1.1.4, for every point $p \in U$ we have a biholomorphism $\phi := (g, f)$, with inverse h, between a suitable neighborhood U_p of p and the product of two open subsets $V \times W \subset \mathbb{C} \times \mathbb{C}$ (up to shrinking neighborhoods in Theorem 1.1.4, we can suppose that the image of (g, f) is a product of two suitable open sets in $\mathbb{C}$). We have then that if c is different from c' the images of $h(V \times \{c\})$ and $h(V \times \{c'\})$ are disjoint. So, the level sets of f are a partition of U into subsets of codimension 1.

We state now the theorem on the existence and uniqueness of the solution of holomorphic differential equations (see [IY08, Theorem 1.1]).

Theorem 1.1.6. *Let $U \subseteq \mathbb{C} \times \mathbb{C}^n$ be an open domain, and $F(t,x)\colon U \to \mathbb{C}^n$ a holomorphic map. Let us consider the holomorphic differential equation*

$$\frac{dx}{dt} = F(t,x), \tag{1.2}$$

and a given point $(t_0, x_0) \in U$. Then there exists a sufficiently small polydisk $\mathbb{D}^n_\varepsilon = \{|t - t_0| < \varepsilon, |x_j - x_{0,j}| < \varepsilon, j = 1, \ldots, n\} \subseteq U$, such that solution of (1.2), *with initial value $x(t_0) = x_0$, exists and is unique in this polydisk.*

This solution depends holomorphically on the initial value $x_0 \in \mathbb{C}^n$ and on any additional parameters, provided that the vector function F depends holomorphically on these parameters.

Example 1.1.7. Suppose we have a non-vanishing holomorphic 1-form ω defined on an open subset U of a complex surface M. We say that a (complex) curve γ defined from a subset V of $\mathbb{C}$ in U is **tangent to** ω if $\gamma^*\omega = 0$. If ω is defined in coordinates by $\omega = f(x,y)dx + g(x,y)dy$ these curves can be found by solving the differential equation

$$f\big(\gamma_1(z), \gamma_2(z)\big)\frac{d\gamma_1(z)}{dz} + g\big(\gamma_1(z), \gamma_2(z)\big)\frac{d\gamma_2(z)}{dz} = 0.$$

Theorem 1.1.6 implies that, since ω is non-vanishing, the images of two tangent curves either are disjoint or coincide. Thus ω defines a partition of U in subsets of codimension 1.

Definition 1.1.8. Let M be a complex manifold, $U \subseteq M$ an open domain of M, and X a holomorphic vector field on U. A **complex integral curve** for X is a holomorphic curve $\theta\colon D \to U$, with $0 \in D \subseteq \mathbb{C}$ an open domain, such that

$$\theta'(t) = d\theta_t\left(\frac{d}{dz}\right) = X_{\theta(t)} \tag{1.3}$$

for every $t \in D$.

A **complex flow** associated to X is a holomorphic map $\Theta\colon \mathcal{D} \to M$, where $\mathcal{D} \subseteq \mathbb{C} \times U$ is an open neighborhood of $\{0\} \times U$ and $\theta_p(t) := \Theta(t,p)$ is a complex integral curve for X, with $\theta_p(0) = p$ for every $p \in U$. We shall also denote by $\theta^t(p) := \Theta(t,p)$ the flow map at time t.

Remark 1.1.9. Complex flows exist, thanks to Theorem 1.1.6, and the "maximal" complex flow associated to a holomorphic vector field is essentially unique. Moreover, directly from (1.3), it follows that $\theta^s(\theta^t(p)) = \theta^{t+s}(p)$ (whenever both members are defined), and hence $(\theta^t)^{-1}(p) = \theta^{-t}(p)$. In particular, up to shrinking U, we can suppose that the complex flow associated to X is defined on a neighborhood of $\mathbb{D}_\varepsilon \times U$, with $\mathbb{D}_\varepsilon \subset \mathbb{C}$ the disk of radius $\varepsilon > 0$, and hence θ^t are biholomorphisms from U to $\theta^t(U)$ for every $t \in \mathbb{D}_\varepsilon$. See [IY08, Section 1] for further details.

Example 1.1.10. Suppose we have a non-vanishing holomorphic vector field X defined on an open subset U of a complex surface M. Let Θ be the complex flow associated to X. Then we can consider the integral curves θ_p for every $p \in U$. Since X is non-vanishing, the integral curves θ_p are embeddings; moreover if two integral curves θ_p and θ_q have non-empty intersection for two points $p, q \in U$, then there exist points s, t in $\mathbb{C}$ such that $\theta^t(p) = \theta^s(q)$, or equivalently $q = (\theta^s)^{-1}(\theta^t(p)) = \theta^{t-s}(p)$. It follows that we have a partition of U into subsets of dimension 1.

In some way all these partitions seem related. Is there any connection with foliations? Is there any connection between them? We shall now prove some lemmas that show how they are related to each other. The first one connects holomorphic submersions with foliations.

Lemma 1.1.11. *Let $\mathcal{F}$ be a foliation of the complex surface M given by an atlas $\{(U_i, \phi_i)\}$, and let $\pi_2\colon \mathbb{C}^2 \to \mathbb{C}$ be the projection on the second coordinate. Set $f_i = \pi_2 \circ \phi_i$; then the $f_i\colon U_i \to \mathbb{C}$ are holomorphic submersions, and for every non-empty intersection $U_i \cap U_j$ there exists a biholomorphism $h_{ij}\colon f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ such that*

$$f_j(p) = h_{ij}\big(f_i(p)\big) \tag{1.4}$$

on $U_i \cap U_j$.

Conversely, given an open covering $\{U_i\}$ of M, a family of holomorphic submersions $f_i\colon U_i \to \mathbb{C}$, and biholomorphisms $h_{ij}\colon f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ satisfying (1.4) *whenever $U_i \cap U_j \neq \emptyset$, then up to refining the covering it exists a foliation $\mathcal{F}$ with atlas $\{(U_i, \phi_i)\}$ such that $f_i = \pi_2 \circ \phi_i$.*

Proof. Assume we have a foliation $\mathcal{F}$ given by an atlas $\{(U_i, \phi_i)\}$. Since ϕ_i is a chart, then obviously f_i is a submersion. Moreover the holomorphic map h_{ij} of equation (1.1) satisfies (1.4).

Conversely, we have by Theorem 1.1.4 that for each $p \in U_i$ there exists a neighborhood $U_{i,p} \subseteq U_i$ such that $\phi_{i,p} := (g_{i,p}, f_i|_{U_{i,p}})$ is a biholomorphism between $U_{i,p}$ and an open subset $V \times W$ of $\mathbb{C}^2$. Refining our cover $\{U_i\}$ to the cover $\{U_{i,p}\}$ we have an atlas $\{(U_{i,p}, \phi_{i,p})\}$ of M. We want to show that this atlas defines a holomorphic foliation; we only have to show that (1.1) holds for every couple of charts.

Let us consider two charts $(U_{i,p}, \phi_{i,p})$ and $(U_{j,q}, \phi_{j,q})$, with $U_{i,p} \cap U_{j,q} \neq \emptyset$; let us take $z \in U_{i,p} \cap U_{j,q}$, and $(x, y) = \phi_{i,p}^{-1}(z)$, i.e., $x = g_{j,q}(z)$ and $y = f_i(z)$. Then

$$\begin{aligned}\phi_{j,q} \circ \phi_{i,p}^{-1}(x, y) &= \phi_{j,q}(z) = \big(g_{j,q}(z), f_j(z)\big) = \Big(g_{j,q}(z), h_{ij}\big(f_i(z)\big)\Big) \\ &= \big(g_{j,q} \circ \phi_{i,p}(x, y), h_{ij}(y)\big),\end{aligned}$$

and we are done. □

Now we shall prove a lemma that connects non-vanishing holomorphic 1-forms and holomorphic submersions.

Lemma 1.1.12. *Let M be a complex surface, $\{U_i\}$ an open covering of M, $f_i \colon U_i \to \mathbb{C}$ holomorphic submersions, and $h_{ij} \colon f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ biholomorphisms satisfying* (1.4) *whenever $U_i \cap U_j \neq \emptyset$. Then up to refining the covering there exist a collection of non-vanishing holomorphic* 1*-forms $\omega_i \in \Omega^1(U_i)$ such that the level sets of f_i are tangent curves to ω_i, and for every non-empty intersection $U_i \cap U_j$, then ω_i and ω_j differ on $U_i \cap U_j$ by multiplication by a non-vanishing holomorphic function.*

Proof. The necessity of a refinement comes, as in Lemma 1.1.11, from Theorem 1.1.4. As showed in Lemma 1.1.11, without loss of generality we can assume to have an atlas $\{U_i\}$ such that on each U_i Theorem 1.1.4 gives rise to a biholomorphism (g_i, f_i) between U_i and $V_i \times W_i$ in $\mathbb{C}^2$. If we call h the inverse of (g_i, f_i) we have that the level sets of f_i are given by maps $h_c(z) := h(z, c)$ from V_i to M. So $f_i(h_c(z))) = c$. Differentiating both sides of this equation we get

$$(h_c^* df_i)(z) = df_i\left(\frac{\partial h_c(z)}{\partial z}\right) \equiv 0.$$

Since f_i is a submersion, the differential df_i is of maximal rank, and then it defines a non-vanishing holomorphic 1-form. So, on each U_i there exists a non-vanishing holomorphic 1-form $\omega_i = df_i$ whose tangent curves are the level sets of f_i. By assumption, we have that $f_j = h_{ij} \circ f_i$ on $U_i \cap U_j$. So

$$\omega_j = df_j = \left(\frac{\partial h_{ij}}{\partial z} \circ f_i\right) df_i = \left(\frac{\partial h_{ij}}{\partial z} \circ f_i\right) \omega_i,$$

and ω_i and ω_j differ by multiplication by the holomorphic function $\frac{\partial h_{ij}}{\partial z} \circ f_i$ on $U_i \cap U_j$, that is non-vanishing since h_{ij} is a biholomorphism. □

Now we shall see the correspondence between non-vanishing holomorphic 1-forms and non-vanishing holomorphic vector fields.

Lemma 1.1.13. *Let M be a complex surface and $\{U_i\}$ an atlas for M. Let us consider collections $\{\omega_i\}$ and $\{X_i\}$, with ω_i a non-vanishing holomorphic* 1*-form, and X_i a non-vanishing holomorphic vector field, both defined on U_i, and such that if $U_i \cap U_j$ is not empty then ω_i and ω_j (X_i and X_j respectively) differ on $U_i \cap U_j$ by multiplication by a non-vanishing holomorphic function.*

Then, given such a collection of one of those two kinds, there exists a collection of the other kind such that the tangent curves to ω_i coincide with the integral curves of X_i for every i.

Proof. Let us fix the index i, and set $U = U_i$. Let ω and X be respectively a non-vanishing holomorphic 1-form and a non-vanishing holomorphic vector field on U; we can write them in coordinates as

$$\omega = f(x,y)dx + g(x,y)dy, \qquad X = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}.$$

If $\gamma = (\gamma_1, \gamma_2)$ is a tangent curve to ω, then

$$f\big(\gamma(z)\big)\frac{d\gamma_1(z)}{dz} + g\big(\gamma(z)\big)\frac{d\gamma_2(z)}{dz} = 0.$$

This happens if and only if, up to a reparametrization of γ,

$$\frac{d\gamma(z)}{dz} = \begin{pmatrix} -g\big(\gamma(z)\big) \\ f\big(\gamma(z)\big) \end{pmatrix}. \tag{1.5}$$

On the other hand, γ is a integral curve of X if and only if

$$\frac{d\gamma(z)}{dz} = \begin{pmatrix} a\big(\gamma(z)\big) \\ b\big(\gamma(z)\big) \end{pmatrix}. \tag{1.6}$$

Equations (1.5) and (1.6) are holomorphic differential equations, as in (1.2), with $F(z,\gamma) = (-g(\gamma), f(\gamma))$ and $F(z,\gamma) = (a(\gamma), b(\gamma))$ respectively: thanks to Theorem 1.1.6 there exists local solutions, that coincide for uniqueness if $(-g, f) = (a, b)$.

This correspondence shows also that if we have two open sets of the covering U_i and U_j with non-empty intersection, and two holomorphic 1-forms ω_i and ω_j that differ by multiplication by a non-vanishing holomorphic function, then the corresponding holomorphic vector fields X_i and X_i differ by multiplication by the same non-vanishing holomorphic funtion, and viceversa. □

Finally, let us see how holomophic vector fields give rise to foliations.

Lemma 1.1.14. *Let M be a complex surface, $\{U_i\}$ a covering of M and $\{X_i\}$ a collection of non-vanishing holomorphic vector fields defined on U_i, such that if $U_i \cap U_j$ is not empty then X_i and X_j differ on $U_i \cap U_j$ by multiplication by a non-vanishing holomorphic function. Then, up to refining the covering of M, there exist holomorphic charts $\phi_i \colon U_i \to \mathbb{C}^2$ such that $\{(U_i, \phi_i)\}$ defines a foliation $\mathcal{F}$ on M and the leaves of $\mathcal{F}$ restricted to every U_i are the integral curves for X_i.*

Proof. Let $\{U_i\}$ and $\{X_i\}$ be as in the hypotheses (we can suppose that U_i are charts for M, up to refining the covering). Then for every i, let us consider the complex flow Θ_i associated to X_i; furthermore, for every $p \in U_i$, let us consider a holomorphic embedding $\tau\colon \mathbb{D}_\delta \to U_i$ such that $\tau(0) = p$ and τ is transverse to X_i, i.e., $d\tau_z\left(\frac{d}{dz}\right)$ and $X_{\tau(z)}$ are linearly independent (here $\mathbb{D}_\delta$ denotes the open disk of radius δ in $\mathbb{C}$). Set

$$\begin{aligned}\Theta_{i,p,\tau}\colon \mathbb{D}_\varepsilon \times \mathbb{D}_\delta &\to U_{i,p,\tau} \subset M\\ (t,z) &\mapsto \Theta_i(t,\tau(z)),\end{aligned}$$

where $U_{i,p,\tau}$ simply denotes the image of $\Theta_{i,p,\tau}$. Up to choosing ε and δ small enough, we can suppose that $U_{i,p,\tau} \subseteq U_i$.

First of all, let us see that $\Theta_{i,p,\tau}$ is a biholomorphism. Since

$$(d\Theta_{i,p,\tau})_{(t,z)} = (d\Theta_i)_{(t,\tau(z))} \circ (\mathrm{id}_t \times d\tau_z)$$

we get

$$(d\Theta_{i,p,\tau})_{(t,z)}\left(\frac{\partial}{\partial t}\right) = (d\Theta_i)_{(t,\tau(z))}\left(\frac{\partial}{\partial t}\right) = X_{\Theta_i(t,\tau(z))}, \tag{1.7}$$

$$(d\Theta_{i,p,\tau})_{(t,z)}\left(\frac{\partial}{\partial z}\right) = (d\Theta_i)_{(t,\tau(z))}\left(d\tau_z\left(\frac{d}{dz}\right)\right) = \left(d\theta_i^t\right)_{\tau(z)}\left(d\tau_z\left(\frac{d}{dz}\right)\right), \tag{1.8}$$

where $\theta_i^t(q) := \Theta_i(t,q)$. Then (1.7) and (1.8) are linearly independent because θ_i^t is a biholomorphism (see Remark 1.1.9), and thanks to the transversality condition.

Then define $\phi_{i,p,\tau}\colon U_{i,p,\tau} \to \mathbb{D}_\varepsilon \times \mathbb{D}_\delta$ as the inverse of $\Theta_{i,p,\tau}$. Let us now show that $\{(U_{i,p,\tau}, \phi_{i,p,\tau})\}$ defines a foliation; then the leaves of this foliation would be integral curves for X_i in every U_i for construction.

So let us consider a general transition map $\phi_{j,q,\sigma} \circ \phi_{i,p,\tau}^{-1}$ on the intersection of the two domains $U_{j,q,\sigma} \cap U_{i,p,\tau} =: U$. Let us first suppose $i = j$. If $x \in U$ then

$$u = \Theta_{i,p,\tau}(t,z) = \Theta_{i,q,\sigma}(s,w).$$

If we set

$$(T_{\tau\sigma}(z), \Delta_{\tau\sigma}(z)) = \phi_{i,q,\sigma} \circ \phi_{i,p,\tau}^{-1}(0,z),$$

then we have

$$(s,w) = \phi_{i,q,\sigma} \circ \phi_{i,p,\tau}^{-1}(t,z) = \big(t + T_{\tau\sigma}(z), \Delta_{\tau\sigma}(z)\big),$$

that is of the form (1.1).

Let us now suppose that $i \neq j$, but $p = q$ and $\tau = \sigma$. Since X_i and X_j differ by multiplication by a non-vanishing holomorphic function, the images of the integral curves of X_i and X_j starting at the same point coincide; it follows that the second coordinate of the transition map in this case is the identity (in the z coordinate), and hence of the form (1.1).

The general case is obtained by composing the previous two cases, and we are done. $\square$

Summing up we have obtained the following

Theorem 1.1.15. *A (regular) holomorphic foliation on a complex surface can be equivalently be described by:*

(a) a collection of pairs (U_i, ω_i) where ω_i is a non-vanishing holomorphic 1-form on U_i, and such that if $U_i \cap U_j$ is not empty then ω_i and ω_j differ on $U_i \cap U_j$ by multiplication by a non-vanishing holomorphic function;

(b) a collection of pairs (U_i, X_i) where X_i is a non-vanishing holomorphic vector field on U_i, and such that if $U_i \cap U_j$ is not empty then X_i and X_j differ on $U_i \cap U_j$ by multiplication by a non-vanishing holomorphic function;

(c) a collection of pairs (U_i, f_i) where $f_i \colon U_i \to \mathbb{C}$ is a holomorphic submersion and there exists a collection of biholomorphisms $h_{ij} \colon f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ such that on $U_i \cap U_j$ we have $f_j = h_{ij} \circ f_i$.

We conclude this section with a small digression on the *tangent bundle* of a foliation, and the *Čech cohomology*.

Definition 1.1.16. Let M be a complex surface and $\mathcal{F}$ the regular holomorphic foliation on M. Let $\{(U_i, \omega_i)\}$ and $\{U_i, X_i\}$ collections of holomorphic 1-forms and holomorphic vector fields respectively, associated to $\mathcal{F}$ as in Theorem 1.1.15. Then the **tangent bundle of the foliation** $\mathcal{F}$ is the sub-bundle of the tangent bundle TM given by $\operatorname{Ker} \omega_i$ (or equivalently by $\operatorname{Span}(X_i)$).

Remark 1.1.17. Suppose we are in the same setting as in the latter definition, and denote by $g_{ij} \colon U_i \cap U_j \to \mathbb{C}$ the non-vanishing holomorphic functions such that $\omega_j = g_{ij}\omega_i$.

Suppose now we have the intersection of three of such open sets $U_i \cap U_j \cap U_l \neq \emptyset$, and we look at the function $\gamma_{ijl} := g_{ij} g_{jl} g_{li}$ defined on this intersection. It is an easy computation to prove that $\gamma_{ijk} \equiv 1$ on $U_i \cap U_j \cap U_l$. So the collection $\{U_{ij}, g_{ij}\}$ is what is called a representative of a **cocycle** in **Čech cohomology**. Čech cohomology is an important tool in geometry, for an exposition we refer to [GH78, Chapter 0, Section 3]. In particular $\{U_{ij}, g_{ij}\}$ is an element of $H^1(M, \mathcal{O}^*)$. Each cocycle in this cohomology group is a complex line bundle on M. It is an easy check that this cocycle is the tangent bundle of the foliation $\mathcal{F}$ we defined above.

1.2 Singular foliations

Definition 1.2.1. Let M be a complex manifold of dimension m. A **singular holomorphic foliation** of dimension k (codimension $m-k$) is a pair $\mathcal{F} = (\mathcal{F}', \Sigma)$

where Σ is a proper analytic (non-empty) subset of M and $\mathcal{F}'$ is a regular holomorphic foliation on $M \setminus \Sigma$. The set Σ is called the **singular set** of $\mathcal{F}$. The foliation is called **saturated** if it cannot be extended to any point of the singular locus. A **leaf** (resp., a **plaque**) for $\mathcal{F}$ is a leaf (resp., a plaque) of the regular foliation $\mathcal{F}'$.

The main question of this section is to understand whether there exist equivalent definitions of singular foliations, like the ones we found for regular foliations.

From now on, in this section, we shall focus on complex dimension 2*.*

The natural way to extend Example 1.1.7 and Example 1.1.10 is to toss the *non-vanishing* request for the holomorphic 1-form or for the holomorphic vector field respectively.

Definition 1.2.2. Let U be an open domain in a complex surface M. A point $p \in U$ is a **singular point** of a holomorphic 1-form ω if $\omega_p = 0$. We shall call **singular set**, or **singular locus**, of ω the set of all singular points of ω.

Analogously, $p \in U$ is a **singular point** of a holomorphic vector field X if $X_p = 0$. We shall call **singular set**, or **singular locus**, of X the set of all singular points of X.

Let us show how a collection of holomorphic 1-forms (with singularities) gives rise to a singular holomorphic foliation.

Lemma 1.2.3. *Let M be a complex surface, and $\{U_i\}$ an atlas for M. Let $\{\omega_i\}$ be a collection of not identically zero holomorphic* 1*-forms, with ω_i defined on U_i, and such that if $U_i \cap U_j$ is not empty then ω_i and ω_j differ on $U_i \cap U_j$ by multiplication by a non-vanishing holomorphic function. Then there exists a (singular) holomorphic foliation $\mathcal{F}$ such that the tangent curves to ω_i outside the singular locus are (subsets of) leaves for $\mathcal{F}$.*

Proof. Let Σ be the union of the singular sets of the ω_i: since two such 1-forms ω_i and ω_j differ by multiplication by a non-vanishing function on $U_i \cap U_j$, their singular sets coincide in this intersection. Outside Σ Theorem 1.1.15 holds, so we obtain a holomorphic foliation outside Σ, and hence a singular holomorphic foliation. $\square$

Let us see now how the correspondence we saw in Lemma 1.1.13 can be extended to the non-vanishing case.

Lemma 1.2.4. *Let M be a complex surface and $\{U_i\}$ an atlas for M. Let us consider collections $\{\omega_i\}$ and $\{X_i\}$, with ω_i a not identically zero holomorphic* 1*-form, and X_i a not identically zero holomorphic vector field, both defined on U_i, and such that if $U_i \cap U_j$ is not empty then ω_i and ω_j (X_i and X_j respectively) differ on $U_i \cap U_j$ by multiplication by a non-vanishing holomorphic function.*

Then, given such a collection of one of those two kinds, there exists a collection of the other kind such that the tangent curves to ω_i coincide with the integral curves of X_i for every i.

Proof. The proof of this lemma is perfectly analogous to the proof of Lemma 1.1.13. □

Thanks to the correspondence in Lemma 1.2.4, and to Lemma 1.2.3, we obtain the next corollary (it can be also proved directly as in Lemma 1.2.3).

Corollary 1.2.5. *Let M be a complex surface, and $\{U_i\}$ an atlas for M. Let $\{X_i\}$ be a collection of not identically zero holomorphic vector fields, with X_i defined on U_i, and such that if $U_i \cap U_j$ is not empty then X_i and X_j differ on $U_i \cap U_j$ by multiplication by a non-vanishing holomorphic function. Then there exists a (singular) holomorphic foliation $\mathcal{F}$ such that the integral curves for X_i outside the singular locus are (subsets of) leaves for $\mathcal{F}$.*

Remark 1.2.6. The correspondence stated in Lemma 1.2.4 gives a correspondence between singular sets of a collection of holomorphic 1-forms and a collection of holomorphic vector fields.

In particular, pick a coordinate neighborhood U of M, a holomorphic 1-form ω and the corrisponding holomorphic vector field X, locally defined as

$$\omega = f(x,y)dx + g(x,y)dy, \qquad X = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y},$$

with the correspondence $(a,b) = (-g, f)$. Then the singular loci for X and for ω coincide.

The main difference from the regular case is that not every singular foliation is given by a collection of (singular) holomorphic vector fields (or holomorphic 1-forms), as the following example shows.

Example 1.2.7. Let us consider the holomorphic 1-form on $M' = \mathbb{C}^2 \setminus \{x = 0\}$ defined by

$$X = \frac{\partial}{\partial x} + e^{\frac{1}{x}}\frac{\partial}{\partial y}.$$

Since X is a non-vanishing holomorphic vector field in M', it defines a regular holomorphic foliation $\mathcal{F}'$ in M' (see Theorem 1.1.15), and hence a singular holomorphic foliation on $\mathbb{C}^2$, with singular set $\Sigma = \{x = 0\}$. Since $x = 0$ is a essencial singularity for $e^{1/x}$, for almost every $c \in \mathbb{C}$ the vector field X assumes the value $\frac{\partial}{\partial x} + c\frac{\partial}{\partial y}$ infinitely many times in every open neighborhood of a point $(0,y)$. In particular $\mathcal{F}'$ cannot be locally defined by a holomorphic vector field in a neighborhood of a point $(0,y)$.

Remark 1.2.8. The foliation defined on Example 1.2.7 has another peculiarity: it is saturated, and in particular it cannot be extended to another foliation with singular set of codimension greater than or equal to 2 (i.e. in this case, made by isolated points).

This is not a coincidence: we shall see that if a singular holomorphic foliation is defined by a collection of singular holomorphic vector fields (as in Corollary 1.2.5), then it can be extended to a saturated foliation with singular set of codimension 2 (Theorem 1.2.9), and conversely, if $\mathcal{F}$ is a singular holomorphic foliation with singular set of codimension 2, then it arises from a collection of holomorphic vector fields with (isolated) singularities (Theorem 1.2.12).

Theorem 1.2.9. *Let M be a complex surface, U a coordinate neighborhood in M, and X a holomorphic vector field on U, with singular set Σ. Let $\mathcal{F}$ be the singular holomorphic foliation that arises from X as in Corollary 1.2.5. Then we can find a singular holomorphic foliation $\mathcal{F}'$, with singular set $\Sigma' \subseteq \Sigma$ of codimension at least 2, whose leaves coincide with those of $\mathcal{F}$ outside Σ.*

Proof. The thesis is trivial if Σ has already codimension greater than or equal to 2. Let us suppose then that Σ has codimension 1. Up to composing by charts, we can suppose U to be an open domain in $\mathbb{C}^2$. Let us now consider a smooth point $p \in \Sigma$ (singular points in Σ are already an analitic subset of codimension greater than or equal to 2). Then in a sufficiently small neighborhood U_p of p, the singular set Σ is given by the zero locus of a suitable holomorphic function $h\colon U_p \to \mathbb{C}$, i.e., $\Sigma \cap U_p = \{h = 0\}$. Since p is a smooth point for Σ, h is irreducible. In U_p, choosing local coordinates (x, y) in p, we can write the vector field as

$$X(x,y) = h^m A(x,y)\frac{\partial}{\partial x} + h^n B(x,y)\frac{\partial}{\partial y},$$

with $h \not| A, B$. Set $k := \min\{m, n\}$. Then $X(x,y) = h^k X'(x,y)$, and X' is a holomorphic vector field in U with only isolated zeros, such that the integral curves for X' coincide with those of X in $U_p \setminus \{h = 0\}$ (since there X and X' differ by multiplication by h^k, a non-vanishing holomorphic function). □

To prove Theorem 1.2.12 we need Hartogs' Theorem and Cartan's Theorem, whose proofs can be found on [GH78, Chapter 0, Section 1, Hartog's Theorem] and [GR65, Theorem VIII.A.13] respectively.

Theorem 1.2.10 (Hartogs' Theorem). *Let U be a polydisk in $\mathbb{C}^2$ of radius r: $U = \mathbb{D}_r^2$. Let $V \subset\subset U$ be a polydisk of radius $r' < r$. Then any holomorphic function in a neighborhood of $U \setminus V$ extends to a holomorphic function on U.*

As a easy corollary we get that in $\mathbb{C}^2$ a holomorphic function defined on the complement of a point in an open subset extends to a holomorphic function on all the subset.

Theorem 1.2.11 **(Cartan's Theorem).** *In a neighborhood of a ball in $\mathbb{C}^n$ with $n \geq 2$ any meromorphic function is the quotient of two holomorphic functions A and B such that the intersection of their zero loci has no component of positive dimension.*

Theorem 1.2.12. *Let U be an open neighborhood of 0 in $\mathbb{C}^2$. Let $\mathcal{F}$ be a saturated singular holomorphic foliation on U with singular set $\Sigma = \{0\}$. Then, up to shrinking U, there exists a holomorphic vector field X in U, singular at 0, such that the integral curves of X outside $\{0\}$ coincide with the leaves of $\mathcal{F}$.*

Proof. Let $f\colon U \setminus \{0\} \to \mathbb{CP}^1$ be the function that to each point p in $U \setminus \{0\}$ associates the tangent line to the leaf of the foliation passing through p. This function is holomorphic, due to the holomorphicity of the foliation outside the singular point. This function cannot be constant, since being it constant the foliation would be trivially extended to 0. Using the representation of $\mathbb{CP}^1$ as $\mathbb{C} \cup \{\infty\}$ we can think of f as a meromorphic function on $U \setminus \{0\}$. By Theorem 1.2.11, restricting the neighborhood, we can represent $f(x, y)$ as the ratio of two holomorphic functions $p(x, y)$ and $q(x, y)$, relatively prime. By Theorem 1.2.10 we can extend p and q to the whole of U. The holomorphic vector field

$$X(x, y) = q(x, y)\frac{\partial}{\partial x} + p(x, y)\frac{\partial}{\partial y}$$

is such that its integral curves are leaves of the given foliation. If it was non-vanishing in 0 then we could extend the foliation in 0 by Theorem 1.1.15, contradicting the hypothesis. □

Then, for singular holomorphic foliations with singular set of codimension 2, we obtain this result, an analogue of Theorem 1.1.15.

Theorem 1.2.13. *A singular holomorphic foliation on a complex surface, with singular set of codimension 2 (i.e., made by isolated points) can be equivalently be described by:*

(a) a collection of pairs (U_i, ω_i) where ω_i is a holomorphic 1-form on U_i with singular locus made by isolated points, so that there exists a collection of non-vanishing functions g_{ij} such that we have $\omega_j = g_{ij}\omega_i$ on $U_i \cap U_j$;

(b) a collection of pairs (U_i, X_i) where X_i is a holomorphic vector field on U_i with singular locus made by isolated points, so that there exists a collection of non-vanishing functions g_{ij} such that we have $X_j = g_{ij}X_i$ on $U_i \cap U_j$.

In the following chapters, by a ***(singular) holomorphic foliation*** *we shall mean a singular holomorphic foliation on a complex surface, of complex dimension (and codimension) 1, and with isolated singular points.*

1.3 Holonomy of a foliation

Let M be a complex manifold of complex dimension m and let $\mathcal{F}$ be a codimension k holomorphic foliation (possibly with singularities). If F is a leaf of $\mathcal{F}$, and $\gamma \subset F$ a path in this leaf, we shall introduce in this section the *holonomy* of $\mathcal{F}$ along the path γ. When the foliation is given by a holomorphic vector field X, we shall see that the holonomy will be striclty correlated to the flow of X; as a matter of fact the holonomy is a sort of replacement for the flow of a vector field, when considered as the foliation it defines, that is when the parametrization is ignored.

Definition 1.3.1. Let M be a complex manifold of complex dimension m, $\mathcal{F}$ a regular holomorphic foliation of codimension k on an open set $U \subset M$, F a leaf of $\mathcal{F}$ and $p \in F$ a point. A **(parametrized) transverse section** of the leaf F at p is a holomorphic map $\tau\colon (\mathbb{C}^k, 0) \to (U, p)$ transverse to F, i.e., such that $T_pU = d\tau_0(T_0\mathbb{C}^k) \oplus T_pF$. We shall call the image of τ a **transverse section** of F at p.

With an abuse of notation, we shall often denote by τ, or by (τ, p), the transverse section associated to a parametrized transverse section τ at p.

Remark 1.3.2. Let us consider a chart (U, ϕ) of the foliation $\mathcal{F}$. Then in these local coordinates, the plaques are of the form $F_c = \phi^{-1}(V \times \{c\})$, with V a suitable open set in $\mathbb{C}^{m-k}$. Let τ_p be a transverse section at a point $p = \phi^{-1}(x_0, c)$. Then we have that

$$\phi \circ \tau_p(w) = \big(g_p(w), h_p(w)\big),$$

with $\det d(h_p)_0 \neq 0$ (and $(g_p(0), h_p(0)) = (x_0, c)$); in particular h_p is invertible near $h_p(0) = c$.

Suppose now we have two points $p = \phi^{-1}(x_0, c)$, $q = \phi^{-1}(y_0, c)$ on the same plaque F_c, and two transverse sections $\tau_p = \phi^{-1} \circ (g_p, h_p)$ and $\tau_q = \phi^{-1} \circ (g_q, h_q)$ at p and q respectively.

Since h_p and h_q are invertible at $h_p(0) = h_q(0) = c$, the map $\Delta_{\tau_p,\tau_q}\colon (\tau_p, p) \to (\tau_q, q)$ given by

$$\Delta_{\tau_p,\tau_q}(w) = \tau_q \circ h_q^{-1} \circ h_p \circ \tau_p^{-1}(w) \tag{1.9}$$

is a well-defined biholomorphism (near p).

This germ does not depend on the parametrizations chosen. Suppose we have another transverse section $\tilde{\tau}_p$ such that $\tau_p = \tilde{\tau}_p \circ \psi_p$, with ψ_p a biholomorphism of $(\mathbb{C}^k, 0)$. Then, if we set

$$\phi \circ \tilde{\tau}_p = (\tilde{g}_p, \tilde{h}_p),$$

we have $\phi \circ \tau_p = \phi \circ \tilde{\tau}_p \circ \psi_p$, and hence $h_p = \tilde{h}_p \circ \psi_p$, and $h_p \circ \tau_p^{-1} = \tilde{h}_p \circ \tilde{\tau}_p^{-1}$.

Moreover Δ_{τ_p,τ_q} does not depend on the chart of the foliation chosen. Indeed, let (U_1, ϕ_1) and (U_2, ϕ_2) be two charts of $\mathcal{F}$, and p, q two points in the same leaf of

$\mathcal{F}$, and in the same connected component of the intersection of domains $U_1 \cap U_j$. If we write the transverse sections using (U_j, ϕ_j) as local coordinates, we have

$$\phi_j \circ \tau_p(w) = \big(g_{p,j}(w), h_{p,j}(w)\big), \qquad \phi_j \circ \tau_q(w) = \big(g_{q,j}(w), h_{q,j}(w)\big).$$

Denote by π_2 the canonical projection $\mathbb{C}^{m-k} \times \mathbb{C}^k \to \mathbb{C}^k$; then thanks to (1.1), we have that there exists a biholomorphism h such that

$$\pi_2 \circ \phi_2 \circ \phi_1^{-1} = h \circ \pi_2.$$

Then

$$h_{p,2} = \pi_2 \circ \phi_2 \circ \tau_p = \pi_2 \circ \phi_2 \circ \phi_1^{-1} \circ \phi_1 \circ \tau_p = h \circ \pi_2 \circ \phi_1 \circ \tau_p = h \circ h_{p,1},$$

and the same for the point q. It follows that

$$\Delta^2_{\tau_p,\tau_q} = \tau_q \circ h_{q,2}^{-1} \circ h_{p,2} \circ \tau_p^{-1} = \tau_q \circ h_{q,1}^{-1} \circ h^{-1} \circ h \circ h_{p,1} \circ \tau_p^{-1} = \Delta^1_{\tau_p,\tau_q}.$$

If we have another point $r \in F_c$, and a transverse section τ_r at r, then we have

$$\Delta_{\tau_p,\tau_q} = \Delta_{\tau_r,\tau_q} \circ \Delta_{\tau_p,\tau_r}. \tag{1.10}$$

Indeed, write τ_r in local coordinates: $\phi \circ \tau_r = (g_r, h_r)$. Then we have

$$\Delta_{\tau_r,\tau_q} \circ \Delta_{\tau_p,\tau_r} = \tau_q \circ h_q^{-1} \circ h_r \circ \tau_r^{-1} \circ \tau_r \circ h_r^{-1} \circ h_p \circ \tau_p^{-1} = \tau_q \circ h_q^{-1} \circ h_p \circ \tau_p^{-1} = \Delta_{\tau_p,\tau_q}.$$

Definition 1.3.3. Let M be a complex manifold of complex dimension m, $\mathcal{F}$ a regular holomorphic foliation of codimension k on M, (U, ϕ) a chart of the foliation $\mathcal{F}$, and F a plaque. Given two points p and q in F, and two transverse sections τ_p, τ_q in p and q respectively, the map $\Delta_{\tau_p,\tau_q}\colon (\tau_p, p) \to (\tau_q, q)$ defined as in Remark 1.3.2 by (1.9) is called the **correspondence map** between τ_p and τ_q.

Remark 1.3.4. We have already seen transverse sections and correspondence maps on the proof of Lemma 1.1.14 (the notations in that proof are coherent with the one we are using here): you can see there the connection between correspondence maps of a foliation given by a holomorphic vector field and its holomorphic flow.

Let us consider now a fixed leaf F of our foliation $\mathcal{F}$, two points $p, q \in F$, and choose a path $\gamma\colon [0,1] \to F$ with $\gamma(0) = p$ and $\gamma(1) = q$. Since the image of γ in F is compact we can find a finite cover $\{U_k\}_{k=1,\dots,n}$ of $\gamma([0,1])$ by charts of $\mathcal{F}$. Choose $0 = t_0 < t_1 < \cdots < t_{l-1} < t_l = 1$ such that $\gamma([t_j, t_{j+1}]) \subset U_k$ for some k, for every $j = 0, \dots, l-1$, and set $p_j = \gamma(t_j)$.

Let τ_j be a transverse section at p_j for every $j = 0, \dots, l$. Since p_j and p_{j+1} belongs to some chart U_k of the foliation, and the correspondence map does not depend on the chart chosen (see Remark 1.3.2), the correspondence maps $\Delta_{\tau_j,\tau_{j+1}}$ are well-defined for each $j = 0, \dots, l-1$. Their composition

$$\Delta_\gamma := \bigcirc_{j=1}^{l} \Delta_{\tau_{l-j},\tau_{l-j+1}} := \Delta_{\tau_{l-1},\tau_l} \circ \cdots \circ \Delta_{\tau_0,\tau_1} \tag{1.11}$$

is a biholomorphism between (τ_0, p) and (τ_l, q).

Remark 1.3.5. The map Δ_γ defined as in (1.11) does not depend either on the covering $\{U_j\}$, or on the intermediate points p_j and transverse sections τ_j (with $j = 1, \cdots, l-1$).

It does not depend on the covering since we have already seen that correspondence maps do not. Suppose then that we have another partition $0 = s_0 < s_1 < \cdots < s_m = 1$ such that $\gamma([s_j, s_{j+1}]) \subset U_k$ for some k, for every $j = 0, \dots m-1$, set $q_j = \gamma(s_j)$, and let $\sigma_0, \dots, \sigma_m$ be transverse sections at $q_0, \dots, q_m$ respectively (with $\sigma_0 = \tau_0 = \tau_p$ and $\sigma_m = \tau_l = \tau_q$).

Up to refining these partitions (taking all t_j and s_j for both of them), we can suppose that $m = l$ and $s_j = t_j$ for all $j = 0, \dots, l$; the composition Δ_γ does not change up to refining, thanks to (1.10).

Moreover, thanks to (1.10) again, if we set $\Delta_{\tau_j,\sigma_j} = \theta_j$ (and hence $\Delta_{\sigma_j,\tau_j} = \theta_j^{-1}$), we have that

$$\Delta_{\tau_j,\tau_{j+1}} = \theta_{j+1}^{-1} \circ \Delta_{\sigma_j,\sigma_{j+1}} \circ \theta_j.$$

It follows that

$$\begin{aligned}
\bigcirc_{j=1}^{l} \Delta_{\tau_{l-j},\tau l-j+1} &= \bigcirc_{j=1}^{l} \theta_{l-j+1}^{-1} \circ \Delta_{\sigma_{l-j},\sigma_{l-j+1}} \circ \theta_{l-j} \\
&= \theta_l^{-1} \circ \bigcirc_{j=1}^{l-1} \left(\Delta_{\sigma_{l-j},\sigma_{l-j+1}} \circ \theta_{l-j} \circ \theta_{l-j}^{-1}\right) \circ \Delta_{\sigma_0,\sigma_1} \circ \theta_0 \\
&= \theta_l^{-1} \circ \bigcirc_{j=1}^{l} \left(\Delta_{\sigma_{l-j},\sigma_{l-j+1}}\right) \circ \theta_0.
\end{aligned} \tag{1.12}$$

Since $\tau_0 = \sigma_0$ and $\tau_l = \sigma_l$, then $\theta_0 = \mathrm{id}$ and $\theta_l = \mathrm{id}$, and Δ_γ does not depend on the choices we made.

Definition 1.3.6. Let M be a complex manifold of complex dimension m, $\mathcal{F}$ a holomorphic foliation of codimension k, F a leaf of $\mathcal{F}$, and $\gamma\colon [0,1] \to F$ a path. Consider two transverse sections τ_p and τ_q at $p = \gamma(0)$ and $q = \gamma(1)$ respectively. The map Δ_γ defined by (1.11), computed with respect to τ_p and τ_p is called the **holonomy map** associated to γ (with respect to τ_p and τ_q).

Remark 1.3.7. If we allow the transverse sections at the endpoints p and q to change, say from τ_p, τ_q to σ_p, σ_q, then, denoting by Δ_γ^τ the holonomy of γ computed with respect to τ_p and τ_q and by Δ_γ^σ the one computed with respect to σ_p and σ_q, with computations as in (1.12), we get

$$\Delta_\gamma^\tau = \theta_q^{-1} \circ \Delta_\gamma^\sigma \circ \theta_p,$$

with $\theta_p = \Delta_{\tau_p,\sigma_p}$ and $\theta_q = \Delta_{\tau_q,\sigma_q}$.

When γ is a loop, i.e., $p = q$, it is natural to choose the same transverse section at γ_0 and γ_1; in this case $\theta_p = \theta_q$ and the two holonomies are conjugated.

Theorem 1.3.8. *Let M be a complex manifold of complex dimension m, $\mathcal{F}$ a holomorphic foliation of codimension k and F a leaf of $\mathcal{F}$. Let us consider two points $p, q \in F$ and two transverse sections τ_p and τ_q at p and q respectively. If γ_0 and γ_1 are two homotopic paths from p to q in F, then $\Delta_{\gamma_0} = \Delta_{\gamma_1}$, where the holonomies are computed with respect to τ_p and τ_q.*

Proof. Set $I = [0,1]$, and let $\Gamma\colon I \times I \to F$ be an homotopy from γ_0 to γ_1, i.e., $\Gamma(s,\cdot) = \gamma_s$ for $s = 0,1$, and, if we denote $\gamma_s(t) = \Gamma(s,t)$, then $\gamma_s(0) = p$ and $\gamma_s(1) = q$ for every $s \in I$. Let $\{U_k\}$ with $k = 1,\dots,l$ be a finite cover of $\Gamma(I \times I)$, made by charts of the foliation. Pick $0 = s_0 \le s_1 < \cdots < s_{u-1} \le s_u = 1$ and $0 = t_0 \le t_1 < \cdots < t_{v-1} \le t_v = 1$ partitions of I such that $\Gamma([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \subset U_k$ for some k, for every $i = 0,\dots,u-1$ and $j = 0,\dots,v-1$.

For every $i = 0,\dots,u$ and $j = 0,\dots,v$, fix transverse sections $\tau_{i,j}$ at $p_{i,j} := \Gamma(s_i,t_j)$, such that $\tau_{i,0} = \tau_p$ and $\tau_{i,v} = \tau_q$ for every i. Then we can compute the holonomy along γ_{s_i} with respect to the transverse sections $\tau_{i,j}$ with $j = 0,\dots,v$, i.e.,

$$\Delta_{\gamma_{s_i}} = \bigcirc_{j=1}^{v} \delta_{i,v-j},$$

where $\delta_{i,j} = \Delta_{\tau_{i,j},\tau_{i,j+1}}$.

For our choices of s_i and t_j, we also have that the correspondence map $\theta_{i,j} := \Delta_{\tau_{i,j},\tau_{i+1,j}}$ is well-defined for every $i = 0,\dots u-1$ and for every $j = 0,\dots,v$, with $\theta_{i,0} = \Delta_{\tau_p,\tau_p} = \mathrm{id}$ and $\theta_{i,v} = \Delta_{\tau_q,\tau_q} = \mathrm{id}$ for every $i = 0,\dots,u-1$, and

$$\delta_{i+1,j} = \theta_{i,j+1} \circ \delta_{i,j} \circ \theta_{i,j}^{-1}$$

for every $i = 0, \ldots, u-1$ and $j = 0, \ldots, v-1$. Then

$$\begin{aligned}
\Delta_{\gamma_{s_{i+1}}} &= \bigcirc_{j=1}^{v} \delta_{i+1,v-j} \\
&= \bigcirc_{j=1}^{v} \theta_{i,v-j+1} \circ \delta_{i,v-j} \circ \theta_{i,v-j}^{-1} \\
&= \theta_{i,v} \circ \bigcirc_{j=1}^{v-1} \left(\delta_{i,v-j} \circ \theta_{i,v-j}^{-1} \circ \theta_{i,v-j}\right) \circ \delta_{i,0} \circ \theta_{i,0}^{-1} \\
&= \theta_{i,v} \circ \bigcirc_{j=1}^{v} \left(\delta_{i,v-j}\right) \circ \theta_{i,0}^{-1} \\
&= \Delta_{\gamma_{s_i}},
\end{aligned}$$

for $i = 0, \ldots, u-1$, and we are done. $\square$

Thanks to Theorem 1.3.8, we can give the next definitions.

Definition 1.3.9. Let M be a complex manifold of complex dimension m, $\mathcal{F}$ a holomorphic foliation of codimension k, F a leaf of $\mathcal{F}$, p a point in F and $\gamma\colon [0,1] \to F$ a loop. Then the conjugacy class of the holonomy Δ_γ is called the **holonomy** of the class $[\gamma] \in \pi_1(F,p)$, and denoted by $\Delta_{[\gamma]}$. The set

$$\mathrm{Hol}(F,p) = \{\Delta_{[\gamma]} \mid [\gamma] \in \pi_1(F,p)\}$$

is called the **holonomy group** of F based at p.

If τ is a transverse section at p, then the set $\mathrm{Hol}(F,p,\tau)$ of the holonomy maps associated with the loops in F based at p, computed with respect to τ, is called the **holonomy group** of F based at p with respect to τ.

Corollary 1.3.10. *Let $\mathcal{F}$ be a holomorphic foliation on a complex manifold M, F a leaf of $\mathcal{F}$, and p a point in F. Let $\pi_1(F,p)$ be the fundamental group of F with base point p. The map which associates to $[\gamma] \in \pi_1(F,p)$ the conjugacy class of its holonomy map Δ_γ is an antihomomorphism of groups (i.e., is an homomorphism, but with the multiplication reversed).*

Proof. Suppose we have $\gamma, \beta \in \pi_1(F,p)$. By Theorem 1.3.8, the holonomy map depends only on the homotopy class of the curve, and by definition $\Delta_{\gamma\cdot\beta} = \Delta_\beta \circ \Delta_\gamma$, where with $\cdot$ we denoted the operation in the homotopy group $\pi_1(F,p)$. $\square$

Remark 1.3.11. In the same setting as Corollary 1.3.10, given a transverse section τ at p, then the map which associates to $[\gamma] \in \pi_1(F,p)$ the holonomy map $\Delta_\gamma \in \mathrm{Hol}(F,p,\tau)$, computed with respect to a transverse section τ in p, is well-defined, thanks to Theorem 1.3.8 and Remark 1.3.5, and an antihomomorphism (it can be proved exactly as Corollary 1.3.10).

Corollary 1.3.12. *Let $\mathcal{F}$ be a holomorphic foliation on a complex manifold M and F a leaf of $\mathcal{F}$. Then for any $p_0, p_1 \in F$ the holonomy groups* $\mathrm{Hol}(F, p_0)$ *and* $\mathrm{Hol}(F, p_1)$ *are conjugated.*

Proof. Let γ be a path connecting p_0 and p_1 in F (leaves of a foliation are pathwise-connected), and let Δ_γ be the corresponding holonomy map. Consider the isomorphism $\gamma^*\colon \pi_1(F, p_1) \to \pi_1(F, p_0)$ induced by γ on the fundamental groups, i.e., given by $\gamma^*[\sigma] = [\gamma \cdot \sigma \cdot \gamma^{-1}]$. Thanks to Corollary 1.3.10, we have

$$\Delta_{\gamma^*[\sigma]} = \Delta_{[\gamma\cdot\sigma\cdot\gamma^{-1}]} = \Delta_{[\gamma]}^{-1} \circ \Delta_{[\sigma]} \circ \Delta_{[\gamma]}.$$

Hence $\Delta_{[\gamma]}$ defines the conjugation between $\mathrm{Hol}(F, p_0)$ and $\mathrm{Hol}(F, p_1)$. $\square$

Since the holonomy group of a leaf F based at a point p is a set of conjugacy classes, thanks to Corollary 1.3.12 $\mathrm{Hol}(F, p)$ does not depend on the base point p. Thus we can give the next definition.

Definition 1.3.13. Let $\mathcal{F}$ be a holomorphic foliation on a complex manifold M, and F a leaf of $\mathcal{F}$. Then we call **holonomy group** of F the group $\mathrm{Hol}\, F = \mathrm{Hol}(F, p)$, where p is an arbitrary point in F.

Remark 1.3.14. Let $\mathcal{F}$ be a foliation of a manifold M, F a leaf of $\mathcal{F}$ and p a point of F. The holonomy group $\mathrm{Hol}(F, p)$ defines a local action of $\pi_1(F, p)$ on τ_0, a transverse section in p, by setting

$$[\gamma] \cdot w \mapsto \Delta_{[\gamma]}(w).$$

1.4 Conjugated holonomies and consequences

In the previous sections we have studied foliations in complex manifolds, but foliations can be studied even in a less regular contest. In particular we can give an analogous definition for a **(regular)** C^r **foliation** $\mathcal{F}$ on a smooth (real) manifold M as in Definition 1.1.1, just replacing "complex manifold" with "smooth manifold", "holomorhic" with "C^r", "biholomorphism" with "C^r diffeomorphism" and $\mathbb{C}$ with $\mathbb{R}$ when you deal with local coordinates and transition maps. All we have seen in the last section for regular holomorphic foliations can be generalized to C^r foliations: we shall not give explicit definitions since they are analogous to the holomorphic case.

Definition 1.4.1. Let $(M, \mathcal{F})$,$(M', \mathcal{F}')$ be two codimension k C^r foliations (resp., holomorphic foliations) of two smooth manifolds (resp., complex manifolds) M and M' respectively. Let F and F' be leaves of $\mathcal{F}$ and $\mathcal{F}'$ respectively, $p \in F$ and $p' \in F'$ two points. We say that the holonomy groups $\mathrm{Hol}(F)$ and $\mathrm{Hol}(F')$ of F

and F' are C^r **conjugated** (resp., **holomorphically conjugated**) if there exist transverse sections τ at p and τ' at p' and a homeomorphism $\phi\colon F \cup \tau \to F' \cup \tau'$ such that $\phi(p) = p'$, $\phi|_F$ and ϕ_τ are C^r diffeomorphisms (resp., biholomorphisms) and for each $[\gamma] \in \pi_1(F, p_0)$ one has

$$\phi \circ \Delta_\gamma \circ \phi^{-1}(s') = \Delta_{\phi\circ\gamma}(s'),$$

for every $s' \in \tau'$ sufficiently near p'.

Remark 1.4.2. In the setting of the latter definition, if there exists τ, τ' and ϕ that defines a conjugacy, then for every transverse sections σ at p and σ' at p' there exists a homeomorphism $\psi\colon F \cup \sigma \to F' \cup \sigma'$ that also defines a conjugacy between holonomy groups (with respect to σ and σ').

Indeed, if we denote by Δ correspondence maps and holonomies with respect to $\mathcal{F}$ and by Δ' correspondence maps and holonomies with respect to $\mathcal{F}'$, setting

$$\psi = \Delta'_{\tau'\sigma'} \circ \phi \circ \Delta_{\sigma\tau}$$

on σ, and $\psi|_F = \phi|_F$, we have

$$\begin{aligned} \psi \circ \Delta^\sigma_\gamma \circ \psi^{-1} &= \Delta'_{\tau'\sigma'} \circ \phi \circ \Delta_{\sigma\tau} \circ \Delta_{\tau\sigma} \circ \Delta^\tau_\gamma \circ \Delta_{\sigma\tau} \circ \Delta_{\tau\sigma} \circ \phi \circ \Delta'_{\sigma'\tau'} \\ &= \Delta'_{\tau'\sigma'} \circ \Delta^{\tau'}_{\phi\circ\gamma} \circ \Delta'_{\sigma'\tau'} = \Delta^{\sigma'}_{\psi\circ\gamma}, \end{aligned}$$

where we used the notation Δ^τ_γ for the holonomy associated to γ, computed using τ as the transverse section at the base point.

Whilst in the C^r case, conjugated holonomies imply conjugated foliations, this is not true in the holomorphic case. For the sake of completeness we prove this result in the C^r case here; in the following chapters we shall see what can be said in the holomorphic case (see Theorem 4.4.4, Remark 4.4.5 and Counterexample 4.4.6).

As a matter of fact the construction fails due to the fact that in the holomorphic case we lack the following lemma, which relies on the existence of tubular neighborhoods, whereas in general holomorphic tubular neighborhoods of complex submanifolds do not exist.

Lemma 1.4.3. *Let M be a C^r-manifold of dimension m let $\mathcal{F}$ be a codimension k foliation of M and let F be a leaf of $\mathcal{F}$ and K a compact set in F. Then there exist an open neighborhood U of K in M, an open neighborhood W of K in F, and a C^r retraction $\pi\colon U \to W$ such that $\pi^{-1}(x)$ is transverse to $\mathcal{F}|_U$ for any $x \in W$.*

Proof. Since K is compact in F we can cover it by a finite number of plaques W_i of $\mathcal{F}$, whose union we shall call W. Since W is a C^r submanifold of M there exists a C^r tubular neighborhood $\pi\colon \tilde{W} \to W$ of W. Since each fiber $\pi^{-1}(y)$ meets W transversally and at y only, and since transversality is an open condition, if we take $x \in \pi^{-1}(y)$ sufficiently near y then $\pi^{-1}(y)$ meets the leaf of $\mathcal{F}$ through x transversally at x. We can then obtain a neighborhood $U \subset \tilde{W}$ of W such that $\pi^{-1}(y)$ meets $\mathcal{F}|_U$ trasversally for all $y \in U \cap F$. □

Lemma 1.4.4. *Let M and M' be complex manifolds (resp., smooth manifolds) of dimension m, and $\mathcal{F}$ and $\mathcal{F}'$ codimension k holomorphic foliations (resp., C^r foliations) of M and M' respectively. Let F be a leaf of $\mathcal{F}$ and let F' be a leaf of $\mathcal{F}'$. If there exist V neighborhood of F, V' neighborhood of F' and a biholomorphism (resp., C^r diffeomorphism) $\Phi\colon V \to V'$ such that $\Phi(F) = F'$ and leaves of $\mathcal{F}$ are mapped by Φ to leaves of $\mathcal{F}'$ then the holonomy groups of F and F' are holomorphically (resp., C^r) conjugated.*

Proof. The proof is formally the same in the C^r and in the holomorphic case. Let p be a point in F, and set $p' = \Phi(p)$. Let γ be a loop in F with base point p, and τ a transverse section in p contained in V. Then $\Phi(\gamma)$ is a curve in F' and $\Phi \circ \tau$ is a transverse section in p' to the leaves of $\mathcal{F}'$, since Φ maps leaves in leaves. Choose intermediate points $p_0, \ldots, p_n$ in γ and transverse sections $\tau_0, \ldots, \tau_n$ (with $p_0 = p_n = p$ and $\tau_0 = \tau_n = \tau$). Let now $\Delta_{\tau_i,\tau_{i+1}}$ be the correspondence map between τ_i and τ_{i+1}: since Φ maps leaves into leaves and since $\Delta_{\tau_i,\tau_{i+1}}(q)$, for q in τ_i is the point of intersection of the only leaf passing through q with τ_{i+1} and since $\Phi \circ \tau_i$ and $\Phi \circ \tau_{i+1}$ are transverse sections to F we have that $\Delta_{\Phi(\tau_i),\Phi(\tau_{i+1})}(\Phi(q)) = \Phi(\Delta_{\tau_i,\tau_{i+1}}(q))$. From the construction of the holonomy map for $\Phi(\gamma)$ we get that

$$\Phi^{-1} \circ \Delta_{\gamma'} \circ \Phi = \Delta_\gamma.$$

Thanks to the arbitrariness of γ we obtain the assertion. □

Theorem 1.4.5. *Let $\mathcal{F}$ and $\mathcal{F}'$ be two foliations of codimension k in C^r manifolds M and M' of dimension m, and let F and F' be compact leaves of $\mathcal{F}$, $\mathcal{F}'$ respectively. The holonomy groups of F and F' are C^r conjugated if and only if there exist neighborhoods V of F, V' of F' and a C^r diffeomorphism $\Phi\colon V \to V'$, with $\Phi(F) = F'$ and sending leaves of $\mathcal{F}|_V$ into leaves of $\mathcal{F}'|_{V'}$*

Proof. One direction follows directly from Lemma 1.4.4. Conversely, assume that the holonomy groups of F and F' are conjugated, i.e., there exists a C^r diffeomorphism $\psi\colon F \to F'$, a point $p_0 \in F$, two transverse sections τ_0 at p_0 and τ_0' at $p_0' = \psi(p_0)$, and a C^r diffeomorphism $\phi\colon (\tau_0, p_0) \to (\tau_0', p_0')$ such that

$$\phi \circ \Delta_\gamma \circ \phi^{-1} = \Delta'_{\psi\circ\gamma} \tag{1.13}$$

for every loop γ in F with base point p_0; we denoted the holonomy with respect to $\mathcal{F}$ by Δ and the holonomy with respect to $\mathcal{F}'$ by Δ'.

Since F and F' are compact there exist, using Lemma 1.4.3, neighborhoods V of F and V' of F' and retractions $\pi\colon V \to F$ with fibers transverse to $\mathcal{F}$ and $\pi'\colon V' \to F'$ with fibers transverse to $\mathcal{F}'$. Pick two transverse sections τ_0 and τ_0' Thanks to Remark 1.4.2, we can suppose $\tau_0 = \pi^{-1}(p_0)$ and $\tau_0' = (\pi')^{-1}(p_0')$. Let $p \in F$, $p \neq p_0$; choose a path γ joining p_0 to p in F, and let Δ_γ and $\Delta'_{\psi\circ\gamma}$ be the holonomy maps associated to γ and $\psi\circ\gamma$. Let us denote by τ_p the transverse section in p defined by $\pi^{-1}(p)$; if $x \in \tau_p$, then $\Delta_{\gamma^{-1}}(x)$ is well-defined for x sufficiently close to p. Thus we define

$$\Phi(x) = \Delta'_{\psi\circ\gamma} \circ \phi \circ \Delta_{\gamma^{-1}}(x).$$

Suppose now we have another path μ connecting p_0 and p and note that $[\mu \cdot \gamma^{-1}]$ is in $\pi_1(F, p_0)$. Thanks to (1.13), we have

$$\phi \circ \Delta_{\mu\cdot\gamma^{-1}} = \Delta'_{\psi\circ(\mu\cdot\gamma^{-1})} \circ \phi.$$

Now, the map that associates to a path its holonomy map is an antihomomorphism of groups (see Corollary 1.3.10), so we have

$$\phi \circ \Delta_\gamma^{-1} \circ \Delta_\mu = \left(\Delta'_{\psi\circ\gamma}\right)^{-1} \circ \Delta'_{\psi\circ\mu} \circ \phi,$$

and hence

$$\Delta'_{\psi\circ\gamma} \circ \phi \circ \Delta_{\gamma^{-1}}(x) = \Delta'_{\psi\circ\mu} \circ \phi \circ \Delta_{\mu^{-1}}(x) = \Phi(x);$$

thus our definition of Φ does not depend on the path γ we choose.

We only have to show that Φ is a C^r diffeomorphism.

It is invertible (up to shrinking the tubular neighborhoods V and V'), since ψ is a homeomorphism itself; let us show that Φ is a C^r map. Let p be a point in F, and γ a path in F that connects p_0 to p. Choose a coordinate neighborhood $U \subset V$ for the foliation $\mathcal{F}$, and a coordinate neighborhood $U' \subset V'$ for the foliation $\mathcal{F}'$, with $p \in U$ and $p' \in U'$.

Since Φ does not depend on the path chosen, and the holonomy along a path in a plaque is simply the correspondence map between the transverse sections, we have

$$\Phi(y) = \Delta'_{\tau_{\psi(p)},\tau_{\psi(q)}} \circ \Delta'_{\psi\circ\gamma} \circ \phi \circ \Delta_{\gamma^{-1}} \circ \Delta_{\tau_q,\tau_p}(y),$$

where $q = \pi(y)$, for every $y \in U \cap \Phi^{-1}(U')$. Thus Φ is a composition of C^r maps, (that correspondence maps have the same regularity as the foliation follows directly from the definition (1.9)) and hence it is a C^r map (near p). For arbitrariness of p, we obtain the assertion. Since Φ^{-1} has the same form of ϕ, it is a C^r map too, and we are done. □

Chapter 2

Reduction of singularities of foliations

Tiziano Casavecchia[1]

In this chapter we are going to present some classical results about desingularization of the germ of a given holomorphic foliation around a singular point. We shall start by giving some basic definitions of blow-up in the first section, inspired by algebraic geometry. Then we shall define the multiplicity of a foliation and its properties in the second section. Finally we shall prove the reduction of singularities Theorem 2.3.3 in the last section. We shall confine ourselves to the two dimensional case. We follow the presentation given in [IY08], [Żoł06] and [MM80].

2.1 Desingularization of analytic subsets and of foliations

Our idea is to proceed as follows: we shall first introduce the blow-up of $\mathbb{C}^2$ at the origin, then the blow-up of a complex surface around a point and then for a foliation around a singularity. We refer to the books [IY08], [Żoł06] for a quick and straightforward exposition, particularly useful for our purpose; in [GLS07] there is a deeper study of blow-ups.

Let 0 be the origin of $\mathbb{C}^2$, and $\mathbb{CP}^1$ the projective space. Let $\widetilde{\mathbb{C}}^2$ be the following subset of $\mathbb{C}^2 \times \mathbb{CP}^1$:

$$\widetilde{\mathbb{C}}^2 := \big\{(z, w, [u : t]) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid zt = uw\big\}. \tag{2.1}$$

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For every point $[u:t] \in \mathbb{CP}^1$, this set contains all the points $(z, w, [u:t]) \in \mathbb{C}^2 \times \{[u:t]\}$ such that (z,w) lies inside the line determined by $[u:t]$ in $\mathbb{C}^2$. We note that $\widetilde{\mathbb{C}}^2$ is a complex surface; indeed consider the set

$$V = \left\{(z, w, [u:t]) \in \widetilde{\mathbb{C}}^2 \mid u \neq 0\right\} \tag{2.2}$$

and

$$W = \left\{(z, w, [u:t]) \in \widetilde{\mathbb{C}}^2 \mid t \neq 0\right\}; \tag{2.3}$$

they are open in $\widetilde{\mathbb{C}}^2$ because intersections of it with open sets. Furthermore set

$$\varphi\colon \mathbb{C}^2 \ni (z, t) \mapsto (z, zt, [1:t]) \in V \tag{2.4}$$

and

$$\psi\colon \mathbb{C}^2 \ni (w, u) \mapsto (uw, w, [u:1]) \in W. \tag{2.5}$$

The maps φ and ψ are holomorphic, invertible, with continuous inverse as can be easily checked; furthermore the composition $\varphi \circ \psi^{-1}$ is a biholomorphism; so they give $\widetilde{\mathbb{C}}^2$ a structure of complex 2-manifold.

We remark that φ sends the set $\{(z, w) \in \mathbb{C}^2 \mid z = 0\}$ one-to-one onto the set $\{(z, w, [u:t] \in \widetilde{\mathbb{C}}^2 \mid z = 0, w = 0, u \neq 0\}$, that is one of the standard coordinate chart of the projective space $\mathbb{CP}^1$. Analogously ψ sends $\{(z, w) \in \mathbb{C}^2 \mid w = 0\}$ one-to-one onto the set $\{(z, w, [u:t] \in \widetilde{\mathbb{C}}^2 \mid z = 0, w = 0, t \neq 0\}$.

Set

$$\sigma\colon \widetilde{\mathbb{C}}^2 \ni (z, w, [u:t]) \mapsto (z, w) \in \mathbb{C}^2; \tag{2.6}$$

then σ is a biholomorphism between $\widetilde{\mathbb{C}}^2 \setminus \sigma^{-1}(0)$ and $\mathbb{C}^2 \setminus \{0\}$ and

$$S := \sigma^{-1}(0) \cong \mathbb{CP}^1. \tag{2.7}$$

In particular, S is a compact complex submanifold of $\widetilde{\mathbb{C}}^2$.

In the coordinate chart φ the map σ has the form

$$\sigma \circ \varphi\colon \mathbb{C}^2 \ni (z, w) \mapsto (z, zw) \in \mathbb{C}^2, \tag{2.8}$$

while in the coordinate chart ψ it has the form

$$\sigma \circ \psi\colon \mathbb{C}^2 \ni (z, w) \mapsto (zw, w) \in \mathbb{C}^2. \tag{2.9}$$

Definition 2.1.1. Let $\widetilde{\mathbb{C}}^2$ given by (2.1). We call the map $\sigma\colon \widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$ defined by (2.6) the **(elementary) blow-down**; the set S in (2.7) is called the **exceptional divisor** of $\widetilde{\mathbb{C}}^2$, and $\sigma^{-1}\colon \mathbb{C}^2 \setminus \{0\} \to \widetilde{\mathbb{C}}^2 \setminus S$ is the **(elementary) blow-up**. We shall refer to $(\widetilde{\mathbb{C}}^2, S, \sigma)$, or simply to $\widetilde{\mathbb{C}}^2$, as the **blow-up** of $\mathbb{C}^2$ at the origin. We shall refer to the open subset V and W of (2.2), (2.3) as the **standard coordinate domains** of $\widetilde{\mathbb{C}}^2$, and to φ and ψ in (2.4) and (2.5) as **standard coordinate charts**.

The exceptional divisor is really "exceptional". First it cannot be defined by a single holomorphic function near S; indeed if S were the zero locus of a holomorphic function f, defined in a open set around S, the function $g = f \circ \sigma^{-1}$ would be holomorphic and non vanishing in a open subset of $\mathbb{C}^2 \setminus \{0\}$; by Riemann extension theorem, since $\{0\}$ has codimension 2, we could extend g in 0 but then its zero locus should be of codimension 1. Second, $\widetilde{\mathbb{C}}^2$ does not contain any other positive-dimensional compact complex submanifold. Indeed if S' were such a manifold, $\sigma(S')$ should be a compact, connected analytic subset of $\mathbb{C}^2$ and hence a single point; but σ is one-to-one outside S.

As an example, let $\gamma = \{(z, w) \in \mathbb{C}^2 \mid az + bw = 0\}$ be the equation of a straight line through the origin. We want to find the representation of $\sigma^{-1}(\gamma)$ in coordinate charts. Performing the substitution given by (2.8) we get

$$\varphi^{-1}\big(\sigma^{-1}(\gamma)\big) = (\sigma \circ \varphi)^{-1}(\gamma) = \big\{(z, w) \in \mathbb{C}^2 \mid z(a + bw) = 0\big\}.$$

It is the union of S and of the image of the complex line $\{(z, w) \in \mathbb{C}^2 \mid a + bw = 0\}$. If we apply (2.9) we obtain the same result when $b \neq 0$. We are substantially finding homogeneous coordinate to reppresent a complex line.

Remark 2.1.2. It can be shown (see [Sha94, Chapter II, Section 4.2]) that if X is a complex surface such that there exists a map $\rho\colon X \to \mathbb{C}^2$, which sends biholomorphically $X \setminus T$ onto $\mathbb{C}^2 \setminus \{0\}$, where $T = \rho^{-1}\{0\} \cong \mathbb{CP}^1$, then X is biholomorphic to $\widetilde{\mathbb{C}}^2$ via a biholomorphism G such that $\sigma \circ G = \rho$.

Since the automorphism group of $\mathbb{C}^2$ is transitive, we can use a similar procedure to blow-up $\mathbb{C}^2$ at any point, or any open subset $U \subseteq \mathbb{C}^2$ at any point $p \in U$.

Consider now a complex surface M and a finite set Σ of points of M. Let $\bigcup_{i \in I} W_i$ be an open covering of M by domains of coordinate charts such that no W_i contains more than one point of Σ. Then M is biholomorphic to $\bigsqcup_{i \in I} W_i \big/ \sim$, where $\sim$ is the relation that identifies identical points of M in different W_i. Performing the blow-up construction only on the W_i containing points of Σ we prove the following:

Theorem 2.1.3. *Let M be a complex surface and $\Sigma \subset M$ a finite set of points. Then there exist a complex surface $\widetilde{M}$ and a holomorphic map $\pi\colon \widetilde{M} \to M$ such that*

(i) for each $p \in \Sigma$, we have $S_p := \pi^{-1}(p) \cong \mathbb{CP}^1$;

(ii) $\pi\colon \widetilde{M} \setminus \bigcup_{p \in \Sigma} S_p \to M \setminus \Sigma$ is a biholomorphism;

(iii) around each S_p there is an open neighborhood such that π, restricted to it, in local coordinates is the elementary blow down defined above.

Definition 2.1.4. Given a complex surface M and a finite set of its points Σ, we call $\widetilde{M}$ the **blow-up of** M **at points of** Σ and π the projection; if $\Sigma = \{a\}$, we shall refer to $\widetilde{M}$ like the **blow-up of** M **at** a.

Remark 2.1.5. Since a connected open subset of a complex surface is trivially a complex surface, we can perform the blow-up construction around any point in a complex surface and reiterate it since at every step we get a complex surface.

Example 2.1.6. Consider the complex curve in $\mathbb{C}^2$ defined by $\gamma = \{(z, w) \in \mathbb{C}^2 \mid z^2 - w = 0\}$. After a first blow-up (in 0), using (2.8), we get for the inverse image of γ the equation $z(z - w) = 0$; after a second one (in $[1 : 0] \in S$, where S denotes the exceptional divisor of the first blow-up), we get $z^2(1 - w) = 0$.

After having defined the blow-up of a complex surface at a given point we are now going to define the blow-up of a complex analytic subset of dimension one and then of a foliation at a given point. We will start with analytic subsets. Note that dimension one analytic subsets of a complex surfaces is the only interesting case because the blow-ups of dimension zero and two subsets are trivial.

Definition 2.1.7. Let γ be an analytic subset of dimension one of $\mathbb{C}^2$ containing 0. We define the **blow-up** or **strict transform** of γ to be the subset $\widetilde{\gamma} = \overline{\sigma^{-1}(\gamma \setminus \{0\})}$ of $\widetilde{\mathbb{C}}^2$. Using the projection π given in Theorem 2.1.3 instead of the elementary blow-up we get the analogous definition in a complex surface.

Example 2.1.8. The intersection of the strict transform of a line l through 0 with the exceptional divisor is the point of $\mathbb{CP}^1$ corresponding to the line l.

We have to observe that for any analytic subset of dimension one γ containing 0, the set $\sigma^{-1}(\gamma)$ always contains the exceptional divisor, so the previous definition aims to cut off this set from blow-up. As the following theorem is going to show, what really happens is that we are dividing by the largest power of z, the equation that locally defines the blow-up of a given curve.

Theorem 2.1.9. *Let M be a complex surface, γ an analytic subset of dimension one, containing the point $a \in M$, and defined near a by a holomorphic function f; let $(\widetilde{M}, S_a, \pi)$, and $\widetilde{\gamma}$ be, respectively, their blow-up at the point a. Then $\widetilde{\gamma}$ is an analytic subset of dimension one of $\widetilde{M}$.*

Proof. Clearly, since π is a biholomorphism outside S_a, there the statement is obvious. So let p be any fixed point in S_a, and U_p an open subset of $\widetilde{M}$, where S_a is defined as zero locus of a holomorphic function g_p, whose germ at p is irreducible. Let $\widetilde{f}$ be the result of the division of $f \circ \pi$ by the maximal power of g_p dividing it (both $f \circ \pi$ and $\widetilde{f}$ are defined in U_p), and finally let $\gamma_{\widetilde{f}}$ be the zero locus of $\widetilde{f}$.

Outside S_a, $\widetilde{\gamma}$ and $\gamma_{\widetilde{f}}$ coincide. Furthermore $\widetilde{f}_{|S_a} \not\equiv 0$ and hence $(S_a \cap U_p) \nsubseteq \gamma_{\widetilde{f}}$. Now if $\widetilde{f}(p) \neq 0$, then p lies neither in $\widetilde{\gamma}$ nor in $\gamma_{\widetilde{f}}$. Otherwise if $\widetilde{f}(p) = 0$ then $p \in \gamma_{\widetilde{f}} \cap \widetilde{\gamma}$. Hence $\gamma_{\widetilde{f}} = \widetilde{\gamma} \cap U_p$. □

Remark 2.1.10. If γ is an irreducible analytic subset of dimension 1, then $\widetilde{\gamma}$ intersects the exceptional divisor only at one point. The proof, not difficult, depends upon Weierstraß Preparation Theorem and Hansel Lemma (see [GLS07, Lemma 3.19]). A straightforward argument using power expansion at $(0,0)$ in coordinate charts shows that, if γ is a smooth complex curve, then also $\widetilde{\gamma}$ is smooth. More precisely, $\widetilde{\gamma}$ is biholomorphic to γ and intersects the exceptional divisor in the point corresponding to the tangent line of γ at the origin.

Example 2.1.11. Let γ be the zero locus of a homogeneous polynomial $P_n(z,w)$ of degree n. Since $P_n(z,w)$ is reducible, γ is a star of complex lines. Then, counted with their multiplicity, the blow-up $\widetilde{\gamma}$ of γ is a finite set of n lines; furthermore they intersect the exceptional divisor in points all contained in the set V defined by (2.2) if and only if z does not divide $P_n(z,w)$, that is if and only if it contains the term in w^n. A similar assertion holds for the domain W.

The following example will clarify Remark 2.1.10.

Example 2.1.12. Let $\gamma = \{(z,w) \in \mathbb{C}^2 \mid f(z,w) = 0\}$, with $f(z,w) = z^p - w^q$, $p,q \in \mathbb{N}^*$ coprime (we already seen the case $p = q = 1$ in Example 2.1.11, suppose we are not in this case). In particular γ is an irreducible analytic subset of dimension 1. Let us suppose, up to switch z and w, that $p > q$, and set $r = p - q \geq 1$. Then, using (2.8) and (2.9) we get respectively

$$(f \circ \sigma \circ \varphi)(z,w) = z^q(z^r - w^q),$$
$$(f \circ \sigma \circ \psi)(z,w) = w^q(z^p w^r - 1),$$

where φ and ψ are the standard coordinate charts.

After dividing by z^q and w^q respectively we have in the first case that the zero locus of $\widetilde{f} \circ \varphi$ in $S \cap V = \{z = 0\}$ is made by a unique point $(w = 0)$, while the zero locus of $\widetilde{f} \circ \psi$ does not intersect $S \cap W = \{w = 0\}$. In particular, the blow-up of γ is contained in V, where it has equation $z^r - w^q = 0$.

Remark 2.1.13. Example 2.1.12 shows that generally the blow-up of an irreducible curve is not a smooth curve. You can however obtain a smooth curve by performing a finite sequence of blow-ups: this result is known as the **resolution of singularities theorem** (for curves), see [Sha94, Chapter IV, Section 4.1, Theorem 1].

Let us recall a definition.

Definition 2.1.14. Let f be a holomorphic function defined on an open subset U of a complex surface M, and a a point in U. Let (z,w) be local coordinates at a, and write

$$f(z,w) = \sum_{i,j=0}^{\infty} f_{i,j} z^i w^j,$$

the Taylor series of f at a. Then the **order** of f at a is

$$\nu_a(f) := \min\{i+j \ : \ f_{i,j} \neq 0\}.$$

Remark 2.1.15. Clearly the definition of order of a holomorphic function f at a point a does not depend on the local coordinates (z,w) at a chosen (it is a straightforward computation).

From the proof of Theorem 2.1.9, recalling that $\varphi^{-1}(S)$ and $\psi^{-1}(S)$ are defined, in $\mathbb{C}^2$, respectively by the equation $\{z=0\}$ and $\{w=0\}$, we can extract a proof of the following corollary, that summarizes the situation.

Corollary 2.1.16. *Let γ be an analytic subset of dimension one of $\mathbb{C}^2$ containing 0, defined near 0 by the equation $f(z,w)=0$, where f is a holomorphic function, and set $n=\nu_0(f)$, the order of f at 0; furthermore let $f_n(z,w)$ be the homogeneous polynomial of degree n of the terms of order n in the Taylor expansion of f. Let $\widetilde{\gamma}$ be the blow-up of γ at 0 and φ, ψ like in (2.4), (2.5) respectively. Then $\varphi^{-1}(\widetilde{\gamma})$ is given by the zero locus of the holomorphic function*

$$\widetilde{f}_z(z,w) := \frac{f(z,zw)}{z^n}; \tag{2.10}$$

while $\psi^{-1}(\widetilde{\gamma})$ is given by the zero locus of

$$\widetilde{f}_w(z,w) := \frac{f(zw,w)}{w^n}. \tag{2.11}$$

The intersections of $\widetilde{\gamma}$ with S are the (finitely many) solutions in $\mathbb{CP}^1$ of the equation

$$f_n(z,w) = 0,$$

and are given in local coordinates by

$$\varphi^{-1}(\widetilde{\gamma}\cap S) = \{(z,w)\in\mathbb{C}^2 \mid z=0, f_n(1,w)=0\}$$

and

$$\psi^{-1}(\widetilde{\gamma}\cap S) = \{(z,w)\in\mathbb{C}^2 \mid w=0, f_n(z,1)=0\}.$$

Definition 2.1.17. Given a holomorphic function defined in a neighborhood of 0 in $\mathbb{C}^2$, we shall call the holomorphic function $\widetilde{f}$ defined in (2.10) the **blow-up of f in the z-direction**, while $\widetilde{f}$ in (2.11) is the **blow-up of f in the w-direction**.

The proof of Theorem 2.1.9 is substantially a theorem about the extendibility of a holomorphic function around any given point of the exceptional divisor. Using that proof we can easily prove the following corollary.

Corollary 2.1.18. *Let M be a complex surface, $(\widetilde{M}, S_a, \pi)$ its blow-up at a; let f be a holomorphic function defined near a and of order $\nu_a(f)$. Then for any given point c in S_a, if g is an irreducible holomorphic function that defines S_a as its zero locus near c, there exists a holomorphic function $\widetilde{f}_c$ such that*

(i) $\widetilde{f}_c \not\equiv 0$ *in* S_a;

(ii) $g^{\nu_a(f)}\widetilde{f}_c \equiv f \circ \pi$.

Furthermore two such $\widetilde{f}_c$ differs by multiplication by a unit in the ring of germs at c of holomorphic functions.

In virtue of this corollary we can give the following definition.

Definition 2.1.19. Let M be a complex surface, f a holomorphic function defined in M near a and let c be any given point in S_a; we call any of the holomorphic function $\widetilde{f}_c$ in $\widetilde{M}$ whose existence is stated in Corollary 2.1.18 a **blow-up at a of the function f near c**.

Remark 2.1.20. The link between Definition 2.1.19 and Definition 2.1.17 is clear. Indeed let M be a complex surface, f a holomorphic function defined near the point a in M and θ a local chart around a such that $\theta(0) = a$; consider the blow-up $(\widetilde{M}, S_a, \pi)$ of M at a. Then around any given point c of S_a, there are a local chart $\widetilde{\theta}$ with $\pi \circ \widetilde{\theta} = \theta \circ \sigma$, and a blow-up f_c of f near c such that

$$\widetilde{(f \circ \theta)}_z = f_c \circ \widetilde{\theta}$$

or

$$\widetilde{(f \circ \theta)}_w = f_c \circ \widetilde{\theta}$$

depending in which standard coordinate domain $\widetilde{\theta}^{-1}(c)$ lies.

After having defined blow-up of analytic subsets, we are going to extend this definition to holomorphic foliations around a singular point.

Remark 2.1.21. Recall (see Theorem 1.2.9 and the equivalence between foliations and vector fields as stated in Theorem 1.2.13) that each foliation can be extended to a saturated foliation whose singular set is an analytic subset of codimension two, that is, in dimension two, a discrete set. So we can state the following definition.

Definition 2.1.22. Let M be a complex surface, with a saturated foliation $\mathcal{F}$ which has a finite set of singular points Σ. Then the **blow-up of the foliation** is the foliation $\widetilde{\mathcal{F}}$ obtained by extending, as stated in Remark 2.1.21, the foliation $\sigma^{-1}(\mathcal{F})$ defined in $\widetilde{M} \setminus \bigcup_{p\in\Sigma} S_p$.

Remark 2.1.23. In particular the blow-up of a saturated holomorphic foliation $\mathcal{F}$ defined by a holomorphic form ω around a singular point works like this: first we calculate $\sigma^{-1}(\mathcal{F})$ computing the pull-back $\sigma^*(\omega)$; then we divide the coefficient of $\sigma^*(\omega)$ by their maximum common divisor in order to get the form $\widetilde{\omega}$, that locally defines the saturation $\widetilde{\mathcal{F}}$ (see Theorem 1.2.9).

Example 2.1.24. Let $\mathcal{F}$ be defined by the form $\omega = -wdz + zdw$; applying (2.8), we get $\sigma^*(\omega) = -zwdz + z(wdz + zdw) = z^2dw$; then we divide by z^2 and we get $\widetilde{\omega} = dw$. If we instead apply (2.9), we get $\widetilde{\omega} = -dz$.

Before going further we recall the following definition.

Definition 2.1.25. Let ω be a holomorphic 1-form defined in an open subset of a complex surface M, given in local coordinates by $\omega = f(z,w)dz + g(z,w)dw$. The **order** $\nu_a(\omega)$ of ω in a is $\min\{\nu_a(f), \nu_a(g)\}$, where $\nu_a(f)$, $\nu_a(g)$ are the orders of zero in a of the holomorphic functions f and g respectively.

Remark 2.1.26. Let us consider $\mathcal{F}$ a holomorphic foliation in an open subset U of a complex surface M, and a point $a \in U$. Up to shrinking U, we can suppose that the foliation there is given by a holomorphic 1-form ω, and hence we can associate the order of a foliation at a point. This order does not depend on the holomorphic 1-form or on the local coordinates chosen; indeed if we change coordinates, the holomorphic 1-form that defines the foliations change by multiplication by a non-vanishing function (see Theorem 1.2.13), that does not change the order or the 1-form. So we can give the next definition.

Definition 2.1.27. Let $\mathcal{F}$ be a holomorphic foliation in an open subset U of a complex surface M. We define the **order** of $\mathcal{F}$ in a as $\nu_a(\mathcal{F}) = \nu_a(\omega)$, where ω, is a holomorphic 1-form that defines $\mathcal{F}$ locally near a.

The following proposition summarizes the situation. If f is a holomorphic function, we shall denote by f_n the homogeneous component of order n in the Taylor expansion of f.

Proposition 2.1.28. *Let $\mathcal{F}$ be a saturated holomorphic foliation in a open subset U of $\mathbb{C}^2$, singular at 0; let $\omega = f(z,w)dz + g(z,w)dw$ be a form that defines $\mathcal{F}$ near 0, and let $\widetilde{\mathcal{F}}$ be the blow-up foliation at 0. We set $h(z,w) = zf(z,w) + wg(z,w)$ and let $n = \nu_0(\omega)$, $n+1+m = \nu_0(h)$, $n+r = \nu_0(f)$ and $n+s = \nu_0(g)$, with $r = 0$ or $s = 0$. Finally let $\widetilde{f}_z$, $\widetilde{f}_w$, $\widetilde{g}_z$, $\widetilde{g}_w$, $\widetilde{h}_z$ and $\widetilde{h}_w$ be defined as in Definition* 2.1.17. *Then*

(i) if $h_{n+1} \not\equiv 0$, then $\widetilde{\mathcal{F}}$ is defined around any point of $S \setminus \{[0:1]\}$ by

$$\widetilde{\omega} = \widetilde{h}_z(z,w)dz + z^{s+1}\widetilde{g}_z(z,w)dw;$$

its singular locus in $S \setminus \{[0:1]\} = \{z = 0\}$ is given by the equation $h_{n+1}(1,w) = 0$;

(ii) if $h_{n+1} \not\equiv 0$, then $\widetilde{\mathcal{F}}$ is defined around any point of $S \setminus \{[1:0]\}$ by

$$\widetilde{\omega} = w^{r+1}\widetilde{f}_w(z,w)dz + \widetilde{h}_w(z,w)dw;$$

its singular locus in $S \setminus \{[1:0]\} = \{w = 0\}$ is given by the equation $h_{n+1}(z,1) = 0$;

(iii) if $h_{n+1} \equiv 0$, then $\widetilde{\mathcal{F}}$ is defined around any point of $S \setminus \{[0:1]\}$ by

$$\widetilde{\omega} = z^{m-1}\widetilde{h}_z(z,w)dz + \widetilde{g}_z(z,w)dw;$$

its singular locus in $S\setminus\{[0:1]\} = \{z = 0\}$ is given by the equations $g_n(1,w) = 0$ and $z^{m-1}h_{n+1+m}(1,w) = 0$ (the latter equation gives no restrictions if $m \geq 2$);

(iv) if $h_{n+1} \equiv 0$, then $\widetilde{\mathcal{F}}$ is defined around any point of $S \setminus \{[1:0]\}$ by

$$\widetilde{\omega} = \widetilde{f}_w(z,w)dz + w^{m-1}\widetilde{h}_w(z,w)dw;$$

its singular locus in $S\setminus\{[1:0]\} = \{w = 0\}$ is given by the equations $f_n(z,1) = 0$ and $w^{m-1}h_{n+1+m}(z,1) = 0$ (the latter equation gives no restrictions if $m \geq 2$).

Proof. We shall work the details only performing a blow-up in the z-direction (cases (i) and (iii) of the proposition); the computation in the other chart is perfectly analogous. So let us start by applying (2.8) and get, in these local coordinates,

$$\sigma^*(\omega) = [f(z,zw) + wg(z,zw)]dz + zg(z,zw)dw = z^{-1}h(z,zw)dz + zg(z,zw)dw.$$

Thanks to (2.10) then we have

$$\sigma^*(\omega) = z^{n+m}\widetilde{h}_z(z,w)dz + z^{n+s+1}\widetilde{g}_z(z,w)dw.$$

By Remark 2.1.23, to obtain $\tilde{\omega}$, we only have to divide $\sigma^*(\omega)$ by the greatest common factor of the form z^k (for a suitable k). If $h_{n+1} \not\equiv 0$, then $m = 0$ and $k = n$, and we have (i). If $h_{n+1} \equiv 0$, then $m > 0$, that implies $r = s = 0$; then $k = n + 1$ and we have (iii).

For the description of the singular set, let us split the proof in cases (i) and (iii). In the first case, in $\{z = 0\}$ a singular point $(0, w)$ has to satisfy $\widetilde{h}_z(0, w) = 0$. Being $\widetilde{h}_z(z, w) = z^{-n-1}h(z, zw)$, we have that $\widetilde{h}_z(0, w) = h_{n+1}(1, w)$. In the second case, in $\{z = 0\}$ a singular point $(0, w)$ has to satisfy $\widetilde{g}_z(0, w) = 0$, and $\widetilde{h}_z(0, w) = 0$ if $m = 1$. As before, we have that $\widetilde{h}_z(0, w) = h_{n+m+1}(1, w)$ and $\widetilde{g}_z(0, w) = g_n(1, w)$ (since $s = 0$). □

Proposition 2.1.28 can obviously be reformulated in a complex surface (M, a) instead of in $(\mathbb{C}^2, 0)$. It proves the usefulness of the following definition.

Definition 2.1.29. Let $\mathcal{F}$ be a holomorphic foliation in a complex surface M, singular at a, defined near a, in a local chart, by the form $\omega = f(z, w)dz + g(z, w)dw$. We set $h(z, w) = zf(z, w) + wg(z, w)$ and let $n = \nu_0(\omega)$; let $f_n(z, w)$, $g_n(z, w)$ be the terms of degree n in the Taylor expansion of f, g, respectively, at a. Then the form

$$h_{n+1}(z, w) = zf_n(z, w) + wg_n(z, w)$$

is called the **tangent form** of ω at its singularity a.

Before going further we need a couple of definitions.

Definition 2.1.30. A **(complex) separatrix** of a saturated foliation $\mathcal{F}$ singular at a is a leaf whose closure in a neighborhood of a is an analytic subset of dimension one containing a.

Definition 2.1.31. A singularity a of a saturated holomorphic foliation $\mathcal{F}$ of M is called **non-dicritical** if the exceptional divisor S_a at a is a separatrix of the blow-up of the foliation $\widetilde{\mathcal{F}}$. It is called **dicritical** otherwise.

Next proposition will give us a necessary and sufficient condition under which a singularity in a foliation is non-dicritical or dicritical; it is a trivial consequence of Proposition 2.1.28.

Proposition 2.1.32. *Let a be a singularity of a saturated holomorphic foliation $\mathcal{F}$ in a complex surface M; let $\omega = f(z, w)dz + g(z, w)dw$ be a 1-form that defines $\mathcal{F}$ near a, in a local chart. Then a is a non-dicritical singularity if and only if $h_{\nu_a(\omega)+1} \not\equiv 0$, if and only if*

$$\nu_a\bigl(zf(z, w) + wg(z, w)\bigr) = 1 + \nu_a(\omega) = 1 + \min\bigl\{\nu_a(f), \nu_a(g)\bigr\}; \tag{2.12}$$

while a is a dicritical singularity if and only if $h_{\nu_a(\omega)+1} \equiv 0$, if and only if

$$\nu_a\bigl(zf(z, w) + wg(z, w)\bigr) > 1 + \nu_a(\omega).$$

Proof. By Theorem 2.1.3 we can work in $(\widetilde{\mathbb{C}}^2, \sigma)$ instead of $(\widetilde{M}, \pi)$, with $a = (0, 0)$. By Proposition 2.1.28, recalling that the equation of the exceptional divisor is $z = 0$ or $w = 0$, we see that a is non-dicritical if and only if $h_{n+1} \not\equiv 0$ and hence if and only if (2.12) holds. □

Remark 2.1.33. It follows by Proposition 2.1.28 (and by Proposition 2.1.32) that if 0 is a non-dicritical singularity of a foliation given by a holomorphic 1-form ω, and if we set $n = \nu_0(\omega)$, then the singularities of the blow-up foliation in the exceptional divisor $S \cong \mathbb{CP}^1$ are the zeros of the tangent form h_{n+1}.

Before stating the next proposition, we give the definition of elementary singularity. We first deal with holomorphic vector fields.

Definition 2.1.34. Let $X(z, w) = f(z, w)\frac{\partial}{\partial z} + g(z, w)\frac{\partial}{\partial w}$ be a holomorphic vector field, defined in an open subset of a complex surface M and singular at a point a. The **elementary matrix** of X at a is the Jacobian matrix of $F(z, w) = (f(z, w), g(z, w))$ at a.

We say that a is an **elementary singularity** for X if the elementary matrix of X at a has at least one eigenvalue different from 0; we say that a is a **nilpotent singularity** for X if the elementary matrix of X at a is nilpotent but not zero.

We give now equivalent definitions when the foliation is given by a holomorphic 1-form (see Lemma 1.1.13 and Lemma 1.2.4 for the connection between holomorphic 1-forms and holomorphic vector fields).

Definition 2.1.35. Let $\omega(z, w) = f(z, w)dz + g(z, w)dw$ be a holomorphic 1-form, defined in an open subset of a complex surface M and singular at a point a. The **elementary matrix** of ω at a is the Jacobian matrix of $F(z, w) = (-g(z, w), f(z, w))$ at a.

We say that a is an **elementary singularity** for ω if the elementary matrix of X at a has at least one eigenvalue different from 0; we say that a is a **nilpotent singularity** for ω if the elementary matrix of X at a is nilpotent but not zero.

These definitions can be extended from holomorphic vector fields (or from holomorphic 1-forms) to foliations, thanks to Theorem 1.2.13. Indeed, changing the holomorphic vector field (or the holomorphic 1-form respectively) by multiplication by a non-vanishing holomorphic function, the elementary matrix changes by multiplication by a non-zero constant, and hence the number of its non-zero eigenvalues does not change.

Definition 2.1.36. Let $\mathcal{F}$ be a holomorphic foliation in an complex surface M and singular at a, given by a holomorphic vector field X (resp., a holomorphic 1-form ω) near a. The **elementary matrix** of $\mathcal{F}$ at a is $[L] \in \mathbb{P}(\mathrm{Mat}(2 \times 2, \mathbb{C}))$, where L is the elementary matrix of X (resp., of ω) at a. Moreover a is called an **elementary singularity** (resp., a **nilpotent singularity**) for $\mathcal{F}$ if it is an elementary singularity (resp., a nilpotent singularity) for X (resp., for ω).

The following result states a sufficient condition for a singularity in the blow-up foliation to be elementary.

Proposition 2.1.37. *Let $\mathcal{F}$ be a saturated holomorphic foliation in U singular at 0, where U is an open subset of $\mathbb{C}^2$; let $\omega = f(z,w)dz + g(z,w)dw$ be a form that defines $\mathcal{F}$ in U, and $h_{n+1} = zf_n + wg_n$ its tangent form; finally let $\widetilde{\mathcal{F}}$ be the blow-up of F at 0. Then each simple linear factor $az + bw$ of h_{n+1} corresponds to an elementary singularity $(0, 0, [a : b])$ of $\widetilde{\mathcal{F}}$.*

Proof. Let us suppose that the tangent form h_{n+1} has a single linear factor: up to a linear change of coordinates, we can suppose that this factor is w. Then for the tangent form we have $h_{n+1}(1, w) = wk(w)$, with $a_w := k(0) \neq 0$. Thanks to Proposition 2.1.28.(i), the blow-up foliation has a singularity at $[0 : 1]$, and it is given by

$$\widetilde{\omega} = (a_z z + z_w w + \text{h.o.t.})dz + (b_z z + \text{h.o.t.})dw,$$

with suitable $a_z, b_z \in \mathbb{C}$. Then the elementary matrix at $[0 : 1]$ is

$$\begin{pmatrix} -b_z & a_z \\ 0 & a_w \end{pmatrix},$$

that has $-b_z$ and $a_w \neq 0$ as eigevalues, and we are done. $\square$

2.2 Intersection multiplicity

The main tool needed in the proof of the reduction of singularities theorem is the intersection multiplicity of two analytic subset of dimension one. This notion will be used to define the multiplicity of a foliation.

Definition 2.2.1. Let M be a complex surface, γ an analytic subset of dimension one of M and a one of its point; furthermore let f be a holomorphic map that locally defines γ near a as its zero locus. Then a holomorphic injective map $\tau\colon (U, 0) \to (M, a)$ such that $f \circ \tau \equiv 0$ is called a **parametrization** of γ at a.

The existence of parametrizations at a given point is not trivial. In the case of complex surface, a theorem of Puiseaux, whose first ideas come from Newton, gives us an algorithm to find a special kind of parametrization. We do not present its proof here, but we refer to [GLS07], [Fis01], [dJP00] and [Chi89]; for a presentation more oriented to a historical perspective see [BK86]. We want only to recall the following theorem, that is a version of Puiseaux, more suited to our purposes (for a proof see [GLS07, Theorem 3.3]).

Theorem 2.2.2. *Let γ be an irreducible analytic subset of dimension one of $\mathbb{C}^2$ containing 0, f an irreducible holomorphic function that defines γ near 0 as its zero locus, and let $n = \nu_0(f)$ be its order at 0. Then, up to a linear change of coordinates, there exists one and only one injective holomorphic map $(\tau_1, \tau_2) := \tau \colon (U, 0) \to (\mathbb{C}^2, 0)$, from a domain U of $\mathbb{C}$, such that*

(i) $f \circ \tau \equiv 0$ *in* U*;*

(ii) $\tau_1(t) = t^n$ *and* $\nu_0(\tau_2) > n$.

Example 2.2.3. Let $\gamma = \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 - 2z - 2w = 0\}$. Then γ is irreducible in 0 ($\gamma \cap \mathbb{R}^2$ is the circumference with center $(1, 1)$ passing through 0). After the linear change of coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}$$

we have $\gamma = \{x^2 + y^2 - 2y = 0\}$. Here the parametrization of γ will be given by $\tau(t) = (t, 1 - \sqrt{1 - t^2})$ (note that $\nu_0(\tau_2) = 2 > 1$).

Another classical example is $\gamma = \{(z, w) \in \mathbb{C}^2 \mid z^2 - w^3 = 0\}$; the natural parametrization then would be $\tau(t) = (t^3, t^2)$, but here $\nu_0(\tau_2) = 2 < 3$: we have to switch z and w.

Remark 2.2.4. Given an irreducible analytic curve γ, there is an easy (and geometric) way to find out the right change of coordinates that makes Theorem 2.2.2 work. As a matter of fact, if γ is irreducible, then by Remark 2.1.10 there exists a unique point c in the exceptional divisor that belongs to the blow-up $\widetilde{\gamma}$ of γ. Then the linear change of coordinate is such that, in the new coordinates, $c = [1 : 0] \in \mathbb{CP}^1$.

Definition 2.2.5. Let γ be an analytic subset of dimension one of $\mathbb{C}^2$ containing 0, and f an irreducible holomorphic function that defines γ near 0 as its zero locus. Then we call the map τ whose existence is stated in Theorem 2.2.2 a **primitive parametrization** of f (or of γ) at 0.

The existence of primitive parametrizations and algebraic proprieties of the ring of germs of holomorphic functions in a point allows us to define the *intersection multiplicity* of two holomorphic functions. This notion was first introduced in the algebraic setting leading to intersection theory, and then revealed its usefulness also to study singularities of holomorphic functions. The reader can consult the books [AGZV85], [AGZV88] and especially [GLS07]. In [Żoł06, Chapter 2] and [IY08, Section 8] the reader can find a summary of the main statements and some proofs. The proper setting to define the intersection multiplicity is analytic subsets of dimension one, or the ring of germs of holomorphic maps at given point after having identified germs that are obtained from each other by multiplication by unit. In order to avoid introducing divisors, or a heavier algebraic notation, we decided to follow this second less elegant approach.

Definition 2.2.6. Let U be a domain in a complex surface M, a a point of U and f, g germs in a of holomorphic functions. Then we define the **intersection multiplicity** between f and g at a by:

- $\mu_a(f,g) := +\infty$ if f and g have a common factor;
- $\mu_a(hf,kg) := \mu_a(f,g)$ if h and k are units in the ring of germs at a;
- $\mu_a(f,g) := \nu_0(f \circ \tau)$ if g is irreducible and τ is a parametrization of g;
- $\mu_a(f,g) := \sum_{i=1}^n m_i \mu_a(f,g_i)$ if $g = \prod_{i=1}^n g_i^{m_i}$ is the representation of g in irreducible factors.

We state, without proof, some proprieties of intersection multiplicity that will be used later. For proofs we refer to the books cited above.

Proposition 2.2.7. *Let U be a domain in a complex surface M, $a \in U$ a point and f, g and h germs in a of holomorphic functions. Then:*

(i) $\mu_a(f,g) = 0$ *if and only if the zero loci of f and g do not intersect in a;*

(ii) $\mu_a(f,g) = 1$ *if and only if their zero loci meet transversally at a;*

(iii) $\mu_a(f,g) = \mu_a(g,f)$*;*

(iv) $\mu_a(f + hg, g) = \mu_a(f,g)$.

The most important property for us will be the following Proposition 2.2.8 relating intersection multiplicity of functions and of their blow-ups, as defined in Definition 2.1.19.

Proposition 2.2.8. *Let M be a complex surface, f, g two holomorphic functions defined near a point a in M, and for any c in S_a, let $\widetilde{f}_c$, $\widetilde{g}_c$ be their blow-ups near c; let $\nu_a(f)$ and $\nu_a(g)$ be the order of zero of f and g respectively at a. Then we have*

$$\mu_a(f,g) = \nu_a(f)\nu_a(g) + \sum_{c \in S_a} \mu_c(\widetilde{f}_c, \widetilde{g}_c).$$

Proof. First observe that by Theorem 2.1.3 we can work in $(\widetilde{\mathbb{C}}^2, S, \sigma)$ and in the local coordinates there. Note that, by Definition 2.1.19 and Corollary 2.1.18, the choice of $\widetilde{f}_c$, $\widetilde{g}_c$, that is not unique, does not change the intersection multiplicities. Furthermore by Corollary 2.1.16 the summatory in the second member contains only a finite number of terms since only for finitely many $c \in S$ the intersection of the zero loci of $\widetilde{f}_c$ and $\widetilde{g}_c$ is not empty. So let us begin by assuming g to be a irreducible germ at 0, and set $n = \nu_0(f)$, $m = \nu_0(g)$. Up to composing by a biholomorphism, that again does not affect intersection multiplicities, we can suppose that z does not divide $f_n(z, w)$, nor $g_m(z, w)$ (which denote the homogeneous parts of degree n and m of f and g respectively); hence all the points c in S that have non-zero multiplicity, lie inside $S \setminus \{[0 : 1]\}$ and it suffices to work only applying (2.8). By Remark 2.1.20 and Corollary 2.1.16 we have

$$z^n \widetilde{f}(z, w) = f(z, zw) \quad \text{and} \quad z^m \widetilde{g}(z, w) = g(z, zw) \tag{2.13}$$

in local coordinates; by Remark 2.1.10 there is only one c in S such that $\widetilde{g}(c) = 0$. Let $\tau(t) = (t^m, \theta(t))$ be a primitive parametrization of g at 0 defined in $(U, 0)$ (we know it exists from Theorem 2.2.2). Then the map $\widetilde{\tau} = \sigma^{-1} \circ \tau$ is holomorphic and injective in $U \setminus \{0\}$; setting $\widetilde{\tau}(0) = c$ it can be extended continuously and hence holomorphically in U, preserving injectivity. Furthermore, by (2.8), we have

$$\widetilde{\tau}(t) = \left(t^m, \frac{\theta(t)}{t^m} \right).$$

Since again by (2.8) in local coordinates $(f \circ \sigma)(z, w) = f(z, zw)$ we have

$$\mu_0(f, g) = \nu_0(f \circ \tau) = \nu_0(f \circ \sigma \circ \widetilde{\tau}).$$

But by equation (2.13)

$$(f \circ \sigma \circ \widetilde{\tau})(t) = t^{mn} (\widetilde{f} \circ \widetilde{\tau})(t)$$

and thus

$$\mu_0(f, g) = \nu_0(t^{nm}) + \nu_0((\widetilde{f} \circ \widetilde{\tau})(t)) = \nu_0(f)\nu_0(g) + \mu_c(\widetilde{f}, \widetilde{g}).$$

To remove the assumption on g to be irreducible and get the general case, we have to apply Proposition 2.2.7.(iii) and sum up every irreducible factor of g. □

2.3 Reduction of singularities theorem

The main goal of this section is to prove the reduction of singularities theorem for holomorphic foliations. We follow the exposition in [Żoł06, Section 2 of Chapter

9], [MM80] and [IY08, Section 8 of Chapter 1]. The first modern proof was given by Seidenberg in [Sei68], in the formal category. The proof we present here uses intersection multiplicity and was given by van den Essen in [vdE79]. The link between foliations and intersection multiplicity is given by the following definition.

Definition 2.3.1. Let $\mathcal{F}$ be a saturated holomorphic foliation in a complex surface M, singular at a, and let $\omega = f(z,w)dz + g(z,w)dw$ be a holomorphic 1-form that defines $\mathcal{F}$ near a. Then the **multiplicity** of $\mathcal{F}$ at a is

$$\mu_a(\mathcal{F}) := \mu_a(f,g).$$

So computing multiplicity of a foliation at a given singular point, is a matter of computing intersection multiplicity of two holomorphic functions. Since the intersection multiplicity of two holomorphic functions is invariant under multiplication by units, the multiplicity of a foliation does not depend on the 1 form we choose. Even more, we can replace f and g by any couple of holomorphic functions having the same intersection multiplicity at a.

Before stating the main theorem we shall prove a proposition, interesting on its own. For a given holomorphic function f, we use the notation f_n to denote the homogeneous polynomial of terms of degree n in the Taylor series of f at any given point (when otherwise not stated, at 0).

Proposition 2.3.2. *Let $\mathcal{F}$ be a saturated holomorphic foliation in a complex surface M, singular at a, and of order $n = \nu_a(\mathcal{F})$. Furthermore let $\widetilde{\mathcal{F}}$ be the blow-up of $\mathcal{F}$ at a. Then the following relations hold:*

(i) if a is a non-dicritical singularity of $\mathcal{F}$, then

$$\mu_a(\mathcal{F}) = n^2 - n - 1 + \sum_{c\in S_a} \mu_c(\widetilde{\mathcal{F}}); \tag{2.14}$$

(ii) if a is a dicritical singularity of $\mathcal{F}$, then

$$\mu_a(\mathcal{F}) = n^2 + n - 1 + \sum_{c\in S_a} \mu_c(\widetilde{\mathcal{F}}). \tag{2.15}$$

Proof. By Theorem 2.1.3 we can work in a domain around 0. So let $\mathcal{F}$ be a holomorphic foliation singular in 0, of a domain U, there defined by a form $\omega = f(z,w)dz+g(z,w)dw$ of order $n = \nu_0(\omega)$; we put $h(z,w) = zf(z,w)+wg(z,w)$. We know by Proposition 2.1.32 that 0 is a non-dicritical (resp., dicritical) singularity if $h_{n+1} \not\equiv 0$ (resp., $h_{n+1} \equiv 0$).

Suppose first that 0 is a non-dicritical singularity; then we can perform a linear change of coordinate in such a way that $\nu_0(f) = \nu_0(g) = n$, and z does not divide

g_n: this by Proposition 2.1.28 implies that all the singularities of $\widetilde{\mathcal{F}}$ lie inside $S \setminus \{[0:1]\}$, and thus it is sufficient to perform a blow-up in the z-direction. So let $\widetilde{f} = \widetilde{f}_z$ and $\widetilde{g} = \widetilde{g}_z$ be respectively the blow-ups of f and g in the z-direction; we shall also consider them as germs $\widetilde{f}_c$, $\widetilde{g}_c$ in $c \in S \setminus \{[0:1]\}$, using a suitable translation of the standard projection given by (2.4) as a chart centered in c for Definition 2.1.19. Recall that, again by Proposition 2.1.28 and Remark 2.1.20, and arguing similarly, $\widetilde{\mathcal{F}}$ can be defined near any given c in $S \setminus \{[0:1]\}$ by the 1-form

$$\widetilde{\omega}_c = \big(\widetilde{f}(z,w) + w\widetilde{g}(z,w)\big)dz + z\widetilde{g}(z,w)dw.$$

We have by Proposition 2.2.8

$$\mu_0(\mathcal{F}) = \mu_0(f,g) = \nu_0(f)\nu_0(g) + \sum_{c\in S} \mu_c(\widetilde{f}_c, \widetilde{g}_c) = n^2 + \sum_{c\in S} \mu_c(\widetilde{f}, \widetilde{g}).$$

Next

$$\mu_c(\widetilde{\mathcal{F}}) = \mu_c(\widetilde{f} + w\widetilde{g}, z\widetilde{g}) = \mu_c(\widetilde{f} + w\widetilde{g}, z) + \mu_c(\widetilde{f}, \widetilde{g}).$$

But $\mu_c(\widetilde{f} + w\widetilde{g}, z)$ is the order of zero of $\widetilde{f} + w\widetilde{g}$ at c; so summing up over S we get

$$\sum_{c\in S} \mu_c(\widetilde{f}, \widetilde{g}) = \sum_{c\in S} \Big(\mu_c(\widetilde{g}) - \mu_c(\widetilde{f} + w\widetilde{g}, z)\Big) = \sum_{c\in S} \mu_c(\widetilde{\omega}) - n - 1.$$

Hence

$$\mu_0(\mathcal{F}) = \mu_0(\omega) = n^2 - n - 1 + \sum_{c\in S} \mu_c(\widetilde{\mathcal{F}}),$$

and (2.14) is proved.

Let us now suppose 0 is a dicritical singularity; then $h_{n+1} \equiv 0$ by Proposition 2.1.32. Since $h_{n+1} \equiv 0$, necessarily $f_n, g_n \not\equiv 0$. By Proposition 2.2.8

$$\mu_0(\mathcal{F}) = \mu_0(f,g) = n^2 + \sum_{c\in S} \mu_c(\widetilde{f}_c, \widetilde{g}_c). \tag{2.16}$$

We shall first analyze the contributions to the multiplicity given by singularities in $S \setminus \{[0:1]\}$ in equation (2.16) and then the contribution of the point $[0:1]$ to the total multiplicity. Set $p := [0:1]$ and $\nu_0(h) := n + 1 + m$. Computing the blow-up in the z-direction (dropping the index z in computations) we have

$$z^n \widetilde{f}(z,w) = f(z, zw), \qquad z^n \widetilde{g}(z,w) = g(z, zw) \tag{2.17}$$

and

$$z^{n+1+m} \widetilde{h}(z,w) = h(z, zw). \tag{2.18}$$

Again by Proposition 2.1.28 $\mathcal{F}$ is defined in $S \setminus \{[0:1]\}$ by the form

$$\widetilde{\omega} = z^{m-1}\widetilde{h}(z,w)dz + \widetilde{g}(z,w)dw. \tag{2.19}$$

Note that by equation (2.17) and (2.18), we have

$$z^m\widetilde{h}(z,w) = \widetilde{f}(z,w) + w\widetilde{g}(z,w), \tag{2.20}$$

and by (2.19) and Theorem 2.2.7

$$\begin{aligned}
\sum_{c\in S\setminus\{p\}} \mu_c(\widetilde{\mathcal{F}}) &= \sum_{c\in S\setminus\{p\}} \big(\mu_c(z^{m-1},\widetilde{g}) + \mu_c(\widetilde{h},\widetilde{g})\big)\\
&= \sum_{c\in S\setminus\{p\}} (m-1)\nu_c\big(\widetilde{g}(0,w)\big) + \sum_{c\in S\setminus\{p\}} \mu_c(\widetilde{h},\widetilde{g})\\
&= \sum_{c\in S\setminus\{p\}} (m-1)\nu_c\big(g_n(1,w)\big) + \sum_{c\in S\setminus\{p\}} \mu_c(\widetilde{h},\widetilde{g})\\
&= (m-1)\deg_w\big(g_n(1,w)\big) + \sum_{c\in S\setminus\{p\}} \mu_c(\widetilde{h},\widetilde{g})\\
&= (m-1)k + \sum_{c\in S\setminus\{p\}} \mu_c(\widetilde{h},\widetilde{g}),
\end{aligned}$$

where $k := \deg_w(g_n(1,w))$. Computing the intersection multiplicity with $\widetilde{g}$ of both sides of equation (2.20) and summing over $S \setminus \{p\}$ we get

$$\begin{aligned}
\sum_{c\in S\setminus\{p\}} \mu_c(\widetilde{f},\widetilde{g}) &= \sum_{c\in S\setminus\{p\}} \mu_c(z^m\widetilde{h},\widetilde{g})\\
&= \sum_{c\in S\setminus\{p\}} \Big(m\nu_c\big(\widetilde{g}(0,w)\big) + \mu_c(\widetilde{h},\widetilde{g})\Big)\\
&= m\deg_w\big(g_n(1,w)\big) + \sum_{c\in S\setminus\{p\}} \mu_c(\widetilde{h},\widetilde{g})\\
&= mk + \sum_{c\in S\setminus\{p\}} \mu_c(\widetilde{h},\widetilde{g}).
\end{aligned}$$

Confronting the two previous equations leads to

$$\sum_{c\in S\setminus\{p\}} \mu_c(\widetilde{f},\widetilde{g}) = \sum_{c\in S\setminus\{p\}} \mu_c(\widetilde{\mathcal{F}}) + k. \tag{2.21}$$

Now we are going to compute $\mu_p(\widetilde{f}_p,\widetilde{g}_p)$. Blowing-up in w-direction we get

$$w^n\widetilde{f}(z,w) = f(zw,w), \qquad w^n\widetilde{g}(z,w) = g(zw,w), \tag{2.22}$$

and
$$w^{n+1+m}\widetilde{h}(z,w) = h(zw,w). \tag{2.23}$$
Again by Proposition 2.1.28 $\mathcal{F}$ is defined in $S \setminus \{[1:0]\}$ by the 1-form
$$\widetilde{\omega}_p = \widetilde{f}(z,w)dz + w^{m-1}\widetilde{h}(z,w)dw.$$
Note that by (2.22) and (2.23), we have
$$w^m\widetilde{h}(z,w) = z\widetilde{f}(z,w) + \widetilde{g}(z,w). \tag{2.24}$$
Now we have, putting $r := \nu_0(f_n(z,1))$
$$\mu_p(\widetilde{\mathcal{F}}) = \mu_p(\widetilde{f}, w^{m-1}\widetilde{h}) = (m-1)\mu_p(\widetilde{f}, w) + \mu_p(\widetilde{f}, \widetilde{h}) = (m-1)r + \mu_p(\widetilde{f}, \widetilde{h}).$$
Then computing the intersection multiplicity with $\widetilde{f}$ of both sides of equation (2.24), we have
$$\mu_p(\widetilde{g}, \widetilde{f}) = m\mu_p(w, \widetilde{f}) + \mu_p(\widetilde{h}, \widetilde{f}) = mr + \mu_p(\widetilde{h}, \widetilde{f}).$$
Confronting the last two equations gives
$$\mu_p(\widetilde{g}, \widetilde{f}) = r + \mu_p(\widetilde{\mathcal{F}}).$$
This last equation with (2.21) gives
$$\mu_0(\mathcal{F}) = \mu_0(f,g) = n^2 + \sum_{c\in S}\mu_c(\widetilde{f}, \widetilde{g}) = n^2 + k + r + \sum_{c\in S}\mu_c(\widetilde{\mathcal{F}}).$$
We are left to prove $k + r = n - 1$ to get (2.15). By $zf_n \equiv -wg_n$ it follows
$$f_n(z,w) = a_0z^{n-1}w + a_1z^{n-2}w^2 + \cdots + a_{n-1}w^n$$
and
$$g_n(z,w) = -a_0z^n - a_1z^{n-1}w + \cdots - a_{n-1}zw^{n-1};$$
thus
$$f_n(z,1) = a_0z^{n-1} + a_1z^{n-2} + \cdots + a_{n-1}$$
and
$$g_n(1,w) = -a_0 - a_1w + \cdots - a_{n-1}w^{n-1}.$$
Hence $\nu_0(f_n(z,1)) = r$, then $\deg_w(g_n(1,w)) = n - 1 - r$, and we are done. □

So we can state and prove the following:

Theorem 2.3.3. *Let $\mathcal{F}$ be a saturated holomorphic foliation with finite many singular points in a complex surface M and let Σ be the finite set of its singularities. Then there exists a complex surface $\overline{M}$, a map $\rho\colon \overline{M} \to M$, and a foliations $\overline{\mathcal{F}}$ such that:*

(i) $\overline{\mathcal{F}}$ has only a finite number of elementary singularity;

(ii) ρ is the composition of finitely many elementary blow-ups.

Proof. It is clear that since $\mathcal{F}$ has only finite singular points it is sufficient to prove the theorem only for one of them, that we can assume to be $0 \in \mathbb{C}^2$. We argue by induction on the multiplicity $\mu_0(\mathcal{F})$. If $\mu_0(\mathcal{F}) = 1$ then, by Proposition 2.2.8, 0 is an elementary singularity, and we are done. Assume the assertion is true for singularities of multiplicity at most $\mu - 1$, and take a foliation $\mathcal{F}$ singular in 0 with multiplicity μ and order $n \geq 1$. If 0 is dicritical, or 0 is non-dicritical and $n > 1$, then by Proposition 2.3.2, the multiplicity of the blow-up foliation at each singular point is strictly less than μ and the assertion follows by induction. If instead 0 is a non elementary singularity of order 1 (i.e., a nilpotent singularity), then $\mathcal{F}$ is defined by a form $\omega = A(z,w)dz + (w + B(z,w))dw$, with A and B holomorphic functions such that $\nu_0(A)$, $\nu_0(B) \geq 2$. We shall write $A = \sum_{i,j} a_{i,j} z^i w^j$ and $B = \sum_{i,j} b_{i,j} z^i w^j$ the Taylor series of A and B respectively. The tangent form is w^2, hence 0 is a non-dicritical singularity, and the only singularity of the blow-up foliation in the exceptional divisor is $[1:0]$ (see Remark 2.1.33). Applying a blow-up in the z-direction (2.8),

$$\begin{cases} z = z_1, \\ w = z_1 w_1, \end{cases}$$

we see that the form $\omega_1 := \widetilde{\omega}$ that defines the blow-up foliation (see Proposition 2.1.28) is

$$\omega_1 = \big(z_1 A_1(z_1, w_1) + w_1^2\big)dz_1 + z_1\big(w_1 + z_1 B_1(z_1, w_1)\big)dw_1, \tag{2.25}$$

where

$$\begin{aligned} A_1(z_1, w_1) &= z_1^{-2}\big(A(z_1, z_1 w_1) + w_1 B(z_1, z_1 w_1)\big), \\ B_1(z_1, w_1) &= z_1^{-2} B(z_1, z_1 w_1), \end{aligned}$$

are such that $\nu_0(A_1), \nu_0(B_1) \geq 0$. Clearly ω_1 has only one singular point 0 since the tangent form of ω is w^2; moreover $\mu_0(\omega_1) = \mu + 1$, thanks to (2.14).

There are two possible cases:

- $A_1(0,0) \neq 0$;

- $A_1(0,0) = 0$.

Let us first consider the situation $A_1(0,0) = a_{2,0} \neq 0$. In this case, $\nu_0(\omega_1) = 1$, and we have already seen that $\mu_0(\omega_1) = \mu+1$. The tangent form of ω_1 is $z_1^2 A_1(0,0)$ and so 0 is again a non-dicritical singularity (but still nilpotent). Moreover, thanks to Remark 2.1.33, the unique singularity of the blow-up foliation on the exceptional divisor is $[0:1]$; so we apply a second blow-up in the w_1-direction,

$$\begin{cases} z_1 = z_2 w_2, \\ w_1 = w_2, \end{cases}$$

and we obtain $\omega_2 := \widetilde{\omega}_1$ in these coordinates as

$$\omega_2 = [z_2 w_2 A_2(z_2, w_2) + w_2^2] dz_2 + [z_2^2 B_2(z_2, w_2) + 2 z_2 w_2] dw_2,$$

where $A_2(z_2, w_2) = A_1(z_2 w_2, w_2)$ and $B_2(z_2, w_2) = A_1(z_2 w_2, w_2) + w_2 B_1(z_2 w_2, w_2)$. In particular,

$$A_2(0,0) = B_2(0,0) = A_1(0,0) \neq 0.$$

We have $\mu_0(\omega_2) = \mu_0(\omega_1) + 1 = \mu + 2$ by (2.14), $\nu_0(\omega_2) = 2$, while the tangent form of ω_2 is

$$z_2 w_2 (2 a_{2,0} z_2 + 3 w_2),$$

which has three different roots, and hence its blow-up foliation in $[0:1]$ has three singular points (let us say p_1, p_2, p_3) on the exceptional divisor (by Remark 2.1.33). Applying (2.14) at the blow-up of ω_2 in $[0:1]$, we have

$$\mu + 2 = 1 + \sum_{j=1}^{3} \mu_{p_j}(\widetilde{\omega}_2),$$

with $\mu_{p_j}(\widetilde{\omega}_2) \geq 1$ for $j = 1,2,3$; therefore $\mu_{p_j}(\widetilde{\omega}_2) \leq \mu - 1$ for $j = 1,2,3$, and we are done by induction.

We are left with the last possibility, $A_1(0,0) = a_{2,0} = 0$. Observing the equation (2.25) we see that $\nu_0(\omega_1) = 2$, while we have already seen that $\mu_0(\omega_1) = \mu + 1$; moreover, the tangent form of ω_1 is

$$z_1 \big(a_{3,0} z_1^2 + (a_{1,1} + 2 b_{2,0}) z_1 w_1 + 2 w_1^2 \big).$$

It has at least two distint roots, and hence its blow-up foliation in $[1:0]$ has at least two singular points on the exceptional divisor (by Remark 2.1.33). Let us call these singular points p_j ($j = 1, \dots, k$, with $k = 2$ or $k = 3$). As before, applying (2.14) at the blow-up of ω_1 in $[1:0]$, we have

$$\mu + 1 = 1 + \sum_{j=1}^{k} \mu_{p_j}(\widetilde{\omega}_1),$$

with $\mu_{p_j}(\widetilde{\omega}_2) \geq 1$ for every j. Then we have again that $\mu_{p_j}(\widetilde{\omega}_1) \leq \mu - 1$, and we are done. □

The blow-up process can be used to simplify some elementary singularities: we shall obtain the so called *final forms*.

Proposition 2.3.4. *Let $\mathcal{F}$ be a saturated holomorphic foliation in a complex surface M, with an elementary singularity at a. Then, up to perform a blow-up at a, we can suppose that the elementary matrix of $\mathcal{F}$ at a is diagonalizable.*

Proof. The assertion is not trivial only if the Jordan form of the elementary matrix L is
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};$$
up to a linear change of coordinates, we can suppose that L coincide with its Jordan form.

Without loss of generality, we can suppose that $\mathcal{F}$ is given by the holomorphic 1-form
$$\omega = (z + w + \text{h.o.t.})dz + (-z + \text{h.o.t.})dw.$$
The tangent form of ω is $h_2 = z^2$, so, thanks to Proposition 2.1.28, the blow-up foliation $\widetilde{\mathcal{F}}$ has only one singularity $c = [0 : 1]$ on the exceptional divisor; moreover, near c, $\widetilde{\mathcal{F}}$ is given by

$$\widetilde{\omega} = w\bigl(1 + f(z,w)\bigr)dz + \bigl(z^2 + wg(z,w)\bigr)dw,$$

for suitable holomorphic functions f and g, such that $f(0,0) = 0$ and $g(0,0) \in \mathbb{C}$. Then the elementary matrix L_c of $\widetilde{\omega}$ at c is

$$L_c = \begin{pmatrix} 0 & 0 \\ g(0,0) & 1 \end{pmatrix},$$

that has 0 and 1 as eigenvalues, and hence it is diagonalizable. □

Remark 2.3.5. Thanks to Theorem 2.3.3 and Proposition 2.3.4, if we have a holomorphic foliation in a complex surface, with finitely many singular points, we can suppose, up to performing first a finite number of blow-ups and then linear changes of coordinates near the singularities, that all this singular points are elementary singularities with diagonal elementary matrix. In particular we can suppose that the blow-up foliation is given by

$$\omega = zdw - \alpha wdz + \text{h.o.t.}$$

at every singular point a, where $\alpha = \frac{\lambda_2}{\lambda_1} \in \mathbb{C}$, and λ_1 and λ_2 are the eigenvalues of the elementary matrix at a. Moreover, the parameter α is almost uniquely defined:

if $\alpha \neq 0$, then it can be replaced by α^{-1} performing the linear change of coordinates $(z, w) \mapsto (w, z)$.

So, for having a complete description of the behavior of a holomorphic foliation, up to considering local coordinates, we have to study the local behavior of foliations in open neighborhoods of $0 \in \mathbb{C}^2$, with an elementary singularity at 0, and with diagonal elementary matrix at 0.

Definition 2.3.6. Let $\mathcal{F}$ be a holomorphic foliation in a open subset U of $\mathbb{C}^2$, such that 0 is an elementary singularity. Thanks to Remark 2.3.5, we can suppose that $\mathcal{F}$ is given by the holomorphic 1-form

$$\omega = zdw - \alpha w dz + \text{h.o.t.},$$

where $\alpha := \lambda_2/\lambda_1$ is the ratios of the eigenvalues of the elementary matrix of $\mathcal{F}$ at 0. We shall call α the **index** of $\mathcal{F}$ at 0 (up to the identification $\alpha \sim \alpha^{-1}$). Then we call the point 0

- a **focus** if α is in $\mathbb{C} \setminus \mathbb{R}$;
- a **saddle** if $\alpha < 0$;
- a **node** if $\alpha > 0$;
- a **saddle-node** if $\alpha = 0$.

In each of the previous cases we call 0 **resonant** if α is in $\mathbb{Q}$.

An elementary singularity whose elementary matrix is non-diagonalizable is called a **Jordan node**: in this case we shall say that the elementary singularity has index 1 (since it is the ratio of the eigenvalues of its elementary matrix).

Proposition 2.3.7. *Let $\mathcal{F}$ be a holomorphic foliation in a open subset U of $\mathbb{C}^2$, such that 0 is an elementary singularity, given by the holomorphic 1-form*

$$\omega = zdw - \alpha w dz + h.o.t.,$$

with $\alpha \in \mathbb{C}$. Let $\widetilde{\mathcal{F}}$ the blow-up foliation of $\mathcal{F}$ at 0. If $\alpha \neq 1$ then $\widetilde{\mathcal{F}}$ has two singular points in the exceptional divisor, whose indexes are $\alpha - 1$ and $\frac{\alpha}{1-\alpha}$. If $\alpha = 1$ then $\widetilde{\mathcal{F}}$ has no singular points in the exceptional divisor.

Proof. The tangent form for ω is $h_2 = (1-\alpha)zw$. If $\alpha \neq 1$, then $h_2 \neq 0$ and 0 is a non-dicritical singularity. Thanks to Proposition 2.1.28, $\widetilde{\mathcal{F}}$ has two singular points, $p := [1 : 0]$ and $q := [0 : 1]$. In the first case, the blow-up foliation is given by

$$\widetilde{\omega}_p = \big((1-\alpha)w + b_z z\big)dz + zdw + \text{h.o.t.},$$

for a suitable $b_z \in \mathbb{C}$. Hence the elementary matrix at p is of the form

$$\begin{pmatrix} -1 & b_z \\ 0 & 1-\alpha \end{pmatrix}$$

and the index is $\alpha - 1$.

In the second case, the blow-up foliation is given by

$$\widetilde{\omega}_q = -\alpha w dz + \big((1-\alpha)z + a_w w\big)dw + \text{h.o.t.},$$

for a suitable $a_w \in \mathbb{C}$. Hence the elementary matrix at q is of the form

$$\begin{pmatrix} \alpha - 1 & 0 \\ -a_w & -\alpha \end{pmatrix}$$

and the index is $\frac{\alpha}{1-\alpha}$.

If $\alpha = 1$, then $h_2 \equiv 0$, thus 0 is a dicritical singularity. We have

$$\omega = f(z,w)dz + g(z,w)dw, \qquad \text{with } f(z,w) = -w + \text{h.o.t.}, \quad g(z,w) = z + \text{h.o.t.}$$

Thanks to Proposition 2.1.28, and since $\tilde{g}_z(z,w) = 1 + \text{h.o.t.}$ and $\tilde{f}_w(z,w) = -1 + \text{h.o.t.}$ (we are using notations of Proposition 2.1.28), there are no singular points in the exceptional divisor, and we are done. $\square$

Proposition 2.3.8. *Let $\mathcal{F}$ be a holomorphic foliation in a open subset U of $\mathbb{C}^2$, such that 0 is an elementary singularity. Then, up to performing finitely many blow-ups, we can suppose that the index α of the blow-up foliation at every singularity (and its inverse α^{-1}) does not belong to $\mathbb{N} \setminus \{0\}$.*

Proof. We shall use an induction argument. If $\alpha = 1$, then if the elementary matrix is non-diagonalizable, thanks to Proposition 2.3.4 there is only one singular point for the blow-up foliation in the exceptional divisor, whose index is 0, while if the elementary matrix is diagonalizable, then there are no singularities in the exceptional divisor, thanks to Proposition 2.3.7.

So suppose that the assertion is proved for any singularity with index less than or equal to $n-1$, and consider an elementary singularity whit index $\alpha = n$. Thanks to Proposition 2.3.7, after a blow-up we have two elementary singularities, with indexes $n-1$ and $\frac{n}{1-n}$. For the first singularity, we can apply the induction hypothesis, while the second index does not belong to $\mathbb{N}$. $\square$

Definition 2.3.9. Let $\mathcal{F}$ be a holomorphic foliation in a open subset U of $\mathbb{C}^2$, such that 0 is an elementary singularity with index α. Then 0 is **in final form** if $\alpha, \alpha^{-1} \notin \mathbb{N} \setminus \{0\}$.

In this chapter we have proved that a holomorphic foliation with a finite number of singularities in a complex surface, up to blow-ups, can be reduced to a foliation with only elementary singularities (in final form).

The next chapter is devoted to the study of focus and node singularities (the so called *Poincaré domain*), while the last chapter is devoted to saddle singularities (the *strict Siegel domain*).

We shall not deal with the saddle-node case: we refer to [Dul04] for the formal classification, to [MR82] and [Mou93] for the analytic classification, and to [Sau06] and [Sau09] for an interesting approach to the saddle-node case using Écalle's mould calculus and resurgent functions.

Chapter 3

Dynamics of foliations in the Poincaré domain

Jasmin Raissy[1]

3.1 Basic definitions

In this chapter and in the next one we shall deal with the local behavior of foliations with an elementary singularity. We start introducing some notations and definitions.

Let $\mathcal{O}_n$ be the ring of the germs at the origin $0 \in \mathbb{C}^n$ of holomorphic functions in n complex variables, and let $\mathfrak{m}_n$ be the unique maximal ideal of $\mathcal{O}_n$, i.e., the set of germs of $\mathcal{O}_n$ vanishing at the origin; if we fix $z = (z_1, \ldots, z_n)$ local coordinates in $0 \in \mathbb{C}^n$, then $\mathfrak{m}_n = \langle z_1, \ldots, z_n \rangle$ is the ideal generated by $z_1, \ldots, z_n$. We shall denote by $\mathbb{C}[\![z_1, \ldots, z_n]\!] = \mathbb{C}[\![z]\!]$ the ring of formal power series in n complex variables, and we set $\widehat{\mathfrak{m}}_n = \langle z_1, \ldots, z_n \rangle$ the maximal ideal of $\mathbb{C}[\![z_1, \ldots, z_n]\!]$.

We shall also denote by $\mathfrak{X}_n$ the module of germs of holomorphic vector fields at $(\mathbb{C}^n, 0)$ with a singularity at the origin.

Since we can identify with $\mathbb{C}^n$ the tangent space of $\mathbb{C}^n$ in each of its points using the canonical basis $\partial_1, \ldots, \partial_n$, where $\partial_j = \frac{\partial}{\partial z_j}$, we can always locally write a holomorphic vector field $X \in \mathfrak{X}_n$ in the form

$$X = \sum_{j=1}^{n} X_j \partial_j$$

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where $X_j \in \mathfrak{m}_n$ for every $j = 1, \dots, n$.

An expression of the form

$$\widehat{X} = \sum_{j=1}^{n} X_j \partial_j, \tag{3.1}$$

where $X_j \in \widehat{\mathfrak{m}}_n$ for $j = 1, \dots, n$, shall be called a **formal vector field singular at the origin of** $\mathbb{C}^n$, and we shall denote by $\widehat{\mathfrak{X}}_n$ the module of germs of formal vector fields at $(\mathbb{C}^n, 0)$ with a singularity at the origin.

We can always locally write $X \in \widehat{\mathfrak{X}}_n$ as

$$X = X^{(1)} + X^{(2)} + \cdots$$

where $X^{(m)}$ is a homogeneous vector field of degree m (i.e., all its monomials are homogeneous of degree m).

Definition 3.1.1. Let $Q = (q_1, \dots, q_n) \in \mathbb{N}^n$ be a multi-index, $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, and $z = (z_1, \dots, z_n)$ be local coordinates in $0 \in \mathbb{C}^n$; then we shall define

$$|Q| = \sum_{j=1}^{n} q_j, \;\; \langle Q, \lambda \rangle = \sum_{j=1}^{n} q_j \lambda_j, \;\text{ and }\; z^Q = \prod_{j=1}^{n} z_j^{q_j}.$$

Definition 3.1.2. The space $\mathfrak{X}_n^m$ of the polynomial vector fields of $\mathfrak{X}_n$ of degree less than or equal to m is called the **space of** m**-jets of vector fields**. We shall denote by $\pi_m \colon \widehat{\mathfrak{X}}_n \to \mathfrak{X}_n^m$ the obvious truncation map.

Then if $X \in \mathfrak{X}_n^m$, locally we have

$$X = \sum_{j=1}^{n} X_j \partial_j,$$

where $X_j \in \mathbb{C}[z_1, \dots, z_n]$, $X_j(0) = 0$ and $\deg(X_j) \leq m$ for every $j = 1, \dots, n$.

It is obvious that $\mathfrak{X}_n^m$ is a finite dimensional complex vector space, and a basis of $\mathfrak{X}_n^m$ is

$$\mathcal{B}_n^m = \big\{ z^Q \partial_k \mid Q \in \mathbb{N}^n, 1 \leq |Q| \leq m, k \in \{1, \dots, n\} \big\}. \tag{3.2}$$

It is also easy to verify that

$$\widehat{\mathfrak{X}}_n = \varprojlim \mathfrak{X}_n^m,$$

where $\varprojlim$ is the **inverse limit** (see [Bou68, p. 191]).

Definition 3.1.3. A formal vector field $\widehat{X} \in \widehat{\mathfrak{X}}_n$ is called k**-flat** if it does not contain terms of order less than or equal to k, or equivalently, if $\widehat{X} \in \widehat{\mathfrak{m}}_n^k \widehat{\mathfrak{X}}_n$, i.e., writing $\widehat{X}$ as in (3.1), if $X_j \in (\widehat{\mathfrak{m}}_n)^{k+1}$ for every $j = 1, \dots, n$.

Definition 3.1.4. Two holomorphic vector fields $X, Y \in \mathfrak{X}_n$ are **holomorphically conjugated** (resp., **formally conjugated**) if there exists a holomorphic (resp., formal) change of variables Φ of $(\mathbb{C}^n, 0)$ such that

$$\Phi_* X = d\Phi \circ X \circ \Phi^{-1} = Y. \tag{3.3}$$

They are **holomorphically equivalent** (resp., **formally equivalent**) if there exist a holomorphic (resp., formal) change of variables Φ of $(\mathbb{C}^n, 0)$ and a holomorphic non-vanishing function (resp., formal non-vanishing power series) Ψ such that

$$\Phi_* X = d\Phi \circ X \circ \Phi^{-1} = \Psi Y. \tag{3.4}$$

3.2 Formal normalization

Definition 3.2.1. A n-tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ is said to be **resonant** if there exists a multi-index $Q = (q_1, \dots, q_n) \in \mathbb{N}^n$ such that $|Q| \geq 2$ and there exists $j \in \{1, \dots, n\}$ such that

$$\lambda_j = \langle Q, \lambda \rangle = \sum_{k=1}^{n} q_k \lambda_k. \tag{3.5}$$

Equation (3.5) is said a **resonance relation** for λ and the number $|Q|$ is the **order of the resonance**.

Let X be a holomorphic vector field, singular at $0 \in \mathbb{C}^2$, and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of its linear part $X^{(1)}$. Then X is called **resonant** if $\lambda = (\lambda_1, \dots, \lambda_n)$ is a resonant n-tuple. Then the monomial vector field $z^Q \partial_j$ is called a **resonant term** for λ (or for X) if $\lambda_j = \langle Q, \lambda \rangle$ is a resonance relation for λ.

Remark 3.2.2. Given $X \in \mathfrak{X}_n$, up to a linear change of coordinates we can assume that its linear term $X^{(1)}$ is in Jordan normal form, that is

$$X^{(1)} = \sum_{j=1}^{n} (\lambda_j z_j + \varepsilon_j z_{j-1}) \partial_j = S + N$$

where

$$S = \sum_{j=1}^{n} \lambda_j z_j \partial_j, \quad N = \sum_{j=1}^{n} \varepsilon_j z_{j-1} \partial_j$$

and $\varepsilon_j \in \{0, 1\}$ can be non-zero only if $\lambda_j = \lambda_{j-1}$ and $j > 1$. Note that the Lie bracket $[S, N]$ vanishes.

Lemma 3.2.3. *Let* $S = \sum_{j=1}^{n} \lambda_j z_j \partial_j \in \mathfrak{X}_n$. *Then for every multi-index* $Q \in \mathbb{N}^n$ *and for every* $k \in \{1, \dots, n\}$, *we have*

$$[S, z^Q \partial_k] = \big(\langle Q, \lambda \rangle - \lambda_k\big) z^Q \partial_k.$$

Proof. We have

$$\begin{aligned} [S, z^Q \partial_k] &= \left[\sum_{j=1}^{n} \lambda_j z_j \partial_j , z^Q \partial_k \right] \\ &= \sum_{j=1}^{n} [\lambda_j z_j \partial_j , z^Q \partial_k] \\ &= \left(\sum_{j=1}^{n} \lambda_j q_j - \lambda_k \right) z^Q \partial_k , \end{aligned}$$

that is the thesis. □

Remark 3.2.4. It follows from Lemma 3.2.3 that each monomial vector field $z^Q \partial_k$ with $Q \in \mathbb{N}^n$, $|Q| \geq 1$ and $k \in \{1, \dots, n\}$, is an eigenvector with eigenvalue $\alpha_{Q,k} = \langle Q, \lambda \rangle - \lambda_k$ of the Lie operator $\mathcal{L}_S = \mathrm{ad}_S \colon \widehat{\mathfrak{X}}_n \to \widehat{\mathfrak{X}}_n$ (recall that $\mathrm{ad}_S = [S, \cdot]$). In particular a monomial vector field $z^Q \partial_k$ with $Q \in \mathbb{N}^n$, $|Q| \geq 2$ and $k \in \{1, \dots, n\}$ belongs to $\ker(\mathcal{L}_S)$ if and only if $\langle Q, \lambda \rangle - \lambda_k = 0$, and hence if and only if it is a resonant term for λ, while $z_h \partial_k$ belongs to $\ker(\mathcal{L}_S)$ if and only if $\lambda_h = \lambda_k$. It follows that $\mathcal{L}_S$ has non-trivial kernel (i.e., a kernel with not only linear terms) if and only if λ is resonant.

The space of m-jets $\mathfrak{X}_n^m$ is a finite dimensional complex vector space, with canonical basis $\mathcal{B}_n^m$ defined by (3.2); furthermore for every linear vector field $A \in \mathfrak{X}_n^1$ it is easy to verify that $\mathcal{L}_A$ is a linear operator on $\mathfrak{X}_n^m$ and $\mathcal{L}_A(\mathfrak{X}_n^m) \subseteq \mathfrak{X}_n^m$ for every $m \geq 1$. Therefore, if we consider $\mathcal{L}_S \colon \mathfrak{X}_n^m \to \mathfrak{X}_n^m$ where $S = \sum_{j=1}^{n} \lambda_j z_j \partial_j$, by Lemma 3.2.3, for every $m \geq 1$ we have

$$\mathfrak{X}_n^m = \bigoplus_{\alpha_{Q,k} \in \mathcal{N}^m} E^m_{\alpha_{Q,k}}$$

where

$$\mathcal{N}^m = \big\{ \alpha_{Q,k} \in \mathbb{C} : \alpha_{Q,k} = \langle Q, \lambda \rangle - \lambda_k ,\, Q \in \mathbb{N}^n ,\, 1 \leq |Q| \leq m, k \in \{1, \dots, n\} \big\} \tag{3.6}$$

and $E^m_{\alpha_{Q,k}}$ is the eigenspace corresponding to the eigenvalue $\alpha_{Q,k}$.

Lemma 3.2.5. *Let $S = \sum_{j=1}^{n} \lambda_j z_j \partial_j \in \mathfrak{X}_n$. Then*

$$\widehat{\mathfrak{X}}_n = \varprojlim \bigoplus_{\alpha_{Q,k} \in \mathcal{N}^m} E^m_{\alpha_{Q,k}}$$

where $\mathcal{N}^m$ is defined by 3.6 and $E^m_{\alpha_{Q,k}} \in \mathfrak{X}^m_n$ is the eigenspace corresponding to the eigenvalue $\alpha_{Q,k}$.

Proof. It is obvious since, for each $m \geq 1$, we have $\mathfrak{X}^m_n = \bigoplus_{\alpha_{Q,k} \in \mathcal{N}^m} E^m_{\alpha_{Q,k}}$, and

$$\widehat{\mathfrak{X}}_n = \varprojlim \mathfrak{X}^m_n.$$

□

Corollary 3.2.6. *Let $S = \sum_{j=1}^{n} \lambda_j z_j \partial_j \in \mathfrak{X}_n$. Then the point spectrum of the Lie operator $\mathcal{L}_S \colon \widehat{\mathfrak{X}}_n \to \widehat{\mathfrak{X}}_n$ is*

$$\mathcal{S} = \big\{\alpha_{Q,k} \in \mathbb{C} : \alpha_{Q,k} = \langle Q, \lambda\rangle - \lambda_k,\, Q \in \mathbb{N}^n,\, |Q| \geq 1, k \in \{1, \ldots, n\}\big\}.$$

Proof. Thanks to Lemma 3.2.3 and to Lemma 3.2.5, each element of the eigenspace $E_\alpha \subseteq \widehat{\mathfrak{X}}_n$ corresponding to the eigenvalue α is the projective limit of finite linear combinations of monomial vector fields $z^Q \partial_k$ with $\alpha = \langle Q, \lambda\rangle - \lambda_k$. □

Remark 3.2.7. In particular, every $Y \in\to \widehat{\mathfrak{X}}_n$ can be uniquely written in the form

$$Y = Y_0 + W$$

with $Y_0 \in E_0$ and $W \in \oplus_{\alpha \in \mathcal{S} \setminus \{0\}} E_\alpha$.

Definition 3.2.8. We shall say that a vector field $N \in \widehat{\mathfrak{X}}_n$ is **nilpotent**, if for every $k \geq 1$ there is $m = m(k) \geq 0$ so that $\pi_k \circ \mathcal{L}^m_N|_{\mathfrak{X}^k_n} \equiv 0$.

Remark 3.2.9. Every 1-flat vector field is nilpotent, because if N is h-flat and X is k-flat then $\mathcal{L}_N(X)$ is $(h+k)$-flat. More generally, if $N_0 \in \mathfrak{X}^1_n$ is nilpotent and N_1 is 1-flat, then $N_0 + N_1$ is nilpotent even when $[N_0, N_1] \neq 0$.

Proposition 3.2.10. *Let $X = S + N \in \widehat{\mathfrak{X}}_n$, where $S = \sum_{j=1}^{n} \lambda_j z_j \partial_j$ and N is a nilpotent vector field such that $[S, N] = 0$. For every α in the point spectrum of $\mathcal{L}_S$ let $E_\alpha \subset \widehat{\mathfrak{X}}_n$ be the corresponding eigenspace. Then the Lie operator $\mathcal{L}_X \colon \widehat{\mathfrak{X}}_n \to \widehat{\mathfrak{X}}_n$ restricted to $\bigoplus_{\alpha \neq 0} E_\alpha$ is invertible with inverse*

$$\mathcal{L}_X^{-1} = \sum_{l=0}^{\infty} (-1)^l (\mathcal{L}_S^{-1})^{(l+1)} \circ (\mathcal{L}_N)^l.$$

In particular, if $W \in \widehat{\mathfrak{X}}_n$ is k-flat, then $\mathcal{L}_X^{-1}(W)$ is also k-flat. Moreover, if we restrict ourselves to the space $\mathfrak{X}^k_n$ of k-jets, it suffices to take the sum up to m where $m = m(k)$ is such that $\pi_k \circ \mathcal{L}^m_N$ restricted to $\mathfrak{X}^k_n$ is zero.

Proof. Thanks to Lemma 3.2.3, if $\langle Q, \lambda\rangle - \lambda_k \neq 0$, i.e., if $z^Q\partial_k \notin E_0$, we have

$$\mathcal{L}_S^{-1}(z^Q\partial_k) = \frac{1}{\langle Q, \lambda\rangle - \lambda_k} z^Q\partial_k.$$

Moreover, note that $\mathcal{L}_X = \mathcal{L}_S + \mathcal{L}_N$ and $\mathcal{L}_S$ commute with $\mathcal{L}_N$; hence $\mathcal{L}_S, \mathcal{L}_S^{-1}, \mathcal{L}_N$ pairwise commute.

Let us consider

$$\mathcal{L} = \sum_{l=0}^{\infty}(-1)^l(\mathcal{L}_S^{-1})^{(l+1)} \circ (\mathcal{L}_N)^l.$$

Notice that $\mathcal{L}$ is well defined because N is nilpotent. We have

$$\begin{aligned}
\mathcal{L}\mathcal{L}_X &= \sum_{l=0}^{\infty}(-1)^l\mathcal{L}_S^{-(l+1)}\mathcal{L}_N^l(\mathcal{L}_S + \mathcal{L}_N) \\
&= \sum_{l=0}^{\infty}(-1)^l\mathcal{L}_S^{-l}\mathcal{L}_N^l + \sum_{l=0}^{\infty}(-1)^l\mathcal{L}_S^{-(l+1)}\mathcal{L}_N^{(l+1)} \\
&= \sum_{l=0}^{\infty}(-1)^l\mathcal{L}_S^{-l}\mathcal{L}_N^l + \sum_{l=1}^{\infty}(-1)^{l+1}\mathcal{L}_S^{-l}\mathcal{L}_N^l \\
&= (-1)^0\mathcal{L}_S^0\mathcal{L}_N^0 \\
&= \mathrm{Id};
\end{aligned}$$

and analogously we verify that $\mathcal{L}_X\mathcal{L} = \mathrm{Id}$. It follows that $\mathcal{L}$ is the inverse operator $\mathcal{L}_X^{-1}$ of $\mathcal{L}_X$.

Note that, if we project to the space $\mathfrak{X}_n^k$ of k-jets and we consider

$$\mathcal{L}_{(m)} = \sum_{l=0}^{m}(-1)^l(\mathcal{L}_S^{-1})^{(l+1)} \circ (\mathcal{L}_N)^l,$$

where $m = m(k)$ is such that $\mathcal{L}_N^m$ restricted to $\mathfrak{X}_n^k$ is zero, we have

$$\begin{aligned}
\pi_k\left(\mathcal{L}_{(m)}\mathcal{L}_X\right) &= \pi_k\left(\sum_{l=0}^{m}(-1)^l\mathcal{L}_S^{-(l+1)}\mathcal{L}_N^l\mathcal{L}_X\right) \\
&= \pi_k\left(\sum_{l=0}^{m}(-1)^l\mathcal{L}_S^{-l}\mathcal{L}_N^l + \sum_{l=1}^{m}(-1)^{l+1}\mathcal{L}_S^{-l}\mathcal{L}_N^l\right) \\
&= \pi_k\big(\mathrm{Id} + (-1)^{m+1}\mathcal{L}_S^{-m}\mathcal{L}_N^m\big) \\
&= \pi_k + (-1)^{m+1}\mathcal{L}_S^{-m} \circ \pi_k \circ \mathcal{L}_N^m \\
&= \pi_k,
\end{aligned}$$

and we are done. □

Proposition 3.2.11. *Let $X = S + N \in \widehat{\mathfrak{X}}_n$, where $S = \sum_{j=1}^n \lambda_j z_j \partial_j$ and N is a nilpotent vector field sucht that $[S, N] = 0$. Let $\mathcal{S}$ be the point spectrum of $\mathcal{L}_S$ and, for every $\alpha \in \mathcal{S}$, let $E_\alpha \subset \widehat{\mathfrak{X}}_n$ be the corresponding eigenspace. Then for every given $Y \in \widehat{\mathfrak{X}}_n$ there exist a unique $Y_0 \in E_0$ and a unique $Z \in \bigoplus_{\alpha \neq 0} E_\alpha$ such that*

$$Y = Y_0 + [X, Z].$$

Moreover, if Y is k-flat then Y_0 and Z are k-flat.

Proof. Let Y_0 be the projection of Y on E_0; thus we have $Y = Y_0 + W$, where $W \in \bigoplus_{\alpha \neq 0} E_\alpha$. Then it is sufficient to take $Z = \mathcal{L}_X^{-1}(W)$, and the last assertion follows from the proof of Proposition 3.2.10. □

Theorem 3.2.12 (**Poincaré-Dulac**, 1904)**.** *Let $\mathfrak{X} \in \widehat{\mathfrak{X}}_n$ and let $\{\lambda_1, \dots \lambda_n\}$ be the spectrum of its linear term. The X is formally conjugated to*

$$X^{\mathrm{PD}} = S + X^{\mathrm{res}} \in \widehat{\mathfrak{X}}_n$$

where $S = \sum_{j=1}^n \lambda_j z_j \partial_j$, $X^{\mathrm{res}} \in \widehat{\mathfrak{X}}_n$ and $[S, X^{\mathrm{res}}] = 0$. In particular, if $\lambda = (\lambda_1, \dots \lambda_n)$ is non-resonant, then X is formally linearizable.

Proof. Up to a linear change of coordinates, we can assume that the linear term $X^{(1)}$ of X is in Jordan normal form, i.e., $X^{(1)} = S + N$, where $S = \sum_{j=1}^n \lambda_j z_j \partial_j$, $N \in \mathfrak{X}_n^1$ and $[S, N] = 0$ (see Remark 3.2.2). Notice that N is nilpotent.

Assume that, for $k \geq 1$, we can write

$$X = S + X_k^{\mathrm{res}} + R^{(k+1)},$$

where $X_k^{\mathrm{res}} \in \mathfrak{X}_n^k$ is nilpotent, $[S, X_k^{\mathrm{res}}] = 0$, and $R^{(k+1)} \in \widehat{\mathfrak{X}}_n$ is k-flat. Then, by Proposition 3.2.11, there exist a unique $X_{k+1}^{\mathrm{r}} \in E_0$ and a unique $U_{k+1} \in \bigoplus_{\alpha \neq 0} E_\alpha$ (where $E_\alpha \subset \widehat{\mathfrak{X}}_n$ is the eigenspace corresponding to the eigenvalue α of $\mathcal{L}_S$), such that X_k^{r} and U_{k+1} are k-flat and

$$R^{(k+1)} = X_{k+1}^{\mathrm{r}} + [S + X_k^{\mathrm{res}}, U_{k+1}]. \tag{3.7}$$

Let $\varphi^{(k+1)} = \exp(U_{k+1})$ be the time-1 flow of U_{k+1}. Then

$$\varphi_*^{(k+1)} X = S + X_k^{\mathrm{res}} + X_{k+1}^{\mathrm{r}} + \tilde{R}^{(2k+1)},$$

where $\tilde{R}^{(2k+1)}$ is $2k$-flat. Indeed

$$(\exp tU_{k+1})_* X = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{L}_{U_{k+1}}^n (X) = X + t[U_{k+1}, X] + \frac{1}{2} t^2 \big[U_{k+1}, [U_{k+1}, X]\big] + \cdots ;$$

hence the coefficient of t is k-flat, the coefficient of t^2 is $2k$-flat, ..., the coefficient of t^p is pk-flat, and so on. It follows that $\varphi_*^{(k+1)}X \equiv X + [U_{k+1}, X]$ modulo $2k$-flat vector fields. Using then equation (3.7), we have

$$\begin{aligned}\varphi_*^{(k+1)}X &\equiv S + X_k^{\mathrm{res}} + R^{(k+1)} + [U_{k+1}, S + X_k^{\mathrm{res}} + R^{(k+1)}] \\ &\equiv S + X_k^{\mathrm{res}} + R^{(k+1)} - [S + X_k^{\mathrm{res}}, U_{k+1}] \\ &\equiv S + X_k^{\mathrm{res}} + X_{k+1}^{\mathrm{r}}\end{aligned}$$

modulo $2k$-flat vector fields. Notice that we obtain the same result if we use the $(2k)$-jet of $\varphi^{(k+1)}$; therefore X is holomorphically conjugated to $S + X_k^{\mathrm{res}} + X_{k+1}^{\mathrm{r}}$ up to $(2k)$-flat vector fields. Iterating this process, we get the assertion.

Notice that if λ is non-resonant, then X^{res} does not have non-linear terms (see Remark 3.2.4, so X is formally conjugated to $X^{(1)} = S + N$. □

Definition 3.2.13. Let $X \in \widehat{\mathfrak{X}}_n$, let $\{\lambda_1, \dots \lambda_n\}$ be the spectrum of its linear term, and let $S = \sum_{j=1}^n \lambda_j z_j \partial_j$. We say that

- X is in **Poincaré-Dulac normal form up to order** k, with $k \geq 1$, if it is of the form
$$X = S + X_k^{\mathrm{res}} + W$$
where $X_k^{\mathrm{res}} \in \mathfrak{X}_n^k$ is nilpotent, $[S, X_k^{\mathrm{res}}] = 0$, and $W \in \widehat{\mathfrak{X}}_n$ is k-flat;
- X is in **Poincaré-Dulac formal normal form** if it is of the form
$$X = S + X^{\mathrm{res}}$$
where $X^{\mathrm{res}} \in \widehat{\mathfrak{X}}_n$ and $[S, X^{\mathrm{res}}] = 0$;
- X is in **Poincaré-Dulac normal form** if it can be written as
$$X = S + X^{\mathrm{res}}$$
where $X^{\mathrm{res}} \in \mathfrak{X}_n$ and $[S, X^{\mathrm{res}}] = 0$.

Remark 3.2.14. Given $X \in \widehat{\mathfrak{X}}_n$, its Poincaré-Dulac normal forms are not unique. Moreover the proof Theorem 3.2.12 implies that, for any fixed $k \geq 1$, we can always conjugate $X \in \mathfrak{X}_n$ to a holomorphic vector field in Poincaré-Dulac normal form up to order k, but in general X is only formally conjugated to *one of its* Poincaré-Dulac *formal* normal forms.

In the next section we shall see that under certain hypotheses on the eigenvalues of the linear term of $X \in \mathfrak{X}_n$, the vector field X is indeed holomorphically equivalent to a holomorphic vector field in Poincaré-Dulac normal form.

3.3 Holomorphic normalization in Poincaré domain

We saw in the previous section that to linearize a given non-resonant vector field singular at the origin $X \in \mathfrak{X}_n$, at each step of the Poincaré-Dulac process we had to compute the inverse of $\mathcal{L}_S = \operatorname{ad}_S$ on the space of homogeneous vector fields. To do that, we had to divide the Taylor coefficients by expression of the form $\lambda_j - \langle Q, \lambda \rangle \in \mathbb{C}$ where $Q \in \mathbb{N}^n$, $|Q| \geq 2$ and $1 \leq j \leq n$; these *denominators* may be *small* even when $\mathcal{L}_S$ is invertible. There are two different cases.

Definition 3.3.1. A n-tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ **belongs to the Poincaré domain**, and we write $\lambda \in \mathfrak{P}$, if the convex hull of the set of points $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}$ does not contain the origin inside or on its boundary. The **large Siegel domain** $\mathfrak{S}$ is the complement of $\mathfrak{P}$ in $\mathbb{C}^n$; we say that λ **belongs to the strict Siegel domain** if the convex hull of $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}$ contains the origin strictly inside.

Let $X \in \mathfrak{X}_n$ be a holomorphic vector field germ, $X^{(1)}$ its linear part, and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ the n-tuple of its eigenvalues. Then X **belongs to the Poincaré domain** (resp., **to the large** or **strict Siegel domain**) if λ does.

Proposition 3.3.2. *If a n-tuple $\lambda \in \mathbb{C}^n$ belongs to the Poincaré domain, then $\lambda_j - \langle Q, \lambda \rangle = 0$ only for a finite number of multi-indices $Q \in \mathbb{N}^n$ with $|Q| \geq 2$ and $1 \leq j \leq n$. Moreover, every non-zero denominator $\lambda_j - \langle Q, \lambda \rangle$, where $Q \in \mathbb{N}^n$, $|Q| \geq 2$ and $1 \leq j \leq n$, is bounded away from the origin, that is the origin is an isolated point of the set $\mathcal{N} = \{\lambda_j - \langle Q, \lambda \rangle \mid Q \in \mathbb{N}^n, |Q| \geq 2, 1 \leq j \leq n\}$.*

On the contrary, if λ belongs to the large Siegel domain, then either there are infinitely many vanishing denominators $\lambda_j - \langle Q, \lambda \rangle$, or the origin of $\mathbb{C}$ is an accumulation point of the set $\mathcal{N}$.

Proof. If the convex hull of $\{\lambda_1, \dots, \lambda_n\}$ in $\mathbb{C}$ does not contain the origin, by the convex separability theorem there exists a real linear functional $l \colon \mathbb{C} \to \mathbb{R}$ such that $l(\lambda_j) \leq -r < 0$ for all $j = 1, \dots, n$, and hence $l(\langle Q, \lambda \rangle) \leq -r|Q|$ for every multi-index Q with $|Q| \geq 2$. Then we have

$$l(\lambda_j - \langle Q, \lambda \rangle) \geq l(\lambda_j) + r|Q| \to +\infty \quad \text{for } |Q| \to +\infty.$$

Since l is bounded on any small neighbourhood of the origin of $\mathbb{C}$, the first two assertions are proved.

To prove the last assertion, notice that in the large Siegel domain, 0 lies on the relative interior of the convex hull of one, two or three eigenvalues.

The first case is the simpler one: if one of the eigenvalues is zero, say $\lambda_1 = 0$, then $q\lambda_1 = 0$ for every $q \in \mathbb{N}$, and hence we have infinitely many resonance relations $\lambda_1 = q\lambda_1$ for every $q \geq 2$.

If the origin lies on the line segment between two non-zero eigenvalues λ_1, λ_2, then there exists $m \in (0,1)$ such that

$$m\lambda_1 + (1-m)\lambda_2 = 0,$$

that is

$$\frac{\lambda_2}{\lambda_1} = -\frac{m}{1-m}.$$

Now, if $\frac{\lambda_2}{\lambda_1} \in \mathbb{Q}^-$, then there are infinitely many vanishing denominators; if $\frac{\lambda_2}{\lambda_1} \in \mathbb{R}^- \setminus \mathbb{Q}^-$, then if we consider its continued fraction expansion (see [Mar03, p. 22 and p. 75]), then, for every $n \geq 1$, its n-th convergent p_n/q_n satisfies

$$\left|\frac{\lambda_2}{\lambda_1} - \frac{p_n}{q_n}\right| \leq \frac{1}{2q_n^2},$$

and hence, the origin is an accumulation point of the set $\mathcal{N}$.

If the origin is inside a triangle formed by three eigenvalues, then up to rename them and up to (non-conformal) affine transformation of the complex plane $\mathbb{R}^2 \simeq \mathbb{C}$, we may assume without loss of generality that $\lambda_1 = 1$, $\lambda_2 = i$ and $-\lambda_3 \in \mathbb{R}^2_+ = \mathbb{R}^+ + i\mathbb{R}^+$. In this case, all "fractional parts" $-\mathbb{N}\lambda_3 (\mathrm{mod}\ \mathbb{Z} + i\mathbb{Z})$ of natural multiples of $-\lambda_3$ either form a finite subset of the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ (in which case all points of this set correspond to infinitely many vanishing denominators), or are uniformly distributed along some 1-torus, or dense. In both latter cases the point $(0,0) \in \mathbb{R}^2/\mathbb{Z}^2$ is the accumulation point of the "fractional parts" which are affine images of the denominators. □

Corollary 3.3.3. *Let $X \in \mathfrak{X}_n$ be in the Poincaré domain. Then any formal Poincaré-Dulac normal form of X is polynomial.*

Remark 3.3.4. Resonant n-tuples $\lambda \in \mathbb{C}^n$ are dense in the large Siegel domain and not dense in the Poincaré domain. For a proof of this see [Arn88, p. 188].

Now we shall prove that if the vector field $X \in \mathfrak{X}_n$ belongs to the Poincaré domain, then X always admits a holomorphic normalization.

Theorem 3.3.5. *(Poincaré normalization theorem) Let $X \in \mathfrak{X}_n$ be in the Poincaré domain. Then X is holomorphically conjugated to a polynomial Poincaré-Dulac normal form.*

In particular, if X is non-resonant, then it is holomorphically conjugated to its linear part.

We shall first prove this result for holomorphic vector fields with a diagonal non-resonant linear part $S = \sum_{j=1}^{n} \lambda_j z_j \partial_j$. The classical proof of Poincaré was achieved by the so-called **majorant method**. In the modern language, it takes the more convenient form of the contracting map principle in an appropriate functional space, the **majorant space**.

Definition 3.3.6. The **majorant operator** is the non-linear operator acting on formal power series $M\colon \mathbb{C}[\![z_1,\dots,z_n]\!] \to \mathbb{R}[\![z_1,\dots,z_n]\!]$, obtained by replacing all Taylor coefficients by their absolute values

$$M\left(\sum_{Q\in\mathbb{N}^n} c_Q z^Q\right) = \sum_{Q\in\mathbb{N}^n} |c_Q| z^Q.$$

If $f \in \mathbb{C}[\![z_1,\dots,z_n]\!]$ is a formal series, then we shall call $M(f)$ the **majorant series** for f.

The action of the majorant operator naturally extends to all formal objects (formal vector fields, formal transformations, etc.)

Definition 3.3.7. The **majorant ρ-norm** is the functional acting on the space of formal power series $\mathbb{C}[\![z_1,\dots,z_n]\!]$ by

$$[\![f]\!]_\rho = \sup_{|z|<\rho} |Mf(z)| = Mf(\rho,\dots,\rho) \le +\infty. \tag{3.8}$$

For a formal map $F = (F_1,\dots,F_n)$ the majorant ρ-norm is

$$[\![F]\!]_\rho = [\![F_1]\!]_\rho + \cdots + [\![F_n]\!]_\rho. \tag{3.9}$$

The **majorant space** is the subspace of formal power series (resp., maps) having finite majorant ρ-norm

$$\mathcal{B}_\rho = \{ f \in \mathbb{C}[\![z_1,\dots,z_n]\!] \mid [\![f]\!]_\rho < +\infty \}.$$

Proposition 3.3.8. *The space $\mathcal{B}_\rho$ with the majorant ρ-norm $[\![\cdot]\!]_\rho$ is complete.*

Proof. If $\rho = 1$ this is obvious since $\mathcal{B}_1$ is the space of infinite absolutely converging sequences $\{c_Q\}$, that is isomorphic to the standard Lebesgue space l^1 which is complete. The general case of an arbitrary ρ follows from the fact that the correspondence $f(\rho z) \mapsto f(z)$ is an isomorphism between $\mathcal{B}_\rho$ and $\mathcal{B}_1$. □

Remark 3.3.9. The space $\mathcal{B}_\rho$ is closely related but not coinciding with the space $\mathcal{A}_\rho$ of functions that are holomorphic in the polydisk $\{|z| < \rho\}$ and continuous on its closure, equipped with the usual sup-norm $\|f\|_\rho = \max_{|z|<\rho} |f(z)|$.

It is obvious that $\mathcal{B}_\rho \subset \mathcal{A}_\rho$, since a series belonging to $\mathcal{B}_\rho$ is absolutely convergent on the closed polydisk $\{|z| \le \rho\}$. On the contrary, if f is holomorphic in $\{|z| < \rho\}$ and continuous on its boundary, then by the Cauchy estimates, the Taylor coefficients c_Q of f satisfy

$$|c_Q| \le \|f\|_\rho \cdot \rho^{|Q|}, \quad Q \in \mathbb{N}^n.$$

The series $[\![f]\!]_\rho = \sum |c_Q|\rho^{|Q|}$ may still diverge, but any other norm $[\![f]\!]_{\rho'}$ with $\rho' < \rho$ shall be finite:

$$[\![f]\!]_{\rho'} \le [\![f]\!]_\rho \cdot \sum_{Q\in\mathbb{N}^n} \delta^{|Q|} < C[\![f]\!]_\rho, \quad C = C(\delta, n), \quad \delta = \frac{\rho'}{\rho} < 1.$$

To construct a counterexample showing that indeed $\mathcal{B}_\rho \subset \mathcal{A}_\rho$, but $\mathcal{B}_\rho \ne \mathcal{A}_\rho$, consider a convergent but not absolutely convergent Fourier series $\sum_{k\in\mathbb{Z}} c_k e^{ikt}$ in one real variable t and let $f(z) = \sum c_k z^k$. Then f converges at all points of the boundary $|z| = 1$ and represents an element of $\mathcal{A}_1$, but by construction its 1-norm is infinite. For other details see [IY08, p. 63] and references therein.

The important properties of the majorant spaces and norms concern operations on functions.

Lemma 3.3.10. *Let $\rho > 0$. Then:*

(i) for any two series $f, g \in \mathbb{C}[\![z_1, \dots, z_n]\!]$ we have

$$[\![fg]\!]_\rho \le [\![f]\!]_\rho \cdot [\![g]\!]_\rho, \tag{3.10}$$

provided that all norms are finite;

(ii) for any $F, G \in (\mathbb{C}[\![z_1, \dots, z_n]\!])^n$ with $F(0) = G(0) = 0$, we have

$$[\![F \circ G]\!]_\rho \le [\![F]\!]_\sigma, \quad \sigma = [\![G]\!]_\rho. \tag{3.11}$$

Proof.

(i) All Taylor coefficients of the product are obtained from the coefficients of the factors by addition and multiplication only, so the assertion is obvious.

(ii) Since all binomial coefficients are non-negative, each component of $M(F\circ G)$ has coefficients less than or equal to the corresponding coefficient of the corresponding component of $M(F) \circ M(G)$. Evaluating at $\rho = (\rho, \dots, \rho)$ yields $M(G)(\rho) = y \le \sigma = (\sigma, \dots, \sigma)$ where $\sigma = [\![G]\!]_\rho$ (and $y \le \sigma$ means that each component of y is less than or equal to σ). By monotonicity,

$$[\![F \circ G]\!]_\rho \le (M(F) \circ M(G))(\rho) \le M(F)(y) \le M(F)(\sigma) = [\![F]\!]_\sigma,$$

and we are done. $\square$

Lemma 3.3.11. *Let* $S = \sum_{j=1}^{n} \lambda_j z_j \partial_j \in \mathfrak{X}_n^1$*, with* $\lambda = (\lambda_1, \dots, \lambda_n)$ *a* n*-tuple of non-resonant complex numbers belonging to the Poincaré domain. Then the adjoint operator* $\mathcal{L}_S = ad_S$ *has a bounded inverse in the space of formal vector fields equipped with the majorant norm.*

Proof. The formal inverse of $\mathcal{L}_S$ is

$$\mathcal{L}_S^{-1} \colon \sum_{k=1}^{n} \sum_{|Q|\geq 2} c_{Q,k} z^Q \partial_k \mapsto \sum_{k=1}^{n} \sum_{|Q|\geq 2} \frac{c_{Q,k}}{\lambda_k - \langle Q, \lambda \rangle} z^Q \partial_k.$$

In the Poincaré domain the absolute values of all denominators are bounded from below by a positive constant $\varepsilon > 0$. Therefore any majorant ρ-norm is increased by no more than ε^{-1}:

$$[\![\mathcal{L}_S^{-1}]\!]_\rho \leq \left(\inf_{Q,k} |\lambda_k - \langle Q, \lambda \rangle| \right)^{-1} \leq \varepsilon^{-1} < +\infty,$$

and this proves that $\mathcal{L}_S^{-1}$ is bounded. □

Definition 3.3.12. Let X be a germ of holomorphic self-map of $(\mathbb{C}^n, 0)$ fixing the origin. The **operator of argument shift** is the operator $S_X \colon \mathcal{O}_n^n \to \mathcal{O}_n^n$ defined by

$$S_X \colon h(z) \mapsto X(z + h(z)),$$

Consider the one-parameter family of majorant Banach spaces $\mathcal{B}_\rho$ as in Definition 3.3.7, indexed by the real parameter $\rho \in \mathbb{R}^+$. We consider $\mathcal{B}_\rho$ as a subspace in $\mathcal{B}_{\rho'}$ for all $0 < \rho < \rho'$ (the natural embedding $\mathrm{Id}_{\rho,\rho'} \colon \mathcal{B}_\rho \to \mathcal{B}_{\rho'}$ is continuous).

Let S be an operator defined on all of these spaces for all sufficiently small values of ρ, considered as a family of operators $S_\rho \colon \mathcal{B}_\rho \to \mathcal{B}_\rho$ which commutes with the restriction operators $\mathrm{Id}_{\rho,\rho'}$ for any $\rho < \rho'$ (but we shall always omit the index ρ in the notation).

Definition 3.3.13. The operator $S = \{S_\rho\}$ is **strongly contracting** if

(i) $[\![S(0)]\!]_\rho = O(\rho^2)$ and

(ii) S is Lipschitz on the ball $B_\rho = \{[\![h]\!]_\rho \leq \rho\} \subset \mathcal{B}_\rho$ of the majorant ρ-norm (with the same ρ), with Lipschitz constant $O(\rho)$ as $\rho \to 0$.

Note that any strongly contracting operator takes, for ρ small enough, the balls B_ρ strictly into themselves, since the center is shifted by $O(\rho^2)$ and the diameter of the image $S(B_\rho)$ does not exceed $2\rho O(\rho) = O(\rho^2)$.

Lemma 3.3.14. *Let X be a germ of holomorphic self-map of $(\mathbb{C}^n, 0)$ fixing the origin and with no linear part. Then the operator of argument shift S_X is strongly contracting.*

Proof. First note that $S_X(0) = X$, which has ρ-norm $O(\rho^2)$ for all sufficiently small ρ, since X has no linear part.

Let us now compute the Lipschitz constant for $S = S_X$ restricted to the ball $B_\rho \subset \mathcal{B}_\rho$. If $h_1, h_2 \in (\mathbb{C}[\![z_1, \dots, z_n]\!])^n$, then the difference

$$g = S(h_1) - S(h_2) = X(\operatorname{Id} + h_1) - X(\operatorname{Id} + h_2)$$

can be represented as the integral

$$g(z) = \int_0^1 \left(\frac{\partial X}{\partial z}\right) \big(z + \tau h_1(z) + (1-\tau)h_2(z)\big) \cdot \big(h_1(z) - h_2(z)\big) d\tau.$$

By Lemma 3.3.10, since $\tau \in [0, 1]$, we have

$$[\![g]\!]_\rho \le \left[\!\!\left[\frac{\partial X}{\partial z}\right]\!\!\right]_{\sigma(\tau)} \cdot [\![h_1 - h_2]\!]_\rho, \quad \sigma(\tau) = \big[\!\big[z + \tau h_1(z) + (1-\tau)h_2(z)\big]\!\big]_\rho.$$

If $h_1, h_2 \in B_\rho$, we have

$$\sigma(\tau) \le [\![z]\!]_\rho + \max\left\{[\![h_1]\!]_\rho, [\![h_2]\!]_\rho\right\} = (n+1)\rho.$$

On the other hand since X is without constant and linear terms, its Jacobian matrix is holomorphic and has no constant term, and hence its $\sigma(\tau)$-norm is no greater than $C\sigma(\tau)$ for all sufficiently small positive $\sigma(\tau)$. Therefore S_X is Lipschitz on the ρ-ball B_ρ, with Lipschitz constant not exceeding $(n+1)C\rho$, and hence S_X is strongly contracting. □

Proof (of Theorem 3.3.5 in the non-resonant case). Now we can prove that a holomorphic vector field X with diagonal non-resonant linear part S with spectrum in the Poincaré domain is holomorphically linearizable in a sufficiently small neighbourhood of the origin.

The map $H = \operatorname{Id} + h$ is a holomorphic change of coordinates in a neighbourhood of the origin linearizing $X = S + \tilde{X}$, i.e.,

$$dH \circ S = X \circ H,$$

where $S = \sum_{j=1}^n \lambda_j z_j \partial_j$, if and only if

$$\left(\frac{\partial h}{\partial z}\right) S - S(h) = \tilde{X}(\operatorname{Id} + h),$$

that is,

$$\mathcal{L}_S(h) = S_{\tilde{X}}(h) \tag{3.12}$$

where $\mathcal{L}_S = \mathrm{ad}_S$ and $S_{\tilde{X}}(h) = \tilde{X}(\mathrm{Id} + h)$, up to considering $h = h_1\partial_1 + \cdots + h_n\partial_n$ as a vector field instead of the map $h = (h_1, \ldots, h_n)$. We shall show that $\mathcal{L}_S^{-1} \circ S_{\tilde{X}}$ restricted to $\mathcal{B}_\rho$ has a fixed point h for $\rho > 0$ sufficiently small, implying the thesis.

Consider this operator $\mathcal{L}_S^{-1} \circ S_{\tilde{X}}$ in the space $\mathcal{B}_\rho$ with sufficiently small $\rho > 0$. The operator $\mathcal{L}_S^{-1}$ is bounded by Lemma 3.3.11, and its norm is the reciprocal of the minimum of the small divisors and is independent of ρ. On the other hand, the argument shift operator $S_{\tilde{X}}$ is strongly contracting by Lemma 3.3.14 with contraction rate going to zero with ρ as $O(\rho)$. Thus the composition shall be contracting on the ρ-ball B_ρ in the ρ-majorant norm with contraction rate going to zero with ρ as $O(1) \cdot O(\rho) = O(\rho)$. By the contracting map principle, there exists a unique fixed point of the operator equation (3.12) in the space $\mathcal{B}_\rho$, for ρ small enough, as desired. □

Now we deal with the resonant case.

Theorem 3.3.15 (Lyapunov-Dulac). *Let $X \in \mathfrak{X}_n$ be in the Poincaré domain. Then X is locally holomorphically conjugated to any holomorphic vector field with the same $(r-1)$-jet.*

Proof. Since the spectrum $\lambda_1, \ldots, \lambda_n$ of the linear part of X belongs to the Poincaré domain, there exist $\theta \in \mathbb{R}$ and $0 < a < A$ such that

$$0 < a < \mathrm{Re}(e^{i\theta}\lambda_j) < A \quad \forall j = 1, \ldots, n. \tag{3.13}$$

We take

$$r = \left\lfloor \frac{A}{a} \right\rfloor + 1$$

where $\lfloor x \rfloor$ denotes the integer part of x.

A holomorphic conjugacy $H = \mathrm{Id} + h$ between the vector fields $X = X^{(1)} + \tilde{X}$ and $X + Z$, where the holomorphic vector field $Z \in \mathfrak{X}_n$ is $(r-1)$-flat, has to satisfy the functional equation $dH \circ X = (X + Z) \circ H$, which can be expanded as

$$\left(\frac{\partial h}{\partial z}\right) X^{(1)} - X^{(1)} h = \left(\tilde{X} \circ (\mathrm{Id} + h) - \tilde{X}\right) + Z \circ (\mathrm{Id} + h) - \left(\frac{\partial h}{\partial z}\right) \tilde{X}. \tag{3.14}$$

Using the three operators

$$T_{\tilde{X}} \colon h \mapsto \tilde{X} \circ (\mathrm{Id} + h) - \tilde{X}, \quad S_Z \colon h \mapsto Z \circ (\mathrm{Id} + h), \quad \Psi \colon h \mapsto \left(\frac{\partial h}{\partial z}\right) \tilde{X},$$

we can write equation (3.14) in the form

$$\mathcal{L}_{X^{(1)}}(h) = T_{\tilde{X}}(h) + S_Z(h) - \Psi(h), \tag{3.15}$$

where, as before, $\mathcal{L}_{X^{(1)}} = \mathrm{ad}_{X^{(1)}}$. The key differences with the non-resonant case are: first, due to the presence of resonances, the operator $\mathcal{L}_{X^{(1)}}$ is not invertible anymore, and second, since the field $\tilde{X}$ is non-linear, we have the additional operator Ψ in the right hand side. Note that this operator is a derivation of h, thus is unbounded in any majorant norm $[\![\cdot]\!]_\rho$.

Let, for any $m \in \mathbb{N}$ and $\rho > 0$,

$$\mathcal{B}_{m,\rho} = \{f \in (\mathbb{C}[\![z_1, \ldots, z_n]\!])^n \mid f \text{ has no terms of order } \leq m-1\} \cap \mathcal{B}_\rho$$

equipped with the same majorant norm $[\![\cdot]\!]_\rho$. Since $\tilde{X}$ has no linear term, all the operators $T_{\tilde{X}}, S_Z, \Psi$ map the subspace $\mathcal{B}_{m,\rho}$ into itself for any $m \in \mathbb{N}$. Moreover, by Lemma 3.3.14, the argument shift operator S_Z is strongly contracting, regardless of the choice of m. Since $T_{\tilde{X}}(h) = S_{\tilde{X}}(h) - S_{\tilde{X}}(0)$, the operator $T_{\tilde{X}}$ differs from the argument shift $S_{\tilde{X}}$ by the constant operator $\tilde{X} = S_{\tilde{X}}(0)$, which does not affect the Lipschitz constant; since $[\![\tilde{X}]\!]_\rho = O(\rho^2)$, the operator $T_{\tilde{X}}$ is also strongly contracting.

The operator $\mathcal{L}_{X^{(1)}}$ preserves the order of all monomial terms, hence it also maps $\mathcal{B}_{m,\rho}$ into itself for all m and all ρ, and it is invertible on these spaces if $m \geq r$. Indeed, if $|Q| \geq r$, then by (3.13), we have

$$\begin{aligned}
\left|\lambda_k - \langle Q, \lambda\rangle\right| &= \left|e^{i\theta}\lambda_k - \langle Q, e^{i\theta}\lambda\rangle\right| \\
&\geq \sum_{j=1}^{n} Q_j \mathrm{Re}(e^{i\theta}\lambda_j) - \mathrm{Re}(e^{i\theta}\lambda_k) \\
&\geq a|Q| - A \\
&\geq a\left(1 - \frac{A}{a+A}\right)|Q|.
\end{aligned}$$

So, since

$$\mathcal{L}^{-1}_{X^{(1)}}\big|_{\mathcal{B}_{m,\rho}} \colon \sum_{k=1}^{n} \sum_{|Q|\geq m} c_{Q,k} z^Q \partial_k \mapsto \sum_{k=1}^{n} \sum_{|Q|\geq m} \frac{c_{Q,k}}{\lambda_k - \langle Q, \lambda\rangle} z^Q \partial_k,$$

the restriction of $\mathcal{L}^{-1}_{X^{(1)}}$ on $\mathcal{B}_{m,\rho}$ is bounded and

$$[\![\mathcal{L}^{-1}_{X^{(1)}}(h)]\!]_\rho \leq O\left(\frac{1}{m}\right)[\![h]\!]_\rho, \tag{3.16}$$

uniformly over all $h \in \mathcal{B}_{m,\rho}$ of order $m \geq r$.

Thus the two compositions, $\mathcal{L}^{-1}_{X^{(1)}} \circ S_Z$ and $\mathcal{L}^{-1}_{X^{(1)}} \circ T_{\tilde{X}}$ are strongly contracting. To prove the theorem, it remains to show that the linear operator $\mathcal{L}^{-1}_{X^{(1)}} \circ \Psi \colon \mathcal{B}_{m,\rho} \to \mathcal{B}_{m,\rho}$ is strongly contracting for $m \geq r$. Let us consider the $[\![\cdot]\!]_\rho$-normalized monomial vector fields

$$h_{P,k} = \rho^{-|P|} z^P \partial_k$$

for all $k = 1, \dots, n$ and $|P| \geq m$, spanning the entire space $\mathcal{B}_{m,\rho}$. We prove that

$$[\![\mathcal{L}^{-1}_{X^{(1)}} \circ \Psi(h_{P,k})]\!]_\rho = O(\rho) \quad \text{as } \rho \to 0 \tag{3.17}$$

uniformly over $|P| \geq m$ and all k. Since $\mathcal{L}^{-1}_{X^{(1)}} \circ \Psi$ is linear, this would imply that $\mathcal{L}^{-1}_{X^{(1)}} \circ \Psi$ is strongly contracting. The direct computation yields

$$\Psi(h_{P,k}) = \sum_{j=1}^{n} \rho^{-|P|} \frac{p_j}{z_j} z^P \tilde{X}_j \partial_k.$$

Since $\tilde{X}$ is non-linear, $[\![\tilde{X}_j]\!]_\rho = O(\rho^2)$; substituting this into the definition of the majorant norm, we obtain

$$[\![\Psi(h_{P,k})]\!]_\rho \leq \sum_{j=1}^{n} p_j \rho^{-1} O(\rho^2) = |P| O(\rho),$$

where $O(\rho)$ is uniform over P and k. Since the order of the products $\frac{z^P}{z_j} \tilde{X}_j$ is at least $|P| + 1$, by (3.16) we have

$$[\![\mathcal{L}^{-1}_{X^{(1)}} \circ \Psi(h_{P,k})]\!]_\rho \leq \frac{|P|}{|P|} O(\rho) = O(\rho)$$

uniformly over k and P with $|P| \geq m \geq r$. Thus the last remaining composition $\mathcal{L}^{-1}_{X^{(1)}} \circ \Psi$ is also strongly contracting, which implies the existence of a solution for the fixed point equation

$$h = \mathcal{L}^{-1}_{X^{(1)}} \circ (T_{\tilde{X}} + S_Z - \Psi)(h)$$

equivalent to (3.15), in a sufficiently small polydisk $\{|z| < \rho\}$. □

Now we can easily complete the proof of the holomorphic normalization theorem in the Poincaré domain in the resonant case.

Proof (of Theorem 3.3.5 in the resonant case). By the proof of Poincaré-Dulac formal normalization Theorem 3.2.12 (see Remark 3.2.14), we can eliminate all non-resonant terms up to any finite order m by a polynomial transformation. Therefore, if $m \geq r + 1$, by Theorem 3.3.15, we can eliminate all the terms of order greater than m with a holomorphic transformation, and we are done. □

If $n = 2$ the Poincaré domain $\mathfrak{P}$ coincides with

$$\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid \lambda_1\lambda_2 \neq 0 \text{ and } \alpha = \lambda_2/\lambda_1 \in \mathbb{C}^* \setminus \mathbb{R}^-\},$$

and $(\lambda_1, \lambda_2) \in \mathfrak{P}$ is resonant if and only if there exists $n \in \mathbb{N}\backslash\{0\}$ such that $\lambda_2 = n\lambda_1$ or $\lambda_2 = \frac{1}{n}\lambda_1$. Theorem 3.3.5 gives then the following classification.

Corollary 3.3.16. *Let $X \in \mathfrak{X}_2$ be in the Poincaré domain, and let $\lambda_1, \lambda_2 \in \mathbb{C}^*$ be the eigenvalues of the linear part of X. Let $\alpha = \lambda_2/\lambda_1 \in \mathbb{C}^* \setminus \mathbb{R}^-$ be the index of X at 0. Then*

(i) if $\alpha, 1/\alpha \notin \mathbb{N}$ then X is holomorphically conjugated to

$$\lambda_1 z_1 \partial_1 + \lambda_2 z_2 \partial_2, \tag{3.18}$$

i.e., it is holomorphically equivalent to

$$z_1\partial_1 + \alpha z_2 \partial_2;$$

(ii) if $\alpha = n$ for some integer $n \geq 2$ then X is holomorphically conjugated to

$$\lambda_1 z_1 \partial_1 + (\lambda_2 z_2 + a z_1^n)\partial_2,$$

i.e., it is holomorphically equivalent to

$$z_1\partial_1 + (n z_2 + a z_1^n)\partial_2,$$

where $a \in \mathbb{C}$ is a holomorphic invariant; analogously, if $\alpha = \frac{1}{n}$ for some integer $n \geq 2$ then X is holomorphically conjugated to

$$(\lambda_1 z_1 + a z_2^n)\partial_1 + \lambda_2 z_2 \partial_2,$$

i.e., it is holomorphically equivalent to

$$(n z_1 + a z_2^n)\partial_1 + z_2\partial_2; \tag{3.19}$$

(iii) if $\alpha = 1$ then, if the linear part of X is diagonalizable then X is holomorphically conjugated to (3.18), otherwise, it is holomorphically conjugated to

$$\lambda(z_1 + z_2)\partial_1 + \lambda z_2 \partial_2,$$

i.e., it is holomorphically equivalent to

$$(z_1 + z_2)\partial_1 + z_2\partial_2.$$

Proof. Cases (i) and (iii) follow directly from the proof of Theorem 3.3.5 in the non-resonant case. Case (ii) follows from Theorem 3.3.5 in the resonant case as soon as we prove that the complex number a is a holomorphic invariant. It suffices to prove that if $X = (\lambda_1 u + av^n)\partial_1 + \lambda_2 v\partial_2$ is holomorphically conjugated to $Y = (\lambda_1 x + by^n)\partial_1 + \lambda_2 y\partial_2$, then $a = b$ (we make computations for $\alpha = 1/n$, the case $\alpha = n$ is perrfectly analogous). In fact, let $(u, v) = (x, y)+\xi(x, y)$ be a holomorphic change of coordinates of $(\mathbb{C}^2, 0)$ tangent to the identity conjugating X to Y, i.e., such that

$$X \circ (I + \xi) = d(I + \xi) \circ Y. \tag{3.20}$$

Then the first coordinate of (3.20) is

$$\lambda_1\big(x + \xi_1(x, y)\big) + a\big(y + \xi_2(x, y)\big)^n = (\lambda_1 x + by^n)\left(1 + \frac{\partial\xi_1}{\partial x}\right) + \lambda_2 y\frac{\partial\xi_1}{\partial y},$$

that is, writing $\xi_1(x, y) = \sum_{h+k\geq 2} \xi^{(1)}_{hk} x^h y^k$, we have

$$\begin{aligned}\lambda_1 x + \lambda_1 \sum_{h+k\geq 2} \xi^{(1)}_{hk} x^h y^k + a\big(y + \xi_2(x, y)\big)^n \\ = \lambda_1 x + by^n + \lambda_1 \sum_{h+k\geq 2} \xi^{(1)}_{hk} h x^h y^k + b \sum_{h+k\geq 2} \xi^{(1)}_{hk} h x^{h-1} y^{n+k} + \lambda_2 \sum_{h+k\geq 2} \xi^{(1)}_{hk} k x^h y^k.\end{aligned}$$

The coefficient of y^n in the left hand side is $\lambda_1\xi^{(1)}_{0n} + a$ while the coefficient of y^n in the right hand side is $b + \lambda_2 n\xi^{(1)}_{0n}$. Since $\lambda_1 = n\lambda_2$, then $a = b$, that is a is a holomorphic invariant. □

3.4 Topology of the leaves in the 2-dimensional Poincaré case

In this section we shall describe the topology of the leaves for vector fields in the Poincaré domain. It shall be sufficient to study the normal forms given by Corollary 3.3.16.

Foliation induced by $X = \lambda_1 z_1\partial_1 + \lambda_2 z_2\partial_2 \in \mathfrak{X}_2$ with $(\lambda_1, \lambda_2) \in \mathfrak{P}$.

Proposition 3.4.1. *Let $X = \lambda_1 z_1\partial_1 + \lambda_2 z_2\partial_2 \in \mathfrak{X}_2$ with $(\lambda_1, \lambda_2) \in \mathfrak{P}$ and let $\mathcal{F}$ be the induced foliation in a neighbourhood of the origin. Then*

(i) the leaves of $\mathcal{F}$ are locally transverse to the spheres $\mathbb{S}_R = \{|z_1|^2 + |z_2|^2 = R^2\}$ for $R > 0$;

(ii) there exists a real linear flow having leaves included in the leaves of $\mathcal{F}$ and for which the origin is an attractor.

Proof. Let $\alpha = \frac{\lambda_2}{\lambda_1}$ be the index of $\mathcal{F}$ at 0. We can parametrize a leaf of $\mathcal{F}$ passing through $(x_0, y_0) \in \mathbb{C}^2$ by

$$\varphi\big(T, (x_0, y_0)\big) = \big(x_0 e^{\lambda_1 T}, y_0 e^{\lambda_2 T}\big)$$

with $T \in \mathbb{C}$. Since $\alpha \in \mathbb{C}^* \setminus \mathbb{R}^-$, there exists $\beta_0 \in \mathbb{C}^*$ such that

$$\begin{cases} \operatorname{Re}(\beta_0 \lambda_1) < -c < 0 \\ \operatorname{Re}(\beta_0 \lambda_2) < -c < 0 \end{cases}$$

for some $c \in \mathbb{R}^+$. Let us consider the real flow

$$\varphi_{\beta_0}\big(t, (x_0, y_0)\big) = \big(x_0 e^{\beta_0 \lambda_1 t}, y_0 e^{\beta_0 \lambda_2 t}\big)$$

for $t \in \mathbb{R}$. Then we have

$$\big|\varphi_{\beta_0}(t, (x_0, y_0))\big|^2 = |x_0|^2 e^{2\operatorname{Re}(\beta_0 \lambda_1) t} + |y_0|^2 e^{2\operatorname{Re}(\beta_0 \lambda_2) t} \leq \big(|x_0|^2 + |y_0|^2\big) e^{-2ct},$$

and hence the assertion follows. □

Then in this case every leaf of $\mathcal{F}$ is topologically *a cone over its intersection with* $\mathbb{S}_1$ *and vertex the origin.* To describe the base of this cone let us first note that the flow φ_{β_0} allows us to define a diffeomorphism between the solid torus $\mathbb{T}_1 = \{|z_1| = 1\}$ and $\mathbb{S}_1 \setminus (\mathbb{S}_1 \cap \{z_1 = 0\})$ mapping every point (x_0, y_0) having $|x_0| = 1$ to the unique point in which $\varphi_{\beta_0}(t, (x_0, y_0))$ intersects the sphere $\mathbb{S}_1$.

We shall now describe the intersections of $\mathcal{F}$ with $\mathbb{T}_1$. Let us choose $\beta_1 = \frac{i}{\lambda_1}$ and consider the real flow

$$\varphi_{\beta_1}\big(t, (x_0, y_0)\big) = \big(x_0 e^{it}, y_0 e^{\alpha i t}\big), \quad t \in \mathbb{R}$$

for which $\mathbb{T}_1$ is invariant. The axis of $\mathbb{T}_1$, i.e., $\{|z_1| = 1, z_2 = 0\}$, is a leaf.

Theorem 3.4.2. *Let $X = \lambda_1 z_1 \partial_1 + \lambda_2 z_2 \partial_2 \in \mathfrak{X}_2$ with $(\lambda_1, \lambda_2) \in \mathfrak{P}$ and let $\mathcal{F}$ be the induced foliation in a neighbourhood of the origin. Denote by $\alpha = \frac{\lambda_2}{\lambda_1} \in \mathbb{C}^* \setminus \mathbb{R}^-$ the index of $\mathcal{F}$ at 0. Then*

(i) if $\alpha \in \mathbb{C} \setminus \mathbb{R}$ then the induced foliation on $\mathbb{S}_1$ has only two closed leaves, $\{|z_1| = 1, z_2 = 0\}$ and $\{z_1 = 0, |z_2| = 1\}$, and the other leaves accumulate spiralizing on the closed ones;

(ii) if $\alpha = 1$ then the leaves of $\mathcal{F}$ are of the form $\{x_0 z_2 = y_0 z_1\} \setminus \{0\}$, with $(x_0, y_0) \in \mathbb{C}^2 \setminus \{0\}$, and they intersect circles on $\mathbb{S}_1$;

(iii) if $\alpha \in \mathbb{Q}^+$ then all the leaves of the foliation induced by $\mathcal{F}$ on $\mathbb{S}_1$ are closed;

(iv) if $\alpha \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ then $\mathbb{S}_1$ is decomposed into invariant tori on which the leaves of the foliation induced by $\mathcal{F}$ on $\mathbb{S}_1$ are dense, and the unique closed leaves of such foliation on $\mathbb{S}_1$ are $\{|z_1| = 1, z_2 = 0\}$ and $\{z_1 = 0, |z_2| = 1\}$.

Proof. Recall that

$$\varphi_{\beta_1}\big(t, (x_0, y_0)\big) = (x_0 e^{it}, y_0 e^{\alpha it}), \quad t \in \mathbb{R},$$

where $\beta_1 = \frac{i}{\lambda_1}$.
(i) We have

$$\begin{array}{l} \lim\limits_{t\to+\infty} \left|e^{\alpha it}\right| = 0 \ \text{ and } \ \lim\limits_{t\to-\infty} \left|e^{\alpha it}\right| = +\infty \ \text{ if } \ \mathrm{Im}\,(\alpha) > 0 \\ \lim\limits_{t\to+\infty} \left|e^{\alpha it}\right| = +\infty \ \text{ and } \ \lim\limits_{t\to-\infty} \left|e^{\alpha it}\right| = 0 \ \text{ if } \ \mathrm{Im}\,(\alpha) < 0. \end{array}$$

Then we have leaves spiralizing around the axis of $\mathbb{T}_1$ in one direction and going away indefinitely in the other direction.
(ii) It is obvious, since in this case the real flow is

$$\varphi_{\beta_1}\big(t, (x_0, y_0)\big) = (x_0 e^{it}, y_0 e^{it}), \quad t \in \mathbb{R}.$$

(iii) It is obvious, since in this case we have

$$\alpha = \frac{m}{n}$$

with $m, n \in \mathbb{N} \setminus \{0, 1\}$ and $(m, n) = 1$, and hence

$$\varphi_{\beta_1}\big(t, (x_0, y_0)\big) = \left(x_0 e^{it}, y_0 e^{\frac{m}{n} it}\right), \quad t \in \mathbb{R}.$$

(iv) In this case, we have

$$|\varphi_{\beta_1}(t, (x_0, y_0)| = |x_0|^2 + |y_0|^2$$

for every $t \in \mathbb{R}$, and hence $\varphi_{\beta_1}(t, (x_0, y_0) \in \mathbb{S}_1$ if and only if $|x_0|^2 + |y_0|^2 = 1$.

If $x_0 = 0$ (resp., $y_0 = 0$), then the corresponding leaf is the vertical (resp., horizontal) complex separatrix, that intersect $\mathbb{S}_1$ on the circle $\{z_1 = 0, |z_2| = 1\}$ (resp., $\{|z_1| = 1, z_2 = 0\}$).

Now pick a point $(x_0, y_0) \in \mathbb{S}_1$ such that $x_0 y_0 \neq 0$, and take the sequence $\{\varphi_{\beta_1}(2\pi k, (x_0, y_0))\}_{k\in\mathbb{Z}}$: it is dense in $\{z_1 = x_0, |z_2| = |y_0|\}$; it follows that the intersection of the leaf of $\mathcal{F}$ passing through (x_0, y_0) and $\mathbb{S}_1$ is dense in the (real) invariant tori $\{|z_1| = |x_0|, |z_2| = |y_0|\}$. □

Remark 3.4.3. In the linear case, there is a simpler way to study the leaves, simply solving the differential equation given by the associated holomorphic 1-form

$$\omega = z_1 dz_2 - \alpha z_2 dz_1,$$

where as always α denotes the index of ω at 0. Here the solution of $\omega = 0$ is $z_2 = k z_1^{\alpha}$, where $k \in \mathbb{C}$ is a constant value, or $z_1 = 0$ the vertical complex separatrix. In particular, if $\alpha = \frac{p}{q} \in \mathbb{Q}^+$, then the leaves are of the form $\{z_2^q - k^q z_1^p = 0\}$, and hence they are all complex separatrices.

Foliation induced by $X = (nz_1 + az_2^n)\partial_1 + z_2\partial_2 \in \mathfrak{X}_2$ with $n \in \mathbb{N}\setminus\{0\}$ and $a \neq 0$.

Theorem 3.4.4. *Let $X = (nz_1 + az_2^n)\partial_1 + z_2\partial_2 \in \mathfrak{X}_2$ with $n \in \mathbb{N} \setminus \{0\}$ and $a \neq 0$ and let $\mathcal{F}$ be the induced foliation in a neighbourhood of the origin. Then*

(i) the leaves of $\mathcal{F}$ are locally transverse to the spheres $\mathbb{S}_R = \{|z_1|^2 + |z_2|^2 = R^2\}$ for $R > 0$ small enough;

(ii) the foliation induced by $\mathcal{F}$ on $\mathbb{S}_R$ has only one closed leaf, corresponding to $z_2 = 0$, while the other leaves accumulate on the closed one spiralizing in both directions.

Proof.
(i) If $(x_0, y_0) \in \mathbb{S}_R$, the tangent space $T_{(x_0,y_0)}\mathbb{S}_R$ to $\mathbb{S}_R$ in (x_0, y_0) is given by

$$\eta_{(x_0,y_0)} = 2\mathrm{Re}(\overline{x_0} dz_1 + \overline{y_0} dz_2) = 0.$$

If $|(x_0, y_0)| > 0$ is small enough, then (recall that for $n = 1$ we have $a = 1$)

$$\begin{aligned}\eta_{(x_0,y_0)}(nx_0 + ay_0^n, y_0) &= 2\mathrm{Re}(n|x_0|^2 + a\overline{x_0}y_0^n + |y_0|^2)\\ &\geq 2\left(n|x_0|^2 + |y_0|^2 - |a||x_0||y_0|^n\right)\\ &> 0\end{aligned}$$

and we are done.
(ii) We can parametrize the leaves of X by

$$\begin{cases} z_1(T) = (az_2^n T + z_1)e^{nT} \\ z_2(T) = z_2 e^T \end{cases}$$

with $T \in \mathbb{C}$. If $z_2 \neq 0$, we have

$$\frac{|z_1(T)|}{|z_2(T)^n|} = \frac{|az_2^n T + z_1|}{|z_2|^n}.$$

Now let $\{T_m\}$ be a sequence with $|T_m| \to +\infty$ as $m \to +\infty$ and such that $(z_1(T_m), z_2(T_m)) \in \mathbb{S}_1$ for all m. Then

$$\lim_{|T_m|\to+\infty} \frac{|z_1(T_m)|}{|z_2(T_m)^n|} = \lim_{|T_m|\to+\infty} \frac{|az_2^n T_m + z_1|}{|z_2|} = +\infty$$

implies, since $|z_2(T_m)| = \sqrt{1 - |z_1(T_m)|^2}$,

$$\lim_{|T_m|\to+\infty} |z_1(T_m)| = 1 \quad \text{and} \quad \lim_{|T_m|\to+\infty} |z_2(T_m)| = 0,$$

and we are done. □

Holonomy

We end this chapter by computing the holonomy of some complex separatrices of foliations in the Poincaré domain. We first need a remark and a definition.

Remark 3.4.5. Let us consider a holomorphic foliation $\mathcal{F}$ in a neighborhood U of the unique singular point $0 \in \mathbb{C}^2$. A complex separatrix L is biholomorphic to a punctured disk $\mathbb{D}^*$, so $\pi_1(L) \cong \mathbb{Z}$. Moreover there is an induced orientation in L, that gives us the notion of “sign” of a loop.

Definition 3.4.6. Let L be a complex separatrix of a singular foliation $\mathcal{F}$. Then the **holonomy** of L is (the conjugacy class of) the holonomy along a loop $\gamma \subset L$ such that $[\gamma] = 1 \in \pi_1(L)$.

If $X = \lambda_1 z_1 \partial_1 + \lambda_2 z_2 \partial_2 \in \mathfrak{X}_2$ with $(\lambda_1, \lambda_2) \in \mathfrak{P}$, we know that the induced foliation $\mathcal{F}$ has a singularity at the origin and the coordinate axes are complex separatrices. We would like to compute the holonomy of the horizontal separatrix $L = \{z_2 = 0\}$. Fix a point $p_0 = (z_1^0, 0) \in L$; the generator of $\pi_1(L)$ is represented by the loop $\gamma(t) = (z_1^0 e^{it}, 0)$. A transverse section over $\gamma(t)$ is parametrized by $\tau_t(\zeta) = (z_1^0 e^{it}, \zeta)$, and the leaf passing through $\tau_0(\zeta)$ is parametrized by $\varphi_\zeta(T) = (z_1^0 e^{\lambda_1 T}, \zeta e^{\lambda_2 T})$. So the leaf through $\tau_0(\zeta)$ intersects the transverse section over $\gamma(t)$ at the point

$$\varphi_\zeta\left(\frac{it}{\lambda_1}\right) = (z_1^0 e^{it}, \zeta e^{i\alpha t}).$$

The holonomy is then given by the second coordinate of $\varphi_\zeta(2\pi i/\lambda_1)$, that is

$$h(\zeta) = e^{2\pi i \alpha}\zeta.$$

Analogously, the holonomy of the vertical separatrix is

$$h(\zeta) = e^{2\pi i/\alpha}\zeta.$$

In particular, if $\alpha \in \mathbb{Q}^+$, the holonomy maps corresponding to $\{z_1 = 0\}$ and $\{z_2 = 0\}$ are periodic.

Theorem 3.4.7. *Let $X = (nz_1 + az_2^n)\partial_1 + z_2\partial_2 \in \mathfrak{X}_2$ with $n \in \mathbb{N} \setminus \{0\}$ and $a \neq 0$ and let $\mathcal{F}$ be the induced foliation in a neighbourhood of the origin. Then the holonomy h of the unique separatrix $L = \{z_2 = 0\}$, computed with respect to the standard transverse section $\tau = \{z_1 = 1\}$, is tangent to a rotation by the rational angle $\frac{2\pi}{n}$ and its n-th iterate has an isolated fixed point at the origin of multiplicity $n + 1$.*

Proof. The generator of $\pi_1(L)$ is represented by the loop $\gamma(t) = (e^{it}, 0)$. As transverse section over $\gamma(t)$ we take $\tau_t(\zeta) = (e^{it}, \zeta)$; so the leaf passing through $\tau_0(\zeta)$ is parametrized by

$$\varphi_\zeta(T) = \big((1 + a\zeta^n)e^{nT}, \zeta e^T\big),$$

with $T \in \mathbb{C}$. The leaf through $\tau_0(\zeta)$ intersects the transverse section over $\gamma(t)$ at the point

$$\varphi_\zeta\left(T_\zeta(t)\right) = \left(e^{it}, \zeta e^{T_\zeta(t)}\right),$$

where $T_\zeta(t)$ is a solution of the equation

$$(1 + a\zeta^n T)e^{nT} = e^{it}.$$

In particular, since $\varphi_0(T_0(t))$ must be $\gamma(t)$, we get $T_0(t) = it/n$. The holonomy map is then given by

$$h(\zeta) = e^{T_\zeta(2\pi)}\zeta,$$

where $T_\zeta(2\pi)$ solves the equation

$$1 + a\zeta^n T = e^{-nt},$$

with $T_0(2\pi) = 2\pi i/n$. Write $T_\zeta(2\pi) = \frac{2\pi i}{n} + \delta(\zeta)$; then $\delta(\zeta)$ solves the equation

$$1 + a\zeta^n\left(\frac{2\pi i}{n} + \delta(\zeta)\right) = e^{-n\delta(\zeta)}. \tag{3.21}$$

By the implicit function theorem, this equation for ζ small has a unique holomorphic solution with $\delta(0) = 0$. Expanding δ in Taylor series and comparing coefficients in both sides of (3.21) we get

$$\delta(\zeta) = -\frac{2\pi i a}{n^2}\zeta^n + o(\zeta^n);$$

hence

$$\begin{aligned} h(\zeta) &= \zeta \, \exp\left(\frac{2\pi i}{n} - \frac{2\pi i a}{n^2}\zeta^n + o(\zeta^n)\right) \\ &= e^{2\pi i/n}\zeta \exp\left(-\frac{2\pi i a}{n^2}\zeta^n + o(\zeta^n)\right) \\ &= e^{2\pi i/n}\zeta\left(1 - \frac{2\pi i a}{n^2}\zeta^n + o(\zeta^n)\right). \end{aligned}$$

In particular,

$$h^n(\zeta) = \zeta\left(1 - \frac{2\pi i a}{n}\zeta^n + o(\zeta^n)\right),$$

and we are done. □

Chapter 4

Dynamics of foliations in the Siegel domain

Matteo Ruggiero[1]

4.1 Basic definitions

In the previous chapter we have studied singular foliations with a (elementary) singularity in $0 \in \mathbb{C}^n$, through their equivalent description as integral flow of a holomorphic vector field.

Here we shall focus on dimension $n = 2$, and on holomorphic vector fields X in the strict Siegel domain (see Definition 3.3.1). We shall present some results that can be found on [MM80] and [PMY94], while we refer to [MR83] for further details on the resonant case (i.e., a negative rational index).

Remark 4.1.1. For our purposes it shall be more convenient to work with holomorphic 1-forms instead of holomorphic vector fields, so we recall here the connection already seen in the first chapter (Theorem 1.2.13) between them, and give the equivalent concepts of the ones we gave in Definition 3.1.4 (of holomorphic and formal conjugation and equivalence).

So a holomorphic foliation $\mathcal{F}$ shall be given by a vector field of the form

$$X = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2},$$

with $X_1, X_2 \in \mathfrak{m}_2$, or equivalently, by the 1-form

$$\omega = \omega_1 dx_1 + \omega_2 dx_2,$$

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where $\omega_1 = -X_2$ and $\omega_2 = X_1$.

Definition 4.1.2. Let ω, ω' be two holomorphic 1-forms with an isolated singular point $0 \in \mathbb{C}^2$. We shall say that they are **holomorphically conjugated** (resp., **formally conjugated**), and we shall denote it by $\omega \cong \omega'$ (resp., $\omega \overset{for}{\cong} \omega'$) if there exists a biholomorphism (resp. an invertible formal map) Φ such that

$$\Phi^*\omega' = \omega. \tag{4.1}$$

We shall say that they are **holomorphically equivalent** (resp., **formally equivalent**), and we shall denote this by $\omega \sim \omega'$ (resp., $\omega \overset{for}{\sim} \omega'$) if there exist a biholomorphism (resp., an invertible formal map) Φ and a holomorphic non-vanishing function (resp., formal non-vanishing power series) Ψ such that

$$\Phi^*\omega' = \Psi\omega. \tag{4.2}$$

Remark 4.1.3. Equations (4.1) and (4.2) correspond respectively to (3.3) and (3.4) of Definition 3.1.4.

Definition 4.1.4. Let X a holomorphic vector field, or ω a holomorphic 1-form, with an isolated singular point in $0 \in \mathbb{C}^2$, as in Remark 4.1.1. Let $X^{(1)}$ be the linear part of X, and denote by $\lambda = (\lambda_1, \lambda_2)$ the eigenvalues of $X^{(1)}$.

Then we say that λ, or X, or ω, belongs to the **Siegel domain**, if $\lambda_1, \lambda_2 \neq 0$ and the index $\alpha := \lambda_2/\lambda_1 \in (-\infty, 0) = \mathbb{R}^-$.

Remark 4.1.5. This definition of the Siegel domain is equivalent to the Definition 3.3.1 of strict Siegel domain (in dimension 2).

Remark 4.1.6. In particular, if X belongs to the Siegel domain, the two eigenvalues λ_1, λ_2 as in Definition 4.1.4 are distinct. Therefore $X^{(1)}$ is diagonalizable; up to a (linear) change of coordinates, we can suppose

$$X_j = \lambda_j x_j + f_j(x), \tag{4.3}$$

or equivalently

$$\begin{aligned}\omega_1 &= -\big(\lambda_2 x_2 + f_2(x)\big)\\ \omega_2 &= \lambda_1 x_1 + f_1(x),\end{aligned}$$

with $f_j \in \mathfrak{m}_2^2$ for $j = 1, 2$.

4.2 Formal normalization

Let us recall the formal classification for vector fields in the Siegel domain. This result follows directly from the formal classification we have already seen in the previous chapter (see Theorem 3.2.12 and Proposition 3.3.2), but here we present a more direct proof, that shall be useful for proving results in the next section.

Proposition 4.2.1. *Let X be a holomorphic vector field in the Siegel domain, i.e., of the form*

$$X = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2},$$

with $X_j = \lambda_j x_j + f_j(x)$, $f_j \in \mathfrak{m}_2^2$, for $j = 1, 2$, and let $\alpha = \frac{\lambda_2}{\lambda_1} \in \mathbb{R}^-$ be the index at 0. If $\alpha \in \mathbb{R}^- \setminus \mathbb{Q}^-$, then

$$X \overset{for}{\cong} \lambda_1 x_1 \frac{\partial}{\partial x_1} + \lambda_2 x_2 \frac{\partial}{\partial x_2}.$$

If $\alpha = -\frac{p_1}{p_2} \in \mathbb{Q}^-$, with $p_1, p_2 \in \mathbb{N}^*$ coprime, then

$$X \overset{for}{\cong} \lambda_1 x_1 \big(1 + a_1(x_1^{p_1} x_2^{p_2})\big) \frac{\partial}{\partial x_1} + \lambda_2 x_2 \big(1 + a_2(x_1^{p_1} x_2^{p_2})\big) \frac{\partial}{\partial x_2}, \tag{4.4}$$

with suitable $a_1, a_2 \in \hat{\mathfrak{m}}_1$.

Proof. We want to perform a (formal) change of coordinates tangent to the identity, i.e., of the form

$$x_j = y_j + \phi_j(y),$$

with $\phi_j \in \hat{\mathfrak{m}}_2^2$, for $j = 1, 2$. In these new coordinates, we obtain

$$X = Y_1 \frac{\partial}{\partial y_1} + Y_2 \frac{\partial}{\partial y_2},$$

with $Y_j = \lambda_j y_j + g_j(y)$, $g_j \in \mathfrak{m}_2^2$, for $j = 1, 2$.

We can compute X_j with respect to the x coordinates, and then perform the change of coordinates, obtaining

$$X_j = \lambda_j x_i + f_j(x) = \lambda_j \big(y_j + \phi_j(y)\big) + f_j\big(y + \phi(y)\big), \tag{4.5}$$

or we can compute X with respect to the y coordinates, and then consider the component along $\frac{\partial}{\partial x_j}$, obtaining

$$X_j = \sum_{k=1}^{2} \frac{\partial x_j}{\partial y_k} Y_k = \sum_{k=1}^{2} \left(\delta_{kj} + \frac{\partial \phi_j}{\partial y_k} \right) \big(\lambda_k y_k + g_k(y)\big), \tag{4.6}$$

where δ denotes the Kronecker's delta function. Comparing (4.5) and (4.6) we get

$$-\lambda_j\phi_j(y)+g_j(y)+\sum_{k=1}^{2}\lambda_k\frac{\partial\phi_j}{\partial y_k}y_k=f_j\big(y+\phi(y)\big)-\sum_{k=1}^{2}\frac{\partial\phi_j}{\partial y_k}g_k(y).$$

Expliciting coefficients, we obtain

$$\sum_{|I|\geq 2}(\delta_{j,I}\phi_{j,I}+g_{j,I})y^I=f_j\big(y+\phi(y)\big)-\sum_{k=1}^{2}\frac{\partial\phi_j}{\partial y_k}g_k(y)=:\chi_j(y), \tag{4.7}$$

where $\delta_{j,I}:=\lambda_1 i_i+\lambda_2 i_2-\lambda_j$, and $\chi_j(y)=\sum_{|I|\geq 2}\chi_{j,I}y^I$ is such that $\chi_{j,I}$ depends only on $\phi_{k,H}$ and $g_{k,H}$ with $|H|<|I|$.

Hence we can solve (4.7) by recursion, by setting

$$\phi_{j,I}=\begin{cases}\frac{\chi_{j,I}}{\delta_{j,I}}\\ \\ 0\end{cases},\qquad g_{j,I}=\begin{cases}0 & \text{if } \delta_{j,I}\neq 0\\ \\ \chi_{j,I} & \text{if } \delta_{j,I}=0\end{cases}.$$

If $\alpha\in\mathbb{R}^-\setminus\mathbb{Q}^-$, then $\delta_{j,I}\neq 0$ for every $j=1,2$, $|I|\geq 2$, and X is linearizable. If $\alpha=-\frac{p_1}{p_2}\in\mathbb{Q}^-$, then $\delta_{1,I}=0$ if and only if $p_2(i_1-1)=p_1 i_2$, i.e., if and only if $i_1-1=kp_1$, $i_2=kp_2$, with $k\in\mathbb{N}^*$, and analogously with $\delta_{2,I}$; it follows that we can reduce the vector field to the form (4.4). □

4.3 Holomorphic conjugation and complex separatrices

The main result of this section is Theorem 4.3.3, that shall imply the existence of (at least) two complex separatrices for a foliation that belongs to the Siegel domain (see Remark 4.3.5).

Definition 4.3.1. Let $f=\sum f_I x^I\in\mathbb{C}[\![x_1,\dots,x_n]\!]$, and denote by $M(f):=\sum|f_I|\,x^I$ the majorant series for f (see Definition 3.3.6). We shall call

$$N(f):=\sum|f_I|\,z^{|I|}=M(f)(z,\dots,z)\in\mathbb{R}[\![z]\!]$$

the **norm series** for f. In particular $N(f)(\rho)=[\![f]\!]_\rho$, see Definition 3.3.7.

If $g=\sum g_I x^I\in\mathbb{C}[\![x_1,\dots,x_n]\!]$, we shall say that g is a **majorant** for f, and denote this by $f\prec g$, if $|f_I|\leq|g_I|$ for every I.

Lemma 4.3.2. *Let f be in $\mathfrak{m}_1^k\subseteq\mathbb{C}\{z\}$, with $k\in\mathbb{N}$. Then there exist $M,a>0$ such that*

$$f\prec\frac{Mz^k}{1-az}.$$

Proof. From the hypothesis, $f/z^k \in \mathbb{C}\{z\}$. Set $1/\rho = \limsup \sqrt[n]{|a_n|} < \infty$, where $f/z^k = \sum a_n z^n$. For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n > N$ we have $\sqrt[n]{|a_n|} < 1/\rho + \varepsilon$. Fix ε (for example $\varepsilon = 1$), and set $a := 1/\rho + \varepsilon$. Then $(1-az)^{-1} = \sum_{n\geq 0} a^n z^n$ is such that $|a_n| < a^n$ for every $n > N$. On the other hand, for $n \leq N$ the finite sequence $|a_n|/a^n$ admits a maximum M. Up to replacing M by $\max\{M,1\}$, we obtain $|a_n| \leq Ma^n$ for all $n \in \mathbb{N}$, that is the assertion. □

Theorem 4.3.3. *Let $X = X_1\partial_1 + X_2\partial_2$ be a holomorphic vector field, with X_j of the form* (4.3)*, and let $\alpha = \frac{\lambda_2}{\lambda_1} \in \mathbb{C}^*$ be the index at* 0*.*

If $\alpha, \alpha^{-1} \notin \mathbb{N}\setminus\{1\}$, then, up to holomorphic conjugacy, we can suppose that $x_1x_2x_j \mid f_j$ for $j = 1, 2$.

Proof.

(**Step** 1)**.** Let us show first that, up to holomorphic conjugacy, we can suppose that $x_1x_2 \mid f_j$ for $j = 1, 2$. For $I \in \mathbb{N}^2$ and $j = 1, 2$ put $\delta_{j,I} = i_1\lambda_1 + i_2\lambda_2 - \lambda_j$, and notice that if $y^I \notin \langle y_1y_2\rangle$, i.e., if $i_1i_2 = 0$, then $\delta_{j,I} \neq 0$. Indeed, if $i_1 = 0$ then $\delta_{1,I} = \lambda_2 i_2 - \lambda_1 = 0$ for some $i_2 \geq 2$ if and only if $\alpha^{-1} \in \mathbb{N}\setminus\{0,1\}$, while $\delta_{2,I} = \lambda_2 i_2 - \lambda_2 = 0$ for some $i_2 \geq 2$ if and only if $\lambda_2 = 0 = \alpha$; the case $i_2 = 0$ is perfectly analogous. So we can set

$$\phi_{j,I} = \begin{cases} \frac{\chi_{j,I}}{\delta_{j,I}} \\ \\ 0 \end{cases} \qquad g_{j,I} = \begin{cases} 0 & \text{if } y^I \notin \langle y_1y_2\rangle \\ \\ \chi_{j,I} & \text{if } y^I \in \langle y_1y_2\rangle \end{cases}.$$

With this definition (4.7) holds, so we have the (for now only formal) conjugation we wanted; let us prove that this conjugation is holomorphic.

First of all, we can easily see that there exists $\delta > 0$ such that $|\delta_{j,I}| > \delta$ for every $|I| \geq 2$, $j = 1, 2$, $y^I \notin \langle y_1y_2\rangle$.

From (4.7), taking majorant series, we obtain

$$\delta M(\phi_j) \prec \sum_{|I|\geq 2} |\delta_{j,I}\phi_{j,I}|\, y^I \prec M(g_j) + M(f_j)\big(y + M(\phi)\big) + \sum_{k=1}^{2} \frac{\partial M(\phi_j)}{\partial y_k} M(g_k). \quad (4.8)$$

If $y^I \in \langle y_1y_2\rangle$, then $\phi_{j,I} = 0$ and these terms do not enter the estimates; if $y^I \notin \langle y_1y_2\rangle$, then $g_{j,I} = 0$ for $j = 1, 2$. So we can omit $M(g_1)$ and $M(g_2)$ in (4.7), obtaining

$$\delta M(\phi_j) \prec M(f_j)\big(y + M(\phi)\big). \quad (4.9)$$

Taking norm series, and summing for $j = 1, 2$, we have

$$\begin{aligned}\delta N(\phi) = \delta\bigl(N(\phi_1) + N(\phi_2)\bigr) &\prec \sum_{k=1}^{2} M(f_k)\bigl(z + N(\phi_1), z + N(\phi_2)\bigr)\\ &\prec \sum_{k=1}^{2} M(f_k)\bigl(z + N(\phi_1) + N(\phi_2), z + N(\phi_2) + N(\phi_1)\bigr),\end{aligned}$$

and then

$$N(\phi) \prec \delta^{-1} N(f)\bigl(z + N(\phi)\bigr).$$

Now we apply Lemma 4.3.2 at $\delta^{-1}N(f) \in \mathfrak{m}_1^2$: there exist $M, a > 0$ such that

$$\frac{N(f)}{\delta} \prec \frac{Mz^2}{1 - az}.$$

Now set $u := N(\phi)/z \in \hat{\mathfrak{m}}_1$. If we show that u is holomorphic, then ϕ (and g) will be holomorphic too, finishing the proof of the first step. Putting together the estimates, we obtain

$$\begin{aligned}\frac{N(\phi)}{z} &\prec \frac{1}{\delta z} N(f)\bigl(z + N(\phi)\bigr)\\ &\prec \frac{M\bigl(z + N(\phi)\bigr)^2}{z\bigl(1 - a(z + N(\phi))\bigr)} = \frac{Mz^2\bigl(1 + N(\phi)/z\bigr)^2}{z\bigl(1 - az(1 + N(\phi)/z)\bigr)},\end{aligned}$$

and hence

$$u \prec \frac{Mz(1 + u)^2}{1 - az(1 + u)}. \tag{4.10}$$

Let us compare u with the solution $v \in \hat{\mathfrak{m}}_1$ of

$$v = \frac{Mz(1 + v)^2}{1 - az(1 + v)}. \tag{4.11}$$

First of all, let us see that v is holomorphic: directly from (4.11) we obtain

$$(M + a)zv^2 + \bigl((2M + a)z - 1\bigr)v + Mz = 0,$$

and then

$$v = \frac{1 - (2M + a)z - \sqrt{1 - 2(2M + a)z + a^2z^2}}{2(M + a)z};$$

so $v \in \mathfrak{m}_1$ is holomorphic.

Set $1+u=\sum_{n\geq 0} u_n z^n$ and $1+v=\sum_{n\geq 0} v_n z^n$ (in particular $u_0=v_0=1$). Then we have

$$\frac{Mz(1+u)^2}{1-az(1+u)} = M\sum_{n=1}^{\infty} z^n \sum_{k=2}^{n+1} a^{k-2} \sum_{\substack{J\in\mathbb{N}^k \\ |J|=n-k+1}} u_{j_1}\cdots u_{j_k},$$

and hence the coefficient of z^n is of the form $P_n(u_0,\dots,u_{n-1})$ for every $n\geq 1$, with P_n a suitable polynomial with striclty positive coefficients (analogous deductions can be made for v).

Then from (4.10) and (4.11) we get

$$u_n \leq P_n(u_0,\dots u_{n-1}), \qquad v_n = P_n(v_0,\dots,v_{n-1}).$$

Now, using and induction argument, we shall show that $u_n \leq v_n$ for every n. For the basis of the induction, $u_0=v_0=1$. Let us suppose that $u_j\leq v_j$ for every $j=1,\dots n-1$; then

$$u_n \leq P_n(u_1,\dots,u_{n-1}) \leq P_n(v_1,\dots v_{n-1}) = v_n,$$

where the last inequality arises from the positivity of coefficents of P_n, and from the induction hypothesis. Hence $u \prec v$, and u is holomorphic.

(**Step** 2)**.** Thanks to the first step, we can suppose that $x_1x_2 \mid f_j$ for $j=1,2$. As before, if $|I|\geq 2$ e $y^I \notin \langle y_1y_2y_j\rangle$, then $\delta_{j,I}\neq 0$. We have to show it only for $i_j=1$: if $j=1$ then $0=\delta_{1,I}=\lambda_1+\lambda_2 i_2-\lambda_1$ if and only if $\lambda_2=0$, and the same for $j=2$ (for simmetry).

So we can define

$$\phi_{j,I} = \begin{cases} \frac{\chi_{j,I}}{\delta_{j,I}} \\ \\ 0 \end{cases}, \qquad g_{j,I} = \begin{cases} 0 & \text{if } y^I \notin \langle y_1y_2y_j\rangle \\ \\ \chi_{j,I} & \text{if } y^I \in \langle y_1y_2y_j\rangle \end{cases}.$$

We can as before estimate $\delta_{j,I}$ from below for $y^I \notin \langle y_1y_2y_j\rangle$ with a $\delta>0$, obtaining again an estimate as in (4.8). We want to obtain an estimate as in (4.9), omitting the terms $M(g_j)$ and $\frac{\partial M(\phi_j)}{\partial y_k}M(g_k)$ for $k=1,2$.

As before, only terms with I such that $y^I\notin\langle y_1y_2y_j\rangle$ are involved in the estimates, while $M(g_k)\in\langle y_1y_2y_k\rangle$; so we can surely omit $M(g_j)$ and $\frac{\partial M(\phi_j)}{\partial y_j}M(g_j)$.

Concerning $\frac{\partial M(\phi_j)}{\partial y_i}M(g_i)$ with $i\neq j$, thanks to our hypothesis on f, we know that the right-hand side of (4.7) is a multiple of y_1y_2, so the same should be true for the left-hand side; but g_j is also a multiple of y_1y_2, and then ϕ_j for $j=1,2$ should be too. It follows also that $\frac{\partial M(\phi_j)}{\partial y_i}$ is a multiple of y_j, so $\frac{\partial M(\phi_j)}{\partial y_i}M(g_i)\in\langle y_1y_2y_j\rangle$ as desired.

So (4.9) holds for this change of coordinates too, and the argument used in the first step implies that ϕ is holomorphic, as claimed.

□

Directly from Theorem 4.3.3, we have

Corollary 4.3.4. *Let $\omega = X_1 dx_2 - X_2 dx_1$ be a holomorphic 1-form, with X_j of the form (4.3), and let $\alpha = \frac{\lambda_2}{\lambda_1} \in \mathbb{C}^*$ be the index at 0.*

If $\alpha, \alpha^{-1} \notin \mathbb{N} \setminus \{1\}$, then

$$\omega \sim x_1 dx_2 - \alpha x_2 \big(1 + f(x)\big) dx_1, \tag{4.12}$$

with $x_1 x_2 \mid f$.

Proof. Thanks to Theorem 4.3.3, we have

$$\omega \cong \lambda_1 x_1 \big(1 + f_1(x)\big) dx_2 - \lambda_2 x_2 \big(1 + f_2(x)\big) dx_1,$$

with $x_1 x_2 \mid f_j$, for $j = 1, 2$. Since $\lambda_1(1 + f_1(x))$ is invertible (in $\mathbb{C}\{x\}$), up to holomorphic equivalence we can divide by that factor. Being $x_1 x_2 \mid f_1$, the inverse of this factor is of the form $\lambda_1^{-1}(1 + x_1 x_2 g(x))$, with $g \in \mathbb{C}\{x\}$. Multiplying this by $\lambda_2(1 + f_2(x))$, we obtain the assertion. □

Remark 4.3.5. It follows directly from Corollary 4.3.4 that every foliation in the Siegel domain admits at least two complex separatrices (of the form $\{x_j = 0\}$ for $j = 1, 2$).

4.4 Holomorphic equivalence and holonomy

In this section we shall prove Mattei-Moussu's Theorem 4.4.4 (see [MM80]), a sort of converse of Lemma 1.4.4 for a foliation in the Siegel domain. For the proof, the computations in the next remark shall be useful.

Remark 4.4.1. Now we try to compute first terms of the holonomy of the horizontal complex separatrix of a foliation given by a 1-form ω as in (4.12). Let $L_0 = \{x_2 = 0\} \setminus \{0\}$ be the horizontal complex separatrix. We can suppose, up to conjugating by a suitable linear map, that f is holomorphic in a neighborhood U of the closed polydisk $P_1 := \overline{\mathbb{D} \times \mathbb{D}}$, and that 0 is the unique singular point of $\mathcal{F}$ in U. Consider an analytic curve $\gamma\colon [0, 1] \to L_0$. For every $a \in \overline{\mathbb{D}}$, we shall consider the vertical transverse section $\{x_1 = a\}$. For every y small enough, there is a lift γ_y of γ such that $\gamma_y(0) = (\gamma(0), y)$ and $\gamma(t, y) := \gamma_y(t) \in L_y$ for a suitable leaf L_y. By analiticity, we can write $\gamma(t, y) = (\gamma(t), h(t, y))$ for a suitable analytic map h (such that $h(0, y) = y$). Directly from definitions, we have then that $h_\gamma(y) = h(1, y)$.

The condition $\gamma(t,y) \in L_y$ is equivalent to

$$\omega\left(\frac{\partial\gamma}{\partial t}(t,y)\right) = 0.$$

We can write

$$h(t,y) = \sum_{k\geq 0} h_k(t)y^k, \qquad \text{with } h_k(0) = \delta_{k,1} = \begin{cases} 1 & \text{if } k=1 \\ 0 & \text{otherwise.} \end{cases}$$

Then putting all together, we obtain

$$\frac{\partial\gamma}{\partial t}(t,y) = \gamma'(t)\frac{\partial}{\partial x_1} + \frac{\partial h}{\partial t}(t,y)\frac{\partial}{\partial x_2};$$

$$\begin{gathered} 0 = \omega\left(\tfrac{\partial\gamma}{\partial t}(t,y)\right) = \gamma(t)\tfrac{\partial h}{\partial t}(t,y) - \alpha h(t,y)\Big(1 + f\big(\gamma(t,y)\big)\Big)\gamma'(t) \\ \Downarrow \\ \tfrac{\partial h}{\partial t}(t,y) = \alpha h(t,y)\Big(1 + f\big(\gamma(t,y)\big)\Big)\tfrac{\gamma'(t)}{\gamma(t)}. \end{gathered} \tag{4.13}$$

Example 4.4.2. Let us compute the holonomy for $\gamma(t) = e^{2\pi i t}$, and hence of the horizontal complex separatrix L_0 (we still suppose that $x_2 \mid f$). Expanding in power series (in y) both sides in (4.13), and comparing terms of the same degree, we have $h_0 \equiv 0$, and

$$\begin{cases} h_1'(t) = 2\pi i\alpha h_1(t), \\ h_1(0) = 1, \end{cases}$$

and then $h_1(t) = e^{2\pi i\alpha t}$. In particular we obtain $h(y) = h(1,y) = e^{2\pi i\alpha}y + \sum_{k\geq 2} h_k(1)y^k$; it follows that h has rotation number equal to α, and hence h^{-1} has rotation number equal to $-\alpha$.

Example 4.4.3. Let us make the same computation but for $\gamma\colon [0,1-\varepsilon] \to L_0$, defined by $\gamma(t) = (1-t)e^{2\pi i\theta}$ (and suppose as before that $x_2 \mid f$). Arguing as before from (4.13) we obtain ($h_0 \equiv 0$ and)

$$\begin{cases} h_1'(t) = -\frac{\alpha}{1-t}h_1(t), \\ h_1(1) = 1; \end{cases}$$

then $h_1(t) = (1-t)^\alpha$.

In particular we obtain $h_\gamma(y) = h(1-\varepsilon, y) = \varepsilon^\alpha y + \sum_{k\geq 2} h_k(1-\varepsilon)y^k$. If we are in the Siegel domain, the main problem we have to deal with is that $\varepsilon^\alpha \to \infty$ as $\varepsilon \to 0$.

Theorem 4.4.4 (**Mattei-Moussu**, 1980)**.** *Let ω and $\tilde{\omega}$ be two 1-forms in the Siegel domain:*

$$\begin{aligned}\omega &= x_1 dx_2 - \alpha x_2\big(1+f(x)\big)dx_1,\\ \tilde{\omega} &= x_1 dx_2 - \alpha x_2\big(1+\tilde{f}(x)\big)dx_1,\end{aligned}$$

with $\alpha \in \mathbb{R}^-$. Then $\omega \sim \tilde{\omega}$ if and only if $h \cong \tilde{h}$, where h and $\tilde{h}$ are the holonomies of the horizontal complex separatrices of ω and $\tilde{\omega}$ respectively.

Proof. The direct implication is Lemma 1.4.4. Let us prove the other implication.

Thanks to Corollary 4.3.4, we can suppose that $x_1x_2 \mid f, \tilde{f}$. In particular we can write $f = x_1 g$, $\tilde{f} = x_1\tilde{g}$; up to conjugacy by a linear map, we can suppose $g, \tilde{g}$ to be holomorphic in neighborhoods U, $\tilde{U}$ of the closed polydisk $P_1 = \overline{\mathbb{D}}^2$ of radius 1, with $|g|, |\tilde{g}| < 1/2$ in P_1, and that 0 is the unique singular point of $\omega, \tilde{\omega}$ in $U, \tilde{U}$ respectively. Now suppose that $h \cong \tilde{h}$, with $h, \tilde{h}$ calculated on $t \mapsto e^{2\pi i t}$, with base point $x_1 = 1$. Then there exists $\phi\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that

$$h = \phi^{-1} \circ \tilde{h} \circ \phi. \tag{4.14}$$

For $x \neq 0$, let us define

$$\Phi(x, y) = \left(x, \tilde{h}_\gamma \circ \phi \circ h_\gamma^{-1}(y)\right),$$

where γ is a curve from 1 to x with support in $L_0 := \overline{\mathbb{D}} \times \{0\}$, and h_γ and $\tilde{h}_\gamma$ are the holonomies along γ (of $\mathcal{F}_\omega$ and $\mathcal{F}_{\tilde{\omega}}$ respectively).

We first notice that this definition does not depend on the choice of γ. We already know that holonomies depend only on homotopy classes (see Theorem 1.3.8). Let γ_1, γ_2 be two curves from 1 to x; then we want to prove that

$$\tilde{h}_{\gamma_1} \circ \phi \circ h_{\gamma_1}^{-1} = h'_{\gamma_2} \circ \phi \circ h_{\gamma_2}^{-1},$$

which is equivalent to

$$\tilde{h}_{\gamma_2}^{-1} \circ \tilde{h}_{\gamma_1} \circ \phi \circ h_{\gamma_1}^{-1} \circ h_{\gamma_2} = \phi. \tag{4.15}$$

If we denote by $\gamma_1 \cdot \gamma_2^{-1}$ the loop obtained following γ_1 and γ_2 (on the opposite direction), then (4.15) is equivalent to

$$\tilde{h}_{\gamma_1\cdot\gamma_2^{-1}} \circ \phi \circ h^{-1}_{\gamma_1\cdot\gamma_2^{-1}} = \phi. \tag{4.16}$$

Being $\gamma_1 \cdot \gamma_2^{-1}$ a loop, we have $h_{\gamma_1\cdot\gamma_2^{-1}} = h^k$, with $k = [\gamma_1 \cdot \gamma_2^{-1}]$; the same is true for $\tilde{h}_{\gamma_1\cdot\gamma_2^{-1}} = \tilde{h}^k$. Then (4.16) is equivalent to

$$\tilde{h}^k \circ \phi \circ h^{-k} = \phi,$$

which follows directly from (4.14).

So Φ is well defined and holomorphic on $\{x_1 \neq 0\}$; moreover it defines an holomorphic equivalence between the two foliations outside $\{x_1 \neq 0\}$, sending leaves into leaves. We would like to extend Φ on $\{x_1 = 0\}$ too, in order to have the equivalence on the whole neighborhood of 0, and hence the thesis. We shall then prove that Φ is bounded in $\{x_1 \neq 0\}$, implying that Φ is analitically extendable and concluding the proof.

To estimate $\Phi(x, y)$, we compute it using the curve $\gamma = \gamma_1 \cdot \gamma_2$, where, if $x = |x|\, e^{2\pi i\theta}$, then $\gamma_1 \colon [0, \theta] \to L_0$ is given by $t \mapsto e^{2\pi i t}$ (a curve along the unit circle), and $\gamma_2 \colon [0, -\log|x|] \to L_0$ is given by $t \mapsto \frac{x}{|x|} e^{-t}$ (a curve along a radius of the unit circle).

Since $\Phi(x, y)$ is bounded in $\{|x| = 1\} \times \overline{\mathbb{D}}$ (by compactness), there exists $M > 0$ independent of θ such that

$$\left| \tilde{h}_{\gamma_1} \circ \phi \circ h_{\gamma_1}^{-1}(y) \right| \leq M\, |y|\, .$$

Now we have to estimate holonomies along γ_2.

From (4.13), applied to $\tilde{h}_{\gamma_2} =: \tilde{h}_2$, we get

$$\frac{\partial \tilde{h}_2}{\partial t}(t, y) = -\alpha \tilde{h}_2(t, y)\big(1 + \tilde{f}(\gamma_2(t, y))\big);$$

solving this equation and taking the real part we have

$$\log \left| \frac{\tilde{h}_2(t, y)}{y} \right| + \alpha t \leq \tilde{k}(t, y),$$

where

$$\begin{aligned} \tilde{k}(t, y) &:= \left| \alpha \int_0^t \tilde{f}\left(\frac{x}{|x|} e^{-s}, \tilde{h}_2(s, y) \right) ds \right| \\ &\leq |\alpha| \int_0^t e^{-s} \left| \tilde{g}\left(\frac{x}{|x|} e^{-s}, \tilde{h}_2(s, y) \right) \right| ds; \end{aligned}$$

then, since $|\tilde{g}| < 1/2$, we have

$$\tilde{k}(t, y) \leq \frac{|\alpha|\,(1 - e^{-t})}{2}.$$

It follows that

$$\left| \tilde{h}_2(t, y) \right| \leq |y|\, e^{-\alpha t} e^{\frac{|\alpha|(1-e^{-t})}{2}}.$$

Analogously for $h_2 := h_{\gamma_2}^{-1}$, we consider $\gamma_2^{-1}\colon [0, -\log|x|] \to L_0$ given by $t \mapsto xe^t$, and we obtain from (4.13)

$$\frac{\partial h_2}{\partial t}(t,y) = \alpha h_2(t,y)\Big(1 + \tilde{f}\big(\gamma_2(t,y)\big)\Big).$$

Solving this equation and taking the real part we have

$$\log\left|\frac{h_2(t,y)}{y}\right| - \alpha t \le k(t,y),$$

where

$$\begin{aligned} k(t,y) &:= \left|\alpha \int_0^t f\big(xe^s, h_2(s,y)\big)ds\right| \\ &\le |\alpha| \int_0^t |x|\, e^s \left|g\big(xe^s, h_2(s,y)\big)\right| ds; \end{aligned}$$

then, since $|g| < 1/2$, we have

$$k(t,y) \le \frac{|\alpha|\,|x|\,(e^t - 1)}{2}.$$

It follows that

$$|h_2(t,y)| \le |y|\, e^{\alpha t} e^{\frac{|\alpha||x|(e^t-1)}{2}}.$$

Putting together all the estimates, with $t = -\log|x|$, we get

$$\begin{aligned} \left|\tilde{h}_{\gamma_2} \circ \tilde{h}_{\gamma_1} \circ \phi \circ h_{\gamma_1}^{-1} \circ h_{\gamma_2}^{-1}(y)\right| &\le \left|\tilde{h}_{\gamma_1} \circ \phi \circ h_{\gamma_1}^{-1} \circ h_{\gamma_2}^{-1}(y)\right| |x|^{\alpha}\, e^{\frac{|\alpha|(1-|x|)}{2}} \\ &\le \left|h_{\gamma_2}^{-1}(y)\right| M\, |x|^{\alpha}\, e^{\frac{|\alpha|(1-|x|)}{2}} \\ &\le |y|\, M\, |x|^{-\alpha}\, e^{\frac{|\alpha||x|(|x|^{-1}-1)}{2}}\, |x|^{\alpha}\, e^{\frac{|\alpha|(1-|x|)}{2}} \\ &\le |y|\, M e^{|\alpha|(1-|x|)}, \end{aligned}$$

which is uniformly bounded for $x \in \mathbb{D}^*$. □

Remark 4.4.5. Checking the hypothesis we used in Theorem 4.4.4, assuming ω and $\tilde{\omega}$ of the given form, we see that this theorem is also valid for every α such that $\alpha, \alpha^{-1} \notin \mathbb{N} \setminus \{1\}$, or for every $\alpha \in \mathbb{C}^*$ if we suppose that $x_1 x_2 \mid f, \tilde{f}$ a priori (up to replacing α with $\mathrm{Re}(\alpha)$ where necessary). This result cannot be extended to the case α or $\alpha^{-1} \in \mathbb{N} \setminus \{1\}$, as the following example shows.

Counterexample 4.4.6. Let $n \in \mathbb{N} \setminus \{0,1\}$, $a \in \mathbb{C}^*$ and consider the foliations given by

$$(nz_1 + az_2^n)\partial_1 + z_2\partial_2,$$

as in (3.19). We have seen in Corollary 3.3.16 that two such foliations associated to different a are not holomorphically equivalent. We have also computed the first terms of the Taylor series of the holonomy of the unique complex separatrix (the horizontal one, see Theorem 3.4.4) in Theorem 3.4.7: let us repeat that computation, and focus our attention on the dependence on the parameter a. Recalling the proof of Theorem 3.4.7, the holonomy $h_a(s)$ is such that

$$h_a(s) = \mu s e^{\delta_a(s)},$$

with $\mu = e^{\frac{2\pi i}{n}}$ and $\delta_a \in \mathfrak{m}_1$ such that

$$1 + as^n\left(\frac{2\pi i}{n} + \delta_a(s)\right) = e^{-n\delta_a(s)} = \sum_{j=0}^{\infty} \frac{\big(-n\delta_a(s)\big)^j}{j!}. \tag{4.17}$$

Let us write $\delta_a(s) = \sum_{l=1}^{\infty} b_l s^l$, where b_l depends on a, and set $b_0 = \frac{2\pi i}{n}$. Then (4.17) is equivalent to

$$as^n\left(\sum_{l=0}^{\infty} b_l s^l\right) = \sum_{j=1}^{\infty} \frac{(-n)^j}{j!}\left(\sum_{l=1}^{\infty} b_l s^l\right)^j;$$

if we rearrange terms in order to have power series (in s) in both members, we obtain

$$a\sum_{m=n}^{\infty} b_{m-n}s^m = \sum_{m=1}^{\infty} s^m\left(\sum_{j=1}^{m}\frac{(-n)^j}{j!}\sum_{\substack{L\in(\mathbb{N}^*)^j\\ |L|=m}} b_{l_1}\cdots b_{l_j}\right). \tag{4.18}$$

From (4.18) we obtain the recursion

$$nb_m = -ab_{m-n} + \sum_{j=2}^{m}\frac{(-n)^j}{j!}\sum_{\substack{L\in(\mathbb{N}^*)^j\\ |L|=m}} b_{l_1}\cdots b_{l_j},$$

for $m \geq n$, and $b_1 = \cdots b_{n-1} = 0$.

We can see by induction that $b_{ln+j} = 0$ for $j = 1, \dots n-1$ and every $l \in \mathbb{N}$, while $b_{ln} = a^l c_l$ for a suitable c_l that does not depend on a. Then $\delta_a(s) := \sum_{j=1}^{\infty} b_j s^j = \sum_{l=1}^{\infty} c_l a^l s^{nl}$, and for the holonomy we have

$$h_a(s) = \mu s e^{\delta_a(s)} = \mu s\left(1 + \sum_{l=1}^{\infty} d_l a^l s^{nl}\right),$$

with suitable d_l that does not depend on a.

By performing the linear change of coordinate $s \mapsto a^{-1}s$, we obtain that $h_a(s)$ is holomorphically conjugated to $h_1(s)$, so the holonomies are conjugated even though the foliations are not.

For having a complete answer for the holomorphic classification of foliations in the Siegel domain, we have to discover how many germs can be holonomies of the horizontal complex separatrix. The answer (all germs with rotation number α can be obtained as holonomies), given by Yoccoz and Perez-Marco, is a little hard to prove, and we shall present it in the last section (see Theorem 4.6.1).

4.5 Topology of leaves

We have seen in Example 4.4.2 that the holonomy of the horizontal complex separatrix given by a 1-form as in (4.12), with index $\alpha \in \mathbb{R}^-$, has α as rotation number. As a references for local dynamics in this case, see [Mil06, Section 10] or [BH09, Chapter 4] for the parabolic case ($\alpha \in \mathbb{Q}$), and [Mil06, Section 11], [Mar03] or [BH09, Chapter 5] for the irrational case ($\alpha \in \mathbb{R} \setminus \mathbb{Q}$). Here we give a list of results that will be useful for drawing our conclusions on the topology of the leaves of Siegel foliations.

Let us denote by

$$S_\alpha := \left\{ f \in \mathbb{C}\{z\} \ : \ f'(0) = e^{2\pi i \alpha} \right\}$$

the set of germs with rotation number α.

First suppose that $\alpha = p/q \in \mathbb{Q}$; then $f \in S_{p/q}$ is linearizable if and only if $f^q = \mathrm{id}$. Moreover, if $f^q \neq \mathrm{id}$, then we have Leau-Fatou's theorem (see [Mil06, Theorem 10.7]): there exist attracting and repelling petals that cover a pointed neighborhood of the origin. In particular, the orbit of a point can accumulate the origin forward, backward, or both, depending whether the point lies in an attracting petal, in a repelling one, or in the intersection of two consecutive petals.

If $\alpha \in \mathbb{R}^- \setminus \mathbb{Q}^-$, we consider its continuous fractions expansion, and we denote by p_n/q_n the n-th convergent (see [Mar03], or [BH09, Section 3 of Chapter 5]). Then we say that

- α satisfies the **Brjuno condition** if $\sum_n \frac{\log q_{n+1}}{q_n} < \infty$;
- α satisties the **Perez-Marco condition** if $\sum_n \frac{\log \log q_{n+1}}{q_n} < \infty$.

Then α satisfies the Brjuno condition if and only if every $f \in S_\alpha$ is linearizable; if α satisfies the Perez-Marco condition, then for every non-linearizable germ $f \in S_\alpha$ there exist periodic orbits arbitrarily close to the origin. If α does not satisfies the

Perez-Marco condition, then there exists a germ $f \in S_\alpha$ without periodic orbits arbitrarily close to the origin.

Let us study now the topology of the leaves.

Example 4.5.1. We study the linear case, and hence the leaves of the foliation given by

$$\omega = x_1 dx_2 - \alpha x_2 dx_1.$$

In this case we can explicitly solve the equation $\omega = 0$, obtaining $y = kx^\alpha$. In particular, if $\alpha \in \mathbb{Q}^-$, then every leaf is closed, while if $\alpha \in \mathbb{R}^- \setminus \mathbb{Q}^-$, then the only closed leaves are the horizontal and vertical complex separatrices, and the other leaves are bounded away from the complex separatrices (they are dense in $\{|y| = |k|\,|x|^\alpha\}$).

For the general case we have to use another approach. Set $C_j(R) := \{|x_j| = R\}$ for $j = 1, 2$. The idea is to consider the intersections of the leaves with $C_1(R)$, and then to describe leaves by gluing these intersections along radial curves, through holonomies. Moreover, to study intersections with $C_1(R)$, we will use the holonomy h of the horizontal complex separatrix L_0.

Let us denote by h_R the holonomy along the circle $\{|x_1| = R\}$, with vertical transverse sections, for every $R > 0$. These holonomies are conjugated to one another.

The idea is the following: start from the point (R, y), and then follow the holonomy along the circle $\{|x_1| = R\}$. We will come back to the point R, but generally on a different height: $(R, h_R(y))$. We can follow the circle once again, obtaining $(R, h_R^2(y))$, and so on. Checking the behavior of the orbit of h_R, we can find properties for the intersection of the leaves with $C_1(R)$: for example, if y has a periodic orbit, then this intersection is a closed curve, while if the orbit is not periodic, this intersection is an open curve (homeomorphic to $\mathbb{R}$).

We can also repeat this study inverting the role of x_1 and x_2, using the holonomy of the vertical complex separatrix with rotation number α^{-1}.

Example 4.5.2. We already treated the linear (and hence linearizable) case; here we deal with the non linearizable case. Let us suppose:

- $\alpha = p/q \in \mathbb{Q}^-$, and the holonomy h (of the horizontal complex separatrix) not linearizable. Then we can have attracting orbits, repelling orbits (that escapes from the given neighborhood of the origin), and orbits that are forward and backward attracted by the origin (when we take a point on the intersection of two petals). It follows that we can have open leaves tending to both horizontal and vertical complex separatrices, or, in the last case, leaves tending to the horizontal (or the vertical) complex separatrix, while being bounded away from the vertical one (the horizontal one respectively).

- $\alpha \in \mathbb{R}^- \setminus \mathbb{Q}^-$, and suppose that h is not linearizable. If α satisfies the Perez-Marco condition, then there are periodic orbits arbitrarily close to the origin, and hence there are closed leaves arbitrarily near the horizontal (respectively, the vertical) complex separatrix. If the Perez-Marco condition is not satisfied, and h is without periodic orbits near the origin, then we do not have closed leaves near the origin (besides the horizontal and the vertical complex separatrices).

4.6 Construction of foliations with prescribed holonomy

This last section will be dedicated to the following important result (see [PMY94]), that gives us a complete answer to the holomorphic classification of foliations in the Siegel domain.

Theorem 4.6.1 (**Perez-Marco-Yoccoz**, (1994))**.** *Let $h \in S_\alpha$ be a holomorphic germ with rotation number $\alpha \in \mathbb{R}^-$. Then there exists $f \in \mathfrak{m}_2$ so that h is the holonomy of the horizontal complex separatrix of the foliation given by the 1-form*

$$\omega = w_1 dw_2 - \alpha w_2\big(1 + f(w)\big) dw_1.$$

We shall first describe the main idea of the proof, then we shall introduce some technical tools, and finally we shall give the detailed proof.

Idea of the proof. We know that every foliation on the Siegel domain has at least 2 complex separatrices (see Remark 4.3.5). So in order to study such foliations (let us say generated by the 1-form ω as in Theorem 4.6.1), we can consider $D \setminus \{w_1 w_2 = 0\}$, where $D \subseteq \mathbb{C}^2$ is a neighborhood of the origin, take its universal covering given by $w = E(y) = (e^{2\pi i y_1}, e^{2\pi i y_2})$, and consider the pull-back $\Omega_2 = E^*\omega$.

In the linear case, we have

$$\begin{aligned} \omega &= w_1 dw_2 - \alpha w_2 dw_1, \\ \Omega_2 &= 2\pi i e^{2\pi i (y_1 + y_2)}(dy_2 - \alpha dy_1) \sim dy_2 - \alpha dy_1; \end{aligned}$$

thus we have a foliation made by parallel (complex) lines. In the general case we will have a foliation with leaves "near to" the linear case.

Projecting through E is equivalent to taking the quotient under the action of

$$\begin{aligned} T_1(y) &= (y_1 + 1, y_2), \\ T_2(y) &= (y_1, y_2 + 1). \end{aligned}$$

The idea is to consider the foliation arising from the linear case on the universal covering space, and quotienting for a different action, let us say one generated by suitable F_1, F_2, in order to obtain a different foliation on the base space. It will be very important to connect this point of view and the original one, through a diffeomorphism $y = v(x)$ that allows us to switch from the linear foliation to the original (lifted) foliation, such that $T_j \circ v = v \circ F_j$ for $j = 1, 2$.

It is not so difficult to obtain a C^∞ foliation with the prescribed holonomy; but to obtain a holomorphic foliation we shall have to solve a $\overline{\partial}$-equation, with a bound for the norm of the solution: in this way we will manage to deform the C^∞ foliation and to transform it into a holomorphic one, without changing the holonomy.

For computations, it will be easier to find a foliation with a prescribed inverse of the holonomy of the horizontal complex separatrix, and to set $\beta = -\alpha \in \mathbb{R}^+$.

We set here some definitions and notations.

Definition 4.6.2. Let M be a complex manifold of complex dimension n. Let $\omega_1, \ldots, \omega_n$ be C^∞ $(1,0)$-forms on M, forming a basis for the holomorphic cotangent space on M.

For any $g \in C^\infty(M)$, we shall denote by $\partial_j g$ the j-th coordinate of ∂g with respect to the basis $\{\omega_1, \ldots, \omega_n\}$, and by $\overline{\partial_j} g$ the j-th coordinate of $\overline{\partial} g$ with respect to the basis $\{\overline{\omega_1}, \ldots, \overline{\omega_n}\}$, i.e.,

$$dg = \partial g + \overline{\partial} g = \sum_j \big((\partial_j g)\omega_j + (\overline{\partial_j} g)\overline{\omega_j}\big).$$

We shall also denote by $g_{j,\overline{k}}$ the coordinates of $\partial\overline{\partial} g$ with respect to those bases, so that

$$\partial\overline{\partial} g = \sum_{j,k} g_{j,\overline{k}} \omega_j \wedge \overline{\omega_k}.$$

If $f = \sum_j f_{\overline{j}} \overline{\omega_j}$ is a $(0,1)$-form, we set $|f|^2 = \sum_j \left|f_{\overline{j}}\right|^2$.

Remark 4.6.3. Not every $\overline{\partial}$-equation can be solved in a generic complex manifold. During the proof, we shall show that, up to shrinking the neighborhood D, we can work on a Stein manifold, where $\overline{\partial}$-equations can be solved.

Definition 4.6.4. Let X be a complex manifold. Denote by $\mathcal{O}(X)$ the set of holomorphic functions $f \colon X \to \mathbb{C}$. We say that X is a **Stein manifold** if:

(i) $\mathcal{O}(X)$ **separates points**, i.e., for every $x \neq y \in X$ there exists $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$;

(ii) X is **holomorphically convex**, i.e., for every compact set $K \subseteq X$, the **holomorphic envelope** $\hat{K} = \{x \in X \;:\; |f(x)| \leq \max_K |f| \;\; \forall f \in \mathcal{O}(X)\}$ is compact.

Stein manifolds are strictly related to plurisubharmonic functions, as Theorem 4.6.7 shows.

Definition 4.6.5. Let X be a complex manifold, and $\phi\colon X \to \mathbb{R}$ a C^2 function. The **Levi form** of ϕ is the hermitian form $L(\phi, x)$ on T_xX defined in local coordinates by

$$L(\phi,x)(\zeta) = \sum_{j,k} \frac{\partial^2\phi}{\partial z_j \partial \overline{z_k}}(x)\zeta_j\overline{\zeta_k}.$$

Definition 4.6.6. Let X be a complex manifold. A C^2 function $\phi\colon X \to \mathbb{R}$ is **strictly plurisubharmonic** if the Levi form $L(\phi, x)$ is positive-definite for every $x \in X$. It is called an **exaustion** if $\phi^{-1}((-\infty, a])$ is relatively compact in X for every $a \in \mathbb{R}$.

We shall see now a classical charaterization of Stein manifold, whose proof can be found in [Hör73, Theorem 5.2.10].

Theorem 4.6.7 (**Grauert**, 1958)**.** *A complex manifold X is a Stein manifold if and only if there exists a strictly plurisubharmonic function $\phi\colon X \to \mathbb{R}$ which is an exhaustion.*

We shall need to solve the $\overline{\partial}$-equation with bounds, in order to control the deformation. The following theorem will be fundamental: for the proof, see [Hör73, Chapters IV and V], or [PMY94, Section III].

Theorem 4.6.8 (**Hörmander**, 1965-1973)**.** *Let M be a Stein manifold of complex dimension n, endowed with a Hermitian metric. Let $\omega_1, \dots, \omega_n$ be C^∞ $(1,0)$-forms on M, forming a basis for the holomorphic cotangent space on M. Set $dV = \left(\frac{i}{2}\right)^n \omega_1 \wedge \overline{\omega_1} \wedge \cdots \wedge \omega_n \wedge \overline{\omega_n}$.*

Let $a^l_{j,k}, c^l_{j,\overline{k}} \in C^\infty(M)$ be such that

$$\partial\omega_l = \sum_{j,k} a^l_{j,k}\omega_j \wedge \omega_k \qquad (\textit{with } a^l_{j,k} = -a^l_{k,j});$$
$$\overline{\partial}\omega_l = \sum_{j,k} c^l_{j,\overline{k}}\omega_j \wedge \overline{\omega_k}.$$

Suppose that there exist continuous functions $\theta_0, \theta_1\colon M \to \mathbb{R}$ such that

$$\left|a^l_{j,k}\right| \le \theta_0, \qquad \left|c^l_{j,\overline{k}}\right| \le \theta_0,$$
$$\left|\partial_e a^l_{j,k}\right| \le \theta_1, \qquad \left|\partial_e c^l_{j,\overline{k}}\right| \le \theta_1,$$
$$\left|\overline{\partial_e} a^l_{j,k}\right| \le \theta_1, \qquad \left|\overline{\partial_e} c^l_{j,\overline{k}}\right| \le \theta_1,$$

for every $e = 1, \dots, n$. Then there exists $A > 0$ (depending only on n) with the following property: given a function $\theta\colon M \to \mathbb{R}^+$, a strictly plurisubharmonic function $\chi \in C^2(M)$, and a $\overline{\partial}$-closed C^∞ $(0,1)$-form f on M such that

$$\sum_{j,k} \chi_{j,\overline{k}} \zeta_j \overline{\zeta_k} \geq (\theta + A(\theta_0^2 + \theta_1)) \, |\zeta|^2 \tag{4.19}$$

for some $A > 0$, and

$$\int_M \theta^{-1} |f|^2 e^{-\chi} dV < \infty, \tag{4.20}$$

then there exists $u \in C^\infty(M)$ such that

$$\overline{\partial} u = f,$$
$$\int_M |u|^2 e^{-\chi} dV \leq \int_M \theta^{-1} |f|^2 e^{-\chi} dV.$$

Now we can prove Theorem 4.6.1.

Proof (of Theorem 4.6.1). Fix a C^∞ function $\eta\colon \mathbb{R} \to [0,1]$ such that $\eta \equiv 1$ in $(-\infty, 1/3]$ and $\eta \equiv 0$ in $[2/3, +\infty)$, and denote $\beta = -\alpha > 0$.

We have h^{-1} of the form $h^{-1}(z) = e^{2\pi i \beta} z + O(z^2)$ by assumption. Up to homotheties, we can suppose that h^{-1} is holomorphic in the unitary disk $\mathbb{D}$, and that $h^{-1}(\mathbb{D}^*) \subset \mathbb{C}^*$. Let $z \mapsto H(z) = z + \beta + \phi(z)$ be the lift of h^{-1} through $\pi\colon \mathbb{H} \to \mathbb{D}^*$ given by $z \mapsto e^{2\pi i z}$, with H holomorphic in $\mathbb{H}$, where ϕ is $\mathbb{Z}$-periodic, and $\lim_{\mathrm{Im} z \to \infty} \phi(z) = 0$. Again up to homotheties in $\mathbb{D}^*$, we may assume that

$$\left| D^i \phi(z) \right| \leq C_1 e^{-2\pi \mathrm{Im} z} \qquad \forall i = 0, \dots, 4, \forall z \in \mathbb{H}. \tag{4.21}$$

We split the proof into seven steps.

(**Step** 1)**.** Let us construct the C^∞ 1-form Ω_0. Set

$$U = \{x \in \mathbb{C}^2 \mid \mathrm{Im}(x_2 + \beta x_1) > 0\},$$

and define $F_1, F_2\colon U \to \mathbb{C}^2$ by

$$F_1(x) = \big(x_1 + 1, x_2 + \phi(x_2 + \beta x_1)\big),$$
$$F_2(x) = (x_1, x_2 + 1).$$

We can easily see that

$$\begin{array}{rcl} F_1 \circ F_2 & = & F_2 \circ F_1 \\ & \Updownarrow & \\ (x_1 + 1, x_2 + 1 + \phi(x_2 + 1 + \beta x_1)) & = & (x_1 + 1, x_2 + 1 + \phi(x_2 + \beta x_1)), \end{array}$$

and the last equation holds since ϕ is $\mathbb{Z}$-periodic.

Then we can quotient (a subset of) U by F_1, F_2. Define $v_0, v_1\colon U \to \mathbb{C}$ by setting

$$\begin{aligned} v_0(x) &= \eta(\mathrm{Re}x_1)\phi(x_2+\beta x_1),\\ v_1(x) &= \eta(\mathrm{Re}x_1)\log\big(1+D\phi(x_2+\beta x_1)\big). \end{aligned}$$

Remark 4.6.9. Both v_0 and v_1 are $\mathbb{Z}$-periodic on the second coordinate, thanks to the $\mathbb{Z}$-periodicity of ϕ.

Now let us define $v\colon U \to \mathbb{C}^2$ by

$$y = v(x) = \big(x_1, x_2 + v_0(x)\big),$$

and set

$$\Omega_0 = e^{v_1(x)}(dx_2 + \beta dx_1),$$

so that Ω_0 is a C^∞ $(1,0)$-form.

Lemma 4.6.10. *Let ∂^l be a partial derivative of order $l \le 3$. There exists $C_2 > 0$ such that*

$$\big|\partial^l v_j(x)\big| \le C_2 e^{-2\pi \mathrm{Im}(x_2+\beta x_1)} \qquad \forall x \in U,\ j = 0, 1.$$

Proof. It follows from definitions and (4.21) □

Let $T_1.T_2\colon \mathbb{C}^2 \to \mathbb{C}^2$ be given by

$$\begin{aligned} T_1(y) &= (y_1+1, y_2),\\ T_2(y) &= (y_1, y_2+1). \end{aligned}$$

Lemma 4.6.11. *We have*

(i) $v\circ F_2 = T_2 \circ v$ and $F_2^*\Omega_0 = \Omega_0$ in U;

(ii) $v\circ F_1 = T_1\circ v$ and $F_1^*\Omega_0 = \Omega_0$ for $x \in U \cap F_1^{-1}(U)$ and $|\mathrm{Re}x_1| \le 1/3$.

Proof.

(i) We have

$$\begin{aligned} v\circ F_2(x) &= v(x_1, x_2+1) = \big(x_1, x_2+1+v_0(x_1,x_2+1)\big)\\ T_2\circ v(x) &= T_2\big(x_1, x_2+v_0(x)\big) = \big(x_1, x_2+1+v_0(x)\big). \end{aligned}$$

They are equal thanks to the $\mathbb{Z}$-periodicity of $v_0(x_1,\cdot)$.

For the pull back of Ω_0 we have

$$(F_2^*\Omega_0)_x = e^{v_1(x_1,x_2+1)}(dx_2+\beta dx_1) = (\Omega_0)_x,$$

again thanks to the $\mathbb{Z}$-periodicity of $v_1(x_1,\cdot)$.

(ii) Let us write $F_1(x) =: (f_1(x), f_2(x))$; then we have

$$\begin{aligned}
v \circ F_1(x) &= v\big(x_1 + 1, x_2 + \phi(x_2 + \beta x_1)\big) \\
&= \big(x_1 + 1, x_2 + \phi(x_2 + \beta x_1) + \eta(\mathrm{Re} x_1 + 1)\phi(f_2 + \beta f_1)\big) \\
T_1 \circ v(x) &= T_1\big(x_1, x_2 + v_0(x)\big) = \big(x_1 + 1, x_2 + \eta(\mathrm{Re} x_1)\phi(x_2 + \beta x_1)\big).
\end{aligned}$$

The condition $x \in U \cap F^{-1}(U)$ is necessary to have all members well defined; moreover if $|\mathrm{Re} x_1| \leq 1/3$ then $\eta(\mathrm{Re} x_1) = 1$ and $\eta(\mathrm{Re} x_1 + 1) = 0$, and hence the two members coincide.

For the pull back of Ω_0 we have

$$(F_1^* \Omega_0)_x = (\Omega_0)_{F_1(x)} \circ d(F_1)_x;$$

But

$$d(F_1)_x = \begin{pmatrix} 1 & 0 \\ \beta D\phi(x_2 + \beta x_1) & 1 + D\phi(x_2 + \beta x_1) \end{pmatrix};$$

it follows

$$\begin{aligned}
(F_1^* \Omega_0)_x &= e^{v_1 \circ F_1(x)} \Big(\big(1 + D\phi(x_2 + \beta x_1)\big) dx_2 + \beta D\phi(x_2 + \beta x_1) dx_1 + \beta dx_1 \Big) \\
&= e^{\eta(\mathrm{Re} x_1 + 1) \log(1 + D\phi(f_2 + \beta f_1))} \big(1 + D\phi(x_2 + \beta x_1)\big)(dx_2 + \beta dx_1) \\
&= \big(1 + D\phi(x_2 + \beta x_1)\big)(dx_2 + \beta dx_1),
\end{aligned}$$

since $\eta(\mathrm{Re} x_1 + 1) = 0$ when $|\mathrm{Re} x_1| \leq 1/3$. On the other hand,

$$\Omega_x = e^{\eta(\mathrm{Re} x_1) \log(1 + D\phi(x_2 + \beta x_1))}(dx_2 + \beta dx_1) = \big(1 + D\phi(x_2 + \beta x_1)\big)(dx_2 + \beta dx_1),$$

since $\eta(\mathrm{Re} x_1) = 1$ when $|\mathrm{Re} x_1| \leq 1/3$. $\square$

Fix $\varepsilon \in (0, 1)$, and set $\beta_1 = \beta(1 + \varepsilon)$ and $\beta_2 = \beta(1 - \varepsilon)$ (both in $\mathbb{R}^+$). Set $I(y) = \mathrm{Im}(y_2 + \beta y_1)$, and $I_j(y) = \mathrm{Im}(y_2 + \beta_j y_1)$ for $j = 1, 2$. For every $R > 0$, we define

$$\begin{aligned}
V_R &= \left\{ y \in \mathbb{C}^2 \ : \ -\frac{1}{4} < \mathrm{Re} y_j < \frac{5}{4} \text{ for } j = 1, 2; I(y) > 0, I_1(y) I_2(y) > R \right\}, \\
U_R &= v^{-1}(V_R) \subset U.
\end{aligned}$$

Note that I_1 and I_2 are invariant under the action of T_1 and T_2.

Remark 4.6.12. If $y \in V_R$, we have

$$4I(y)^2 = \big(I_1(y) + I_2(y)\big)^2 = I_1(y)^2 + I_2(y)^2 + 2I_1(y)I_2(y) \geq 4I_1(y)I_2(y) > 4R,$$

and hence

$$I(y) > \sqrt{R}.$$

Lemma 4.6.13. *There exists $R_1 > 0$ such that for every $R > R_1$:*

(i) $v|_{U_R}$ *is a C^∞ diffeomorphism on V_R;*

(ii) ${}^{U_R}\!/\!{}_{\langle F_1,F_2\rangle} =: M_R$ *is a complex manifold, and v induces a C^∞ diffeomorphism* $\overline{v}\colon M_R \to {}^{V_R}\!/\!{}_{\mathbb{Z}^2} = {}^{V_R}\!/\!{}_{\langle T_1,T_2\rangle}$*;*

(iii) Ω_0 *induces a C^∞ $(1,0)$-form Ω_1 on M_R.*

Proof. Suppose $x', x'' \in U_R$ such that $v(x') = v(x'')$. Then $x'_1 = x''_1$ and $x'_2+v_0(x') = x''_2 + v_0(x'')$, and thanks to Lemma 4.6.10 we get

$$|x'_2 - x''_2| \leq |v_0(x') - v_0(x'')| \leq C_2 e^{-2\pi I(x)} |x'_2 - x''_2| \,, \tag{4.22}$$

with x a suitable point on the segment between x'_2 and x''_2.

If $y \in U_R$, thanks to Remark 4.6.12 we have that $I(y) = I(v(x)) > \sqrt{R}$.

For $I(x)$ we have

$$I(x) = I(y) - \operatorname{Im}(v_0(x)) \geq I(y) - C_2 e^{-2\pi I(x)}$$

thanks to Lemma 4.6.10; it follows that, up to choose R large enought, we can suppose that $I(x) > \sqrt{R}/2$. From (4.22) we obtain

$$|x'_2 - x''_2| \leq C_2 e^{-\pi\sqrt{R}} |x'_2 - x''_2| \,,$$

and hence $x'_2 = x''_2$ as soon as $C_2 e^{-\pi\sqrt{R}} < 1$, and v is injective. The first point easily follows.

For the second one, M_R is a complex manifold, since F_1 and F_2 are holomorphic.

To take the quotient by the action of F_1 in U_R means that $x \sim x'$ if and only if $x' = F_1(x)$ (or the symmetric relation); in particular, $|\mathrm{Re} x_1| = |\mathrm{Re} y_1| \leq \frac{1}{4} < \frac{1}{3}$, and from Lemma 4.6.11 we obtain the well-definiteness and the regularity for $\overline{v}$, and the third result. □

(**Step** 2)**.** From now on, we have to focus on the hypotheses on Theorem 4.6.8, and on Theorem 4.6.7. In particular now we shall discuss properties of a specific basis (ω_1, ω_2) for the holomorphic cotangent space on M_R.

Denote by $\pi_R\colon U_R \to M_R$ the canonical projection, and define on M_R (or U_R)

$$\begin{aligned} h_j(x) &= \operatorname{Im}(x_2 + v_0(x) + \beta_j x_1) = \operatorname{Im}(y_2 + \beta_j y_1) = I_j(y) \text{ for } j = 1,2, \\ h_m(x) &= \frac{h_1(x) + h_2(x)}{2} = \operatorname{Im}\bigl(x_2 + v_0(x) + \beta x_1\bigr) = \operatorname{Im}(y_2 + \beta y_1) = I(y); \end{aligned}$$

finally set $\omega_j = \partial h_j$ for $j = 1, 2$.

Lemma 4.6.14. *For every $R > R_1$:*

(i) *the* $(1,0)$*-forms* ω_1, ω_2 *form a basis of the holomorphic cotangent space to* M_R*;*

(ii) *if* $dV = \left(\frac{i}{2}\right)^2 \omega_1 \wedge \overline{\omega_1} \wedge \omega_2 \wedge \overline{\omega_2}$*, and* $dV_0 = \left(\frac{i}{2}\right)^2 dx_1 \wedge d\overline{x_1} \wedge dx_2 \wedge d\overline{x_2}$*, then there exists* $C_3 > 0$ *such that*

$$C_3^{-1} dV_0 \le \pi_R^* dV \le C_3 dV_0$$

in U_R*.*

(iii) $\partial\omega_1 = \partial\omega_2 = 0$*, and there exists* $C_4 > 0$ *such that*

$$\overline{\partial}\omega_1 = \overline{\partial}\omega_2 = \overline{\partial}\partial \mathrm{Im} v_0 = \sum_{j,k} c_{j,\overline{k}} \omega_j \wedge \overline{\omega_k},$$

with $c_{j,\overline{k}} \in C^\infty(M_R)$ *and*

$$\left|c_{j,\overline{k}}\right|, \left|\partial_l c_{j,\overline{k}}\right|, \left|\overline{\partial}_l c_{j,\overline{k}}\right| \le C_4 e^{-2\pi h_m}, \tag{4.23}$$

for every j, k, l*.*

Proof.
(i) Since v is a diffeomorphism for $R > R_1$, we can compute derivatives in the y coordinates. Then

$$\omega_j = \partial h_j = \frac{1}{2i}(dy_2 + \beta_j dy_1);$$

thus ω_1 and ω_2 are not proportional and never zero.
(ii) For $j = 1, 2$, we have

$$\omega_j = \partial h_j = \frac{1}{2i}(dx_2 + \beta_j dx_1) + \partial \mathrm{Im} v_0(x); \tag{4.24}$$

$$\overline{\partial} h_j = -\frac{1}{2i}(d\overline{x_2} + \beta_j d\overline{x_1}) + \overline{\partial} \mathrm{Im} v_0(x);$$

so we have to show only that $|\partial \mathrm{Im} v_0|$ and $\left|\overline{\partial} \mathrm{Im} v_0\right|$ are bounded in U_R. But

$$|v_0| \le |\phi(x_2 + \beta x_1)| \le \frac{1}{2} e^{-2\pi \mathrm{Im}(x_2 + \beta x_1)} \le \frac{1}{2},$$

and similar estimates hold for ∂v_0 and $\overline{\partial} v_0$.
(iii) We obviously have $\partial^2 = \overline{\partial}^2 = 0$, while from (4.24) we have

$$\overline{\partial}\omega_j = \overline{\partial}\partial(\mathrm{Im} v_0).$$

Thanks to point (i), we can consider $\{\omega_1, \omega_2\}$ as a basis of the holomorphic cotangent space. The estimates for the coefficients follow then from Lemma 4.6.10. We note that we can estimate $e^{-2\pi I(x)} \le \tilde{C} e^{-2\pi h_m(x)}$, by the same argument we used in the proof of Lemma 4.6.13.(i). □

(**Step** 3)**.** Here we shall prove that, up to choosing R large enough, M_R is a Stein manifold.

Thanks to Theorem 4.6.7, we have to construct an exhaustive striclty plurisubharmonic function on M_R.

Now we shall describe a procedure to obtain functions on M_R, for which we can easily estimate the Levi form with respect to the basis $\{\omega_1, \omega_2\}$ (see Lemma 4.6.14.(i)).

Set $B_R := \{h \in \mathbb{R}^2 \mid h_1, h_2 > 0, h_1 h_2 > R\}$. Let $\tilde{\psi} \colon B_R \to \mathbb{R}$ be a C^2 function. Let us suppose $R > R_1$. We can associate to $\tilde{\psi}$ a function $\psi \colon U_R \to \mathbb{R}$ defined by

$$\psi(z) = \tilde{\psi}\big(h_1(z), h_2(z)\big).$$

Since $R > R_1$, we can use the basis $\{\omega_1, \omega_2\}$ (see Lemma 4.6.14.(i)):

$$\partial\overline{\partial}\psi = \sum_{j,k} \psi_{j,\overline{k}} \omega_j \wedge \overline{\omega_k}. \tag{4.25}$$

Lemma 4.6.15. *We have*

$$\psi_{j,\overline{k}} = \left[\frac{\partial^2 \tilde{\psi}}{\partial h_j \partial h_k} + c_{j,\overline{k}} \left(\frac{\partial \tilde{\psi}}{\partial h_1} + \frac{\partial \tilde{\psi}}{\partial h_2} \right) \right] \circ (h_1, h_2). \tag{4.26}$$

Proof. Let us compute all coefficients in the standard basis:

$$\begin{aligned}
\omega_j &= \sum_l \frac{\partial h_j}{\partial x_l} dx_l, \\
\omega_j \wedge \overline{\omega}_k &= \sum_{l,m} \frac{\partial h_j}{\partial x_l} \frac{\partial h_k}{\partial \overline{x}_m} dx_l \wedge d\overline{x}_m, \\
\sum_{j,k} c_{j,\overline{k}} \omega_j \wedge \overline{\omega}_k &= \sum_{l,m} \sum_{j,k} c_{j,\overline{k}} \frac{\partial h_j}{\partial x_l} \frac{\partial h_k}{\partial \overline{x}_m} dx_l \wedge d\overline{x}_m \\
&= \sum_{l,m} \frac{\partial^2 h_s}{\partial x_l \partial \overline{x}_m} dx_l \wedge d\overline{x}_m,
\end{aligned}$$

where $s = 1, 2$, thanks to Lemma 4.6.14.(iii).

Moreover

$$\frac{\partial\psi}{\partial\overline{x}_m} = \sum_p \frac{\partial\tilde{\psi}}{\partial h_p}(h_1,h_2)\cdot\frac{\partial h_p}{\partial\overline{x}_m},$$

$$\begin{aligned}\frac{\partial^2\psi}{\partial x_l\partial\overline{x}_m} &= \sum_q\left(\sum_p \frac{\partial^2\tilde{\psi}}{\partial h_p\partial h_q}(h_1,h_2)\cdot\frac{\partial h_p}{\partial\overline{x}_m}\cdot\frac{\partial h_q}{\partial x_l} + \frac{\partial\tilde{\psi}}{\partial h_p}(h_1,h_2)\cdot\frac{\partial^2 h_p}{\partial x_l\partial\overline{x}_m}\right)\\ &= \sum_{p,q}\frac{\partial^2\tilde{\psi}}{\partial h_p\partial h_q}(h_1,h_2)\cdot\frac{\partial h_p}{\partial\overline{x}_m}\cdot\frac{\partial h_q}{\partial x_l}\\ &\quad+\left(\sum_p\frac{\partial\tilde{\psi}}{\partial h_p}(h_1,h_2)\right)\cdot\left(\sum_{j,k}c_{j,\overline{k}}\frac{\partial h_j}{\partial x_l}\cdot\frac{\partial h_k}{\partial\overline{x}_m}\right).\end{aligned}$$

It follows that

$$\begin{aligned}\partial\overline{\partial}\psi =&\sum_{l,m}\frac{\partial^2\psi}{\partial x_l\partial\overline{x}_m}dx_l\wedge d\overline{x}_m\\ =&\sum_{p,q}\left(\frac{\partial^2\tilde{\psi}}{\partial h_p\partial h_q}(h_1,h_2)\sum_{l,m}\frac{\partial h_p}{\partial\overline{x}_m}\cdot\frac{\partial h_q}{\partial x_l}dx_l\wedge d\overline{x}_m\right)\\ &+\left(\sum_p\frac{\partial\tilde{\psi}}{\partial h_p}(h_1,h_2)\right)\cdot\sum_{j,k}\left(c_{j,\overline{k}}\sum_{l,m}\frac{\partial h_j}{\partial x_l}\cdot\frac{\partial h_k}{\partial\overline{x}_m}dx_l\wedge d\overline{x}_m\right)\\ =&\sum_{p,q}\frac{\partial^2\tilde{\psi}}{\partial h_p\partial h_q}(h_1,h_2)\omega_q\wedge\overline{\omega}_p+\left(\sum_p\frac{\partial\tilde{\psi}}{\partial h_p}(h_1,h_2)\right)\left(\sum_{j,k}c_{j,\overline{k}}\omega_j\wedge\overline{\omega}_k\right).\end{aligned}$$

Comparing (4.25) with the latter equation we get (4.26). □

Now apply this construction three times:

$$\begin{aligned}\tilde{\psi}^{(0)}(h_1,h_2) =& -\log(h_1h_2-R) = -\log\rho, \qquad \text{where } \rho := h_1h_2-R;\\ \tilde{\psi}^{(1)}(h_1,h_2) =& h_1^2+h_2^2;\\ \tilde{\psi}^{(2)}(h_1,h_2) =& -\pi(h_1+h_2) = -2\pi h_m.\end{aligned}$$

We want to estimate Levi forms in this three cases, but first we need a lemma:

Lemma 4.6.16. *For $(h_1,h_2)\in B_R$ we have that*

$$\begin{aligned}h_2^2\left|\zeta_1\right|^2+h_1^2\left|\zeta_2\right|^2+R(\zeta_1\overline{\zeta_2}+\zeta_2\overline{\zeta_1}) \geq& \frac{h_1^2h_2^2-R^2}{h_1^2+h_2^2}\left(\left|\zeta_1\right|^2+\left|\zeta_2\right|^2\right)\\ \geq& \frac{\rho R}{2h_m^2}\left(\left|\zeta_1\right|^2+\left|\zeta_2\right|^2\right),\end{aligned} \tag{4.27}$$

for every $\zeta_1,\zeta_2\in\mathbb{C}$.

Proof. For the first inequality, we equivalently have to show that

$$h_2^4 |\zeta_1|^2 + h_1^4 |\zeta_2|^2 + 2R \operatorname{Re}\left(\zeta_1 \overline{\zeta_2}\right) \left(h_1^2 + h_2^2\right) + R^2 \left(|\zeta_1|^2 + |\zeta_2|^2\right) \geq 0.$$

If $\zeta_1 = 0$ then the latter inequality is simply

$$h_1^4 |\zeta_2|^2 + R^2 |\zeta_2|^2 \geq 0,$$

which is obviously true.

If $\zeta_1 \neq 0$, set $\mu = \zeta_2/\zeta_1 \in \mathbb{C}$; in this case we obtain

$$h_2^4 |\zeta_1|^2 + h_1^4 |\mu|^2 |\zeta_1|^2 + 2R \operatorname{Re}\mu |\zeta_1|^2 \left(h_1^2 + h_2^2\right) + R^2 |\zeta_1|^2 \left(1 + |\mu|^2\right) \geq 0,$$

and we can simplify $|\zeta_1|^2$. Then

$$\begin{aligned} h_2^4 + h_1^4 |\mu|^2 + 2R \operatorname{Re}\mu \left(h_1^2 + h_2^2\right) + R^2 \left(1 + |\mu|^2\right) \\ \geq h_2^4 + h_1^4 (\operatorname{Re}\mu)^2 + 2R \operatorname{Re}\mu \left(h_1^2 + h_2^2\right) + R^2 \left(1 + (\operatorname{Re}\mu)^2\right) \\ = (h_2^2 + R \operatorname{Re}\mu)^2 + (\operatorname{Re}\mu\ h_1^2 + R)^2 \geq 0. \end{aligned}$$

For the second inequality, we only have to show that

$$\frac{h_1^2 h_2^2 - R^2}{h_1^2 + h_2^2} \geq \frac{\rho R}{2h_m^2},$$

that is

$$\left(h_1^2 h_2^2 - R^2\right) (h_1 + h_2)^2 \geq 2 \left(h_1^2 + h_2^2\right) \rho R.$$

Using that $h_1 h_2 > R$ in B_R and the definition of ρ, we obtain

$$2R(h_1 + h_2)^2 \geq 2R \left(h_1^2 + h_2^2\right),$$

which holds since $h_1 h_2 \geq 0$. □

Lemma 4.6.17. *There exists $C_5 > 0$ such that*

$$\sum_{j,k} \psi^{(0)}_{j,\bar{k}} \zeta_j \overline{\zeta_k} \geq \rho^{-1} \left((2h_m^2)^{-1} R - C_5 h_m e^{-2\pi h_m}\right) |\zeta|^2 ;$$

$$\sum_{j,k} \psi^{(1)}_{j,\bar{k}} \zeta_j \overline{\zeta_k} \geq \left(2 - C_5 h_m e^{-2\pi h_m}\right) |\zeta|^2 ;$$

$$\sum_{j,k} \psi^{(2)}_{j,\bar{k}} \zeta_j \overline{\zeta_k} \geq -C_5 e^{-2\pi h_m} |\zeta|^2 .$$

Proof. We want to apply Lemma 4.6.15 to $\psi^{(0)}$, $\psi^{(1)}$ and $\psi^{(2)}$.

In the first case we have

$$D\tilde{\psi}^0 = \rho^{-1} \begin{pmatrix} -h_2 \\ -h_1 \end{pmatrix}, \qquad D^2\tilde{\psi}^0 = \rho^{-2} \begin{pmatrix} h_2^2 & R \\ R & h_1^2 \end{pmatrix};$$

hence

$$\sum_{j,k} \psi^{(0)}_{j,\bar{k}} \zeta_j \overline{\zeta_k} = \rho^{-2}\big(h_2^2 |\zeta_1|^2 + h_1^2 |\zeta_2|^2 + R(\zeta_1\overline{\zeta_2} + \zeta_2\overline{\zeta_1})\big) - 2\rho^{-1} h_m \sum_{j,k} c_{j,\bar{k}} \zeta_j \overline{\zeta_k}.$$

But then, using (4.27) from Lemma 4.6.16 and (4.23) from Lemma 4.6.14, we get

$$\sum_{j,k} \psi^{(0)}_{j,\bar{k}} \zeta_j \overline{\zeta_k} \geq \rho^{-1} \left(\frac{R}{2h_m^2} - 4h_m C_4 e^{-2\pi h_m} \right) |\zeta|^2 .$$

In the second case we have

$$D\tilde{\psi}^1 = \begin{pmatrix} 2h_1 \\ 2h_2 \end{pmatrix}, \qquad D^2\tilde{\psi}^1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix};$$

so from Lemma 4.6.15 we have

$$\sum_{j,k} \psi^{(1)}_{j,\bar{k}} \zeta_j \overline{\zeta_k} = 2 |\zeta|^2 + 4h_m \sum_{j,k} c_{j,\bar{k}} \zeta_j \overline{\zeta_k}.$$

Thanks to (4.23) from Lemma 4.6.14, we obtain

$$\sum_{j,k} \psi^{(1)}_{j,\bar{k}} \zeta_j \overline{\zeta_k} \geq (2 - 8h_m C_4 e^{-2\pi h_m}) |\zeta|^2 .$$

In the last case we have

$$D\tilde{\psi}^2 = \begin{pmatrix} -\pi \\ -\pi \end{pmatrix}, \qquad D^2\tilde{\psi}^1 = 0;$$

hence from Lemma 4.6.15 we have

$$\sum_{j,k} \psi^{(2)}_{j,\bar{k}} \zeta_j \overline{\zeta_k} = -\pi \sum_{j,k} c_{j,\bar{k}} \zeta_j \overline{\zeta_k} \geq (-2\pi C_4 e^{-2\pi h_m}) |\zeta|^2 ,$$

using (4.23) from Lemma 4.6.14 again, and we are done. □

Lemma 4.6.18. *There exists $R_2 \geq R_1$ such that M_R is a Stein manifold for every $R > R_2$.*

Proof. Set $\psi = \psi^{(0)} + \psi^{(1)} \in C^\infty(M_R)$. We want to show that ψ is an exhaustive plurisubharmonic function for M_R (for R large enough). To show that ψ is plurisubharmonic, thanks to Lemma 4.6.17, we only have to ask for $h_m e^{-2\pi h_m}$ (for $\psi^{(1)}$) and $h_m^3 e^{-2\pi h_m}$ (for $\psi^{(0)}$) to be small enough. So we only have to ask for an R large enough (since $h_m > \sqrt{R}$ in M_R).

Now we want to prove that $\psi^{-1}(-\infty, a]$ is relatively compact in M_R for every $a \in \mathbb{R}$. We have $\psi = -\log\rho + h_1^2 + h_2^2$; we have $h_1 h_2 = \rho + R$, and hence $h_1^2 + h_2^2 \geq 2h_1h_2 = 2(\rho + R)$. Now if we suppose that $\psi(h_1, h_2) \leq a$, then we obtain

$$a \geq \psi(h_1, h_2) = h_1^2 + h_2^2 - \log\rho \geq 2R + 2\rho - \log\rho,$$

and hence ρ has to be bounded. Then $\|(h_1, h_2)\|^2 = h_1^2 + h_2^2 \leq a + \log\rho$ is bounded. $\square$

(**Step** 4)**.** Here we solve the problem $\overline{\partial} u_i = -\overline{\partial} v_i =: f_i$, for $i = 0, 1$, using twice Theorem 4.6.8.

Remark 4.6.19. We could obviously take $u_i = -v_i$, but we want a "small" solution, for which we can control the norm, as in Hörmander's estimates.

We must show that the hypotheses of Theorem 4.6.8 holds for certain θ, χ, while we have fixed f_i, and we showed that $\{\omega_1, \omega_2\}$ is a basis for the holomorphic cotangent space in M_R (for $R > R_1$, see Lemma 4.6.14.(i)). Moreover, $a^i_{j,k} \equiv 0$ and $c^1_{j,\overline{k}} = c^2_{j,\overline{k}} = c_{j,\overline{k}}$.

Thanks to Lemma 4.6.14.(iii), if $R > R_1$ then we can choose $\theta_0 = \theta_1 = C_4 e^{-2\pi h_m}$.

Now we want to apply Hörmander's estimates for the plurisubharmonic weight

$$\chi = \psi^{(0)} + \psi^{(2)}.$$

Lemma 4.6.20. *There exists $R_3 \geq R_2$ such that for $R > R_3$ we have that χ is a plurisubharmonic function on M_R and* (4.19) *holds.*

Proof. From Lemma 4.6.17 we have

$$\sum_{j,k} \chi_{j,\overline{k}} \zeta_j \overline{\zeta_k} \geq \left(\frac{R}{2\rho h_m^2} - C_5 e^{-2\pi h_m} \left(\frac{h_m}{\rho} + 1 \right) \right) |\zeta|^2;$$

in particular, χ is plurisubharmonic for R large enough.

Set $\theta = \frac{R}{3\rho h_m^2}$; we have

$$\frac{R}{2\rho h_m^2} - C_5 e^{-2\pi h_m} \left(\frac{h_m}{\rho} + 1 \right) = \theta + \frac{R}{6\rho h_m^2} - C_5 e^{-2\pi h_m} \left(\frac{h_m}{\rho} + 1 \right).$$

We want to choose R large enough to have

$$\frac{R}{6\rho h_m^2} - C_5 e^{-2\pi h_m}\left(\frac{h_m}{\rho}+1\right) \geq A(\theta_0^2+\theta_1),$$

or equivalently

$$\frac{R}{6} - C_5\rho h_m^2 e^{-2\pi h_m}\left(\frac{h_m}{\rho}+1\right) \geq AC_4\rho h_m^2 e^{-2\pi h_m}\left(1+C_4e^{-2\pi h_m}\right). \tag{4.28}$$

Studying the behavior of both members as h_m tends to ∞, since $p(x)e^{-2\pi x} \to 0$ when $x \to \infty$ for every polynomial p, and remembering that $0 \leq \rho \leq h_m^2 - R$, we have the first member in (4.28) tends to $R/6$ while the second one tends to 0 (whatever is the value of A), and hence (4.28) holds for R large enough. □

Lemma 4.6.21. *There exists $R_4 \geq 0$ such that for $R \geq R_4$ the integral estimate* (4.20) *holds. In particular we have*

$$\int_{M_R} \theta^{-1}\left|f\right|^2 e^{-\chi} dV \leq 1.$$

Proof. Directly from definitions we have that

$$\theta^{-1}e^{-\chi}\left|f_j\right|^2 = \frac{3h_m^2\rho}{R}\rho e^{2\pi h_m}\left|f_j\right|^2.$$

Thanks to Lemma 4.6.10, and arguing as in Lemma 4.6.14, we find that there exists $\tilde{C} > 0$ such that

$$|f_j| \leq \tilde{C}e^{-2\pi h_m}.$$

Then we obtain

$$\theta^{-1}e^{-\chi}\left|f_j\right|^2 \leq \frac{3\tilde{C}^2h_m^2\rho^2}{R}e^{-2\pi h_m},$$

which decays exponentially to 0 when $R \to \infty$. Then for R large enough we have the thesis. □

Corollary 4.6.22. *Set $R_5 = \max\{R_3, R_4\}$. Then for every $R > R_5$ there exists u_j such that $\overline{\partial}u_j = f_j = -\overline{\partial}v_j$, and*

$$\int_{M_R} \left|u_j\right|^2 \rho e^{2\pi h_m} dV \leq 1, \tag{4.29}$$

for $j = 0, 1$.

Proof. We only have to apply Theorem 4.6.8, recalling Lemma 4.6.20 and Lemma 4.6.21 □

(**Step** 5). Now we construct a holomorphic 1-form Ω_2 from Ω_0, using the distorsion u_1, and connecting it with the linear case (where we quotient by the action of F_1, F_2) through the biholomorphism $\tilde{v}$.

For $R > R_5$, we set

$$\begin{aligned} \hat{V}_R &= \left\{ y \in \mathbb{C}^2 \mid 0 \le \mathrm{Re} y_j \le 1, I(y) > 0, I_j(y) > \frac{1+\beta_j}{2} + \sqrt{R} \right\}, \\ \tilde{V}_R &= \left\{ y \in \mathbb{C}^2 \mid 0 \le \mathrm{Re} y_j \le 1, I(y) > 0, I_j(y) > 1+\beta_j + \sqrt{R} \right\}, \\ \hat{U}_R &= v^{-1}(\hat{V}_R). \end{aligned}$$

For $i = 0, 1$, we set $\tilde{v}_i = v_i + u_i \circ \pi_R$, defined on U_R. We obviously have that the $\tilde{v}_i$ are holomorphic in U_R, by construction.

Lemma 4.6.23. *There exists $R_6 > R_5$ such that for every $R > R_6$ and $i = 0, 1$, $j = 1, 2$ we have*

$$|\tilde{v}_i| \le C_6 e^{-\pi h_m}, \qquad \left| \frac{\partial \tilde{v}_i}{\partial x_j} \right| \le C_6 e^{-\pi h_m}, \qquad \text{in } \hat{U}_R,$$

for a suitable constant $C_6 > 0$.

Proof. Let $x^0 \in \hat{U}_R$, and consider the ball

$$\mathbb{B}^2_\delta = \{ \left| x - x^0 \right| < \delta \}.$$

We claim that $\mathbb{B}^2_\delta \subseteq U_R$ for R large enough.

Indeed from Lemma 4.6.10, $|v_0(x)| \le C_2 e^{-2\pi h_m}$ tends to 0 as R tends to ∞, so v is uniformly close to the identity when R is large enough, and the claim follows from definitions of U_R and $\hat{U}_R$.

Moreover, for $x \in \mathbb{B}^2_\delta$ and R large enough, then there exists $\tilde{C}$ (independent of x^0) such that

$$\left| h_m(x) - h_m(x^0) \right| \le \tilde{C}.$$

Since

$$h_j(x) > \frac{1+\beta_j}{2} + \sqrt{R} \text{ for } j = 1, 2,$$

it follows

$$\rho(x) = h_1(x) h_2(x) - R > R + \sqrt{R}(1+\beta) + \frac{(1+\beta_1)(1+\beta_2)}{4} - R > \sqrt{R}.$$

Since $\tilde{v}_i$ is holomorphic, $|\tilde{v}_i|^2$ is plurisubharmoinic and hence the submean inequality holds:

$$\left| \tilde{v}_i(x^0) \right|^2 \le \frac{1}{\mathrm{vol}(\mathbb{B}^2_\delta)} \int_{\mathbb{B}^2_\delta} |\tilde{v}_i(x)|^2 \, dV_0 \le \frac{2}{\mathrm{vol}(\mathbb{B}^2_\delta)} \int_{\mathbb{B}^2_\delta} \left(|v_i(x)|^2 + |u_i(x)|^2 \right) dV_0.$$

From the integral estimate (4.29), also using Lemma 4.6.14, we get

$$1 \geq \int_{M_R} |u_i|^2 \rho e^{2\pi h_m} dV \geq \int_{\mathbb{B}^2_\delta} |u_i|^2 \sqrt{R} e^{2\pi(h_m(x^0)-\tilde{C})} C_3^{-1} dV_0,$$

and hence

$$\frac{2}{\mathrm{vol}(\mathbb{B}^2_\delta)} \int_{\mathbb{B}^2_\delta} |u_i(x)|^2 dV_0 \leq \frac{2C_3}{\mathrm{vol}(\mathbb{B}^2_\delta)\sqrt{R}} e^{2\pi(\tilde{C}-h_m(x^0))}.$$

Moreover, from Lemma 4.6.10, we have

$$|v_i|^2 \leq C_2^2 e^{-4\pi h_m} \leq C_2^2 e^{-2\pi\sqrt{R}} e^{-2\pi h_m} \leq C_2^2 e^{-2\pi\sqrt{R}} e^{2\pi\tilde{C}} e^{-2\pi h_m(x^0)},$$

and hence

$$\frac{2}{\mathrm{vol}(\mathbb{B}^2_\delta)} \int_{\mathbb{B}^2_\delta} |v_i|^2 dV_0 \leq 2C_2^2 e^{-2\pi\sqrt{R}} e^{2\pi\tilde{C}} e^{-2\pi h_m(x^0)}.$$

Putting together all the estimates, we obtain the assertion.

For the derivatives, we use a Cauchy estimate

$$\left| \frac{\partial \tilde{v}_i}{\partial x_j}(x^0) \right| \leq \frac{\max\{|\tilde{v}_i(x)| \mid x \in \mathbb{B}^2_\delta\}}{\delta} \leq \delta^{-1} e^{-\pi(h_m(x^0)-\tilde{C})},$$

and we are done. □

Set:

$$\begin{aligned} \tilde{v}(x) =& \big(x_1, x_2 + \tilde{v}_0(x)\big); \\ \tilde{\Omega}_0 =& e^{\tilde{v}_1(x)}(dx_2 + \beta dx_1); \\ \tilde{U}_R =& \tilde{v}^{-1}(\tilde{V}_R) \cap \hat{U}_R; \\ \widetilde{M}_R =& \pi(\tilde{U}_R). \end{aligned}$$

Lemma 4.6.24. *There exists $R_7 \geq R_6$ such that for every $R > R_7$*

(i) $\tilde{v}|_{\tilde{U}_R}$ is a biholomorphism on $\tilde{V}_R$, and it induces a biholomorphism between $\widetilde{M}_R$ and ${}^{\tilde{V}_R}/_{\mathbb{Z}^2} =: \widetilde{W}_R$;

(ii) $\tilde{\Omega}_0$ induces a holomorphic $(1,0)$-form $\Omega_2 = (\tilde{v}^{-1})^*\tilde{\Omega}_0$ on $\widetilde{W}_R$, and we have

$$\Omega_2 \sim A_1(y)\beta dy_1 + A_2(y)dy_2,$$

with A_j such that

$$|A_j(y) - 1| \leq C_7 e^{-\pi h_m(\tilde{v}^{-1}(y))}. \tag{4.30}$$

Proof.

(i) Thanks to Lemma 4.6.23, we have that $\tilde{v}_i$ is close to the identity as R grows to ∞; arguing as in the proof of Lemma 4.6.13, we have that $\tilde{v}$ is a biholomorphism on $\widetilde{V}_R$ for R large enough. Moreover, it induces a well defined biholomorphism on $\widetilde{M}_R$, since $\tilde{v}_0 = v_0 + \underline{u}_0$, where v_0 induces a map on $\widetilde{M}_R$, while u_0 is already defined as a map in $M_R \supset \widetilde{M}_R$.

(ii) Using the same argument as in point (i), but with $\tilde{v}_1$ instead of $\tilde{v}_0$, we see that Ω_2 is well defined.

If we set $x = \tilde{v}^{-1}(y)$, then

$$d(\tilde{v})_x = \begin{pmatrix} 1 & 0 \\ \frac{\partial \tilde{v}_0}{\partial x_1} & 1 + \frac{\partial \tilde{v}_0}{\partial x_2} \end{pmatrix}_x \implies d(\tilde{v}^{-1})_y = \left(1 + \frac{\partial \tilde{v}_0}{\partial x_2}\right)^{-1} \cdot \begin{pmatrix} 1 + \frac{\partial \tilde{v}_0}{\partial x_2} & 0 \\ -\frac{\partial \tilde{v}_0}{\partial x_1} & 1 \end{pmatrix}_x ,$$

and hence

$$\begin{aligned}
(\Omega_2)_y &= (\tilde{\Omega}_0)_x \circ d((\tilde{v})^{-1})_y \\
&= e^{\tilde{v}_1(x)} \left[\beta dy_1 \left(1 - \beta^{-1} \frac{\frac{\partial \tilde{v}_0}{\partial x_1}}{1 + \frac{\partial \tilde{v}_0}{\partial x_2}} \right) + dy_2 \frac{1}{1 + \frac{\partial \tilde{v}_0}{\partial x_2}} \right] \\
&\sim \beta dy_1 \left(1 - \beta^{-1} \frac{\frac{\partial \tilde{v}_0}{\partial x_1}}{1 + \frac{\partial \tilde{v}_0}{\partial x_2}} \right) + dy_2 \left(1 - \frac{\frac{\partial \tilde{v}_0}{\partial x_2}}{1 + \frac{\partial \tilde{v}_0}{\partial x_2}} \right),
\end{aligned}$$

Applying Lemma 4.6.23, we obtain

$$\begin{aligned}
|A_1 - 1| &= \left| \beta^{-1} \frac{\frac{\partial \tilde{v}_0}{\partial x_1}}{1 + \frac{\partial \tilde{v}_0}{\partial x_2}} \right| \le \frac{C_6 \beta^{-1} e^{-\pi h_m}}{1 - C_6 e^{-\pi h_m}} \le \frac{C_6 e^{-\pi h_m}}{\beta(1 - C_6 e^{-\pi\sqrt{R}})}; \\
|A_2 - 1| &= \left| \frac{\frac{\partial \tilde{v}_0}{\partial x_2}}{1 + \frac{\partial \tilde{v}_0}{\partial x_2}} \right| \le \frac{C_6 e^{-\pi h_m}}{1 - C_6 e^{-\pi h_m}} \le \frac{C_6 e^{-\pi h_m}}{1 - C_6 e^{-\pi\sqrt{R}}},
\end{aligned}$$

because $h_m \ge \sqrt{R}$, and we are done. □

(**Step** 6)**.** We finally define our candidate Ω for the 1-form whose foliation has the prescribed holonomy (for the horizontal complex separatrix).

We simply take Ω_2, project it through E outside the horizontal and vertical complex separatrices, and extend it to the whole neighborhood of the origin. Let us describe the details.

Since the A_j are $\mathbb{Z}^2$-periodic and thanks to (4.30), we can write

$$A_j(y) = 1 + B_j(y),$$

with B_j a suitable bounded $\mathbb{Z}^2$-periodic holomorphic function (for $j = 1, 2$). We also notice that $B_j(y) \to 0$ when $h_m(\tilde{v}^{-1}(y)) \to \infty$ (see (4.30)), or, equivalently, when $I(y) \to \infty$ (since $\tilde{v}$ tends to the identity when R grows to ∞). Set

$$\begin{aligned} D_R &= \{w \in \mathbb{C}^2 \ : \ |w_2|\,|w_1|_j^{\beta} < e^{-2\pi(1+\beta_j+\sqrt{R})}\}, \\ D_R^* &= \{w \in D_R \ : \ w_1 w_2 \neq 0\}, \end{aligned}$$

and denote

$$w = E(y) = (e^{2\pi i y_1}, e^{2\pi i y_2}).$$

Notice that $D_R^* = E(\tilde{V}_R)$, and $E\colon \widetilde{W}_R \to D_R^*$ is a biholomorphism.

If we set $\overline{B}_j = B_j \circ E$ and $\Omega = (E^{-1})^*\Omega_2$, then

$$\begin{aligned} \Omega &\sim \frac{\beta}{2\pi i w_1}(1+\overline{B}_1)dw_1 + \frac{1}{2\pi i w_2}(1+\overline{B}_2)dw_2 \\ &\sim \beta w_2(1+\overline{B}_1)dw_1 + w_1(1+\overline{B}_2)dw_2. \end{aligned}$$

This foliation on D_R^* can be extended to a holomorphic foliation on D_R.

(**Step** 7)**.** Now we only have to check that Ω has holonomy of the horizontal complex separatrix equal to h.

Let $\mathcal{F}_0$ be the foliation defined by $\tilde{\Omega}_0$ on $\tilde{U}_R$, and $\mathcal{F}$ the foliation defined by Ω on D_R. Leaves of $\mathcal{F}_0$ are defined by points with a costant value for $x_2 + \beta x_1$, and $\mathcal{F}$ is the image of $\mathcal{F}_0$ through $E \circ \tilde{v}$.

Set:

$$\begin{aligned} &\Sigma = \{w \in D_R \ : \ w_1 = 1\} \text{ (the vertical section on 1);} \\ &\Sigma^* = \Sigma \setminus \{(1,0)\} = \Sigma \cap D_R^*, \\ &\Sigma_0 = \{x \in U_R \ : \ x_1 = 0\} \text{ (the corresponding vertical section on } U_R\text{);} \\ &\Sigma_1 = \{x \in U_R \ : \ x_1 = 1\} = F_1(\Sigma_0). \end{aligned}$$

The map $E \circ \tilde{v}|_{\Sigma_0}$ is an universal covering of Σ^*, while $z \mapsto e^{2\pi i z}$ is a coordinate on Σ^*.

We have that the images of $(0, x_2)$ and $F_1(0, x_2) = (1, x_2 + \phi(x_2))$ through $E \circ \tilde{v}$ coincide; but $(1, x_2 + \phi(x_2))$ and $(0, x_2 + \beta + \phi(x_2)) = (0, H(x_2))$ belong to the same leaf of $\mathcal{F}_0$.

It follows that the holonomy with respect to Σ for $\mathcal{F}$ along $[0,1] \ni t \mapsto (e^{-2\pi i t}, 0)$ on the horizontal complex separatrix is h^{-1}, and hence h is the holonomy of the horizontal complex separatrix for $\mathcal{F}$.

$\square$

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