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LOCAL ENERGY ESTIMATES FOR THE MAXWELL–LANDAU–LIFSHITZ SYSTEM AND APPLICATIONS

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Abstract. We study the Maxwell–Landau–Lifshitz system without exchange energy. First, we prove an $L^p(L^2_{\text{loc}})$ estimate for the linear wave equation and apply this local energy estimate to obtain a bound on the curl of the electromagnetic field, uniformly in time and locally in space. Next, we prove strong convergence results, when the time t tends to ∞ or when the speed of light tends to ∞ (which corresponds to the quasi-stationary approximation). Finally, we establish a stability result with respect to the damping parameter of the Landau–Lifshitz equation.

Keywords: Ferromagnetism; Maxwell's equations; local energy; quasi-stationary limit.

1. Introduction

1.1. *Presentation of the Maxwell–Landau–Lifshitz system*

The Maxwell–Landau–Lifshitz system reads

$$\begin{cases} \partial_T(\varepsilon_0 \mathbf{E}) - \mathbf{curl}\,\mathbf{H} = 0 \\ \partial_T(\mu_0(\mathbf{H}+\mathbf{M})) + \mathbf{curl}\,\mathbf{E} = 0 \\ \partial_T \mathbf{M} = \gamma_0 \left(\mathbf{M} \times \mathbf{H}_T + \dfrac{\alpha}{|\mathbf{M}|} \mathbf{M} \times (\mathbf{M} \times \mathbf{H}_T) \right) \\ (\mathbf{E}(0), \mathbf{H}(0)) = (\mathbf{E}_0, \mathbf{H}_0) \\ \mathbf{M}(0) = \mathbf{M}_0 \\ \operatorname{div}(\varepsilon_0 \mathbf{E}) = 0 \\ \operatorname{div}(\mu_0(\mathbf{H}+\mathbf{M})) = 0 \end{cases} \tag{1.1}$$

where the electric field $\mathbf{E}$, the magnetic field $\mathbf{H}$ and the magnetization $\mathbf{M}$ depend on the time $T \in \mathbb{R}_+$ and the space-variable $X \in \mathbb{R}^d$ and take values into $\mathbb{R}^3$. Here, the dielectric and magnetic permittivities ε_0 and μ_0 are constants.

Although in most physical applications $d = 3$, the cases $d = 1, 2$ are also of interest. $\mathbf{M}(T)$ is assumed to be supported in a compact $\bar{\Omega}$ included in B_R. $|\mathbf{M}|$ is assumed to be a constant M_s on $\bar{\Omega}$. α is a non-dimensional constant called the damping parameter, between 0 and 1. Physically, we have α of order 10^{-1} or 10^{-2}. Here and below we denote by $B_\rho = \{x \in \mathbb{R}^d \mid |x| \leq \rho\}$ the ball in $\mathbb{R}^d$ centered in 0 with radius ρ, and $B = B_1$.

The effective magnetic field $\mathbf{H}_T$, is defined by

$$\mathbf{H}_T = \mathbf{H} + \mathbf{H}_a(\mathbf{M}) + \mathbf{H}_e(\mathbf{M}) + \mathbf{H}_{\text{ext}},$$

where (see also [7, 14])

- for M in $\mathbb{R}^3$, $\mathbf{H}_a(M) = -\nabla_M \tilde{\Phi}(M)$, where the datum $\tilde{\Phi}$ is a non-negative convex function from $\mathbb{R}^3$ which vanishes at 0. $\mathbf{H}_a(\mathbf{M})$ is called the anisotropy energy.
- $\mathbf{H}_e(\mathbf{M}) = -K\mathbb{1}_\Omega \Delta\mathbf{M}$, is the exchange energy.
- and $\mathbf{H}_{\text{ext}}$ the Zemann energy or exterior energy, is given and does not depend on T.

Here, we consider the case $K = 0$. Mathematical results are very different in the other case (see, for instance, [2]). Note that the last two equations in (1.1) are satisfied for all times if they hold at $T = 0$. The non-dimensionalized system is:

$$\begin{cases} \eta\, \partial_t \mathbf{e} - \mathbf{curl}\, \mathbf{h} = 0 \\ \eta\, \partial_t (\mathbf{h} + \mathbf{m}) + \mathbf{curl}\, \mathbf{e} = 0 \\ \partial_t \mathbf{m} = -\mathbf{m} \times \mathbf{h}_T - \alpha\, \mathbf{m} \times (\mathbf{m} \times \mathbf{h}_T) \\ (\mathbf{e}(0), \mathbf{h}(0)) = (\mathbf{e}_0, \mathbf{h}_0) \\ \mathbf{m}(0) = \mathbf{m}_0 \\ \operatorname{div}(\mathbf{e}) = 0 \\ \operatorname{div}(\mathbf{h} + \mathbf{m}) = 0 \\ |\mathbf{m}| = \mathbb{1}_\Omega \quad \Omega \subset B \\ \mathbf{h}_T = \mathbf{h} + \mathbf{h}_{\text{ext}} - \nabla\Phi(\mathbf{m}) \end{cases} \tag{1.2}$$

where $\eta = v/c$ is the quotient of two characteristic speeds of the system, i.e. the giromagnetic ratio in the Landau–Lifshitz equation $v = R|\gamma_0|\, M_s$, and the speed of light $c = (\varepsilon_0 \mu_0)^{-1/2}$.

1.2. *Main results*

In the case with no exchange energy, the system was studied by Joly–Métivier–Rauch in [12]. They established the existence of energy solutions, i.e. weak solutions satisfying natural energy estimates.

Theorem 1.1 (Joly–Métivier–Rauch). *For $d \leq 3$, assume that $\mathbf{e}_0$, $\mathbf{h}_0$ are in $L^2(\mathbb{R}^d)$, and that $\mathbf{m}_0$, $\mathbf{h}_{\text{ext}} \in L^\infty(\mathbb{R}^d)$ and* $\operatorname{supp} \mathbf{m}_0$ *is compact. Then there exists an energy solution of* (1.2), *such that the fields $\mathbf{e}$, $\mathbf{h}$, $\mathbf{m}$ are in $\mathcal{C}^0(\mathbb{R}_+; L^2(\mathbb{R}^d))$.*

Moreover, if we assume that $\mathbf{curl}\,\mathbf{e}_0$ *and* $\mathbf{curl}\,\mathbf{h}_0$ *are in* $L^2(\mathbb{R}^d)$, *then the energy solution is unique and the fields* $\mathbf{curl}\,\mathbf{e}$, $\mathbf{curl}\,\mathbf{h}$ *are also in* $\mathcal{C}^0(\mathbb{R}_+; L^2(\mathbb{R}^d))$.

The proof of this theorem can be found in [12]. The term $\Phi(\mathbf{m})$ does not appear there, but its addition does not modify the proof substantially.

When $d = 2$, Haddar [7] has generalized the above result to variable ε in (1.1). When $d = 3$, Jochmann [10] proved the existence of weak energy solution of (1.1) in a more general situation, when ε and μ are non-constant and the first equation is replaced with

$$\partial_T(\varepsilon\mathbf{E}) - \mathbf{curl}\,\mathbf{H} = -\sigma\mathbf{E} - \mathbf{J}, \tag{1.3}$$

where σ is a bounded non-negative function from $\mathbb{R}^3$, and $\mathbf{J}$ belongs to $L^1(L^2(\mathbb{R}^3))$. No other result is known in the literature for strong solutions without exchange term.

Other results are available in the presence of the exchange term $\mathbf{h}_e(\mathbf{m})$ (see [3]).

We are interested in the strong solution of the problem (1.2) given by Theorem 1.1. We consider only the spatial dimensions $d = 1, 2, 3$, in order for the system (1.2) to be physically meaningful. However, the results on the wave equation are clearly true in any dimension.

We use the orthogonal decomposition of $L^2(\mathbb{R}^d)$: for $\mathbf{h} \in L^2$, $\mathbf{h}_\perp$ (respectively, $\mathbf{h}_\parallel$) is the orthogonal projection on $\ker \operatorname{div}$ (respectively, $\ker \mathbf{curl}$). The orthogonal component of $\mathbf{h}$ satisfies the following wave equation:

$$\left(\eta^2\partial_t^2 - \Delta\right)\mathbf{h}_\perp = -\eta^2\partial_t^2\mathbf{m}_\perp. \tag{1.4}$$

Consider a function P which is $\mathcal{C}^\infty$ on $\mathbb{R}^d\backslash\{0\}$ and homogeneous of degree 0, and denote by $P(D)v$ the function such that $\widehat{P(D)v}(\xi) = P(\xi)\hat{v}(\xi)$, where $\hat{f}(\xi)$ is the Fourier transform of f defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,\mathrm{d}x.$$

This work is based on a local energy estimate for solution of (1.4). Our first result is:

Theorem 1.2. *Let* v *be a real-valued function from* $\mathbb{R}_+$ *to* $\mathbb{R}^d$ *and let* $1 \le p \le +\infty$. *Let us assume that* $\operatorname{supp} v \subset \mathbb{R}_+ \times B$, *and* $v \in L^p(\mathbb{R}_+, L^2(\mathbb{R}^d))$, *and* $d \ge 2$. *Denote by* u_{Pv} *the solution of the Cauchy problem*

$$\begin{cases} \left(\eta^2\partial_t^2 - \Delta\right)u = P(D)v = Pv \\ u_{|t=0} = 0 \\ \partial_t u_{|t=0} = 0. \end{cases}$$

Then $\nabla_{t,x}u_{Pv} \in L^p(\mathbb{R}_+, L^2_{\mathrm{loc}}) \cap L^\infty(\mathbb{R}_+; L^2_{\mathrm{loc}})$, *and there exists a constant* $C > 0$ *depending only on* P, *such that for* $v \in L^p(\mathbb{R}_+, L^2(\mathbb{R}^d))$ α, q *with* $p \le q \le +\infty$ *and* $\rho \ge 1$

$$\eta\|\partial_t u_{Pv}\|_{L^q(L^2(B_\rho))} + \|\nabla u_{Pv}\|_{L^q(L^2(B_\rho))} \le C\rho^{1+\frac{1}{q}-\frac{1}{p}}\|v\|_{L^p(L^2(\mathbb{R}^d))}.$$

This theorem can be used to prove for the system (1.2):

Proposition 1.3. *Assume that* $\mathbf{e}_0$, $\mathbf{h}_0$, $\mathbf{m}_0$, $\mathbf{curl}\,\mathbf{e}_0$, $\mathbf{curl}\,\mathbf{h}_0$ *are in* $L^2(\mathbb{R}^d)$, $\mathbf{h}_{\mathrm{ext}}$ *is in* $L^\infty(B)$ *and let* $(\mathbf{e}, \mathbf{h}, \mathbf{m})$ *be the strong solution of* (1.2). *Then, when* $\alpha > 0$ *the fields* $\mathbf{e}$ *and* $\mathbf{h}_\perp$ *are in* $L^2(\mathbb{R}_+; L^2_{\mathrm{loc}})$.

We can obtain:

Theorem 1.4. *Assume that* $\alpha > 0$ *and* $\mathbf{e}_0$, $\mathbf{h}_0$ *are in* $L^2(\mathbb{R}^d)$, $\mathbf{h}_{\mathrm{ext}} \in L^\infty(B)$; *then* $\mathbf{e}(t)$, $\mathbf{h}_\perp(t) \to 0$ *when* $t \to +\infty$ *in* L^2_{loc}.

Next, we derive a uniform bound on first derivatives when η is sufficiently small.

Theorem 1.5. *Assume that* $\mathbf{e}_0$, $\mathbf{h}_0$, $\mathbf{m}_0$, $\mathbf{curl}\,\mathbf{e}_0$, $\mathbf{curl}\,\mathbf{h}_0$ *are in* $L^2(\mathbb{R}^d)$, $\mathbf{h}_{\mathrm{ext}}$ *is in* $L^\infty(B)$. *Then for* η *sufficiently small,* $\mathbf{curl}\,\mathbf{e}$ *and* $\mathbf{curl}\,\mathbf{h}$ *are in* $L^\infty(L^2_{\mathrm{loc}})$.

This improves the results of [12], giving uniform estimates with respect to t of first derivatives of the electromagnetic field.

In [11], when $t \to +\infty$, the weak convergence in L^2 of $\mathbf{E}(t)$ and $\mathbf{H}_\perp(t)$ is established in (1.1)–(1.3), even in the case of variable ε and μ. A strong convergence result on $\mathbf{E}(t)$ is proved only when $\sigma \geq \sigma_0 > 0$. The description of the ω-limit set is only obtained with an exchange term which leads to an H^1 bound for $\mathbf{m}$; cf. [3, 5].

We prove next the following quasi-stationary convergence result:

Theorem 1.6. *The Cauchy problem*

$$\begin{cases} \partial_t \mathbf{m} = -\mathbf{m} \times \mathbf{h}_T - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{h}_T) \\ \mathbf{curl}\,\mathbf{h} = 0 \\ \operatorname{div}(\mathbf{h} + \mathbf{m}) = 0 \\ \mathbf{m}_{|t=0} = \mathbf{m}_0 \end{cases}$$

has a unique solution $\mathbf{m} \in \mathcal{C}^1(\mathbb{R}_+; L^2)$. *Moreover, denoting by* $(\mathbf{e}_\eta, \mathbf{h}_\eta, \mathbf{m}_\eta)$ *the solution of* (1.2) *for* $\alpha \geq 0$ *fixed, the fields* $\mathbf{e}_\eta$ *and* $\mathbf{h}_{\eta\perp}$ *converge as* $\eta \to 0$ *to* 0 *in* $L^2(\mathbb{R}_+; L^2_{\mathrm{loc}})$ *while* $\mathbf{m}_\eta$ *converges strongly to* $\mathbf{m}$ *in* $\mathcal{C}(\mathbb{R}_+; L^2)$.

Without damping term (i.e. when $\alpha = 0$) the uniqueness has been proved by Jochmann [10], assuming that μ is non-constant. Again in [10], the weak quasi-stationary limit has been established.

In the last section of this paper, η being fixed and $(\mathbf{e}_\alpha, \mathbf{h}_\alpha, \mathbf{m}_\alpha)$ being the solution of (1.2), we prove:

Proposition 1.7. *When* α *tends to* 0, $(\mathbf{e}^\alpha, \mathbf{h}^\alpha, \mathbf{m}^\alpha)$ *converges in* $L^2_{\mathrm{loc}}(L^2_{\mathrm{loc}}) \times L^2_{\mathrm{loc}}(L^2_{\mathrm{loc}}) \times \mathcal{C}(L^2)$ *to a strong solution* $(\mathbf{e}^0, \mathbf{h}^0, \mathbf{m}^0)$ *of the system* (1.2) *with* $\alpha = 0$.

2. Transformation and Non-Dimensionalization

2.1. *Transformation*

We consider the system (1.1). We assume that

- $\operatorname{supp} \mathbf{M}_0 \subset B_R$;

- $\forall\, x \in \operatorname{supp} \mathbf{M}_0$, $|\mathbf{M}_0(x)| = M_s \in \mathbb{R}_+^*$;

Let $(\mathbf{E}, \mathbf{H}, \mathbf{M})$ be an energy solution of this system. Then we have

$$\partial_t \mathbf{M}(t,x)\, \mathbf{M}(t,x) = 0 \quad \text{f.a.e. } x \in \mathbb{R}^d.$$

This implies that:

$$\forall\, t \in \mathbb{R}_+, \quad |\mathbf{M}(t,\,\cdot)| = M_s \mathbb{1}_{\operatorname{supp}\mathbf{M}_0}.$$

In particular, $(\mathbf{E}, \mathbf{H}, \mathbf{M})$ is a solution of the (polynomial) PDE system:

$$\begin{cases} \partial_T \varepsilon_0 \mathbf{E} - \mathbf{curl}\, \mathbf{H} = 0 \\ \partial_T \mu_0 (\mathbf{H} + \mathbf{M}) + \mathbf{curl}\, \mathbf{E} = 0 \\ \partial_T \mathbf{M} = \gamma_0 \left(\mathbf{M} \times \mathbf{H} + \dfrac{\alpha}{M_s} \mathbf{M} \times (\mathbf{M} \times \mathbf{H}) \right) \\ (\mathbf{E}(0), \mathbf{H}(0)) = (\mathbf{E}_0, \mathbf{H}_0) \\ \mathbf{M}(0) = \mathbf{M}_0 \\ \operatorname{div}(\varepsilon_0 \mathbf{E}) = 0 \\ \operatorname{div} \mu_0 (\mathbf{H} + \mathbf{M}) = 0. \end{cases}$$

Conversely, by the same method, a solution of this PDE system is also solution of the system (1.1).

2.2. *Non-dimensionalization*

Make the following scalings:

$$\begin{cases} \mathbf{e} = \sqrt{\dfrac{\varepsilon_0}{\mu_0}} \dfrac{\mathbf{E}}{M_s} & \Phi(m) = \dfrac{\tilde{\Phi}(M_s m)}{{M_s}^2} \\ \mathbf{h} = \dfrac{\mathbf{H}}{M_s} & \mathbf{m} = \dfrac{\mathbf{M}}{M_s} \\ x = \dfrac{X}{R} & t = |\gamma_0|\, M_s\, T. \end{cases}$$

Define

$$\eta = R |\gamma_0|\, M_s \sqrt{\varepsilon_0 \mu_0}.$$

Remark 2.1. η is nothing but the quotient v/c of two characteristic system speed: first, the giromagnetic ratio in Landau–Lifshitz equation $v = R|\gamma_0|\, M_s$, next, the speed of light $c = (\varepsilon_0 \mu_0)^{-1/2}$. In particular, the limit $\eta \to 0$ corresponds to the quasi-stationary approximation of the electromagnetic field.

The non-dimensionalized system is now exactly (1.2), and we know that $\operatorname{supp} \mathbf{m}_0 \subset B$ and $|\mathbf{m}_0| = 1$ on $\operatorname{supp} \mathbf{m}_0$.

Remark 2.2. We do not see the geometry of $\operatorname{supp} \mathbf{m}_0$; we only use that it is bounded, so has a finite Lebesgue measure.

Notation 2.3. Let $f(m,h) = -m \times h - \alpha m \times (m \times h)$.

Properties 2.4. The function f satisfies:

- $f(0,h)=0$, $h\in\mathbb{R}^3$;
- f is linear with respect to h;
- f is locally Lipschitzian with respect to m;
- $f(m,h)\cdot m=0$, h, $m\in\mathbb{R}^3$;
- $f(m,h)\cdot h=\alpha|m\times h|^2$, h, $m\in\mathbb{R}^3$;
- $|f(m,h)|^2=(1+\alpha^2|m|^2)|m\times h|^2$, h, $m\in\mathbb{R}^3$.

2.3. *Classical energy estimates*

Notation 2.5. Let $(\mathbf{e},\mathbf{h},\mathbf{m})$ be a weak solution of system (1.2). We note

$$\mathcal{E}(t)=\frac{1}{2}\big(\|\mathbf{e}(t)\|^2_{L^2(\mathbb{R}^d)}+\|\mathbf{h}(t)\|^2_{L^2(\mathbb{R}^d)}\big)+\int_B \Phi(\mathbf{m}(t))+\frac{1}{2}|\mathbf{h}_{\mathrm{ext}}-\mathbf{m}(t)|^2\,\mathrm{d}x.$$

$\mathcal{E}$ is the usual electromagnetic energy in the Maxwell system, completed with the different energies coming from Landau–Lifshitz equation.

Let $(\mathbf{e},\mathbf{h},\mathbf{m})$ be a strong solution of system (1.2). Recall that

$$|\mathbf{m}(t,x)|=|\mathbf{m}_0(x)|\quad \text{a.e.} \tag{2.2}$$

Take the $L^2(\mathbb{R}^d)$ scalar product of the first (respectively, second) equation in (1.2) with $\mathbf{e}(t)$ (respectively, $\mathbf{h}(t)$). Add the two expressions. Because **curl** is self-adjoint, we find

$$\eta\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)+\eta\int_{\mathbb{R}^d}\partial_t\mathbf{m}(t)\cdot\mathbf{h}(t)\,\mathrm{d}x=0.$$

Now, $\mathbf{h}=\mathbf{h}_T+\nabla\Phi(\mathbf{m})-\mathbf{h}_{\mathrm{ext}}$. Thus,

$$\begin{aligned}\partial_t\mathbf{m}\cdot\mathbf{h}&=\partial_t\mathbf{m}\cdot\nabla\Phi(\mathbf{m})-\partial_t\mathbf{m}\cdot\mathbf{h}_{\mathrm{ext}}+\partial_t\mathbf{m}\cdot\mathbf{h}_T\\&=\partial_t\Phi(\mathbf{m})-\partial_t\mathbf{m}\cdot\mathbf{h}_{\mathrm{ext}}+f(\mathbf{m},\mathbf{h}_T)\cdot\mathbf{h}_T.\end{aligned}$$

Then, with (2.2), we have $-\partial_t\mathbf{m}\cdot\mathbf{h}_{\mathrm{ext}}=\partial_t\frac{1}{2}|\mathbf{h}_{\mathrm{ext}}-\mathbf{m}|^2$. Now, thanks to the two last properties in 2.4,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)+\frac{\alpha}{1+\alpha^2}\|\partial_t\mathbf{m}(t)\|^2_{L^2}\le 0,\\&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)+\alpha\|\mathbf{m}(t)\times\mathbf{h}(t)\|^2_{L^2}\le 0.\end{aligned}$$

With a time-integration, we obtain:

Proposition 2.6. *Assume that* $(\mathbf{e},\mathbf{h},\mathbf{m})$ *is an energy solution of system* (1.2); *then for all* $t\ge 0$,

$$\mathcal{E}(t)+\frac{\alpha}{1+\alpha^2}\int_0^t\|\partial_t\mathbf{m}(s)\|^2_{L^2}\,\mathrm{d}s\le\mathcal{E}(0),$$

$$\mathcal{E}(t)+\alpha\int_0^t\|\mathbf{m}(s)\times\mathbf{h}(s)\|^2_{L^2}\,\mathrm{d}s\le\mathcal{E}(0).$$

In particular, $\mathbf{m}\times\mathbf{h}$ *and* $\partial_t\mathbf{m}$ *belong to* $L^2\big(\mathbb{R}_+^{1+d}\big)$.

Definition 2.7. Let $(\mathbf{e}, \mathbf{h}, \mathbf{m})$ be a weak solution, that is a solution in the distribution sense, of the system (1.2). We say that it is an energy solution if $\mathcal{E}(0) < +\infty$ and if the estimates (2.2), and Proposition 2.6 are satisfied.

3. Local Energy for the Wave Equation

3.1. *An L^2 orthogonal decomposition*

The system (1.2) uses the curl and div operators. We decompose the system, taking the curl part and the div part. Introduce the orthogonal decomposition of $L^2(\mathbb{R}^d)$:

Notation 3.1. Define

$$L^2_\perp(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) \mid \operatorname{div} u = 0\}$$
$$\text{and} \quad L^2_\parallel(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) \mid \mathbf{curl}\, u = 0\}.$$

Proposition 3.2. *We have an orthogonal sum*

$$L^2(\mathbb{R}^d) = L^2_\perp(\mathbb{R}^d) \oplus L^2_\parallel(\mathbb{R}^d).$$

We denote by $P_\perp : u \mapsto P_\perp u = u_\perp$ and $P_\parallel : u \mapsto P_\parallel u = u_\parallel$ the two projectors associated to this decomposition.

Those projectors are both Fourier multipliers, with a symbols defined, respectively by

$$\widehat{P_\perp f}(\xi) = -\frac{\xi \times (\xi \times \hat{f}(\xi))}{|\xi|^2} \quad \textit{and} \quad \widehat{P_\parallel f}(\xi) = \frac{\xi \cdot (\xi \cdot \hat{f}(\xi))}{|\xi|^2}.$$

The equation $\operatorname{div} \mathbf{e} = 0$ implies that $\mathbf{e}_\perp = \mathbf{e}$, that $\mathbf{e}_\parallel = 0$. We also have $\mathbf{h}_\parallel + \mathbf{m}_\parallel = 0$. We know that $\mathbf{m}$ has a bounded and time-invariant support in space, and $|\mathbf{m}| = 1$ where $\mathbf{m} \neq 0$. Moreover, $\partial_t \mathbf{m} = f(\mathbf{m}, \mathbf{h})$. This implies that $\operatorname{supp} \partial_t \mathbf{m} \subset \mathbb{R}_+ \times B$.

This provides some information about $\mathbf{h}_\parallel = -\mathbf{m}_\parallel$. In order to know $\mathbf{h}_\perp$, write the wave equation satisfied by $\mathbf{h}$:

$$\left(\eta^2 \partial_t^2 - \Delta\right)\mathbf{h} = -\eta^2 \partial_t^2 \mathbf{m} - \nabla \operatorname{div} \mathbf{m}.$$

Take the projection on $L^2_\perp(\mathbb{R}^d)$:

$$\left(\eta^2 \partial_t^2 - \Delta\right)\mathbf{h}_\perp = -\eta^2 \partial_t^2 \mathbf{m}_\perp.$$

We obtain similarly

$$\left(\eta^2 \partial_t^2 - \Delta\right)\mathbf{e} = -\eta\, \mathbf{curl}\, \partial_t \mathbf{m}_\perp.$$

We can consider for the moment $\mathbf{m}$ as a datum. We know that $\partial_t \mathbf{m} \in L^2\left(\mathbb{R}_+^{1+d}\right)$ thanks to Eq. (2.6). Next, we note that, if u is a solution of $\left(\eta^2 \partial_t^2 - \Delta\right)u = -\partial_t \mathbf{m}_\perp$, then $\mathbf{e} - \mathbf{curl}\, u$ and $\mathbf{h}_\perp - \partial_t u$ are solutions of the linear homogeneous wave equation $\left(\eta^2 \partial_t^2 - \Delta\right)g = 0$.

3.2. L^2 local estimates for the non-homogeneous linear wave equation

In this section, d is any positive integer (not necessary less or equal than 3).

Notations 3.3. We denote by $S_\rho = \{x \in \mathbb{R}^d \mid |x| = \rho\}$ the sphere centered in 0 with radius ρ, and $\Gamma_{a,b} = \{x \in \mathbb{R}^d \mid a \le |x| \le b\}$ the annulus with radii a and b.

Notation 3.4. We denote by $E = E(t,x) \in \mathcal{D}'(\mathbb{R}^{1+d})$ the fundamental solution of the wave equation ($\Box E = \delta_{t,x} \in \mathcal{D}'(\mathbb{R}^{1+d})$, where $\Box = \partial_t^2 - \Delta$) supported in $\{t \ge 0\}$, and $\mathbf{E} : t \mapsto \mathbf{E}(t)$ the fundamental solution valued at the time t, i.e. $\mathbf{E} \in \mathcal{C}^\infty(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}^d))$ is defined by:

$$\forall\, \phi \in \mathcal{D}(\mathbb{R}^{1+d}), \quad \langle E, \phi\rangle_{\mathcal{D}'(\mathbb{R}^{1+d}),\mathcal{D}(\mathbb{R}^{1+d})} = \int_{\mathbb{R}_+} \langle \mathbf{E}(t), \phi(t,\,\cdot\,)\rangle_{\mathcal{D}'(\mathbb{R}^d),\mathcal{D}(\mathbb{R}^d)}\, \mathrm{d}t.$$

We have for $t \ge 0$

$$\widehat{\mathbf{E}(t)}(\xi) = \frac{\sin(t|\xi|)}{|\xi|}.$$

This formula implies that

Proposition 3.5. *For $t \ge 0$, $\mathbf{E}'(t)$ defines by convolution in $\mathbb{R}^d_x$ a continuous mapping $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with norm equal to 1. Similarly, $\nabla\mathbf{E}(t)$ defines a continuous mapping from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ with norm equal to 1.*

When d is even, $\mathbf{E}(t)$ is a smooth distribution outside S_t, defined by (see [6, 16])

$$\begin{aligned}\mathbf{E}(t)(x) = E(t,x) &= \frac{\left(\frac{d}{2}-1\right)!}{2\pi^{d/2}\,(d-1)!} \frac{\mathbb{1}_{\{|x|<t\}}}{\left(\sqrt{t^2-|x|^2}\right)^{d-1}} \\ &= \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-2}{2}} \frac{t^d \mathbb{1}_{\{|x|<t\}}}{\left(\sqrt{t^2-|x|^2}\right)^{1/2}}.\end{aligned}$$

This implies:

$$\text{for } |x| < t, \quad \partial_t E(t,x) = \mathbf{E}'(t)(x) = -\frac{\left(\frac{d}{2}-1\right)!}{2\pi^{d/2}(d-2)!} \frac{t}{(\sqrt{t^2-|x|^2})^{d+1}},$$

$$\text{for } |x| < t, \quad \nabla_x E(t,x) = \nabla\mathbf{E}(t)(x) = -\frac{\left(\frac{d}{2}-1\right)!}{2\pi^{d/2}(d-2)!} \frac{x}{(\sqrt{t^2-|x|^2})^{d+1}}.$$

When d is odd, the distributions $\mathbf{E}'(t)$ and $\nabla\mathbf{E}(t)$ are supported in S_t (Huygens principle).

We consider the solution u of the Cauchy problem:

$$\begin{cases} \left(\eta^2\partial_t^2 - \Delta\right)u = v \\ u_{|t=0} = 0 \\ \partial_t u_{|t=0} = 0. \end{cases} \tag{3.2}$$

When $\eta = 1$, u is given by

$$u(t) = \int_0^t \mathbf{E}(t-s) *_x v(s)\,\mathrm{d}s := E *_{t,x} v(t).$$

Therefore,

$$\partial_t u(t) = \int_0^t \mathbf{E}'(t-s) *_x v(s)\,\mathrm{d}s = \partial_t E *_{t,x} v(t).$$

For locally integrable functions supported in $\{t \geq 0\}$, we use the notation

$$f *_t g(t) = \int_0^t f(t-s)\,g(s)\,\mathrm{d}s.$$

First, we prove the following theorem:

Theorem 3.6. *Let* $1 \leq p \leq +\infty$. *Assume that* $\operatorname{supp} v \subset \mathbb{R}_+ \times B$, *and* $v \in L^p(\mathbb{R}_+, L^2(\mathbb{R}^d))$. *Let* u_v *be the solution of the Cauchy problem* (3.2). *Then* $\partial_t u$ *and* ∇u_v *are in* $L^p(\mathbb{R}_+, L^2_{\mathrm{loc}}) \cap L^\infty(\mathbb{R}_+; L^2_{\mathrm{loc}})$. *More precisely, there exists a constant* C *which depends only on* d, *such that for* $v \in L^p(\mathbb{R}_+, L^2(\mathbb{R}^d))$ *with* $\operatorname{supp} v \subset \mathbb{R}_+ \times B$, *for all* q *with* $p \leq q \leq +\infty$, *for all* $T \in \mathbb{R}_+ \cup \{+\infty\}$ *and* $\rho \geq R$, *one has*:

$$\eta\,\|\partial_t u\|_{L^q(L^2(B_\rho))} + \|\nabla u_v\|_{L^q(L^2(B_\rho))} \leq C\rho^{1+\frac{1}{q}-\frac{1}{p}}\|v\|_{L^p(L^2(\mathbb{R}^d))}.$$

Notation 3.7. Let P be a smooth function outside the origin, homogeneous with a degree 0 on $\mathbb{R}^d$. Let

$$\|P\|_{\mathcal{C}^d(S^{d-1})} = \sum_{j=0}^{d} \|P^{(j)}\|_{L^\infty(S^{d-1})}.$$

Proof of Theorem 3.6. Consider $\tilde{u}$ defined by $\tilde{u}(t,x) = u(\eta\rho t, \rho x)$; we can check that:

- u is solution of $\Box\tilde{u} = \rho^2\tilde{v}$ where $\tilde{v}(t,x) = v(\rho t, \rho x)$.
- $\operatorname{supp}\tilde{v} \supset \mathbb{R}_+ \times B$.
- $\|\tilde{v}\|_{L^p(L^2)} = \eta^{-\frac{1}{p}}\rho^{-\frac{1}{2}-\frac{1}{p}}\|v\|_{L^p(L^2)}$.
- $\|\nabla\tilde{u}\|_{L^q(L^2(B_r))} = \eta^{-\frac{1}{p}}\rho^{1-\frac{1}{2}-\frac{1}{p}}\|\nabla u\|_{L^q(L^2(B_{\rho r}))}$.
- $\|\partial_t\tilde{u}\|_{L^q(L^2(B_r))} = \eta^{1-\frac{1}{p}}\rho^{1-\frac{1}{2}-\frac{1}{p}}\|\partial_t u\|_{L^q(L^2(B_{\rho r}))}$.

Hence, it suffices to prove the theorem for $R = \rho = 1$ and $\eta = 1$, which we now assume.

The principle is to decompose v into several pieces, and next to bound each of them by the convolution of $t \mapsto \|v(t)\|_{L^2(\mathbb{R}^d)}$ with a function of $[L^1 \cap L^\infty](\mathbb{R}_+)$.

We write the details of the proof for $\partial_t u$. The estimates for ∇u are similar; in particular, we use in even dimension, that

$$|\nabla_x E(t,x)| \leq |\partial_t E(t,x)| \qquad |x| < t.$$

First with $(t-4)^+ := \max\{t-4, 0\}$ write

$$u_v(t) = \int_0^{(t-4)^+} \mathbf{E}'(t-s) *_x v(s)\,\mathrm{d}s + \int_{(t-4)^+}^t \mathbf{E}'(t-s) *_x v(s)\,\mathrm{d}s.$$

The second integral in the right-hand side can be bounded:

$$\begin{aligned}
&\left\| \int_{(t-4)^+}^t \mathbf{E}'(t-s) *_x v(s)\,\mathrm{d}s \right\|_{L^2(B)} \\
&\leq \left\| \int_{(t-4)^+}^t \mathbf{E}'(t-s) *_x v(s)\,\mathrm{d}s \right\|_{L^2(\mathbb{R}^d)} \\
&\leq \int_{(t-4)^+}^t \|\mathbf{E}'(t-s) *_x v(s)\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}s \\
&\leq \int_{(t-4)^+}^t \|v(s)\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}s \\
&\leq (\mathbb{1}_{[0,4]} *_t \|v\|_{L^2(\mathbb{R}^d)})(t).
\end{aligned}$$

Let us consider $\mathbf{E}'(t-s) *_x v(s)$ on B when $t-s \geq 4$. In odd dimension, the support property of convolution shows that, when $t-s \geq 4$.

$$\mathbf{E}'(t-s) *_x v(s) = 0 \text{ on } B_3 \quad \text{thus on } B.$$

This completes the proof in this case. In even dimension, $\mathbf{E}'(t-s) *_x v(s)$ is on B a smooth function, and for $x \in B$:

$$[\mathbf{E}(t-s) *_x v(s)](x) = \int_B \frac{C_d\,(t-s)}{\sqrt{(t-s)^2 - |x-y|^2}^{d+1}} \cdot v(s,y)\,\mathrm{d}y\,\mathrm{d}s$$

The Schwarz inequality yields thanks to the fact that $(t-s)^2 - |x-y|^2 \geq \frac{1}{2}(t-s)^2$ for $x \in B$ and $t-s \geq 4$:

$$\begin{aligned}
\|\mathbf{E}'(t-s) *_x v(s)\|_{L^2(B)} &\leq \sqrt{\sigma_d}\|\mathbf{E}'(t-s) *_x v(s)\|_{L^\infty(B)} \\
&\leq 2^{d/2} C_d\,\sigma_d \cdot (t-s)^{-d} \cdot \|v(s)\|_{L^2(B)} \\
&\leq 2^{d/2} C_d\,\sigma_d \cdot (t-s)^{-d} \cdot \|v(s)\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

where σ_d is the Lebesgue measure of the unit ball in $\mathbb{R}^d$.

Noting that $t \mapsto t^{-d}\mathbb{1}_{[4,+\infty[}(t)$ belongs to $L^1 \cap L^\infty(\mathbb{R}_+)$, this completes the proof. □

Proof of Theorem 1.2. The scheme of the proof is the same as the proof of Theorem 3.6. Write

$$u_{Pv}(t) = \int_0^{(t-6)^+} \mathbf{E}'(t-s) *_x Pv(s)\,\mathrm{d}s + \int_{(t-6)^+}^{t} \mathbf{E}'(t-s) *_x Pv(s)\,\mathrm{d}s.$$

The second term in the left hand side is bounded by $\mathbb{1}_{[0,6]} *_t \|v\|_{L^2(\mathbb{R}^d)}$. Write, for $t-s \geq 6$,

$$Pv(s) = \underbrace{\mathbb{1}_{B_2} Pv(s)}_{v_1(t,s)} + \underbrace{\mathbb{1}_{\Gamma_{2,t-s-4}} Pv(s)}_{v_2(t,s)} + \underbrace{\mathbb{1}_{\Gamma_{t-s-4,t-s+1}} Pv(s)}_{v_3(t,s)} + \underbrace{\mathbb{1}_{B^c_{t-s+1}} Pv(s)}_{v_4(t,s)}.$$

We want to show that $\|\mathbf{E}'(t-s) *_x v_j(t,s)\|_{L^2(B)} \leq f_j(t-s)\|v(s)\|_{L^2}$ where $f_j \in L^1 \cap L^\infty(\mathbb{R}_+)$, $j = 1,2,3,4$. The support properties imply:

$$[\mathbf{E}'(t-s) *_x v_4(t,s)]_{|B} = 0$$

and, in odd space dimension,

$$[\mathbf{E}'(t-s) *_x v_1(t,s)]_{|B} = [\mathbf{E}'(t-s) *_x v_2(t,s)]_{|B} = 0.$$

Moreover, because $\mathbf{E}'(t-s) *_x : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is continuous,

$$\|[\mathbf{E}(t-s) *_x v_3(t,s)]\|_{L^2(B)} \leq \|v_3(t,s)\|_{L^2(\mathbb{R}^d)} = \|Pv(s)\|_{L^2(\Gamma_{t-s-4,t-s+1})}. \tag{3.3}$$

In order to bound the last quantity, we use the following theorem:

Theorem 3.8. *Let $P = P(D)$ be a Fourier multiplier, with symbol $P(\xi)$ smooth on $\mathbb{R}^d \backslash \{0\}$ and homogeneous of degree 0. The inverse Fourier transform of $P(\xi)$, denoted $\tilde{P}(x)$ is a smooth function outside the origin, homogeneous of degree $-d$, and there exists a constant γ (which does not depend of P) such that*

$$\forall\, x \neq 0, \qquad |\tilde{P}(x)| \leq \frac{\gamma\,\|P\|_{\mathcal{C}^d(S^{d-1})}}{|x|^d}.$$

For a proof of this result, see [17]. From now on, γ will be the constant given by this theorem.

Corollary 3.9. *Let $f \in L^1(\mathbb{R}^d)$ supported in B, and P as above. Then Pf is smooth on $\mathbb{R}^d \backslash B$, and, when $|x| > 1$, we have*:

$$|Pf(x)| \leq \frac{\gamma\,\|P\|_{\mathcal{C}^d(S^{d-1})}\|f\|_{L^1(\mathbb{R}^d)}}{(|x|-1)^d}.$$

Proof. Since the singular support of $\tilde{P}$ is contained in $\{0\}$ the singular support of Pf is contained in the support of f and therefore in B (see, for instance, [9, Chap. 4]). The inequality is a consequence of Theorem 3.8 and of the following equality

$$Pf(x) = \int_B \tilde{P}(x-y) f(y)\,\mathrm{d}y. \qquad \square$$

Corollary 3.10. *There is a constant C which depends only of d such that for all $f \in L^2(\mathbb{R}^d)$ with* $\operatorname{supp} f \in B$ *and* $t \geq 6$,

$$\|Pf\|_{L^2(\Gamma_{t-4,t})} \leq \frac{C\gamma\,\|P\|_{\mathcal{C}^d(S^{d-1})}\|f\|_{L^2(\mathbb{R}^d)}}{t^{\frac{d+1}{2}}}.$$

Proof. We have $\|Pf\|_{L^2(\Gamma_{t-4,t})} = \|\mathbb{1}_{\Gamma_{t-4,t}} P\,\mathbb{1}_B\,f\|_{L^2(\mathbb{R}^d)}$. The kernel of $\mathbb{1}_{\Gamma_{t-4,t}} P \mathbb{1}_B$ is

$$K(x,y) = \mathbb{1}_{\Gamma_{t-4,t}}(x)\tilde{P}(x-y)\mathbb{1}_B(y).$$

Therefore, we have

$$\begin{aligned}
M_1 &:= \sup_{x\in\mathbb{R}^d} \|K(x,\cdot\,)\|_{L^1(\mathbb{R}^d)} = \sup_{x\in\Gamma_{t-4,t}} \|\tilde{P}(x-\,\cdot\,)\|_{L^1(B)} \leq \frac{C_d\,\gamma\|P\|_{\mathcal{C}^d(S^{d-1})}}{(t-5)^d},\\
M_2 &:= \sup_{y\in\mathbb{R}^d} \|K(\,\cdot\,,y)\|_{L^1(\mathbb{R}^d)} = \sup_{y\in B} \|\tilde{P}(\,\cdot\,-y)\|_{L^1(\Gamma_{t-4,t})}\\
&\leq \frac{C\,\gamma\|P\|_{\mathcal{C}^d(S^{d-1})}\,[t^d-(t-5)^d]}{(t-5)^d} \leq \frac{\gamma\|P\|_{\mathcal{C}^d(S^{d-1})}C'_d}{t-5}.
\end{aligned}$$

The operator is bounded on $L^2(\mathbb{R}^d)$ with norm smaller than $\sqrt{M_1\,M_2}$ thanks to Schur's lemma, implying the corollary. □

The last corollary bounds the second member in the inequality (3.3) by $\frac{\|v(s)\|_{L^2(\mathbb{R}^d)}}{(t-s)^{\frac{d+1}{2}}}$. This finishes the proof of Theorem 1.2 in odd dimension.

It remains to study contributions of v_1 and v_2 in even dimension. In this case:

$$\text{for } x\in B,\quad [\mathbf{E}'(t-s)*_x v_1(t,s)](x) = \int_{B_2} \frac{C_d(t-s)\mathbb{1}_{\{t-s\geq x-y\}}}{\sqrt{(t-s)^2-|x-y|^2}^{d+1}} Pv(s,y)\,\mathrm{d}y.$$

Use the inequality $(t-s)^2-|x-y|^2 \geq \frac{3}{4}(t-s)^2$ on the domain of integration

$$|[\mathbf{E}'(t-s)*_x v_1(t,s)](x)| \leq \left(\frac{4}{3}\right)^{\frac{d+1}{2}} \frac{C_d}{(t-s)^d}\int_{B_2} |Pv(s,y)|\,\mathrm{d}y.$$

Schwarz inequality and the boundedness of P in $\mathcal{L}(L^2(\mathbb{R}^d))$ imply:

$$\|\mathbf{E}'(t-s)*_x v_1(t,s)\|_{L^\infty(B)} \leq \frac{2}{\sqrt{3}}\left(\frac{8}{3}\right)^{d/2} \frac{\sqrt{\sigma_d}C_d}{(t-s)^d}\cdot\|v(s)\|_{L^2(\mathbb{R}^d)}$$

which concludes for the contribution of v_1.

Now, we can apply Corollary 3.9 to have an estimate for the contribution of v_2.

$$[\mathbf{E}'(t-s)*_x v_2(t,s)](x) = \int_{\Gamma_{2,t-s-4}} \frac{C_d(t-s)\,\mathbb{1}_{\{t-s\geq|x-y|\}}}{\sqrt{(t-s)^2-|x-y|^2}^{d+1}} Pv(s,y)\,\mathrm{d}y$$

$$|[\mathbf{E}'(t-s)*_x v_2(t,s)](x)| \leq \int_{\Gamma_{2,t-s-4}} \frac{C_d\,(t-s)}{\sqrt{(t-s)^2-(1+|y|)^2}^{d+1}}\frac{\|v(s)\|_{L^2(\mathbb{R}^d)}}{|y|^d}\mathrm{d}y$$

On the domain of integration, $t-s+1+|y| \geq t-s$, so in polar coordinates:

$$\begin{aligned}
|[\mathbf{E}'(t-s)*_x v_2(t,s)](x)| &\leq \int_{\Gamma_{2,t-s-4}} \frac{C_d\,(t-s)\,\|v(s)\|_{L^2(\mathbb{R}^d)}}{\sqrt{((t-s)-(1+|y|))(t-s+1+|y|)}^{d+1}} \frac{\mathrm{d}y}{|y|^d} \\
&\leq C'_d \|v(s)\|_{L^2(\mathbb{R}^d)} \int_2^{t-s-4} \frac{1}{(t-s)^{\frac{d-1}{2}}(t-s-(1+r))^{\frac{d+1}{2}}} \frac{\mathrm{d}r}{r} \\
&\leq C''_d \|v(s)\|_{L^2(\mathbb{R}^d)} \int_3^{t-s-3} \frac{1}{(t-s)^{\frac{d-1}{2}}(t-s-r)^{\frac{d+1}{2}}} \frac{\mathrm{d}r}{r}.
\end{aligned}$$

Thus, taking $u = t-s$, it is sufficient to prove the integrability and the boundedness of the function $u \mapsto \frac{1}{u^{\frac{d-1}{2}}} \int_3^{u-3} \frac{1}{(u-r)^{\frac{d+1}{2}}} \frac{\mathrm{d}r}{r}$ on $[6,+\infty[$. To do this, note that $u-r \geq u/2$ when $r \leq u/2$:

$$\begin{aligned}
\frac{1}{u^{\frac{d-1}{2}}} \int_3^{u-3} \frac{1}{(u-r)^{\frac{d+1}{2}}} \frac{\mathrm{d}r}{r} &\leq \frac{2^{\frac{d+1}{2}}}{u^{\frac{d-1}{2}}} \int_3^{u/2} \frac{1}{u^{\frac{d+1}{2}}} \frac{\mathrm{d}r}{r} + \frac{2}{u^{\frac{d-1}{2}}} \int_{u/2}^{u-3} \frac{1}{u} \frac{\mathrm{d}r}{(u-r)^{\frac{d+1}{2}}} \\
&\leq \frac{2^{\frac{d+1}{2}} \ln u}{u^d} + \frac{2}{u^{\frac{d+1}{2}}} \int_3^{u/2} \frac{\mathrm{d}r}{r^{\frac{d+1}{2}}} \\
&\leq \frac{2^{\frac{d+1}{2}} \ln u}{u^d} + \frac{2}{u^{\frac{d+1}{2}}} \frac{1}{(\frac{d+3}{2})3^{\frac{d+3}{2}}}.
\end{aligned}$$

The last member is a sum of two functions of u integrable on $[6,+\infty[$ when $d \geq 2$, this finishes the proof of Theorem 1.2. $\square$

3.3. *Estimates for the Cauchy problem for the homogeneous linear wave equation*

Proposition 3.11. *Let $u_0 \in H^1(\mathbb{R}^d)$ and $u_1 \in L^2(\mathbb{R}^d)$ be two data supported in B_R, and u such that*

$$\begin{cases} (\eta^2 \partial_t^2 - \Delta) u = 0 \\ u_{|t=0} = u_0 \\ \partial_t u_{|t=0} = u_1. \end{cases}$$

Then exists a constant C which depends only on d such that for all $1 \leq p \leq +\infty$, and $\rho \geq R$, we have:

$$\begin{aligned}
&\sqrt{\eta}(\eta \|\partial_t u\|_{L^p(\mathbb{R}_+;L^2(B_\rho))} + \|\nabla u\|_{L^p(\mathbb{R}_+;L^2(B_\rho))}) \\
&\quad \leq C\rho^{1+\frac{1}{p}} [\|\nabla u_0\|_{L^2(\mathbb{R}^d)} + \eta \|u_1\|_{L^2(\mathbb{R}^d)}].
\end{aligned}$$

Proof. After rescaling, it suffices to prove the theorem for $R = \rho = \eta = 1$.

Choose a function $\chi(t)$, smooth on $\mathbb{R}_+$, vanishing when $t \leq 1/2$, equal to 1 when $t \geq 1$. Let $w(t,x) = \chi(t)u(t,x)$. The function w is the solution of the following Cauchy problem:

$$\begin{cases} \Box w = \chi'' u + \chi' \partial_t u \\ w_{|t=0} = 0 \\ \partial_t w_{|t=0} = 0. \end{cases}$$

The function $\chi'' u + \chi' \partial_t u$ is supported by $\left[\frac{1}{2}, 1\right] \times B_2$, and

$$\begin{aligned} \|\chi'' u + \chi' \partial_t u\|_{L^\infty(\mathbb{R}_+;L^2)} \leq \|\chi''\|_{L^\infty} \|u\|_{L^\infty((0,1);L^2)} \\ + \|\chi'\|_{L^\infty} \|\partial_t u\|_{L^\infty((0,1);L^2)}. \end{aligned}$$

Since $u(t) = u_0 + \int_0^t \partial_t u(s)\,\mathrm{d}s$, we have, when $t \in [0,1]$,

$$\|u\|_{L^\infty((0,1);L^2)} \leq \|u_0\|_{L^2} + \int_0^t \|\partial_t u(s)\|_{L^2}\,\mathrm{d}s \leq \|u_0\|_{L^2} + \|\partial_t u\|_{L^\infty(\mathbb{R}_+;L^2)}.$$

With the energy conservation for the wave equation

$$\|\partial_t u(t)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla_x u(t)\|^2_{L^2(\mathbb{R}^d)} = \|u_1\|^2_{L^2(\mathbb{R}^d)} + \|\nabla u_0\|^2_{L^2(\mathbb{R}^d)},$$

this implies that

$$\|\chi'' u + \chi' \partial_t u\|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^d))} \leq C(\|u_1\|_{L^2(\mathbb{R}^d)} + \|\nabla u_0\|_{L^2(\mathbb{R}^d)}).$$

Moreover, $\chi'' u + \chi' \partial_t u$ is compactly supported in time, and for all $p \geq 1$,

$$\|\chi'' u + \chi' \partial_t u\|_{L^p(\mathbb{R}_+;L^2(\mathbb{R}^d))} \leq \|\chi'' u + \chi' \partial_t u\|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^d))}.$$

Thus, by Theorem 3.6,

$$\|\partial_t w\|_{L^p(\mathbb{R}_+;L^2(B_\rho))} \leq C\rho^{1+\frac{1}{p}}(\|u_1\|_{L^2(\mathbb{R}^d)} + \|\nabla u_0\|_{L^2(\mathbb{R}^d)}).$$

To conclude, use the estimate:

$$\begin{aligned} \|\partial_t u\|^p_{L^p(\mathbb{R}_+;L^2(B_\rho))} &= \int_0^\infty \|\partial_t u(s)\|^p_{L^2(B_\rho)}\,\mathrm{d}s \\ &= \int_0^1 \|\partial_t u(s)\|^p_{L^2(B_\rho)}\,\mathrm{d}s + \int_1^\infty \|\partial_t w(s)\|^p_{L^2(B_\rho)}\,\mathrm{d}s \\ &\leq \int_0^1 \left(\|u_1\|^2_{L^2(\mathbb{R}^d)} + \|\nabla_x u_0\|^2_{L^2(\mathbb{R}^d)}\right)^{p/2}\mathrm{d}s + \|\partial_t w\|^p_{L^p(\mathbb{R}_+;L^2(B_\rho))}. \end{aligned}$$

□

Remark 3.12. We have analogous results for initial data of the form Pu_0 and Pu_1. Indeed, let v the solution of homogeneous wave equation with initial data Pu_0 and Pu_1. The new $w = \chi v$ satisfies $\Box w = \chi'' v + \chi' v = P[\chi'' u + \chi' u]$ because P commutes with differential operators. The remainder of the proof is exactly the same, using Theorem 1.2 instead of Theorem 3.6.

Remark 3.13. For the case $p = +\infty$, the conservation of energy

$$\mathcal{E}(t) = \frac{1}{2}\|\nabla_{t,x} u(t)\|^2_{L^2}$$

gives already the result, with no restriction on the localization of the initial values, nor on the evaluation.

In the case $p = 2$, for non-compactly supported initial data, we use the following estimate:

Theorem 3.14. *Let $u_0 \in \mathcal{D}'(\mathbb{R}^d)$ such that $\nabla u_0 \in L^2(\mathbb{R}^d)$ and $u_1 \in L^2(\mathbb{R}^d)$. Let u the solution of $\big(\eta^2 \partial_t^2 - \Delta\big) u = 0$ such that $u_{|t=0} = u_0$ and $\partial_t u_{|t=0} = u_1$. Then, for all ball B_R, there is a constant $C = C(R) > 0$ such that*

$$\eta^2 \|\partial_t u\|^2_{L^2(\mathbb{R}_+; L^2(B_R)} + \|\nabla u\|^2_{L^2(\mathbb{R}_+; L^2(B_R)} \leq C(\|\nabla u_0\|^2_{L^2(\mathbb{R}^d)} + \eta^2 \|u_1\|^2_{L^2(\mathbb{R}^d)}).$$

Proof. Once again, it suffices to prove theorem when $\eta = 1$.

When d is odd, the idea is to write explicitly the integral to be calculated, and to use firstly Huygens principle ($\|\nabla_{t,x} u(t)\|_{L^2(B_R)} \leq \|\nabla_{t,x} u(0)\|_{L^2(\Gamma_{t-R,t+R})}$), secondly Fubini's theorem.

$$\begin{aligned}
\|\nabla_{t,x} u\|^2_{L^2(\mathbb{R}_+, L^2(B_R))} &= \int_{\mathbb{R}_+} \|\nabla_{t,x} u(t)\|^2_{L^2(B_R)} \,\mathrm{d}t \\
&\leq \int_{\mathbb{R}_+} \|\nabla_{t,x} u(0)\|^2_{L^2(\Gamma_{t-R,t+R})} \,\mathrm{d}t \\
&= \int_{\mathbb{R}_+} \int_{t-R \leq |x| \leq t+R} |\nabla_{t,x} u(0,x)|^2 \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_{\mathbb{R}^d} |\nabla_{t,x} u(0,x)|^2 \,\mathrm{d}x \int_{|x|-R \leq t \leq |x|+R} \mathrm{d}t \\
&\leq \int_{\mathbb{R}^d} |\nabla_{t,x} u(0,x)|^2 \,\mathrm{d}x \, 2R \\
&= 2R\big(\|\nabla u_0\|^2_{L^2(\mathbb{R}^d)} + \|u_1\|^2_{L^2(\mathbb{R}^d)}\big).
\end{aligned}$$

When d is even, write

$$\begin{aligned}
u(t) &= \mathbf{E}'(t) *_x u_0 + \mathbf{E}(t) *_x u_1 \\
\nabla u(t) &= \mathbf{E}'(t) *_x \nabla u_0 + \nabla \mathbf{E}(t) *_x u_1 \\
\partial_t u(t) &= \mathbf{E}''(t) *_x u_0 + \mathbf{E}'(t) *_x u_1 \\
&= \Delta \mathbf{E}(t) *_x u_0 + \mathbf{E}'(t) *_x u_1 \\
&= -\nabla \mathbf{E}(t) *_x \nabla u_0 + \mathbf{E}'(t) *_x u_1.
\end{aligned}$$

We only give details for the contribution of $\mathbf{E}'(t) *_x u_1$. The other terms are similar.

$$\begin{aligned}
\|\mathbf{E}'(t) *_x u_1\|^2_{L^2(B)} &\leq \|\mathbf{E}'(t) *_x u_1 \mathbb{1}_{B_{t-2}}\|^2_{L^2(B)} + \|\mathbf{E}'(t) *_x u_1 \mathbb{1}_{B^c_{t-2}}\|^2_{L^2(B)} \\
&\leq \|\mathbf{E}(t) *_x u_1 \mathbb{1}_{B_{t-2}}\|^2_{L^2(B)} + \|u_1\|_{L^2(\Gamma_{t-2,t+1})}.
\end{aligned} \tag{3.4}$$

The second term of the right-hand side can be bounded as in odd dimension. Consider the first term. For $|x| \leq 1$, one has:

$$|\mathbf{E}'(t) *_x u_1 \mathbb{1}_{B_{t-2}}(x)|^2 = C_d \left(\int_{|y|\leq t-2} \frac{t}{\sqrt{t^2 - |x-y|^2}^{d+1}} u(y) \mathrm{d}y \right)^2 .$$

Using Schwarz inequality, and the inequality $|x - y| \leq |y| + 1$,

$$|\mathbf{E}'(t) *_x u_1 \mathbb{1}_{B_{t-2}}(x)|^2 \leq C t^{d-1} \int_{|y|\leq t-2} \frac{|u_1(y)|^2 \, t^2}{(t^2 - (|y|+1)^2)^{d+1}} \, \mathrm{d}y.$$

So, integrating in $x \in B$,

$$\|\mathbf{E}'(t) *_x u_1 \mathbb{1}_{B_{t-2}}\|^2_{L^2(B)} \leq C \int_{|y|\leq t-2} \frac{|u_1(y)|^2 \, t^{d+1}}{(t - (|y|+1))^{d+1} (t + |y| + 1)^{d+1}} \, \mathrm{d}y.$$

Let us denote $v_1(t) = \mathbf{E}'(t) *_x u_1 \mathbb{1}_{B_{t-2}}$. We want to prove

$$\|v_1\|_{L^2(L^2(B))} \leq C \|u_1\|_{L^2}.$$

A time integration and Fubini's theorem, give:

$$\|v_1\|^2_{L^2(\mathbb{R}_+; L^2(B))} \leq C \int_{\mathbb{R}^d} |u_1(y)|^2 \mathrm{d}y \int_{t=|y|+2}^{+\infty} \frac{t^{d+1}}{(t - (|y|+1))^{d+1} (t + |y| + 1)^{d+1}} \, \mathrm{d}t.$$

Thus, we just need to bound uniformly in $r = |y|$ the following integral

$$F(r) = \int_{t=r+2}^{+\infty} \frac{t^{d+1}}{(t - (r+1))^{d+1} (t + r + 1)^{d+1}} \, \mathrm{d}t.$$

We have (with $u = t - r + 1$)

$$F(r) = \int_{u=1}^{+\infty} \frac{1}{u^{d+1}} \left(\frac{u + r + 1}{u + 2r + 2} \right)^{d+1} \mathrm{d}u$$

$$F(r) \leq \int_1^{+\infty} \frac{\mathrm{d}u}{u^{d+1}} = \frac{1}{d} < +\infty.$$

This finishes the proof. □

4. Application to the Maxwell–Landau–Lifshitz System

In this section, we apply to the system (1.2) the new results on the wave equation proved in the previous section. Let $(\mathbf{e}, \mathbf{h}, \mathbf{m})$ be a strong solution of the system (1.2). The key point is the fact that $\mathbf{h}_\perp$ and $\mathbf{e}$ are solutions of the following Cauchy problems.

$$\begin{cases} (\eta^2 \partial_t - \Delta) \tilde{\mathbf{h}} = -\eta^2 \partial_t^2 \mathbf{m}_\perp \\ \tilde{\mathbf{h}}_{|t=0} = P_\perp \mathbf{h}_0 \\ \partial_t \tilde{\mathbf{h}}_{|t=0} = -\dfrac{1}{\eta} \mathbf{curl}\, \mathbf{e}_0 - f(\mathbf{m}_0, \mathbf{h}_0 - \nabla\Phi(\mathbf{m}_0) + \mathbf{h}_{\mathrm{ext}})_\perp \end{cases} \tag{4.1}$$

$$\begin{cases} (\eta^2 \partial_t - \Delta) \tilde{\mathbf{e}} = -\mathbf{curl}\, \eta \partial_t \mathbf{m}_\perp \\ \tilde{\mathbf{e}}_{|t=0} = \mathbf{e}_0 \\ \partial_t \tilde{\mathbf{e}}_{|t=0} = -\dfrac{1}{\eta} \mathbf{curl}\, \mathbf{h}_0 \end{cases} \tag{4.2}$$

4.1. *Boundedness of* $\mathbf{h}_\perp$ *and* $\mathbf{e}$ *with respect to* η

In order to study $\mathbf{h}_\perp$ and $\mathbf{e}$, decompose the problem in three independent problems: let u, ϕ and ψ be the solutions of the following Cauchy problems:

$$\begin{cases} \left(\eta^2\partial_t^2 - \Delta\right)u = -\eta\partial_t\mathbf{m}_\perp \\ u_{|t=0} = 0 \\ \partial_t u_{|t=0} = 0 \end{cases} \tag{4.3}$$

$$\begin{cases} \left(\eta^2\partial_t^2 - \Delta\right)\phi = 0 \\ \phi_{|t=0} = 0 \\ \partial_t \phi_{|t=0} = \dfrac{1}{\eta}P_\perp\mathbf{h}_0 \end{cases} \tag{4.4}$$

$$\begin{cases} \left(\eta^2\partial_t^2 - \Delta\right)\psi = 0 \\ \psi_{|t=0} = 0 \\ \partial_t \psi_{|t=0} = \dfrac{1}{\eta}\mathbf{e}_0 \end{cases} \tag{4.5}$$

Lemma 4.1. *Define* u, ϕ, ψ *as solutions of the Cauchy problems* (4.3)–(4.5). *Then* $\mathbf{h}_\perp = \eta\partial_t u + \eta\partial_t\phi - \mathbf{curl}\,\psi$ *and* $\mathbf{e} = \mathbf{curl}\,u + \mathbf{curl}\,\phi - \partial_t\psi$.

Proof. Denote $\tilde{\mathbf{h}} = \eta\partial_t u + \eta\partial_t\phi - \mathbf{curl}\,\psi$. Clearly, $\tilde{\mathbf{h}}$ is solution of wave equation in (4.1), and $\tilde{\mathbf{h}}_{|t=0} = P_\perp\mathbf{h}_0$. It remains to check that

$$\partial_t\tilde{\mathbf{h}}_{|t=0} = \eta\, f(\mathbf{m}_0, \mathbf{h}_0 - \nabla\Phi(\mathbf{m}_0) + \mathbf{h}_{\text{ext}})_\perp - \mathbf{curl}\,\mathbf{e}_0.$$

Indeed,

$$\begin{aligned} \partial_t\tilde{\mathbf{h}}_{|t=0} &= \partial_t^2 u_{|t=0} + \partial_t^2\phi_{|t=0} + \partial_t\mathbf{curl}\,\psi_{|t=0} \\ &= (\Delta u)_{|t=0} - P_\perp(\partial_t\tilde{\mathbf{m}})_{|t=0} + (\Delta\phi)_{|t=0} + [\mathbf{curl}\,(\partial_t\psi)]_{|t=0} \\ &= \Delta(u_{|t=0}) - \eta f(\mathbf{m}_0, \mathbf{h}_0 - \nabla\Phi(\mathbf{m}_0) + \mathbf{h}_{\text{ext}})_\perp + \Delta(\phi_{|t=0}) - \mathbf{curl}\,\mathbf{e}_0 \\ &= \eta f(\mathbf{m}_0, \mathbf{h}_0 - \nabla\Phi(\mathbf{m}_0) + \mathbf{h}_{\text{ext}})_\perp - \mathbf{curl}\,\mathbf{e}_0. \end{aligned}$$

The proof for $\mathbf{e}$ is similar. □

Proposition 4.2. *Let* $(\mathbf{e}, \mathbf{h}, \mathbf{m})$ *be a solution of the system* (1.2). *Then for all* $R \geq 1$, *there is a constant* C_R *which does not depend of* $\mathbf{m}$ *and* η *such that*:

$$\|\mathbf{h}_\perp\|_{L^2(\mathbb{R}_+;L^2(B_R))} + \|\mathbf{e}\|_{L^2(\mathbb{R}_+;L^2(B_R))} \leq C_R\sqrt{\eta}\left(\sqrt{\mathcal{E}(0)} + \sqrt{\eta}\|\partial_t\mathbf{m}\|_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^d))}\right)$$

Proof. By Theorem 1.2, the solution u of (4.3) satisfies:

$$\|\eta\partial_t u\|_{L^2(L^2(B_R))} + \|\nabla u\|_{L^2(L^2(B_R))} \leq C_R\eta\|\partial_t\mathbf{m}\|_{L^2(L^2)}$$

By Theorem 3.14, the solutions ϕ and ψ of (4.4) and (4.5) satisfies:

$$\begin{gathered} \|\eta\partial_t\phi\|_{L^2(L^2(B_R))} + \|\nabla\phi\|_{L^2(L^2(B_R))} \leq C_R\sqrt{\eta}\|\mathbf{h}_0\|_{L^2} \\ \|\eta\partial_t\psi\|_{L^2(L^2(B_R))} + \|\nabla\psi\|_{L^2(L^2(B_R))} \leq C\sqrt{\eta}\|\mathbf{e}_0\|_{L^2}. \end{gathered}$$

We conclude by using Lemma 4.1. □

Proof of Theorem 1.4. It suffices again to prove Theorem with $\eta = 1$.

We showed in the proof of Theorem 1.2 that there exists $h_R \in L^1 \cap L^\infty(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$ such that the solution u of (4.3) satisfies

$$\|\nabla_{t,x} u(t)\|_{L^2(B_R)} \le h_R *_t \|\partial_t \mathbf{m}(t)\|_{L^2(\mathbb{R}^d_x)}.$$

The property of convolution $L^2 * L^2$ implies that the right-hand side is continuous and tends to 0 at infinity. Thus, when $t \to +\infty$,

$$\|\nabla_{t,x} u(t)\|_{L^2(B_R)} \to 0. \tag{4.6}$$

By Huygens principle, for $t > R$, and Φ, Ψ solutions of (4.4) and (4.5),

$$\begin{aligned}\|\nabla_{t,x}(\Phi,\Psi)(t)\|_{L^2(B_R)} &\le \|\nabla_{t,x}(\Phi,\Psi)(0)\|_{L^2(\Gamma_{t-R,t+R})} \\ &= \|\mathbb{1}_{\Gamma_{t-R,t+R}} \nabla_{t,x}(\Phi,\Psi)(0)\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Thus, by Lebesgue's theorem, when $t \to +\infty$,

$$\|\nabla_{t,x}(\Phi,\Psi)(t)\|_{L^2(B_R)} \to 0. \tag{4.7}$$

We conclude again by using (4.6) and (4.7) with Lemma 4.1. □

4.2. *Bounds for the curl fields*

Notation 4.3. We denote $\mathcal{E}_{\mathrm{curl}}(t) = \|\mathbf{curl}\,\mathbf{e}(t)\|^2_{L^2(\mathbb{R}^d)} + \|\mathbf{curl}\,\mathbf{h}(t)\|^2_{L^2(\mathbb{R}^d)}$.

The aim of this paragraph is to prove Theorem 1.5. More precisely, we establish the following result:

Proposition 4.4. *There is a constant K such that, for all positive real fixed $\mathcal{E}_r$, there exists $\eta_0 > 0$ such that for all $0 < \eta \le \eta_0$ and all solution of the problem* (1.2) *such that $\mathcal{E}_{\mathrm{curl}}(0) \le \mathcal{E}_r$, we have*

$$\mathcal{E}_{\mathrm{curl}}(t) \le \mathcal{E}_{\mathrm{curl}}(0) + K(\mathcal{E}(0)^4 + 1).$$

Proof. We search to estimate $\partial_t \mathbf{h}_\perp$. Decompose $\mathbf{h}_\perp = \mathbf{h}_1 + \mathbf{h}_2$, where $\mathbf{h}_1$ and $\mathbf{h}_2$ are the solutions of the following Cauchy problems:

$$\begin{cases} \left(\eta^2\partial_t^2 - \Delta\right)\mathbf{h}_1 = -\eta^2\partial_t^2\mathbf{m}_\perp \\ \mathbf{h}_{1|t=0} = 0 \\ \partial_t \mathbf{h}_{1|t=0} = 0, \end{cases} \tag{4.8}$$

$$\begin{cases} \left(\eta^2\partial_t^2 - \Delta\right)\mathbf{h}_2 = 0 \\ \mathbf{h}_{2|t=0} = \mathbf{h}_{0\perp} \\ \partial_t \mathbf{h}_{2|t=0} = \dfrac{1}{\eta}\mathbf{curl}\,\mathbf{e}_0 + f(\mathbf{m}_0, \mathbf{h}_0 - \nabla\Phi(\mathbf{m}_0) + \mathbf{h}_{\mathrm{ext}})_\perp. \end{cases} \tag{4.9}$$

In order to apply Theorem 1.2 to $\mathbf{h}_1$ solution of (4.8), estimate $\partial_t^2\mathbf{m}_\perp$. We have

$$\partial_t^2\mathbf{m} = \partial_t f(\mathbf{m}, \mathbf{h}_T) = D_m f(\mathbf{m}, \mathbf{h}_T)\partial_t\mathbf{m} + f(\mathbf{m}, \partial_t\mathbf{h}_T) \tag{4.10}$$

$$|D_m f(\mathbf{m},\mathbf{h}_T)\partial_t\mathbf{m}| \le 2|\mathbf{h}_T|\,|\partial_t\mathbf{m}| \le 4\,|\mathbf{h}_T|^2\,|\mathbf{m}| = 4\,|\mathbf{h}_\perp - \mathbf{m}_\parallel - \nabla\Phi(\mathbf{m}) + \mathbf{h}_{\text{ext}}|^2\,\mathbb{1}_B$$

$$|D_m f(\mathbf{m},\mathbf{h}_T)\partial_t\mathbf{m}|^2 \lesssim \left[|\mathbf{h}_\perp|^4 + |\mathbf{m}_\parallel|^4 + |\nabla\Phi(\mathbf{m})|^4 + |\mathbf{h}_{\text{ext}}|^4\right]\mathbb{1}_B \tag{4.11}$$

where we wrote $f \lesssim g$ in order to say that there is a constant C which is independent of η and $(\mathbf{e},\mathbf{h},\mathbf{m})$, such that $f \le Cg$. Since $\mathbf{m}(t)$ is supported in B,

$$\|\mathbf{m}_\parallel(t)\|^4_{L^4(\mathbb{R}^d)} \lesssim \|\mathbf{m}(t)\|^4_{L^4(\mathbb{R}^d)} \lesssim \|\mathbf{m}\|_{L^\infty} = 1.$$

Next, because $\nabla\Phi$ is continuous, $\|\Phi(\mathbf{m})\|_{L^\infty} \le \sup_{m\in B}|\Phi(m)| < +\infty$. Moreover, $\Phi(0) \le \Phi(m)$ for $m \in \mathbb{R}^3$. Thus, $\nabla\Phi(0) = 0$ and $\operatorname{supp}\nabla\Phi(\mathbf{m})$ is supported in B. Consequently,

$$\|\nabla\Phi(\mathbf{m}(t))\|_{L^4(\mathbb{R}^d)} \lesssim \|\nabla\Phi(\mathbf{m}(t))\|_{L^\infty(\mathbb{R}^d)} \lesssim 1.$$

Integrate in x on B in (4.11), then

$$\|D_m f(\mathbf{m}(t),\mathbf{h}(t))\partial_t\mathbf{m}(t)\|^2_{L^2(\mathbb{R}^d)} \lesssim \|\mathbf{h}_\perp(t)\|^4_{L^4(B)} + 1.$$

Now, because $d \le 3$, by Sobolev inequality (see [1]):

$$\|h\|_{L^4(B)} \lesssim \|h\|_{L^2(B)} + \|\nabla h\|_{L^2(B)}.$$

Thus

$$\|D_m f(\mathbf{m}(t),\mathbf{h}_T(t))\partial_t\mathbf{m}(t)\|^2_{L^2(\mathbb{R}^d)} \lesssim \|\nabla_x\mathbf{h}_\perp(t)\|^4_{L^2(B)} + \|\mathbf{h}_\perp(t)\|^4_{L^2(B)} + 1.$$

Similarly for the second term in (4.10),

$$\begin{aligned}
|f(\mathbf{m},\partial_t\mathbf{h}_T)| &\le 2|\mathbf{m}|\,|\partial_t\mathbf{h}_T| \le 2|\partial_t\mathbf{h}_T|\mathbb{1}_B,\\
|f(\mathbf{m},\partial_t\mathbf{h}_T)|^2 &\lesssim (|\partial_t\mathbf{h}_\perp|^2 + |\partial_t\mathbf{m}_\parallel|^2 + |\nabla\Phi(\mathbf{m})|)\mathbb{1}_B,\\
\|f(\mathbf{m}(t),\partial_t\mathbf{h}_T(t))\|^2_{L^2(\mathbb{R}^d)} &\lesssim \|\partial_t\mathbf{h}_\perp(t)\|^2_{L^2(B)} + \|\partial_t\mathbf{m}(t)\|^2_{L^2(\mathbb{R}^d)} + 1\\
&\lesssim \|\partial_t\mathbf{h}_\perp(t)\|^2_{L^2(B)} + \|\mathbf{h}(t)\|^2_{L^2(\mathbb{R}^d)} + 1.
\end{aligned}$$

Taking the essential supremum in t on $(0,T)$:

$$\begin{aligned}
\|\partial_t f(\mathbf{m},\mathbf{h})\|^2_{L^\infty((0,T);L^2(\mathbb{R}^d))} \lesssim \big[&\|\nabla_x\mathbf{h}_\perp\|^4_{L^\infty((0,T);L^2(B))}\\
&+ \|\partial_t\mathbf{h}_\perp\|^2_{L^\infty((0,T);L^2(B_R))} + C(\mathbf{h}_\perp,\partial_t\mathbf{m})\big]
\end{aligned} \tag{4.12}$$

with

$$C(\mathbf{h}_\perp,\partial_t\mathbf{m}) = \|\mathbf{h}_\perp\|^4_{L^\infty(\mathbb{R}_+;L^2(B))} + 1 + \|\mathbf{h}\|^2_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^d))} \lesssim \mathcal{E}(0)^2 + 1 < +\infty. \tag{4.13}$$

By Theorem 1.2,

$$\|(\eta\partial_t\mathbf{h}_1,\nabla\mathbf{h}_1)\|_{L^\infty((0,T);L^2(B))} \lesssim \eta^2\|\partial_t f(\mathbf{m},\mathbf{h})\|_{L^\infty((0,T);L^2(\mathbb{R}^d))}.$$

We have also conservation of energy for the linear wave equation

$$\begin{aligned}\|(\eta\partial_t \mathbf{h}_2, \nabla \mathbf{h}_2)\|_{L^\infty((0,T);L^2(B))} &\leq \|\eta\partial_t \mathbf{h}_2(0), \nabla \mathbf{h}_2(0)\|_{L^2(\mathbb{R}^d)^4} \\ &\leq \sqrt{\|\mathbf{curl}\,\mathbf{h}_0\|_{L^2}^2 + \|\mathbf{curl}\,\mathbf{e}_0\|_{L^2}^2}.\end{aligned}$$

Adding the last two inequalities yields:

$$\begin{aligned}\|(\eta\partial_t \mathbf{h}_\perp, \nabla \mathbf{h}_\perp)\|_{L^\infty((0,T);L^2(B))^4} &\lesssim \eta^2 \|\partial_t f(\mathbf{m}, \mathbf{h})\|_{L^\infty((0,T);L^2)} \\ &\quad + \sqrt{\mathcal{E}_{\mathrm{curl}}(0)}.\end{aligned}$$

Finally, we use the inequalities (4.12) and (4.13). There is a constant C_1 such that

$$\begin{aligned}\|(\eta\partial_t \mathbf{h}_\perp, \nabla \mathbf{h}_\perp)\|_{L^\infty((0,T);L^2(B))^4} &\leq C_1\eta^2 \big[\|\nabla_x \mathbf{h}_\perp\|_{L^\infty((0,T);L^2(B))}^2 \\ &+ \|\partial_t \mathbf{h}_\perp\|_{L^\infty((0,T);L^2(B_R))} + \mathcal{E}(0)^2 + 1\big] + \sqrt{\mathcal{E}_{\mathrm{curl}}(0)}.\end{aligned}$$

In order to put the term $\|\partial_t \mathbf{h}_\perp\|_{L^\infty((0,T);L^2(B_R))}$ in (4.12) in the left-hand side, assume that $\eta \leq \frac{1}{2C_1}$.

Let

$$X(T) = \|(\eta\partial_t \mathbf{h}_\perp, \nabla \mathbf{h}_\perp)\|_{L^\infty((0,T);L^2(B_R))}.$$

Then we have, with $C_2 = 4C_1$:

$$X(T) \leq \eta^2 C_2 (X(T)^2 + \mathcal{E}(0)^2 + 1) + \sqrt{\mathcal{E}_{\mathrm{curl}}(0)}.$$

Now, remark that $\|\nabla_x \mathbf{h}_\perp(t)\|_{L^2(\mathbb{R}^d)} = \|\mathbf{curl}\,\mathbf{h}\|_{L^2(\mathbb{R}^d)}$. Now, because we are looking for solution of the (1.2) with regularity $\mathcal{C}(\mathbb{R}_+, H(\mathbf{curl}, \mathbb{R}^d))$, we obtain that X is a continuous function. Thus, if $X(0) \leq X_1$, where X_1 is the smaller root of the quadratic polynomial $\eta^2\, C_2\, (X^2 + \mathcal{E}(0)^2 + 1) - X$, then $X(t) \leq X_1$ for all $t \geq 0$. We are in this case when:

$$\eta \leq \frac{1}{4C_2(C_2(\mathcal{E}(0)^2 + 1) + \sqrt{\mathcal{E}_{\mathrm{curl}}(0)})}.$$

To conclude, remark that $\|\nabla \mathbf{h}_\perp(t)\|_{L^2} = \|\mathbf{curl}\,\mathbf{h}(t)\|_{L^2}$ and, thanks to the second equation and the condition $\mathrm{div}(\mathbf{h} + \mathbf{m}) = 0$ in (1.2),

$$\|\mathbf{curl}\,\mathbf{e}(t)\|_{L^2} \leq \|\eta\partial_t \mathbf{h}_\perp(t)\| + \eta\|\mathbf{h}(t)\|_{L^2}. \qquad \square$$

Remark 4.5. The L^2 estimates so obtained on $\mathbf{curl}\,\mathbf{e}$ and $\mathbf{curl}\,\mathbf{h}$ does not depend on the parameter $\alpha \in [0, 1]$ in the definition of $f(m, h)$. So, we can obtain a convergence result, in the next section, when $\alpha \to 0$.

5. Convergence Toward Quasistationary System; Existence and Uniqueness of Weak Solutions in Quasistationary Problem

We prove a convergence result when η tends to 0 in (1.2), which is the quasi-stationary limit. In [10], Jochmann proved a weak quasi-stationary limit for weak energy solution of (1.1). Here, we consider a strong limit for strong solution. We first study $\mathbf{e} = \mathbf{e}_\eta$ and $\mathbf{h}_{\eta\perp}$ using Proposition 4.2, next we study the convergence of $\mathbf{m} = \mathbf{m}_\eta$.

5.1. *Convergence of* $\mathbf{h}_{\eta\perp}$ *and* $\mathbf{e}_\eta$ *to* 0

Proposition 5.1. *For fixed* $\alpha \geq 0$. *Let* $(\mathbf{h}_\eta, \mathbf{e}_\eta, \mathbf{m}_\eta)$ *be the solution of the Cauchy problem* (1.2). *Then, for all* $R > 0$, *when* $\alpha > 0$

$$\lim_{\eta\to 0^+} \|\mathbf{h}_{\eta\perp}\|_{L^2(\mathbb{R}_+;L^2(B_R))} + \|\mathbf{e}_\eta\|_{L^2(\mathbb{R}_+;L^2(B_R))} = 0.$$

When $\alpha = 0$, *we have, for all* $T > 0$,

$$\lim_{\eta\to 0^+} \|\mathbf{h}_{\eta\perp}\|_{L^2((0,T);L^2(B_R))} + \|\mathbf{e}_\eta\|_{L^2((0,T);L^2(B_R))} = 0.$$

Proof. First, recall Proposition 4.2 (when $0 < T \leq +\infty$)

$$\|\mathbf{h}_{\eta\perp}\|_{L^2(\mathbb{R}_+;L^2(B_R))} + \|\mathbf{e}_\eta\|_{L^2(\mathbb{R}_+;L^2(B_R))} \leq C_R\sqrt{\eta}\left(\sqrt{\mathcal{E}(0)} + \sqrt{\eta}\|\partial_t \mathbf{m}\|_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^d))}\right).$$

By Proposition 2.6, when $\alpha > 0$,

$$\|\partial_t \mathbf{m}_\eta\|_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^d))} \leq \sqrt{\frac{1+\alpha^2}{\alpha}\mathcal{E}(0)}.$$

This gives the proof in this case. When $\alpha = 0$, the proof follows from the inequality

$$\|\partial_t \mathbf{m}(t)\|_{L^\infty((0,T);L^2)} \leq \|\mathbf{h}(t)\|_{L^\infty((0,T);L^2)} \leq \sqrt{2\mathcal{E}(0)}$$

given by the Landau–Lifshitz equation. □

5.2. *Convergence of* $\mathbf{m}_\eta$

In this section, we prove the following result in several steps:

Theorem 5.2. *There exists a unique solution in* $\mathcal{C}^1(\mathbb{R}_+; L^2)$ *of the following Cauchy problem*:

$$\begin{aligned} \partial_t \mathbf{m} &= f(\mathbf{m}, -\mathbf{m}_\parallel) \\ \mathbf{m}(0) &= \mathbf{m}_0. \end{aligned} \tag{5.1}$$

Moreover, if $(\mathbf{h}_\eta, \mathbf{e}_\eta, \mathbf{m}_\eta)$ *is a weak solution of* (1.2), *then as* $\eta \to 0$ $\mathbf{m}_\eta \to \mathbf{m}$ *strongly in* $\mathcal{C}((0,T); L^2)$.

5.2.1. *Weak convergence*

Proposition 5.3. *There exists* $\mathbf{m} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ *and a subsequence of* $(\mathbf{m}_\eta)_\eta$ *such that*:

- $\mathbf{m}_\eta$ *converges* $*$*-weakly in* L^∞ *to* $\mathbf{m}$.
- $\mathbf{m}$ *is continuous from* $\mathbb{R}_+$ *to* $L^2_{\text{weak}}(\mathbb{R}^d)$.
- *for all* $t \geq 0$, $\mathbf{m}_\eta(t)$ *converges weakly in* $L^2(\mathbb{R}^d)$ *to* $\mathbf{m}(t)$.

Proof. $(\mathbf{m}_\eta)_{\eta>0}$ is bounded in $L^\infty([0,T] \times \mathbb{R}^d)$, so, extracting a subsequence, converges $*$-weakly to a $\mathbf{m} \in L^\infty([0,T] \times \mathbb{R}^d)$. Moreover, we know that

$$\partial_t \mathbf{m}_\eta = f(\mathbf{m}_\eta, \mathbf{h}_{\eta\perp} - \mathbf{m}_{\eta\|} - \nabla\Phi(\mathbf{m}_\eta) + \mathbf{h}_{\text{ext}}),$$

so

$$\begin{aligned}
\|\partial_t \mathbf{m}_\eta\|_{L^\infty([0,T];L^2)} &\leq \|\mathbf{m}_\eta\|_{L^\infty}(\|\mathbf{h}_{\eta\perp}\|_{L^\infty([0,T];L^2)} + \|\mathbf{m}_{\eta\|}\|_{L^\infty([0,T];L^2)} \\
&\qquad + \|\nabla\Phi(\mathbf{m}_\eta)\|_{L^\infty([0,T];L^2)} + \|\mathbf{h}_{\text{ext}}\|_{L^2(B)}) \\
&\leq \sqrt{\mathcal{E}(0)} + C.
\end{aligned}$$

Consequently, the set $(\mathbf{m}_\eta)_\eta$ is equi-continuous from $[0,T]$ to L^2.

L^2 with its weak topology is a metric space such that bounded sets are compact. According to Ascoli's theorem, $\mathbf{m}_\eta$ converges, extracting further a subsequence, in $C([0,T]; L^2_{\text{weak}})$, necessarily to $\mathbf{m}$ by uniqueness of the limit in the distribution sense. In particular, for all $t \in [0,T]$, $\mathbf{m}_\eta(t)$ converges weakly to $\mathbf{m}(t)$ in $L^2(\mathbb{R}^d)$. □

5.2.2. *First estimates*

Let $\eta_1, \eta_2 > 0$ two positive reals, intended to converge to 0.

$$\begin{aligned}
\partial_t \mathbf{m}_{\eta_1} - \partial_t \mathbf{m}_{\eta_2} &= f(\mathbf{m}_{\eta_1}, \mathbf{h}_{\eta_1\perp} - \mathbf{m}_{\eta_1\|} - \nabla\Phi(\mathbf{m}_{\eta_1}) + \mathbf{h}_{\text{ext}}) \\
&\quad - f(\mathbf{m}_{\eta_2}, \mathbf{h}_{\eta_2\perp} - \mathbf{m}_{\eta_2\|} - \nabla\Phi(\mathbf{m}_{\eta_2}) + \mathbf{h}_{\text{ext}}) \\
&= (f(\mathbf{m}_{\eta_1}, -\mathbf{m}_\| - \nabla\Phi(\mathbf{m}_{\eta_1}) + \mathbf{h}_{\text{ext}}) \\
&\quad - f(\mathbf{m}_{\eta_2}, -\mathbf{m}_\| - \nabla\Phi(\mathbf{m}_{\eta_1}) + \mathbf{h}_{\text{ext}}) \\
&\quad + f(\mathbf{m}_{\eta_2}, \nabla\Phi(\mathbf{m}_{\eta_1}) - \nabla\Phi(\mathbf{m}_{\eta_2})) \\
&\quad + f(\mathbf{m}_{\eta_1}, \mathbf{h}_{\eta_1\perp}) - f(\mathbf{m}_{\eta_2}, \mathbf{h}_{\eta_2\perp}) \\
&\quad + f(\mathbf{m}_{\eta_1}, P_\|(\mathbf{m} - \mathbf{m}_{\eta_1})) - f(\mathbf{m}_{\eta_2}, P_\|(\mathbf{m} - \mathbf{m}_{\eta_2})). \qquad (5.2)
\end{aligned}$$

Take the scalar product of this equality by $\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2}$, to find a bound, using the fact that $|\mathbf{m}_{\eta_1}|, |\mathbf{m}_{\eta_2}| \leq 1$:

$$\begin{aligned}
\frac{1}{2}\partial_t|\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2}|^2 \leq\ & 2|\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2}|^2\left(|\mathbf{m}_\parallel| + |\nabla\Phi(\mathbf{m}_{\eta_1})| + |\mathbf{h}_{\text{ext}}|\right) \\
& + 2|\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2}|\,|\nabla\Phi(\mathbf{m}_{\eta_1}) - \nabla\Phi(\mathbf{m}_{\eta_2})|\,|\mathbf{m}_{\eta_2}| \\
& + 2|\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2}|\,|\mathbf{m}_{\eta_1}|\,|\mathbf{h}_{\eta_1\perp}| + 2|\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2}|\,|\mathbf{m}_{\eta_2}|\,|\mathbf{h}_{\eta_2\perp}| \\
& + |f(\mathbf{m}_{\eta_1}, P_\parallel(\mathbf{m} - \mathbf{m}_{\eta_1}))\cdot(\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2})| \\
& + |f(\mathbf{m}_{\eta_2}, P_\parallel(\mathbf{m} - \mathbf{m}_{\eta_2}))\cdot(\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2})| \\
\leq\ & 2|\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2}|^2\left(|\mathbf{m}_\parallel| + C\right) \\
& + 8(|\mathbf{h}_{\eta_1\perp}| + |\mathbf{h}_{\eta_2\perp}|) \\
& + |f(\mathbf{m}_{\eta_1}, P_\parallel(\mathbf{m} - \mathbf{m}_{\eta_1}))\cdot(\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2})| \\
& + |f(\mathbf{m}_{\eta_2}, P_\parallel(\mathbf{m} - \mathbf{m}_{\eta_2}))\cdot(\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2})|. \qquad (5.3)
\end{aligned}$$

We use a Gronwall's lemma to absorb the first term in the left-hand side. $\mathbf{m}$ is bounded, and supported in B, hence in $L^2([0,T];L^2(\mathbb{R}^d))$, thus $\mathbf{m}_\parallel$ is in $L^2([0,T];L^2(\mathbb{R}^d))$, therefore in $L^1([0,T]\times B)$. By Fubini's theorem, the function $x \mapsto \|\mathbf{m}_\parallel(\,\cdot\,,x)\|_{L^1([0,T])}$ is integrable, thus finite almost everywhere. Hence, we can define, for almost every $x \in B$:

$$a(t,x) = |x|^2 + 2\int_0^t (|\mathbf{m}_\parallel(s,x)| + C)\,\mathrm{d}s.$$

Remark 5.4. We need the term $|x|^2$ in order to have $e^{-a(t)} \in L^4(\mathbb{R}^d)$, which is used later.

There holds

$$\begin{aligned}
\frac{1}{2}\partial_t|e^{-2a}(\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2})|^2 \leq\ & 8e^{-2a}(|\mathbf{h}_{\eta_1\perp}| + |\mathbf{h}_{\eta_2\perp}|) \\
& + \left|f(\mathbf{m}_{\eta_1}, e^{-a}P_\parallel(\mathbf{m} - \mathbf{m}_{\eta_1}))\cdot\left(e^{-a}(\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2})\right)\right| \\
& + \left|f(\mathbf{m}_{\eta_2}, e^{-a}P_\parallel(\mathbf{m} - \mathbf{m}_{\eta_2}))\cdot\left(e^{-a}(\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2})\right)\right|.
\end{aligned}$$

Integrate on $[0,t]\times B$. Because $\mathbf{m}_{\eta_1}(0) = \mathbf{m}_0 = \mathbf{m}_{\eta_2}(0)$, we have

$$\begin{aligned}
& \frac{1}{2}\|e^{-a(t)}(\mathbf{m}_{\eta_1}(t) - \mathbf{m}_{\eta_2}(t))\|_{L^2}^2 \\
& \quad \leq 8\|e^{-2a}(|\mathbf{h}_{\eta_1\perp}| + |\mathbf{h}_{\eta_2\perp}|)\|_{L^1([0,T]\times B)} \\
& \qquad + \int_0^t\int_B \left|f(\mathbf{m}_{\eta_1}, e^{-a}P_\parallel(\mathbf{m} - \mathbf{m}_{\eta_1}))\cdot\,(e^{-a}(\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2}))\right|\mathrm{d}x\mathrm{d}t \\
& \qquad + \int_0^t\int_B \left|f(\mathbf{m}_{\eta_2}, e^{-a}P_\parallel(\mathbf{m} - \mathbf{m}_{\eta_2}))\cdot(e^{-a}(\mathbf{m}_{\eta_1} - \mathbf{m}_{\eta_2}))\right|\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Assume for the moment the following result, which is proved in the next section.

Proposition 5.5. *There is a constant C and a function $D(\eta_1, \eta_2)$ converging to 0 when (η_1, η_2) tends to $(0,0)$ such that, for η which is η_1 or η_2, and all $t \in [0,T]$,*

$$\int_B \left| f\big(\mathbf{m}_\eta(t), e^{-a(t)} P_\parallel(\mathbf{m}(t) - \mathbf{m}_\eta(t))\big) \cdot \big(e^{-a(t)}(\mathbf{m}_{\eta_1}(t) - \mathbf{m}_{\eta_2}(t))\big) \right| \mathrm{d}x \\ \leq C \| e^{-a(t)} (\mathbf{m}_{\eta_1}(t) - \mathbf{m}_{\eta_2}(t)) \|_{L^2} \| e^{-a(t)} (\mathbf{m}(t) - \mathbf{m}_\eta(t)) \|_{L^2} + D(\eta_1, \eta_2).$$

Let $D'(\eta_1, \eta_2) = D(\eta_1, \eta_2) + 8 \| e^{-2a} (|\mathbf{h}_{\eta_1 \perp}| + |\mathbf{h}_{\eta_2 \perp}|) \|_{L^1([0,T] \times B)}$. Then, by Proposition 5.1.

$$\lim_{\eta_1, \eta_2 \to 0} D'(\eta_1, \eta_2) = 0.$$

Hence,

$$\frac{1}{2} \| e^{-a(t)} \big(\mathbf{m}_{\eta_1}(t) - \mathbf{m}_{\eta_2}(t)\big) \|_{L^2}^2 \\ \leq D'(\eta_1, \eta_2) + C \int_0^t \| e^{-a(s)} (\mathbf{m}_{\eta_1}(s) - \mathbf{m}_{\eta_2}(s)) \|_{L^2} \| e^{-a(s)} (\mathbf{m}(s) - \mathbf{m}_\eta(s)) \|_{L^2} \, \mathrm{d}s. \tag{5.4}$$

We use the following nonlinear Gronwall's lemma (for a proof, see for instance the annex.C in [7]),

Lemma 5.6 (Square Gronwall). *Let y be a function in $H^1(0,T)$, $C \geq 0$ and f in $L^1(0,T)$ such that*:

$$\forall t \in [0,T], \quad y^2(t) \leq C + \int_0^t f(s)\, y(s) \, \mathrm{d}s.$$

Then

$$\forall t \in [0,T], \quad y(t) \leq \sqrt{C} + \frac{1}{2} \int_0^t f(s) \, \mathrm{d}s.$$

Applying this lemma to $y(t) = \| e^{-a(t)} (\mathbf{m}_{\eta_1}(t) - \mathbf{m}_{\eta_2}(t)) \|_{L^2}$ and

$$f(t) = \| e^{-a(t)} (\mathbf{m}(t) - \mathbf{m}_\eta(t)) \|_{L^2}$$

in (5.4), we obtain

$$\frac{1}{2} \| e^{-a(t)} (\mathbf{m}_{\eta_1}(t) - \mathbf{m}_{\eta_2}(t)) \|_{L^2} \leq \sqrt{D'(\eta_1, \eta_2)} + \frac{1}{2} \int_0^t \| e^{-a(s)} (\mathbf{m}(s) - \mathbf{m}_\eta(s)) \|_{L^2}.$$

We take the limit for $\eta = \eta_1$ fixed and $\eta_2 \to 0$. Because the norm is lower semi-continuous for the weak topology we have

$$\| e^{-a(t)} (\mathbf{m}_{\eta_1}(t) - \mathbf{m}(t)) \|_{L^2} \leq 2 \liminf_{\eta_2 \to 0} \sqrt{D'(\eta_1, \eta_2)} \\ + \int_0^t \| e^{-a(s)} (\mathbf{m}_{\eta_1}(s) - \mathbf{m}(s)) \|_{L^2} \, \mathrm{d}s.$$

The usual Gronwall's lemma implies

$$\| e^{-a(t)} (\mathbf{m}_{\eta_1}(t) - \mathbf{m}(t)) \|_{L^2} \leq 2 \liminf_{\eta_2 \to 0} \sqrt{D(\eta_1, \eta_2)} e^T$$

thus, the convergence of $(e^{-a} \mathbf{m}_\eta)_\eta$ to $e^{-a} \mathbf{m}$ in $L^\infty([0,T]; L^2)$.

Because $e^{-a} > 0$ a.e, modulo a subsequence, we can further assume that $\mathbf{m}_\eta(t,x)$ converges to $\mathbf{m}(t,x)$ almost everywhere in $[0,T] \times B$. Using the boundedness in $L^\infty(\mathbb{R}_+^{1+d})$ of the family $(\mathbf{m}_\eta)_\eta > 0$, the Lebesgue theorem prove the convergence of $\mathbf{m}_\eta$ to $\mathbf{m}$ in $L^p([0,T] \times B)$, $p < +\infty$.

This convergence implies in particular the convergence of $\Phi(\mathbf{m}_\eta)$ to $\Phi(\mathbf{m})$, in L^2. Finally, we obtain a strong convergence in $L^1([0,T]\times B)$ of $f(\mathbf{m}_\eta, \mathbf{h}_{\eta\perp} - \mathbf{m}_{\eta\|})$ of $f(\mathbf{m}, -\mathbf{m}_\|)$. Moreover, $\partial_t \mathbf{m}_\eta$ converge to $\partial_t \mathbf{m}$ (in the distribution sense), we obtain that $\mathbf{m}$ is a solution of (5.1).

Remark 5.7. This proves also the existence of solution of the Cauchy problem (5.1).

5.3. *Uniqueness of the Cauchy problem*

We show the uniqueness of the solution to this Cauchy problem. This shows that there is only one limit of $(\mathbf{m}_\eta)$ (the solution), so the full sequence $(\mathbf{m}_\eta)$ converges to $\mathbf{m}$.

Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be two solutions in $L^\infty([0,T] \times \mathbb{R}^d) \cap H^1([0,T]; L^2(\mathbb{R}^d))$ of the Cauchy problem (5.1). The computations are similar to (5.2) and (5.3).

$$\partial_t \mathbf{m}_1 - \partial_t \mathbf{m}_2 = f(\mathbf{m}_1, -\mathbf{m}_{1\|} - \nabla\Phi(\mathbf{m}_1) + \mathbf{h}_{\text{ext}}) - f(\mathbf{m}_2, -\mathbf{m}_{2\|} - \nabla\Phi(\mathbf{m}_2) + \mathbf{h}_{\text{ext}}).$$

We have

$$\frac{1}{2}\partial_t |\mathbf{m}_1 - \mathbf{m}_2|^2 \leq 2|\mathbf{m}_1 - \mathbf{m}_2|\,|P_\|(\mathbf{m}_1 - \mathbf{m}_2)| + (|\mathbf{m}_{1\|}| + C)\,|\mathbf{m}_1 - \mathbf{m}_2|^2. \quad (5.5)$$

If $\mathbf{m}_{1\|}$ were bounded, the Cauchy–Schwarz inequality and Gronwall's lemma would yield that $\mathbf{m}_1(t) = \mathbf{m}_2(t)$ for all $t \geq 0$. Unfortunately, this is not the case, but we use the following substitute (see [17] for a proof):

Proposition 5.8. *There exists a constant C such that for all $p \in\,]2, +\infty[$, the operator $P_\|$ is bounded from $L^p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ with norm less than Cp.*

We cut $P_\| \mathbf{m}_1$ in a bounded part, and a small remainder in L^1.

Definition 5.9. For $M > 0$, let $P_\|^M$ be defined in $\bigcup_p L^p(\mathbb{R}^d)$ by:

$$P_\|^M(f) = f_\| \times \mathbb{1}_{\{|f_\|| > M\}}.$$

Let also $P'^M_\|(f) = f_\| \times \mathbb{1}_{\{|f_\|| \leq M\}}$, so that $\|P'^M_\|(f)\|_{L^\infty} \leq M$ and $P_\| = P_\|^M + P'^M_\|$.

We use Proposition 5.8 to prove

Lemma 5.10. *Let $f \in [L^1 \cap L^\infty](\mathbb{R}^d)$. Then exists two constants c, $C > 0$ such that for all $M > 1/c$, we have*:

$$\|P_\|^M(f)\|_{L^1} \leq C \exp(-cM).$$

Proof. Let $C_1 = [\|f\|_{L^1} + \|f\|_{L^\infty}]$, so that $\|f\|_{L^p} \leq C_1$ for all p. For a borelian Ω in $\mathbb{R}^d$, note $|\Omega|$ its Lebesgue's measure. First, we have, thanks to the Bienaymé–Tchebytchev inequality

$$|\{|f_\parallel| > M\}| \leq \left(\frac{\|f_\parallel\|_{L^r}}{M}\right)^r.$$

By Proposition 5.8, $\|f_\parallel\|_{L^r} \lesssim r\|f\|_{L^r} \lesssim r$. Thus, for all $r \geq 1$,

$$|\{|f_\parallel| > M\}| \leq \left(\frac{rC_1}{M}\right)^r.$$

If we assume M sufficiently large, we can choose $r = \frac{M}{Ce}$ so that:

$$|\{|f_\parallel| > M\}| \lesssim e^{-cM}$$

where $c = \frac{1}{Ce}$. Secondly, Hölder's inequality implies that for all q (with $\frac{1}{q} + \frac{1}{q'} = 1$):

$$\begin{aligned}\|P_\parallel^M(f)\|_{L^1} &\leq \|f_\parallel\|_{L^q}\|\mathbb{1}_{\{|f|>M\}}\|_{L^{q'}} \\ &\leq qC_1|\{|f| > M\}|^{1-\frac{1}{q}} \\ &\leq qC_1C^{1-\frac{1}{q}}e^{-\frac{cM}{q}}e^{-cM}.\end{aligned}$$

Choose $q = cM$ (assuming that $M > \frac{1}{c}$). Thus,

$$\|P_\parallel^M(f)\|_{L^1} \lesssim Me^{-cM} \lesssim e^{-c'M}$$

with c' such that $0 < c' < c$.

We now finish the proof of uniqueness in problem (5.1). In inequality (5.5), decompose $\mathbf{m}_{1\parallel} = P_\parallel^M(\mathbf{m}_1) + P'^M_\parallel(\mathbf{m}_1)$. Using $\|P'^M_\parallel(\mathbf{m}_1)\|_{L^\infty} \leq M$, we obtain

$$\frac{1}{2}\partial_t|\mathbf{m}_1 - \mathbf{m}_2|^2 \leq 2(M + C)|\mathbf{m}_1 - \mathbf{m}_2|^2 + 2|\mathbf{m}_2 - \mathbf{m}_2|\,|\mathbf{m}_{1\parallel} - \mathbf{m}_{2\parallel}| + 4|P_\parallel^M(\mathbf{m}_2)|.$$

Integrate on B (using Cauchy–Schwarz inequality for the second term in the right-hand side, and the fact that $\|P_\parallel\|_{\mathcal{L}(L^2)} = 1$).

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{m}_1(t) - \mathbf{m}_2(t)\|_{L^2} \leq 2(M + C + 1)\|\mathbf{m}_1(t) - \mathbf{m}_2(t)\|^2 + 4Ce^{-cM}.$$

Using Gronwall's lemma, we obtain that, for $T > 0$, which will be fixed later, for all $t \in [0, T]$,

$$\|\mathbf{m}_1(t) - \mathbf{m}_2(t)\|_{L^2}^2 \lesssim e^T \exp(M(T - c)).$$

Fix $T < c$, and let M tend to $+\infty$. There holds $\mathbf{m}_1(t) = \mathbf{m}_2(t)$ for all $t < T$. We have uniqueness on $[0, T[$, therefore, global uniqueness since it is an autonomous system. ∎

5.4. *Proof of Proposition 5.5*

The aim of this subsection is to prove Proposition 5.5. We start by giving another formulation which is equivalent because of the symmetry of the indices η_1 and η_2.

Proposition 5.11. *There is a constant C which only depends of T such that for all $\varepsilon > 0$, exists $\eta_0 > 0$ such that for all η_1, $\eta_2 \leq \eta_0$ and $t \in [0,T]$, we have*

$$\begin{aligned}\int_B \left| f\big(\mathbf{m}_{\eta_1}(t), e^{-a(t)} P_{\|}(\mathbf{m}(t) - \mathbf{m}_{\eta_1}(t))\big) \cdot \big(e^{-a(t)}(\mathbf{m}_{\eta_1}(t) - \mathbf{m}_{\eta_2}(t))\big)\right| \mathrm{d}x \\ \leq C\|e^{-a(t)}(\mathbf{m}_{\eta_1}(t) - \mathbf{m}_{\eta_2}(t))\|_{L^2}\|e^{-a(t)}(\mathbf{m}(t) - \mathbf{m}_{\eta_1}(t))\|_{L^2} + \varepsilon.\end{aligned}$$

Proof. The first step is to write $e^{-a(t)}P_{\|} = P_{\|}\, e^{-a(t)} + [e^{-a(t)}, P_{\|}]$ and use the linearity of f with respect to its second argument. So we have

$$\begin{aligned}&\int_B \left| f\big(\mathbf{m}_{\eta_1}(t), e^{-a(t)} P_{\|}(\mathbf{m}(t) - \mathbf{m}_{\eta_1}(t))\big) \cdot \big(e^{-a(t)}(\mathbf{m}_{\eta_1}(t) - \mathbf{m}_{\eta_2}(t))\big)\right| \mathrm{d}x \\ &\quad\leq \int_B \left| f\big(\mathbf{m}_{\eta_1}(t), P_{\|} e^{-a(t)}(\mathbf{m}(t) - \mathbf{m}_{\eta_1}(t))\big) \cdot \big(e^{-a(t)}(\mathbf{m}_{\eta_1}(t) - \mathbf{m}_{\eta_2}(t))\big)\right| \mathrm{d}x \\ &\qquad + \int_B \left| f\big(\mathbf{m}_{\eta_1}(t), [e^{-a(t)}, P_{\|}](\mathbf{m}(t) - \mathbf{m}_{\eta_1}(t))\big) \cdot \big(e^{-a(t)}(\mathbf{m}_{\eta_1}(t) - \mathbf{m}_{\eta_2}(t))\big)\right| \mathrm{d}x.\end{aligned}$$

Using the fact that $|f(m,h)| \leq 2|h|$ when $|m| \leq 1$, the first term in the right-hand side is bounded by

$$\begin{aligned}2\int_B \big(P_{\|} e^{-a(t)}(\mathbf{m}(t) - \mathbf{m}_{\eta_1}(t))\big) \cdot \big(e^{-a(t)}(\mathbf{m}_{\eta_1}(t) - \mathbf{m}_{\eta_2}(t))\big)\, \mathrm{d}x \\ \leq 2\|e^{-a(t)}(\mathbf{m}(t) - \mathbf{m}_{\eta_1}(t))\|_{L^2}\|e^{-a(t)}(\mathbf{m}_{\eta_1}(t) - \mathbf{m}_{\eta_2}(t))\|_{L^2}\end{aligned}$$

according to Cauchy–Schwarz inequality and the property $\|P_{\|}\|_{\mathcal{L}(L^2)} = 1$.

For the second term, we show the strong convergence of $[e^{-a(t)}, P_{\|}](\mathbf{m}_{\eta_1}(t) - \mathbf{m}(t))$ to 0, uniformly in $t \in [0,T]$. Since $P_{\|}$ is a linear continuous map from L^4 to itself (and from L^2 to itself), we have, for $G \in L^2 \cap L^4$,

$$\|[G, P_{\|}](\mathbf{m}(t) - \mathbf{m}_{\eta_1}(t))\| \leq C\|G\|_{L^2\cap L^4}\|\mathbf{m}(t) - \mathbf{m}_{\eta_1}(t)\|_{L^4} \leq C'\|G\|_{L^2\cap L^4}. \tag{5.6}$$

Since $a(t,x) \geq 0$ almost everywhere,

$$|e^{-a(t,x)} - e^{-a(t',x)}| \leq 2e^{-|x|^2}\left|\int_t^{t'} |\mathbf{m}(s,x)| + C\,\mathrm{d}s\right|^p.$$

Use Hölder's inequality for the last term.

$$|e^{-a(t,x)} - e^{-a(t',x)}|^p \leq C_p^1|t-t'|^{p-1}e^{-p|x|^2}\int_0^t \big(|\mathbf{m}_{\|}(s,x)|^p + C_p^2\big)\,\mathrm{d}s.$$

So, $\|e^{-a(t)} - e^{-a(t')}\|_{L^p(\mathbb{R}^d)} \leq C_p|t-t'|^{(p-1)/p}$. In particular, $t \mapsto e^{-a(t)}$ is uniformly continuous from $[0,T]$ to L^p for $p > 1$.

Let ϕ and χ be two non-negative smooth functions supported in a compact on $\mathbb{R}^+$, equal to 1 near 0. Let, for $\rho > 0$, $\chi^\rho(x) = \frac{1}{\rho^d}\chi\left(\frac{x}{\rho}\right)$ and $X^\rho f(x) = \phi(\rho x)\, \chi * f(x)$. Then, for all $t \in [0,T]$, $X^\rho e^{-a(t)}$ converges in $L^4 \cap L^2$ to $e^{-a(t)}$ when $\rho \to 0$.

Secondly, as the family of operators (X^ρ) is bounded on $\mathcal{L}(L^4)$, we have the following result: $(t \mapsto X^\rho e^{-a(t)})_\rho$ is an equicontinuous family from $[0,T]$ to L^4. By Ascoli's theorem, the sequence $X^\rho e^{-a}$ converges to e^{-a} in $L^\infty([0,T]; L^4 \cap L^2)$.

Fix ρ such that $C'\|e^{-a(t)} - X^\rho e^{-a(t)}\|_{L^4} \leq \varepsilon/2$ for all $t \in [0,T]$. Use inequality (5.6), then we have $\|[e^{-a(t)} - X^\rho e^{-a(t)}, P_\|](\mathbf{m}(t) - \mathbf{m}_{\eta_1}(t))\|_{L^2} \leq \varepsilon/2$.

Now, $\rho > 0$ being fixed, note that the function $t \mapsto [X^\rho e^{-a(t)}, P_\|]$ is continuous from $[0,T]$ to the space of compact operators in L^2, so uniformly continuous thanks to Heine's theorem. Hence we have, for $t, t' \in [0,T]$:

$$
\begin{aligned}
&[X^\rho e^{-a(t)}, P_\|]\mathbf{m}^{\eta_1}(t) - [X^\rho e^{-a(t')}, P_\|]\mathbf{m}^{\eta_1}(t') \\
&\quad = [X^\rho e^{-a(t)}, P_\|](\mathbf{m}^{\eta_1}(t) - \mathbf{m}^{\eta_1}(t')) + \big([X^\rho e^{-a(t)}, P_\|] - [X^\rho e^{-a(t')}, P_\|]\big)\mathbf{m}^{\eta_1}(t').
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|[X^\rho e^{-a(t)}, P_\|]\mathbf{m}^{\eta_1}(t) - [X^\rho e^{-a(t')}, P_\|]\mathbf{m}^{\eta_1}(t')\|_{L^2 \cap L^4} \\
&\quad \leq \|[X^\rho e^{-a}, P_\|]\|_{L^\infty([0,T];\mathcal{L}(L^2;L^2\cap L^4))}\|(\mathbf{m}^{\eta_1}(t) - \mathbf{m}^{\eta_1}(t'))\|_{L^2} \\
&\qquad + \|[X^\rho e^{-a(t)}, P_\|] - [X^\rho e^{-a(t')}, P_\|]\|_{\mathcal{L}(L^2\cap L^4;L^2)} \cdot C.
\end{aligned}
$$

Therefore, the sequence $(t \mapsto [X^\rho e^{-a(t)}, P_\|]\mathbf{m}^{\eta_1}(t))_{\eta_1}$ is equicontinuous, and strongly converges to $t \mapsto [X^\rho e^{-a(t)}, P_\|]\mathbf{m}(t)$; the convergence is uniform in t by Ascoli's theorem. Therefore, there exists η_0 such that for all $\eta_1 \leq \eta_0$, we have $\|[X^\rho e^{-a(t)}, P_\|](\mathbf{m}^{\eta_1}(t) - \mathbf{m}(t))\|_{L^2} < \varepsilon/2$ for all $t \in [0,T]$.

Finally, we obtain that $\|[e^{-a(t)}, P_\|](\mathbf{m}^{\eta_1}(t) - \mathbf{m}(t))\|_{L^2} \leq \varepsilon$ for all $\eta_1 \leq \eta_0$. Therefore we have strong convergence to 0 in $L^\infty([0,T]; L^2)$ of $t \mapsto [e^{-a(t)}, P_\|](\mathbf{m}^{\eta_1}(t) - \mathbf{m}(t))$. □

6. The Damping Parameter

We prove here a strong convergence result of the system (1.2) when the parameter α tends to 0. In [8], Hamdache and Tiloua establish a weak convergence result for the system, with an exchange term.

Theorem 6.1. *Assume that* $\mathbf{e}_0$, $\mathbf{h}_0$, $\mathbf{m}_0$, $\mathbf{curl}\,\mathbf{e}_0$, $\mathbf{curl}\,\mathbf{h}_0$ *are in* $L^2(\mathbb{R}^D)$, $\mathbf{h}_{\text{ext}}$ *is in* $L^\infty(B)$ *and* η *is small. Let, for a damping paramater* α *not fixed* $(\mathbf{e}_\alpha, \mathbf{h}_\alpha, \mathbf{m}_\alpha)$ *be the solution of the system* (1.2). *Then* $(\mathbf{e}_\alpha, \mathbf{h}_\alpha, \mathbf{m}_\alpha)$ *converges strongly in* $\mathcal{C}^0(L^2) \times \mathcal{C}^0(L^2) \times \mathcal{C}^0(L^2)$ *to the solution* $(\mathbf{e}, \mathbf{h}, \mathbf{m})$ *of the system* (1.2) *with* $\alpha = 0$ *when* α *tends to* 0.

Proof. First, we can assume, after extracting a subsequence, that $\mathbf{e}^\alpha$ and $\mathbf{h}^\alpha$ converge weakly in $L^2((0,T)\times\mathbb{R}^d)$. Now the uniform estimates in t on $\mathbf{h}_\perp$ in Propositions 4.2 and 4.4 are independant of α. Consequently, because $\eta>0$ is fixed, the family $(\nabla_{t,x}\mathbf{h}_\perp^\alpha)$ is bounded in $L^\infty((0,T);L^2(B))\subset L^2((0,T)\times B)$, so $(\mathbf{h}_\perp^\alpha)$ is bounded $H^1((0,T)\times B)$ and, after extracting a subsequence, converges strongly in $L^1((0,T)\times B)$.

Let α_1, $\alpha_2>0$ intended to tend to 0. We have

$$\begin{aligned}\partial_t\mathbf{m}^{\alpha_1}-\partial_t\mathbf{m}^{\alpha_2} &= (\mathbf{m}^{\alpha_1}-\mathbf{m}^{\alpha_2})\times(\mathbf{h}^{\alpha_1}+\mathbf{h}_{\text{ext}}+\nabla\Phi(\mathbf{m}^{\alpha_1}))\\ &\quad+\mathbf{m}^{\alpha_1}\times(\nabla\Phi(\mathbf{m}^{\alpha_1})-\nabla\Phi(\mathbf{m}^{\alpha_2}))\\ &\quad+\mathbf{m}^{\alpha_1}\times(\mathbf{m}_\parallel^{\alpha_1}-\mathbf{m}_\parallel^{\alpha_2})\\ &\quad+\mathbf{m}^{\alpha_1}\times(\mathbf{h}_\perp^{\alpha_1}-\mathbf{h}_\perp^{\alpha_2})\\ &\quad+\alpha_1\,\mathbf{m}^{\alpha_1}\times(\mathbf{m}^{\alpha_1}\times\mathbf{h}^{\alpha_1})\\ &\quad-\alpha_2\,\mathbf{m}^{\alpha_2}\times(\mathbf{m}^{\alpha_2}\times\mathbf{h}^{\alpha_2}).\end{aligned}$$

So, the estimate on $|\mathbf{m}^{\alpha_1}-\mathbf{m}^{\alpha_2}|$ is written as:

$$\begin{aligned}\frac{1}{2}\partial_t|\mathbf{m}^{\alpha_1}-\mathbf{m}^{\alpha_2}|^2 &\le 0+C|\mathbf{m}^{\alpha_1}-\mathbf{m}^{\alpha_2}|^2\\ &\quad+|\mathbf{m}^{\alpha_1}-\mathbf{m}^{\alpha_2}|\,|P_\parallel(\mathbf{m}^{\alpha_1}-\mathbf{m}^{\alpha_2})|\\ &\quad+2\left|\mathbf{h}_\perp^{\alpha_1}-\mathbf{h}_\perp^{\alpha_2}\right|+\alpha_1|\mathbf{h}^{\alpha_1}|+\alpha_2|\mathbf{h}^{\alpha_2}|.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{m}^{\alpha_1}(t)-\mathbf{m}^{\alpha_2}(t)\|_{L^2}^2 &\le (C+1)\|\mathbf{m}^{\alpha_1}(t)-\mathbf{m}^{\alpha_2}(t)\|_{L^2}^2\\ &\quad+\|\mathbf{h}_\perp^{\alpha_1}(t)-\mathbf{h}_\perp^{\alpha_2}(t)\|_{L^2(B)}+D(\alpha_1,\alpha_2)\end{aligned}$$

which gives, after an integration in time,

$$\begin{aligned}\frac{1}{2}\|\mathbf{m}^{\alpha_1}(t)-\mathbf{m}^{\alpha_2}(t)\|_{L^2}^2 &\le (C+1)\int_0^t\|\mathbf{m}^{\alpha_1}(s)-\mathbf{m}^{\alpha_2}(s)\|_{L^2}^2\,\mathrm{d}s\\ &\quad+\|\mathbf{h}^{\alpha_1}-\mathbf{h}^{\alpha_2}\|_{L^1((0,T\times B))}+TD(\alpha_1,\alpha_2).\end{aligned}$$

We conclude with Gronwall's lemma that the subsequence $(\mathbf{m}^\alpha)_\alpha$ is a Cauchy sequence in $L^\infty_{\text{loc}}(L^2)$. It has a strong limit, which makes to take the limit in the nonlinear terms; consequently, the limit is a solution of the Landau–Lipschitz equation without damping parameter. $\square$

Remark 6.2. In fact, we can prove with the arguments developped in these two last sections, that $(\alpha,\eta)\mapsto(\mathbf{e},\mathbf{h},\mathbf{m})$ is continuous from $[0,1]\times[0,1]$ to $L^2([0,T];L^2_{\text{loc}})\times L^2([0,T];L^2_{\text{loc}})\times\mathcal{C}([0,T];L^2)$.

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References

[1] H. Brezis, *Analyse Fonctionnelle, Théorie et Applications* (Masson, Paris, 1983).

[2] G. Carbou and P. Fabrie, Comportement asymptotique des solutions faibles des équations de Landau-Lifshitz, *C.R. Acad. Sci. Paris* **325** (1997) 717–720.

[3] G. Carbou and P. Fabrie, Time average in micromagnetism, *J. Differential Equations* **147** (1998) 383–409.

[4] G. Carbou and P. Fabrie, Regular solutions for Landau-Lifshitz equation in a bounded domain, *Differential Integral Equations* **14** (2001) 213–229.

[5] G. Carbou, P. Fabrie and F. Jochmann, A remark on the weak ω-limit set for micromagnetism equation, *Appl. Math. Lett.* **15** (2002) 95–99.

[6] L. C. Evans, *Partial Differential Equations*, Graduates Studies in Mathematics, Vol. 19 (American Mathematical Society, 1997).

[7] H. Haddar, Modèles asymptotiques en ferromagnétisme; couches minces et homogénéisation, PhD thesis (December 2000).

[8] K. Hamdache and M. Tilioua, The Landau–Lifshitz equations and the damping Parameter, to appear in *Boll. Unione. Mat. Ital. Sez. B.*

[9] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Grundlehren der Mathematischen Wissenschaften (Springer-Verlag, 1990).

[10] F. Jochmann, Existence of solutions and quasi-stationary limit for a hyperbolic system describing ferromagnetism, *SIAM. J. Math. Anal.* **34** (2002) 315–340.

[11] F. Jochmann, Asymptotic behavior of the electromagnetic field for a micromagnetism equation without exchange energy, preprint (2004).

[12] J.-L. Joly, G. Métivier and J. Rauch, Global solution to Maxwell equation in a ferromagnetic medium, *Ann. Inst. H. Poincaré* **1** (2000) 307–340.

[13] P. Joly and O. Vacus, Mathematical and numerical studies of 1D non linear ferromagnetic material, in *Numerical Methods in Engineering '96* (1996).

[14] S. Labbé, Simulation numérique du comportement hyperfréquence des matériaux ferromagnétiques, PhD thesis (December 1998).

[15] L. Landau and E. Lifshitz, *Electrodynamique des Milieux Continus* (Mir, Moscou, 1969).

[16] L. Schwartz, *Théorie des Distributions* (Hermann, Paris, 1966).

[17] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions* (Princeton University Press, 1970).