Torus actions in the normalization problem

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Abstract. Let f be a germ of biholomorphism of $\mathbb{C}^n$, fixing the origin. We show that if the germ commutes with a torus action, then we get information on the germs that can be conjugated to f, and furthermore on the existence of a holomorphic linearization or of a holomorphic normalization of f. We find out in a complete and computable manner what kind of structure a torus action must have in order to get a Poincaré-Dulac holomorphic normalization, studying the possible torsion phenomena. In particular, we link the eigenvalues of $\mathrm{d}f_O$ to the weight matrix of the action. The link and the structure we found are more complicated than what one would expect; a detailed study was needed to completely understand the relations between torus actions, holomorphic Poincaré-Dulac normalizations, and torsion phenomena. We end the article giving an example of techniques that can be used to construct torus actions.

1. Introduction

We consider a germ of biholomorphism f of $\mathbb{C}^n$ at a fixed point p, which we may place at the origin O. One of the main questions in the study of local holomorphic dynamics (see [A1], [A2], and [Bra] for general surveys on this topic) is when f is *holomorphically linearizable*, i.e., when there exists a local holomorphic change of coordinates such that f is conjugated to its linear part. The answer to this question depends on the set of eigenvalues of $\mathrm{d}f_O$, usually called the *spectrum* of $\mathrm{d}f_O$. In fact if we denote by $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ the eigenvalues of $\mathrm{d}f_O$, then it may happen that there exists a multi-index $K = (k_1, \ldots, k_n) \in \mathbb{N}^n$ with $|K| = k_1 + \cdots + k_n \geq 2$ and such that

$$\lambda^K - \lambda_j := \lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j = 0 \tag{1}$$

for some $1 \leq j \leq n$; a relation of this kind is called a *multiplicative resonance* of f, and K is called a *resonant multi-index*. A *resonant monomial* is a monomial $z^K = z_1^{k_1} \cdots z_n^{k_n}$ in the j-th coordinate such that $\lambda^K = \lambda_j$. From the formal point of view, we have the following classical result (see [Ar] pp. 192–193 for a proof):

Theorem 1.1. *Let f be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ fixing the origin O with no resonances. Then f is formally conjugated to its differential $\mathrm{d}f_O$.*

In presence of resonances, even the formal classification is not easy, as the following result of Poincaré-Dulac, [Po], [D], shows

Theorem 1.2. (Poincaré-Dulac) *Let f be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ fixing the origin O. Then f is formally conjugated to a formal power series $g \in \mathbb{C}[\![z_1, \ldots, z_n]\!]^n$ without constant term such that $\mathrm{d}g_O$ is in Jordan normal form, and g has only resonant monomials.*

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The formal series g is called a *Poincaré-Dulac normal form* of f; a proof of Theorem 1.2 can be found in [Ar] p. 194.

The problem with Poincaré-Dulac normal forms is that they are not unique. In particular, one may wonder whether it could be possible to have such a normal form including finitely many resonant monomials only. This is indeed the case (see, e.g., Reich [Re]) when $\mathrm{d}f_O$ belongs to the so-called *Poincaré domain*, that is when $\mathrm{d}f_O$ is invertible and O is either *attracting*, i.e., all the eigenvalues of $\mathrm{d}f_O$ have modulus less than 1, or *repelling*, i.e., all the eigenvalues of $\mathrm{d}f_O$ have modulus greater than 1 (when $\mathrm{d}f_O$ is still invertible but does not belong to the Poincaré domain, we shall say that it belongs to the *Siegel domain*).

Even without resonances, the holomorphic linearization is not guaranteed. The easiest positive result is due to Poincaré [Po] who, using majorant series, proved the following

Theorem 1.3. (Poincaré, 1893 [Po]) *Let f be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ with an attracting or repelling fixed point. Then f is holomorphically linearizable if and only if it is formally linearizable. In particular, if there are no resonances then f is holomorphically linearizable.*

When O is not attracting or repelling, even without resonances, the formal linearization might diverge. In [Ra2] we found, under certain arithmetic conditions on the eigenvalues and some restrictions on the resonances, a necessary and sufficient condition for holomorphic linearization in presence of resonances, that in fact has as corollaries most of the known linearization results. In [Ra3] we found that, given $m \geq 2$ germs $f_1, \ldots, f_m$ of biholomorphisms of $\mathbb{C}^n$, fixing the origin, with $(\mathrm{d}f_1)_O$ diagonalizable and such that f_1 commutes with f_h for any $h = 2, \ldots, m$, under certain arithmetic conditions on the eigenvalues of $(\mathrm{d}f_1)_O$ and some restrictions on their resonances, $f_1, \ldots, f_m$ are simultaneously holomorphically linearizable if and only if there exists a particular complex manifold invariant under $f_1, \ldots, f_m$.

In any cases, there are germs not holomorphically linearizable, for instance when $\mathrm{d}f_O$ is not diagonalizable (see also [PM] for related results):

Theorem 1.4. (Yoccoz, 1995 [Y]) *Let $A \in \mathrm{GL}(n, \mathbb{C})$ be an invertible matrix such that its eigenvalues have no resonances and such that its Jordan normal form contains a non-trivial block associated to an eigenvalue of modulus one. Then there exists a germ of biholomorphism f of $\mathbb{C}^n$ fixing the origin, with $\mathrm{d}f_O = A$ which is not holomorphically linearizable.*

Then, since every germ of biholomorphism is formally normalizable, studying the *holomorphic normalization problem*, i.e., when there exists a local holomorphic change of coordinates such that f is conjugated to one of its Poincaré-Dulac normal forms, could be very useful to understand the dynamics of non-linearizable germs.

In [Zu], Zung found that to find a Poincaré-Dulac holomorphic normalization for a germ of holomorphic vector field is the same as to find (and linearize) a suitable torus action which preserves the vector field. Following this idea, we found that commuting with a linearizable germ gives us information on the germs conjugated to a given one, and also on the linearization. More precisely we have the following results (for the definition of weight matrix see section 2).

Theorem 1.5. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Then f commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $1 \leq r \leq n$ with weight matrix $\Theta \in M_{n \times r}(\mathbb{Z})$ if and only there exists a local holomorphic change of coordinates conjugating f to a germ with linear part in Jordan normal form and containing only Θ-resonant monomials.*

Theorem 1.6. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Then f is holomorphically linearizable if and only if it commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $1 \leq r \leq n$ with weight matrix Θ having no resonances.*

We want to face and to solve the following problem: to find out in a clear (and possibly computable) manner what kind of structure a torus action must have in order to get a Poincaré-Dulac holomorphic normalization from Theorem 1.5. In particular, to do so we need to link in a clever way the eigenvalues of $\mathrm{d}f_O$ to the weight matrix of the action. Zung dealt with this problem in the case of holomorphic vector fields (see [Zu]), introducing the notion of *toric degree* of a vector field. The following definition is a reformulation of Zung's original one, clearer and more suitable to our needs.

Definition 1.1. The *toric degree* of a germ of holomorphic vector field X of $(\mathbb{C}^n, O)$ singular at the origin is the minimum $r \in \mathbb{N}$ such that $X^{\mathrm{dia}} = \sum_{j=1}^n \varphi_j z_j \partial_j$, the diagonalized semi-simple part of the first jet of X, can be written as linear combination with complex coefficients of r diagonal vector fields with integer coefficients, i.e.,

$$X^{\mathrm{dia}} = \sum_{k=1}^{r} \alpha_k Z_k,$$

where $\alpha_1, \dots, \alpha_r \in \mathbb{C}^*$ and $Z_k = \sum_{j=1}^n \rho_j^{(k)} z_j \partial_j$ with $\rho^{(k)} \in \mathbb{Z}^n$. The r-tuple $Z_1, \dots, Z_k$ is called a *r-tuple of toric vector fields associated to* X, and the numbers $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ are a r-tuple of *toric coefficients* of the toric r-tuple.

Then he found that

Theorem 1.7. (Zung, 2002 [Zu]) *Let X be a germ of holomorphic vector field X of $(\mathbb{C}^n, O)$ singular at the origin, of toric degree $1 \leq r \leq n$. Then X admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r preserving X and such that the columns of the weight matrix of the action are a r-tuple of toric vectors associated to X.*

It is a common thinking that once something can be done with germs of vector fields, i.e., for continuous local dynamical systems, then it can be translated analogously for germs of biholomorphisms, i.e., for discrete local dynamical systems. This is not completely true. At the very least there are torsion phenomena to be considered, preventing a straightforward translation from additive resonances (see below for the definition) to multiplicative resonances, and giving rise to new behaviors. One of our aims is exactly to understand up to which point one can push the analogies between continuous and discrete dynamics in the normalization problem. Following Écalle [É], we shall use the following definition of torsion.

Definition 1.2. Let $\lambda \in (\mathbb{C}^*)^n$. The *torsion* of λ is the natural integer τ such that

$$\frac{1}{\tau} 2\pi i \mathbb{Z} = (2\pi i \mathbb{Q}) \cap \left((2\pi i \mathbb{Z}) \bigoplus_{1 \leq j \leq n} (\log(\lambda_j)\mathbb{Z}) \right).$$

To understand what kind of structure a torus action must have in the case of germs of biholomorphisms to get a result equivalent to Theorem 1.7, we first need a right notion of toric degree for germs of biholomorphisms, and to link it to the torsion we introduced above. The link and the structure we found are more complicated than what one would expect: torsion is not enough to measure the difference between germs of holomorphic vector fields and germs of biholomorphisms. We therefore need a more detailed study.

Notice that given $\lambda \in (\mathbb{C}^*)^n$, there is a unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i [\varphi]}$, i.e., $\lambda_j = e^{2\pi i [\varphi_j]}$ for every $j = 1, \dots, n$. The right definition of toric degree for maps is then the following

Definition 1.3. Let $[\varphi] = ([\varphi_1], \dots, [\varphi_n]) \in (\mathbb{C}/\mathbb{Z})^n$, where $[\,\cdot\,]\colon \mathbb{C}^n \to (\mathbb{C}/\mathbb{Z})^n$ denotes the standard projection. The *toric degree of* $[\varphi]$ is the minimum $r \in \mathbb{N}$ such that there exist $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ and $\theta^{(1)}, \dots, \theta^{(r)} \in \mathbb{Z}^n$ such that

$$[\varphi] = \left[\sum_{k=1}^{r} \alpha_k \theta^{(k)} \right]. \tag{2}$$

The r-tuple $\theta^{(1)}, \dots, \theta^{(r)}$ is called a *r-tuple of toric vectors associated to* $[\varphi]$, and the numbers $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ are *toric coefficients* of the toric r-tuple.

Definition 1.4. Given $\theta \in \mathbb{C}^n$ and $j \in \{1, \dots, n\}$, we say that a multi-index $Q \in \mathbb{N}^n$, with $|Q| = \sum_{h=1}^{n} q_h \geq 2$, gives an *additive resonance relation for θ relative to the j-th coordinate* if

$$\langle Q, \theta \rangle = \sum_{h=1}^{n} q_h \theta_h = \theta_j$$

and we put

$$\mathrm{Res}_j^+(\theta) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \langle Q, \theta \rangle = \theta_j\}.$$

Given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$, the set

$$\mathrm{Res}_j([\varphi]) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, [\langle Q, \varphi \rangle - \varphi_j] = [0]\}$$

of multiplicative resonances of $[\varphi]$ is well-defined and we have $\mathrm{Res}_j(\lambda) = \mathrm{Res}_j([\varphi])$, where $\lambda = e^{2\pi i [\varphi]}$.

We shall find relations between the additive resonances of toric vectors associated to $[\varphi]$ and the multiplicative resonances of $[\varphi]$. One of the advantages of the approach we found is that we shall be able to easily compute the multiplicative resonances, passing through the additive resonances of r-tuples of toric vectors (see Lemma 7.1).

Given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ of toric degree $1 \leq r \leq n$, even when the r-tuple of toric vectors associated to $[\varphi]$ is not unique, we can always say whether the toric coefficients are rationally independent with 1 or not.

Definition 1.5. Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$. We say that $[\varphi]$ is in the *torsion-free case*, or simply $[\varphi]$ *is torsion-free*, if its r-tuples of toric vectors have toric coefficients rationally independent with 1.

As a first application of our methods, we have the following characterization of the vectors $\lambda \in (\mathbb{C}^*)^n$ without torsion

Theorem 1.8. *Let $\lambda = e^{2\pi i [\varphi]} \in (\mathbb{C}^*)^n$. Then $[\varphi]$ is torsion-free if and only if the torsion of λ is* 1.

In the torsion case, we can always find a more useful toric r-tuple.

Definition 1.6. Let $[\varphi] = ([\varphi_1], \dots, [\varphi_n]) \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$ in the torsion case. We say that a r-tuple $\eta^{(1)}, \dots, \eta^{(r)}$ of toric vectors associated to $[\varphi]$ with toric coefficients $\beta_1, \dots, \beta_r$ rationally dependent with 1 is *reduced* if $\beta_1 = 1/m$ with $m \in \mathbb{N} \setminus \{0, 1\}$ and $m, \eta_1^{(1)}, \dots, \eta_n^{(1)}$ coprime. In this case the toric vectors $\eta^{(2)}, \dots, \eta^{(r)}$ are called *reduced torsion-free toric vectors* associated to $[\varphi]$.

We have explicit (and easy to use) techniques to compute the toric degree and toric r-tuples (reduced in the torsion case) of $[\varphi]$. Furthermore, we can also prove that, in the torsion case, the torsion of $e^{2\pi i [\varphi]}$ always divides m (see Proposition 5.1).

As expected, we are able to show that the torsion-free case behaves as the vector fields case, proving the following analogue of Theorem 1.7 (which works even when $\mathrm{d}f_O$ is not diagonalizable).

Theorem 1.9. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Assume that, denoted by $\lambda = \{\lambda_1, \dots, \lambda_n\}$ the spectrum of $\mathrm{d}f_O$, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i [\varphi]}$ is of toric degree $1 \le r \le n$ and torsion-free. Then f admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with f and such that the columns of the weight matrix of the action are a r-tuple of toric vectors associated to $[\varphi]$.*

The torsion case is more delicate and difficult to deal. First of all, given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ with toric degree $1 \le r \le n$ and torsion $\tau \ge 2$, and a reduced toric r-tuple $\eta^{(1)}, \dots, \eta^{(r)}$, we always have

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) \supseteq \mathrm{Res}_j([\varphi]) \supseteq \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\eta^{(k)}). \tag{3}$$

This suggest a subdivision in several subcases, all realizable (we have examples for all of them) and, surprisingly, having very different behaviours. We have cases similar to the case of germs of vector fields (even if we have torsion!), and cases that are indeed different. In particular, considering iterates of f to reduce to the torsion-free case hides very interesting phenomena, and it does not allow to see that some torsion cases can be directly studied. Moreover, we have explicit (and computable) techniques to decide in which subcase a given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ belongs to.

Definition 1.7. Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$ and in the torsion case. We say that $[\varphi]$ is in the *impure torsion case* if, for one (and hence any: see Lemma 7.6) reduced r-tuple $\eta^{(1)}, \dots, \eta^{(r)}$ of toric vectors associated to $[\varphi]$ we have

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}), \tag{4}$$

for all $j \in \{1, \dots, n\}$. Otherwise we say that $[\varphi]$ is in the *pure torsion case.*

The impure torsion case is the subcase behaving as the case of germs of vector fields, and in which, again, we do not need $\mathrm{d}f_O$ diagonalizable. In fact, we can prove the following

Theorem 1.10. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Assume that, denoted by $\lambda = \{\lambda_1, \dots, \lambda_n\}$ the spectrum of $\mathrm{d}f_O$, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i [\varphi]}$ is of toric degree $1 \le r \le n$ and in the impure torsion case. Then it admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $r-1$ commuting with f, and such that the columns of the weight matrix of the action are reduced torsion-free toric vectors associated to $[\varphi]$.*

The next subcase is

Definition 1.8. Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$ and in the pure torsion case. We say that $[\varphi]$ *can be simplified* if it admits a reduced r-tuple of toric vectors $\eta^{(1)}, \dots, \eta^{(r)}$ such that

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\eta^{(k)}), \tag{5}$$

for all $j = 1, \dots, n$. The r-tuple $\eta^{(1)}, \dots, \eta^{(r)}$ is said a *simple reduced* r-tuple associated to $[\varphi]$.

As we shall see in section 7, condition (5) depends on the chosen toric r-tuple. However, we have techniques to decide whether $[\varphi]$ can be simplified or not.

The case in which $[\varphi]$ can be simplified is similar to the case of germs of vector fields, but we have a distinction between the case of $\mathrm{d}f_O$ diagonalizable and $\mathrm{d}f_O$ not diagonalizable, as we see in the following result (for the definition of compatibility see section 3):

Theorem 1.11. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Assume that, denoted by $\lambda = \{\lambda_1, \dots, \lambda_n\}$ the spectrum of $\mathrm{d}f_O$, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$ is of toric degree $1 \leq r \leq n$ and in the pure torsion case and it can be simplified. Then:*

- *(i) if $\mathrm{d}f_O$ is diagonalizable, f admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with f and such that the columns of the weight matrix Θ of the action are a simple reduced r-tuple of toric vectors associated to $[\varphi]$;*
- *(ii) if $\mathrm{d}f_O$ is not diagonalizable and there exists a simple reduced r-tuple of toric vectors associated to $[\varphi]$ such that its vectors are the columns of a matrix Θ compatible with $\mathrm{d}f_O$, f admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with f and with weight matrix Θ.*

The case in which $[\varphi]$ cannot be simplified is the furthest from the case of germs of vectors fields, because we cannot reduce the multiplicative resonances to additive ones. In fact, we only have a sufficient condition for holomorphic normalization.

Proposition 1.12. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Assume that, denoted by $\lambda = \{\lambda_1, \dots, \lambda_n\}$ the spectrum of $\mathrm{d}f_O$, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$ is of toric degree $1 \leq r \leq n$ and in the pure torsion case and it cannot be simplified. If there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with f and such that the columns of the weight matrix of the action are a reduced r-tuple of toric vectors associated to $[\varphi]$, then f admits a holomorphic Poincaré-Dulac normalization.*

We have then completely understood the relations between torus actions, holomorphic Poincaré-Dulac normalizations, and torsion phenomena. We end the article giving an example of techniques to construct torus actions.

Definition 1.9. Let $1 \leq m \leq n$. A *set of m integrable vector fields* of $(\mathbb{C}^n, O)$ is a set $X_1, \dots, X_m$ of germs of holomorphic vector fields of $(\mathbb{C}^n, O)$ singular at the origin, of order 1 and such that:

- (i) $X_1, \dots, X_m$ commute pairwise and are linearly independent;
- (ii) there exist $n - m$ germs of holomorphic functions $g_1, \dots, g_{n-m}$ in $(\mathbb{C}^n, O)$ which are common first integrals of $X_1, \dots, X_m$, and they are functionally independent almost everywhere.

Definition 1.10. A germ of biholomorphism f of $(\mathbb{C}^n, O)$ fixing the origin *commutes with a set of integrable vector fields* if there exists a positive integer $1 \leq m \leq n$, such that there exists a set of m germs of holomorphic integrable vector fields $X_1, \dots, X_m$ such that

$$\mathrm{d}f(X_j) = X_j \circ f$$

for each $j = 1, \dots, m$.

A germ of biholomorphism f of $(\mathbb{C}^n, O)$ commutes with a vector field X according to the previous definition if and only if it commutes with the flow generated by X. Then (see also Section 8 for more general statements):

Theorem 1.13. *Let f be a germ of biholomorphism of $(\mathbb{C}^n, O)$ fixing the origin and commuting with a set of integrable holomorphic vector fields $X_1, \ldots, X_m$. Then f commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension equal to the toric degree r of X_1 and such that the columns of the weight matrix of the action are a r-tuple of toric vectors associated to X_1.*

If the semi-simple part of the linear term of X_1 has the same resonances of the eigenvalues of $\mathrm{d}f_O$, then we get a Poincaré-Dulac holomorphic normalization. Moreover, the commutation condition implies that f preserves the foliation generated by $X_1, \ldots, X_m$; hence this is a condition similar to condition A of [Brj] for the convergence of a normalization in the case of vector fields. Finally, commuting with a torus action implies that f preserves the orbits of the action, hence a condition of the kind of Definition 1.10 is close to be necessary and sufficient for the existence of a commuting torus action.

The structure of this paper is as follows.

In section 2 we shall recall some basic facts about linear torus actions and we shall fix some notations. In section 3 we shall describe the relations between the existence of torus actions with certain properties and the possibility to conjugate a given germ of biholomorphism to another one of a particular form, and we shall prove Theorem 1.5 and Theorem 1.6. In section 4 we shall study the toric degree and the toric r-tuples associated to the eigenvalues of the differential $\mathrm{d}f_O$ of a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin, and the weight matrices of torus actions. In section 5 we shall study the notion of torsion and we shall prove Theorem 1.8. In section 6 we shall study the relations between the resonances of the eigenvalues of the differential $\mathrm{d}f_O$ of a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and the additive resonances of an associated toric r-tuple, in the torsion-free case, and we shall prove Theorem 1.9. In section 7 we shall study the relations between the resonances of the eigenvalues of the differential $\mathrm{d}f_O$ of a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and the additive resonances of an associated toric r-tuple, in the torsion case, and we shall prove Theorem 1.10, Theorem 1.8 and Proposition 1.12. In the last section we shall give some geometric conditions to construct the torus actions we need, and we shall prove Theorem 1.13 and other similar results.

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2. Preliminaries

Let $A\colon \mathbb{T}^r \times M \to M$ be a torus action on a complex manifold M, with a fixed point $p_0 \in M$ (that is $A(x, p_0) = A_x(p_0) = p_0$ for all $x \in \mathbb{T}^r$). The differential $\mathrm{d}(A_x)_{p_0}\colon T_{p_0}M \to T_{p_0}M$ is then well-defined, and thus we have a linear torus action on $T_{p_0}M$. A linear torus action can be thought of as a Lie group homomorphism $A\colon \mathbb{T}^r \to \mathrm{Aut}(T_{p_0}M)$, that is as a representation of $\mathbb{T}^r$ on the vector space $V = T_{p_0}M$.

Characters and weights of $\mathbb{T}^r$ are well known. All characters of $\mathbb{T}^1 = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ are of the form

$$\chi_\theta(x) = \exp(2\pi i x\theta)$$

with $\theta \in \mathbb{Z}$; hence the character group of $\mathbb{T}^1$ is isomorphic to $\mathbb{Z}$. Since $\mathbb{T}^r = \mathbb{T}^1 \times \cdots \times \mathbb{T}^1$, the characters of $\mathbb{T}^r$ are obtained multiplying characters of $\mathbb{T}^1$, that is they are of the form

$$\chi_\theta(x) = \exp\left(2\pi i \sum_{k=1}^{r} x_k \theta^k\right),$$

with $\theta = (\theta^1, \dots, \theta^r) \in (\mathbb{Z}^r)^*$, where the * denotes the dual. In particular, θ should be thought of as a row vector. The weights of $\mathbb{T}^r$ are then the differential of the characters computed at the identity element, and thus are given by

$$w_\theta(v) = 2\pi i \sum_{k=1}^{r} v_k \theta^k$$

with $\theta \in (\mathbb{Z}^r)^*$ and $v \in \mathbb{R}^r$. If we write $\theta_j = (\theta_j^1, \dots, \theta_j^r) \in (\mathbb{Z}^r)^*$, then the matrix representation of the linear action A in the eigenvector basis is given by

$$A(x) = \operatorname{diag}\left(\chi_{\theta_j}(x)\right) = \operatorname{diag}\left(\exp\left(2\pi i \sum_{k=1}^{r} x_k \theta_j^k\right)\right).$$

We have then associated to our torus action a matrix $\Theta = (\theta_j^k) \in M_{n\times r}(\mathbb{Z})$, whose columns do not depend on the particular coordinates chosen to express the torus action, but can be uniquely (up to order) recovered by the action itself.

Definition 2.1. The matrix Θ just defined is called the *weight matrix* of the torus action.

Definition 2.2. Let $\theta \in \mathbb{C}^n$ and let $j \in \{1, \dots, n\}$. We say that a multi-index $Q \in \mathbb{N}^n$, with $|Q| = \sum_{h=1}^{n} q_h \geq 2$, gives an *additive resonance relation for θ relative to the j-th coordinate* if

$$\langle Q, \theta \rangle = \sum_{h=1}^{n} q_h \theta_h = \theta_j$$

and we put

$$\operatorname{Res}_j^+(\theta) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \langle Q, \theta \rangle = \theta_j\}.$$

Let $\lambda \in (\mathbb{C}^*)^n$ and let $j \in \{1, \dots, n\}$. We say that a multi-index $Q \in \mathbb{N}^n$, with $|Q| \geq 2$, gives a *multiplicative resonance relation for λ relative to the j-th coordinate* if

$$\lambda^Q = \lambda_1^{q_1} \cdots \lambda_n^{q_n} = \lambda_j$$

and we put

$$\operatorname{Res}_j(\lambda) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \lambda^Q = \lambda_j\}.$$

Remark 2.1. Given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$, where $[\,\cdot\,]\colon \mathbb{C}^n \to (\mathbb{C}/\mathbb{Z})^n$ is the standard projection, the set

$$\{Q \in \mathbb{N}^n \mid |Q| \geq 2, \langle Q, \varphi \rangle - \varphi_j \in \mathbb{Z}\}.$$

does not depend on the specific representative $\varphi \in \mathbb{C}^n$ but only on the class $[\varphi]$, and so it is well defined the set $\operatorname{Res}_j([\varphi])$ as

$$\operatorname{Res}_j([\varphi]) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, [\langle Q, \varphi \rangle - \varphi_j] = [0]\}.$$

Remark 2.2. Notice that given $\lambda \in (\mathbb{C}^*)^n$, we can always find a unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$, i.e., $\lambda_j = e^{2\pi i[\varphi_j]}$ for every $j = 1, \ldots, n$. Then $\mathrm{Res}_j(\lambda) = \mathrm{Res}_j([\varphi])$, thus justifying the definitions and the terminology.

3. Torus Actions and Normal Forms of germs of biholomorphisms

In this section we shall describe the relations between the existence of torus actions with certain properties and the possibility to conjugate a given germ of biholomorphism to another of a particular form.

Definition 3.1. Let $\theta^{(1)}, \ldots, \theta^{(r)} \in \mathbb{Z}^n$. We say that a monomial $z^Q e_j$, with $Q \in \mathbb{N}^n$, $|Q| \geq 1$ and $j \in \{1, \ldots, n\}$, is Θ-*resonant*, where Θ is the $n \times r$ matrix whose columns are $\theta^{(1)}, \ldots, \theta^{(r)}$, if

$$\langle Q, \theta^{(k)} \rangle = \theta_j^{(k)}$$

for every $k = 1, \ldots, r$. In other words, $z_h e_j$ is Θ-resonant if $\theta_h^{(k)} = \theta_j^{(k)}$, for all $k = 1, \ldots, r$, and $z^Q e_j$, with $|Q| \geq 2$ is Θ-resonant, if

$$Q \in \mathcal{R}_j(\Theta) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\theta^{(k)}). \tag{6}$$

We say that Θ has *no resonances* if $\mathcal{R}_j(\Theta) = \varnothing$ for every $j = 1, \ldots, n$.

Definition 3.2. Let $\theta^{(1)}, \ldots, \theta^{(r)} \in \mathbb{Z}^n$ and let T be a linear map of $\mathbb{C}^n$. We say that the matrix Θ, with columns $\theta^{(1)}, \ldots, \theta^{(r)}$, is *compatible with* T if and only if we can write T in Jordan form with all monomials Θ-resonant. In other words, a matrix $T = (t_{ij})$ in Jordan form is compatible with Θ if and only if $\theta_j^{(k)} = \theta_{j+1}^{(k)}$ for all $k = 1, \ldots, r$ when $t_{j,j+1} \neq 0$, that is in a Jordan block of dimension at least 2.

Theorem 3.1. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Then f commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $1 \leq r \leq n$ with weight matrix $\Theta \in M_{n\times r}(\mathbb{Z})$ if and only there exists a local holomorphic change of coordinates conjugating f to a germ with linear part in Jordan normal form and containing only Θ-resonant monomials.*

Proof. Let us suppose that the linear part of f is in Jordan normal form and f contains only Θ-resonant monomials. Then we claim that f commutes with the linear effective torus action

$$A\colon \mathbb{T}^r \times (\mathbb{C}^n, O) \to (\mathbb{C}^n, O),$$

defined by

$$A(x, z) = \mathrm{Diag}\left(e^{2\pi i \sum_{k=1}^{r} x_k \theta_j^{(k)}}\right) z.$$

In fact in these hypotheses the j-th coordinate of f is

$$\lambda_j z_j + \varepsilon_j z_{j-1} + \sum_{\substack{|Q|\geq 2\\ Q\in\mathcal{R}_j(\Theta)}} f_{Q,j} z^Q$$

where $\varepsilon_j \in \{0, 1\}$ can be different from 0 only if $\lambda_j = \lambda_{j-1}$, the set $\mathcal{R}_j(\Theta)$ is defined in (6) and the assumption that $\varepsilon_j z_{j-1} e_j$ is Θ-resonant implies $\theta_{j-1}^{(k)} = \theta_j^{(k)}$ for $k = 1, \ldots, r$ if $\varepsilon_j \neq 0$. Then for every $x \in \mathbb{T}^r$ we have

$$
\begin{aligned}
f_j(A(x,z)) &= \lambda_j e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} z_j + \varepsilon_j e^{2\pi i \sum_{k=1}^r x_k \theta_{j-1}^{(k)}} z_{j-1} + \sum_{\substack{|Q|\geq 2\\ Q\in\mathcal{R}_j(\Theta)}} f_{Q,j} e^{2\pi i \sum_{k=1}^r x_k \langle Q, \theta^{(k)}\rangle} z^Q \\
&= \lambda_j e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} z_j + \varepsilon_j e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} z_{j-1} + \sum_{\substack{|Q|\geq 2\\ Q\in\mathcal{R}_j(\Theta)}} f_{Q,j} e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} z^Q \\
&= e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} \left(\lambda_j z_j + \varepsilon_j z_{j-1} + \sum_{\substack{|Q|\geq 2\\ Q\in\mathcal{R}_j(\Theta)}} f_{Q,j} z^Q \right) \\
&= e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} (f_j(z)) \\
&= A(x, f(z))_j.
\end{aligned}
$$

Conversely, let us suppose that f commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $1 \leq r \leq n$ with weight matrix Θ. Then, by Böchner linearization theorem [B], there exists a tangent to the identity holomorphic change of variables ψ linearizing the torus action. Furthermore, up to a linear change of coordinates we can assume that in the new coordinates the action is given by

$$
A(x,z) = \mathrm{Diag}\left(e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}}\right) z,
$$

and that f (still commuting with the torus action) has linear part in Jordan normal form compatible with Θ, and thus its j-th coordinate is

$$
\lambda_j z_j + \varepsilon_j z_{j-1} + \sum_{|Q|\geq 2} f_{Q,j} z^Q
$$

where $\varepsilon_j \in \{0,1\}$ can be different from 0 only if $\lambda_{j-1} = \lambda_j$ and $\theta_{j-1} = \theta_j$. For every $x \in \mathbb{T}^r$, we have

$$
f_j(A(x,z)) = \lambda_j e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} z_j + \varepsilon_j e^{2\pi i \sum_{k=1}^r x_k \theta_{j-1}^{(k)}} z_{j-1} + \sum_{|Q|\geq 2} f_{Q,j} e^{2\pi i \sum_{k=1}^r x_k \langle Q, \theta^{(k)}\rangle} z^Q,
$$

and

$$
A(x, f(z))_j = e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}} \left(\lambda_j z_j + \varepsilon_j z_{j-1} + \sum_{|Q|\geq 2} f_{Q,j} z^Q \right).
$$

Then $f_j(A(x,z)) = A(x, f(z))_j$ if and only if

$$
f_{Q,j}\left(e^{2\pi i \sum_{k=1}^r x_k \langle Q, \theta^{(k)}\rangle} - e^{2\pi i \sum_{k=1}^r x_k \theta_j^{(k)}}\right) = 0
$$

for every $x \in \mathbb{T}^r$, $j = 1, \cdots, n$, $Q \in \mathbb{N}^n$ with $|Q| \geq 2$, i.e., $f_{Q,j}$ can be non-zero only when

$$
\sum_{k=1}^r x_k (\langle Q, \theta^{(k)}\rangle - \theta_j^{(k)}) \in \mathbb{Z} \quad \forall x \in \mathbb{T}^r,
$$

which is equivalent to

$$
\langle Q, \theta^{(k)}\rangle - \theta_j^{(k)} = 0
$$

for every $k = 1, \ldots, r$, meaning that f contains only Θ-resonant monomials. □

As a consequence of this result we have

Corollary 3.2. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O. Then f is holomorphically linearizable if and only if it commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $1 \le r \le n$ with weight matrix Θ having no resonances.*

Proof. If f is linear and in Jordan normal form, then it commutes with any linear action of $\mathbb{T}^1$ with compatible weight matrix Θ; so it suffices to choose Θ with $\mathcal{R}_1(\Theta) = \ldots = \mathcal{R}_n(\Theta) = \varnothing$.

Conversely, if f commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $1 \le r \le n$ with weight matrix Θ, then, by the previous result, Θ is compatible with the linear part of f and there exists a local holomorphic change of coordinates such that f is conjugated to a germ with the same linear part and containing only Θ-resonant monomials. But each $\mathcal{R}_j(\Theta)$ is empty; hence there are no Θ-resonant monomials of degree at least 2, and thus f is holomorphically linearizable. □

4. Toric degree

We want to study the relations between the resonances of the eigenvalues of the differential $\mathrm{d}f_O$ of a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin, and the weight matrices of torus actions to understand in which cases Theorem 3.1 gives us a Poincaré-Dulac holomorphic normalization. Thanks to Remark 2.2 we have to deal with vectors of $(\mathbb{C}/\mathbb{Z})^n$. A concept that turns out to be crucial for this study is that of *toric degree*.

Definition 4.1. Let $[\varphi] = ([\varphi_1], \ldots, [\varphi_n]) \in (\mathbb{C}/\mathbb{Z})^n$. The *toric degree of* $[\varphi]$ is the minimum $r \in \mathbb{N}$ such that there exist $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$ and $\theta^{(1)}, \ldots, \theta^{(r)} \in \mathbb{Z}^n$ such that

$$[\varphi] = \left[\sum_{k=1}^{r} \alpha_k \theta^{(k)}\right]. \tag{7}$$

The r-tuple $\theta^{(1)}, \ldots, \theta^{(r)}$ is called a *r-tuple of toric vectors associated to* $[\varphi]$, and the numbers $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$ are *toric coefficients* of the toric r-tuple.

Remark 4.1. Note that the toric degree is necessarily at most n, since

$$[\varphi] = \left[\sum_{k=1}^{n} \varphi_k e_k\right].$$

We did not say *the* toric coefficients because we have the following result.

Lemma 4.2. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$ and let $\theta^{(1)}, \ldots, \theta^{(r)}$ be a r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$. Then $\beta_1, \ldots, \beta_r \in \mathbb{C}$ satisfy*

$$[\varphi] = \left[\sum_{k=1}^{r} \beta_k \theta^{(k)}\right]$$

if and only if

$$\Theta \begin{pmatrix} \alpha_1 - \beta_1 \\ \vdots \\ \alpha_r - \beta_r \end{pmatrix} \in \mathbb{Z}^n$$

where Θ is the $n \times r$ matrix whose columns are $\theta^{(1)}, \dots, \theta^{(r)}$.

Proof. We have

$$\left[\sum_{k=1}^{r} \alpha_k \theta^{(k)}\right] = \left[\sum_{k=1}^{r} \beta_k \theta^{(k)}\right]$$

if and only if

$$\sum_{k=1}^{r} \alpha_k \theta^{(k)} - \sum_{k=1}^{r} \beta_k \theta^{(k)} \in \mathbb{Z}^n,$$

that is

$$\Theta \begin{pmatrix} \alpha_1 - \beta_1 \\ \vdots \\ \alpha_r - \beta_r \end{pmatrix} \in \mathbb{Z}^n,$$

which is the assertion. □

Thanks to Remark 2.2 the following definition makes sense.

Definition 4.2. Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and denote by $\lambda = \{\lambda_1, \dots, \lambda_n\}$ the spectrum of $\mathrm{d}f_O$. The *toric degree of f* is the toric degree of the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i [\varphi]}$.

Toric r-tuples and toric coefficients have to satisfy certain arithmetic properties, as the following result shows.

Lemma 4.3. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$ and let $\theta^{(1)}, \dots, \theta^{(r)}$ be a r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $\alpha_1, \dots, \alpha_r \in \mathbb{C}$. Then:*

(i) $\alpha_1, \dots, \alpha_r$ is a set of rationally independent complex numbers;

(ii) every r-tuple of toric vectors associated to $[\varphi]$ is a set of $\mathbb{Q}$-linearly independent vectors.

Proof. (i) Let us suppose by contradiction that $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ are rationally dependent. Then there exists $(c_1, \dots, c_r) \in \mathbb{Z}^r \setminus \{O\}$ such that

$$c_1\alpha_1 + \cdots + c_r\alpha_r = 0.$$

Up to reordering we may assume $c_1 \neq 0$. Then

$$\alpha_1 = -\frac{1}{c_1}(c_2\alpha_2 + \cdots + c_r\alpha_r),$$

and hence

$$\begin{aligned}
[\varphi] &= \left[\sum_{k=1}^{r} \alpha_k \theta^{(k)}\right] \\
&= \left[-\frac{1}{c_1}(c_2\alpha_2 + \cdots + c_r\alpha_r)\theta^{(1)} + \alpha_2\theta^{(2)} + \cdots + \alpha_r\theta^{(r)}\right] \\
&= \left[\frac{\alpha_2}{c_1}(c_1\theta^{(2)} - c_2\theta^{(1)}) + \cdots + \frac{\alpha_r}{c_1}(c_1\theta^{(r)} - c_r\theta^{(1)})\right],
\end{aligned}$$

and this contradicts the definition of toric degree.

(ii) The proof is analogous to the previous one. □

Remark 4.4. Given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$, of toric degree $1 \leq r \leq n$, if $\theta^{(1)}, \dots, \theta^{(r)}$ is a r-tuple of toric vectors associated to $[\varphi]$, the $n \times r$ matrix Θ whose columns are $\theta^{(1)}, \dots, \theta^{(r)}$, has maximal rank r.

Remark 4.5. Note that, if $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ has toric degree $1 \leq r \leq n$, and $\theta^{(1)}, \dots, \theta^{(r)}$ is a r-tuple of toric vectors associated to $[\varphi]$, up to change the toric coefficients $\alpha_1, \dots, \alpha_r$, we can always assume $\theta_1^{(k)}, \dots, \theta_n^{(k)}$ coprime for each $1 \leq k \leq r$. In fact, if $d_k \in \mathbb{Z}$ is the greatest common divisor of $\theta_1^{(k)}, \dots, \theta_n^{(k)}$, then

$$[\varphi] = \left[\sum_{k=1}^{r} \alpha_k \theta^{(k)} \right] = \left[\sum_{k=1}^{r} d_k \alpha_k \widetilde{\theta}^{(k)} \right],$$

where

$$\widetilde{\theta}^{(k)} = \begin{pmatrix} \theta_1^{(k)}/d_k \\ \vdots \\ \theta_n^{(k)}/d_k \end{pmatrix}$$

for $k = 1, \dots, r$.

Remark 4.6. *Given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$, of toric degree $1 \leq r \leq n$, the r-tuple of toric vectors associated to $[\varphi]$ is not necessarily unique.* Let us consider, for example

$$[\varphi] = \begin{bmatrix} 3\sqrt{2} + 4i \\ 2\sqrt{2} + 6i \\ -\sqrt{2} + 2i \end{bmatrix}.$$

The toric degree cannot be 1, since it is immediate to verify that φ cannot be written as the product of a complex number times an integer vector. The toric degree is in fact 2, since we have

$$[\varphi] = \left[\sqrt{2} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + 2i \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right].$$

However we can also write $[\varphi]$ as

$$[\varphi] = \left[\frac{-3\sqrt{2} + 16i}{6} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{3\sqrt{2} + 4i}{6} \begin{pmatrix} 6 \\ 5 \\ -1 \end{pmatrix} \right].$$

Note that, in both cases, the toric coefficients are rationally independent with 1.

Example 4.7. The vector of $(\mathbb{C}/\mathbb{Z})^2$

$$[\varphi] = \begin{bmatrix} (1 + 6\sqrt{2})/6 \\ (1 - 2\sqrt{2})/2 \end{bmatrix},$$

has toric degree 2, since we have

$$[\varphi] = \left[\frac{1 + 6\sqrt{2}}{6} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1 - 2\sqrt{2}}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right],$$

and it is not difficult to verify that it cannot have toric degree 1. We can also write $[\varphi]$ as

$$[\varphi] = \left[\frac{1}{6} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \sqrt{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right].$$

Note that this time, in both cases, the toric coefficients are rationally dependent with 1.

We shall prove that, given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ of toric degree $1 \leq r \leq n$, even when the r-tuple of toric vectors associated to $[\varphi]$ is not unique, we can always say whether the toric coefficients are rationally independent with 1 or not, so this will be an intrinsic property of the vector. Before proving this, we shall need the following result that gives us a way to find a more useful toric r-tuple when the toric coefficients are rationally dependent with 1.

Remark 4.8. Note that $\alpha \in \mathbb{C}$ is rationally dependent with 1 if and only if it belongs to $\mathbb{Q}$.

Lemma 4.9. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$, and let $\theta^{(1)}, \ldots, \theta^{(r)}$ be a r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$ rationally dependent with 1. Then there exists a r-tuple of toric vectors $\eta^{(1)}, \ldots, \eta^{(r)}$ associated to $[\varphi]$ with toric coefficients $\beta_1, \ldots, \beta_r \in \mathbb{C}$ such that $\beta_1 = 1/m$ with $m \in \mathbb{N}\backslash\{0,1\}$ and $m, \eta_1^{(1)}, \ldots, \eta_n^{(1)}$ coprime. Moreover $\beta_2, \ldots, \beta_r$ are rationally independent with 1.*

Proof. If $r = 1$, then α is rationally dependent with 1 if and only if it belongs to $\mathbb{Q}$, i.e.,

$$[\varphi] = \left[\frac{p}{q} \theta \right]$$

where we may assume without loss of generality p and q coprime and $q, \theta_1, \ldots, \theta_n$ coprime. Then

$$[\varphi] = \left[\frac{1}{q} \eta \right]$$

where $\eta = p \cdot \theta \in \mathbb{Z}^n$ and we are done.

Let us suppose now $r \geq 2$. Since $\alpha_1, \ldots, \alpha_r$ are (rationally independent and) rationally dependent with 1, we can consider the minimum positive integer $m_0 \in \mathbb{N} \setminus \{0\}$ so that there exists $(m_1, \ldots, m_r) \in \mathbb{Z}^r \setminus \{O\}$ such that

$$m_1 \alpha_1 + \cdots + m_r \alpha_r = m_0.$$

Thanks to the minimality of m_0, we have that $m_1, \ldots, m_r, m_0$ are coprime. Up to reordering we may assume $m_1 \neq 0$. Then

$$\begin{aligned} \alpha_1 &= \frac{m_0}{m_1} - \left(\frac{m_2}{m_1} \alpha_2 + \cdots + \frac{m_r}{m_1} \alpha_r \right) \\ &= \frac{m_0'}{m_1'} - \left(\frac{m_2}{m_1} \alpha_2 + \cdots + \frac{m_r}{m_1} \alpha_r \right), \end{aligned}$$

where $\frac{m_0}{m_1} = \frac{m_0'}{m_1'}$ with $(m_0', m_1') = 1$ and $m_1' \in \mathbb{N} \setminus \{0, 1\}$. Let d be the greatest common divisor of m_1' and the components of $\theta^{(1)}$, and consider

$$\widetilde{\theta}^{(1)} = \frac{1}{d} \theta^{(1)}, \quad \widetilde{m}_1 = \frac{m_1'}{d}.$$

Hence

$$
\begin{aligned}
[\varphi] &= \left[\frac{m_0'}{m_1'} \theta^{(1)} + \sum_{k=2}^{r} \frac{\alpha_k}{m_1} (m_1 \theta^{(k)} - m_k \theta^{(1)}) \right] \\
&= \left[\frac{m_0'}{\widetilde{m}_1} \widetilde{\theta}^{(1)} + \sum_{k=2}^{r} \frac{\alpha_k}{m_1} (m_1 \theta^{(k)} - m_k \theta^{(1)}) \right] \\
&= \left[\frac{1}{\widetilde{m}_1} m_0' \widetilde{\theta}^{(1)} + \sum_{k=2}^{r} \frac{\alpha_k}{m_1} (m_1 \theta^{(k)} - m_k \theta^{(1)}) \right] \\
&= \left[\sum_{k=1}^{r} \beta_k \eta^{(k)} \right],
\end{aligned}
$$

where

$$
\beta_1 = \frac{1}{\widetilde{m}_1}, \beta_2 = \frac{\alpha_2}{m_1}, \ldots, \beta_r = \frac{\alpha_r}{m_1},
$$

and

$$
\eta^{(1)} = m_0' \widetilde{\theta}^{(1)},\ \eta^{(2)} = m_1 \theta^{(2)} - m_2 \theta^{(1)},\ \ldots,\ \eta^{(r)} = m_1 \theta^{(r)} - m_r \theta^{(1)}.
$$

Notice that $\widetilde{m}_1$ is necessarily greater than 1, because otherwise the toric degree of $[\varphi]$ would be less than r.

Now, if $\beta_2, \ldots, \beta_r$ were rationally dependent with 1, then we would have $(k_2, \ldots, k_r) \in \mathbb{Z}^{r-1} \setminus \{O\}$ such that

$$
k_2 \beta_2 + \cdots + k_r \beta_r = k \in \mathbb{Z} \setminus \{0\},
$$

then

$$
-k\, \widetilde{m}_1 \cdot \frac{1}{\widetilde{m}_1} + k_2 \beta_2 + \cdots + k_r \beta_r = 0,
$$

contradicting Lemma 4.3. This concludes the proof. □

Definition 4.3. Let $[\varphi] = ([\varphi_1], \ldots, [\varphi_n]) \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$. We say that a r-tuple $\eta^{(1)}, \ldots, \eta^{(r)}$ of toric vectors associated to $[\varphi]$ with toric coefficients $\beta_1, \ldots, \beta_r$ rationally dependent with 1 is *reduced* if $\beta_1 = 1/m$ with $m \in \mathbb{N} \setminus \{0, 1\}$ and $m, \eta_1^{(1)}, \ldots, \eta_n^{(1)}$ coprime. In this case the toric vectors $\eta^{(2)}, \ldots, \eta^{(r)}$ are called *reduced torsion-free toric vectors* associated to $[\varphi]$.

Now we can prove that the rational independence with 1 of the coefficients of toric r-tuples associated to a given vector $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ of toric degree $1 \le r \le n$ is an intrinsic property of $[\varphi]$.

Proposition 4.10. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$, and let $\theta^{(1)}, \ldots, \theta^{(r)}$ be a r-tuple of toric vectors associated to $[\varphi]$, with toric coefficients $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$ rationally independent with 1. Then every other r-tuple of toric vectors associated to $[\varphi]$ has toric coefficients rationally independent with 1.*

Proof. Let us assume by contradiction that there exists a r-tuple $\eta^{(1)}, \ldots, \eta^{(r)}$ of toric vectors associated to $[\varphi]$ with toric coefficients $\beta_1, \ldots, \beta_r$ rationally dependent with 1. Thanks to Lemma 4.9, we may assume without loss of generality $\beta_1 = 1/m$ with $m \in \mathbb{N} \setminus \{0, 1\}$ and $m, \eta_1^{(1)}, \ldots, \eta_n^{(1)}$ coprime. Let N be the matrix with columns $\eta^{(1)}, \ldots, \eta^{(r)}$, and let Θ be the matrix with columns $\theta^{(1)}, \ldots, \theta^{(r)}$. We have

$$
[\varphi] = \left[N \cdot \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_r \end{pmatrix} \right] = \left[\Theta \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} \right],
$$

that is, there exists an integer vector $\mathbf{k} \in \mathbb{Z}^n$ such that

$$N \cdot \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_r \end{pmatrix} = \Theta \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} + \mathbf{k}.$$

Since N has maximal rank r, the linear map $N \colon \mathbb{Q}^r \to \mathbb{Q}^n$ is injective and, for every $U \subseteq \mathbb{Q}^n$ such that $\mathbb{Q}^n = \mathrm{Im}(N) \oplus U$, there is a linear map $L_U \colon \mathbb{Q}^n \to \mathbb{Q}^r$ such that $\ker(L_U) = U$ and $L_U N = \mathrm{Id}$; hence there is a linear map $\widetilde{L}_U \colon \mathbb{Z}^n \to \mathbb{Z}^r$ that $\widetilde{L}_U N = h \,\mathrm{Id}$, with $h \in \mathbb{Z} \setminus \{0\}$. Then

$$h \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_r \end{pmatrix} = \widetilde{L}_U \Theta \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} + \widetilde{L}_U \mathbf{k}.$$

Moreover, we can choose U so that the first row of $\widetilde{L}_U \Theta$ is not identically zero. In fact, the first row of $\widetilde{L}_U \Theta$ is identically zero if and only if the first vector e_1 of the standard basis belongs to $\ker(\Theta^T \widetilde{L}_U^T)$, and hence it is orthogonal to $\mathrm{Im}(\widetilde{L}_U \Theta)$, because for any $u \in \mathbb{Q}^r$ we have

$$0 = \langle u, \Theta^T \widetilde{L}_U^T e_1 \rangle = \langle \Theta u, \widetilde{L}_U^T e_1 \rangle = \langle \widetilde{L}_U \Theta u, e_1 \rangle.$$

In particular $\mathrm{Im}(\Theta) \cap U \neq \{O\}$; otherwise $\widetilde{L}_U|_{\mathrm{Im}(\Theta)}$ would be injective, thus $\mathrm{Im}(\widetilde{L}_U \Theta) = \mathbb{Q}^r$, and e_1 could not be orthogonal to $\mathrm{Im}(\widetilde{L}_U \Theta)$. Now, it is a well-known fact of linear algebra that given two subspaces V, W of a vector space T having the same dimension there exists a subspace U such that $T = V \oplus U = W \oplus U$. Hence choosing U so that $\mathbb{Q}^n = \mathrm{Im}(N) \oplus U = \mathrm{Im}(\Theta) \oplus U$, we have $\mathrm{Im}(\Theta) \cap U = \{O\}$, and thus the first row of $\widetilde{L}_U \Theta$ is not identically zero.

Then

$$h \frac{1}{m} = (\widetilde{L}_U \Theta)_1 \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} + (\widetilde{L}_U \mathbf{k})_1$$

and this gives us a contradiction since $\alpha_1, \ldots, \alpha_r$ are rationally independent with 1 by assumption. □

We have then two cases to deal with: the rationally independent with 1 case, and the rationally dependent with 1 case.

Definition 4.4. Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$. We say that $[\varphi]$ is in the *torsion-free case*, or simply $[\varphi]$ *is torsion-free*, if its r-tuples of toric vectors have toric coefficients rationally independent with 1.

A notion of torsion-free germ of biholomorphism was firstly introduced by Écalle in [É]. We shall show in the next section that our notion is equivalent to his; our approach however gives more information on the normalization problem.

We end this section with a couple of results showing how to compute the toric degree, starting with toric degree 1.

Proposition 4.11. *Let* $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$. *Then:*

(i) $[\varphi]$ *has toric degree* 1 *with rational toric coefficient if and only if it belongs to* $(\mathbb{Q}/\mathbb{Z})^n$;

(ii) $[\varphi]$ *has toric degree* 1 *with toric coefficient in* $\mathbb{C} \setminus \mathbb{Q}$ *if and only if* $[\varphi] \notin (\mathbb{Q}/\mathbb{Z})^n$, *and there exists* $\theta \in \mathbb{Z}^n \setminus \{O\}$, *with* $\theta_k = 0$ *if* $[\varphi_k] = [0]$, *such that there is* $j_0 \in \{1, \ldots, n\}$ *so that*

(a) $[\varphi_{j_0}] \notin (\mathbb{Q}/\mathbb{Z})^n$ *and*

(8) $$\theta_k[\varphi_{j_0}] - \theta_{j_0}[\varphi_k] = [0]$$

for any k *so that* $[\varphi_k] \neq [0]$; *and*

(b) *for any representatives* φ_k *of* $[\varphi_k]$, *the integer vector* $\varphi_{j_0}\theta - \theta_{j_0}\varphi$ *belongs to the subspace* $\mathrm{Span}_{\mathbb{Z}}\{\widehat{\theta}, -\theta_{j_0}e_1, \ldots, \widehat{-\theta_{j_0}e_{j_0}}, \ldots, -\theta_{j_0}e_n\}$, *where* $\widehat{\theta} = \theta - \theta_{j_0}e_{j_0}$.

Proof. (i) If $\alpha = p/q \in \mathbb{Q}$ then

$$[\varphi] = \left[\frac{p}{q}\theta\right],$$

hence $[\varphi] \in (\mathbb{Q}/\mathbb{Z})^n$.

Conversely, if $[\varphi_j] = [p_j/q_j]$ with $p_j/q_j \in \mathbb{Q}$ for $j = 1, \ldots, n$, then, considering $q = q_1 \cdots q_n$ we get

$$[\varphi] = \begin{bmatrix} \frac{p_1 q_2 \cdots q_n}{q} \\ \vdots \\ \frac{p_n q_1 \cdots q_{n-1}}{q} \end{bmatrix} = \left[\frac{1}{q}\theta\right],$$

and we are done.

(ii) If

$$[\varphi] = \left[\alpha \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}\right],$$

with $\alpha \in \mathbb{C} \setminus \mathbb{Q}$ and $\theta \in \mathbb{Z}^n \setminus \{O\}$ then it is immediate to verify that $[\varphi] \notin (\mathbb{Q}/\mathbb{Z})^n$, and θ satisfies (a). By assumption, once we choose arbitrarily representatives φ_k of $[\varphi_k]$ we can write $\varphi_k = \alpha\theta_k + m_k$ for suitable $m_k \in \mathbb{Z}$. Then

$$\theta_k\varphi_j - \theta_j\varphi_k = \theta_k(\alpha\theta_j + m_j) - \theta_j(\alpha\theta_k + m_k) = \theta_k m_j - \theta_j m_k,$$

for any j and k, thus (b) is verified.

Conversely, let $\theta \in \mathbb{Z}^n \setminus \{O\}$ satisfy the hypotheses. By assumption $[\varphi] \notin (\mathbb{Q}/\mathbb{Z})^n$ and there is $j_0 \in \{1, \ldots, n\}$ such that $[\varphi_{j_0}] \notin (\mathbb{Q}/\mathbb{Z})^n$ satisfies (a) and (b); for the sake of simplicity, we may assume, without loss of generality, $j_0 = 1$. Let us choose a representative φ of $[\varphi]$ and set

$$\theta_j\varphi_1 - \theta_1\varphi_j = k_j \in \mathbb{Z}$$

for $j = 2, \ldots, n$. Condition (b) means that we can find $m_1, \ldots, m_n \in \mathbb{Z}$ so that

(9) $$\begin{pmatrix} \theta_2 & -\theta_1 & & \\ \vdots & & \ddots & \\ \theta_n & & & -\theta_1 \end{pmatrix} \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} k_2 \\ \vdots \\ k_n \end{pmatrix},$$

that is

$$k_j = \theta_j m_1 - \theta_1 m_j.$$

Now we put

$$\alpha = \frac{\varphi_1 - m_1}{\theta_1} \notin \mathbb{Q}.$$

Then $[\varphi] = [\alpha\theta]$; indeed

$$\alpha\theta_j = \frac{\theta_j(\varphi_1 - m_1)}{\theta_1} = \frac{\theta_j\varphi_1 - k_j - \theta_1 m_j}{\theta_1} = \varphi_j - m_j.$$

□

Remark 4.12. Condition (b) of the previous Proposition is necessary. In fact, if we just assume that condition (a) holds, then it is always possible to solve (9) in $\mathbb{Q}$, but this does not imply that it is solvable in $\mathbb{Z}$. For example the vector

$$[\varphi] = \begin{bmatrix} (2i+1)/3 \\ i \\ (11+10i)/6 \end{bmatrix}$$

has toric degree 2, but if we consider

$$\theta = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

we get condition (a) for $j = 1$. Moreover, choosing $((2i+4)/3, i, (11+10i)/6)$ as representative of $[\varphi]$, we get

$$\begin{pmatrix} k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

and it is not difficult to verify that

$$\begin{pmatrix} 3 & -2 & 0 \\ 5 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

has no solution $(m_1, m_2, m_3) \in \mathbb{Z}^3$.

Example 4.13. The vector of $(\mathbb{C}/\mathbb{Z})^3$

$$[\varphi_1] = \begin{bmatrix} (\sqrt{2}+i)/6 \\ (\sqrt{2}+i)/3 \\ 5(\sqrt{2}+i)/6 \end{bmatrix}$$

has toric degree 1, since it can be written as

$$[\varphi_1] = \left[\frac{\sqrt{2}+i}{6} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \right].$$

In general, to compute the toric degree of a vector one starts from the trivial representation of Remark 4.1, and then uses (the proof of) Lemma 4.3 to obtain rationally independent toric coefficients and toric vectors. Then the toric degree is computed as follows (see also Proposition 5.5)

Proposition 4.14. *Let $\alpha_1, \ldots, \alpha_r$ be $1 \le r \le n$ rationally independent complex numbers and let $\theta^{(1)}, \ldots, \theta^{(r)} \in \mathbb{Z}^n$ be $\mathbb{Q}$-linearly independent integer vectors. Then:*

- *(i) if $\alpha_1, \ldots, \alpha_r$ are rationally independent with 1, then $[\varphi] = \left[\sum_{k=1}^{r} \alpha_k \theta^{(k)}\right]$ has toric degree r;*
- *(ii) if $\alpha_1, \ldots, \alpha_r$ are rationally dependent with 1, then $[\varphi] = \left[\sum_{k=1}^{r} \alpha_k \theta^{(k)}\right]$ has toric degree $r-1$ or r.*

Proof. (i) Let $\alpha_1, \dots, \alpha_r$ be rationally independent with 1. The toric degree of $[\varphi]$ is not greater than r. Let us suppose by contradiction that $[\varphi]$ has toric degree $s < r$. Then there exist $\eta^{(1)}, \dots, \eta^{(s)} \in \mathbb{Z}^n$ and $\beta_1, \dots, \beta_s \in \mathbb{C}$ rationally independent such that

$$\left[\sum_{k=1}^{r} \alpha_k \theta^{(k)}\right] = \left[\sum_{k=1}^{s} \beta_k \eta^{(k)}\right].$$

Let N be the matrix with columns $\eta^{(1)}, \dots, \eta^{(s)}$, and Θ the matrix with columns $\theta^{(1)}, \dots, \theta^{(r)}$. We have

$$[\varphi] = \left[N \cdot \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_s \end{pmatrix}\right] = \left[\Theta \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix}\right],$$

that is there exists an integer vector $\mathbf{k} \in \mathbb{Z}^n$ such that

$$N \cdot \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_s \end{pmatrix} = \Theta \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} + \mathbf{k}.$$

Since Θ has maximal rank r, the linear map $\Theta\colon \mathbb{Q}^r \to \mathbb{Q}^n$ is injective and, for every $U \subseteq \mathbb{Q}^n$ such that $\mathbb{Q}^n = \operatorname{Im}(\Theta) \oplus U$, there is a linear map $L_U\colon \mathbb{Q}^n \to \mathbb{Q}^r$ such that $\ker(L_U) = U$ and $L_U \Theta = \mathrm{Id}$; hence there is a linear map $\widetilde{L}_U\colon \mathbb{Z}^n \to \mathbb{Z}^r$ such that $\widetilde{L}_U \Theta = h\,\mathrm{Id}$, with $h \in \mathbb{Z} \setminus \{0\}$. Then

$$\widetilde{L}_U N \cdot \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_s \end{pmatrix} - \widetilde{L}_U \mathbf{k} = h \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix}.$$

Now, $\dim(\ker(\widetilde{L}_U N)^T) \geq 1$. In particular, there exists $\xi \in \mathbb{Z}^r \setminus \{O\}$ such that $(\widetilde{L}_U N)^T \xi = O$, that is $\xi^T \widetilde{L}_U N = O$; therefore

$$\mathbb{Z} \ni -\xi^T \widetilde{L}_U \mathbf{k} = h \xi^T \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = h \langle \xi, \alpha \rangle,$$

which is an absurdum, because $\alpha_1, \dots, \alpha_r, 1$ are rationally independent.

(ii) Now we have $\alpha_1, \dots, \alpha_r$ rationally dependent with 1, and, arguing as in the proof of Lemma 4.9, we can suppose, without loss of generality, $\alpha_1 = 1/m$ and $\alpha_2, \dots, \alpha_r$ rationally independent with 1. If m divides $\theta_1^{(1)}, \dots, \theta_n^{(1)}$, then $[\varphi] = [\sum_{k=2}^{r} \alpha_k \theta^{(k)}]$ has toric degree $r-1$ thanks to (i). Otherwise, we may assume, without loss of generality, $m, \theta_1^{(1)}, \dots, \theta_n^{(1)}$ coprime. The toric degree of $[\varphi]$ is not greater than r. Let us suppose that $[\varphi]$ has toric degree $s < r$. Then there exist $\eta^{(1)}, \dots, \eta^{(s)} \in \mathbb{Z}^n$ and $\beta_1, \dots, \beta_s \in \mathbb{C}$ such that

$$\left[\sum_{k=1}^{r} \alpha_k \theta^{(k)}\right] = \left[\sum_{k=1}^{s} \beta_k \eta^{(k)}\right],$$

thus we have

$$[m\varphi] = \left[\sum_{k=2}^{r} \alpha_k \cdot m\theta^{(k)}\right] = \left[\sum_{k=1}^{s} \beta_k \cdot m\eta^{(k)}\right],$$

and, since $\alpha_2, \dots, \alpha_r$ are rationally independent with 1, by (i) we get $s = r - 1$. □

Remark 4.15. Note that both cases in (ii) can occur. In fact, it is not difficult to verify that

$$[\varphi_1] = \begin{bmatrix} 1/2 \\ \sqrt{2} \\ i \end{bmatrix} = \left[\frac{1}{2}e_1 + \sqrt{2}e_2 + ie_3\right]$$

has toric degree 3. However, if we consider,

$$[\varphi_2] = \left[\frac{1}{2}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{\sqrt{2}-1}{2}\begin{pmatrix}1\\1\\3\end{pmatrix} + \frac{i}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}\right],$$

then

$$[\varphi_2] = \begin{bmatrix} \sqrt{2}/2 \\ (\sqrt{2}+i)/2 \\ (-2+3\sqrt{2}+i)/2 \end{bmatrix} = \left[\frac{\sqrt{2}}{2}\begin{pmatrix}1\\1\\3\end{pmatrix} + \frac{i}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}\right],$$

so the toric degree is 2. Proposition 5.5 will show how to distinguish between the two cases of Proposition 4.14.(ii).

5. Torsion

In [É], Écalle introduced the following notion.

Definition 5.1. Let $\lambda \in (\mathbb{C}^*)^n$. The *torsion* of λ is the natural integer τ such that

$$\frac{1}{\tau}2\pi i\mathbb{Z} = (2\pi i\mathbb{Q}) \cap \left((2\pi i\mathbb{Z}) \bigoplus_{1\le j\le n} ((\log \lambda_j)\mathbb{Z})\right). \tag{10}$$

Translated in our notation, (10) becomes

$$\frac{1}{\tau}\mathbb{Z} = \mathbb{Q} \cap \left(\mathbb{Z} \bigoplus_{1\le j\le n} \varphi_j\mathbb{Z}\right),$$

where φ is a representative of the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = \exp(2\pi i[\varphi])$.

The torsion is well-defined, as the following result shows (and whose proof describes how to explicitly compute the torsion).

Proposition 5.1. *The torsion of a n-tuple $(\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$ is a well-defined natural integer. Furthermore, writing $\lambda = e^{2\pi i[\varphi]}$, if $[\varphi]$ is torsion-free, then $\tau = 1$; otherwise τ divides the denominator of the first toric coefficient in a reduced representation of $[\varphi]$.*

Proof. Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be the unique vector such that $\lambda = \exp(2\pi i[\varphi])$, let $1 \le r \le n$ be its toric degree and let $\theta^{(1)}, \dots, \theta^{(r)}$ be a r-tuple of toric vectors associated to $[\varphi]$ with coefficients $\alpha_1, \dots, \alpha_r$.

Our aim is to determine the structure of the set

$$R = \mathbb{Q} \cap \left(\mathbb{Z} \bigoplus_{j=1}^{n} \varphi_j\mathbb{Z}\right),$$

that is of the set of rational numbers x that can be expressed in the form

$$\mathbb{Q} \ni x = m_0 + m_1\varphi_1 + \cdots + m_n\varphi_n$$

with $m_0, \ldots, m_n \in \mathbb{Z}$. Write, as usual,

$$\varphi_j = h_j + \sum_{k=1}^{r} \alpha_k \theta_j^{(k)},$$

with $h_j \in \mathbb{Z}$. Then

$$\begin{aligned} x &= (m_0 + m_1 h_1 + \cdots + m_n h_n) + m_1 \sum_{k=1}^{r} \alpha_k \theta_1^{(k)} + \cdots + m_n \sum_{k=1}^{r} \alpha_k \theta_n^{(k)} \\ &= \widetilde{m} + \sum_{k=1}^{r} \alpha_k \langle M, \theta^{(k)} \rangle, \end{aligned}$$

where $\widetilde{m} \in \mathbb{Z}$ and $M \in \mathbb{Z}^n$ are generic. If $\alpha_1, \ldots, \alpha_r$ are rationally independent with 1, it follows that $x \in \mathbb{Q}$ if and only if $\langle M, \theta^{(1)} \rangle = \cdots = \langle M, \theta^{(r)} \rangle = 0$, and thus $R = \mathbb{Z}$ and $\tau = 1$.

If $\alpha_1, \ldots, \alpha_r$ are not rationally independent with 1, let us use instead the reduced representation, with $\beta_1 = 1/m$, the remaining coefficients $\beta_2, \ldots, \beta_r$ rationally independent with 1, and with $\eta^{(1)}, \ldots, \eta^{(r)}$ as toric vectors. We get

$$x = \widetilde{m} + \frac{1}{m}\langle M, \eta^{(1)} \rangle + \sum_{k=2}^{r} \beta_k \langle M, \eta^{(k)} \rangle.$$

Therefore $x \in \mathbb{Q}$ if and only if $\langle M, \eta^{(2)} \rangle = \cdots = \langle M, \eta^{(r)} \rangle = 0$, and moreover in that case

$$x = \widetilde{m} + \frac{1}{m}\langle M, \eta^{(1)} \rangle.$$

Now, the set

$$S = \{\langle M, \eta^{(1)} \rangle \mid M \in \mathbb{Z}^n, \langle M, \eta^{(2)} \rangle = \cdots = \langle M, \eta^{(r)} \rangle = 0\}$$

is an ideal of $\mathbb{Z}$; therefore $S = q\mathbb{Z}$ for some $q \in \mathbb{N}$. It follows that

$$R = \mathbb{Z} \oplus \frac{q}{m}\mathbb{Z} = \mathbb{Z} \oplus \frac{\widetilde{q}}{\widetilde{m}}\mathbb{Z} = \frac{1}{\widetilde{m}}\mathbb{Z},$$

where $\widetilde{q}$ and $\widetilde{m}$ are coprime, and $q/m = \widetilde{q}/\widetilde{m}$. Hence $\tau = \widetilde{m}$, and we are done. $\square$

Remark 5.2. Note that, in the previous proof, $S \neq \{O\}$, i.e., $q \neq 0$. Indeed, $S = \{O\}$ if and only if the kernel in $\mathbb{Z}^n$ of the linear form $(\eta^{(1)})^T$ contains the intersection of the kernels in $\mathbb{Z}^n$ of the linear forms $(\eta^{(2)})^T, \ldots, (\eta^{(r)})^T$. It is easy to see that this implies that the kernel in $\mathbb{Q}^n$ of the linear form $(\eta^{(1)})^T$ contains the intersection of the kernels in $\mathbb{Q}^n$ of the linear forms $(\eta^{(2)})^T, \ldots, (\eta^{(r)})^T$. But this implies that the linear form $(\eta^{(1)})^T$ is a $\mathbb{Q}$-linear combination of $(\eta^{(2)})^T, \ldots, (\eta^{(r)})^T$, and so $\eta^{(1)}, \ldots, \eta^{(r)}$ are $\mathbb{Q}$-linearly dependent, impossible.

The next result explains the terminology of Definition 4.4.

Theorem 5.3. *Let $\lambda = e^{2\pi i[\varphi]} \in (\mathbb{C}^*)^n$. Then $[\varphi]$ is torsion-free if and only if the torsion of λ is* 1.

Proof. If $[\varphi]$ is torsion-free, then the toric coefficients of a toric r-tuple associated to $[\varphi]$ are rationally independent with 1, and the torsion τ is 1, by Proposition 5.1.

Conversely, let $\eta^{(1)}, \dots, \eta^{(r)}$ be a reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/m, \beta_2, \dots, \beta_r$. Let us assume by contradiction that the torsion τ of $[\varphi]$ is 1. From the proof of Proposition 5.1 it is clear that we have $\tau = 1$ if and only if $\langle P, \eta^{(1)} \rangle \in m\mathbb{Z}$, for any $P \in \mathbb{Z}^n$ such that $\langle P, \eta^{(k)} \rangle = 0$ for $k = 2, \dots, r$.

Since $\eta^{(1)}, \dots, \eta^{(r)}$ are a toric r-tuple, we may assume, without loss of generality, that the matrix A of $M_{n\times n}(\mathbb{Z})$ with columns $\eta^{(2)}, \dots, \eta^{(r)}, e_r, \dots, e_n$ is invertible in $M_{n\times n}(\mathbb{Q})$. Denote by N' the matrix in $M_{(r-1)\times(r-1)}(\mathbb{Z})$

$$N' = \begin{pmatrix} \eta_1^{(2)} & \dots & \eta_1^{(r)} \\ \vdots & & \vdots \\ \eta_{r-1}^{(2)} & \dots & \eta_{r-1}^{(r)} \end{pmatrix},$$

and by N'' the matrix in $M_{(n-r+1)\times(r-1)}(\mathbb{Z})$

$$N'' = \begin{pmatrix} \eta_r^{(2)} & \dots & \eta_r^{(r)} \\ \vdots & & \vdots \\ \eta_n^{(2)} & \dots & \eta_n^{(r)} \end{pmatrix}.$$

Then

$$A = \begin{pmatrix} N' & O \\ N'' & I_{n-r+1} \end{pmatrix}$$

and $\det(A) = \det(N') \neq 0$.

We claim that, up to pass to another toric r-tuple $\widehat{\eta}^{(1)}, \eta^{(2)}, \dots, \eta^{(r)}$, we may assume that $m = \det(N')$ and $\widehat{\eta}^{(1)} \in \{0\}^{r-1} \times \mathbb{Z}^{n-r+1}$. In fact, $\eta^{(k)} = A^{-1} e_{k-1}$ for $k = 2, \dots, r$, with $A^{-1} \in M_{n\times n}(\mathbb{Q})$. Hence $P \in \mathbb{Z}^n$ is such that $\langle P, \eta^{(k)} \rangle = 0$ for $k = 2, \dots, r$ if and only if $\langle A^T P, e_j \rangle = 0$ for $j = 1, \dots, r-1$, that is $A^T P \in \{0\}^{r-1} \times \mathbb{Z}^{n-r+1}$. Now, we have

$$A^T P = \begin{pmatrix} N'^T & N''^T \\ O & I_{n-r+1} \end{pmatrix} \begin{pmatrix} P' \\ P'' \end{pmatrix} \in \{0\}^{r-1} \times \mathbb{Z}^{n-r+1}$$

if and only if

$$P = \begin{pmatrix} -(N'^T)^{-1} N''^T P'' \\ P'' \end{pmatrix} \quad \text{with} \quad P'' \in \mathbb{Z}^{n-r+1} \text{ and } (N'^T)^{-1} N''^T P'' \in \mathbb{Z}^{r-1},$$

that is

$$P'' \in \mathbb{Z}^{n-r+1} \text{ and } (N'^+)^T N''^T P'' \in \det(N') \mathbb{Z}^{r-1}$$

where $(N'^+)^T \in M_{(r-1)\times(r-1)}(\mathbb{Z})$ and $(N'^+)^T N' = \det(N') I_{r-1}$. In particular, since we are assuming

$$\langle P, \eta^{(k)} \rangle = 0 \text{ for } k = 2, \dots, r \implies \langle P, \eta^{(1)} \rangle \in m\mathbb{Z}, \tag{11}$$

we get

$$\langle A^T P, A^{-1} \eta^{(1)} \rangle = \left\langle \begin{pmatrix} O \\ P'' \end{pmatrix}, A^{-1} \eta^{(1)} \right\rangle \in m\mathbb{Z}$$

for any $P'' \in \det(N')\mathbb{Z}^{n-r+1}$. Then there exist $q_1, \dots, q_{r-1} \in \mathbb{Q}$ and $\widehat{\eta}^{(1)} \in \{0\}^{r-1} \times \mathbb{Z}^{n-r+1}$ such that

$$A^{-1}\eta^{(1)} = q_1 e_1 + \cdots + q_{r-1}e_{r-1} + \frac{m}{\det(N')}\widehat{\eta}^{(1)},$$

that is

$$\eta^{(1)} = q_1 \eta^{(2)} + \cdots + q_{r-1}\eta^{(r)} + \frac{m}{\det(N')}\widehat{\eta}^{(1)},$$

thus we get

$$\begin{aligned}
[\varphi] &= \left[\frac{1}{m}\eta^{(1)} + \sum_{k=2}^{r} \beta_k \eta^{(k)}\right] \\
&= \left[\frac{1}{m}\frac{m}{\det(N')}\widehat{\eta}^{(1)} + \sum_{k=2}^{r}\left(\beta_k + \frac{q_{k-1}}{m}\right)\eta^{(k)}\right] \\
&= \left[\frac{1}{\det(N')}\widehat{\eta}^{(1)} + \sum_{k=2}^{r}\widetilde{\beta}_k \eta^{(k)}\right].
\end{aligned}$$

Note that $\widetilde{\beta}_2, \dots, \widetilde{\beta}_r$ are rationally independent with 1.

Now we can assume that (11) holds with $m = \det(N')$ and $\eta^{(1)} \in \{0\}^{r-1} \times \mathbb{Z}^{n-r+1}$. We claim that there exist $\gamma_2, \dots, \gamma_r \in \mathbb{C}^*$ such that $[\varphi] = [\sum_{k=2}^{r} \gamma_k \eta^{(k)}]$, i.e., $[\varphi]$ has toric degree $r-1$, contradicting the hypotheses. We can have $[\varphi] = [\sum_{k=2}^{r} \gamma_k \eta^{(k)}]$ with $\gamma_2, \dots, \gamma_r \in \mathbb{C}^*$, if there exists $\theta' \in \mathbb{Z}^{r-1}$ such that

$$\begin{pmatrix} \gamma_2 \\ \vdots \\ \gamma_r \end{pmatrix} = \begin{pmatrix} \beta_2 \\ \vdots \\ \beta_r \end{pmatrix} + N'^{-1}\theta',$$

and $\theta' \in \mathbb{Z}^{r-1}$ is a solution

$$N''N'^{+}\begin{pmatrix} x_1 \\ \vdots \\ x_{r-1} \end{pmatrix} \equiv \begin{pmatrix} \eta_r^{(1)} \\ \vdots \\ \eta_n^{(1)} \end{pmatrix} \mod m\mathbb{Z}^{n-r+1}. \tag{12}$$

In fact, since $N''N'^{-1} = (1/m)N''N'^{+}$, this implies

$$\frac{1}{m}\eta^{(1)} = \frac{1}{m}\begin{pmatrix} O \\ \eta''^{(1)} \end{pmatrix} \equiv \begin{pmatrix} O \\ N''N'^{-1}\theta' \end{pmatrix}, \tag{13}$$

modulo $\mathbb{Z}$, where $\eta''^{(1)} = (\eta_r^{(1)}, \dots, \eta_n^{(1)})$, hence

$$\begin{aligned}
[\varphi] &= \left[\frac{1}{m}\eta^{(1)} + N \begin{pmatrix} \beta_2 \\ \vdots \\ \beta_r \end{pmatrix} \right] \\
&= \left[\begin{pmatrix} O \\ N''N'^{-1}\theta' \end{pmatrix} + \begin{pmatrix} N' \\ N'' \end{pmatrix} \left(\begin{pmatrix} \gamma_2 \\ \vdots \\ \gamma_r \end{pmatrix} - N'^{-1}\theta' \right) \right] \\
&= \left[\begin{pmatrix} O \\ N''N'^{-1}\theta' \end{pmatrix} + \begin{pmatrix} N' \\ N'' \end{pmatrix} \begin{pmatrix} \gamma_2 \\ \vdots \\ \gamma_r \end{pmatrix} - \begin{pmatrix} \theta' \\ N''N'^{-1}\theta' \end{pmatrix} \right] \\
&= \left[N \begin{pmatrix} \gamma_2 \\ \vdots \\ \gamma_r \end{pmatrix} \right].
\end{aligned}$$

Now we prove that, if (11) holds with $m = \det(N')$ and $\eta^{(1)} \in \{0\}^{r-1} \times \mathbb{Z}^{n-r+1}$, then there exists a solution $\theta' \in \mathbb{Z}^{r-1}$ of (12). In fact, if $P'' \notin m\mathbb{Z}^{n-r+1}$ is a multi-index such that $P''^T N'' N'^+ \in m\mathbb{Z}^{r-1}$, then by (11) we have $P''^T \eta''^{(1)} \in m\mathbb{Z}$, where we use the same notation of (13); thus, since up to reorder the indices we may assume that the last coordinate of P'' is not in $m\mathbb{Z}$, we can substitute $P''^T N'' N'^+ x \equiv P''^T \eta''^{(1)}$ to the last equation of (12), and we have to solve a system with one equation less. We iterate this procedure for a set of generators of a complement of $m\mathbb{Z}^{n-r+1}$ in the lattice of P'' until, up to reordering, we get

$$B \begin{pmatrix} x_1 \\ \vdots \\ x_{r-1} \end{pmatrix} \equiv \begin{pmatrix} \eta_r^{(1)} \\ \vdots \\ \eta_{r+h-1}^{(1)} \end{pmatrix} \mod m\mathbb{Z}^h$$

where $1 \le h \le n - r + 1$, $B \in M_{h\times(r-1)}(\mathbb{Z})$ is the matrix of the first h rows of $N''N''^+$, and for any $R \notin m\mathbb{Z}^h$, we have $R^T B \notin m\mathbb{Z}^{r-1}$, that is B has maximal rank modulo m.

If $h = 1$, then we have

$$(14) \qquad b_1 x_1 + \cdots + b_{r-1}x_{r-1} \equiv \eta_1^{(1)} \mod m\mathbb{Z}.$$

If $b_1, \dots, b_{r-1}, m$ are coprime it is obvious that (14) is solvable. If the greatest common divisor of $b_1, \dots, b_{r-1}, m$ is $p > 1$, then $m = qp$ and $q(b_1, \dots, b_{r-1}) \in m\mathbb{Z}^{r-1}$, hence, by (11), we must have $\eta_1^{(1)} \in p\mathbb{Z}$ too, thus

$$\frac{b_1}{p}x_1 + \cdots + \frac{b_{r-1}}{p}x_{r-1} \equiv \frac{\eta_1^{(1)}}{p} \mod \frac{m}{p}\mathbb{Z}$$

is solvable.

Let us now suppose $1 < h \le n - r + 1$. Since B has maximal rank modulo m, there exists B^+ in $M_{(r-1)\times h}(\mathbb{Z})$ such that $B^+B \equiv dI_{r-1}$, modulo $m\mathbb{Z}$ where $d \ne m$. Thus we have

$$d \begin{pmatrix} x_1 \\ \vdots \\ x_{r-1} \end{pmatrix} \equiv B^+ \begin{pmatrix} \eta_r^{(1)} \\ \vdots \\ \eta_{r+h-1}^{(1)} \end{pmatrix} \mod m\mathbb{Z}^h.$$

If d and m are coprime, we are done. Otherwise, let p be greatest common divisor of d and m, and let $q = m/p$. Since $B^+B \equiv dI_{r-1}$ modulo $m\mathbb{Z}$, we have $qB^+B \equiv O$ modulo $m\mathbb{Z}$, thus, since we are assuming that for any $R \notin m\mathbb{Z}^h$, we have $R^TB \notin m\mathbb{Z}^{r-1}$, it has to be $qB^+ \equiv O$ modulo $m\mathbb{Z}$, that is $B^+ \equiv p\widetilde{B}$ modulo $m\mathbb{Z}$. Therefore we have

$$\frac{d}{p}\begin{pmatrix} x_1 \\ \vdots \\ x_{r-1} \end{pmatrix} \equiv \widetilde{B}\begin{pmatrix} \eta_r^{(1)} \\ \vdots \\ \eta_{r+h-1}^{(1)} \end{pmatrix} \mod \frac{m}{p}\mathbb{Z}^h,$$

which is solvable, as we wanted. □

The torsion case is more delicate and difficult to deal. First, given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ with toric degree $1 \leq r \leq n$ and torsion $\tau \geq 2$, and a reduced toric r-tuple $\eta^{(1)}, \dots, \eta^{(r)}$, we have

$$\text{(15),} \qquad \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) \supseteq \mathrm{Res}_j([\varphi]) \supseteq \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\eta^{(k)}),$$

yielding a subdivision in more subcases, all realizable (we have examples for all of them) and, surprisingly, having very different behaviours ones from the others; we have cases similar to the case of germs of vector fields (even if we have torsion!), and cases that are indeed different. In particular, considering iterates of f to reduce to the torsion-free case hides very interesting phenomena, and it does not allow to see that some torsion cases can be directly studied. Moreover, we have explicit (and computable) techniques to decide in which subcase a given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ belongs to.

Example 5.4. Let us consider the vector

$$[\varphi] = \left[\frac{1}{6}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \sqrt{2}\begin{pmatrix} 1 \\ -6 \end{pmatrix}\right] \in (\mathbb{C}/\mathbb{Z})^2,$$

of toric degree 2. We have

$$\langle P, \eta^{(2)} \rangle = p_1 - 6p_2 = 0$$

if and only if

$$P \in \begin{pmatrix} 6 \\ 1 \end{pmatrix}\mathbb{Z},$$

hence

$$\mathrm{Res}_1^+(\eta^{(2)}) = \{(6h+1, h) \mid h \in \mathbb{N} \setminus \{0\}\} \quad \text{and} \quad \mathrm{Res}_2^+(\eta^{(2)}) = \{(6h, h+1) \mid h \in \mathbb{N} \setminus \{0\}\},$$

and

$$\langle (6h, h), \eta^{(1)} \rangle \in 9\mathbb{Z},$$

that is

$$S = 9\mathbb{Z},$$

and the torsion is clearly 2. Moreover, we have

$$[\varphi] = \left[\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3\sqrt{2}-1}{3}\begin{pmatrix} 1 \\ -6 \end{pmatrix}\right] \in (\mathbb{C}/\mathbb{Z})^2.$$

Using the torsion τ of a vector, we obtain a complete criterion to compute the toric degree of a vector, as next result shows.

Proposition 5.5. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ and let τ be its torsion. If*

$$[\varphi] = \left[\frac{1}{m}\eta^{(1)} + \sum_{k=2}^{r} \beta_k \eta^{(k)} \right],$$

with $\eta^{(1)} \notin m\mathbb{Z}^n$, then $[\varphi]$ has toric degree r if and only if the torsion of $[\varphi]$ is $\tau > 1$, the coefficients $\beta_2, \ldots, \beta_r$ are rationally independent with 1, and the integer vectors $\eta^{(1)}, \ldots, \eta^{(r)}$ are $\mathbb{Q}$-linearly independent.

Proof. It follows from Lemma 4.9, Proposition 5.1 and from the proof of Theorem 5.3.

6. Poincaré-Dulac Normal Form in the torsion-free case

In the torsion-free case, it is not difficult to show that we can compute the resonances of $[\varphi]$, which are multiplicative, using the additive resonances of one of its associated r-tuples of toric vectors, as the next result shows.

Lemma 6.1. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$ and torsion-free. Then for any r-tuple of toric vectors, $\theta^{(1)}, \ldots, \theta^{(r)}$, associated to $[\varphi]$ we have*

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=1}^{r} \mathrm{Res}_j^{+}(\theta^{(k)})$$

for every $j = 1, \ldots, n$.

Proof. We have

$$[\langle Q, \varphi \rangle - \varphi_j] = \left[\sum_{k=1}^{r} \alpha_k \left(\langle Q, \theta^{(k)} \rangle - \theta_j^{(k)} \right) \right] \tag{16}$$

and, since $\alpha_1, \ldots, \alpha_r$ are rationally independent with 1, the right-hand side of (16) vanishes if and only if $\langle Q, \theta^{(k)} \rangle - \theta_j^{(k)} = 0$ for every $k = 1, \ldots, r$. □

Example 6.2. Let us consider the torsion-free vector

$$[\varphi] = \left[\sqrt{2} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + 2i \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right] \in (\mathbb{C}/\mathbb{Z})^3,$$

of toric degree 2. Then

$$\begin{cases} \langle P, \theta^{(1)} \rangle = 3p_1 + 2p_2 - p_3 = 0 \\ \langle P, \theta^{(2)} \rangle = 2p_1 + 3p_2 + p_3 = 0 \end{cases}$$

for some $P \in \mathbb{Z}^n$, if and only if

$$P \in \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \mathbb{Z}.$$

Hence in this case

$$\mathrm{Res}_1([\varphi]) = \mathrm{Res}_3([\varphi]) = \varnothing \quad \text{and} \quad \mathrm{Res}_2([\varphi]) = \{(1,0,1)\}.$$

Example 6.3. Let us consider the vector

$$[\varphi] = \left[\sqrt{2} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + 2i \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right] \in (\mathbb{C}/\mathbb{Z})^3.$$

Again, $[\varphi]$ has toric degree 2 and it is torsion-free. In this case, we have

$$\begin{cases} \langle P, \theta^{(1)} \rangle = 3p_1 + 2p_2 - p_3 = 0 \\ \langle P, \theta^{(2)} \rangle = 2p_1 - 3p_2 + p_3 = 0 \end{cases}$$

for some $P \in \mathbb{Z}^n$, if and only if

$$P \in \begin{pmatrix} 1 \\ 5 \\ 13 \end{pmatrix} \mathbb{Z}.$$

Hence

$$\begin{aligned} \mathrm{Res}_1([\varphi]) &= \{(q+1, 5q, 13q) \mid q \in \mathbb{N} \setminus \{0\}\} \\ \mathrm{Res}_2([\varphi]) &= \{(q, 5q+1, 13q) \mid q \in \mathbb{N} \setminus \{0\}\} \\ \mathrm{Res}_3([\varphi]) &= \{(q, 5q, 13q+1) \mid q \in \mathbb{N} \setminus \{0\}\}. \end{aligned}$$

We have the following immediate corollary of Lemma 6.1.

Corollary 6.4. *Let $\lambda \in (\mathbb{C}^*)^n$ and let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be such that $\lambda = e^{2\pi i [\varphi]}$. If $[\varphi]$ is torsion-free and has toric degree $1 \le r \le n$, then for every r-tuple $\theta^{(1)}, \ldots, \theta^{(r)}$ of toric vectors associated to $[\varphi]$ we have*

$$\mathrm{Res}_j(\lambda) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\theta^{(k)})$$

for every $j = 1, \ldots, n$.

Lemma 6.5. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$ and torsion-free. Then for any r-tuple of toric vectors, $\theta^{(1)}, \ldots, \theta^{(r)}$, associated to $[\varphi]$ we have $\theta_j^{(k)} = \theta_h^{(k)}$ whenever $[\varphi_j] = [\varphi_h]$, for every $k = 1, \ldots, r$.*

Proof. If $[\varphi_j] = [\varphi_h]$, then

$$\left[\alpha_1 \theta_j^{(1)} + \cdots + \alpha_r \theta_j^{(r)}\right] = \left[\alpha_1 \theta_h^{(1)} + \cdots + \alpha_r \theta_h^{(r)}\right];$$

hence there exists $m \in \mathbb{Z}$, such that

$$\alpha_1 \left(\theta_j^{(1)} - \theta_h^{(1)}\right) + \cdots + \alpha_r \left(\theta_j^{(r)} - \theta_h^{(r)}\right) = m,$$

and, since $\theta_j^{(k)} - \theta_h^{(k)} \in \mathbb{Z}$ for $k = 1, \ldots, r$, the assertion follows from the rational independence with 1 of $\alpha_1, \ldots, \alpha_r$. □

Definition 6.1. Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin. We say that f is *torsion-free* if, denoted by $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ the spectrum of $\mathrm{d}f_O$, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$ is in the torsion-free case.

We have then the following complete description of Poincaré-Dulac holomorphic normalization in the torsion-free case.

Theorem 6.6. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O, of toric degree $1 \leq r \leq n$ and in the torsion-free case. Then f admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with f and such that the columns of the weight matrix of the action are a r-tuple of toric vectors associated to f.*

Proof. It follows from Theorem 3.1, Lemma 6.5 and Corollary 6.4. □

7. Poincaré-Dulac Normal Form in presence of torsion

Let us consider now $[\varphi] \in \mathbb{C}/\mathbb{Z}$, of toric degree $1 \leq r \leq n$ and let $\theta^{(1)}, \ldots, \theta^{(r)}$ be a r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $\alpha_1, \ldots, \alpha_r$ rationally dependent with 1. We shall put

$$\mathcal{D}(\alpha_1, \ldots, \alpha_r) = \{M \in \mathbb{Z}^r \mid m_1\alpha_1 + \cdots + m_r\alpha_r \in \mathbb{Z}\},$$

and

$$\mathrm{Adm}(\theta^{(1)}, \ldots, \theta^{(r)}) = \bigcup_{j=1}^{n} \mathrm{Adm}_j(\theta^{(1)}, \ldots, \theta^{(r)}),$$

where

$$\mathrm{Adm}_j(\theta^{(1)}, \ldots, \theta^{(r)}) = \{M \in \mathbb{Z}^r \mid \exists Q \in \mathbb{N}^n, |Q| \geq 2 \text{ s.t. } m_k = \langle Q - e_j, \theta^{(k)} \rangle \;\forall k = 1, \ldots, r\} \cup \{O\},$$

for all $j \in \{1, \ldots, n\}$.

Even if, in this case, it is not always true that we can compute the resonances of $[\varphi]$ as intersection of additive resonances, we can say many things on the resonant multi-indices using reduced r-tuples associated to $[\varphi]$.

Lemma 7.1. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$ and let $\eta^{(1)}, \ldots, \eta^{(r)}$ be a reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/m, \beta_2, \ldots, \beta_r$. Then*

(i) $\mathcal{D}(1/m, \beta_2, \ldots, \beta_r) = \{(hm, 0, \ldots, 0) \mid h \in \mathbb{Z}\} \subset \mathbb{Z}^r$;

(ii) we have

$$\mathcal{D}(1/m, \beta_2, \ldots, \beta_r) \cap \mathrm{Adm}(\eta^{(1)}, \ldots, \eta^{(r)}) \neq \{O\}$$

if and only there exist $Q \in \mathbb{N}^n$, with $|Q| \geq 2$ and $j \in \{1, \ldots, n\}$ such that

$$\langle Q - e_j, \eta^{(1)} \rangle \in m\mathbb{Z} \setminus \{0\} \quad \text{and} \quad Q \in \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)});$$

(iii) we have

$$\mathrm{Res}_j([\varphi]) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \langle Q - e_j, \eta^{(1)} \rangle \in m\mathbb{Z}\} \cap \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}),$$

for any $j \in \{1, \dots, n\}$. In particular,

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) \supseteq \mathrm{Res}_j([\varphi]) \supseteq \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\eta^{(k)}), \tag{17}$$

for all $j \in \{1, \dots, n\}$.

(iv) $[\varphi_j] = [\varphi_h]$ implies that m divides $\eta_j^{(1)} - \eta_h^{(1)}$, and that $\eta_j^{(k)} = \eta_h^{(k)}$ for any $k = 2, \dots, r$.

Proof. (i) One inclusion is obvious. Conversely, let $M \in \mathcal{D}(1/m, \beta_2, \dots, \beta_r)$; then

$$m_1 \frac{1}{m} + m_2\beta_2 + \cdots + m_r\beta_r \in \mathbb{Z}.$$

Since $\beta_2, \dots, \beta_r$ are rationally independent with 1, this implies $m_2 = \cdots = m_r = 0$, thus we must have $m_1/m \in \mathbb{Z}$, and we are done.

(ii) It is immediate from the definitions of $\mathcal{D}(1/m, \beta_2, \dots, \beta_r)$ and $\mathrm{Adm}(\eta^{(1)}, \dots, \eta^{(r)})$ and from (i).

(iii) It is immediate from (ii) and from

$$[\langle Q, \varphi\rangle - \varphi_j] = \left[\frac{1}{m}\langle Q - e_j, \eta^{(1)}\rangle + \sum_{k=2}^{r} \beta_k \langle Q - e_j, \eta^{(k)}\rangle\right]. \tag{18}$$

(iv) If $[\varphi_j] = [\varphi_h]$, then

$$\left[\frac{1}{m}\eta_j^{(1)} + \beta_2\eta_j^{(2)} + \cdots + \beta_r\eta_j^{(r)}\right] = \left[\frac{1}{m}\eta_h^{(1)} + \beta_2\eta_h^{(2)} + \cdots + \beta_r\eta_h^{(r)}\right],$$

hence

$$\frac{1}{m}\left(\eta_j^{(1)} - \eta_h^{(1)}\right) + \beta_2\left(\eta_j^{(2)} - \eta_h^{(2)}\right) \cdots + \beta_r\left(\eta_j^{(r)} - \eta_h^{(r)}\right) \in \mathbb{Z},$$

and, since $\eta_j^{(k)} - \eta_h^{(k)} \in \mathbb{Z}$ for $k = 1, \dots, r$, the assertion follows as in (i). □

Remark 7.2. Note that, given $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ of toric degree $1 \le r \le n$, if $\eta^{(1)}, \dots, \eta^{(r)}$ is a reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/m, \beta_2, \dots, \beta_r$, and such that $[\varphi_j] = [\varphi_h]$ for some distinct coordinates j and h, but $\eta_j^{(1)} \neq \eta_h^{(1)}$, then, since m divides $\eta_j^{(1)} - \eta_h^{(1)}$, we have

$$\frac{1}{m}\eta_j^{(1)} = \frac{1}{m}\eta_h^{(1)} + \frac{1}{m}\left(\eta_j^{(1)} - \eta_h^{(1)}\right);$$

thus

$$[\varphi] = \left[\frac{1}{m}\widetilde{\eta}^{(1)} + \sum_{k=2}^{r} \beta_k\eta^{(k)}\right]$$

where, $\widetilde{\eta}_p^{(1)} = \eta_p^{(1)}$ for any $p \neq j, h$ and $\widetilde{\eta}_j^{(1)} = \widetilde{\eta}_h^{(1)}$, that is $\widetilde{\eta}^{(1)} = \eta^{(1)} - (\eta_j^{(1)} - \eta_h^{(1)})e_j$, obtaining a compatible reduced r-tuple.

Even in the torsion case, toric r-tuples associated to a same vector $[\varphi]$ have to verify certain properties on the resonances, as next result shows.

Lemma 7.3. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$ and in the torsion case. Let $\eta^{(1)}, \ldots, \eta^{(r)}$ be a reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/m, \beta_2, \ldots, \beta_r$ and let $\xi^{(1)}, \ldots, \xi^{(r)}$ be a reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/\widetilde{m}, \gamma_2, \ldots, \gamma_r$. Then we have*

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\xi^{(k)}),$$

for all $j = 1, \ldots, n$.

Proof. We have

$$[\varphi] = \left[\frac{1}{m}\eta^{(1)} + \sum_{k=2}^{r} \beta_k \eta^{(k)}\right] = \left[\frac{1}{\widetilde{m}}\xi^{(1)} + \sum_{k=2}^{r} \gamma_k \xi^{(k)}\right].$$

Then

$$[m\widetilde{m}\varphi] = \left[\sum_{k=2}^{r} m\widetilde{m}\beta_k \eta^{(k)}\right] = \left[\sum_{k=2}^{r} m\widetilde{m}\gamma_k \xi^{(k)}\right],$$

and, by Proposition 4.14, $[m\widetilde{m}\varphi]$ has toric degree $r-1$ and is torsion-free, because $\beta_2, \ldots, \beta_r$ and $\gamma_2, \ldots, \gamma_r$ are rationally independent with 1. Therefore, by Lemma 6.1, we have

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) = \mathrm{Res}_j([m\widetilde{m}\varphi]) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\xi^{(k)}),$$

for any $j = 1, \ldots, n$, and we are done. □

As Theorem 5.3 shows, it is not possible that $\langle P, \eta^{(1)} \rangle \in m\mathbb{Z}$ for any $P \in \mathbb{Z}^n$ such that $\langle P, \eta^{(k)} \rangle = 0$ for $k = 2, \ldots, r$. However, it is possible that

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)})$$

for all $j \in \{1, \ldots, n\}$, as next example shows.

Example 7.4. Let us consider the vector

$$[\varphi] = \left[\frac{1}{3}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + \sqrt{2}\begin{pmatrix}-12\\0\\0\\1\end{pmatrix} + \sqrt{3}\begin{pmatrix}0\\5\\2\\0\end{pmatrix}\right] \in (\mathbb{C}/\mathbb{Z})^4,$$

of toric degree 3. In this case $\mathcal{D}(1/3, \sqrt{2}, \sqrt{3}) = \{(3h, 0, 0) \mid h \in \mathbb{Z}\}$. We have

$$\langle P, \eta^{(2)} \rangle = -12p_1 + p_4 = 0$$

if and only if

$$P \in \begin{pmatrix}1\\0\\0\\12\end{pmatrix}\mathbb{Z} \oplus e_2\mathbb{Z} \oplus e_3\mathbb{Z},$$

and

$$\langle P, \eta^{(3)} \rangle = 5p_2 + 2p_3 = 0$$

if and only if

$$P \in \begin{pmatrix} 0 \\ -2 \\ 5 \\ 0 \end{pmatrix} \mathbb{Z} \oplus e_1\mathbb{Z} \oplus e_4\mathbb{Z}.$$

We have

$$\begin{aligned}
\mathrm{Res}_1^+(\eta^{(2)}) &= \{(q_1, q_2, q_3, 12(q_1 - 1)) \mid q_1, q_2, q_3 \in \mathbb{N}, 13q_1 + q_2 + q_3 \geq 14\} \\
\mathrm{Res}_2^+(\eta^{(2)}) &= \{(q_1, q_2, q_3, 12q_1) \mid q_1, q_2, q_3 \in \mathbb{N}, 13q_1 + q_2 + q_3 \geq 2\} \\
\mathrm{Res}_3^+(\eta^{(2)}) &= \mathrm{Res}_2^+(\eta^{(2)}) \\
\mathrm{Res}_4^+(\eta^{(2)}) &= \{(q_1, q_2, q_3, 12q_1 + 1) \mid q_1, q_2, q_3 \in \mathbb{N}, 13q_1 + q_2 + q_3 \geq 1\},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Res}_1^+(\eta^{(3)}) &= \{(q_1, 0, 0, q_4) \mid q_1, q_4 \in \mathbb{N}, q_1 + q_4 \geq 2\} \\
\mathrm{Res}_2^+(\eta^{(3)}) &= \{(q_1, 1, 0, q_4) \mid q_1, q_4 \in \mathbb{N}, q_1 + q_4 \geq 1\} \\
\mathrm{Res}_3^+(\eta^{(3)}) &= \{(q_1, 0, 1, q_4) \mid q_1, q_4 \in \mathbb{N}, q_1 + q_4 \geq 1\} \\
\mathrm{Res}_4^+(\eta^{(3)}) &= \mathrm{Res}_1^+(\eta^{(3)}).
\end{aligned}$$

Moreover for each multi-index of the form $(p, 0, 0, 12p)$ with $p \geq 1$, we get

$$\left\langle \begin{pmatrix} p \\ 0 \\ 0 \\ 12p \end{pmatrix}, \eta^{(1)} \right\rangle = 12p \in 3\,\mathbb{Z}.$$

Then it is easy to verify that

$$\mathrm{Res}_j([\varphi]) = \mathrm{Res}_j^+(\eta^{(2)}) \cap \mathrm{Res}_j^+(\eta^{(3)}),$$

for $j = 1, \ldots, 4$.

Remark 7.5. Last example shows that, even in the torsion case, there are vectors $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that, for any j, $\mathrm{Res}_j([\varphi])$ can be written as intersection of sets of additive resonances.

We have then the following definition.

Definition 7.1. Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \leq r \leq n$ and in the torsion case. We say that $[\varphi]$ is in the *impure torsion case* if, given $\eta^{(1)}, \ldots, \eta^{(r)}$ a reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/m, \beta_2, \ldots, \beta_r$, we have

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}), \tag{19}$$

for all $j \in \{1, \ldots, n\}$. Otherwise we say that $[\varphi]$ is in the *pure torsion case.*

The next result shows that the impure torsion case is well-defined, i.e., it does not depend on the chosen toric r-tuple.

Lemma 7.6. *Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$ and in the torsion case. Let $\eta^{(1)}, \dots, \eta^{(r)}$ be a reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/m, \beta_2, \dots, \beta_r$. If*

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}), \tag{20}$$

for all $j \in \{1, \dots, n\}$, then (20) holds for any other reduced toric r-tuple associated to $[\varphi]$.

Proof. Let $\xi^{(1)}, \dots, \xi^{(r)}$ be another reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/\widetilde{m}, \gamma_2, \dots, \gamma_r$. Since $\eta^{(1)}, \dots, \eta^{(r)}$ is in the impure torsion case, we have

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}),$$

but, thanks to Lemma 7.3, we have

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\xi^{(k)}),$$

for any $j = 1, \dots, n$, that is $\xi^{(1)}, \dots, \xi^{(r)}$ satisfy (20). □

Definition 7.2. Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin. We say that f is in the *impure torsion case* [resp., *in the pure torsion case*] if, denoting with $\lambda = \{\lambda_1, \dots, \lambda_n\}$ the spectrum of $\mathrm{d}f_O$, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$ is in the impure torsion case [resp., in the pure torsion case].

Theorem 7.7. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O of toric degree $1 \le r \le n$ and in the impure torsion case. Then it admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension $r-1$ commuting with f, and such that the columns of the weight matrix of the action are reduced torsion-free toric vectors associated to f.*

Proof. It follows from Theorem 3.1, Lemma 7.1 and Lemma 7.6. □

The next examples show that, in case of pure torsion there are more possible cases.

Example 7.8. Let us consider the vector

$$[\varphi] = \left[\frac{1}{6}\begin{pmatrix}1\\3\end{pmatrix} + \sqrt{2}\begin{pmatrix}1\\6\end{pmatrix}\right] \in (\mathbb{C}/\mathbb{Z})^2,$$

of toric degree 2. In this case $\mathcal{D}(1/6, \sqrt{2}) = \{(6h, 0) \mid h \in \mathbb{Z}\}$. We have

$$\langle P, \eta^{(2)} \rangle = p_1 + 6p_2 = 0$$

if and only if

$$P \in \begin{pmatrix}-6\\1\end{pmatrix}\mathbb{Z},$$

hence

$$\mathrm{Res}_1^+(\eta^{(2)}) = \varnothing \quad \text{and} \quad \mathrm{Res}_2^+(\eta^{(2)}) = \{(6,0)\}.$$

Since

$$\langle (6,-1), \eta^{(1)} \rangle = 3 \notin 6\mathbb{Z},$$

we have

$$\mathcal{D}(1/6, \sqrt{2}) \cap \mathrm{Adm}_j(\eta^{(1)}, \eta^{(2)}) = \{O\}$$

for $j = 1, 2$, so we have

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\eta^{(k)}) = \varnothing,$$

for $j = 1, 2$. Moreover, it is evident that the torsion is 2.

Example 7.9. Let us consider the vector

$$[\varphi] = \left[\frac{1}{7} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \sqrt{2} \begin{pmatrix} 1 \\ -6 \end{pmatrix} \right] \in (\mathbb{C}/\mathbb{Z})^2,$$

of toric degree 2. In this case $\mathcal{D}(1/7, \sqrt{2}) = \{(7h, 0) \mid h \in \mathbb{Z}\}$. We have

$$\mathrm{Res}_1^+(\eta^{(1)}) = \varnothing \quad \text{and} \quad \mathrm{Res}_2^+(\eta^{(1)}) = \{(3,0)\},$$

and

$$\mathrm{Res}_1^+(\eta^{(2)}) = \{(6h+1, h) \mid h \in \mathbb{N} \setminus \{0\}\} \quad \text{and} \quad \mathrm{Res}_2^+(\eta^{(2)}) = \{(6h, h+1) \mid h \in \mathbb{N} \setminus \{0\}\};$$

then

$$\mathrm{Res}_1^+(\eta^{(1)}) \cap \mathrm{Res}_1^+(\eta^{(2)}) = \varnothing \quad \text{and} \quad \mathrm{Res}_2^+(\eta^{(1)}) \cap \mathrm{Res}_2^+(\eta^{(2)}) = \varnothing.$$

However, we have

$$\langle (6h, h), \eta^{(1)} \rangle \in 9\mathbb{Z};$$

hence we have

$$\begin{aligned} \mathrm{Res}_1^+(\eta^{(2)}) \supset \mathrm{Res}_1([\varphi]) = \{(42h+1, 7h) \mid h \in \mathbb{N} \setminus \{0\}\} \supset \mathrm{Res}_1^+(\eta^{(1)}) \cap \mathrm{Res}_1^+(\eta^{(2)}) \\ \mathrm{Res}_2^+(\eta^{(2)}) \supset \mathrm{Res}_2([\varphi]) = \{(42h, 7h+1) \mid h \in \mathbb{N} \setminus \{0\}\} \supset \mathrm{Res}_2^+(\eta^{(1)}) \cap \mathrm{Res}_2^+(\eta^{(2)}). \end{aligned}$$

Moreover, it is not difficult to verify that the torsion is 7.

In the pure torsion case, one could ask whether, given a toric r-tuple $\eta^{(1)}, \ldots, \eta^{(r)}$ associated to $[\varphi]$ such that

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) \supset \mathrm{Res}_j([\varphi]) \supset \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\eta^{(k)}), \tag{21}$$

for some $j \in \{1, \ldots, n\}$, then this is true for any other toric r-tuple associated to $[\varphi]$. This is not always true, as next example shows.

Example 7.10. Let us consider the vector

$$[\varphi] = \left[\frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \sqrt{2} \begin{pmatrix} 1 \\ 6 \\ 0 \\ 0 \end{pmatrix} + \sqrt{3} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 5 \end{pmatrix} \right] \in (\mathbb{C}/\mathbb{Z})^4,$$

of toric degree 3. In this case $\mathcal{D}(1/3, \sqrt{2}, \sqrt{3}) = \{(3h, 0, 0) \mid h \in \mathbb{Z}\}$. We have

$$\mathrm{Res}_j^+(\eta^{(1)}) = \varnothing,$$

for $j = 1, \ldots, 4$,

$$\begin{aligned}
\mathrm{Res}_1^+(\eta^{(2)}) &= \{(1,0,p,q) \mid p,q \in \mathbb{N}, p+q \geq 1\} \\
\mathrm{Res}_2^+(\eta^{(2)}) &= \{(6,0,p,q) \mid p,q \in \mathbb{N}\} \cup \{(0,1,p,q) \mid p,q \in \mathbb{N}, p+q \geq 1\} \\
\mathrm{Res}_3^+(\eta^{(2)}) &= \{(0,0,p,q) \mid p,q \in \mathbb{N}, p+q \geq 2\} \\
\mathrm{Res}_4^+(\eta^{(2)}) &= \mathrm{Res}_3^+(\eta^{(2)}),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Res}_1^+(\eta^{(3)}) &= \{(h,k,5q,q) \mid h,k,q \in \mathbb{N}, h+k+6q \geq 2\} \\
\mathrm{Res}_2^+(\eta^{(3)}) &= \mathrm{Res}_1^+(\eta^{(3)}) \\
\mathrm{Res}_3^+(\eta^{(3)}) &= \{(h,k,5q+1,q) \mid h,k,q \in \mathbb{N}, h+k+6q \geq 1\} \\
\mathrm{Res}_4^+(\eta^{(3)}) &= \{(h,k,5(q-1),q) \mid h,k,q \in \mathbb{N}, h+k+6q \geq 7\}.
\end{aligned}$$

Then we have

$$\bigcap_{k=1}^{3} \mathrm{Res}_j^+(\eta^{(k)}) = \varnothing,$$

for $j = 1, \ldots, 4$, but it is not difficult to verify that

$$\mathrm{Res}_2([\varphi]) = \{(0,1,5q,q) \mid q \in \mathbb{N}^*\} \neq \varnothing \quad \text{and} \quad \mathrm{Res}_j([\varphi]) = \mathrm{Res}_j^+(\eta^{(2)}) \cap \mathrm{Res}_j^+(\eta^{(3)}) \;\; j = 1,3,4.$$

Then, since

$$\mathrm{Res}_2^+(\eta^{(2)}) \cap \mathrm{Res}_2^+(\eta^{(3)}) = \{(6,0,5q,q) \mid q \in \mathbb{N}\} \cup \{(0,1,5q,q) \mid q \in \mathbb{N}^*\} \neq \mathrm{Res}_2([\varphi]),$$

we are in the pure torsion case, but we cannot write all the resonances of $[\varphi]$ as intersection of the additive resonances of $\eta^{(1)}$, $\eta^{(2)}$ and $\eta^{(3)}$. However, we can write

$$[\varphi] = \left[\frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \\ -5 \end{pmatrix} + \sqrt{2} \begin{pmatrix} 1 \\ 6 \\ 0 \\ 0 \end{pmatrix} + \sqrt{3} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 5 \end{pmatrix} \right],$$

and it is not difficult to verify that, in this representation, we have

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=1}^{3} \mathrm{Res}_j^+(\xi^{(k)}),$$

for $j = 1, \ldots, 4$.

Example 7.11. If $[\varphi] \in (\mathbb{C}/\mathbb{Z})^2$ is given by Example 7.9, we saw that we can write it in the form

$$[\varphi] = \left[\frac{1}{\tau} \eta^{(1)} + \beta \eta^{(2)} \right]$$

so that

$$\mathrm{Res}_j^+(\eta^{(2)}) \supset \mathrm{Res}_j([\varphi]) \supset \mathrm{Res}_j^+(\eta^{(1)}) \cap \mathrm{Res}_j^+(\eta^{(2)}), \tag{22}$$

for all j. Furthermore, it is easy to check that $[\varphi]$ does not admit any reduced representation

$$[\varphi] = \left[\frac{1}{\tau q}\xi^{(1)} + \gamma\xi^{(2)}\right]$$

such that for all j we have

$$\mathrm{Res}_j([\varphi]) = \mathrm{Res}_j^+(\xi^{(1)}) \cap \mathrm{Res}_j^+(\xi^{(2)}). \tag{23}$$

We are then led to the following

Definition 7.3. Let $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ be of toric degree $1 \le r \le n$ and in the pure torsion case. We say that $[\varphi]$ *can be simplified* if it admits a reduced r-tuple of toric vectors $\eta^{(1)}, \dots, \eta^{(r)}$ such that

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\eta^{(k)}), \tag{24}$$

for all $j = 1, \dots, n$. The r-tuple $\eta^{(1)}, \dots, \eta^{(r)}$ is said a *simple reduced* r-tuple associated to $[\varphi]$.

Definition 7.4. Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin in the pure torsion case. We say that f *can be simplified* if, denoting with $\lambda = \{\lambda_1, \dots, \lambda_n\}$ the spectrum of $\mathrm{d}f_O$, the unique $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ such that $\lambda = e^{2\pi i[\varphi]}$ can be simplified.

Theorem 7.12. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O of toric degree $1 \le r \le n$ and in the pure torsion case, such that it can be simplified. Then:*

- *(i) if $\mathrm{d}f_O$ is diagonalizable, f admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with f and such that the columns of the weight matrix Θ of the action are a simple reduced r-tuple of toric vectors associated to f;*
- *(ii) if $\mathrm{d}f_O$ is not diagonalizable and there exists a simple reduced r-tuple of toric vectors associated to f such that its vectors are the columns of a matrix Θ compatible with $\mathrm{d}f_O$, f admits a holomorphic Poincaré-Dulac normalization if and only if there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with f and with weight matrix Θ.*

Proof. It follows from Theorem 3.1. □

Remark 7.13. Note that we cannot get rid of the compatibility hypothesis in the case of $\mathrm{d}f_O$ non diagonalizable, because if we change a simple reduced toric r-tuple as in Remark 7.2, it is not true that we obtain another simple reduced r-tuple. In fact, if $[\varphi] \in (\mathbb{C}/\mathbb{Z})^n$ has toric degree $1 \le r \le n$, and $\eta^{(1)}, \dots, \eta^{(r)}$ is a simple reduced r-tuple of toric vectors associated to $[\varphi]$ with toric coefficients $1/m, \beta_2, \dots, \beta_r$, but we have $[\varphi_j] = [\varphi_h]$ for some distinct coordinates j and h, and $\eta_j^{(1)} \neq \eta_h^{(1)}$, then for every $P \in \mathrm{Res}_l([\varphi])$, the equality

$$[\varphi] = \left[\frac{1}{m}\widetilde{\eta}^{(1)} + \sum_{k=2}^{r} \beta_k \eta^{(k)}\right]$$

with $\widetilde{\eta}^{(1)} = \eta^{(1)} - (\eta_j^{(1)} - \eta_h^{(1)})e_j$, only implies

$$\frac{\eta_j^{(1)} - \eta_h^{(1)}}{m}(\delta_{lh} - p_h) \in \mathbb{Z}$$

and there well can be resonant multi-indices with $p_h \neq 1$.

In case of pure torsion that cannot be simplified, we have the following result.

Proposition 7.14. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O of toric degree $1 \leq r \leq n$ and in the pure torsion case, such that it cannot be simplified. If there exists a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension r commuting with f and such that the columns of the weight matrix of the action are a reduced r-tuple of toric vectors associated to f, then f admits a holomorphic Poincaré-Dulac normalization.*

Proof. It follows from Theorem 3.1 and Lemma 7.1. □

We end this section describing an algorithm to decide when a vector $[\varphi]$ can be simplified.

We want to know when, given $[\varphi]$ in the torsion case,

$$[\varphi] = \left[\frac{1}{\tau p}\eta^{(1)} + \sum_{k=2}^{r}\beta_k\eta^{(k)}\right]$$

of toric degree r, torsion $\tau \geq 2$, and such that there is $j \in \{1, \ldots, n\}$ so that

$$\bigcap_{k=1}^{r} \mathrm{Res}_j^+(\eta^{(k)}) \subset \mathrm{Res}_j([\varphi]) \subset \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}), \tag{25}$$

there is another reduced representation

$$[\varphi] = \left[\frac{1}{\tau q}\xi^{(1)} + \sum_{k=2}^{r}\gamma_k\xi^{(k)}\right]$$

such that for all $j = 1, \ldots, n$ we have

$$\mathrm{Res}_j([\varphi]) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\xi^{(k)}). \tag{26}$$

We know that there must be $H \in \mathbb{Z}^n \setminus \{O\}$ such that

$$\frac{1}{\tau p}\eta^{(1)} + \sum_{k=2}^{r}\beta_k\eta^{(k)} = \frac{1}{\tau q}\xi^{(1)} + \sum_{k=2}^{r}\gamma_k\xi^{(k)} + H.$$

Since

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) = \bigcap_{k=2}^{r} \mathrm{Res}_j^+(\xi^{(k)}),$$

for any $j = 1, \ldots, n$, we have that

$$\frac{1}{\tau p}\langle \eta^{(1)}, P - e_j\rangle = \frac{1}{\tau q}\langle \xi^{(1)}, P - e_j\rangle + \langle H, P - e_j\rangle$$

for any $P \in \bigcap_{k=2}^r \mathrm{Res}_j^+(\eta^{(k)})$. Now, if $\langle \xi^{(1)}, P - e_j \rangle = 0$ it must be $\langle \eta^{(1)}, P - e_j \rangle \in \tau p \mathbb{Z}$. On the contrary, if $\langle \eta^{(1)}, P - e_j \rangle \in \tau p \mathbb{Z}$, then we would like to find H such that $\langle \xi^{(1)}, P - e_j \rangle = 0$ that is, for any $j = 1, \dots, n$,

$$\frac{1}{\tau p} \langle \eta^{(1)}, P - e_j \rangle = \langle H, P - e_j \rangle$$

for any $P \in \bigcap_{k=2}^r \mathrm{Res}_j^+(\eta^{(k)})$ with $\langle \eta^{(1)}, P - e_j \rangle \in \tau p \mathbb{Z}$. In fact, if such a vector exists, then, setting $q = p$, $\xi^{(1)} = \eta^{(1)} - \tau p H$, $\gamma_k = \beta_k$ and $\eta^{(k)} = \xi^{(k)}$ for $k = 2, \dots, r$, we get

$$[\varphi] = \left[\frac{1}{\tau p} \xi^{(1)} + \sum_{k=2}^r \gamma_k \xi^{(k)} \right],$$

and for any $P \in \mathrm{Res}_j([\varphi])$ we have $P \in \bigcap_{k=2}^r \mathrm{Res}_j^+(\xi^{(k)})$, and

$$\langle \xi^{(1)}, P - e_j \rangle = \langle \eta^{(1)}, P - e_j \rangle - \langle H, P - e_j \rangle = 0,$$

that is (26).

We then have to study the structure of the intersection of a submodule of $\mathbb{Z}^n$ with $\mathbb{N}^n$. It turns out that such a structure is the following. We thank Jean Écalle for suggesting the gist of the following argument.

Let $\mathcal{A} \subset \mathbb{Z}^n$ be a sub-module of $\mathbb{Z}^n$ where $n \in \mathbb{N}^*$, and let us denote by $\mathcal{A}^+$ the set $\mathcal{A} \cap \mathbb{N}^n$. For any vector $A = (a_1, \dots, a_n) \in \mathcal{A}$, we denote by

$$\mathrm{red}(A) = \frac{1}{\alpha} A = \left(\frac{a_1}{\alpha}, \dots, \frac{a_n}{\alpha} \right) \tag{27}$$

where α is the greatest common divisor of $a_1, \dots, a_n$. The *support* of a vector $A \in \mathbb{Z}^n$ is the set

$$\mathrm{supp}(A) = \{ j \in \{1, \dots, n\} \mid a_j \neq 0 \} \subseteq \{1, \dots, n\}.$$

Using the support we can then define a partial order on $\mathcal{A}^+$ as follows: we say that $A \subseteq B$ if $\mathrm{supp}(A) \subset \mathrm{supp}(B)$, or the supports are equal and $A \leq B$ in the usual lexicographic order.

Definition 7.5. Let $\mathcal{A} \subset \mathbb{Z}^n$ be any sub-module of $\mathbb{Z}^n$, where $n \in \mathbb{N}^*$, and let $\mathcal{A}^+$ be the set $\mathcal{A} \cap \mathbb{N}^n$. For any $A, B \in \mathcal{A}^+$ we define

$$A/B = \mathrm{red}(qA - pB) \tag{28}$$

where

$$\frac{p}{q} = \min_{j \in \mathrm{supp}(B)} \left(\frac{a_j}{b_j} \right).$$

Obviously, if $\mathrm{supp}(B) \subseteq \mathrm{supp}(A)$, then $A/B \in \mathcal{A}^+$ and $A/B \subseteq A$.

Definition 7.6. Let $\mathcal{A} \subset \mathbb{Z}^n$ be any sub-module of $\mathbb{Z}^n$, where $n \in \mathbb{N}^*$, and let $\mathcal{A}^+$ be the set $\mathcal{A} \cap \mathbb{N}^n$. An element M of $\mathcal{A}^+$ is said *minimal* if it is minimal with respect to the partial order $\subseteq$. An element C of $\mathcal{A}^+$ is said *cominimal* if for any minimal element M of $\mathcal{A}^+$ we have $C - M \notin \mathcal{A}^+$.

Minimal elements have to satisfy certain properties.

Lemma 7.15. *Let $\mathcal{A} \subset \mathbb{Z}^n$ be any sub-module of $\mathbb{Z}^n$, where $n \in \mathbb{N}^*$, and let $\mathcal{A}^+$ be the set $\mathcal{A} \cap \mathbb{N}^n$. Two minimal elements of $\mathcal{A}^+$ have distinct supports.*

Proof. Let M and P be two distinct minimal elements of $\mathcal{A}^+$ and suppose by contradiction that $\mathrm{supp}(M) = \mathrm{supp}(P)$. Then $A = M/P$ and $B = P/M$ both have supports strictly contained in the ones of M and P contradicting their minimality with respect to $\subseteq$. □

Corollary 7.16. *Let $\mathcal{A} \subset \mathbb{Z}^n$ be any sub-module of $\mathbb{Z}^n$, where $n \in \mathbb{N}^*$, and let $\mathcal{A}^+$ be the set $\mathcal{A} \cap \mathbb{N}^n$. Then $\mathcal{A}^+$ contains only a finite number of minimal elements.*

Minimal elements are a sort of generators of $\mathcal{A}^+$ in a sense that next result clarifies.

Lemma 7.17. *Let $\mathcal{A} \subset \mathbb{Z}^n$ be any sub-module of $\mathbb{Z}^n$, where $n \in \mathbb{N}^*$, and let $\mathcal{A}^+$ be the set $\mathcal{A} \cap \mathbb{N}^n$. Then every element A of $\mathcal{A}^+$ can be written in the form*

$$A = \frac{1}{\delta}(\alpha_1 M_1 + \cdots + \alpha_d M_d) \tag{29}$$

where $\alpha_1, \ldots, \alpha_d \in \mathbb{N}$, $M_1, \ldots, M_d$ are the minimal elements, and $\delta = \delta(\mathcal{A}^+) \in \mathbb{N}^*$ depends only on $\mathcal{A}^+$.

Proof. If A is non minimal, then there exists a minimal element $M_{j_1} \subseteq A$, and there exist $\gamma_1, \delta_1 \in \mathbb{Q}^+$ such that

$$A = \gamma_1 M_{j_1} + \delta_1 A_1,$$

where

$$A_1 = A/M_{j_1},$$

and $\mathrm{supp}(A_1) \subset \mathrm{supp}(A)$. If A_1 is not minimal, we can iterate this procedure getting

$$A_1 = \gamma_2 M_{j_2} + \delta_2 A_2,$$

with $\mathrm{supp}(A_2) \subset \mathrm{supp}(A_1) \subset \mathrm{supp}(A)$. The chain $\mathrm{supp}(A) \supset \mathrm{supp}(A_1) \supset \mathrm{supp}(A_3) \supset \cdots$ has to end because $\mathcal{A}^+ \subset \mathbb{N}^n$, then we eventually arrive to a decomposition of the form (29). Now $\delta = \delta(\mathcal{A}^+)$ cannot be greater than the least common multiple of all $|\det(\mathrm{M}^*)|$ where M^* varies in the square submatrices of order equal to the rank of the matrix having as columns all the minimal elements $M_1, \ldots, M_d$ of $\mathcal{A}^+$. ☐

The cominimal elements are finite too.

Lemma 7.18. *Let $\mathcal{A} \subset \mathbb{Z}^n$ be any sub-module of $\mathbb{Z}^n$, where $n \in \mathbb{N}^*$, and let $\mathcal{A}^+$ be the set $\mathcal{A} \cap \mathbb{N}^n$. Then $\mathcal{A}^+$ contains only a finite number of cominimal elements.*

Proof. Let us assume by contradiction that there is an infinite sequence of distinct cominimal elements $\{C_j\}$. Thanks to Lemma 7.17, for each $j \geq 1$, we have

$$C_j = \frac{1}{\delta} \sum_{k=1}^{d} \gamma_{jk} M_k$$

where $\gamma_{jk} \in \mathbb{N}$. Then there is an infinite subsequence $\{C_{j'}\}$ such that all the corresponding $(\gamma_{j',1}, \ldots, \gamma_{j',d})$ belong to a same class $(\gamma_1^*, \ldots, \gamma_d^*)$ modulo $\delta\mathbb{Z}^d$. Hence there is an infinite subsequence $\{C_{j''}\}$ such that at least one component γ_{j'',k_0} diverges as j'' tends to infinity, and such that the other components $\gamma_{j'',k}$ with $k \neq k_0$ do not decrease. Then there exist at least two cominimal elements $C_{j_1} \leq C_{j_2}$ such that

$$C_{j_2} - C_{j_1} = \sum_{k=1}^{d} \tilde{\gamma}_k M_k$$

with

$$\tilde{\gamma}_k = \frac{1}{\delta} (\gamma_{j_2,k} - \gamma_{j_1,k}) \in \mathbb{N}$$

meaning that C_{j_2} is not cominimal against the assumption. ☐

For each element of $\mathcal{A}^+$, we want to find a decomposition with natural coefficients into linear combination of a finite number of elements of $\mathcal{A}^+$. This is possible using minimal and cominimal elements, as shown in next result.

Proposition 7.19. *Let $\mathcal{A} \subset \mathbb{Z}^n$ be any sub-module of $\mathbb{Z}^n$, where $n \in \mathbb{N}^*$, and let $\mathcal{A}^+$ be the set $\mathcal{A} \cap \mathbb{N}^n$. Then for any $A \in \mathcal{A}^+$ there exist $l_1, \ldots, l_d \in \mathbb{N}$ such that*

$$A = \sum_{j=1}^{d} l_j M_j \tag{30}$$

or

$$A = C_h + \sum_{j=1}^{d} l_j M_j \tag{31}$$

for some $h \in \{1, \ldots, e\}$, where $M_1, \ldots, M_d$ are the minimal elements of $\mathcal{A}^+$, and $C_1, \ldots, C_e$ are the cominimal elements of $\mathcal{A}^+$.

Proof. If A is non cominimal, there exists a minimal element $M_{j_1} \leq A$; thus if $A_1 = A - M_{j_1}$ is not cominimal, we iterate the procedure. The chain $A \geq A_1 \geq A_3 \geq \cdots$ has to end with a zero, i.e., we get a decomposition of the form (30), or with a cominimal element, i.e., we get a decomposition of the form (31). □

Remark 7.20. Note that it can happen that the number of minimal elements of $\mathcal{A}^+$ is not equal to the maximum number of $\mathbb{Q}$-linearly independent elements of $\mathcal{A}^+$. In fact, if we consider the submodule $\mathcal{A}$ of $\mathbb{Z}^4$ orthogonal to $(1, -1, -1, 1)^T$, and $\mathcal{A}^+$, such a maximum is clearly 3, but we have four minimal elements

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

and we need all of them (and no cominimal) to ensure (30) and (31).

Returning to our problem, if now we consider

$$\mathcal{A} = \{Q \in \ \mathbb{Z}^n \mid \langle Q, \eta^{(k)} \rangle = 0, \text{ for } k = 2, \ldots, r\},$$

it is easy to verify that

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^+(\eta^{(k)}) = \mathcal{B}_0^+ \cup \mathcal{B}_j^+$$

where

$$\mathcal{B}_0^+ = \{P \in \mathbb{N}^n \mid P = Q + e_j, Q \in \mathcal{A}^+, |Q| \geq 1\}$$

and

$$\mathcal{B}_j^+ = \{P \in \mathbb{N}^n \mid P = Q + e_j, Q \in \mathcal{A}, q_h \geq 0, \text{ for } h \neq j, \ q_j = -1, |Q| \geq 1\}.$$

Notice that $Q \in \mathcal{B}_j^+$ if and only if we have

$$\langle \widehat{\eta}^{(k)}, \widehat{Q} \rangle = \eta_j^{(k)} \quad \text{for } k = 2, \ldots, r \tag{32}$$

where $\widehat{Q} = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n) \in \mathbb{N}^{n-1}$ and $\widehat{\eta}^{(k)} = (\eta_1^{(k)}, \ldots, \eta_{j-1}^{(k)}, \eta_{j+1}^{(k)}, \ldots, \eta_n^{(k)})$, i.e., $\widehat{Q}$ is a solution in $\mathbb{N}^{n-1}$ of the linear system with integer coefficients (32). Moreover, since $\mathcal{A}$ is a submodule of $\mathbb{Z}^n$, Proposition 7.19 applies to $\mathcal{A}^+$. Let $\mathfrak{M} = \{M_1, \ldots, M_d\}$ be the set of minimal elements of $\mathcal{A}^+$ and let $\mathfrak{C} = \{C_1, \ldots, C_e\}$ be the set of cominimal elements of $\mathcal{A}^+$ (recall that they all are different from O, hence their modulus is at least 1). We can thus consider the subsets $\{M'_1, \ldots, M'_s\} \subset \mathfrak{M}$ and $\{C'_1, \ldots, C'_t\} \subset \mathfrak{C}$ of the minimal and cominimal elements R of $\mathcal{A}^+$ such that $\langle \eta^{(1)}, R\rangle \in \tau p\mathbb{Z}$. Then $[\varphi]$ can be simplified if and only if there exists $H \in \mathbb{Z}^n$ such that

$$\langle H, M'_h\rangle = \frac{1}{\tau p}\langle \eta^{(1)}, M'_h\rangle,$$

for $1 \leq h \leq s$,

$$\langle H, C'_l\rangle = \frac{1}{\tau p}\langle \eta^{(1)}, C'_l\rangle,$$

for $1 \leq l \leq t$, and such that, for any $j = 1, \ldots, n$, we have

$$\langle \widehat{H}, \widehat{Q}\rangle - h_j = \frac{1}{\tau p}\left(\langle \widehat{\eta}^{(1)}, \widehat{Q}\rangle - \eta_j^{(1)}\right),$$

for every solution $\widehat{Q} \in \mathbb{N}^n$ of (32), with $|Q| \geq 1$, such that $\langle \widehat{Q}, \widehat{\eta}^{(1)}\rangle \in \tau p\, \eta_j^{(1)}\mathbb{Z}$.

8. Construction of torus actions

In this last section we shall see some conditions assuring the existence of the torus actions we need.

Let $X \in \mathfrak{X}_n$ be a germ of holomorphic vector field of $(\mathbb{C}^n, O)$ singular at the origin, in Poincaré-Dulac normal form, i.e.,

$$X = X^{\mathrm{dia}} + X^{\mathrm{nil}} + X^{\mathrm{res}}$$

where, denoting with ∂_j the partial derivative $\partial/\partial z_j$,

$$X^{\mathrm{dia}} = \sum_{j=1}^{n} \varphi_j z_j \partial_j,$$

X^{nil} is a linear nilpotent vector field singular at the origin such that

$$[X^{\mathrm{dia}}, X^{\mathrm{nil}}] = 0,$$

and X^{res} is a holomorphic vector field singular at the origin with no linear part and such that

$$[X^{\mathrm{dia}}, X^{\mathrm{res}}] = 0.$$

In particular

$$[X^{\mathrm{dia}}, X^{\mathrm{nil}} + X^{\mathrm{res}}] = 0.$$

Recall that the flows of two commuting vector fields also commute (see [Le] Prop. 18.5). We have

$$\exp(X^{\mathrm{dia}}) = \mathrm{Diag}(e^{\varphi_1}, \ldots, e^{\varphi_n})z.$$

and, in general for a linear vector field $X^{\rm lin} = \sum_{j=1}^{n} \left(\sum_{h=1}^{n} a_{hj} z_h\right) \partial_j$, we have

$$\exp(X^{\rm lin}) = e^A z,$$

where A is the matrix (a_{hj}). If Y is a holomorphic vector field singular at the origin with no linear part, then we have

$$\exp(tY)z = \sum_{k\geq 0} \frac{t^k}{k!} Y^k(z). \tag{33}$$

In fact, defining $K_t(z) = z + tY(z)$, we get $K_0(z) = z$ and $\frac{\partial}{\partial t} K_t(z)|_{t=0} = Y(z)$, then we have $\exp(tY)z = \lim_{m\to\infty}(K_{1/m})^m$, (see [AMR] Theorem 4.1.26), that is (33). Moreover, if V, W are two commuting vector fields, we have

$$\exp(t(V+W)) = \left(\sum_{k\geq 0} \frac{(tV)^k}{k!}\right)\left(\sum_{k\geq 0} \frac{(tW)^k}{k!}\right) = \exp(tV)\exp(tW).$$

Then we have the following result.

Proposition 8.1. *Let X be a germ of holomorphic vector field of $(\mathbb{C}^n, O)$, singular at the origin, and in Poincaré-Dulac normal form. Then its flow is a germ of biholomorphism of $(\mathbb{C}^n, O)$ in Poincaré-Dulac normal form.*

Proof. The flow of $X^{\rm nil} + X^{\rm res}$ is unipotent, then the linear part of the flow of X is Az with A triangular matrix with diagonal $\mathrm{Diag}(e^{\varphi_1}, \ldots, e^{\varphi_n})$, and the flow of X has to commute with the flow of $X^{\rm dia}$. □

In [Zu], Zung found that to find a Poincaré-Dulac holomorphic normalization for a germ of holomorphic vector field is the same as to find (and linearize) a suitable torus action which preserves the vector field. To deal with this problem he introduced the notion of *toric degree* of a vector field. The following definition is a reformulation of Zung's original one, clearer and more suitable to our needs.

Definition 8.1. The *toric degree* of a germ of holomorphic vector field X of $(\mathbb{C}^n, O)$ singular at the origin is the minimum $r \in \mathbb{N}$ such that the diagonalized semi-simple part $X^{\rm dia} = \sum_{j=1}^{n} \varphi_j z_j \partial_j$ of the linear term of X can be written as linear combination with complex coefficients of r diagonal vector fields with integer coefficients, i.e.,

$$X^{\rm dia} = \sum_{k=1}^{r} \alpha_k Z_k,$$

where $\alpha_1, \ldots, \alpha_r \in \mathbb{C}^*$ and $Z_k = \sum_{j=1}^{n} \rho_j^{(k)} z_j \partial_j$ with $\rho^{(k)} \in \mathbb{Z}^n$. The r-tuple $Z_1, \ldots, Z_k$ is called a *r-tuple of toric vector fields associated to* X, and the numbers $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$ are a r-tuple of *toric coefficients* of the toric r-tuple.

In particular, we have

$$\varphi = \sum_{k=1}^{r} \alpha_k \rho^{(k)}.$$

One can prove (see [Ra1] pp. 55–57) that the toric coefficients $\alpha_1, \ldots, \alpha_r$ are rationally independent, and $\rho_j^{(k)} = \rho_h^{(k)}$ whenever $\varphi_j = \varphi_h$, for every $k = 1, \ldots, r$, implying that

$$\mathrm{Res}_j^+(\varphi) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\rho^{(k)})$$

for all $j = 1, \ldots, n$.

Remark 8.2. A vector field has toric degree 1 if and only if, chosen a non-zero eigenvalue of its linear part, all the other eigenvalues are rational multiples of it; then in this case we have uniqueness of the toric vector field associated to X up to multiplication by a non-zero integer.

We recall the following definition from [Zu]

Definition 8.2. A germ of holomorphic vector field X of $(\mathbb{C}^n, O)$ singular at the origin is *integrable* if it has order 1 and there exists a positive integer $1 \leq m \leq n$ such that there exist m germs of holomorphic vector fields $X_1 = X, X_2, \ldots, X_m$ of $(\mathbb{C}^n, O)$ singular at the origin and of order 1, and $n - m$ germs of holomorphic functions $g_1, \ldots, g_{n-m}$ in $(\mathbb{C}^n, O)$ satisfying:

(i) $X_1, \ldots, X_m$ commute pairwise and are linearly independent, i.e., $X_1 \wedge \cdots \wedge X_m \not\equiv 0$;

(ii) $g_1, \ldots, g_{n-m}$ are common first integrals of $X_1, \ldots, X_m$, i.e., $X_j(g_k) = 0$ for any j and k, and they are functionally independent almost everywhere, i.e., $\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_{n-m} \not\equiv 0$.

Noticing that all the vector fields in the previous definition are integrable, we can define

Definition 8.3. Let $1 \leq m \leq n$. A *set of m integrable vector fields* of $(\mathbb{C}^n, O)$ is a set $X_1, \ldots, X_m$ of germs of holomorphic vector fields of $(\mathbb{C}^n, O)$ singular at the origin, of order 1 and such that:

(i) $X_1, \ldots, X_m$ commute pairwise and are linearly independent;

(ii) there exist $n - m$ germs of holomorphic functions $g_1, \ldots, g_{n-m}$ in $(\mathbb{C}^n, O)$ which are common first integrals of $X_1, \ldots, X_m$, and they are functionally independent almost everywhere.

Theorem 8.3. (Zung, 2002 [Zu]) *Let X be a germ of holomorphic vector field of $(\mathbb{C}^n, O)$ singular at the origin which is integrable. Then X admits a holomorphic Poincaré-Dulac normalization.*

As a corollary of Proposition 8.1, we obtain

Corollary 8.4. *The flow of a germ of integrable holomorphic vector field of $(\mathbb{C}^n, O)$ singular at the origin admits a holomorphic Poincaré-Dulac normalization.*

Moreover we have the following result

Theorem 8.5. (Zung, 2002 [Zu]) *Let $1 \leq m \leq n$. Every set of m integrable vector fields admits a simultaneous holomorphic Poincaré-Dulac normalization.*

Thus we have the following corollary

Corollary 8.6. *Let $1 \leq m \leq n$. The flows of a set of m integrable vector fields admit a simultaneous holomorphic Poincaré-Dulac normalization.*

Remark 8.7. The last corollary means that we can conjugate $X_1, \ldots, X_m$ to a m-tuple of vector fields containing only monomials belonging to the intersection of the additive resonances of the eigenvalues of the linear terms of $X_1, \ldots, X_m$.

Now, we introduce an analogous for germs of biholomorphisms of the notion of integrability we described above.

Definition 8.4. A germ of biholomorphism f of $(\mathbb{C}^n, O)$ fixing the origin *commutes with a set of integrable vector fields* if there exists a positive integer $1 \leq m \leq n$, such that there exists a set of m germs of holomorphic integrable vector fields $X_1, \ldots, X_m$ such that

$$\mathrm{d}f(X_j) = X_j \circ f$$

for each $j = 1, \ldots, m$.

Remark 8.8. A germ of biholomorphism f of $(\mathbb{C}^n, O)$ commutes with a vector field X according to the previous definition if and only if it commutes with the flow generated by X.

Theorem 8.9. *Let f be a germ of biholomorphism of $(\mathbb{C}^n, O)$ fixing the origin and commuting with a set of integrable holomorphic vector fields $X_1, \ldots, X_m$. Then f commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a torus of dimension equal to the toric degree r of X_1 and such that the columns of the weight matrix of the action are a r-tuple of toric vectors associated to X_1.*

Proof. From the proof of Theorem 8.3 (see [Zu]) we get r holomorphic vector field $\mathcal{Z}_1, \ldots, \mathcal{Z}_r$ which generate a $\mathbb{T}^r$ action preserving $X_1, \ldots, X_m$. Moreover we get $a_{1,k}, \ldots, a_{m,k}$ holomorphic functions constant on the connected components of each level set $L_y = \mathbf{g}^{-1}(\mathbf{g}(y))$, where we denote by $\mathbf{g} = (g_1, \ldots, g_{n-m})$ the $(n-m)$-tuple of common first integrals of $X_1, \ldots, X_m$, such that

$$\mathcal{Z}_k = \sum_{j=1}^{m} a_{j,k} X_j,$$

for each $k = 1, \ldots, r$. Then, for each $k = 1, \ldots, r$, we have

$$\begin{aligned}
\mathrm{d}f(\mathcal{Z}_k) &= \mathrm{d}f\left(\sum_{j=1}^{m} a_{j,k} X_j\right) \\
&= \sum_{j=1}^{m} (a_{j,k} \circ f)\, \mathrm{d}f(X_j) \\
&= \sum_{j=1}^{m} (a_{j,k} \circ f)\, (X_j \circ f) \\
&= \mathcal{Z}_k \circ f.
\end{aligned}$$

Thus the torus action commutes with f as we wanted. □

Corollary 8.10. *Let f be a germ of biholomorphism of $(\mathbb{C}^n, O)$ fixing the origin and commuting with a set of integrable holomorphic vector fields. Then f is holomorphically conjugated to a germ containing only monomials belonging to the intersection of the additive resonances of the eigenvalues of the linear terms of $X_1, \ldots, X_m$.*

Proof. The assertion follows from Theorem 8.9, Corollary 8.6 and Theorem 3.1. □

Then we also have the following

Corollary 8.11. *Let f be a germ of biholomorphism of $(\mathbb{C}^n, O)$ fixing the origin and commuting with a set of integrable holomorphic vector fields, such that the intersection of the additive resonances of the eigenvalues of the linear terms of $X_1, \ldots, X_m$ is equal or contained in the set of resonances of the spectrum of $\mathrm{d}f_O$. Then f admits a holomorphic Poincaré-Dulac normalization.*

Remark 8.12. A slight generalization of the proof of Theorem 8.3 shows that in the statement of Theorem 8.9 it is not necessary for all the vector fields $X_2, \dots, X_m$ to have order 1; however, it is still necessary that the whole set of vector fields $X_1, \dots, X_m$ commutes with f. We refer to [Ra4] for the precise statement and detailed proof.

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9. Construction of torus actions (long version)

In this last section we shall see some conditions assuring the existence of the torus actions we need. **sistemare**

Let $X \in \mathfrak{X}_n$ be a germ of holomorphic vector field of $(\mathbb{C}^n, O)$ singular at the origin, in Poincaré-Dulac normal form, i.e.,

$$X = X^{\rm dia} + X^{\rm nil} + X^{\rm res}$$

where, denoting with ∂_j the partial derivative $\partial/\partial z_j$,

$$X^{\rm dia} = \sum_{j=1}^{n} \varphi_j z_j \partial_j,$$

$X^{\rm nil}$ is a linear nilpotent vector field singular at the origin such that

$$[X^{\rm dia}, X^{\rm nil}] = 0,$$

$X^{\rm res}$ is a holomorphic vector field singular at the origin with no linear part and such that

$$[X^{\rm dia}, X^{\rm res}] = 0.$$

In particular

$$[X^{\rm dia}, X^{\rm nil} + X^{\rm res}] = 0.$$

Recall that the flows of two commuting vector fields also commute (see [Le] Prop. 18.5). We have

$$\exp(X^{\rm dia}) = {\rm Diag}(e^{\varphi_1}, \ldots, e^{\varphi_n})z.$$

and, in general for a linear vector field $X^{\rm lin} = \sum_{j=1}^{n} \left(\sum_{h=1}^{n} a_{hj} z_h\right) \partial_j$, we have

$$\exp(X^{\rm lin}) = e^{A} z,$$

where A is the matrix (a_{hj}). If Y is a holomorphic vector field singular at the origin with no linear part, then we have

$$\exp(tX)z = \sum_{k\geq 0} \frac{t^k}{k!} X^k(z). \tag{34}$$

In fact, defining $K_t(z) = z + tX(z)$, we get $K_0(z) = z$ and $\frac{\partial}{\partial t}K_t(z)|_{t=0} = X(z)$, then we have $\exp(tX)z = \lim_{m\to\infty}(K_{1/m})^m$, (see [AMR] Theorem 4.1.26), that is (34). Moreover, if V, W are two commuting vector fields, we have

$$\exp(t(V+W)) = \left(\sum_{k\geq 0}\frac{(tV)^k}{k!}\right)\left(\sum_{k\geq 0}\frac{(tW)^k}{k!}\right) = \exp(tV)\exp(tW).$$

Then we have the following result.

Proposition 9.1. *Let X be a germ of holomorphic vector field of $(\mathbb{C}^n, O)$, singular at the origin, and in Poincaré-Dulac normal form. Then its flow is a germ of biholomorphism of $(\mathbb{C}^n, O)$ in Poincaré-Dulac normal form.*

Proof. The flow of $X^{\mathrm{nil}} + X^{\mathrm{res}}$ is unipotent, then the linear part of the flow of X is Az with A triangular matrix with diagonal $\mathrm{Diag}(e^{\varphi_1}, \ldots, e^{\varphi_n})$, and the flow of X has commute with the flow of X^{dia}. □

In [Zu], Zung found that to find a Poincaré-Dulac holomorphic normalization for a germ of holomorphic vector field is the same as to find (and linearize) a suitable torus action which preserves the vector field. To dealt with this problem he introducing the notion of *toric degree* of a vector field. The following definition is a reformulation of Zung's original one, clearer and more suitable to our needs.

Definition 9.1. The *toric degree* of a germ of holomorphic vector field X of $(\mathbb{C}^n, O)$ singular at the origin is the minimum $r \in \mathbb{N}$ such that the semi-simple part $X^{\mathrm{dia}} = \sum_{j=1}^n \varphi_j z_j \partial_j$ of the linear term of X can be written as linear combination with complex coefficients of r diagonal vector fields with integer coefficients, i.e.,

$$X^{\mathrm{dia}} = \sum_{k=1}^r \alpha_k Z_k,$$

where $\alpha_1, \ldots, \alpha_r \in \mathbb{C}^*$ and $Z_k = \sum_{j=1}^n \rho_j^{(k)} z_j \partial_j$ with $\rho^{(k)} \in \mathbb{Z}^n$. The r-tuple $Z_1, \ldots, Z_k$ is called a *r-tuple of toric vector fields associated to* X, and the numbers $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$ are a r-tuple of *toric coefficients* of the toric r-tuple.

In particular, we have

$$\varphi = \sum_{k=1}^r \alpha_k \rho^{(k)}.$$

One can prove that the toric coefficients $\alpha_1, \ldots, \alpha_r$ are rationally independent, and $\rho_j^{(k)} = \rho_h^{(k)}$ whenever $\varphi_j = \varphi_h$, for every $k = 1, \ldots, r$, implying that

$$\mathrm{Res}_j^+(\varphi) = \bigcap_{k=1}^r \mathrm{Res}_j^+(\rho^{(k)})$$

for all $j = 1, \ldots, n$.

Remark 9.2. A vector field has toric degree 1 if and only if, chosen a non-zero eigenvalue of its linear part, all the other eigenvalues are rational multiplies of it; then in this case we have uniqueness of the toric vector field associated to X up to multiplication by a non-zero integer.

We recall the following definition from [Zu]

Definition 9.2. A germ of holomorphic vector field X of $(\mathbb{C}^n, O)$ singular at the origin is *integrable* if it has order 1 and there exists a positive integer $1 \le m \le n$ such that there exist m germs of holomorphic vector fields $X_1 = X, X_2, \dots, X_m$ of $(\mathbb{C}^n, O)$ singular at the origin and of order 1, and $n - m$ germs of holomorphic functions $g_1, \dots, g_{n-m}$ in $(\mathbb{C}^n, O)$ satisfying:

(i) $X_1, \dots, X_m$ commute pairwise and are linearly independent, i.e., $X_1 \wedge \cdots \wedge X_m \not\equiv 0$;

(ii) $g_1, \dots, g_{n-m}$ are common first integrals of $X_1, \dots, X_m$, i.e., $X_j(g_k) = 0$ for any j and k, and they are functionally independent almost everywhere, i.e., $\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_{n-m} \not\equiv 0$.

Dire che tutti i campi vettoriali hanno ordine 1

Noticing that all the vector fields in the previous definition are integrable, we can define

Definition 9.3. Let $1 \le m \le n$. A *set of m integrable vector fields* of $(\mathbb{C}^n, O)$ is a set $X_1, \dots, X_m$ of germs of holomorphic vector fields of $(\mathbb{C}^n, O)$ singular at the origin, of order 1 and such that:

(i) $X_1, \dots, X_m$ commute pairwise and are linearly independent;

(ii) there exist $n - m$ germs of holomorphic functions $g_1, \dots, g_{n-m}$ in $(\mathbb{C}^n, O)$ which are common first integrals of $X_1, \dots, X_m$, and they are functionally independent almost everywhere.

it is not necessary for all of them to have order 1, at least in Zung's theorem and in the statements you have up to now; I'll shall indicate you later the changes needed in the proof if you have some field vanishing at a higher order at the origin. If I'm not mistaken, you need that the field you use to build the torus action has order 1, because you use the linear part of the field to control the weights of the torus action (so to get them compatible with the linear part of the field). Furthermore, it is not that interesting to find a Poincaré-Dulac normal form of a vector field with vanishing linear part: in that case all monomial are resonant, and so the field is already in normal form. . .

Theorem 9.3. (Zung, 2002 [Zu]) *Let X be a germ of holomorphic vector field of $(\mathbb{C}^n, O)$ singular at the origin which is integrable. Then X admits a holomorphic Poincaré-Dulac normalization.*

As a corollary of Proposition 9.1, we obtain

Corollary 9.4. *The flow of a germ of integrable holomorphic vector field of $(\mathbb{C}^n, O)$ singular at the origin admits a holomorphic Poincaré-Dulac normalization.*

Moreover we have the following result

Theorem 9.5. (Zung, 2002 [Zu]) *Let $1 \le m \le n$. Every set of m integrable vector fields admits a simultaneous holomorphic Poincaré-Dulac normalization.*

Thus we have the following corollary

Corollary 9.6. *Let $1 \le m \le n$. The flows of a set of m integrable vector fields admit a simultaneous holomorphic Poincaré-Dulac normalization.*

Remark 9.7. The last corollary means that we can conjugate $X_1, \dots, X_m$ to a m-tuple of vector fields containing only monomials belonging to the intersection of the additive resonances of the eigenvalues of the linear terms of $X_1, \dots, X_m$.

Questo significa, che non solo li ho normalizzati ma, se i loro insiemi di risonanze sono distinti, sono riuscita a togliere anche dei monomi risonanti, perché, ad esempio, φ_1 dovrà essere in forma normale di P-D sia rispetto all'azione relativa a X_1 di cui è il flusso, sia rispetto all'azione relativa a X_2 e così via.

Now, we introduce an analogous for germs of biholomorphisms of the notion of integrability we described above.

Definition 9.4. A germ of biholomorphism f of $(\mathbb{C}^n, O)$ fixing the origin *commutes with a set of integrable vector fields* if there exists a positive integer $1 \leq m \leq n$, such that there exists a set of m germs of holomorphic integrable vector fields $X_1, \ldots, X_m$ such that

$$\mathrm{d}f(X_j) = X_j \circ f$$

for each $j = 1, \ldots, m$.

Remark 9.8. A germ of biholomorphism f of $(\mathbb{C}^n, O)$ commutes with a vector field X according to the previous definition if and only if it commutes with the flow generated by X.

Theorem 9.9. *Let f be a germ of biholomorphism of $(\mathbb{C}^n, O)$ fixing the origin. If f commutes with a set of n integrable holomorphic vector fields, then it commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a 1-torus.*

Proof. Let us fix a holomorphic system of coordinates $\mathbf{z} = (z_1, \ldots, z_n)$ in a neighbourhood of the origin of $\mathbb{C}^n$ in which X_1 is in Poincaré-Dulac normal form up to order $D \in \mathbb{N}$, with D sufficiently large. Let

$$Z^D = \sum_{j=1}^{n} i\theta_j z_j \partial_j$$

be a toric vector field associated to X_1. Since $[X_1, X_h] = 0$ for $h = 2, \ldots, n$, from Lemma **di Zung, inserire con dimostrazione**, we have

$$[Z^D, X_j] = O(|\mathbf{z}|^D), \tag{35}$$

for $j = 1, \ldots, n$. Thanks to the hypotheses, there exist $a_1^D, \ldots, a_n^D \colon \mathbb{C}^n \to \mathbb{C}$ holomorphic functions such that

$$Z^D(\mathbf{z}) = \sum_{j=1}^{n} a_j^D(\mathbf{z}) X_j(\mathbf{z});$$

then (35) implies, since $X_j = O(|\mathbf{z}|)$, that

$$|a_j^D(\mathbf{z}) - a_j^D(0)| = O(|\mathbf{z}|^D).$$

Now we normalize X_1 up to order $D+1$ via a holomorphic, tangent to the identity, change of coordinate $\mathbf{w} = \varphi^{D+1}(\mathbf{z})$ in a neighbourhood of O (we can always do it, up to shrinking the neighbourhood, and $\varphi^{D+1} - \mathrm{Id}$ will be of order $D+1$). Setting

$$Z^{D+1}(\mathbf{w}) = \sum_{j=1}^{n} i\theta_j w_j \frac{\partial}{\partial w_j},$$

as before, from Lemma **di Zung, inserire con dimostrazione**, we have

$$[Z^{D+1}, X_j](\mathbf{w}) = O(|\mathbf{w}|^{D+1}), \tag{36}$$

for $j = 1, \ldots, n$, and there exist $a_1^{D+1}, \ldots, a_n^{D+1} \colon \mathbb{C}^n \to \mathbb{C}$ holomorphic functions such that

$$Z^{D+1}(\mathbf{w}) = \sum_{j=1}^{n} a_j^{D+1}(\mathbf{w}) X_j(\mathbf{w}),$$

with

$$|a_j^{D+1}(\mathbf{w}) - a_j^{D+1}(0)| = O(|\mathbf{w}|^{D+1}).$$

if the highest order of vanishing at the origin of the fields $x_1, \ldots, X_n$ is $\ell \geq 1$, then here we should replace "$O(|\mathbf{z}|^D)$" by "$O(|\mathbf{z}|^{D-\ell+1})$". Similar changes are needed in line 6 of page 16, in equations (7) and (8), and possibly elsewhere; but as soon as $D > \ell$ there is no problem and the rest of the proof goes on unchanged.

In the new coordinates, **the $\hat{\varphi}_j$ have not been defined... mi sembra fosse il pezzo non lineare del cambio di coordinate**

$$\begin{aligned} Z^D(\mathbf{w}) &= \sum_{j=1}^n \left(i\theta_j w_j + i\theta_j \widehat{\varphi}_j(\mathbf{w})\right) \sum_{k=1}^n \frac{\partial w_k}{\partial z_j} \frac{\partial}{\partial w_k} \\ &= \sum_{k=1}^n \left(i\theta_k w_k + \psi_k(\mathbf{w})\right) \frac{\partial}{\partial w_k} \end{aligned}$$

where $\psi_k(\mathbf{w}) = O(|\mathbf{w}|^{D+1})$. Moreover the Z^D are defined in a uniform neighbourhood of the origin because they are obtained by polynomial changes of variables. We also have

$$Z^D(\mathbf{w}) = \sum_{j=1}^n a_j^D(\mathbf{w}) X_j(\mathbf{w}),$$

with

$$|a_j^D(\mathbf{w}) - a_j^D(0)| = O(|\mathbf{w}|^D). \tag{37}$$

Then, since Z^{D+1} coincides with Z^D up to order D,

$$\begin{aligned} |Z^{D+1}(\mathbf{w}) - Z^D(\mathbf{w})| &= \left| \sum_{j=1}^n (a_j^{D+1}(\mathbf{w}) - a_j^D(\mathbf{w})) X_j(\mathbf{w}) \right| \\ &= \left| \sum_{j=1}^n -\psi_k(\mathbf{w}) \frac{\partial}{\partial w_k} \right| \\ &= O(|\mathbf{w}|^{D+1}) \end{aligned}$$

thus

$$|a_j^{D+1}(\mathbf{w}) - a_j^D(\mathbf{w})| = O(|\mathbf{w}|^D) \tag{38}$$

for $j = 1, \ldots, n$. Then $a_j^D(0) = a_j(0)$ does not depend on D for any $j = 1, \ldots, n$. Set

$$Z = \sum_{j=1}^n a_j(0) X_j.$$

The holomorphic vector field Z is 2π-periodic because, from (37), it is arbitrarily close to a 2π-periodic vector field. Then we have

$$\begin{aligned} \mathrm{d}f(Z) &= \mathrm{d}f\left(\sum_{j=1}^{n} a_j(0) X_j\right) \\ &= \sum_{j=1}^{n} a_j(0) \mathrm{d}f(X_j) \\ &= \sum_{j=1}^{n} a_j(0) X_j \circ f \\ &= Z \circ f, \end{aligned}$$

which is equivalent to the thesis because Z is 2π-periodic. □

Theorem 9.10. *Let f be a germ of biholomorphism of $(\mathbb{C}^n, O)$ fixing the origin and commuting with a set of integrable holomorphic vector fields. Then f commutes with a holomorphic effective action on $(\mathbb{C}^n, O)$ of a 1-torus.*

Proof. From the hypothesis, there exists a positive integer $1 \leq m \leq n$, such that there exists a set of m germs of holomorphic integrable vector fields $X_1, X_2, \dots, X_m$ of $(\mathbb{C}^n, O)$ singular at the origin such that

$$\mathrm{d}f(X_j) = X_j \circ f$$

for each $j = 1, \dots, m$.

We dealt with the case $n = m$ in Theorem 9.9. Let us now consider the case $1 \leq m < n$.

Let us fix a holomorphic system of coordinates $\mathbf{z} = (z_1, \dots, z_n)$ in a neighbourhood of the origin of $\mathbb{C}^n$, a standard Hermitian metric in $\mathbb{C}^n$ and a positive sufficiently small number ε_0. Let S be the singular locus of the n-tuple of vector fields $X_1, \dots, X_m$ and of the functions $g_1, \dots, g_{n-m}$, i.e.,

$$\{\mathbf{z} \in \mathbb{C}^n : |\mathbf{z}| < \varepsilon_0,\, X_1 \wedge \cdots \wedge X_m(\mathbf{z}) = 0\} \cup \{\mathbf{z} \in \mathbb{C}^n : |\mathbf{z}| < \varepsilon_0,\, \mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_{n-m}(\mathbf{z}) = 0\}.$$

Thanks to the hypotheses, S is a complex analytic set of complex codimension at least 1; then it is possible to write it locally as the zero locus of a finite number of complex holomorphic functions, $S = \{h_1 = 0, \dots, h_l = 0\}$, and, using Łojasiewicz inequalities (**inserire riferimento a libro**), there exist a positive integer $N > 0$ and a positive constant $C > 0$ such that, for any $\mathbf{z}$ with $|\mathbf{z}| < \varepsilon_0$ we have the following Łojasiewicz inequalities

$$\begin{aligned} &\|X_1 \wedge \cdots \wedge X_m(\mathbf{z})\| \geq C\, d(\mathbf{z}, S)^N \\ &\|\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_{n-m}(\mathbf{z})\| \geq C\, d(\mathbf{z}, S)^N, \end{aligned} \tag{39}$$

where the norms are the standard norms on the considered spaces and the distance is the Euclidean distance.

For each positive integer d and small positive number $\varepsilon(d)$ (which shall be chosen later in function of d with $\lim_{d\to\infty} \varepsilon(d) = 0$), let us define the following open subset of $\mathbb{C}^n$

$$U_{d,\varepsilon(d)} = \{\mathbf{z} \in \mathbb{C}^n : |\mathbf{z}| < \varepsilon(d),\, d(\mathbf{z}, S) > |\mathbf{z}|^d\}.$$

We will define a holomorphic vector field $\mathcal{Z}$ in $U_{d,\varepsilon(d)}$, periodic with period 2π, and in such a way that, for any two positive distinct integers d_1, d_2, the vector field $\mathcal{Z}$ defined in $U_{d_1,\varepsilon(d_1)}$ coincides, in the intersection $U_{d_1,\varepsilon(d_1)} \cap U_{d_2,\varepsilon(d_2)}$, with the one defined in $U_{d_2,\varepsilon(d_2)}$.

Up to holomorphic, tangent to the identity, changes of coordinates, we may assume X_1 to be in Poincaré-Dulac normal form up to order $D(d) \in \mathbb{N}$, with $D(d) = 4dN + 2$ (in particular $\lim_{d\to\infty} D(d) = +\infty$). Let

$$Z^d = \sum_{j=1}^{n} i\theta_j z_j \partial_j$$

be a toric vector field associated to X_1. Since $[X_1, X_h] = 0$ for $h = 2, \dots, n$, from Lemma **di Zung, inserire con dimostrazione**, we have

$$[Z^d, X_j] = O(|\mathbf{z}|^{D(d)}),$$

for any $j = 1, \dots, m$, and

$$Z^d(\mathbf{g})(\mathbf{z}) = O(|\mathbf{z}|^{D(d)})$$

where $\mathbf{g} = (g_1, \dots, g_{n-m})$ is the $(n-m)$-tuple of common first integrals of $X_1, \dots, X_m$.

Let y be an arbitrary point in $U_{d,\varepsilon(d)}$. Then, thanks to inequalities (39) and to the definition of $U_{d,\varepsilon(d)}$, we have

$$\begin{aligned} &\|X_1 \wedge \cdots \wedge X_m(y)\| \geq C\,|y|^{dN} \\ &\|\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_{n-m}(y)\| \geq C\,|y|^{dN}. \end{aligned} \tag{40}$$

Let us denote by $\Gamma^d(t,y) = \Gamma^d(t)$ the closed curve, $t \in [0, 2\pi]$, which is the orbit of the periodic vector field $Z^d f$ starting at y. Then we have $\Gamma^d(0) = y$ and, for $\varepsilon(d)$ small enough, we have $\frac{1}{2}|y| \leq |\Gamma^d(t)| \leq 2|y|$ for any t in $[0, 2\pi]$. Then, for any x in Γ^d we have

$$\begin{aligned} &\|X_1 \wedge \cdots \wedge X_m(x)\| > \frac{C}{2^{dN}}\,|y|^{dN} \\ &\|\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_{n-m}(x)\| > \frac{C}{2^{dN}}\,|y|^{dN}. \end{aligned} \tag{41}$$

Since Z^d commutes with $X_1, \dots, X_m$ up to order $D(d)$ and $\mathbf{g}$ is a first integral of Z^d up to order $D(d)$, for $\varepsilon(d)$ small, we have the following inequalities

$$\begin{aligned} &|\mathbf{g}(x) - \mathbf{g}(y)| < |y|^{D_1(d)} \\ &|[X_j, Z^d](x)| < |y|^{D_1(d)} \quad \forall j = 1, \dots, m \end{aligned} \tag{42}$$

fo any x belonging yo Γ^d_k, where $D_1(d) = dN + 3$, (which is larger than $dN + 2$ and verifies $D_1(d) < D(d) - 1 = 4dN + 1$ for every d). In fact

$$|\mathbf{g}(x) - \mathbf{g}(y)| \leq C_1 |Z^d(\mathbf{g})(y)| \leq C_2 |y|^{D(d)} < |y|^{D_1(d)}$$

and, for any $j = 1, \dots, m$, we have

$$|[X_j, Z^d](x)| \leq C_3 |x|^{D(d)} \leq 2^{D(d)} C_3 |y|^{D(d)} < |y|^{D_1(d)}.$$

The inequalities (41) and (42) imply the following facts:

a) For any point y the regular part of the level set $L_y = \mathbf{g}^{-1}(\mathbf{g}(y))$ has complex dimension m, and its tangent space at each point is spanned by $X_1, \ldots, X_m$. Moreover, the regular part of L_y has an affine flat structure given by the vector fields $X_1, \ldots, X_m$, because they commute.

b) The curve Γ^d can be projected orthogonally on a smooth closed curve $\widehat{\Gamma}^d(t)$ lying on L_y and close to Γ^d in the C^1-topology: the distance between $\widehat{\Gamma}^d$ and Γ^d in the C^1-topology is bounded from above by $|y|^{D_2(d)}$, where $D_2(d) = dN + 1$.

c) We can write $\mathrm{d}\widehat{\Gamma}^d(t)/\mathrm{d}t$ in the form $\sum_{j=1}^m \mathrm{Re}(a^j(t)X_j(\widehat{\Gamma}^d(t)))$, and the holomorphic functions $a^j(t)$ are almost constant, in the sense that

$$|a^j(t) - a^j_y(0)| \leq |y|^{D_3(d)},$$

for $t \in [0, 2\pi]$, where $D_3(d)$ is positive, for example $D_3(d) = D_2(d) - 1 = dN$. This follows from the almost commutativity of $X_1, \ldots, X_m$ with Z^d and from the fact that, thanks to b), we have $|\mathrm{d}\widehat{\Gamma}^d(t)/\mathrm{d}t - \mathrm{Re}(Z^d(\widehat{\Gamma}^d_k(t)))| < |y|^{D_2(d)}$. In fact, since $X_1, \ldots, X_m$ commute, in a suitable system of coordinates $z_1, \ldots, z_n$ we may assume that each X_j coincides with ∂_j for $j = 1, \ldots, m$. Writing Z^d in the form $\sum_{j=1}^n \zeta_j(\mathbf{z})\partial_j$ in these coordinates, since Z^d almost commutes with $X_1, \ldots, X_m$ and it is almost tangent to the level sets, the functions $\zeta_1, \ldots, \zeta_m$, are almost constant along the chosen orbit of Z^d, whereas $\zeta_{m+1}, \ldots, \zeta_n$ are almost zero. Projecting on the level set those functions remain almost constant.

d) Arguing analogously to what we did in the proof of Theorem 9.9, there exist complex numbers $a^1, \ldots, a^m$ such that $|a^j - a^j_y(0)| \leq |y|^{D_3(d)}$, and the time-$2\pi$ flow of the vector field $\sum_{j=1}^m a^j X_j$ in L_y fixes y. Then the real vector field $\mathrm{Re}(\sum_{j=1}^m a^j X_j)$ has a periodic orbit of period 2π passing through y, and this orbit is C^1-close to $\widehat{\Gamma}^d(t, y)$.

e) Thanks to the affine flat structure of L_y, the numbers $a^1, \ldots, a^m$ are well-defined, i.e., unique, and they do not depend, at least locally, on the choice of y in L_y. We can consider $a^1, \ldots, a^m$ as functions of y: $a^1(y), \ldots, a^m(y)$. These functions are holomorphic, due to the holomorphic implicit function theorem, constant on the connected components in $U_{d,\varepsilon(d)}$ of the level sets of $\mathbf{g}$, and they are uniformly bounded in $U_{d,\varepsilon(d)}$ by a constant, provided that $\varepsilon(d)$ is small enough.

Let us now define the vector field $\mathcal{Z}$ as follows

$$\mathcal{Z}(y) = \sum_{j=1}^m a^j(y)X_j(y).$$

Then $\mathcal{Z}$ is a holomorphic vector field in $U_{d,\varepsilon(d)}$ with the following properties:

(a) $\mathcal{Z}$ is uniformly bounded by a constant, and it is periodic with period 2π, at least in an open subset of $U_{d,\varepsilon(d)}$.

(b) If $\mathcal{Z}$ is a vector field defined as above for $U_{d,\varepsilon(d)}$, and $\mathcal{Z}'$ is another vector field defined as above but for $U_{d',\varepsilon(d')}$, with $d \neq d'$, then $\mathcal{Z}$ and $\mathcal{Z}'$ coincide in $U_{d,\varepsilon(d)} \cap U_{d',\varepsilon(d')}$. In fact, the vector field $\mathcal{Z}$ commutes with $\mathcal{Z}'$ on $U_{d,\varepsilon(d)} \cap U_{d',\varepsilon(d')}$ by construction, and $\mathcal{Z} - \mathcal{Z}'$ is tangent to the level sets of $\mathbf{g}$ in $U_{d,\varepsilon(d)} \cap U_{d',\varepsilon(d')}$ and it is a constant vector field with respect to the affine flat structure on each level set. Moreover $\mathcal{Z} - \mathcal{Z}'$ is periodic of period 2π on the considered intersection; but the coefficients of $\mathcal{Z} - \mathcal{Z}'$, when they are written as a linear combination of $X_1, \ldots, X_m$, are bounded from above by $|y|^{\min(D_3(d), D_3(d'))}$, therefore $\mathcal{Z} - \mathcal{Z}'$ is too small to be 2π-periodic unless it is zero. Thus $\mathcal{Z} = \mathcal{Z}'$ in $U_{d,\varepsilon(d)} \cap U_{d',\varepsilon(d')}$.

We have then defined a bounded holomorphic vector field $\mathcal{Z}$ on the open set $U = \bigcup_{d=1}^{\infty} U_{d,\varepsilon(d)}$, which is constant on each L_y with respecto to the affine flat structure. Moreover $\mathcal{Z}$ is 2π-periodic, and there exist $a_1, \ldots, a_m$ holomorphic functions constant on the connected components of each level set, such that

$$\mathcal{Z} = \sum_{j=1}^{m} a_j X_j$$

in U. Then

$$\begin{aligned} \mathrm{d}f(\mathcal{Z}) &= \mathrm{d}f\left(\sum_{j=1}^{m} a_j X_j\right) \\ &= \sum_{j=1}^{m} (a_j \circ f)\, \mathrm{d}f(X_j) \\ &= \sum_{j=1}^{m} (a_j \circ f)\, (X_j \circ f) \\ &= \mathcal{Z} \circ f. \end{aligned}$$

Applying **Lemma Zung Annals**, there exists a holomorphic vector field defined in a whole neighbourhood of the origin, coinciding with $\mathcal{Z}$ on U. Thus we have found a generator of an effective action of a 1-torus commuting with f as we wanted □

Corollary 9.11. *Let f be a germ of biholomorphism of $(\mathbb{C}^n, O)$ fixing the origin and commuting with a set of integrable holomorphic vector fields. Then f is holomorphically conjugated to a germ containing only monomials belonging to the intersection of the additive resonances of the eigenvalues of the linear terms of $X_1, \ldots, X_m$.*

Proof. It follows from the previous proof that we can find r holomorphic periodic vector fields, such that their linear terms form a r-tuple of toric vector fields associated to X_1, which commute pairwise, are linearly independent, and they commute with f. Then the assertion follows from Corollary 9.6 and Theorem 3.1. □

automaticamente commuta con la f. Nell'articolo, puoi invece partire dall'esistenza del campo Z che genera l'azione di toro, ottenuto semplicemente citando il teorema di Zung, fai notare che dalla dimostrazione del teorema di Zung segue che Z è una combinazione opportuna dei campi $X_1, \ldots, X_m$, e dimostri che allora Z commuta con f.

Then we also have the following

Corollay 9.12. *Let f be a germ of biholomorphism of $(\mathbb{C}^n, O)$ fixing the origin and commuting with a set of integrable holomorphic vector fields, such that the intersection of the additive resonances of the eigenvalues of the linear terms of $X_1, \ldots, X_m$ is equal or contained in the set of resonances of the spectrum of $\mathrm{d}f_O$. Then f admits a holomorphic Poincaré-Dulac normalization.*

Una volta che avro' inserito la def. di grado torico per campi vettoriali (e' diversa da quella di Zung, quindi ci vuole), allora richiamo anche i lemmi che uso nella dimostrazione