# Introduction to Algebraic Geometry

Jilong Tong

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# Introduction

The starting point of the algebraic geometry is trying to study the solutions of systems of polynomials: for simplicity, let $k$ be a field, let $P_1, \cdots, P_m \in k[X_1, \cdots, X_n]$ to be $m$ polynomials with $n$ variables. Then we want to study the solutions of the following system:

$$\begin{cases} P_1(X_1, \cdots, X_n) = 0 \\ P_2(X_, \cdots, X_n) = 0 \\ \cdots \\ P_m(X_1, \cdots, X_n) = 0 \end{cases}$$

One could ask the following questions:

- Does this system have at least a solution in $k$?
- If we have a positive answer, then what can we say about these solutions (*i.e.*, are there just finitely many or there are infinite?). Or further more, can we know exactly what are the solutions?

Of course, when each polynomial $P_i$ is of degree 1, we know in the course of linear algebra that, at least theoretically, there is a way to find explicitly all the solutions (if there is any) or the show the contrast. But in the general case, these become very difficult questions. One can consider the following famous example

**Fermet's last theorem**

We consider $k = \mathbb{Q}$, and we want to understand the solutions of the following equations: for $n \geq 3$ an integer, what can way say about the solutions in $\mathbb{Q}$ of the following single equation with only two variables

$$X^n + Y^n = 1.$$

Sure, one can easily find some solutions

$$(x, y) \in \{(1, 0), (0, 1)\}$$

or several more like

$$(-1, 0), \quad (0, -1)$$

when the integer $n$ is even. After these more or less trivial solutions, it becomes difficult to find some new one (maybe you can try with the computers). In fact, one has the following

**Conjecture 0.0.0.1** (Fermat's last theorem, 1637)**.** There is no non trivial solutions for $n \geq 3$.

Fermat himself solved this conjecture when $n = 4$. In fact, Fermet claimed also that he had found a proof for this conjecture for all $n \geq 3$. But today we believe that Fermet must make a mistake in his *proof*. In fact, the human had to spend the next 358 years to completely prove this statement![1] And the proof involves in fact very deep mathematical subjects such as algebraic geometry and number theory, and it uses in a very impressed way the mathematical tools and methods that are developed in the recent years.

**Elliptic curves**

Let again $k = \mathbb{Q}$ (or more generally a number field), and $a, b \in k$. We consider the following equation

$$y^2 = x^3 + ax + b. \tag{1}$$

We suppose that $4a^3 + 27b^2 \neq 0$. Let $E_{\text{aff}}$ the set of solutions in $k$ of the previous equation. Of course, there are three possibilities:

- (1) has no solutions in $\mathbb{Q}$, *i.e.*, $E_{\text{aff}} = \emptyset$.
- (1) has finitely many solutions in $\mathbb{Q}$, *i.e.*, $E_{\text{aff}}$ is a nonempty finite set.
- (1) has infinitely many solutions in $\mathbb{Q}$, *i.e.*, $E_{\text{aff}}$ is an infinite set.

Of course, by saying this, we get nothing. To get something interesting, we add an extra point, denoted by $\infty$, to $E_{\text{aff}}$, we let

$$E = E_{\text{aff}} \cup \{\infty\}.$$

The equation (1) together the point at infinite $\infty$ give an *elliptic curve*, which amounts to consider the projective version of the equation (1) above. The set $E$ is then the $k$-points of this curve. Now the miracle does happen: we have the following result

**Theorem 0.0.0.2.** *One can endow a natural group structure to $E$ so that $E$ becomes an abelian group, and $\infty$ is the neutral element of this group.*[2]

Moreover, we have the following Mordell-Weil theorem

**Theorem 0.0.0.3** (Mordell-Weil)**.** *With the group structure above, $E$ is a finitely generated abelian group.*

Hence, by the structure theorem of abelian groups of finite type, we find

$$E \simeq \mathbb{Z}^r \oplus E_{\text{tor}}$$

with $E_{\text{tor}}$ the torsion part of $E$. The torsion part is more or less known according to the famous result of Mazur, in fact, there are only finitely many possibilities be the torsion part of $E$. The free part of $E$, or equivalently the rank of $E$, is still very mysterious. If we could completely understand the free part of $E$, we can earn one millon dollars by proving the so-called Birch and Swinnerton-Dyer conjecture, which relates the rank of $E$ with some analytic invariant attached to the equation (1).

[1] This is finally done by Andrew Wiles in 1995, with the help of his former student Richard Taylor.

[2] The more correct statement should be the following: the elliptic curve has a natural structure of group scheme over the field of definition (in the example, this is $\mathbb{Q}$), so that $\infty$ gives the neutral element of this group scheme. In fact, what we get is an abelian variety of dimension 1.

**Remark 0.0.0.4.** 1. Note that, an elliptic curve is *not* the same as an ellipse. Here the latter is given in general by equation of the follow form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In fact, we will see later that an elliptic curve is a projective smooth curve of genus 1, while an ellipse is a projective smooth curve of genus 0, these two kinds of curves are of completely different nature.

2. The solution of Fermat's Last Theorem is built on a very sophisticate study of the elliptic curves. In fact, elliptic curve is a very important object both in algebraic geometry and modern number theory.

**This course**

Since a direct way is impossible, one have to find another way. We will put some extra structures on the set of solutions (Zariski topology, variety, sheaf, cohomology), and then try to understand this set together with the structure. That is what we do in algebraic geometry. In this course, we will try to follow the approach of Grothendieck by using the language of schemes. The main technique that we will use is then the commutative algebra. Hence, we will first review some basic notions in commutative algebra.

## Review of some basic notions in commutative algebra

The commutative algebra is the basic tool to study algebraic geometry. In this section, we will fix some notations by reviewing some well-known things in commutative algebra. Recall first the following definition

**Definition 0.0.0.5.** A *commutative unitary ring* is the given of the following datum $(R, +, \cdot, 0, 1)$, where $R$ is a non empty set, $+, \cdot$ are two binary operations defined on $R$, and $1, 0 \in R$ are two distinguish elements (maybe coincide) such that

– $(R, +, 0)$ is an abelian group, while $(R, \cdot, 1)$ is a *commutative unitary* monoid;

– The two binary operations are compatible: for any $x, y, z \in R$, we have

$$(x + y) \cdot z = x \cdot z + y \cdot z.$$

A commutative unitary ring $R$ will be called *trivial* if $1 = 0$, or equivalently, if $R$ has only one element.

**Convention 0.0.0.6.** Unless specifically stated to the contrast, in this course,

1. the word "ring" means a *commutative unitary ring.* Same thing for the subrings etc;
2. the word "field" means a *commutative field.*

Recall the following definition

**Definition 0.0.0.7.** Let $A$ be a ring.

1. An *ideal* $I$ of $A$ is a subgroup (for the addition) of $A$ such that
$$\forall a \in A, \quad \forall x \in I, \quad a \cdot x \in I.$$
An ideal $I \subset A$ is called of finite type if it can be generated as an ideal by finitely many elements.

2. An ideal $I$ is called *prime* if $I \neq A$, and whenever $x \cdot y \in I$ for some $x, y \in A$, we must have either $x \in I$ or $y \in I$. The set of prime ideals of $A$ will be denoted by $\mathrm{Spec}(A)$.

3. An ideal $I$ is called *maximal* if the following condition holds: if $I \neq A$, and for any ideal $J \subset A$ such that $I \subset J \subset A$, then we have either $I = J$ or $A = J$. The set of maximal ideal will be denoted by $\mathrm{Max}(A)$.

The proof of the following lemma uses Zorn's lemma, or equivalently, the axiom of choice.

**Lemma 0.0.0.8.** *We have* $\mathrm{Max}(A) \subset \mathrm{Spec}(A)$. *Moreover, if $A$ is non trivial, then* $\mathrm{Max}(A) \neq \emptyset$.

**Lemma 0.0.0.9.** *Let $A$ be a ring such that $0 \neq 1$, and $I \subset A$ be an ideal. Then*

- *$I$ is a prime ideal if and only if the quotient $A/I$ is an integral domain.*[3]
- *$I$ is a maximal ideal if and only if the quotient $A/I$ is a field.*

**Definition 0.0.0.10.** A ring $A$ is call *noetherian* if it satisfies the following increasing chain condition: for any increasing family of ideals of $A$:
$$I_0 \subset I_1 \subset \cdots I_r \subset I_{r+1} \subset \cdots$$
there exists some sufficiently large integer $r_0 \gg 0$ such that $I_r = I_{r_0}$ for any $r \geq r_0$.

In an equivalent way, a ring $A$ is noetherian if and only if any ideal $I \subset A$ is finitely generated.

**Lemma 0.0.0.11.** *Let $A$ be a ring.*

1. *If $A$ is a field, or a principal ideal domain, then it's noetherian.*
2. *Let $I$ be an ideal of $A$. Then if $A$ is noetherian, so is the quotient $A/I$.*

**Exercise 0.0.0.12.** In general, a subring of a noetherian ring is not necessarily still noetherian. Could you find an example?

**Lemma 0.0.0.13.** *Suppose $A$ noetherian, and $M$ an $A$-module of finite type. Then $M$ is noetherian as $A$-module, namely, any $A$-submodule of $M$ is finitely generated.*

**Theorem 0.0.0.14** (Hilbert's basis theorem)**.** *If $A$ is noetherian, so is $A[X]$.*

*Proof.* Let $(0) \neq J \subset A[X]$ be an ideal, and let $I$ be the set of leading coefficient of polynomials in $J$.[4] Then $I \subset A$ is an ideal. As $A$ is noetherian, $I$ is finitely generated, say by $\mathrm{LC}(P_1), \cdots, \mathrm{LC}(P_n)$. Let
$$J_1 = (P_1, \cdots, P_n) \subset A[X]$$

[3] By definition, a *domain* is always non trivial.

[4] By convention, the leading coefficient of zero polynomial is 0. For $f \in A[X]$, we denote by $\mathrm{LC}(f)$ the leading coefficient of $f$.

the ideal generated by these $P_i$, and $d = \max_i(\deg(P_i))$. Then one verifies easily that

$$J = J \cap M_d + J_1$$

with $M_d$ the set of polynomials in $A[X]$ with degree $\leq d-1$. As $A$ is noetherian, and as $M_d$ is a $A$-module of finite type, according to the previous lemma, we find that $M_d$ is noetherian as $A$-module. In particular, the $A$-submodule $J \cap M_d \subset M_d$ is of finite type as $A$-module. Let $Q_1, \cdots, Q_r \in J \cap M_d \subset J$ be a family of generators, then we have

$$J = J \cap M + J_1 \subset (Q_1, \cdots, Q_r) + (P_1, \cdots, P_n) = (Q_1, \cdots, Q_r, P_1, \cdots, P_n) \subset J.$$

As a result, $J = (Q_1, \cdots, Q_r, P_1, \cdots, P_n)$ is finitely generated. This finishes the proof.[5] □

**Corollary 0.0.0.15.** *For $k$ a field. The polynomial rings $k[X_1, \cdots, X_n]$ are all noetherian. Hence any $k$-algebra of finite type is noetherian.*

**Remark 0.0.0.16.** Let $k$ be a field. Then $k[X_1, \cdots, X_n]$ are the so-called *unique factorization domain* (UFD for short). But in general, it is not a principal domain. In fact, it's principal if and only if $n = 1$.

**Definition 0.0.0.17.** Let $k$ be a field, then $k$ is called algebraically closed, if any polynomial $P \in k[X]$ of degree $\geq 1$ has at least one solution in $k$ (hence has exactly $\deg(P)$ solutions).

**Example 0.0.0.18.** $\mathbb{C}$ is algebraically closed by the fundamental theorem of algebra.[6] Moreover, for any field $k$, one can always find an algebraically closed field $K$ containing $k$, the smallest one is called the algebraic closure of $k$.

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[5] An alternative way to avoiding the previous lemma: after we get the equality $J = J \cap M + J_1$, let $I_1$ be the set of leading coefficients of polynomials in $J \cap M$. Then $I_1 \subset A$ is again an ideal, hence there is $Q_1, \cdots, Q_r \in J \cap M$ such that $I_1 = (\mathrm{LC}(Q_1), \cdots, \mathrm{LC}(Q_r))$. Then similarly, we have

$$J \cap M_d \subset J \cap M_{d_1} + (Q_1, \cdots, Q_r)$$

where $d_1$ is the maximum of $\deg(Q_i)$ for $1 \leq i \leq r$. Clearly, $d_1 < d$, and we have

$$J = J \cap M_{d_1} + (P_1, \cdots, P_n, Q_1, \cdots, Q_r).$$

Now, we continue with this process, and we can conclude.

[6] Despite the name of this theorem, there exists *no* purely algebraic proof of this result. The reason is that the construction of $\mathbb{C}$ is not completely algebraic.

# Chapter 1

# Algebraic sets and morphisms

The aim of this section is trying to gives some geometric background of the algebraic geometry. So here $k = \overline{k}$ is an algebraically closed field.

## 1.1 Affine algebraic sets

### 1.1.1 Some definitions

**Definition 1.1.1.1.** Let $k$ be an algebraically closed field as before.

- The affine space $\mathbb{A}_k^n$ of dimension $n$ is just
$$k^n = \{(x_1, \cdots, x_n) \colon x_i \in k\}.^1$$
- Let $S \subset k[X_1, \cdots, X_n]$ be a subset of polynomials of $n$ variables. Define
$$V(S) := \{(x_1, \cdots, x_n) \in \mathbb{A}_k^n \colon P(x_1, \cdots, x_n) = 0 \text{ for any } P \in S\}$$
Such a subset of $\mathbb{A}_k^n$ is called an *algebraic set* of $\mathbb{A}_k^n$.

**Remark 1.1.1.2.** Keeping the notations as before.

- If $S_1 \subset S_2 \subset k[X_1, \cdots, X_n]$ are two subsets, then $V(S_2) \subset V(S_1)$.
- Recall that for $A$ a ring and $S \subset A$ a subset. The ideal generated by $S$ (denoted by $< S >$) consists of *all finite sums*
$$\sum_i a_i s_i, \quad a_i \in A, \ s_i \in S.$$
Hence $V(< S >) \subset V(S) \subset V(< S >)$. As a result, we find $V(S) = V(< S >)$. So for algebraic sets, we only need to consider those of the form $V(\mathfrak{a})$ with $\mathfrak{a} \subset k[X_1, \cdots, X_n]$ an ideal.
- Recall also that since $k$ is a field, $k[X_1, \cdots, X_n]$ is noetherian. Hence any ideal of $k[X_1, \cdots, X_n]$ is finitely generated, say by $P_1, \cdots, P_m \in \mathcal{A}$. So to define an algebraic set, we only need finitely many equations
$$V(\mathfrak{a}) = \{(x_1, \cdots, x_n) \in k^n \colon P_1(x_1, \cdots, x_n) = \cdots = P_m(x_1, \cdots, x_n) = 0\}.$$

[1] The more correct notation should be $\mathbb{A}_k^n(k)$, but here we just write $\mathbb{A}_k^n$ for simplicity.

**Proposition 1.1.1.3.** *The union of two algebraic sets (of $\mathbb{A}_k^n$) is again algebraic; the intersection of any family of algebraic sets is again algebraic. Moreover, empty set and the total space $\mathbb{A}_k^n$ are algebraic.*

*Proof.* In fact, we have (i) $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cdot \mathfrak{b})$; (ii) $\cap_i V(\mathfrak{a}_i) = V(\sum_i \mathfrak{a}_i)$; (iii) $V((1)) = \emptyset$, and $V((0)) = \mathbb{A}_k^n$. $\square$

**Remark 1.1.1.4.** We remark that we have always

$$V(\mathfrak{a}) = V(\mathfrak{a}^2) = V(\sqrt{\mathfrak{a}}),$$

here

$$\sqrt{\mathfrak{a}} := \{a \in A \colon \exists r \in \mathbb{Z}_{\geq 1} \text{ s.t. } a^r \in \mathfrak{a}\}$$

is the *radical* of the ideal $\mathfrak{a}$. Hence one can not expect to recover the ideal $\mathfrak{a}$ from its corresponding algebraic set $V(\mathfrak{a})$.

### 1.1.2 Hilbert's Nullstellensatz

**Definition 1.1.2.1.** Let $Y \subset \mathbb{A}_k^n$ be a subset. Define the *ideal of* $Y$ to be

$$I(Y) := \{P \in k[X_1, \cdots, X_n] \colon P(x) = 0 \ \forall x \in Y\}.$$

**Lemma 1.1.2.2.** *Keeping the notations as before:*

- *For any subset $Y \subset \mathbb{A}_k^n$, we have $\sqrt{I(Y)} = I(Y)$.*[2]
- *For any subsets $Y_1 \subset Y_2 \subset \mathbb{A}_k^n$, we have $I(Y_2) \subset I(Y_1)$.*
- *For any subsets $Y_1, Y_2 \subset \mathbb{A}_k^n$, we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.*
- *For any ideal $\mathfrak{a} \subset k[X_1, \cdots, X_n]$, $\mathfrak{a} \subset I(Z(\mathfrak{a}))$.*

Now, we may state and prove the first fundamental result in this course

**Theorem 1.1.2.3** (Hilbert's Nullstellensatz)**.** *For any ideal $\mathfrak{a} \subset k[X_1, \cdots, X_n]$, $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

To prove this theorem, we begin with the following lemma:

**Lemma 1.1.2.4.** *Let $K$ be a field, $L/K$ be a field extension which is finitely generated as $K$-algebra. Then $L/K$ is an algebraic extension.*

*Proof.* [3] Up to replace $K$ by its algebraic closure in $L$, we may assume that $L/K$ is an transcendant field extension, *i.e.*, any non zero element of $L$ is transcendant over $K$.

Suppose first that $L/K$ is of transcendant degree 1, namely $L$ contains a copy of $K(x)$ (:=the fraction field of the polynomial ring $K[x]$) such that $L$ is algebraic over $K(x)$. Since $L$ is of finite type as $K$-algebra, it's also of finite type as $K(x)$-algebra. In particular, $L$ is of finite dimension

[2] Such an ideal is called *radical*.

[3] When $K$ is a uncountable field, this lemma can also be proved as follows: remark first that since $L/K$ is finitely generated as $K$-algebra, then $L$ is of *countable dimension* over $K$ as a $K$-vector space. On the other hand, once the field $K$ is *uncountable*, and once one can find $x \in L$ which is transcendant over $K$, then the following family $\{1/(x-a) \colon a \in K\}$ is linearly independent and of cardinality uncountable. From this, we find that the $K$-space $L$ must has uncountable dimension, whence a contradiction. This gives the lemma when $K$ is *uncountable*.

over $K(x)$. Choose $e_1, \cdots e_n$ a basis of $L$ over $K(x)$, and write down the multiplication table for $L$:

$$e_i \cdot e_j = \sum_k \frac{a_{ijk}(x)}{b_{ijk}(x)} e_k \tag{1.1}$$

with $a_{ijk}(x), b_{ijk}(x) \in K[x]$. Now suppose $L = K[f_1, \cdots, f_m]$, and we write each $f_j$ under the basis $\{e_i\}$:

$$f_j = \sum_i \frac{c_{ij}(x)}{d_{ij}(x)} e_i$$

Since for any element $x \in L$, $x$ can be written as a polynomial in the $f_j$'s with coefficients in $K$, it follows that $x$ can also be written as a $K(x)$-combination of 1 and the product of $e_i$'s such that the denominator of each coefficient involves only the products of the polynomials $d_{ij}$'s. Now, by applying the multiplication table (1.1), we find that $x$ can be written as a $K(x)$-combination of 1 and the $e_i$'s such that the denominator of each coefficient involves only the products of the polynomials $d_{ij}$'s and $b_{ijk}$'s. But there exists infinitely many irreducible polynomials in $K[x]$, there exists some polynomial $p(x) \in K[x]$ which does not divide any $d_{ij}$ nor $b_{ijk}$. As a result, the fraction $1/p(x)$ cannot be written in the form described above, which means $1/p(x) \notin K[f_1, \cdots, f_m] = L$, a contradiction since we suppose that $L$ is a field, hence à priori, $1/p(x) \in L$. This finishes the case when $L/K$ is of transcendant degree 1.

For the general case (*i.e.* tr.deg$(L/K) \geq 1$) , one can always find a subfield $K'$ of $L$ containing $K$ such that $L/K'$ is of transcendant degree 1. In particular, the previous argument shows that $L$ cannot be of finite type as $K'$-algebra. A priori, $L$ cannot be of finite type as $K$-algebra. This gives a contradiction. □

From this lemma, we get the *Weak Nullstellensatz*:

**Theorem 1.1.2.5** (Weak Nullstellensatz)**.** *Let $k$ be an algebraically closed field, then every maximal ideal in $A = k[X_1, \cdots, X_n]$ has the form $(X_1 - x_1, \cdots, X_n - x_n)$ for some $(x_1, \cdots, x_n) \in k^n$. As a consequence, a family of polynomials functions on $k^n$ with no common zero generates the unit ideal of $A$.*

*Proof.* Let $\mathfrak{m} \subset A$ be a maximal ideal, then the corresponding quotient $K := A/\mathfrak{m}$ is a field extension of $k$. Moreover, since $A$ is finitely generated as $k$-algebra, so is $K$. Hence the previous lemma tells us that $K/k$ is in fact an algebraic extension. But since the base field $k$ is algebraically closed, the inclusion $k \hookrightarrow K$ must be an isomorphism. Let now $x_i \in k$ be the image of $X_i$ in the quotient $k \simeq K = A/\mathfrak{m}$, then $X_i - x_i \in \mathfrak{m}$ for all $i$. As a result, we get

$$(X_1 - x_1, \cdots, X_n - x_n) \subset \mathfrak{m} \subset A.$$

On the other hand, we can verify directly that the ideal $(X_1 - x_1, \cdots, X_n - x_n)$ is itself maximal, hence the first inclusion above is an equality: $(X_1 - x_1, \cdots, X_n - x_n) = \mathfrak{m}$. In this way, we get the first assertion. For the last assertion, let $J$ be the ideal generated by this family. Since this family of polynomials functions has no common zero on $k^n$, $J$ is not contained in any maximal ideal of $A$. As a result, we must have $J = A$. This finishes the proof. □

**Exercise 1.1.2.6.** Prove directly (*i.e.*, without using Weak Nullstellensatz) that $I(\mathbb{A}_k^n) = (0)$.

*Proof of Hilbert's Nullstellensatz.* Suppose $\mathfrak{a} = (f_1, \cdots, f_m)$, and let $g \in I(V(\mathfrak{a})) - \{0\}$. We consider now the ring of polynomials in $n+1$ variables $k[X_1, \cdots, X_n, X_{n+1}]$, and the polynomials

$f_i, g$ can be viewed naturally as polynomials in $n+1$ variables. Now, consider following family of polynomials

$$\{f_1, \cdots, f_m, X_{n+1}g - 1\},$$

it has no common zero in $k^{n+1}$. Hence by the Weak Nullstellensatz, we get

$$(f_1, \cdots, f_m, X_{n+1}g - 1) = k[X_1, \cdots, X_n, X_{n+1}].$$

Hence, there are polynomials $Q_1, \cdots, Q_{n+1} \in k[X_1, \cdots, X_{n+1}]$ such that

$$1 = Q_1 \cdot f_1 + \cdots Q_n \cdot f_n + Q_{n+1} \cdot (X_{n+1}g - 1)$$

Now, take the image of this equality by the morphism

$$k[X_1, \cdots, X_n, X_{n+1}] \to k(X_1, \cdots, X_n), \quad X_{n+1} \mapsto 1/g.$$

We get the following equality in $k(X_1, \cdots, X_n)$

$$1 = \frac{P_1 \cdot f_1 + \cdots P_n \cdot f_n}{g^s}$$

with $P_i \in k[X_1, \cdots, X_n]$. In particular, $g^s = P_1 \cdot f_1 + \cdots + P_n \cdot f_n \in \mathfrak{a}$. Hence $g \in \sqrt{\mathfrak{a}}$, this finishes the proof. □

**Exercise 1.1.2.7.** Show that $V(I(Y)) = \overline{Y}$ for any subset $Y \subset \mathbb{A}^n_k$.

**Corollary 1.1.2.8.** *There is one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbb{A}^n_k$ and radical ideals $\mathfrak{a} \subset k[X_1, \cdots, X_n]$.*

*Proof.* The correspondence is given by $Y \mapsto I(Y)$, and conversely by $\mathfrak{a} \mapsto V(\mathfrak{a})$. □

### 1.1.3 Zariski topology on an affine algebraic set

**Definition 1.1.3.1** (Topology)**.** A topology on a set $X$ is defined by giving a family $\mathcal{T}$ of subsets of $X$ such that

- $\emptyset, X \in \mathcal{T}$;
- $\mathcal{T}$ is stable by finite intersection;
- $\mathcal{T}$ is stable by any union.

In this case, a subset $U \subset X$ is called *open* iff $U \in \mathcal{T}$, and $F \subset X$ is called *closed* iff its complement $F^c = X - F$ is open in $X$.

As a corollary of the Proposition 1.1.1.3, we have the following

**Definition 1.1.3.2** (Zariski topology)**.** We define the *Zariski topology* on $\mathbb{A}^n_k$ by taking the open subsets to be the complements of algebraic sets. For an algebraic subset $V \subset \mathbb{A}^n_k$, then the *Zariski topology* on $V$ is the subspace topology induced from $\mathbb{A}^n_k$.

**Example 1.1.3.3.**

- A subset $F \subset \mathbb{A}^1_k$ is closed iff $F$ is a finite set. As a result, a subset $U \subset \mathbb{A}^1_k$ is open iff its complement $\mathbb{A}^n_k - U$ is a finite set.
- The Zariski topology on $\mathbb{A}^n_k$ is *not* Hausdorff. It's just a $T_1$ space: for any two different points $x, y \in \mathbb{A}^n_k$, one can find an open $U \subset \mathbb{A}^n_k$ such that $x \in U$ and $y \notin U$.

**Exercise 1.1.3.4.** Determine the closed subsets of $\mathbb{A}_k^2$, and then prove that the topology of $\mathbb{A}_k^2$ is *not* the product topology on $\mathbb{A}_k^2 = \mathbb{A}_k^1 \times \mathbb{A}_k^1$ (here, this is an equality as sets).

**Definition 1.1.3.5.** A non empty subset $Y$ of a topological space $X$ is called *irreducible* if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two *proper* subsets such that each $Y_i$ is closed in $Y$.

**Remark 1.1.3.6.** Thought the empty set $\emptyset$ satisfies vacuously the condition of 1.1.3.5, in this course, we *don't* consider it as an irreducible set.

**Example 1.1.3.7.** $V(X_1 \cdot X_2) = V(X_1) \cup V(X_2) \subset \mathbb{A}_k^2$ is not irreducible, while $\mathbb{A}_k^1$ is irreducible.

**Proposition 1.1.3.8.** *A closed subset $F \subset \mathbb{A}_k^n$ is irreducible iff its ideal $I(F) \subset k[X_1, \cdots, X_n]$ is a prime ideal.*

*Proof.* Let $F \subset \mathbb{A}_k^n$ be irreducible, and let $f, g \in k[X_1, \cdots, X_n]$ be two elements such that $f \cdot g \in I(F)$. We have in particular $(f \cdot g) \subset I(F)$, hence $F = V(I(F)) \subset V(f \cdot g) = V(f) \cup V(g)$, and we get

$$F = V(f \cdot g) \cap F = (V(f) \cap F) \cup (V(g) \cap F)\,.$$

As $V(f)$ and $V(g)$ are both closed, and $F$ is irreducible, we have either $V(f) \cap F = F$ or $V(g) \cap F = F$. Suppose for example $V(f) \cap F = F$, in particular, we get $F \subset V(f)$, hence $f \in I(V(f)) \subset I(F)$. This shows that $I(F) \subset k[X_1, \cdots, X_n]$ is prime. Conversely, suppose $I(F)$ is a prime ideal, and show that $F$ is irreducible. Let

$$F = F_1 \cup F_2$$

with $F_i \subset F$ two closed subsets. Then we have $I(F) \subset I(F_i)$. Now suppose $F_1 \neq F$, in particular, by Hilbert's Nullstellensatz, $I(F) \subsetneq I(F_1)$. Let $f \in I(F_1) - I(F)$, then for any $g \in I(F_2)$, then

$$f \cdot g \in I(F_1 \cup F_2) = I(F).$$

As $I(F)$ is a prime ideal, we get $g \in I(F)$. In particular, $I(F) = I(F_2)$, hence $F = I(V(F)) = I(V(F_2)) = F_2$. This shows that $F$ is irreducible. In this way, we get the proposition. $\square$

**Example 1.1.3.9.** The affine space $\mathbb{A}_k^n$ is irreducible since its ideal is $(0)$ which is of course prime.

**Definition 1.1.3.10.** A topological space $X$ is called *noetherian*, if it satisfies the *descending chain condition* for closed subsets: for any sequence

$$Y_0 \supset Y_1 \supset \cdots Y_i \supset Y_{i+1} \supset \cdots$$

of closed subsets, there exists some integer $i_0 \gg 0$ such that $Y_i = Y_{i_0}$ for any $i \geq i_0$.

**Proposition 1.1.3.11.** *Any algebraic subset together with the Zariski topology is a noetherian topological space.*

*Proof.* Since any closed subset of a noetherian space is again noetherian, we only need to show that $\mathbb{A}_k^n$ is noetherian. Recall that the set of closed subsets of $\mathbb{A}_k^n$ is in inclusion-reversing one-to-one correspondence with the set of radical ideal of $k[X_1, \cdots, X_n]$. Hence to show that $\mathbb{A}_k^n$ is noetherian, we only need to show that $k[X_1, \cdots, X_n]$ satisfies the increasing chain condition, or equivalently, $k[X_1, \cdots, X_n]$ is noetherian. This is exactly Hilbert's basis theorem. As a result, $\mathbb{A}_k^n$ is noetherian. $\square$

**Lemma 1.1.3.12.** *Let $X$ be a neotherian topological space, and $Y \subset X$ be a closed subset. Then $Y$ can be expressed as a finite union $Y = Y_1 \cup Y_2 \cdots \cup Y_r$ of irreducible closed subsets $Y_i$. If we require moreover $Y_i \nsubseteq Y_j$ whenever $i \neq j$, then the family $\{Y_i \colon 1 \leq i \leq r\}$ is uniquely determined. They are called the* irreducible components *of $Y$.*

*Proof.* We will first show the existence of such decomposition. Let $\mathfrak{S}$ be the set of closed subset $Y \subset X$ which can *not* be written as a finite union of irreducible closed subsets, we need to show that $\mathfrak{S} = \emptyset$. If not, let $Y_0 \in \mathfrak{S}$ be an arbitrary element of this set, then $Y_0$ is *not* irreducible, hence we have $Y_0 = Y_0' \cup Y_0''$, with $Y_0', Y_0'' \subsetneq Y_0$ two proper closed subsets of $Y_0$. By the choice of $Y_0$, either $Y_0' \in \mathfrak{S}$ or $Y_0'' \in \mathfrak{S}$. For simplicity, suppose $Y_0' \in \mathfrak{S}$, and we note $Y_1 = Y_0'$. Then we continue with this construction, and we get in this way a infinite sequence of closed subsets of $X$:

$$Y_0 \supsetneq Y_1 \supsetneq Y_2 \supsetneq \cdots \supsetneq Y_r \supsetneq Y_{r+1} \supsetneq \cdots.$$

This gives a contradiction with the assumption that $X$ is noetherian. As a result, $\mathfrak{S} = \emptyset$, and any closed $Y \subset X$ can be written as a finite union of irreducible closed subsets

$$Y = Y_1 \cup Y_2 \cup \cdots \cup Y_r$$

By throwing away a few if necessary, we may assume that $Y_i \nsubseteq Y_j$ whenever $i \neq j$. This gives the existence.

For the uniqueness, suppose

$$Y = \bigcup_{i=1}^{r} Y_i = \bigcup_{j=1}^{s} Y_j'$$

such that $Y_i \nsubseteq Y_{i'}$ and $Y_j' \nsubseteq Y_{j'}$ whenever $i \neq i'$ and $j \neq j'$. Then $Y_1' \subset \cup_i Y_i$, hence there exists some $i$ such that $Y_1' \subset Y_i$. Similarly, we have $Y_i \subset Y_j'$ for some $j$, hence $Y_1' \subset Y_j'$. We must have $1 = j$, and $Y_1' = Y_i$. After renumbering the index $i$, we may assume $i = 1$. Now to finish the proof, it remains remark that

$$\bigcup_{i=2}^{r} Y_i = \overline{Y - Y_1} = \overline{Y - Y_1'} = \bigcup_{j=2}^{s} Y_j'.$$

Hence the uniqueness follows from an induction on $\min(r, s)$. □

**Corollary 1.1.3.13.** *Any closed subset $F \subset \mathbb{A}_k^n$ can be expressed as a finite union of the following form*

$$F = \bigcup_{i=1}^{s} F_i$$

*of irreducible closed subsets, no one containing another.*

**Exercise 1.1.3.14.** Show that a noetherian topological space $X$ is always quasi-compact. If moreover it's Hausdorff, then $X$ is a finite set with the discrete topology.

### 1.1.4 Coordinate ring of an affine algebraic set

Let $V \subset \mathbb{A}_k^n$ be an algebraic set, we define its coordinate ring

$$k[V] := k[X_1, \cdots, X_n]/I(V).$$

**Lemma 1.1.4.1.** *An element $f \in k[X_1, \cdots, X_n]$ defines a function*

$$\mathbb{A}_k^n \to k, \quad x \mapsto f(x),$$

*whose restriction to $V \subset \mathbb{A}_k^n$ depends only on the coset $f + I(V)$.*

In particular, any element of its coordinate ring $k[V]$ can be viewed naturally as a function defined on $V$, which is called a *regular function* of $V$. We call also $k[V]$ the *ring of regular functions*. Moreover, we endow $V$ with the induced topology as $V$ is a subset of the topological space $\mathbb{A}_k^n$. As a corollary of Hilbert's Nullstellensatz, we have the following

**Proposition 1.1.4.2.** *(a) The points of $V$ are in one-to-one correspondence with the maximal ideals of $k[V]$.*

*(b) The closed subsets of $V$ are in one-to-one correspondence with the radical ideals of $k[V]$, such that a closed subset $F \subset V$ is irreducible iff the corresponding radical ideal under this correspondence is a prime ideal of $k[V]$.*[4]

*(c) For any $f \in k[V]$, let $D(f) := \{x \in V : f(x) \neq \emptyset\}$. Then the family $\{D(f) : f \in k[V]\}$ forms a basis for the topology of $V$:* i.e., *each $D(f)$ is open, and any open $U \subset V$ is a union of $D(f)$'s.*

*Proof.* (a) Let $P = (x_1, \cdots, x_n) \in V \subset \mathbb{A}_k^n$ be an element of $V$, then $I(\{P\}) = \mathfrak{m}_P = (X_1 - x_1, \cdots, X_n - x_n) \subset k[X_1, \cdots, X_n]$ is a maximal ideal which contains $I(V)$. Hence its image $\overline{\mathfrak{m}_P} \subset k[V] = k[X_1, \cdots, X_n]/I(V)$ is again a maximal ideal. Now we claim that the correspondence

$$V \to \mathrm{Max}(k[V]), \quad P \mapsto \overline{\mathfrak{m}_P}$$

is bijective. Indeed, let $\mathfrak{n}$ be a maximal ideal of $k[V]$, its inverse image $\mathfrak{m}$ in $k[X_1, \cdots, X_n]$ is again maximal, hence of the form $\mathfrak{m}_{P_\mathfrak{n}}$ for a unique point $P_\mathfrak{n} \in \mathbb{A}_k^n$ by Nullstellensatz. One verifies easily that $P_\mathfrak{n}$ is contained in $V$ (since $I(V) \subset \mathfrak{m}_{P_\mathfrak{n}}$), and the map

$$\mathrm{Max}(k[V]) \to V, \quad \mathfrak{n} \mapsto P_\mathfrak{n}$$

gives an inverse of the previous correspondence.

(b) Similar proof as (a).

(c) First of all $D(f) \subset V$ is open. Indeed, let $\tilde{f}$ be an arbitrary lifting of $f$ in $k[X_1, \cdots, X_n]$, and consider

$$D(\tilde{f}) := \{x \in \mathbb{A}_k^n : \tilde{f}(x) \neq 0\}.$$

Then $D(f) = D(\tilde{f}) \cap V$, hence, we only need to show $D(\tilde{f}) \subset \mathbb{A}_k^n$ is open. But its complement can be described in the following way,

$$\mathbb{A}_k^n - D(\tilde{f}) = \{x \in \mathbb{A}_k^n : \tilde{f}(x) = 0\} = V(\{\tilde{f}\})$$

hence this is closed. As a result, $D(\tilde{f})$ is open. This gives the openness of $D(f)$. For the last assertion, suppose $U = \widetilde{U} \cap V$ be an open of $V$ with $\widetilde{U}$ is open in $\mathbb{A}_k^n$. Hence, there is some ideal $\mathfrak{a} \subset k[X_1, \cdots, X_n]$ such that $\mathbb{A}_k^n - \widetilde{U} = V(\mathfrak{a}) = \cap_{g \in \mathfrak{a}} V(g)$. Hence $\widetilde{U} = \cup_{g \in \mathfrak{a}} D(g)$. As a result, $U = \widetilde{U} \cap V = \cup_{g \in \mathfrak{a}}(D(g) \cap V) = \cup_{g \in \mathfrak{a}} D(\bar{g})$ with $\bar{g}$ the image of $g$ in $k[V]$. This finishes the proof of (c).

□

[4] Note that by our convention 1.1.3.6, the empty set $\emptyset$ is *not* irreducible.

**Exercise 1.1.4.3.** 1. Consider the *affine twisted cubic curve*:

$$C = \{(t, t^2, t^3) : t \in k\} \subset \mathbb{A}^3_k.$$

Show that $C$ is an irreducible closed subset of $\mathbb{A}^3_k$, and find generators of the ideal $I(C) \subset k[X, Y, Z]$.

2. Let $V = V(X^2 - YZ, XZ - X) \subset \mathbb{A}^3_k$. Show that $V$ consists of three irreducible components, and determine the corresponding prime ideals.

3. We identify the space $\mathrm{M}_2(k)$ of $2 \times 2$-matrices over $k$ with $\mathbb{A}^4_k$ with coordinates $a, b, c, d$:

$$\mathrm{M}_2(k) \simeq \mathbb{A}^4_k, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d).$$

Show that the set of nilpotent matrices is an algebraic subset of $\mathbb{A}^4_k$, and determine its ideal.

**Exercise 1.1.4.4.** Consider $C = \{(t^3, t^4, t^5) : t \in k\} \subset \mathbb{A}^3_k$. Show that $C$ is an irreducible algebraic set, et determine $I(C)$. Can $I(C)$ be generated by 2 elements?

*Solution.* Consider the ideal $J$ generated by the following three elements

$$XZ - Y^2, \quad X^2Y - Z^2, \quad YZ - X^3.$$

Then $J \subset I(C)$, and $V(J) = C$. In particular, $C$ is an algebraic set. Next, we claim that $J$ is a prime ideal. Indeed, consider the following morphism

$$\phi \colon k[X, Y, Z] \to k[T], \qquad X \mapsto T^3, \ \ Y \mapsto T^4, \ \ Z \mapsto T^5.$$

Its kernel contains $J$, and we only need to show that $\ker(\phi) = J$. Let $f \in k[X, Y, Z]$, using the three generators of $J$ as above, we have

$$f(X, Y, Z) \equiv a_0(X) + a_1(X) \cdot Y + a_2(X) \cdot Z \bmod J$$

with $a_i(X) \in k[X]$. Now $f \in \ker(\phi)$ means that $a_0(T^3) + a_1(T^3) \cdot T^4 + a_2(T^3) \cdot T^5 = 0$, which is possible only when $a_i(X) = 0$, that is $f \in J$. This proves that $J = \ker(\phi)$ is prime. In particular, $J = I(C)$ by Hilbert's Nullstellensatz. Moreover, $I(C)$ cannot be generated by two elements. In fact, if so, the $k$-vector space

$$\overline{J} := J \otimes_{k[X,Y,Z]} k[X, Y, Z]/(X, Y, Z) = J/(X, Y, Z) \cdot J$$

must have dimension at most 2. On the other hands, we claim that the image of the three generators above in $\overline{J}$ is $k$-linearly independent: let $a, b, c \in k$ such that

$$a \cdot (XZ - Y^2) + b \cdot (X^2Y - Z^2) + c \cdot (YZ - X^3) \in (X, Y, Z) \cdot J$$

Since each term of all non zero element of $(X, Y, Z) \cdot J$ has degree at least 3, we must have

$$a \cdot (XZ - Y^2) - b \cdot Z^2 + c \cdot YZ = 0$$

As a result, $a = b = c = 0$, this gives the claim. In particular, $J = I(C)$ cannot be generated by 2 elements. □

## 1.2 Projective algebraic sets

### 1.2.1 Definitions

Let $n \geq 1$ be an integer, and we consider the following equivalent relation $\sim$ on $k^{n+1} - \{0\}$: for $x = (x_0, \cdots, x_n)$ and $y = (y_0, \cdots, y_n)$, we say $x \sim y$ iff there exists some $\lambda \in k$ (must be non zero) such that $x_i = \lambda \cdot y_i$ for each $i$.

**Definition 1.2.1.1.** We define the *projective space of dimension* $n$ to be the set of equivalent classes:

$$\mathbb{P}_k^n := \left(k^{n+1} - \{0\}\right) / \sim \text{[5]}$$

An element of $\mathbb{P}_k^n$ is called a point of the projective space $\mathbb{P}_k^n$. For each element $(x_0, \cdots, x_n) \in k^{n+1} - \{0\}$, its equivalent class in $\mathbb{P}_k^n$ will be denoted by $(x_0 : x_1 : \cdots : x_n)$. The $n+1$-tuple $(x_0, \cdots, x_n)$ is called the *homogeneous coordinate* of this equivalent class.

Recall that for a polynomial $P \in k[X_0, \cdots, X_n]$, it's called *homogeneous* if there exists some integer $d \geq 0$, such that $P$ can be written under the following form

$$P(X_0, \cdots, X_n) = \sum_{i_0, \cdots, i_n \in \mathbb{Z}_{\geq 0}, i_0 + \cdots + i_n = d} a_{i_0, \cdots, i_n} X_0^{i_0} \cdots X_n^{i_n}$$

In particular,

$$P(\lambda \cdot X_0, \lambda \cdot X_1, \cdots, \lambda \cdot X_n) = \lambda^d \cdot P(X_0, X_1, \cdots, X_n).$$

for any $\lambda \in k$. Let now $P$ be such a homogeneous polynomial, and let $x = (x_0, \cdots, x_n)$ be a zero of $P$. For any $\lambda \in k$, $\lambda \cdot x = (\lambda x_0, \cdots, \lambda x_n)$ is again a zero of $P$. Hence it makes sense to say that for a point $x \in \mathbb{P}_k^n$, whether the value $P(x)$ is zero or not. Moreover, an ideal $\mathfrak{a} \subset k[X_0, \cdots, X_n]$ is called *homogeneous* if it can be generated as an ideal by homogeneous elements, or equivalently, if

$$\mathfrak{a} = \bigoplus_{d \geq 0} \mathfrak{a} \cap k[X_0, \cdots, X_n]_d.$$

**Definition 1.2.1.2.** Let $P \in k[X_0, \cdots, X_n]$ be a homogeneous polynomial. Let $\mathfrak{a} = (f_1, \cdots, f_m)$ be a homogeneous ideal with $\{f_1, \cdots, f_m\}$ a family of homogeneous generators. We define the corresponding *projective algebraic set* to be

$$V_+(\mathfrak{a}) := \{(x_0 : \cdots x_n) \in \mathbb{P}_k^n : f_i(x_0, \cdots, x_n) = 0 \quad \forall i \ \}$$

One can verify that this definition is independent of the choice of the homogeneous generators $\{f_1, \cdots, f_m\}$

**Example 1.2.1.3.** Let $(a_0 : \cdots : a_n) \in \mathbb{P}_k^n$ be a point of the projective space, we consider the ideal $\mathfrak{p}$ generated by $(X_i a_j - X_j a_i)_{i,j}$. Then it's homogeneous, and $V_+(\mathfrak{p}) = \{(a_0 : \cdots : a_n)\} \subset \mathbb{P}_k^n$. Note that this ideal is prime but not maximal. Moreover, $V(\mathfrak{p}) \subset \mathbb{A}_k^{n+1}$ is the line passing through the origin of $\mathbb{A}_k^{n+1}$ and also the point $(a_0, \cdots, a_n) \in \mathbb{A}_k^{n+1}$.

**Example 1.2.1.4** (Hypersurfaces)**.** Let $f(X_0, \cdots, X_n)$ be an irreducible homogeneous polynomial. Then $f = 0$ defines an irreducible algebraic set in $\mathbb{P}_k^n$, called a *hypersurface*.

It may happen that $V_+(\mathfrak{a}) = \emptyset$ for a homogeneous ideal $\mathfrak{a}$ even if $\mathfrak{a} \neq k[X_0, \cdots, X_n]$. In fact, we have

[5] Similar to the affine case, the more correct notation here is $\mathbb{P}_k^n(k)$, here we write $\mathbb{P}_k^n$ just for simplicity.

**Proposition 1.2.1.5.** *Let $\mathfrak{a} \subset k[X_0, \cdots, X_n]$ be a homogeneous ideal. The following two assertions are equivalent*

1. $V_+(\mathfrak{a}) = \emptyset$;
2. $\mathfrak{a}$ *contains* $k[X_0, \cdots, X_n]_d$ *for some* $d > 0$.

*Proof.* [6] Suppose $V_+(\mathfrak{a}) = \emptyset$, which implies that $V(\mathfrak{a}) \subset \mathbb{A}_k^{n+1}$ has *at most* the single point $\{0\}$, that is $V(\mathfrak{a}) \subset \{0\}$. As a result, we find $(X_0, \cdots, X_n) = I(\{0\}) \subset I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. In particular, there exists some $s > 0$ such that $X_i^s \in \mathfrak{a}$, hence $k[X_0, \cdots, X_n]_d \subset \mathfrak{a}$ with $d = (n+1) \cdot s$. This gives (2). Conversely, suppose $\mathfrak{a}$ contains $k[X_0, \cdot, X_n]_d$ for some $d > 0$, then $V(\mathfrak{a}) \subset \{0\}$, hence $V_+(\mathfrak{a}) = \emptyset$. $\square$

**Proposition 1.2.1.6.** *The union of two algebraic sets is algebraic. The intersection of any family of algebraic sets is algebraic. The empty set and the whole space $\mathbb{P}_k^n$ are both algebraic.*

**Definition 1.2.1.7.** We define the Zariski topology on $\mathbb{P}_k^n$ by taking open subsets to be the complements of algebraic sets. For a subset $Y \subset \mathbb{P}_k^n$, the *Zariski topology* is the subspace topology induced from $\mathbb{P}_k^n$.

**Exercise 1.2.1.8.** Describe the closed subsets of $\mathbb{P}_k^1$.

**Definition 1.2.1.9.** A subset $V \subset \mathbb{P}_k^n$ is called a *quasi-projective algebraic set*, if it can be realized as an open of a projective algebraic set of $\mathbb{P}_k^n$.

**Example 1.2.1.10.** For each integer $i \in [0, n]$, let $H_i = Z(X_i) = \{(x_0 \colon \cdots \colon x_n) \colon x_i = 0\} \subset \mathbb{P}_k^n$. Then it's a projective algebraic set, and we will denote by $U_i$ its complement. There is a canonical map of sets

$$U_i \to \mathbb{A}_k^n, \quad (x_0 \colon \cdots \colon x_n) \mapsto \left( \frac{x_0}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_n}{x_i} \right)$$

In fact, this is a homoeomorphism of topological spaces. Note that $\bigcap_i H_i = \emptyset$, hence $\bigcup_i U_i = \mathbb{P}_k^n$.

**Example 1.2.1.11.** [Any affine algebraic set is quasi-projective] Let $V = V(\mathfrak{a}) \subset \mathbb{A}_k^n$ be an affine algebraic set. We identify $\mathbb{A}_k^n$ with the open subset $U_0$ of $\mathbb{P}_k^n$ of elements with homogeneous coordinate $(x_0 : x_1 : \cdots : x_n)$ such that $x_0 \neq 0$. In this way, $V$ can be seen as a subset of $\mathbb{P}_k^n$. Since $V \subset U_0$ is closed, we have $V = \overline{V} \cap U_0$ with $\overline{V} \subset \mathbb{P}_k^n$ the closure of $V$ in $\mathbb{P}_k^n$.

**Exercise 1.2.1.12.** Keeping the notations of Example 1.2.1.11. Describe explicitly from $\mathfrak{a}$ the closure $\overline{V}$ of $V$.

*Solution.* We consider the following construction: for each polynomial $f \in k[X_1, \cdots, X_n]$, et define $\beta(f) \in k[x_0, \cdots, x_n]$ the homogeneous polynomial given by

$$\beta(f) = x_0^{\deg(f)} \cdot f\left( \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right).$$

Now, we consider $\mathfrak{b} \subset k[x_0, \cdots, x_n]$ the homogeneous ideal generated by $\beta(\mathfrak{a}) \subset k[x_0, \cdots, x_n]$, then clearly $V \subset V_+(\mathfrak{b})$, hence $\overline{V} \subset V_+(\mathfrak{b})$. Conversely, for $G \in k[x_0, \cdots, x_n]$ a homogeneous polynomial such that $V \subset V_+(G)$, or equivalently $V \subset V(g)$ with $g = G(1, X_1, \cdots, X_n)$, we have $g \in \sqrt{\mathfrak{a}}$ by Hilbert's Nullstellensatz. In particular, there exists some integer $r > 0$ such that $g^r \in \mathfrak{a}$. In particular, $\beta(g)^r = \beta(g^r) \in (\beta(\mathfrak{a})) = \mathfrak{b}$. Moreover, we have $G = x_0^{\deg(G) - \deg(g)} \beta(g)$, hence $G^r \in \mathfrak{b}$, so $G \in \sqrt{\mathfrak{b}}$. As a result, we find $V_+(\mathfrak{b}) \subset V_+(G)$. In this way, we find $V_+(\mathfrak{b})$ is contained in any closed subset of $\mathbb{P}_k^n$ which contains $V$. From this, we find $\overline{V} = V_+(\mathfrak{b})$. $\square$

[6] In the course, it was said that $V_+(\mathfrak{a}) = \emptyset$ iff $V(\mathfrak{a}) = \{0\}$, but of course this equivalence works *only* when $\mathfrak{a} \subsetneq k[X_0, \cdots, X_n]$. Hence here the more correct statement is that $V_+(\mathfrak{a}) = \emptyset$ iff $V(\mathfrak{a}) \subset \{0\}$.

**Example 1.2.1.13** (Twisted cubic in $\mathbb{P}^3_k$)**.** Recall that the *affine twisted cubic* is the affine algebraic set given by

$$C = \{(t, t^2, t^3) : t \in k\} \subset \mathbb{A}^3_k$$

In terms of equations, this is $V(X^2 - Y, X^3 - Z)$. Now, we identify $\mathbb{A}^3_k$ with an open of $\mathbb{P}^3_k$ by

$$\mathbb{A}^3_k \hookrightarrow \mathbb{P}^3_k, \quad (x, y, z) \mapsto (1 : x : y : z).$$

Then the closure $\overline{C}$ of $C$ is an projective algebraic set of $\mathbb{P}^3_k$, called the *(projective) twisted cubic.* Now we want to give an explicitly description of $\overline{C}$ in terms of equations. Note that

$$(X^2 - Y, X^3 - Y) = (X^2 - Y, XY - Z) = (X^2 - Y, XY - Z, XZ - Y^2)$$

Hence, if we consider $I = (X^2 - YW, XY - ZW, XZ - Y^2) \subset k[W, X, Y, Z]$, then $V_+(I) \cap \mathbb{A}^3_k = C$. Moreover, this ideal is prime: indeed, we can consider the following morphism of rings

$$\alpha \colon k[W, X, Y, Z] \to k[S, T], \quad W \mapsto S^3,\ X \mapsto S^2T,\ Y \mapsto ST^2,\ Z \mapsto T^3.$$

Then $I \subset \alpha$, hence to show that $I$ is prime, we only need to show that $I = \ker(\alpha)$. Now, any polynomial $f \in k[W, X, Y, Z]$ can be written as

$$f \equiv a_0(W, Z) + a_1(W, Z) \cdot X + a_2(W, Z) \cdot Y \bmod I$$

with $a_i \in k[W, Z]$. If $f \in \ker(\alpha)$, we then have

$$0 = a_0(S^3, T^3) + a_1(S^3, T^3) \cdot S^2T + a_2(S^3, T^3) \cdot ST^2$$

which is possible iff $a_i = 0$ for $i \in \{0, 1, 2\}$. Hence $f \in I$. This shows $I$ is prime, hence $V_+(I)$ is irreducible, and it contains $V$ as an open subset, hence $V_+(I) = \overline{V}$. One shows also that this ideal $I$ cannot be generated by two elements.

**Exercise 1.2.1.14.** Consider the map

$$\nu \colon \mathbb{P}^1_k \to \mathbb{P}^3_k, \quad (x; y) \mapsto (x^3 : x^2y : xy^2 : y^3).$$

Prove that the image of this map is a projective algebraic set of $\mathbb{P}^3_k$. Can you recognize this algebraic set?

### 1.2.2 Homogeneous Nullstellensatz

**Definition 1.2.2.1.** Let $Y \subset \mathbb{P}^n_k$ be a subset. We define its homogeneous ideal $I(Y)$ to be the ideal generated by the homogeneous polynomial $P \in k[X_0, \cdots, X_n]$ such that $P(x) = 0$ for any $x \in Y$.

**Proposition 1.2.2.2.** *Keeping the notations as before.*

1. *We have $I(V_+(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, and $V_+(I(Y)) = \overline{Y}$ for $Y \hookrightarrow \mathbb{P}^n_k$ a subset.*

2. *The mappings $\mathfrak{a} \mapsto V(\mathfrak{a})$ and $Y \mapsto I(Y)$ set up a bijection between the set of closed algebraic subsets of $\mathbb{P}^n_k$, and the set of all* radical *homogeneous ideal $\mathfrak{a} \subset k[X_0, \cdots, X_n]$ such that $\mathfrak{a} \neq (X_0, \cdots, X_n)$.*[7]

3. *Under the previous correspondence, the irreducible closed subsets (resp. the points) of $\mathbb{P}^n_k$ correspond to the prime homogeneous ideals not equal to $(X_0, \cdots, X_n)$ (resp. the prime homogeneous ideals $\mathfrak{p} \subsetneq (X_0, \cdots, X_n)$ which is maximal with respect to this property).*

4. *$\mathbb{P}^n_k$ is neotherian with the Zariski topology.*

[7] Note that the radical of a homogeneous ideal is still homogeneous.

**Exercises: affine cone over a projective subset**

Let $U = \mathbb{A}_k^{n+1} - \{0\}$, and $p : U \to \mathbb{P}_k^n$ be the canonical projection. Let $Y \subset \mathbb{P}_k^n$ be a projective algebraic subset, let

$$\widetilde{Y} := p^{-1}(Y) \cup \{0\} \subset \mathbb{A}_k^{n+1}.$$

**Exercise 1.2.2.3.** Show that $\widetilde{Y} \subset \mathbb{A}_k^{n+1}$ is an algebraic subset of $\mathbb{A}_k^{n+1}$, whose ideal is $I(Y)$ considered as an ordinary ideal of $k[X_0, \cdots, X_n]$. Moreover show that $Y$ is irreducible if and only if $\widetilde{Y}$ is irreducible. The affine algebraic set $\widetilde{Y}$ is called the *affine cone over* $Y$.

**Exercise 1.2.2.4.** Use the notion of affine cone to establish 1.2.2.2.

### 1.2.3 Homogeneous coordinate ring

Recall first that a *graded ring* is a ring $A$, together with a decomposition of $A$ into a direct sum of its subgroups

$$A = \bigoplus_{d \geq 0} A_d$$

such that $A_d \cdot A_{d'} \subset A_{d+d'}$. The decomposition above is called a *gradation* of $A$.

Let $V = V_+(\mathfrak{a}) \subset \mathbb{P}_k^n$ be a projective algebraic set, with $\mathfrak{a} \subset k[X_0, \cdots, X_n]$ a homogeneous ideal. Its *homogeneous coordinate ring* is then define to be the quotient $k[X_0, \cdots, X_n]/I(V)$, with $I(V)$ the ideal of $V$. Since $I(V)$ is homogeneous, the quotient $A := k[X_0, \cdots, X_n]/I(V)$ is naturally graded: $A = \oplus_{d \geq 0} A_d$ where

$$A_d := k[X_0, \cdots, X_n]_d / k[X_0, \cdots, X_n]_d \cap I(V)$$

with $k[X_0, \cdots, X_n] = \bigoplus_{d \geq 0} k[X_0, \cdots, X_n]_d$ the usual gradation of $k[X_0, \cdots, X_n]$. Let $A_+ := \oplus_{d>0} A_d$. Let $f \in A$ be a homogeneous element, which is the image of a homogeneous element $\tilde{f} \in k[X_0, \cdots, X_n]$. It's easy to see that the set

$$\{x \in V \subset \mathbb{P}_k^n \colon \tilde{f}(x) \neq 0\}$$

is independent of the lifting $\tilde{f}$ of $f$, hence we will denote it by $D_+(f)$. For the future use, we record the following proposition, which is a projective analogue of 1.1.4.2.

**Proposition 1.2.3.1.**

1. *There is a one-to-one correspondence between the set of closed subsets of $V$, and the set of homogeneous radical ideals not equal to $A_+$.*

2. *Under the previous correspondence, the irreducible closed subsets (resp. points) correspond to the homogeneous prime ideals of $A$ not equal to $A_+$ (resp. prime homogeneous ideals $\mathfrak{p} \subsetneq A_+$ which are maximal with respect to this property).*

3. *For $f \in A = k[X_0, \cdots, X_n]/I(V)$, the subset $D_+(f)$ is open in $V$. Moreover, the family $\{D_+(f) \colon f \in A \text{ homogeneous}\}$ forms a basis for the Zariski topology of $V$. Such an open subset will be called* principal.

### 1.2.4 Exercise: plane curves

We begin with a general ressult.

**Exercise 1.2.4.1.** Let $\mathfrak{a} \subset k[X_1, \cdots, X_n]$ be an ideal. Show that $V(\mathfrak{a})$ is a finite set if and only if the quotient $k[X_1, \cdots, X_n]/\mathfrak{a}$ is of finite dimensional over $k$.

Now, an *affine plane curve*, is the set of solutions in $\mathbb{A}_k^2$ of a nonconstant polynomial $f \in k[X, Y]$.

**Exercise 1.2.4.2.** Let $C = V(f), D = V(g) \subset \mathbb{A}_k^2$ be two plane curves with $f, g$ two irreducible polynomial of degree respectively $m$ and $n$, such that $f \nmid g$. We want to show that $C \cap D$ is a finite set in the following way.

1. For each integers $d \geq 0$, let $P_d$ be the set of polynomials $\in k[X, Y]$ of total degree $\leq d$. Compute the dimension of this $k$-vector space $P_d$.

2. Show that for $d \geq \mathrm{Max}\{m, n\}$, the $k$-vector space $P_d/(f, g) \cap P_d$ is of dimension $\leq mn$. For this, one could use the following sequence of maps

$$P_{d-m} \times P_{d-n} \xrightarrow{\alpha} P_d \xrightarrow{\beta} P_d/(f, g) \cap P_d \tag{1.2}$$

   where $\alpha(u, v) = uf + vg$, and $\beta$ is the natural projection, and note that $\beta \circ \alpha = 0$.

3. Show that $\dim_k(k[X, Y]/(f, g)) \leq mn$ and then conclude.

But it might happen that $C \cap D = \emptyset$. To remedy this, we consider the projective plane curves. By definition, a *projective plane curve*, or just a *plane curve* for short, is the set of solutions in $\mathbb{P}_k^2$ of a nonconstant homogeneous polynomial $F \in k[X, Y, Z]$.

**Exercise 1.2.4.3.** Show that for any two plane curves $C, D \in \mathbb{P}_k^2$, their intersection $C \cap D$ is not empty. One could try to use the sequence (1.2) for help.

Now, we will examiner the case where $C = V_+(F)$ is a plane curve with $F \in k[X, Y, Z]$ a homogenenous irreducible polynomial, and $D = L$ is a line in $\mathbb{P}_k^2$. Let's first define the *intersction multiplicity* of $L$ and $C$ at a point $P \in \mathbb{P}_k^2$. We may assume that $P = (x_0 : y_0 : 1) \in \mathbb{A}_k^2 = D_+(Z) \subset \mathbb{P}_k^2$. Hence $L \cap \mathbb{A}_k^2$ is given by a polynomial $f$ in $x, y$ of degree 1:

$$g(x, y) = a \cdot (x - x_0) + b \cdot (y - y_0)$$

and $C \cap \mathbb{A}_k^2$ is given by a polynomial $f(x, y) = F(x, y, 1) \in k[x, y]$ such that $f(x_0, y_0) = 0$. Without loss of generality, we assume $b \neq 0$, then replace $y$ by

$$-\frac{a}{b}(x - x_0) + y_0$$

in the expression of $f$, and we get a polynomial $\tilde{f}(x) = f(x, -\frac{a}{b}(x - x_0) + y_0) \in k[x]$. By definition, $x_0$ is a root of this polynomial. Now, we define the intersection multiplicity $i(P; L, C)$ to be the multiplicity of the root $x = x_0$ of the polynomial $\tilde{f}$. Moreover, by convention, for $P \notin L \cap C$, we define $i(P; L, C) = 0$.

**Exercise 1.2.4.4** (A special case of Bézout's theorem)**.** [8] Let $L$ be a line of $\mathbb{P}_k^2$ and $C = V_+(F)$ be a plane curve given by an irreducible homogeneous polynomial of degree $n$. Suppose $C \neq L$, then the following equality holds

$$n = \sum_{P \in \mathbb{P}_k^2} i(P; L, C)$$

The actual Bézout's theorem is the following

[8] Of course, the affine analogue of this proposition is not true.

**Theorem 1.2.4.5.** *Let $C$ and $C'$ be two plane curves of degree respectively $m$ and $n$. Suppose that $C$ or $C'$ doesnot contain any irreducible component of the other curve as its irreducible component, then the following equality holds:*

$$m \cdot n = \sum_{P \in \mathbb{P}^2_k} i(P; C, C')$$

## 1.3 Morphisms of algebraic sets

We will need to know when 2 algebraic sets are to be considered isomorphic. More generally, we will need to define not just the *set* of all algebraic sets, but the *category* of algebraic sets.

### 1.3.1 Affine case

Let $U \subset \mathbb{A}^n_k$ and $V \subset \mathbb{A}^m_k$ be two affine algebraic sets.

**Definition 1.3.1.1.** A map $\sigma\colon U \to V$ is called a *morphism* if there exist $m$ polynomials in $n$ variables $P_1, \cdots, P_m \in k[X_1, \cdots, X_n]$ such that for any $x = (x_1, \cdots, x_n)$, we have $\sigma(x) = (P_1(x), \cdots, P_m(x))$. A morphism $\sigma\colon X \to Y$ is called an *isomorphism* if it's bijective,[9] with inverse $\sigma^{-1}$ again a morphism. In this case, we say that $U$ and $V$ are *isomorphic*.

**Remark 1.3.1.2.** If we replace in the previous definition "polynomials" by "regular functions on $U$", we get the same notion. Hence a morphism $\sigma\colon U \to V \subset \mathbb{A}^m_k$ is given by $m$ regular functions $f_1, \cdots, f_m \in k[U]$ such that for any $x \in U$, $\sigma(x) = (f_1(x), \cdots, f_m(x)) \in V$.

**Example 1.3.1.3.** Look at $\mathbb{A}^1_k$ the affine line, and the parabola $C = V(y - x^2) \subset \mathbb{A}^2_k$. The projection $\pi\colon (x, y) \mapsto x$ of the parabola onto the $x$-axis should surely be an isomorphism between these algebraic sets. Indeed, we consider the following map

$$\sigma\colon \mathbb{A}^1_k \to C, \quad x \mapsto (x, x^2).$$

Then we have $\sigma \circ \pi = \mathrm{id}$ and $\pi \circ \sigma = \mathrm{id}$.

**Proposition 1.3.1.4.** *Let $f\colon U \to V$ be a morphism of affine algebraic sets. Then $f$ induces a continuous map between the underlying topological spaces.*

*Proof.* Suppose $U \subset \mathbb{A}^n_k$, and $V \subset \mathbb{A}^m_k$. By the definition, $f$ is just the restriction to $U$ of a regular function $F\colon \mathbb{A}^n_k \to \mathbb{A}^m_k$. Since $U$ and $V$ are endow with the subspace topology, hence we are reduced to show that $F$ is continuous. We write $F = (P_1, \cdots, P_m)$ with $P_i \in k[X_1, \cdots, X_n]$. Let $Y = V(I) \subset \mathbb{A}^m_k$ be a closed subset with $I = (g_1, \cdots, g_r) \subset k[Y_1, \cdots, Y_m]$ an ideal. Then $F^{-1}(V(I)) = V(f_1, \cdots, f_r)$ with $f_i = g_i(P_1, \cdots, P_m) \in k[X_1, \cdots, X_n]$. In particular, $F^{-1}(V(I))$ is closed. Hence, $F$ is continuous. This finishes the proof. □

Let $\sigma\colon U \to V$ be a morphism of affine algebraic sets. For any regular function $f\colon V \to U$, we claim the composition $f \circ \sigma\colon U \to k$ gives a regular function on $U$. Indeed, as $f$ is regular, we may assume that $f(y) = F(y)$ for $y \in V$ with $F \in k[Y_1, \cdots, Y_m]$ a polynomial. Moreover, let $P_1, \cdots, P_m \in k[X_1, \cdots, X_n]$ such that $\sigma(x) = (P_1(x), \cdots, P_m(x))$ for $x \in U$. Then $f \circ \sigma(x) = F(P_1, \cdots, P_n)(x)$ for $x \in U$. As $F(P_1, \cdots, P_m) \in k[X_1, \cdots, X_n]$ is a polynomial, hence the composition $f \circ \sigma$ is a regular function on $U$. In this way, we get a map, denoted by $\sigma^*$:

$$\sigma^*\colon k[V] \to k[U].^{10}$$

[9] Note that $k = \bar{k}$.

[10] The proof given here is slicely different the proof give in the lecture.

**Proposition 1.3.1.5.** *The map*

$$\iota\colon \mathrm{Hom}(U,V) \to \mathrm{Hom}_k(k[V],k[U]), \quad \sigma \mapsto \sigma^*$$

*is bijective.*

*Proof.* We remark first that, for any $x \in U$, if we note $\mathfrak{m}_x = \{f \in k[U]\colon f(x) = 0\} \subset k[U]$ the maximal ideal corresponding to $x \in X$, and $\mathfrak{n}_{\sigma(x)} \subset k[V]$ the maximal ideal corresponding to $\sigma(x)$. Then we have

$$\sigma^{*-1}(\mathfrak{m}_x) = \mathfrak{n}_{\sigma(x)}.$$

This remark implies immediately that the map $\iota$ is injective. To establish the surjectivity, let $\tau\colon k[V] \to k[U]$ be a morphism of $k$-algebras. Let $P_i \in k[X_1, \cdots, X_n]$ be a lifting of $\tau(y_i)$ for each $1 \le i \le m$, we have then the following commutative diagram

$$\begin{tikzcd}
k[Y_1,\cdots,Y_m] \arrow[r, "Y_i\mapsto P_i"] \arrow[d, "Y_i\mapsto y_i"'] & k[X_1,\cdots,X_n] \arrow[d, "X_i\mapsto x_i"] \\
k[V] \arrow[r, "\tau"] & k[U].
\end{tikzcd}$$

Let

$$\sigma = (P_1, \cdots, P_m)\colon \mathbb{A}^n_k \to \mathbb{A}^m_k$$

be the morphism given by the polynomials $P_i$'s. Then $\sigma$ sends $U$ to $V$, which means that for any $x \in U$, $(P_1(x), \cdots, P_m(x)) \in V$. Indeed, we need to show that $f(P_1(x), \cdots, P_m(x)) = 0$ for any $f \in I(V)$. But since the polynomial $P_i$ is a lifting of $\tau(y_i)$, we have $f(P_1(\underline{X}), \cdots, P_m(\underline{X})) \in I(U)$, hence $f(P_1(x), \cdots, P_m(x)) = 0$ for any $x \in U$. In this way, we get a morphism from $U$ to $V$, still denoted by $\sigma$. Finally, it remains to show that $\sigma^* = \tau$. We only need to show $\sigma^*(y_i) = \tau(y_i)$ for each $i$: in fact, we have

$$\sigma^*(y_i) = P_i(x_1, \cdots, x_n) = \overline{P_i(X_1, \cdots, X_n)} = \tau(y_i)$$

This finishes then the proof. □

**Corollary 1.3.1.6.** *A morphism of affine algebraic sets $\sigma\colon U \to V$ is an isomorphism iff the induced maps between the coordinate rings is an isomorphism. In particular, two affine algebraic sets are isomorphic if and only if there coordinate rings are isomorphic as $k$-algebras.*

But note that a bijective morphism $\sigma : U \to V$ is not an isomorphism in general.

**Example 1.3.1.7.** Suppose $k$ of characteristic $p$. Define the morphism

$$f\colon \mathbb{A}^1_k \to \mathbb{A}^1_k, \quad t \mapsto t^p$$

which is bijective. Then in the level of coordinate rings, $f$ induces the map

$$f^*\colon k[X] \to k[X], \quad X \mapsto X^p.$$

Clearly, this map is not an isomorphism. Hence, $f$ is not an isomorphism of affine algebraic sets.

**Example 1.3.1.8.** We consider the following map

$$\sigma\colon \mathbb{A}^1_k \to \mathbb{A}^2_k, \quad t \mapsto (t^2, t^3).$$

Let $C$ be its image, then $C$ is an affine algebraic set, and $\sigma$ induces a bijection between $\mathbb{A}^1_k$ and $C$. But this morphism is not an isomorphism. In fact, these two affine algebraic sets are not isomorphic.

**Exercise 1.3.1.9.** Verify the assertions in the previous exercise.

### 1.3.2 Quasi-projective case

The definition of morphism in the projective case is more subtle. We begin with the notion of regular functions.

**Remark 1.3.2.1.** Even for a homogeneous polynomial $P$, its value at a point $x$ of projective space $\mathbb{P}^n_k$ is not well-defined. But for $P, Q$ two homogeneous polynomials *of the same degree*, such that $Q(x) \neq 0$, then we have a well-defined function on the neighborhood $D(Q) := \{y \in \mathbb{P}^n_k \colon Q(y) \neq 0\}$:

$$\frac{P}{Q} \colon D(Q) \to k, \quad y \mapsto \frac{P(y)}{Q(y)}.$$

**Definition 1.3.2.2.** For a quasi-projective algebraic set $V \subset \mathbb{P}^n_k$, and $x \in V$ a point. A function $f \colon V \to k$ is called *regular at* $x$ if there exist two *homogeneous* polynomials $P, Q \in k[X_0, \cdots, X_n]$ of the same degree such that $Q(x) \neq 0$, and that $f$ agrees with $\frac{P}{Q}$ on a neighborhood of $V$ at $x$. The set of regular functions will be denoted by $k[V]$. This has set naturally a ring structure.

Since an affine algebraic set can also be viewed as a quasi-projective algebraic set, we get two versions of regular functions on it. But in fact, these two versions coincide. More precisely, we have

**Proposition 1.3.2.3.** *Let $V \subset \mathbb{A}^n_k$ be an algebraic set. Then the notion of regular functions on $V$ when $V$ is viewed as an affine algebraic set is the same as the notion of regular function on $V$ when $V$ is viewed as a quasi-projective algebraic set via $V \subset \mathbb{A}^n_k = D_+(X_0) \subset \mathbb{P}^n_k = \{(x_0 : \cdots : x_n) | x_i \in k$ not all zero$\}$.*

*Proof.* We identify $\mathbb{A}^n_k$ as an open of $\mathbb{P}^n_k$ by the usual way:

$$\mathbb{A}^n_k \hookrightarrow \mathbb{P}^n_k, \quad (a_1, \cdots, a_n) \mapsto (1 : a_1 : \cdots : a_n).$$

Let $V \subset \mathbb{A}^n_k$ be a closed subset, and $f : V \to k$ a regular function in the sense of affine algebraic set, then $f = P|_V$ with $P \in k[X_1, \cdots, X_n]$ a polynomial. Now, take $d = \deg(P)$, and $Q = X_0^d P(X_1/X_0, \cdots, X_n/X_0) \in k[X_0, \cdots, X_n]$. Then $Q$ is homogeneous of degree $d$, and we have $f = (Q/X_0^d)|_V$, hence $f$ is a regular function in the sense of projective algebraic set.

Conversely, let $f : V \to k$ a regular function when $V$ is viewed as a quasi-projective set, then for each point $x \in V$, there exist $P_x, Q_x \in k[X_0, \cdots, X_n]$ homogeneous polynomials of the same degree such that $Q_x(x) \neq 0$, and that $f = P_x/Q_x$ in a neighborhood $U_x$ of $x$. Let $P'_x = P_x(1, X_1, \cdots, X_n)$, and $Q'_x = Q_x(1, X_1, \cdots, X_n)$, then $P'_x = f \cdot Q'_x$ in $U_x$. Now, up to multiply on both sides some regular function (in the affine sense) of $V$ which is vanishing on $V - U_x$ but not at the point $x$, we may assume that the equality $P'_x = f \cdot Q'_x$ holds in $V$. Since the family $\{Q'_x : x \in V\}$ has no common zero in $V$, as a result, we must have $(Q'_x : x \in V) = k[V]$. Hence $1 \in k[V]$ can be expressed as the following finite sum:

$$1 = \sum_x g_x \cdot Q'_x$$

with $g_x \in k[V]$. Hence $f = \sum_x g_x \cdot f \cdot Q'_x = \sum_x g_x \cdot P'_x \in k[V]$. This finishes the proof. □

In contrast to the affine case, the ring of regular functions may consist only of constants. For example,

**Example 1.3.2.4.** $k[\mathbb{P}^1_k] = k$. Indeed, recall that, let $x, y$ be the coordinate of $\mathbb{P}^1_k$, then $\mathbb{P}^1_k = U_0 \cup U_1$ with $U_0 = \{(x, y) \colon x \neq 0\}$ and $U_1 = \{(x : y) \colon y \neq 0\}$. Moreover, we can identify $U_0$ with $\mathbb{A}^1_k$:

$$U_0 \simeq \mathbb{A}^1_k, \quad (x : y) \mapsto y/x,$$

and similarly for $U_1$:

$$U_1 \simeq \mathbb{A}^1_k, \quad (x : y) \mapsto x/y.$$

Let now $f \colon \mathbb{P}^1_k \to k$ a regular function, its restriction to $U_0$ is a regular function on $U_0 \simeq \mathbb{A}^1_k$, hence $f|_{U_0} = P_0(y/x)$ with $P_0 \in k[T]$ a polynomial. Similarly, $f|_{U_1} = P_1(x/y)$ with $P_1 \in k[S]$. On the other hand, since $f|_{U_0 \cap U_1} = P_0(y/x)|_{U_0 \cap U_1} = P_1(x/y)|_{U_0 \cap U_1}$. That is, for any $(x : y) \in \mathbb{P}^1_k$ such that $x \cdot y \neq 0$, we have

$$P_0(y/x) = P_1(x/y)$$

But since $P_0$ and $P_1$ are polynomials, the previous equality is possible only when $P_0$ and $P_1$ are constants. Hence we have $f = c$ is a constant function on $\mathbb{P}^1_k$. This finishes the proof.

**Definition 1.3.2.5.** For $V$ a quasi-projective algebraic set. A map

$$f = (f_1, \cdots, f_n) \colon V \to \mathbb{A}^n_k$$

is called *regular* if each component $f_i : V \to k$ is a regular function of $V$.

**Definition 1.3.2.6.** Let $V$ be a quasi-projective algebraic set.

1. A map $f \colon V \to \mathbb{P}^n_k$ is called a *morphism* if for any $x \in V$, there exists some affine piece $\mathbb{A}^n_k$ containing of $f(x) \in \mathbb{P}^n_k$ and a open neighborhood $U$ of $x$ in $V$ such that $f(U) \subset \mathbb{A}^n_k$, and that $f|_U \colon U \to \mathbb{A}^n_k$ is regular.

2. For $W \subset \mathbb{P}^n_k$ a quasi-projective algebraic set. A map $f : V \to W$ is called a *morphism* if the composite map
$$V \to W \hookrightarrow \mathbb{P}^n_k$$
is a morphism in the previous sense.

3. A morphism $f : V \to W$ is called an *isomorphism* if it's bijective, with $f^{-1}$ still a morphism of quasi-projective algebraic set.

**Example 1.3.2.7.** Consider the map $\sigma \colon \mathbb{P}^1_k \to \mathbb{P}^2_k$ given by $(x : y) \mapsto (x^2 : xy : y^2)$. Then this map is a morphism. Indeed, let $P = (x : y) \in \mathbb{P}^1_k$ be a point in $\mathbb{P}^1_k$. Without loss of generality, we may assume that $x \neq 0$. Let $U = \{(x : y) \in \mathbb{P}^1_k \colon x \neq 0\}$, and $V = \{(a : b : c) \colon a \neq 0\}$, and identify them to the affine space in the usual way. Then $\sigma(U) \subset V$. Moreover, we have the following diagram

$$\begin{tikzcd}
\mathbb{P}^1_k \arrow[r] & \mathbb{P}^2_k \\
U \arrow[u, hook] \arrow[r, "\sigma|_U"] & V \arrow[u, hook] \\
\mathbb{A}^1_k \arrow[u, "t\mapsto(1:t)"] \arrow[r] & \mathbb{A}^2_k \arrow[u, "(c,d)\mapsto(1:c:d)"']
\end{tikzcd}$$

Now as this diagram is commutative, the lowest horizontal map is then given by $t \mapsto (t, t^2)$, which is regular since the two components $t$ and $t^2$ are regular functions on $\mathbb{A}^1_k$. This gives the result.

**Exercise 1.3.2.8.** Let $Y$ be the image of morphism of the previous example. Show that $Y \subset \mathbb{P}_k^2$ is a closed subset. Prove that $\mathbb{P}_k^1 \simeq Y$, while their homogeneous coordinate rings are not isomorphic.

Suppose $V \subset \mathbb{P}_k^n = \{(x_0 : \cdots : x_n)\}$ and $W \subset \mathbb{P}_k^m = \{(y_0 : \cdots : y_m)\}$ be two quasi-projective sets. Let $\sigma \colon V \to W$ be a morphism of quasi-projective sets, then $\sigma$ induces a morphism of the ring of regular functions $\sigma^* \colon k[W] \to k[V]$ in the following way: for $f \colon W \to k$ a regular function, then $f \circ \sigma \colon V \to k$.

**Lemma 1.3.2.9.** *The function $f \circ \sigma \colon V \to k$ is regular.*

*Proof.* For any $v \in V$, we write $w = \sigma(v)$. So we need to show that $f \circ \sigma$ is regular at $v$. This is a local question around $v$. Without loss of generality, we suppose that $w$ is contained in the affine piece $\mathbb{A}_k^m = D_+(Y_0) \subset \mathbb{P}_k^m$. Up to replace $V$ by some small neighborhood around $v$, we may assume that $\sigma(V) \subset W \cap D_+(Y_0)$, hence we can write $\sigma = (\sigma_1, \cdots, \sigma_m)$ with $\sigma_i$ regular functions defined on $V$. Up to replace further $V$ by some small open of $v$, we may write $\sigma_i = F_i/G_i$ with $F_i, G_i \in k[X_0, \cdots, X_n]$ homogeneous polynomials of the same degree such that $G_i(x) \neq 0$ for any $x \in V$. The map $\sigma \colon V \to W \subset \mathbb{P}_k^m$ hence can be written as

$$\sigma(x) = (1 : F_1(x)/G_1(x) : \cdots : F_m(x)/G_m(x))$$

Similarly, up to replace $W$ by some small open neighborhood of $w$ (and then replace correspondingly $V$ by some open neighborhood of $v$), we may assume that the regular function $f$ can be written as $P/Q$ with $P, Q \in k[Y_0, \cdots, Y_m]$ two homogeneous polynomials of the same degree such that $Q(y) \neq 0$ for any $y \in W$. As a result,

$$f \circ \sigma = \frac{P(1, F_1/G_1, \cdots, F_m/G_m)}{Q(1, F_1/G_1, \cdots, F_m/G_m)}$$

One verifies that the latter fraction can be written as a quotient $\widetilde{P}/\widetilde{Q}$ with $\widetilde{P}, \widetilde{Q} \in k[X_0, \cdots, X_n]$ be two homogeneous polynomials of the same degree, such that $\widetilde{Q}(v) \neq 0$, which says exactly that $f \circ \sigma$ is regular at $v$. This finishes the proof. $\square$

As a result, a morphism $\sigma \colon V \to W$ induces a morphism between the ring of regular functions

$$\sigma^* \colon k[V] \to k[W].$$

By in general, if we just consider the ring of regular functions $k[V]$ of a quasi-projective algebraic set, we will lost much information about $V$: for example when $V = \mathbb{P}_k^n$, one can show that $k[\mathbb{P}_k^n]$ consists just the constant functions. Because of this, the analogues of 1.3.1.5 and 1.3.1.6 for quasi-projective algebraic sets don't hold in general.

**Exercise 1.3.2.10.** For $X$ a quasi-projective algebraic set, and for any point $x \in X$. Show that there exists a open neighborhood $U$ of $x$ such that $U$ is isomorphic to an affine algebraic set.

**Exercise 1.3.2.11.** Let $V, W$ be two quasi-projective algebraic sets, and $f \colon V \to W$ be a morphism of algebraic sets. Then $f$ is continuous between the underlying topological spaces.

# Chapter 2

# The Language of schemes

## 2.1 Sheaves and locally ringed spaces

### 2.1.1 Sheaves on a topological spaces

Let $X$ be a topological space.

**Definition 2.1.1.1.** A *presheaf* $\mathcal{F}$ of abelian groups (resp. rings) on $X$ consists of the following data:

- for any open subset $U \subset X$, an abelian group $\mathcal{F}(U)$ (resp. a ring $\mathcal{F}(U)$), and
- for any inclusion of open subsets $V \subset U \subset X$, a morphism of groups (resp. of rings)

$$\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$$

subject to the conditions

- $\mathcal{F}(\emptyset) = (0)$;
- $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map for any open $U \subset X$;
- for $W \subset V \subset U \subset X$ three opens of $X$, we have $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$.

**Remark 2.1.1.2.** Very often, for $s \in \mathcal{F}(U)$, and $V \subset U$, the image of $s$ by the map $\rho_{UV}$ is denoted by $s|_V \in \mathcal{F}(V)$.

**Example 2.1.1.3.** Let $X$ be a topological space.

1. For any open $U \subset X$, let $\mathcal{O}_X(U)$ to be the set of real-valued continuous functions defined on $U$. For $V \subset U$, the map $\rho_{UV}$ is defined to be the restriction map. Then $\mathcal{O}_X$ is a presheaf.

2. Let $\mathcal{C}_X$ be the presheaf such that for each open $U \subset X$, $\mathcal{C}_X(U)$ is the set of constant functions defined over $U$, together with the natural restriction. Then this gives us a presheaf.

3. Let $U \subset X$ be a open subset of $X$, and $\mathcal{F}$ be a presheaf on $X$. $\mathcal{F}$ induces in an obvious way, a presheaf $\mathcal{F}|_U$ on $U$ by setting $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for any open subset $V \subset U$. This is the *restriction* of $\mathcal{F}$ to $U$.

**Definition 2.1.1.4.** A presheaf $\mathcal{F}$ on a topological space $X$ is called a *sheaf* if the following two conditions are satisfied:

- For any open $U$ of $X$, and any open covering $U = \bigcup_i U_i$ of $U$. Let $s \in \mathcal{F}(U)$ be a section such that $s|_{U_i} = 0$ for each $i$, then $s = 0$;
- For any open $U$ of $X$, and any open covering $U = \bigcup_i U_i$ of $U$. Suppose that for each $i$, we are given a local section $s_i \in \mathcal{F}(U_i)$ such that these sections verify the following gluing condition: for each $i, j$, we have equality $s_i|_{U_i \cap U_j} = s_j|_{U_j \cap U_i}$. Then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i \in \mathcal{F}(U_i)$.

In a more fancy way, a presheaf $\mathcal{F}$ is a sheaf, if for any open $U$ of $X$, and any open covering $\bigcup_i U_i = U$ of $U$, the following sequence of abelian groups is *exact*:[1]

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{ij} \mathcal{F}(U_i \cap U_j),$$

where the first morphism is $s \mapsto (s|_{U_i})_i$, and the second one is given by

$$(t_i)_i \mapsto (t_i|_{U_i \cap U_j} - t_j|_{U_i \cap U_j})_{ij}.$$

**Example 2.1.1.5.** In the previous example, $\mathcal{O}_X$ is a sheaf, while $\mathcal{C}_X$ is not a sheaf. Moreover, for a sheaf $\mathcal{F}$ on $X$, its restriction $\mathcal{F}|_U$ to an open subset $U \subset X$ is also a sheaf.

**Definition 2.1.1.6.** Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves of groups (resp. of rings) on $X$. A *morphism of presheaves* $f : \mathcal{F} \to \mathcal{G}$ consists of the following data: for each open $U$, a morphism of groups (resp. of rings) $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ such that the following square is commutative: for $V \subset U$,

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "(-)|_V"] \arrow[d, "f(U)"'] & \mathcal{F}(V) \arrow[d, "f(V)"] \\
\mathcal{G}(U) \arrow[r, "(-)|_V"'] & \mathcal{G}(V)
\end{tikzcd}$$

If moreover $\mathcal{F}$ and $\mathcal{G}$ are sheaves, we use the same definition for morphism of sheaves (or presheaves). The set of morphisms of sheaves (or presheaves) of groups from $\mathcal{F}$ to $\mathcal{G}$ will be denoted by $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$.

### $\mathcal{B}$-sheaves

Let $X$ be a topological space, and $\mathcal{B}$ be a basis for the open subsets of $X$.[2] For each open subset $U$ of $X$, let $\mathcal{B}_U$ be the set of elements of $\mathcal{B}$ which is contained in $U$.

**Definition 2.1.1.7.** A $\mathcal{B}$*-presheaf* of abelian groups (resp. rings) on $X$ consists of the following data:

[1] Recall that a sequence of abelian groups

$$\cdots \longrightarrow G_{i-1} \xrightarrow{d_{i-1}} G_i \xrightarrow{d_i} G_{i+1} \longrightarrow \cdots$$

is called a *complex*, if for each $i$, we have $d_i \circ d_{i-1} = 0$. If moreover $\ker(d_i) = \mathrm{im}(d_{i-1})$ for each $i$, then we say that the complex above is *exact*.

[2] This means the following: $\mathcal{B}$ is a set of open subsets of $X$, such that any open subset of $X$ can be written as a union of the open subsets contained in $\mathcal{B}$.

– for any open subset $U \in \mathcal{B}$, an abelian group (resp. a ring) $\mathcal{F}(U)$;

– for two open subsets $V \subset U$ of $X$ contained in $\mathcal{B}$, a morphism of groups (resp. of rings) $\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V)$,

subject to the usual conditions as a presheaf on $X$. A $\mathcal{B}$-presheaf is called a *$\mathcal{B}$-sheaf* if moreover the following condition is verified:

(SC) For any open subset $U \in \mathcal{B}$, any open covering $U = \bigcup_i U_i$ by $U_i \in \mathcal{B}$, the following complex is *exact*

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \xrightarrow{\alpha} \prod_{i,j} \prod_{W \in \mathcal{B}_{U_i \cap U_j}} \mathcal{F}(W)\ , \tag{2.1}$$

where the second morphism $\alpha$ is given by $(s_i)_i \mapsto (s_i|_W - s_j|_W)_{i,j,W}$.

We say that a $\mathcal{B}$-presheaf is *separated* if for any open $U \in \mathcal{B}$ and any open covering $U = \bigcup_{i \in I} U_i$ with $U_i \in \mathcal{B}$, the first map in the sequence (2.1) is injective.

**Remark 2.1.1.8.** Keeping the notations as in the definition, and suppose that the $\mathcal{B}$-presheaf $\mathcal{F}$ is *separated*. Temporarily, we say a family of section $(s_i) \in \prod_i \mathcal{F}(U_i)$ is *compatible* if $(s_i)$ lies in the kernel of $\alpha$. Suppose that, for each $i, j$, we are given an open covering $U_i \bigcap U_j = \bigcup_k W_{ijk}$. Then the family $(s_i) \in \prod_i \mathcal{F}(U_i)$ is compatible if and only if $s_i|_{W_{ijk}} = s_j|_{W_{ijk}}$ for any $k$. Clearly, if $(s_i)$ is compatible, since $W_{ijk} \in \mathcal{B}_{U_i \cap U_j}$, we have $s_i|_{W_{ijk}} = s_j|_{W_{ijk}}$. Conversely, suppose the latter condition is verified, we need to show that for any $W \in \mathcal{B}_{U_i \cap U_j}$, we have $s_i|_W = s_j|_W$. As $U_i \bigcap U_j = \bigcup_k W_{ijk}$, we have $W = \bigcup_l W'_l$ such that $W'_l \in \mathcal{B}$ and that each $W'_l$ is contained in some $W_{ijk}$ for some index $k$. Our hypothesis implies then, for any index $l$,

$$(s_i|_W - s_j|_W)|_{W'_l} = (s_i|_{W_{ijk}} - s_j|_{W_{ijk}})|_{W'_l} = 0$$

where $W_{ijk}$ is such that $W'_l \subset W_{ijk}$. As a result, $s_i|_W = s_j|_W$ since the $\mathcal{B}$-presheaf $\mathcal{F}$ is separated. As a corollary, let $\mathcal{B}'$ be another base of $X$ such that $\mathcal{B}' \subset \mathcal{B}$. Let $\mathcal{F}$ be a $\mathcal{B}$-sheaf. Then the restriction of a $\mathcal{B}$-sheaf to $\mathcal{B}'$ gives a $\mathcal{B}'$-sheaf.

**Lemma 2.1.1.9.** *Let $\mathcal{F}$ be a $\mathcal{B}$-sheaf, then $\mathcal{F}$ can be naturally extended, in a unique way, to a sheaf $\mathcal{G}$ on $X$, such that $\mathcal{F}(U)$ is canonically isomorphic to $\mathcal{G}(U)$ for any $U \in \mathcal{B}$.*

*Proof.* The uniqueness is clear. Let's proceed to the proof of the existence. Let $\mathcal{G}(U)$ be the set of compatible sections $s_V \in \mathcal{F}(V)$ for $V \subset U$ an open in $\mathcal{B}$:

$$\mathcal{G}(U) := \{(s_V)_{V \in \mathcal{B}_U} \colon s_V|_W = s_W \text{ for } W, V \in \mathcal{B}_U \text{ s.t. } W \subset V\} \subset \prod_{V \in \mathcal{B}_U} \mathcal{F}(V),$$

or equivalently, $\mathcal{G}(U) := \varprojlim_{V \in \mathcal{B}_U} \mathcal{F}(V)$. Note that if $U \in \mathcal{B}$, the natural projection $\mathcal{G}(U) \to \mathcal{F}(U)$ is an isomorphism. Let now $U' \subset U$ be two open subsets of $X$, then $\mathcal{B}_{U'} \subset \mathcal{B}_U$. The natural projection induces then a morphism of groups $\mathcal{G}(U) \to \mathcal{G}(U')$. In this way, we get a presheaf $\mathcal{G}$ on $X$ which extends the $\mathcal{B}$-presheaf $\mathcal{F}$.

On the other hand, for another basis $\mathcal{B}'$ of the topology such that $\mathcal{B}' \subset \mathcal{B}$, by the previous remark, the restriction of $\mathcal{F}$ to $\mathcal{B}'$ gives a $\mathcal{B}'$-sheaf on $X$, denoted by $\mathcal{F}'$. Let $\mathcal{G}'$ be the presheaf on $X$ defined from the $\mathcal{B}'$-sheaf $\mathcal{F}$. Since $\mathcal{B}' \subset \mathcal{B}$, the projection induces then a morphism of presheaves $\iota \colon \mathcal{G} \to \mathcal{G}'$. We claim that this latter map is an isomorphism. We first prove that this map is injective. Indeed, let $s = (s_V)_{V \in \mathcal{B}_U} \in \mathcal{G}(U)$ an arbitrary section such that $s_W = 0$ for any $W \in \mathcal{B}'_U$. Since $\mathcal{B}'$ is a basis of $X$, for any $V \in \mathcal{B}_U$, there exists an open covering $V = \bigcup_i V_i$ of

$V$ with $V_i \in \mathcal{B}'$. Now, $s_V|_{V_i} = s_{V_i} = 0$ (as $V_i \in \mathcal{B}'_U$). Now since $\mathcal{F}$ is a $\mathcal{B}$-sheaf, and $V, V_i \in \mathcal{B}$, we get $s_V = 0$, and hence the injectivity. For the surjectivity, let $s' = (s'_W)_{W \in \mathcal{B}'_U} \in \mathcal{G}'(U)$, we need to construct in a compatible way sections $(s_V)_{V \in \mathcal{B}_U}$. Let $V \in \mathcal{B}_U$, then one can find a covering $V = \bigcup_i W_i$ with $W_i \in \mathcal{B}'_U$. For each $W_i$, we have a section $s'_{W_i} \in \mathcal{F}(W_i)$, and these sections are compatible (see the previous remark). As a result, these sections can be glued in a unique way to a section $s_V \in \mathcal{F}(V)$. Moreover, these resulting sections $(s_V)_{V \in \mathcal{B}_U}$ are also compatible, hence give a section $s \in \mathcal{G}(U)$. In this way, we see that the map $\mathcal{G}(U) \to \mathcal{G}'(U)$ is surjective. This proves the claim.

Now to finish the proof of this lemma, one needs to verify the sheaf condition for $\mathcal{G}$. Let $U$ be an open of $X$ with $\bigcup_i U_i$ an open covering of $U$. Up to replace $X$ by $U$, we may suppose that $U = X$. Let $\mathcal{B}' \subset \mathcal{B}$ be the set of elements of $\mathcal{B}$ which is contained in $U_i$ for some $i$. Then $\mathcal{B}'$ is still a basis for the topology. Since the corresponding $\mathcal{B}'$-sheaf extends to the same presheaf as the $\mathcal{B}$-sheaf $\mathcal{F}$, we may also assume $\mathcal{B} = \mathcal{B}'$. In this case, we need to show that the following sequence

$$0 \to \varprojlim_{V \in \mathcal{B}} \mathcal{F}(V) \to \prod_i \varprojlim_{V \in \mathcal{B}_{U_i}} \mathcal{F}(V) \to \prod_{i,j} \varprojlim_{W \in \mathcal{B}_{U_i \cap U_j}} \mathcal{F}(W)$$

By the assumption on $\mathcal{B}$, the first map is injective. To show the exactness in the middle, let $t = (t_{i,V})_{i, V \in \mathcal{B}_{U_i}}$ an element of the middle, then $t$ comes from an element of $\mathcal{G}(X)$ iff for $V \in \mathcal{B}_{U_i} \cap \mathcal{B}_{U_j} = \mathcal{B}_{U_i \cap U_j}$, we have $t_{i,V} = t_{j,V}$, which means exactly that $t$ lies in the kernel of the second map. This gives the exactness at the middle. $\square$

### Stalks

**Definition 2.1.1.10** (Stalk)**.** Let $\mathcal{F}$ be a sheaf on $X$ (or a presheaf on $X$). Let $x \in X$ be a point of $X$. We consider the family $\mathcal{S}$ of the pairs $(U, s)$ where $U \subset X$ is a open neighborhood of $x$, and $s \in \mathcal{F}(U)$. We define the following equivalent relation $\sim$ on $\mathcal{S}$: for $(U, s)$ and $(V, t) \in \mathcal{S}$, we say $(U, s) \sim (V, t)$ if one can find a third open neighborhood $W$ of $x$ such that $W \subset U \cap V$ and that $s|_W = t|_W$. We denote by $\mathcal{F}_x$ the set of equivalent classes $\mathcal{S}/\sim$. It is called the *stalk* of the sheaf $\mathcal{F}$ at $x$.

In a more fancy way, the definition of stalk can be expressed as the following direct limits

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U).$$

**Remark 2.1.1.11.** Let $X$ be a topological space, with $\mathcal{B}$ a basis for the topology. Let $\mathcal{F}$ be a $\mathcal{B}$-sheaf which extends to the sheaf $\mathcal{G}$ on $X$. For $x \in X$, we have

$$\mathcal{G}_x \simeq \mathcal{F}_x := \varinjlim_{x \in U \in \mathcal{B}} \mathcal{F}(U).$$

### Sheaf associated with a presheaf

**Proposition 2.1.1.12.** *For any presheaf $\mathcal{F}$ of groups, there is a canonically associated sheaf of groups $\mathcal{F}^a$ on $X$ together with a morphism $\iota\colon \mathcal{F} \to \mathcal{F}^a$ such that for any* sheaf *$\mathcal{G}$ on $X$, the following map*

$$\mathrm{Hom}(\mathcal{F}^a, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}), \quad f \mapsto f \circ \iota$$

*is bijective. The pair $(\mathcal{F}^a, \iota)$, or simply $\mathcal{F}^a$ is called* the associated sheaf of $\mathcal{F}$*, it is unique up to isomorphisms. Moreover, the canonical map $\iota$ induces an isomorphism between the stalks $\mathcal{F}_x \simeq \mathcal{F}^a_x$ for any $x \in X$.*

*Proof.* (Sketch) For any open $U$ of $X$, we consider $\mathcal{F}^a(U)$ the set of applications $s\colon U \to \coprod_{x\in U}\mathcal{F}_x$ such that

– $s(x) \in \mathcal{F}_x$ for any $x \in U$;

– For any $x_0 \in U$, there exists some open $V$ of $x_0$ contained in $U$ and some element $t \in \mathcal{F}(V)$ such that $s(x) = t_x \in \mathcal{F}_x$.

The collection $\{\mathcal{F}^a(U)\}$ together with the usual restriction maps give a sheaf $\mathcal{F}^a$ on $X$. Moreover, there exists a canonical map $\iota\colon \mathcal{F} \to \mathcal{F}^a$, and the couple $(\mathcal{F}^a, \iota)$ solves the universal problem stated in the proposition. Indeed, let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves with $\mathcal{G}$ a sheaf on $X$. For any element $s \in \mathcal{F}^a(U)$, let $U = \bigcup V_i$ be an open covering of $U$, and $t_i \in \mathcal{F}(V_i)$ such that $s|_{V_i} = \iota(t_i)$. We take $t'_i = \phi(t_i) \in \mathcal{G}(V_i)$, and claim that there exists some $t' \in \mathcal{G}(U)$ such that $t'|_{V_i} = t'_i$. For this, we need to verify that $t'_i|_{V_i\cap V_j} = t'_j|_{V_i\cap V_j}$. As $t_{i,x} = t_{j,x}$ for any $x \in V_i \cap V_j$, we have equally $t'_{i,x} = t'_{j,x}$ for any $x \in V_i \cap V_j$, which implies then $t'_i|_{V_i\cap V_j} = t'_j|_{V_i\cap V_j}$ as $\mathcal{G}$ is a sheaf on $X$. Moreover, it's easy to show that the element $t$ is independent of the choice of the covering $U = \bigcup_i V_i$ and the sections $t_i \in \mathcal{F}(V_i)$. The association $s \mapsto t$ gives then a morphism of abelian sheaves $\phi^a\colon \mathcal{F}^a \to \mathcal{G}$ such that $\phi^a \circ \iota = \phi$. □

Now, we look at some properties related to exact sequences.

**Definition 2.1.1.13.** Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

1. We define the kernel of $f$ to be the presheaf $\ker(f)$ given by $\ker(f)(U) = \ker(f(U) : \mathcal{F}(U) \to \mathcal{G}(U))$ for any open $U \subset X$. One verifies that $\ker(f)$ is indeed a sheaf.
2. The cokernel of $f$ is the sheaf $\mathrm{coker}(f)$ associated with the presheaf on $X$ given by $U \mapsto \mathrm{coker}(f(U)\colon \mathcal{F} \to \mathcal{G}(U))$.
3. $f$ is called a *monomorphism* (resp. an *epimorphism*) if $\ker(f) = 0$ (resp. $\mathrm{coker}(f) = 0$).

**Proposition 2.1.1.14.** *Let $\mathcal{F}, \mathcal{G}$ be two sheaves, and $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.*

1. *$\mathcal{F} = 0$ if and only if $\mathcal{F}_x = 0$ for any $x \in X$.*
2. *$f$ is a monomorphism (resp. an epimorphism, resp. an isomorphism) if and only if $f_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is a monomorphism (resp. an epimorphism, resp. an isomorphism) for any $x \in X$.*

**Some functorialities**

Let $f\colon X \to Y$ be a continuous map of topological spaces. In particular, for any open $V \subset Y$, the preimage $f^{-1}(V) \subset X$ is again open. We begin with the direct image. Let $\mathcal{F}$ be a sheaf on $X$, we define $f_*\mathcal{F}$ to be the presheaf on $Y$ given by

$$V \subset Y \text{ open } \mapsto \mathcal{F}(f^{-1}(V)).$$

This gives indeed a presheaf on $Y$.

**Lemma 2.1.1.15.** *$f_*\mathcal{F}$ is a sheaf. We call it the* direct image of $\mathcal{F}$ by the morphism $f$*.*

Let $\mathcal{G}$ be the sheaf on $Y$. We consider the presheaf $\mathcal{F}$ on $X$ given by

$$U \subset X \text{ open } \mapsto \varinjlim_{V\subset Y \text{ open, s.t. } f(U)\subset V} \mathcal{G}(V)$$

**Lemma 2.1.1.16.** *$\mathcal{F}$ is indeed a presheaf on $X$.*

**Definition 2.1.1.17.** We define the inverse image of $\mathcal{G}$ by $f$ the associated sheaf of the presheaf $\mathcal{F}$, denoted by $f^{-1}\mathcal{G}$.

**Proposition 2.1.1.18.** *Let $\mathcal{F}$ be a sheaf on $X$, and $\mathcal{G}$ be a sheaf on $Y$. There is a canonical bijection of sets*

$$\mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}).$$

**Example 2.1.1.19.** Let $X$ be a topological space, $\mathcal{F}$ be a presheaf on $X$.

1. Let $U \subset X$ be an open subset, et we note by $j\colon U \to X$ the inclusion map. Then $j^{-1}\mathcal{F}$ is just the restriction $\mathcal{F}|_U$ of $\mathcal{F}$ to $U$.

2. Let $x \in X$ be a point, et $i\colon \{x\} \to X$ be the inclusion map. Then $i^{-1}\mathcal{F} \simeq \mathcal{F}_x$.

3. Consider two continuous maps $Z \xrightarrow{g} Y \xrightarrow{f} X$ , then there exists a canonical isomorphism $g^{-1}(f^{-1}\mathcal{F}) \simeq (f \circ g)^{-1}\mathcal{F}$. In particular, we have $(f^{-1}\mathcal{G})_x \simeq \mathcal{G}_{f(x)}$.

### 2.1.2 Ringed space

A *ringed space* is a pair $(X, \mathcal{O}_X)$ consists of

– a topological space $X$; and

– a sheaf of rings $\mathcal{O}_X$ on $X$.

When there is no confusion possible, we will omit $\mathcal{O}_X$ from the notation. A ringed space is called *local* if for any $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.[3] For a locally ringed space $(X, \mathcal{O}_X)$, the quotient $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is called the *residuel field* of $X$ at $x$. A morphism $f = (f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces is the given of

- a continuous map $f : X \to Y$; and
- a morphism of sheaves of rings $f^\sharp\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ (or equivalently a morphism of sheaves of rings $\mathcal{O}_Y \to f_*\mathcal{O}_X$).

In particular, for any $x \in X$, the morphism $(f, f^\sharp)$ induces a morphism of the stalks $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,f(x)}$. If moreover $X, Y$ are both locally ringed spaces, a morphism $(f, f^\sharp)\colon X \to Y$ is called a morphism of locally ringed spaces if for any $x \in X$, the induced map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,f(x)}$ is a local morphism.[4]

**Example 2.1.2.1.** Let $X$ be a topological space, and $\mathcal{O}_X$ be the sheaf of continuous real-valued functions on $X$. The pair $(X, \mathcal{O}_X)$ gives a ringed space. Actually, this is a locally ringed space.

**Example 2.1.2.2.** Let $X$ be a topological space, and $\mathcal{O}'_X$ be the sheaf of *not necessarily continuous functions* on $X$. Then $(X, \mathcal{O}'_X)$ is also a ringed space, which is not local in general.

[3] Recall that a local ring is a ring with only one maximal ideal.

[4] A morphism $\phi\colon A \to B$ of local rings is called *local* if $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$ with $\mathfrak{m} \subset A$ (resp. $\mathfrak{n} \subset B$) be the maximal ideal of $A$ (resp. of $B$).

**Example 2.1.2.3.** Let $X, Y$ be two topological spaces with $\mathcal{O}_X$ and $\mathcal{O}_Y$ be the corresponding sheaves of continuous real-valued functions. Let $f\colon X \to Y$ be a continuous map. It induces the a morphism of ringed spaces

$$f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y).$$

One can show that this morphism is local.

**Example 2.1.2.4.** Let $(X, \mathcal{O}_X)$ be a ringed space (resp. locally ringed space), and $U \subset X$ be an open subset. Then $(U, \mathcal{O}_X|_U)$ is a ringed space (resp. locally ringed space). Let $j\colon U \hookrightarrow X$ be the inclusion map, together with the adjunction map $j^\sharp\colon \mathcal{O}_X \to j_*\mathcal{O}_V$, we get a morphism of ringed spaces

$$j = (j, j^\sharp)\colon (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$$

Such a morphism is called an *open immersion* of ringed spaces.

**Locally ringed space structure on an algebraic set**

Let $V$ be a quasi-projective algebraic set. For any open subset $U \subset V$, it is again quasi-projective. Hence we can consider the regular functions defined over $U$. In this way, we get the sheaf of regular functions $\mathcal{O}_V$ on $V$, and in this way, we get a ringed space $(V, \mathcal{O}_V)$. Moreover, let $\phi\colon V \to W$ be a morphism of algebraic sets, it induces then a morphism of the ringed spaces

$$\phi = (\phi, \phi^\sharp)\colon (V, \mathcal{O}_V) \to (W, \mathcal{O}_W).$$

**Proposition 2.1.2.5.** *Keeping the notations as before. Then the ringed space $(V, \mathcal{O}_V)$ is local, and the morphism $\phi$ above is also local.*

**Exercise 2.1.2.6.** Use the fact that a regular function $f\colon U \to k$ defines a continuous map $f\colon U \to \mathbb{A}^1_k$ to gives a proof of the previous proposition.

**Proposition 2.1.2.7.** *Let $V \subset \mathbb{A}^n_k$ be an affine algebraic set.*

1. *For each $f \in k[V]$, we have $\Gamma(D(f), \mathcal{O}_V) \simeq k[V]_f$.*

2. *For each point $x \in V$ corresponding to the maximal ideal $\mathfrak{m} \subset k[V]$, then we have $\mathcal{O}_{V,x} \simeq k[V]_{\mathfrak{m}}$.*

*In particular, this implies also that $(V, \mathcal{O}_V)$ is a locally ringed space.*

*Proof.* (1) By definition, $\Gamma(D(f), \mathcal{O}_V)$ is the set of regular functions on $D(f)$. Hence, we have a natural map

$$k[V] \to \Gamma(D(f), \mathcal{O}_V)$$

which sends $f$ to an invertible element of $\Gamma(D(f), \mathcal{O}_V)$, hence the previous map factors through the localization $k[V]_f$, and we get in this way a canonical map $\theta_f\colon k[V]_f \to \Gamma(D(f), \mathcal{O}_V)$. It's easy to see that this map is injective, and we only need to prove that it's also surjective. Let $a\colon D(f) \to k$ be a regular function, then for each $x \in D(f)$, there exist some open neighborhood $U_x$ of $x$ contained in $D(f)$, and two polynomials $P_x, Q_x \in k[X_1, \cdots, X_n]$ such that $Q_x(x) \neq 0$, and that $a = P_x/Q_x$ on $U_x$. In particular $P_x = a \cdot Q_x$ on $U_x$. Up to multiply both sides by some element in $k[V]$ which vanishes on $V - U_x$ but not at $x$, then we may assume that the equality holds in $V$. Now if we look at the ideal $I$ generated by the family $\{Q_x \colon x \in D(f)\}$, then $V(I) \cap D(f) = \emptyset$, hence $V(I) \subset V(f)$. As a result, $\sqrt{(f)} \subset \sqrt{I}$, hence there exists some $r > 0$ such that $f^r \in I$, so $f^r$ can be expressed as a finite sum

$$f^r = \sum_{x \in D(f)} R_x \cdot Q_x.$$

In particular, we have $a \cdot f^r = \sum_{x \in D(f)} R_x \cdot P_x \in k[V]$. Hence $a = \theta(a \cdot f^r / f^r)$ lies in the image of $\theta$. This gives the surjectivity of $\theta$, and hence the proof of (1).

(2) Let $D(f) \subset D(g) \subset V$ be two opens of $V$, we have $V(g) \subset V(f)$. Hence $\sqrt{(f)} \subset \sqrt{(g)}$. One can then find some integer $s > 0$ and $u \in k[V]$ such that $f^s = ug$. In particular, the canonical map $k[V] \to k[V]_f$ factors through $k[V]_g$, and we get in this way a map

$$\alpha_{f,g} \colon k[V]_g \to k[V]_f.$$

Moreover, by checking the definition, the following square is commutative

$$\begin{tikzcd} k[V]_g \arrow[r, "\alpha_{f,g}"] \arrow[d, "\theta_g"'] & k[V]_f \arrow[d, "\theta_f"] \\ \Gamma(D(g), \mathcal{O}_V) \arrow[r] & \Gamma(D(f), \mathcal{O}_V) \end{tikzcd} .$$

Now, the condition that $x \in D(f)$ is equivalent to say that $f \in k[V] - \mathfrak{m}$, hence from the previous commutative diagram, the morphisms $\theta_f$'s (for $f \in k[V] - \mathfrak{m}$) pass to the direct limits, and we get a canonical isomorphism

$$\theta = \varinjlim_{f \in k[V] - \mathfrak{m}} \theta_f \colon k[V]_{\mathfrak{m}} = \varinjlim_{f \in k[V] - \mathfrak{m}} k[V]_f \simeq \varinjlim_{f \in k[V] - \mathfrak{m}} \Gamma(D(f), \mathcal{O}_V) = \mathcal{O}_{V,x}$$

This proves (2). $\square$

**Remark 2.1.2.8.** For a projective algebraic set $V \subset \mathbb{P}^n_k$, let $A = k[X_0, \cdots, X_n]/I(V)$ be its homogeneous coordinate ring, with the natural gradation. Then (we refer to later discussion about the precise definition of *homogeneous localization* of a graded ring)

1. For any $f \in A$, $\mathcal{O}_V(D_+(f)) \simeq A_{(f)}$, where $A_{(f)}$ is the corresponding homogeneous localization.

2. For a point $x \in V$, which corresponds to a homogeneous prime ideal $\mathfrak{p} \subset A$. Then $\mathcal{O}_{V,x} \simeq A_{(\mathfrak{p})}$, where $A_{(\mathfrak{p})}$ is the corresponding homogeneous localization.

## 2.2 Schemes

### 2.2.1 Definition of schemes

Let $A$ be a ring, we define the *prime spectrum* of $A$ to be

$$\mathrm{Spec}(A) := \{\mathfrak{p} \subset A \text{ prime ideals}\}$$

An element $\mathfrak{p} \in \mathrm{Spec}(A)$ is called a *point* of $\mathrm{Spec}(A)$. For $\mathfrak{a} \subset A$ an ideal, we put

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(A) \colon \mathfrak{a} \subset \mathfrak{p}\}.$$

**Lemma 2.2.1.1.** *Let $A$ be a ring, and $\mathfrak{a}, \mathfrak{b} \subset A$ be two ideals. The following two things are true.*

1. $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$.

2. $V(\mathfrak{a}) \subset V(\mathfrak{b})$ *iff* $\mathfrak{b} \subset \sqrt{\mathfrak{a}}$. *In particular,* $V(\mathfrak{a}) = V(\mathfrak{b})$ *iff* $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.

*Proof.* (1) Since a prime ideal $\mathfrak{p}$ is radical, we have $\sqrt{\mathfrak{a}} \in \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$. Conversely, up to replace $A$ by $A/\mathfrak{a}$, we may assume $\mathfrak{a} = 0$. We then need to show that $\bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}$ consists of nilpotent elements. Let $f \in \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}$. If $f$ is not nilpotent, the localization $A_f$ is non trivial, hence it has a prime ideal $\mathfrak{q}_0 \subset A_f$, which corresponds then a prime ideal $\mathfrak{p}_0$ of $A$ such that $f \notin \mathfrak{p}_0$, a contradiction. Hence $\sqrt{(0)} = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}$. (2) If $V(\mathfrak{a}) \subset V(\mathfrak{b})$, then $\sqrt{\mathfrak{a}} = \cap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} \supset \cap_{\mathfrak{q} \in V(\mathfrak{b})} \mathfrak{q} = \sqrt{\mathfrak{q}}$. In particular, $\mathfrak{b} \subset \sqrt{\mathfrak{a}}$. Conversely, suppose $\mathfrak{b} \subset \mathfrak{a}$. Let $\mathfrak{p} \in V(\mathfrak{a})$, then $\mathfrak{b} \subset \sqrt{\mathfrak{a}} \subset \sqrt{\mathfrak{p}} = \mathfrak{p}$, that is $\mathfrak{p} \in V(\mathfrak{b})$. This proves (2).

□

**Lemma 2.2.1.2.** *We have (i)* $V(A) = \emptyset$, $V((0)) = \mathrm{Spec}(A)$; *(ii)* $\cap_i V(\mathfrak{a}_i) = V(\sum_i \mathfrak{a}_i)$; *and (iii)* $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b})$.

**Definition 2.2.1.3.** The topology on $\mathrm{Spec}(A)$ in which the closed subsets are given by the subsets of the form $V(\mathfrak{a})$ for some ideal of $A$ is called the *Zariski topology* on $\mathrm{Spec}(A)$.

**Exercise 2.2.1.4.** Let $X = \mathrm{Spec}(A)$.

1. The topological space $X$ is $T_0$.

2. The space $X$ is quasi-compact, namely any open covering of $X$ admits a finite sub-covering.

3. If moreover $A$ is a noetherian ring, then $X$ is a noetherian topological space.

**Proposition 2.2.1.5.** *Let* $X = \mathrm{Spec}(A)$. *The family of opens* $\{D(f) \colon f \in A\}$ *forms a basis for the Zariski topology on* $X$. *Moreover, for* $f, g \in A,$, *then* $D(f) \subset D(g)$ *iff* $f^r \in (g)$ *for some* $r \in \mathbb{Z}_{\geq 1}$.

*Proof.* Let $U = X - V(\mathfrak{a}) \subset X$ be an open subset of $X$. Then we have $V(\mathfrak{a}) = \cap_f V(f)$, hence $U = \cup_f D(f)$. The last assertion follows from Lemma 2.2.1.1. □

**Example 2.2.1.6.** For $k = \overline{k}$ an algebraically closed field. $\mathrm{Spec}(k[X_1, \cdots, X_n])$ is the in one-to-one correspondence with the irreducible closed subsets of $\mathbb{A}^n_k(k)$.[5] Moreover, if we endow $\mathbb{A}^n_k(k)$ with the Zariski topology as we did in the first chapter, the natural map $\mathbb{A}^n_k(k) \to \mathrm{Spec}(k[X_1, \cdots, X_n])$ is then continuous.

**Example 2.2.1.7.** $\mathrm{Spec}(\mathbb{Z}) = \{(p) \colon p \text{ prime}\} \cup \{(0)\}$, where $(0) \in \mathrm{Spec}(\mathbb{Z})$ is the only non closed point of $\mathrm{Spec}(\mathbb{Z})$.

**Remark 2.2.1.8.**

1. As we see in the previous example, in contrast to the algebraic sets, not all the points in $\mathrm{Spec}(A)$ is closed. In fact, let $x \in \mathrm{Spec}(A)$ be a point which corresponds to the prime ideal $\mathfrak{p} \subset A$, then $\overline{\{x\}} = V(\mathfrak{p})$. Hence $x$ is a closed point, iff $\overline{\{x\}} = \{x\}$, or equivalently, iff $\mathfrak{p} \subset A$ is a maximal ideal.

2. The Zariski topology on $\mathrm{Spec}(A)$ is a very "nonclassical" topology, in the sense that it's non-Hausdorff. In fact, as in the case of algebraic sets, we have seen such things. But here, the Zariski topology on $\mathrm{Spec}(A)$ is even "less Hausdorff" since we have included all prime ideal, hence non closed points.

[5] In this first chapter, $\mathbb{A}^n_k(k)$ is denoted by $\mathbb{A}^n_k$ for simplicity.

**Structure sheaf on** Spec$(A)$

Let $X = \mathrm{Spec}(A)$, for $f \in A$, let $D(f) = X - V((f))$.

**Lemma 2.2.1.9.** *The set of principal opens $\mathcal{B} = \{D(f) \colon f \in A\}$ gives an base of the open subsets of $X$.*

Hence, to define a sheaf of rings $\mathcal{O}_X$ on $X$, we only need to define a $\mathcal{B}$-sheaf on $X$. We will begin with some generalities on the localizations of rings. Let $D(f) \subset D(g)$, or equivalently, $f^r = g \cdot a$ for $r \in \mathbb{Z}_{\geq 1}$ and $a \in A$, the element $g \in A_f$ is invertible with inverse given by $1/g = a/ga = a/f^r \in A_f$. By the universal property of localization, we obtain the following commutative diagram of rings

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_f \\ A_g \arrow[ur, dotted, "\exists\ \alpha_{g,f}"'] & \end{tikzcd}$$

The isomorphisms $\alpha_{g,f}$'s satisfy also the transitive conditions: for $D(f) \subset D(g) \subset D(h)$, then (i) $\alpha_{h,f} = \alpha_{g,f} \circ \alpha_{h,g}$; and (ii) $\alpha_{f,f} = \mathrm{id}_{A_f}$. In particular, when $D(f) = D(g)$, the canonical map $\alpha_{gf} \colon A_g \to A_f$ is an isomorphism. In the following, we will use this isomorphism to identify $A_f$ and $A_g$.

Now, for each principal open subset $U \in \mathcal{B}$, we take $f_U \in A$ such that $U = D(f_U)$. We define then

$$\mathcal{O}'_X(U) = A_{f_U},[6]$$

and for $V \subset U$ two principal open subsets, we define

$$\rho_{UV} = \alpha_{f_U, f_V} \colon \mathcal{O}'_X(U) = A_{f_U} \to A_{f_V} = \mathcal{O}'_X(V).$$

The transitivity property of $\alpha_{g,f}$ implies that the family $\{\mathcal{O}'_X(U), \rho_{UV}\}_{U,V \in \mathcal{B}}$ gives a $\mathcal{B}$-presheaf of rings.

**Lemma 2.2.1.10.** *The $\mathfrak{B}$-presheaf $\mathcal{O}'_X$ is a $\mathcal{B}$-sheaf.*

*Proof.* Up to replace $X$ by $D(f)$, we may assume that $X = D(f)$. Suppose $X = \cup_i D(f_i)$, we are reduced to show that the following sequence

$$0 \longrightarrow A \xrightarrow{\text{can}} \prod_i A_{f_i} \longrightarrow \prod_{i,j} A_{f_i f_j} \tag{2.2}$$

is exact, where the second map is given by $(a_i)_i \mapsto (a_i - a_j \in A_{f_i f_j})_{i,j}$.

For an element $a \in A$ in the kernel of the first map, *i.e.*, $a = 0 \in A_{f_i}$ for each $i$, one can find some power $f_i^{r_i} \in A$ such that $f_i^{r_i} \cdot a = 0$ for each $i$. On the other hand, since $X = \bigcup_i D(f_i) = \bigcup_i D(f_i^{r_i})$, we can find $g_i \in A$ for each $i$ such that

$$1 = \sum_i g_i \cdot f_i^{r_i},$$

[6] If we like, for $U \subset \mathrm{Spec}(A)$ a principal open subset, we can define

$$\mathcal{O}'(U) = \left\{ (s_f)_f \in \prod_{f \in A, D(f) = U} A_f : \alpha_{g,f}(s_g) = s_f, \text{ for any } f, g \in A \text{ such that } D(f) = D(g) = U \right\}$$

from where we find $a = \sum_i g_i \cdot f_i^{r_i} \cdot a = 0$. This shows the injectivity.

To show the exactness at the middle term, we can assume that the covering $X = \bigcup D(f_i)$ is finite. Let $(a_i)_i \in \prod_i A_{f_i}$ a family of elements in the kernel of the second map of the sequence (2.2), where $a_i = g_i/f_i^{e_i} \in A_{f_i}$ with $g_i \in A$. Up to replace $f_i$ by $f_i^{e_i}$, we may assume that $e_i = 1$ for all $i$. Moreover, since $a_i = a_j$ in $A_{f_i f_j}$ for all $i, j$, we have

$$(f_i f_j)^{r_{ij}}(g_i f_j - g_j f_i) = 0$$

for some integer $r_{ij}$. We may equally assume that $r_{ij} = r \gg 0$ for any $i, j$. Since the covering $X = \bigcup D(f_i)$ is *finite*, up to replace $r_{ij}$ by a bigger one, we may suppose that $r_{ij} = r$ is independent of $i, j$. The previous equality gives then

$$g_i f_j^{r+1} f_i^r = g_j f_i^{r+1} f_j^r, \quad \forall i, j.$$

Again, as $X = \bigcup_i D(f_i) = \bigcup_i D(f_i^{r+1})$, there exist $h_i \in A$ such that $1 = \sum_i h_i \cdot f_i^{r+1}$. We take then $a = \sum_i h_i g_i f_i^r$, and we find

$$a f_j^{r+1} = \sum_i h_i g_i f_i^r f_j^{r+1} = \sum_i h_i g_j f_i^{r+1} f_j^r = g_j f_j^r$$

which means $a = g_j/f_j$ in $A_{f_j}$ for each $j$. Hence the element $(a_i)$ lies in the image of the first map of (2.2). This finishes the proof. □

**Definition 2.2.1.11.** We call the *structural sheaf* on $X = \operatorname{Spec}(A)$ the sheaf $\mathcal{O}_X$ on $X$ extending the $\mathcal{B}$-sheaf $\mathcal{O}'_X$.

**Proposition 2.2.1.12.** *Let $X = \operatorname{Spec}(A)$ with $\mathcal{O}_X$ its structural sheaf.*

1. *$\mathcal{O}_X(D(f)) \simeq A_f$;*
2. *$\mathcal{O}_{X,x} \simeq A_{\mathfrak{p}_x}$ with $x \in X$ a point which corresponds to the prime ideal $\mathfrak{p}_x$ of $A$.*

In particular, $(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)})$ gives a local ringed space.

**Remark 2.2.1.13.** Let $M$ be an $A$-module, the previous construction allows us also to associate a sheaf on $X$ to $M$: we first define a $\mathcal{B}$-presheaf $\widetilde{M}'$ on $X$: for $U = D(f) \in \mathcal{B}$,

$$\widetilde{M}'(D(f)) := M_f$$

As before, we can show that this gives indeed a $\mathcal{B}$-sheaf on $X$, we can then consider the sheaf $\widetilde{M}$ on $X$ extending $\widetilde{M}'$, and $\widetilde{M}$ is a *sheaf of $\mathcal{O}_X$-modules* (or just an *$\mathcal{O}_X$-module* for short). An $\mathcal{O}_X$-module $\mathcal{F}$ on $X$ is called *quasi-coherent*, if it's isomorphic to $\widetilde{M}$ for some $A$-module $M$.

Now we are in the position to give the definition of schemes.

**Definition 2.2.1.14.**
1. A ringed space $(X, \mathcal{O}_X)$ is called an *affine scheme* is it's isomorphic to some $(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)})$ constructed as above. By abuse of notation, the latter will often be denoted simply by $\operatorname{Spec}(A)$.
2. A ringed space $(X, \mathcal{O}_X)$ is called a *scheme*, if $X$ admits an open covering $X = \bigcup_i U_i$ such that for each $i$, the open sub-ringed space $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme. Very often, a scheme $(X, \mathcal{O}_X)$ will be denoted by $X$ when no confusion is possible.

**Remark 2.2.1.15.** In particular, a scheme is always a locally ringed space, and we will see later that an open subset of a scheme with the natural sub-ringed space structure is also a scheme.

### 2.2.2 Morphisms of schemes

**Definition 2.2.2.1.** Let $X, Y$ be two schemes, then a morphism of *locally* ringed spaces

$$(f, f^\sharp)\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

is called a *morphism of schemes*. When no confusion is possible, we will simply denote the latter by $f\colon Y \to X$. We will use the notation $\mathrm{Mor}(Y, X)$ to denote the set of morphisms from $Y$ to $X$.

We first look at the affine cases. Let $\phi\colon A \to B$ be a morphism of rings. Then for any prime ideal $\mathfrak{p} \subset B$, its inverse image $\phi^{-1}(\mathfrak{p}) \subset A$ is a prime ideal of $A$. In particular, $\phi$ induces a map of sets

$$f_\phi\colon Y = \mathrm{Spec}(B) \to X = \mathrm{Spec}(A), \quad \mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$$

called the *associated map* of $\phi$. Moreover, this map is continuous with respect to the Zariski topology on both sides, and for each $a \in A$, we have $f_\phi^{-1}(D(a)) = D(\phi(a))$. On the other hand, for each $a$, we have the following canonical map $A_a \to B_{\phi(a)}$, from which we get a compatible family of morphisms of rings $\{\mathcal{O}_X(D(a)) \to \mathcal{O}_Y(f^{-1}(D(a)))\}$. By the sheaf properties, this family extends uniquely to a morphism of sheaves of rings $f_\phi^\sharp : \mathcal{O}_X \to f_{\phi,*}\mathcal{O}_Y$. In this way, we get a morphism of ringed spaces:

$$(f_\phi, f_{\phi,\phi}^\sharp)\colon Y = (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to X = (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$$

**Lemma 2.2.2.2.** *The morphism above is local.*

*Proof.* By construction, the following diagram is commutative

$$\begin{tikzcd}
\mathcal{O}_{X,f_\phi(y)} \arrow[r, "f_{\phi,y}"] & \mathcal{O}_{Y,y} \\
\mathcal{O}_X(X) \arrow[r, "f_\phi(X)"] \arrow[u] & \mathcal{O}_Y(Y) \arrow[u] \\
A \arrow[r, "\phi"] \arrow[u, equal] & B \arrow[u, equal]
\end{tikzcd}.$$

Let $\mathfrak{p} \subset B$ be the prime ideal corresponding to $y$, then after identifying $\mathcal{O}_{Y,y}$ with $B_\mathfrak{p}$, and $\mathcal{O}_{X,f_\phi(y)}$ with $A_{\phi^{-1}(\mathfrak{p})}$, the morphism $f_{\phi,y}$ is such that

$$f_{\phi,y}\colon A_{\phi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}, \quad a/1 \mapsto \phi(a)/1$$

for $a \in A$. Hence, for any $s \in A - \phi^{-1}(\mathfrak{p})$, we have

$$\phi(s) f_{\phi,y}(a/s) = f_{\phi,y}(s/1) f_{\phi,y}(a/s) = f_{\phi,y}(s \cdot (a/s)) = f_{\phi,y}(a/1) = \phi(a).$$

As $\phi(s) \notin \mathfrak{p}$, we find $f_{\phi,y}(a/s) = \phi(a)/\phi(s) \in B_\mathfrak{p}$. As a result, $f_{\phi,y}$ is a local morphism. This finishes the proof. □

**Proposition 2.2.2.3.** *The previous construction*

$$\Phi\colon \mathrm{Hom}_{\mathrm{ring}}(A, B) \to \mathrm{Mor}(\mathrm{Spec}(B), \mathrm{Spec}(A)), \quad \phi \mapsto (f_\phi, f_\phi^\sharp)$$

*is bijective.*

*Proof.* For any morphism $f\colon Y = \mathrm{Spec}(B) \to X = \mathrm{Spec}(A)$, it defines then a morphism of sheaves on $X$:

$$f^\sharp\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$$

which gives a morphism of rings between the global sections $\Psi(f)\colon A = \mathcal{O}_X(X) \to (f_*\mathcal{O}_Y)(Y) = B$. In particular, we get another map

$$\Psi\colon \mathrm{Mor}(Y, X) \to \mathrm{Hom}(A, B), \quad f \mapsto \Psi(f).$$

By constructions, we have $\Psi \circ \Phi = \mathrm{id}$, and it remains to show that $\Phi \circ \Psi = \mathrm{id}$. Let $f\colon Y \to X$ a morphism of schemes, with $\phi = \Psi(f)\colon A \to B$. We must show $f = f_\phi$. First, they give the same map on the underlying topological spaces: indeed, let $y \in Y$, and $x = f(y) \in X$, we have then the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, equal] & B \arrow[d, equal] \\
\mathcal{O}_X(X) \arrow[r, "f^\sharp"] \arrow[d] & \mathcal{O}_Y(Y) \arrow[d] \\
\mathcal{O}_{X,x} \arrow[r, "f_y^\sharp"] & \mathcal{O}_{Y,y}
\end{tikzcd}$$

Since $f_y^\sharp$ is a local morphism, we find $\phi^{-1}(\mathfrak{q}_y) = \mathfrak{p}_x$ (with $\mathfrak{p}_x \subset A$ (resp. $\mathfrak{q}_y \subset B$) the prime ideal corresponds to $x$ (resp. to $y$)). In particular, we have $f_\phi(y) = x = f(y)$, as asserted. Moreover, this commutative diagram implies also that $f_y^\sharp = f_{\phi,y}^\sharp\colon \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ for any $y$, hence $f^\sharp = f_\phi^\sharp$. This finishes then the proof. □

**Example 2.2.2.4.** Let $S \subset A$ be a multiplicative system of $A$, and $\iota\colon A \to S^{-1}A$ be the canonical morphism. It induces then the following map of topological spaces

$$f_\iota\colon \mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A).$$

In fact, $\mathrm{Im}(f_\iota) = \bigcap_{f\in S} D(f)$, and the induced map $\mathrm{Spec}(S^{-1}A) \to \bigcap_{f\in S} D(f)$ is a homeomorphism of topological spaces. An important special case is the following: let $f \in A$ which is not nilpotent, then the natural map $f_\iota\colon \mathrm{Spec}(A_f) \to \mathrm{Spec}(A)$ induces a homeomorphism between $\mathrm{Spec}(A_f)$ and the principal open subset $D(f) \subset \mathrm{Spec}(A)$. In fact, this identifies also the scheme $\mathrm{Spec}(A_f)$ with the open affine subscheme $D(f)$ of $\mathrm{Spec}(A)$. Indeed, by abuse of the notations, we still denote by $\iota$ the induced morphism of locally ringed spaces $f_\iota\colon \mathrm{Spec}(A_f) \to D(f)$. As $f_\iota$ is a homeomorphism between the underlying topological spaces, only need to show that the following morphism of sheaves $\mathcal{O}_{D(f)} = \mathcal{O}_{\mathrm{Spec}(A)}|_{D(f)} \to f_{\iota,*}\mathcal{O}_{\mathrm{Spec}(A_f)}$ is an isomorphism. Hence we are reduced to prove that for any prime ideal $\mathfrak{p} \subset A$ such that $f \notin \mathfrak{p}$, the morphism $\iota\colon A \to A_f$ induces an isomorphism of local rings $A_\mathfrak{p} \simeq A_{f,\mathfrak{p}A_f}$ which is easy to establish.

**Proposition 2.2.2.5.** *Let $(X, \mathcal{O}_X)$ be a scheme, and $U \subset X$ an open subset. Then the open sub-ringed space $(U, \mathcal{O}_X|_U)$ is also a scheme.*

*Proof.* Let $X = \bigcup_i U_i$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme. Then $U = \bigcup_i (U \cap U_i)$, hence we are reduced to show that any open of an affine scheme is again a scheme. Hence the proposition follows since such a open can be covered by the principal open subsets, which are affine scheme by the previous example. □

**Definition 2.2.2.6.** Let $X$ a scheme, and $U \subset X$ be an open subset of the underlying topological space of $X$. Then the open sub-ringed space $(U, \mathcal{O}_X|_U)$ is called an *open subscheme* of $X$, and it will often simply denoted by $U$. We will say that $U$ is an *open affine subscheme* of $X$ if the scheme $U$ is affine. A morphism of schemes $f\colon Y \to X$ is called an *open immersion* if there exists an open subset $U \subset X$ such that $f(Y) \subset U$, and that the induced morphism $Y \to U$ is an isomorphism of schemes.

**Example 2.2.2.7.** Let $\mathfrak{a} \subset A$ be an ideal. The projection $A \to A/\mathfrak{a}$ defines then a morphism of affines schemes

$$\mathrm{Spec}(A) \to \mathrm{Spec}(A/\mathfrak{a}).$$

A morphism of schemes $f\colon Y \to X$ is called a *closed immersion* if for each point $x \in X$, there exists some affine open subscheme $U = \mathrm{Spec}(A) \subset X$, such that the restriction $f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$ is isomorphic to $\mathrm{Spec}(A/\mathfrak{a}) \to \mathrm{Spec}(A)$ constructed as above. In this case, $Y$ is also called a *closed subscheme* of $X$. Finally, a morphism $f\colon X \to Y$ is called an *immersion* if $f$ can be decomposed as $f = g \circ h$ with $h$ a closed immersion, and $g$ an open immersion.[7]

**Remark 2.2.2.8.** In the previous example, for $f\colon Y \to X$ a closed immersion with $X = \mathrm{Spec}(A)$, one can find for each point $x \in X$, an principal affine open subset $D(a) \subset X$ containing $x$ such that $f^{-1}(D(a))$ is affine, and that the induced map $\mathcal{O}_X(D(a)) \to \mathcal{O}_Y(f^{-1}(D(a)))$ is surjective. Indeed, the definition gives an open subset $U \subset X$ containing $x$, such that $f^{-1}(U)$ is affine, and that the induced map $\mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U))$ is surjective. Since $U \subset X$ is open, one can find $a \in A$ such that $x \in D(a) \subset U$. Now, we write $U = \mathrm{Spec}(A')$ and $f^{-1}(U) = \mathrm{Spec}(B')$. Let $a'$ be the image of $a$ by the canonical map $A \to A'$, then $D(a) = D(a')$ is also the principal open subset of $U$ defined by the element $a'$. Now, $f^{-1}(D(a)) = f^{-1}(D(a')) = D(f^\sharp(a'))$, where $f^\sharp$ is the following canonical map $f^\sharp\colon A' = \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U)) = B'$. In particuar, $f^{-1}(D(a)) = \mathrm{Spec}(B'_{f^\sharp(a')})$ is affine, and the canonical map $\mathcal{O}_X(D(a)) \to \mathcal{O}_Y(f^{-1}(D(a)))$ is surjective since it can be identified as the localization with respect to $a'$ of the following surjection $A' \to B'$. This proves the assertion.

**Exercise 2.2.2.9.** Show that the composition of two immersions is an immersion.

**Exercise 2.2.2.10.** Let $Y \to \mathrm{Spec}(A)$ be a closed immersion to a affine scheme. Show that $Y$ is affine, and of the form $\mathrm{Spec}(A/\mathfrak{a})$ for some ideal $\mathfrak{a} \subset A$.

**Exercise 2.2.2.11.** Let $X, Y$ be two schemes with $Y$ affine. Show that the canonical map

$$\mathrm{Mor}(X,Y) \to \mathrm{Hom}_{\mathrm{ring}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)), \quad f \mapsto (f^\sharp(X)\colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)).$$

is bijective. Note that when $X$ is also affine, we recover then 2.2.2.3.

We finish this section by the following gluing lemma, which allows us to construct schemes.

**Lemma 2.2.2.12** (Gluing lemma)**.** *Let $\{X_i\}_i$ be a family of schemes, and $X_{ij} \hookrightarrow X_i$ be a open subscheme of $X_i$. Suppose we are given a family of isomorphisms of schemes $f_{ij}\colon X_{ij} \simeq X_{ji}$ such that (i) $X_{ii} = X_i$, and $f_{ii} = \mathrm{id}_{X_i}$ for all $i$; (ii) $f_{ij}(X_{ij} \bigcap X_{ik}) \subset X_{ji} \bigcap X_{jk}$, and $f_{ik} = f_{jk} \circ f_{ij}$ on $X_{ij} \bigcap X_{ik}$ for all index $i, j, k$. Then there exists a scheme $X$, unique up to isomorphism, together with open immersions $g_i\colon X_i \to X$ such that $g_i = g_j \circ f_{ij}$ on $X_{ij}$, and that $X = \bigcup_i g_i(X_i)$.*

[7] This is the definition given in EGA I.4.1.3. Note that if we define immersion as morphism of the form $g \circ h$ with $h$ open and $g$ closed, then this kind of "immersion" is not closed under compositions.

### 2.2.3 Projective schemes

[8]

Let $S$ be a scheme, an *$S$-scheme* (or a *scheme over $S$*) is by definition a morphism of schemes $f : X \to S$. When $S = \operatorname{Spec}(R)$ is affine, an $S$-scheme is also called an *$R$-scheme*. The aim of this section is to define the so called *projective schemes* over an affine scheme $S = \operatorname{Spec}(R)$.

#### Aside: graded rings and homogeneous localization

Let $R$ be a ring. Recall that a *graded $R$-algebra* is an $R$-algebra $A$, together with a decomposition

$$A = \bigoplus_{d \geq 0} A_d$$

of $A$ into a sum of $R$-modules, such that for any $d, d' \in \mathbb{Z}_{\geq 0}$, we have $A_d \cdot A_{d'} \subset A_{d+d'}$. An element $x \in A_d - \{0\}$ is called *homogeneous of degree $d$*. An ideal $\mathfrak{a} \subset A$ is then called *homogeneous* if $\mathfrak{a}$ can be generated by homogeneous elements, or equivalently, if

$$\mathfrak{a} = \bigoplus_{d \geq 0} \mathfrak{a} \cap A_d.$$

For $A = \bigoplus_d A_d$ a graded $R$-algebra, let

$$A_+ := \bigoplus_{d > 0} A_d.$$

**Example 2.2.3.1.** Let $R$ be a ring.

1. The ring of polynomials $R[X_1, \cdots, X_n]$ is naturally graded, with $R[X_1, \cdots, X_d]_d - \{0\}$ the set of homogeneous polynomial of degree $d$.

2. Let $\mathfrak{a} \subset R[X_1, \cdots, X_n]$ be a homogeneous ideal. Then the quotient $R[X_1, \cdots, X_n]/\mathfrak{a}$ has also a natural gradation:

$$R[X_1, \cdots, X_n] = \bigoplus \frac{R[X_1, \cdots, X_n]_d}{\mathfrak{a} \cap R[X_1, \cdots, X_n]_d}$$

**Definition 2.2.3.2** (Homogeneous localization)**.** Let $R$ be a ring, and $A$ be a graded $R$-algebra.

1. Let $f \in A$ be a homogeneous element of degree $d$, the *homogeneous localization* $A_{(f)}$ of $A$ with respect to $f$ is the subring of $A_f$ consists of elements of the form $a/f^m$ with $a \in A_{md}$.

2. Let $\mathfrak{p} \subset A$ be a homogeneous prime ideal, and $T \subset A - \mathfrak{p}$ be the set of homogeneous elements. We define the the *homogeneous localization* $A_{(\mathfrak{p})}$ of $A$ with respect to $\mathfrak{p}$ is the subring of $T^{-1}A$ consists of elements of the form $a/s$ with $a \in A$ and $s \in T$ such that $\deg(a) = \deg(s)$.

[8]NEED TO MODIFY: graded $R$-algebra to graded ring. For a graded $R$-algebra, need to define the structural map.

**The projective spectrum** $\mathrm{Proj}(A)$

Let $R$ be a ring, $A$ be a graded $R$-algebra, we define

$$\mathrm{Proj}(A)\colon = \{\mathfrak{p} \in \mathrm{Spec}(A) \text{ homogeneous such that } A_+ \not\subseteq \mathfrak{p}\}$$

For $\mathfrak{a} \subset A$ a *homogeneous ideal*, let

$$V_+(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Proj}(A) \colon \mathfrak{a} \subset \mathfrak{p}\}$$

**Proposition 2.2.3.3.** *One can endow a topology on the set* $\mathrm{Proj}(A)$ *by taking the closed subsets to be the subsets of the form* $V_+(\mathfrak{a})$ *with* $\mathfrak{a} \subset A$ *some homogeneous ideal. This topology is called* the Zariski topology *on* $\mathrm{Proj}(A)$*. Moreover, for any homogenous element* $f \in A$*, let* $D_+(f) = \mathrm{Proj}(A) - V_+(fA)$ *(such an open subset is called* principal*). Then the family* $\mathcal{B} := \{D_+(f) \colon f \in A_+ \text{ homogeneous}\}$ *is a basis for the Zariski topology on* $\mathrm{Proj}(A)$*.*

*Proof.* For the last statement, we prove first that the family $\mathcal{B}' := \{D_+(f) \colon f \in A\}$ forms a basis for the Zariski topology. Indeed, for $U = \mathrm{Proj}(A) - V_+(\mathfrak{a})$ an open subset with $\mathfrak{a} \subset A$ a homogeneous ideal. Then we have

$$V_+(\mathfrak{a}) = \bigcap_{f \in \mathfrak{a} \text{ homogeneous}} V_+(fA).$$

Hence

$$U = \bigcup_{f \in \mathfrak{a} \text{ homogeneous}} D_+(f).$$

This shows that the family $\mathcal{B}'$ is indeed a basis. On the other hand,

$$\emptyset = \bigcap_{g \in A_+ \text{ homogeneous}} V_+(gA),$$

hence

$$\mathrm{Proj}(A) = \bigcup_{g \in A_+ \text{ homogeneous}} D_+(g),$$

from where, we find

$$D_+(f) = \bigcup_{g \in A_+ \text{ homogeneous}} (D_+(f) \cap D_+(g)) = \bigcup_{g \in A_+ \text{ homogeneous}} D_+(fg).$$

Note that $fg \in A_+$, this gives then the proof. □

**Remark 2.2.3.4.** For $d > 0$ an integer, then the family of opens

$$\{D_+(f) \colon f \in \bigoplus_{n>0} A_{nd} \text{ homogeneous}\}$$

gives also an basis for the topology on $\mathrm{Proj}(A)$.

Let $I \subset A$ be an arbitrary ideal of $A$, we associate to it a homogeneous ideal $I^h$ generated by the homogeneous elements contained in $I$:

$$I^h = \bigoplus_d (I \cap A_d).$$

**Lemma 2.2.3.5.** *Let $I, J$ be two ideal of $A$.*

1. *If $I$ is prime, so is $I^h$.*

2. *If $I, J$ are homogeneous, then $V_+(I) \subset V_+(J)$ iff $J \cap A_+ \subset \sqrt{I}$.*

3. $\mathrm{Proj}(A) = \emptyset$ *iff $A_+$ is nilpotent.*

*Proof.* Suppose $I$ is a prime ideal. Let $a, b \in A$ such that $ab \in I^h$. We need to show that either $a \in I^h$ or $b \in I^h$. We will prove this by contradiction. Hence assume $a, b \notin I^h$. Now, we write $a = a_0 + \cdots + a_n$ and $b = b_0 + \cdots + b_m$ be the decomposition of $a, b$ into the homogeneous components with $a_n \neq 0$ and $b_m \neq 0$. Moreover, up to replace $a$ (resp. $b$) by $a - a_n$ (resp. by $b - b_m$), we may assume that $a_n \notin I^h$ and $b_m \notin I^h$. Now

$$ab = \sum_{i,j} a_i b_j = a_n b_m + \text{termes of lower degree} \in I^h$$

As a result, $a_n b_m \in I^h$ as $I^h$ is homogeneous. Hence $a_n b_m \in I$, which implies that either $a_n \in I$ or $b_m \in I$. A contradiction. This gives (1). To show (2), suppose first that $J \cap A_+ \subset \sqrt{I}$. Let $\mathfrak{p} \in V_+(I)$, *i.e.*, $I \subset \mathfrak{p}$ and $A_+ \not\subset \mathfrak{p}$. In particular, $J \cap A_+ \subset \sqrt{I} \subset \mathfrak{p}$. But as $A_+ \not\subset \mathfrak{p}$, let $\lambda \in A_+ - \mathfrak{p}$, and let $a \in J$, then $\lambda \cdot a \in J \cap A_+ \subset \mathfrak{p}$. So we get $a \in \mathfrak{p}$, hence $J \subset \mathfrak{p}$. That is $\mathfrak{p} \in V_+(J)$, hence $V_+(I) \subset V_+(J)$. Conversely, assume $V_+(I) \subset V_+(J)$. The previous argument shows also that $V_+(I) = V_+(I \cap A_+)$. Hence we have

$$\sqrt{I \cap A_+} = \bigcap_{\mathfrak{p} \in V(I \bigcap A_+)} \mathfrak{p} = \left( \bigcap_{\mathfrak{p} \in V(IA_+), A_+ \not\subset \mathfrak{p}} \mathfrak{p} \right) \bigcap \left( \bigcap_{\mathfrak{p} \in V(I \cap A_+), A_+ \subset \mathfrak{p}} \right) = \left( \bigcap_{\mathfrak{p} \in V_+(I)} \mathfrak{p} \right) \bigcap \sqrt{A_+}$$

In particular, if $V_+(I) \subset V_+(J)$, we get

$$J \cap A_+ \subset \sqrt{J \bigcap A_+} \subset \sqrt{I \bigcap A_+} \subset \sqrt{I}.$$

This finishes the proof of (2). For the last statement, $\mathrm{Proj}(A) = \emptyset$ if and only if $V_+(0) \subset V_+(A_+)$, and by (2), the latter condition is equivalent to say that $A_+ \subset \sqrt{(0)}$. In other words, $A_+$ is nilpotent. $\square$

Now we want to define a structure sheaf on $\mathrm{Proj}(A)$, in a similar way as the affine case. Suppose $f, g \in A_+$ two homogeneous elements, such that $D_+(f) \subset D_+(g)$, or equivalently, $f \in \sqrt{(g)}$. The natural morphism defined in ???? $\alpha_{g,f} \colon A_g \to A_f$, then it sends $A_{(g)}$ to $A_{(f)}$ and we get hence a morphism of rings

$$\alpha_{(g,f)} \colon A_{(g)} \to A_{(f)}.$$

These morphisms satisfy still the transitivity condition: (i) $\alpha_{(f,f)} = \mathrm{id}_{A_{(f)}}$; (ii) $\alpha_{(h,f)} = \alpha_{(g,f)} \circ \alpha_{(h,g)}$ for three opens $D_+(f) \subset D_+(g) \subset D_+(h)$. In particular, we have $\alpha_{(g,f)} : A_{(g)} \simeq A_{(f)}$ for $D_+(f) = D_+(g)$. We will use this canonical isomorphism to identify these two homogeneous localization rings.

Now, we define, for each principal open $U = D_+(f) \subset X := \mathrm{Proj}(A)$ with $f \in A_+$, $\mathcal{O}'_x(U) = A_{(f)}$. This gives clearly a $\mathcal{B}$-presheaf, with $\mathcal{B}$ the basis consists of principal open subset of the form $D_+(f)$ for some homogeneous $f \in A_+$.

**Lemma 2.2.3.6.** *The $\mathcal{B}$-presheaf $\mathcal{O}'_X$ is a $\mathcal{B}$-sheaf.*

As a corollary, the $\mathcal{B}$-sheaf $\mathcal{O}'_X$ extends to a sheaf of rings $\mathcal{O}_X$ on $X$. In this way, we obtain a ringed space $(X, \mathcal{O}_X)$. The next task is then to show that this ringed space is actually a *scheme*. For this, we want to determine the open subringed space $(D_+(f), \mathcal{O}_X|_{D_+(f)})$ for $f \in A_+$ homogeneous of degree $d > 0$. In fact, we have

**Lemma 2.2.3.7.** *There is an canonical isomorphism of ringed spaces* $D_+(f) \simeq \operatorname{Spec}(A_{(f)})$.

*Proof.* We have the two canonical morphisms of rings

$$A_{(f)} \longrightarrow A_f \xleftarrow{\beta} A\ ,$$

hence the diagram of maps of *sets*:

$$\begin{tikzcd} \operatorname{Spec}(A_{(f)}) & \operatorname{Spec}(A_f) \arrow[l] \arrow[r, hook, "\beta^*"] & \operatorname{Spec}(A) \\ & & \operatorname{Proj}(A) \arrow[u, hook] \end{tikzcd}$$

Note that, $\beta^*$ identifies $\operatorname{Spec}(A_f)$ with the open subscheme $D(f) \subset \operatorname{Spec}(A)$, and $D_+(f) = D(f) \cap \operatorname{Proj}(A)$. In this way, we get a continuous map $\theta\colon D_+(f) \to \operatorname{Spec}(A_{(f)})$.

We will first prove that $\theta$ is homeomorphism. We remark first that, for $a \in A_{nd}$, we have

$$\theta^{-1}(D(a/f^n)) = D_+(af).$$

Indeed, for $\mathfrak{p} \in D_+(f)$, $\theta(\mathfrak{p}) \in D(a/f^n)$ iff $\mathfrak{p}_f \cap A_{(f)} \in D(a/f^n)$, or still $a/f^n \notin \mathfrak{p}_f$. Since $f \notin \mathfrak{p}$, the last condition is equivalent to say that $a \notin \mathfrak{p}$, that is $\mathfrak{p} \in D_+(af)$. Now since $\operatorname{Spec}(A_((f)))$ and $D_+(f)$ are both $T_0$-spaces, and since the family $\{D(a/f^n)\colon a/f^n \in A_{(f)}\}$ (resp. the family $\{D_+(af)\colon a \in A_{nd}$ for some $n\}$) is a basis of $\operatorname{Spec}(A_{(f)})$ (resp. of $D_+(f)$), the equality above implies in particular that $\theta$ is injective. Hence to complete the proof of the assertion, it suffices to show that $\theta$ is surjective. Let now $\mathfrak{q} \subset A_{(f)}$ be a prime ideal, and for each $n$, $\mathfrak{p}_n$ be the set of elements $x$ of $A_n$ such that $x^d/f^n \in \mathfrak{q}$. Let $\mathfrak{p} = \bigoplus_n \mathfrak{p}_n$, which is then a subgroup of $A$. Indeed, let $x, y \in \mathfrak{p}_n$, by binomial formula, $(x-y)^{2d}/f^{2n} \in \mathfrak{q}$, hence $(x-y)^d/f^n \in \mathfrak{q}$. In other words, $x - y \in \mathfrak{p}_n$, which implies that $\mathfrak{p}_n \subset A_n$ is a subgroup.

We claim that this is in fact a prime ideal of $A$. First, it's an ideal: let $a \in A_m$, $x \in \mathfrak{p}_n$, then $ax \in A_{m+n}$, moreover,

$$(ax)^d/f^{m+n} = (a^d/f^m) \cdot (x^d/f^n) \in \mathfrak{q}.$$

It's also a prime ideal: let $x = \sum_i x_i \in A$ and $y = \sum_j y_j \in A$ such that $x, y \notin \mathfrak{p}$. Suppose $x, y \notin \mathfrak{p}$. Let $x_{i_0}$ (resp. $y_{j_0}$) be the homogeneous component of $x$ (resp. of $y$) of minimal degree $i_0$ (resp. $j_0$), we may assume that $x_{i_0} \notin \mathfrak{p}$, and $y_{j_0} \notin \mathfrak{p}$, hence

$$xy = x_{i_0}y_{j_0} + \text{ terms of higher degree} \in \mathfrak{p}.$$

So, we must have $x_{i_0}y_{j_0} \in \mathfrak{p}_{i_0+j_0}$. But

$$(x_{i_0}y_{i_0})^d/f^{i_0+j_0} = (x_{i_0}^d/f^{i_0}) \cdot (y_{j_0}^d/f^{j_0}) \in \mathfrak{q}.$$

As $\mathfrak{q} \subset A_{(f)}$ is a prime ideal, we have either $x_{i_0}^d/f^{i_0} \in \mathfrak{q}$ or $y_{j_0}^d/f^{j_0} \in \mathfrak{q}$. Hence either $x_{i_0} \in \mathfrak{p}$ or $y_{j_0} \in \mathfrak{p}$. This gives a contradiction. In this way, we see that $\mathfrak{p} \subset A$ is a homogeneous prime ideal. To finish the proof of surjectivity, since $f \notin \mathfrak{p}$ (otherwise, $1 = f^d/f^d \in \mathfrak{q}$ which is impossible), we find $\mathfrak{p} \in D(f)$, and $A_+ \not\subseteq \mathfrak{p}$. Hence $\mathfrak{p} \in D_+(f)$. Finally,

$$\mathfrak{p}_f \cap A_{(f)} = \{a/f^n\colon a \in \mathfrak{p}_{nd}\},$$

and for any $a \in \mathfrak{p}_{nd}$, we have $a^d/f^{nd} = (a/f^n)^d \in \mathfrak{q}$. As $\mathfrak{q}$ is a prime ideal, we then find $a/f^n \in \mathfrak{q}$. This shows that $\mathfrak{p}_f \cap A_{(f)} \subset \mathfrak{q}$. Conversely, for any $a/f^n \in \mathfrak{q}$ with $a \in A_{nd}$, we find $a^d/f^{nd} \in \mathfrak{q}$, in particular, $a \in \mathfrak{p}_{nd}$, as a result, $a/f^n \in \mathfrak{p}_f \cap A_{(f)}$, this gives $\mathfrak{q} \subset \mathfrak{p}_f \cap A_{(f)}$. So we get finally $\mathfrak{q} = \mathfrak{p}_f \cap A_{(f)} = \theta(\mathfrak{p})$, which gives then the surjectivity. As a corollary, $\theta \colon D_+(f) \to \mathrm{Spec}(A_{(f)})$ is a homeomorphism.

Now to finish the proof, we need to construct an isomorphism of sheaves

$$\mathcal{O}_{\mathrm{Spec}(A_{(f)})} \to \theta_*(\mathcal{O}_{\mathrm{Proj}(A)}|_{D_+(f)})$$

We will first do it in the level of $\mathcal{B}$-sheaves. For each $a/f^n \in A_{(f)}$,

$$\mathcal{O}_{\mathrm{Spec}(A_{(f)})}(D(a/f^n)) \simeq \left(A_{(f)}\right)_{a/f^n}$$

and $\mathcal{O}_{\mathrm{Proj}(A)}(D_+(af)) \simeq A_{(af)}$, it's only need to establish a natural isomorphism

$$(A_{(f)})_{a/f^n} \simeq A_{(af)}.$$

For this, we consider the canonical map $A_{(f)} \to A_{(af)}$ given by $b/f^n \mapsto a^n b/(af)^n$ (for $b \in A_{nd}$). Then the image of $a/f^n$ under this map is $a^{n+1}/(af)^n$ is invertible, with inverse given by $f^{n+1}/af \in A_{(af)}$. In this way, we find a natural map $(A_{(f)})_{a/f^n} \to A_{(af)}$. One verifies easily that this gives indeed an isomorphism. This finishes the proof.

□

**Corollary 2.2.3.8.** *Let $A$ be a graded $R$-algebra. The ringed space* $(\mathrm{Proj}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ *is a scheme.*

**Corollary 2.2.3.9.** *Let $x \in X = \mathrm{Proj}(A)$ with corresponds to the homogeneous prime ideal $\mathfrak{p}$ of $A$. Then $\mathcal{O}_{X,x} \simeq A_{(\mathfrak{p})}$.*

**Definition 2.2.3.10.** Let $A$ be a graded $R$-algebra. The *projective scheme* associated with $A$ to be the ringed space $(\mathrm{Proj}(A), \mathcal{O}_{\mathrm{Proj}(A)})$ constructed as above.

**Example 2.2.3.11.** Let $R$ be a ring, $A = R[T_0, \cdots, T_d]$ with the natural gradation. Then $\mathrm{Proj}(A)$ is called the projective space of dimension $d$ over $R$ (or over $\mathrm{Spec}(R)$), denoted by $\mathbb{P}^d_R$ or $\mathbb{P}^d_{\mathrm{Spec}(R)}$.[9] More general, a morphism $f \colon X \to S$ is called projective, if there exists an open covering $S = \bigcup_{i \in I} S_i$ with $S_i$ affine, such that for each $i \in I$, the restriction $f|_{f^{-1}(S_i)} \colon S_i$ can be realized as a closed immersion to a projective space $\mathbb{P}^n_{S_i}$ for some integer $n$.

### Functorial property of projective spectrum

Let $R$ be a ring, $A, B$ be two graded $R$-algebras. Recall that a morphism of $R$-algebras $\varphi \colon A \to B$ is called *graded* if $\varphi(A_n) \subset B_n$ for any integer $n \geq 0$. Let $\varphi \colon A \to B$ be such a morphism. In particular, we have $\varphi(A_+) \subset B_+$. Let

$$G(\varphi) := \mathrm{Proj}(B) - V_+(\varphi(A_+)),$$

which is an open subset of $\mathrm{Proj}(B)$. Let $\mathfrak{p} \subset B$ be a homogeneous prime ideal such that $\mathfrak{p} \in G(\varphi)$. Then $\varphi^{-1}(\mathfrak{p}) \subset A$ is again a homogeneous prime ideal such that $A_+ \not\subset \varphi^{-1}(\mathfrak{p})$. Hence $\varphi^{-1}(\mathfrak{p}) \in \mathrm{Proj}(A)$. We get in this way a map

$${}^a\varphi \colon G(\varphi) \to \mathrm{Proj}(A).$$

[9]One can define the general projective space over a scheme by gluing the previous definition for affine schemes. We will see later a direct definition using the notion of base change.

**Lemma 2.2.3.12.** *For any $f \in A_+$ homogeneous, $D_+(\varphi(f)) \subset G(\varphi)$. Moreover, ${}^a\varphi^{-1}(D_+(f)) = D_+(\varphi(f))$.*

*Proof.* We show first that $D_+(\varphi(f)) \subset G(\varphi)$. Indeed, for any $\mathfrak{p} \in D_+(\varphi(f))$, that is, an element $\mathfrak{p} \in \mathrm{Proj}(B)$ such that $\varphi(f) \notin \mathfrak{p}$. Since $f \in A_+$, we find in particular $\varphi(A_+) \not\subseteq \mathfrak{p}$. Hence $\mathfrak{p} \notin V_+(\varphi(A_+)B)$, *i.e.*, $\mathfrak{p} \in G(\varphi)$. This gives the first statement. For the second statement, for any $\mathfrak{p} \in G(\varphi)$, we find that $\mathfrak{p} \in {}^a\varphi^{-1}(D_+(f))$ if and only if ${}^a\varphi(\mathfrak{p}) \in D_+(f)$, in other words, $f \notin \phi^{-1}(\mathfrak{p})$. The latter condition is also equivalent to the condition that $\varphi(f) \notin \mathfrak{p}$, that is, $\mathfrak{p} \in D_+(\varphi(f))$. Hence ${}^a\varphi^{-1}(D_+(f)) = D_+(\varphi(f))$. □

As a result, ${}^a\varphi$ is continuous. To define a morphism of schemes $G(\varphi) \to \mathrm{Proj}(A)$, we still need to define a morphism of sheaves of rings

$$\mathcal{O}_{\mathrm{Proj}(A)} \to ({}^a\varphi)_*\mathcal{O}_{G(\varphi)}.$$

Hence, we need to define for each $f \in$, a compatible family of morphisms of rings

$$A_{(f)} = \mathcal{O}_{\mathrm{Proj}(A)}(D_+(f)) \to \left(({}^a\varphi)_*\mathcal{O}_{G(\varphi)}\right)(D_+(f)) = \mathcal{O}_{G(\varphi)}(D_+(\varphi(f))) = B_{(\varphi(f))}.$$

which can be given by the morphism

$$A_{(f)} \to B_{(\varphi(f))}, \quad \frac{a}{f^n} \mapsto \frac{\varphi(a)}{\varphi(f)^n}.$$

In this way, we get a morphism of schemes, which will denoted by $\mathrm{Proj}(\varphi)$:

$$\mathrm{Proj}(\varphi)\colon G(\varphi) \to \mathrm{Proj}(A).$$

**Example 2.2.3.13.** Let $R$ be a ring, and $A$ be a graded $R$-algebra. For $I \subset A$ a homogeneous ideal, we denote by $B = A/I$ the $R$-algebra with the induced gradation, and by $\varphi\colon A \to B$ the canonical projection. Hence, we have $\varphi(A_+) = B_+$, as a result, $V_+(\varphi(A_+)B) = V_+(B_+)$. In particular, $G(\varphi) = \mathrm{Proj}(B)$, and we get a morphism of schemes

$$\mathrm{Proj}(\varphi)\colon \mathrm{Proj}(B) \to \mathrm{Proj}(A).$$

This morphism is actually a closed immersion of schemes. Indeed, for any principal open subset $D_+(f) \subset \mathrm{Proj}(A)$, the inverse image $\mathrm{Proj}(\varphi)^{-1}(D_+(f)) = D_+(\varphi(f))$, which is affine. Moreover, the induced map

$$\mathcal{O}_{\mathrm{Proj}(A)}(D_+(f)) \to \left(\mathrm{Proj}(\varphi)_*\mathcal{O}_{\mathrm{Proj}(B)}\right)(D_+(f)) = \mathcal{O}_{\mathrm{Proj}(B)}(D_+(\varphi f))$$

can be identified with the following morphism of rings

$$A_{(f)} \to B_{(\varphi(f))}, \quad \frac{a}{f^m} \mapsto \frac{\varphi(a)}{\varphi(f)^m}$$

which is surjective. Hence by definition, $\mathrm{Proj}(\varphi)$ is a closed immersion.

## 2.3 First properties of schemes and morphisms of schemes

### 2.3.1 Topological properties

Recall that a topological space $X$ is called *quasi-compact*, if any open covering of $X$ admit a finite subcovering.

**Definition 2.3.1.1.** Let $X$ be a scheme. $X$ is called *connected* (resp. *irreducible*, resp. *quasi-compact*) if its underlying topological space is connected (resp. *irreducible*, resp. quasi-compact).

**Example 2.3.1.2.** Let $k$ be a field. $\mathbb{A}^2_k = \mathrm{Spec}(k[X,Y])$ is irreducible (hence connected), while $\mathrm{Spec}(k[X,Y]/(XY))$ is connected but not irreducible. Affine schemes are all quasi-compact.

Recall that, an element $a \in A$ of a ring is called *idempotent* if $a^2 = a$. Clearly, for any ring, 0 and 1 are two idempotents of $A$.

**Proposition 2.3.1.3.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. The following two assertions are equivalent:*

1. *$X$ is connected;*
2. *The only idempotents of $A$ are $0$ and $1$.*

*Proof.* Suppose $X$ is connected. If $A$ contains a third idempotent $e \in A$, then we have the following decomposition of $A$ into a product of two rings:

$$A = e \cdot A \times (1-e) \cdot A$$

As $e \neq 0, 1$, the two rings $e \cdot A$ and $(1-e)\cdot A$ are both non trivial. Hence, we get $\mathrm{Spec}(A) = \mathrm{Spec}(eA) \coprod \mathrm{Spec}((1-e)A)$. Moreover, these two subsets are all closed, hence $X$ is not connected. Conversely, suppose that $0, 1$ are the only idempotents of $A$. If $X$ is not connected, hence $X = X_1 \coprod X_2$ with $X_i \subset X$ non empty open and closed. As a result, from the sheaf property, we find $\mathcal{O}_X(X) = \mathcal{O}_X(X_1) \times \mathcal{O}_X(X_2)$. As $X_i$ non empty, the two rings $\mathcal{O}_X(X_i)$ are both non trivial. Hence the element $(1, 0)$ gives a non trivial idempotent of $A$. A contradiction. $\square$

**Definition 2.3.1.4.** Let $X$ be a topological space. Let $x, y \in X$ be points of $X$. We say that $y$ is a *specialization* of $x$, or that $x$ *specializes* to $y$ if $y \in \overline{\{x\}}$. We say that $x \in X$ is a *generic point* if $x$ is then unique point of $X$ that specializes to $x$.

**Exercise 2.3.1.5.** Show that a topological space admitting a *unique* generic point must be irreducible.

Recall that for a topological space $Z$, an *irreducible component* of $Z$ is a maximal irreducible closed subset of $Z$.

**Proposition 2.3.1.6.** *Let $X$ be a scheme.*

1. *Any irreducible closed subset of $X$ contains a unique generic point.*
2. *For any generic point $\xi \in X$, $\overline{\{\xi\}}$ is an irreducible component of $X$. This establishes a bijection between the irreducible components of $X$ and the generic points of $X$.*
3. *Let $X$ be a scheme and $x \in X$. Then the irreducible components of $\mathrm{Spec}(\mathcal{O}_{X,x})$ correspond bijectively to the irreducible components of $X$ passing through $x$.*

*Proof.* When $X = \mathrm{Spec}(A)$ is affine. A closed subset $V(\mathfrak{a}) \subset X$ is irreducible iff $\sqrt{\mathfrak{a}} = \mathfrak{p}$ with $\mathfrak{p} \subset A$ a prime ideal. In this case, the point $\mathfrak{p}$ is a generic point of $V(\mathfrak{a}) \subset X$. Now for $X$ an arbitrary scheme, $Z \subset X$ an irreducible subset. Let $x \in X$ be a point contained in $Z$, then $x$ has a affine neighborhood $U \subset X$. Since $Z$ is irreducible, $U \cap Z \subset Z$ is dense and irreducible. Moreover, $U \cap Z \subset U$ is closed and irreducible with $U$ an affine scheme, it contains a generic point, which gives also a generic point of $Z$. The uniqueness follows from the fact that the underlying topological space of a scheme is a $T_0$-space. This gives (1). For (2), let $Z \subset X$ be an irreducible component of $X$, and $\xi \in Z$ be its generic point. Then we claim that $\xi$ is a generic point of $X$, that is, no point other then $\xi$ can specialize to $\xi$: indeed, if $\eta$ specialize to $\xi$, then $\xi \in \overline{\{\eta\}}$, hence $Z = \overline{\{\xi\}} \subset \overline{\{\eta\}}$. As $Z$ is a maximal irreducible closed subset of $X$, we must have $\overline{\{\xi\}} = \overline{\{\eta\}}$, hence $\xi = \eta$. This shows that $\xi \in X$ is a generic point. Then one verifies easily that the correspondence stated in (2). For the last assertion (3), we may assume that $X = \mathrm{Spec}(A)$ is affine, with $x \in X$ corresponds to a prime ideal $\mathfrak{p} \subset A$. By the correspondence between irreducible closed subsets and the prime ideals of $A$, an irreducible component of $X$ corresponds to a minimal prime ideal of $A$. Hence the irreducible components of $X$ passing through $x$ are in one-to-one correspondence with the minimal prime ideals of $A$ which are contained in $\mathfrak{p}$, or still, with the minimal prime ideals of $A_{\mathfrak{p}} = \mathcal{O}_{X,x}$, that is the irreducible components of $\mathrm{Spec}(\mathcal{O}_{X,x})$. This gives (3). □

### 2.3.2 Noetherian schemes

**Definition 2.3.2.1.** A scheme $X$ is called *locally noetherian* if $X$ admits an affine open covering $X = \bigcup X_i$ such that $\mathcal{O}_X(X_i)$ is a noetherian ring for all $i$. $X$ is called *noetherian*, if it's quasi-compact and locally neotherian.

**Proposition 2.3.2.2.** *1. An affine scheme* $\mathrm{Spec}(A)$ *is noetherian iff $A$ is noetherian.*

*2. Let $X$ be a locally noetherian scheme (resp. neotherian scheme), then so is any open subscheme scheme of $X$.*

*3. For $X$ a noetherian scheme, its underlying topological space is a noetherian topological space. In particular, any closed subset of $X$ can be decomposed as a finite union of its irreducible components.*

*Proof.* (1) The latter condition is clearly sufficient. We show that it's also necessary. Let $X = \mathrm{Spec}(A)$ be an affine scheme which is noetherian. Since a localization of a noetherian ring is again noetherian, $X$ contains then a topological basis $\mathcal{B}$ which consists of open principal $D(f) = \mathrm{Spec}(A_f)$ with $A_f$ noetherian. In particular, $X$ can be covered by finitely many principal opens in $\mathcal{B}$: $X = \cup_i X_i$ with $X_i = \mathrm{Spec}(A_{f_i})$. Now, as $A_{f_i}$ is noetherian, $\mathfrak{a}_{f_i} \subset A_{f_i}$ is an ideal of finite type. Let $\{a_{ij}\}_j$ be a family of generators of $I_{f_i}$, we may assume that $a_{ij} \in I$. We claim that $\{a_{ij} : i, j\}$ gives then a family of generators of $\mathfrak{a}$. Indeed, for each $a \in I$, and for each $i$, there exists $\lambda_{ij} \in A$ and $e_{ij} \in \mathbb{Z}_{\geq 1}$ such that

$$a = \sum_j \lambda_{ij} \cdot a_{ij}/f_i^{e_{ij}} \in A_{f_i}$$

Up to replace $e_{ij}$ by some bigger integer, we may assume that $e_{ij} = e$ is independent of $i, j$. Moreover, there exists $m_i \in \mathbb{Z}_{\geq 0}$ such that

$$f_i^{m_i+e} a = \sum_j f_i^{m_i} \lambda_{ij} a_{ij}$$

But, since $X = \bigcup_i D(f_i) = \bigcup_i D(f_i^{m_i+e})$, one can find $\mu_i \in A$ such that $1 = \sum_i \mu_i f_i^{m_i+e}$. So finally, we get

$$a = \sum_i a\mu_i f_i^{m_i+e} = \sum_i \sum_j \mu_i f_i^{m_i} \lambda_{ij} a_{ij}.$$

This gives (1). For (2), as the localizations of a noetherian ring are again neotherian, $X$ contains then a open basis consisting of noetherian affine schemes. This shows that any open subscheme of a locally noetherian scheme is locally noetherian. If moreover $X$ is noetherian, any open subset of $X$ is quasi-compact, in particular, any open subscheme is noetherian. This gives (2). The proof of (3) is left to the readers. □

### 2.3.3 Reduced and integral schemes

Recall that a ring $A$ is called *reduced*, if the only nilpotent element of $A$ is 0, and $A$ is called *integral* if the equality $a \cdot b = 0$ for $a, b \in A$ implies either $a = 0$ or $b = 0$.

**Definition 2.3.3.1.**
1. A scheme $X$ is called *reduced at a point* $x$, if the local ring $\mathcal{O}_{X,x}$ is reduced. $X$ is called *reduced*, if it's reduced at all its points.
2. A scheme $X$ is called *integral at a point* $x$, if the local ring $\mathcal{O}_{X,x}$ is integral. If $X$ is integral at all points of $X$, and $X$ is irreducible, then we say $X$ is *integral*.

**Proposition 2.3.3.2.** *Let $X$ be a scheme. Then*

1. *$X$ is reduced (resp. integral) iff for each non empty open $U \subset X$, the ring $\mathcal{O}_X(U)$ is reduced (resp. integral).*
2. *$X$ is integral iff $X$ is irreducible and reduced.*

*Proof.* Suppose $X$ is reduced, and let $f \in \mathcal{O}_X(U)$ such that $f^n = 0$. We want to show $f = 0$. Indeed, the assumption implies that the image $f_x \in \mathcal{O}_{X,x}$ of $f$ is also nilpotent, hence $f_x = 0$ by the reducedness of $\mathcal{O}_{X,x}$. As a $\mathcal{O}_X$ is a sheaf, we then have $f = 0$. The converse statement is easy since a direct limit of reduced rings is still reduced. Now we suppose $X$ integral. Let $f, g \in \mathcal{O}_X(U) - \{0\}$ such that $f \cdot g = 0$, and let

$$D_f = \{x \in U : f(x) = 0\},^{10} \quad \text{and} \quad D_g = \{x \in U : g(x) = 0\}.$$

These are two closed subsets of $X$. Indeed, for this assertion, we only need to verify that $D_f \cap V$ is closed in $V$ for any $V = \mathrm{Spec}(A) \subset X$ affine open subscheme. In fact, as a set, we have $D_f \cap V = V((f)) \subset \mathrm{Spec}(A)$, which is hence closed in $V$, and this gives the assertion. Moreover, $D_f \bigcup D_g = U$ (as $f \cdot g = 0$), and $U$ being irreducible (as $X$ is), hence either $D_f = U$, or $D_g = U$. By symmetry, we assume that $D_f = U$. Now, we claim $f = 0$. To see this, we only need to show that $f|_V = 0$ for any affine open $V \subset U$. But then $f|_V$ lies in the nilpotent radical of $\mathcal{O}_X(V)$ which is reduced by what we have shown in the beginning of this proof, hence $f|_V = 0$. Conversely, suppose $\mathcal{O}_X(U)$ is integral for any non empty $U$ of $X$. In particular, all local rings $\mathcal{O}_{X,x}$ are integral. It remains to verify that $X$ is irreducible. Indeed, otherwise, $X = X_1 \bigcup X_2$ with $X_i \hookrightarrow X$ two closed subsets of $X$ such that $X_i \subsetneq X$. Now we consider $U_i = X - X_i$ which is open in $X$. Moreover, $U_1 \bigcap U_2 = \emptyset$. Hence we find

$$\mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \oplus \mathcal{O}_X(U_2)$$

[10] Here, $f(x)$ is the image of $f$ in the residuel field of $X$ at $x$.

In particular, $\mathcal{O}_X(U)$ is not integral. A contradiction. This finishes the proof of (1). If we read carefully the proof of (1), it proves actually that if $X$ is irreducible and reduced, then $X$ is integral. This gives partially (2). The other direction is trivial, and hence the proof is finished. □

By definition, for $X$ a scheme, a *closed subscheme* of $X$, is an closed immersion $i : Y \hookrightarrow X$.

**Proposition 2.3.3.3.** *Let $X$ be a scheme, and $|Y| \subset |X|$ be a closed subspace of $|X|$. Then there is a unique closed subscheme structure $Y_{\text{red}}$ on $|Y|$ such that $Y_{\text{red}}$ is reduced. This is called the* reduced subscheme structure on *$Y$.*

*Sketch of the proof.* We first prove this proposition when $X = \operatorname{Spec}(A)$ is affine. In this case, $Y = V(I) \subset X$, then $Y_{\text{red}}$ is defined to be $\operatorname{Spec}(A/\sqrt{I})$, this gives then the *unique* reduced subscheme structure on $Y$.

Next, we consider the case where $X$ is an open of some affine schemes. For this case, we can cover $X$ by the affine opens $X = \bigcup_i U_i$ such that $U_i \cap U_j$ is again affine.[11] Now we take $Y_i = U_i \cap Y \hookrightarrow U_i$. By the previous construction for affine schemes, there is a unique reduced subscheme structure $Y_{i,\text{red}} \hookrightarrow U_i$. Moreover, let $U_{ij} = U_i \cap U_j$, then $Y_{i,\text{red}} \cap U_{ij} \hookrightarrow U_{ij}$ and $Y_{j,\text{red}} \cap U_{ij} \hookrightarrow U_{ij}$ give two reduced subscheme structures on $Y \cap U_{ij}$. Hence, by the uniqueness for the affine case, there exists a unique isomorphism between $Y_{i,\text{red}} \cap U_{ij} \hookrightarrow U_{ij}$ and $Y_{j,\text{red}} \cap U_{ij} \hookrightarrow U_{ij}$. Using these isomorphisms to glue the small pieces, we get a subscheme structure $Y_{\text{red}} \subset X$ on $Y$, which is also reduced. One can also verify that this scheme structure is unique.

It remains to treat the general case. We cover $X$ by affine open subschemes $X = \bigcup_i U_i$, and for each $i$ we take $Y_i = U_i \cap Y$ which is a closed subset of $U_i$. In particular, the previous construction for the affine case gives a canonical reduced subscheme structure $Y_{i,\text{red}}$ on $Y_i$. Now let $U_{ij} := U_i \cap U_j \subset X$, since $U_{ij} \hookrightarrow U_i$ is an open of an affine scheme, hence by the uniqueness in the second case, there exist an unique isomorphism between the two reduced subscheme structures $Y_{i,\text{red}} \cap U_{ij} \hookrightarrow U_{ij}$ and the closed immersion $Y_{j,\text{red}} \cap U_{ij} \hookrightarrow U_{ij}$ on $Y \cap U_{ij}$. Hence the conclusion follows again by gluing. This finishes the proof. □

**Remark 2.3.3.4.**

1. When we take $|Y| = |X|$ in the previous construction, then we get a reduced closed subscheme $X_{\text{red}} \hookrightarrow X$. This is always a homeomorphism on the underlying topological spaces, and it's an isomorphism of schemes iff $X$ is reduced.

2. For $Y \subset X$ a closed subset, then $Y_{\text{red}} \hookrightarrow X$ is also the *minimal* closed subscheme structure on $Y$: for any closed subscheme $Z \hookrightarrow X$ whose underlying topological space contains $Y$, then $Y_{\text{red}} \hookrightarrow X$ can be factorized through $Z$. To show this, by gluing processus, we may reduce to the case where $X$ is affine, then this assertion follows easily.

**Proposition 2.3.3.5.** *Let $X$ be an integral scheme, with generic point $\xi$.*

1. *Let $V$ be an affine open subset of $X$, then $\mathcal{O}_X(V) \to \mathcal{O}_{X,\xi}$ induces an isomorphism* $\operatorname{Frac}(\mathcal{O}_X(V)) \simeq \mathcal{O}_{X,\xi}$.

2. *For any open subset $U$ of $X$, and any point $x \in U$. The canonical morphisms $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\xi}$ are injective.*

3. *By identifying $\mathcal{O}_X(U)$ and $\mathcal{O}_{X,x}$ to subrings of $\mathcal{O}_{X,\xi}$, we have $\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x}$.*

[11] which means that $X$ is separated.

*Proof.* The first assertion (1) is clear since if $V = \mathrm{Spec}(A)$, then $\mathcal{O}_{X,\xi}$ is exactly the fraction field $\mathrm{Frac}(A)$, and the canonical map $\mathcal{O}_X(V) \to \mathcal{O}_{X,\xi}$ is then the natural map $A \subset \mathrm{Frac}(A)$. Hence the conclusion follows. For (2), let $f \in \mathcal{O}_X(U)$ be any element such that its image $f_x \in \mathcal{O}_{X,x}$ is zero. Then there exists an affine open neighborhood $V$ of $x$ such that $f|_V = 0$. In particular, the image $f_\xi \in \mathcal{O}_{X,\xi}$ is zero. Now by applying the first assertion, we see that for any affine open $V'$ of $X$, we have $f|_{V'} = 0$. Hence $f = 0$. This gives the injectivity of the first morphism. For the second injectivity, we take any affine open neighborhood $V = \mathrm{Spec}(A)$ of $x$, and suppose that $x$ corresponds to the prime ideal $\mathfrak{p} \subset A$, then the canonical map $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\xi}$ is just the canonical inclusion $A_\mathfrak{p} \subset \mathrm{Frac}(A)$, where comes the desired injectivity. For (3), we have clearly $\mathcal{O}_X(U) \subset \bigcap_{x \in U} \mathcal{O}_{X,x}$. Conversely, by the sheaf condition and the injectivity proved in (2), we may assume that $U = \mathrm{Spec}(A)$ is affine. Then we are reduced to show that $A = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} A_\mathfrak{p}$, seen as a subring of $\mathrm{Frac}(A)$. Indeed, for a fraction $f \in \mathrm{Frac}(A)$ which is contained in $\bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} A_\mathfrak{p}$, then for each $\mathfrak{p} \in \mathrm{Spec}(A)$, there exists $s_p \in A - \mathfrak{p}$, and $a_p \in A$ such that $f = a_\mathfrak{p}/s_\mathfrak{p}$. As $A$ is integral, we deduce then $f \cdot s_\mathfrak{p} \in A$. Now, if we take the family $\{s_\mathfrak{p} \colon \mathfrak{p} \in \mathrm{Spec}(A)\}$, which generates the unit ideal. Hence, one can find $b_\mathfrak{p} \in A$, almost all zero, such that $1 = \sum_\mathfrak{p} b_\mathfrak{p} s_\mathfrak{p}$. From where, we find $f = \sum_\mathfrak{p} b_\mathfrak{p} f s_\mathfrak{p} = \sum_\mathfrak{p} b_\mathfrak{p} a_\mathfrak{p} \in A$. This gives the result. □

**Definition 2.3.3.6.** Let $X$ be an integral scheme, with generic point $\xi$. We denote the field $\mathcal{O}_{X,\xi}$ by $K(X)$. Sometimes, when $X$ is an algebraic over a field $k$, we also denote $K(X)$ by $k(X)$. An element of $K(X)$ is called a *rational function* on $X$. We call $K(X)$ the *field of rational functions* or *function field* of $X$. We say that $f \in K(X)$ is *regular at* $x \in X$, if $f \in \mathcal{O}_{X,x}$.

With the terminology above, a rational function $f$ is regular at any point of $x \in U$ is contained in $\mathcal{O}_X(U)$.

### 2.3.4 Finiteness conditions

**Definition 2.3.4.1.** Let $f \colon X \to Y$ be a morphism of schemes.

1. $f$ is said to be *quasi-compact*, if for any quasi-compact open subset $V \subset Y$, the inverse image $f^{-1}(U)$ is quasi-compact.

2. $f$ is said to be *locally of finite type*, if for any affine opens $U = \mathrm{Spec}(A) \subset X$ and $V = \mathrm{Spec}(B) \subset Y$ such that $f(U) \subset V$, and that the map $B = \mathcal{O}_Y(V) \to \mathcal{O}_X(U) = A$ makes $A$ into a $B$-algebra of finite type.

3. $f$ is said to be *finite type*, if $f$ is quasi-compact and locally of finite type.

**Example 2.3.4.2.** If $X$ is quasi-compact, $Y$ is affine, then any morphism $f \colon X \to Y$ is quasi-compact. Indeed, since $X$ is quasi-compact, $X$ can be written as a finite union of affine open subsets of $X$. As a result, for any principal subset $U \subset Y$, $f^{-1}(U)$ can be covered by finitely many affine schemes, in particular $f^{-1}(U)$ is quasi-compact. Now, let $V$ be an arbitrary affine open of $Y$. Since $V$ can be covered by finitely many principal open subsets of $Y$, we find that $f^{-1}(V)$ can be equally covered by finitely many affine opens of $X$. From here, we find that $f^{-1}(V)$ is quasi-compact. This gives the result.

**Proposition 2.3.4.3.** *Let $f \colon X \to Y$ be a morphism.*

1. *Suppose that there exists a covering $\{V_i\}_i$ of $Y$ by affine open subsets such that for every $i$, $f^{-1}(Y_i)$ is a union of affine open subsets $U_{ij}$ such that $\mathcal{O}_X(U_{ij})$ is an $\mathcal{O}_Y(V_i)$-algebra of finite type. Then $f$ is locally of finite type.*

2. *Suppose that there exists a covering $\{V_i\}_i$ by* affine *open subsets, $f^{-1}(V_i)$ is quasi-compact. Then $f$ is quasi-compact.*

*Proof.* (1) We begin with the following remark: if $B$ is an $A$-algebra of finite type, then for any $f \in B$, we know that $B_f \simeq B[X]/(fX-1)$ is still of finite type over $A$. Hence to prove (1), we may assume that $X = \mathrm{Spec}(B)$, and $Y = \mathrm{Spec}(A)$ are both affine. The hypothesis says that we can find open covering $Y = \bigcup_i Y_i$ with $Y_i = D(a_i) \subset \mathrm{Spec}(A) = Y$, and affine covering of $f^{-1}(Y_i) = \bigcup_j X_{ij}$ such that $X_{ij}$ is affine, and that $\mathcal{O}_X(X_{ij})$ is finitely generated as $\mathcal{O}_Y(Y_i)$-algebra. Moreover, since $Y_i$ is principal, so is $f^{-1}(Y_i) = D(f^\sharp(a_i))$. Hence we may assume that the coverings $Y = \bigcup_i Y_i$. and $f^{-1}(Y_i) = \bigcup_j X_{ij}$ are all *finite*. We claim first that it suffices to prove that $\mathcal{O}_X(f^{-1}(a_i))$ is finitely generated as $\mathcal{O}_Y(D(a_i))$-algebra. Indeed, we know $\mathcal{O}_Y(Y_i) = A_{a_i}$, and $\mathcal{O}_X(f^{-1}(D(a_i))) = B_{b_i}$ with $b_i = f^\sharp(a_i)$. If $B_{b_i}$ is finitely generated, namely $B_{b_i} = A_{a_i}[c_{i,1}, \cdots, c_{i,r_i}]$ for some $c_{i,j} \in B_{b_i}$. Since $b_i = f^\sharp(a_i)$, we may assume that $c_{i,j} \in B$. Then one can show $B = A[c_{i,j} | i \in I, 1 \le j \le r_i]$. In particular, $B$ is finitely generated as $A$-algebra. Now, to complete the proof, we may then assume $X = \bigcup_{i=1}^n X_i$ such that $X_i \subset X$ is principal affine, and that $\mathcal{O}_X(X_i)$ is a finitely generated $A$-algebra. Suppose $X_i = D(g_i)$, then there exists $c_{i,1}, \cdots, c_{i,s_i} \in B_{g_i}$ such that $B_{g_i} = A[c_{i,1}, \cdots, c_{i,s_i}]$. We may moreover assume $c_{i,j} = c'_{i,j}/g_i^m$ for some $m$ sufficiently large, and $c'_{i,j} \in B$. Since $X = \bigcup_{i=1}^n D(g_i)$, there exists some $\lambda_i \in B$ such that $1 = \sum_{i=1}^n \lambda_i \cdot g_i$. Now, we claim that

$$B = A[g_i, \lambda_i, c'_{i,j} | 1 \le i \le n, 1 \le j \le s_i].$$

For any $b \in B$, its image in $B_{b_i}$ can be written as $f_i(c_{i,1}, \cdots, c_{i,s_i})$ with $f_i \in A[X_{i,1}, \cdots, X_{i,s_i}]$ a polynomial. As a result, there exists a polynomial $F_i \in A[X_{i,1}, \cdots, X_{i,s_i}, Y_i]$ and some integer $N \gg 0$ such that $b = F_i(c'_{i,1}, \cdots, c'_{i,s_i}, g_i)/g_i^N$. Hence, there exists $M \gg 0$, such that

$$g_i^{M+N} b = g_i^M F_i(c'_{i,1}, \cdots, c'_{i,s_i}, g_i) \in B.$$

As $1 = 1^{n(M+N)} = (\sum_i \lambda_i g_i)^{n(M+N)} = \sum_i \lambda'_i g_i^{M+N}$ with $\lambda'_i \in B[\lambda_i, g_i | 1 \le i \le n]$. So finally, we get

$$b = b \cdot 1 = \sum_i \lambda'_i g_i^{M+N} b = \sum_i \lambda'_i g_i^M F_i(c'_{i,1}, \cdots, c'_{i,s_i} . g_i) \in A[g_i, \lambda_i, c'_{i,j}].$$

This gives (1). To prove (2), by Example 2.3.4.2, the restriction $f|_{f^{-1}(V_i)}$ is quasi compact. As a result, if we consider the set $\mathcal{B}$ of quasi-compact opens $V$ in $Y$ such that $f^{-1}(V)$ is also quasi-compact, this set $\mathcal{B}$ gives a base for the open subsets of $Y$. Now for any quasi-compact open $V \subset Y$, it can be written as a finite union $V = \bigcup_{i=1}^n V_i$ with $V_i \in \mathcal{B}$. As a result, $f^{-1}(V) = \bigcup_{i=1}^n f^{-1}(V_i)$ can be written as a finite union of quasi-compact open subsets. In particular, $f^{-1}(V)$ is quasi-compact. This gives (2). □

## 2.4 Dimension

### 2.4.1 Dimension of a topological space

Let $X$ be a topological space. A *chain of irreducible closed subsets of* $X$ is a strictly ascending sequence of irreducible closed subsets

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subset X.$$

The integer $n$ is called the *length of the chain.*

**Definition 2.4.1.1.** Let $X$ be a topological space. We define the (Krull) dimension of $X$, which we denote by $\dim(X)$, to be the supremum of the lengths of the chains of irreducible closed subsets. Note that this number is might be not finite. Moreover, by convention, the empty set is of dimension $-\infty$. $X$ is called *purely of dimension* $n$ if all its irreducible components are of the same dimension $n$.

**Example 2.4.1.2.**

1. A discrete topological space is of dimension 0.
2. For $k$ a field, then the underlying topological space of $\mathbb{A}^1_k$ is of dimension 1.

**Definition 2.4.1.3.** For $X$ a topological space, and $x \in X$ a point. We put

$$\dim_x X = \inf\{\dim U \colon U \text{ an open neighborhood of } x\}$$

**Proposition 2.4.1.4.** *Let $X$ be a topological space.*

1. *For any subset $Y$ of $X$ endowed with the induced topology, then* $\dim Y \leq \dim X$.
2. *Suppose $X$ irreducible of finite dimension, and let $Y \subset X$ be a closed subset. Then $Y = X$ iff* $\dim Y = \dim X$.
3. *The dimension of $X$ is the supremum of the dimensions of its irreducible components.*
4. *We have* $\dim X = \sup\{\dim_x X \colon x \in X\}$.

*Proof.* (1) Note that, for a closed subset $Z$ of $Y$, if we note $\overline{Z}$ the closure of $Z$ in $X$, then $\overline{Z} \cap Y = Z$. Indeed, we only need to show that $\overline{Z} \cap Y \subset Z$. But, as $Z \subset Y$ is closed with respect to the subspace topology on $Y$, hence $Z = F \cap Y$ with $F \subset X$ closed. In particular, we have $\overline{Z} \subset F$, hence $\overline{Z} \cap Y \subset F \cap Y = Z$, this proves the claim. As a corollary, for any two closed subsets of $Y$: $Z_1 \subsetneq Z_2$, their closures verify $\overline{Z_1} \subsetneq \overline{Z_2}$. Then the first assertion follows easily. The assertions (2), (3) and (4) are clear from definition. □

**Definition 2.4.1.5.** Let $Y$ be an *irreducible* closed subset of $X$, we define the *codimension of $Y$ in $X$* to be the supremum of the lengths of the chains of irreducible closed subsets of $X$ containing $Y$:

$$Y = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subset X.$$

Let $Z$ be a closed subset of $X$, the *codimension of $Z$ in $X$*, which we denote by $\operatorname{codim}(Z, X)$, is the infimum of the codimensions in $X$ of the irreducible components.

**Exercise 2.4.1.6.** Let $X$ be a topological space with $Z \subset X$ a closed subset of $X$. Show that

$$\operatorname{codim}(Z, X) + \dim(Z) \leq \dim(X).$$

But the equality doesn't hold in general. Find a conter-example.

### 2.4.2 Dimension of schemes and rings

Let $X$ be a scheme. Its *dimension* is defined to be the dimension of its underlying topological space. For $A$ a ring, we define its (Krull) dimension to be the dimension of the corresponding affine scheme $\operatorname{Spec}(A)$, denoted by $\dim(A)$. According to the inclusion-reversing one-to-one correspondence between the set of irreducible closed subsets and the set of prime ideals of $A$, dim is also the supremum of the lengths of strictly ascending chains of prime ideals of $A$:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subset A.$$

For $\mathfrak{p} \subset A$ a prime ideal, we define the *height of* $\mathfrak{p}$ (resp. *depth of* $\mathfrak{p}$) to be the supremum of the lengths of strictly ascending chains of prime ideals contained in $\mathfrak{p}$ (resp. containing $\mathfrak{p}$), denoted by $\mathrm{ht}(\mathfrak{p})$ (resp. $\mathrm{depth}(\mathfrak{p})$). Via the usual correspondence, we have $\mathrm{ht}(\mathfrak{p}) = \mathrm{codim}(V(\mathfrak{p}), \mathrm{Spec}(A))$. Moreover, since there is a one-to-one correspondence between the set of prime ideals of $A$ contained in $\mathfrak{p}$, and the set the prime ideals of $A_{\mathfrak{p}}$, we find also $\mathrm{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}})$, and we have

$$\mathrm{ht}(\mathfrak{p}) + \mathrm{depth}(\mathfrak{p}) \leq \dim(A).$$

Similarly, $\mathrm{depth}(\mathfrak{p}) = \dim(V(\mathfrak{p}))$.

**Proposition 2.4.2.1.** *Let $A$ be a ring. Then*

1. *Let $\mathfrak{n} \subset A$ be a nilpotent ideal.*[12] *Then* $\dim(A) = \dim(A/\mathfrak{n})$.
2. *We have* $\dim(A) = \sup_{\mathfrak{m} \in \mathrm{Max}(A)} \{\dim(A_{\mathfrak{m}})\}$

**Example 2.4.2.2.** A field $k$ is of dimension 0. A DVR which is not a field is then of dimension 1. Any Dedekind domain is also which is not a field (e.g. the ring of integers of an algebraic number field) is also of dimension 1.

Recall that a morphism of rings $\phi\colon A \to B$ is called *integral* if $B$ is integral over the image of $A$, that is, for any $b \in B$, there exist $a_0, \cdots, a_{n-1} \in A$ such that

$$b^n + \phi(a_{n-1}) \cdot b^{n-1} + \cdots \phi(a_1)b + \phi(a_0) = 0.$$

**Lemma 2.4.2.3.** *Let $\phi\colon A \to B$ be an injective integral morphism. If $B$ is a field, so is $A$.*

*Proof.* First of all, $A$ is a domain as $\phi$ is injective. Let $a \in A - \{0\}$, we consider $1/a \in B$, which satisfies then an integral relation with coefficient: one can find $a_0, \cdots, a_{n-1} \in A$ such that

$$(1/a)^n + a_{n-1} \cdot (1/a)^{n-1} + \cdots a_1 \cdot (1/a) + a_0 = 0.$$

Hence, we get

$$1 + a \cdot (a_{n-1} + a_{n-2} \cdot a + \cdots + a_1 \cdot a^{n-2} + a_0 \cdot a^{n-1}) = 0$$

In particular, $a^{-1} = -(a_{n-1} + a_{n-2} \cdot a + \cdots + a_1 \cdot a^{n-2} + a_0 \cdot a^{n-1}) \in A$. This gives the proof. □

**Proposition 2.4.2.4.** *Let $\phi\colon A \to B$ be an integral homomorphism. For any $\mathfrak{q} \in \mathrm{Spec}(B)$, take $\mathfrak{p} = \phi^{-1}(\mathfrak{q}) \in \mathrm{Spec}(A)$.*

1. *We have* $\mathrm{ht}(\mathfrak{q}) \leq \mathrm{ht}(\mathfrak{p})$. *In particular,* $\dim(B) \leq \dim(A)$.
2. *If moreover $\phi$ is injective. Then then induced map* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is surjective. Moreover, we have* $\mathrm{depth}(\mathfrak{p}) = \mathrm{depth}(\mathfrak{q})$ *and* $\dim(A) = \dim(B)$.

*Proof.* (1) We will show that for any prime ideal $\mathfrak{q}_1 \subsetneq \mathfrak{q}$ of $B$, then $\phi^{-1}(\mathfrak{q}_1) \subsetneq \phi^{-1}(\mathfrak{q})$. Indeed, up to replace $A$ by $A/\phi^{-1}(\mathfrak{q}_1)$, and $B$ by $B/\mathfrak{q}_1$, we may assume that $\phi$ is injective, and that $\mathfrak{q}_1 = 0$. So we need to show that $\phi^{-1}(\mathfrak{q}) \neq 0$. So let $b \in \mathfrak{q} - \{0\}$ be any non zero element, with minimal integral polynomial $X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0 \in A[X]$. We remark that $a_0 \neq 0$, and

$$\phi(a_0) = -(b^n + \phi(a_{n-1}) \cdot b^{n-1} + \cdots + \phi(a_1) \cdot b) \in \mathfrak{q}.$$

[12] It means that any element of $\mathfrak{n}$ is nilpotent. When $\mathfrak{n}$ is of finite type, this means also that $\mathfrak{n}^r = 0$ for some $r > 0$.

In particular, $a_0 \in \phi^{-1}(\mathfrak{q})$, and $\phi^{-1}(\mathfrak{q}) \neq 0$. This gives the assertion. Let now

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n \subset \mathfrak{q}$$

be a strictly ascending chain of prime ideals contained in $\mathfrak{p}$, by applying $\phi^{-1}(-)$ and the previous assertion, we find

$$\phi^{-1}(\mathfrak{q}_0) \subsetneq \phi^{-1}(\mathfrak{q}_1) \subsetneq \cdots \subsetneq \phi^{-1}(\mathfrak{q}_n) \subset \phi^{-1}(\mathfrak{q}) = \mathfrak{p},$$

from where we find $\mathrm{ht}(\mathfrak{q}) \leq \mathrm{ht}(\mathfrak{p})$. In particular, $\dim(B) \leq \dim(A)$. This gives (1).

For the second statement, we show first that $\phi^*\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Since $\phi : A \to B$ is injective, we identify $A$ with a subring of $B$ over which $B$ is integral. Let $\mathfrak{p} \in \mathrm{Spec}(A)$, up to replace $A$ by $A_{\mathfrak{p}}$, and $B$ by $S^{-1}B$ with $S = A - \mathfrak{p}$, we may assume that $\mathfrak{p} \subset A$ is the *only* maximal. Let now $\mathfrak{q}$ be a maximal ideal of $B$ (note that $A \subset B$, hence the ring $B$ is non trivial). Then $\mathfrak{q} \cap A$ is also a maximal ideal according to the previous lemma. So we must have $\mathfrak{q} \cap A = \mathfrak{p}$, and this proves the surjectivity of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. As a result, for any strictly ascending chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subset A,$$

one can find inductively an ascending chain of prime ideals

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n \subset B$$

such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$. As a result, $\dim(A) \leq \dim(B)$, hence $\dim(A) = \dim(B)$ by taking account the first statement. Moreover, since $\mathfrak{p} = \mathfrak{q} \cap A$, the morphism $A/\mathfrak{p} \to B/\mathfrak{q}$ is integral and injective. In particular, $\dim(A/\mathfrak{p}) = \dim(B/\mathfrak{q})$, that is, $\mathrm{depth}(\mathfrak{p}) = \mathrm{depth}(\mathfrak{q})$. □

The previous proof gives also the so-called *going up theorem* (see the book of Atiyah-MacDonald). There exists also a similar going down theorem, which we will include below without proof (see the book of Atiyah-MacDonald for the proof). Recall that an integral domain $A$ is called *normal* if for any $x \in \mathrm{Frac}(A)$ with $x$ integral over $A$, then $x \in A$.

**Theorem 2.4.2.5** (going down theorem)**.** *Let $A \subset B$ be integral domains such that $B$ is integral over $A$, and that $A$ is normal. Let*

$$\mathfrak{p}_1 \supset \mathfrak{p}_2 \supset \cdots \supset \mathfrak{p}_n$$

*and*

$$\mathfrak{q}_1 \supset \mathfrak{q}_2 \supset \cdots \supset \mathfrak{q}_m$$

*be decreasing chains of prime ideals of $A$ and $B$ respectively with $M < n$ such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for $i \leq m$. Then one can extend the chain $\mathfrak{q}_1 \supset \mathfrak{q}_2 \supset \cdots \supset \mathfrak{q}_m$ to a chain*

$$\mathfrak{q}_1 \supset \mathfrak{q}_2 \supset \cdots \supset \mathfrak{q}_m \supset \mathfrak{q}_{m+1} \supset \cdots \supset \mathfrak{q}_n$$

*such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $i \leq n$.*

### 2.4.3 The noetherian case

We start by characterizing the scheme of dimension 0. Recall that a ring $A$ is called *artinian* if every descending sequence of ideals of $A$ is stationary.

**Lemma 2.4.3.1.** *Let $(A, \mathfrak{m})$ be a noetherian local ring. The following conditions are equivalent:*

1. $\dim(A) = 0$;

2. $\mathfrak{m} = \sqrt{0}$;

3. *There exists* $q \geq 1$ *such that* $\mathfrak{m}^q = 0$;

4. *A is artinian.*

*Proof.* Assume (1), then $A$ contains only *one* prime ideal, namely the maximal ideal $\mathfrak{m}$, hence $\sqrt{0} = \bigcap_{\mathfrak{p}\in\mathrm{Spec}(A)} \mathfrak{p} = \mathfrak{m}$. This gives (2), and hence (3) since $\mathfrak{m}$ is of finite type. Now, assume (3), we claim that $\mathfrak{m}$ is of finite length as $A$-modules. Indeed, as $\mathfrak{m}^q = 0$ for some $q > 0$, hence we are only need to show that $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is of finite length as $A$-module. By this is clear since as a $A/\mathfrak{m}$-space, it's of finite dimensional. Now, it's easy to see that $A$ is artinian since any ideal $A$ must have finite length as $A$-modules. This shows (4). It remains to show that (4) implies (1). So we need to show that $\mathfrak{m}$ is the only prime ideal of $A$, hence we only need to prove that any element of $\mathfrak{m}$ is nilpotent: let $a \in \mathfrak{m}$, we consider the descending sequence of ideals $(a) \supset (a^2) \supset \cdots (a^r) \supset \cdots$. Hence there is some integer $r_0 \gg 0$ such that $(a^{r_0}) = (a^r)$ for any $r \geq r_0$. In particular, $a^{r_0} = u \cdot a^{r_0+1}$ for some $u \in A$. As a result $(1-ua)\cdot a^{r_0} = 0$, hence $a^{r_0} = 0$ as $1 - ua \in A^*$. This shows that $\mathfrak{m}$ is nilpotent, hence $\mathfrak{m}$ is the only prime ideal of $A$, that is, $\dim(A) = 0$. □

**Proposition 2.4.3.2.** *Let $X$ be a* noetherian *scheme of dimension* 0*. Then (i) $X$ is affine with finite cardinality; (ii) any point $x \in X$ is open; and (iii) $\mathcal{O}_X(X)$ is a product of artinian rings.*

*Proof.* Let $U = \mathrm{Spec}(A) \subset X$ be an affine open of $X$, then $U$ is also noetherian of dimension 0. Let $U = \cup_i^n Y_i$ its decomposition into the union of its irreducible components. In particular, $Y_{i,\mathrm{red}}$ is affine, hence, its contains at least a closed point $x_i \in Y_i$. But since $Y_i$ has dimension zero, we find that $\{x_i\} = Y_i$ is a single point. In particular, this shows that $U$ is a finite set, and each point is open in $U$, which give immediately the first assertion as $X$ is noetherian. In particular $X$ is affine. Let $X = \{x_1, \cdots, x_n\}$, we have then

$$\mathcal{O}_X(X) = \prod_{i=1}^n \mathcal{O}_X(\{x_i\}) = \prod_{i=1}^n \mathcal{O}_{X,x}.$$

$\mathcal{O}_{X,x}$ is local noetherian of dimension 0, hence, it's artinian. □

**Theorem 2.4.3.3** (Krull's principal ideal theorem)**.** *Let $A$ be a noetherian ring, and $f \in A$ be a non-invertible element. Then for any prime ideal $\mathfrak{p}$ that is minimal among those containing $f$, then* $\mathrm{ht}(\mathfrak{p}) \leq 1$.

*Proof.* Cf de book of Liu. □

**Lemma 2.4.3.4.** *Let $A$ be a noetherian ring, and $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be an ascending sequence of prime ideals of $A$. Let $f \in A$ be an element such that $f \in \mathfrak{p}_n$. Then there exists a sequence of prime ideals $\mathfrak{p}'_1 \subsetneq \mathfrak{p}'_2 \subsetneq \cdots \subsetneq \mathfrak{p}'_n = \mathfrak{p}_n$ such that $f \in \mathfrak{p}'_1$.*

*Proof.* The proof is done by induction on $n$. If $n = 1$, there is nothing to prove. For $n > 1$, we may assume that $f \notin \mathfrak{p}_{n-1}$. Consider the quotient $A/\mathfrak{p}_{n-2}$, then the image $\bar{f} \in A/\mathfrak{p}_{n-2}$ is non zero, and let $\mathfrak{q}$ be the minimal prime ideal containing $\bar{f}$, and contained in $\mathfrak{p}_n/\mathfrak{p}_{n-2} \subset A/\mathfrak{p}_{n-2}$. Let $\mathfrak{p}'_{n-1} \subset A$ be the preimage of $\mathfrak{q}$. Then $f \in \mathfrak{p}'_{n-1}$, and $\mathfrak{p}_{n-2} \subsetneq \mathfrak{p}'_{n-1} \subsetneq \mathfrak{p}_n$. Now, it suffices to apply the induction hypothesis to the ascending sequence $\mathfrak{p}_0 \subsetneq \cdots \mathfrak{p}_{n-2} \subsetneq \mathfrak{p}'_{n-1}$ to conclude. □

**Corollary 2.4.3.5.** *Let $A$ be a noetherian ring, and $I$ an ideal generated by $r$ elements.*

1. *Let* $\mathfrak{p} \subset A$ *be a prime ideal of* $A$*, minimal among those containing* $I$*, then* $\mathrm{ht}(\mathfrak{p}) \leq r$.

2. *If, moreover,* $A$ *is local with maximal ideal* $\mathfrak{m}$*, then* $A$ *is of finite dimension, and* $\dim(A) \leq \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$.

*Proof.* We will prove (1) by induction. When $I$ is generated by one element $f$, and $\mathfrak{p}$ be a prime ideal, minimal among those containing $f$ (note that this implies implicitly that $f$ is non-invertible), the previous theorem says then $\mathrm{ht}(\mathfrak{p}) \leq 1$. Suppose now this statement is proved for an ideal of a noetherian ring generated by at most $r-1$ elements (for $r \geq 2$). Let $I = (f_1, \cdots, f_r)$, $\tilde{A} := A/(f_r)$, and $\tilde{I}$ be the image of $I$ in $\tilde{A}$. Then $\tilde{I}$ can be generated by $r-1$ elements. Let $\mathfrak{p} \subset A$ be a prime ideal, minimal among those containing $I$. In particular, $f_1 \in \mathfrak{p}$, and the image $\tilde{\mathfrak{p}} \subset \tilde{A}$ is then a prime ideal which is minimal among those containing $\tilde{I}$. In particular, $\mathrm{ht}(\tilde{\mathfrak{p}}) \leq r-1$. On the other hand, let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$ be an ascending sequence of prime ideals contained in $\mathfrak{p}$. By the previous lemma, one can find a strictly ascending sequence of prime ideals $\mathfrak{p}_1' \subsetneq \mathfrak{p}_2' \subsetneq \cdots \subsetneq \mathfrak{p}_n' = \mathfrak{p}$ such that $f_r \in \mathfrak{p}_1'$. Hence their images in $\tilde{A}$ give a strictly ascending sequence of prime ideals contained in $\tilde{\mathfrak{p}}$. As a result, we must have $n-1 \leq \mathrm{ht}(\tilde{\mathfrak{p}}) \leq r-1$. Hence $n \leq r$. This gives (1). For the second statement, let $r = \dim_{A/\mathfrak{m}}\mathfrak{m}/\mathfrak{m}^2$. According to (1), we only need to show that $\mathfrak{m}$ then can be generated by $r$ elements. This latter statement follows from Nakayama's lemma. □

**Remark 2.4.3.6.** The geometric interpretation of the previous corollary is that, for $X$ a noetherian scheme, and for $Y \subset X$ a closed subscheme which can locally be defined by $r$ elements, then $\mathrm{codim}(Y, X) \leq r$.

**Theorem 2.4.3.7.** *Let* $(A, \mathfrak{m})$ *be a noetherian local ring, and* $f \in \mathfrak{m}$*. Then* $\dim(A/fA) \geq \dim(A) - 1$*. Moreover, the equality holds if* $f$ *is not contained in any minimal prime ideal of* $A$.

*Proof.* The fist statement is contained in Lemma 2.4.3.4. For the last assertion, suppose that $f$ is not contained in any minimal prime ideal of $A$, and let $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_d$ be a strictly ascending sequence of prime ideals in $A/f$ of longest length (hence $d = \dim(A/f)$), then the preimages of these ideals give a strictly ascending sequence of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d$. Moreover, as $f \in \mathfrak{p}_0$, this prime ideal is not minimal. Hence $\mathfrak{p}_0$ contains at least one prime ideal $\mathfrak{p}_{-1} \subsetneq \mathfrak{p}_0$. In this way, we find a strictly ascending sequence of prime ideals of length $d+1$. Hence $\dim(A) = \dim(A/f) + 1$. □

Finally, we consider the dimension of an affine space.

**Lemma 2.4.3.8.** *Let* $(A, \mathfrak{m})$ *be a noetherian local ring, and* $\mathfrak{n} \subset A[T]$ *be a maximal ideal such that* $\mathfrak{n} \cap A = \mathfrak{m}$*. Then* $\mathrm{ht}(\mathfrak{n}) = \dim(A) + 1$.

*Proof.* Since for any prime ideal $\mathfrak{p} \subset A$ of $A$, $\mathfrak{p}A[T] \subset A[T]$ is a prime ideal contained in $\mathfrak{m}A[T]$. In particular, $\mathrm{ht}(\mathfrak{n}) \geq \mathrm{ht}(\mathfrak{m}A[T]) \geq \dim(A)$. Moreover, as $A[T]/\mathfrak{m}A[T] \simeq (A/\mathfrak{m})[T]$ is not a field, $\mathfrak{m}A[T] \subsetneq \mathfrak{n}$. Hence $\mathrm{ht}(\mathfrak{n}) \geq \dim(A) + 1$. It remains to show that $\mathrm{ht}(\mathfrak{n}) \leq \dim(A) + 1$. We will show this inequality by induction on $\dim(A)$ (which is finite by the noetherian hypothesis). If $\dim(A) = 0$, $\mathfrak{m} \subset A$ is nilpotent, hence $\mathfrak{m}A[T]$ is also nilpotent. As a result, $\dim(A[T]) = \dim(A[T]/\mathfrak{m}A[T]) = \dim((A/\mathfrak{m})[T]) = 1$, and any maximal ideal of $A[T]$ is of height 1. Suppose now $\dim(A) > 0$, in particular, $\mathfrak{m}$ is of height $\geq 1$. Let $f \in \mathfrak{m}$ which is not contained in any of the minimal prime ideals of $A$ (see the exercise below), by the previous theorem, $\dim(A/fA) = \dim(A) - 1$. Moreover, the maximal ideal $\bar{\mathfrak{n}} = \mathfrak{n}/fA[T] \subset (A/f)[T]$ is of height $\leq \dim(A/f) + 1 = \dim(A)$ (induction hypothesis). Now lemma 2.4.3.4 implies then $\mathrm{ht}(\mathfrak{n}) \leq \dim(A) + 1$: otherwise, we find find a strictly ascending sequence of prime ideals of length $n > \dim(A) + 1$: $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{n}$. Since $f \in \mathfrak{n}$, by 2.4.3.4, we find hence a sequence of length $n-1 > \dim A$: $\mathfrak{p}'_1 \subsetneq \cdots \subsetneq \mathfrak{p}'_n = \mathfrak{n}$ such that $f \in \mathfrak{p}'_1$, a contradiction. This gives the inequality, and hence the lemma. □

**Exercise 2.4.3.9.** Let $A$ be a ring, and $\mathfrak{a}, \mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be $n+1$ ideals of $A$ such that $\mathfrak{p}_i$'s are prime ideals. Show that if $\mathfrak{a} \subset \bigcup_{i=1}^n \mathfrak{p}_i$, then $\mathfrak{a}$ is contained in one of the ideals $\mathfrak{p}_i$.

**Lemma 2.4.3.10.** *Let $A$ be any noetherian ring, and $\mathfrak{p} \subset A$ be a prime ideal. Then* $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}R[T])$.

*Proof.* First of all, for any chain of prime ideals of $A$ contained in $\mathfrak{p}$:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p},$$

it gives a chain of prime ideals of $A[T]$ contained in $\mathfrak{p}A[T]$:

$$\mathfrak{p}_0A[T] \subsetneq \mathfrak{p}_1A[T] \subsetneq \cdots \subsetneq \mathfrak{p}_nR[T] = \mathfrak{p}A[T].$$

Hence $\mathrm{ht}(\mathfrak{p}A[T]) \geq \mathrm{ht}(\mathfrak{p})$. It remains to show that $\mathrm{ht}(\mathfrak{p}A[T]) \leq \mathrm{ht}(\mathfrak{p})$. For any chain of prime ideals of $A[T]$ contained in $\mathfrak{p}A[T]$

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n = \mathfrak{p}A[T],$$

we have $\mathfrak{q}_i \cap A \subset \mathfrak{q}_n \cap A = \mathfrak{p}$. Hence the above chain induces also a chain of prime ideals of $S^{-1}A[T] = A_{\mathfrak{p}}[T]$ (where $S = A - \mathfrak{p}$):

$$S^{-1}\mathfrak{q}_0 \subsetneq S^{-1}\mathfrak{q}_1 \subsetneq \cdots \subsetneq S^{-1}\mathfrak{q}_n = \mathfrak{p}A_{\mathfrak{p}}[T],$$

Hence $\mathrm{ht}(\mathfrak{p}A[T]) \leq \mathrm{ht}(\mathfrak{p}A_{\mathfrak{p}}[T])$. But since $\mathfrak{p}A_{\mathfrak{p}}[T]$ is *not* a maximal ideal, we must have $\mathrm{ht}(\mathfrak{p}A_{\mathfrak{p}}[T]) \leq \dim(A_{\mathfrak{p}}[T]) - 1 = \dim(A_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p})$. So, we obtain the desired inequality $\mathrm{ht}(\mathfrak{p}A[T]) \leq \mathrm{ht}(\mathfrak{p})$. □

**Lemma 2.4.3.11.** *Let $A$ be any ring, and $\mathbb{Q}_1 \subsetneq \mathfrak{q}_2$ be two prime ideals of $A[T]$. Suppose $\mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A =: \mathfrak{p}$, then $\mathfrak{q}_1 = \mathfrak{p}A[T]$.*

*Proof.* Up to replace $A$ by $A/\mathfrak{p}$, we may assume that $\mathfrak{p} = 0$. So we need to show that $\mathfrak{q}_1 = 0$. As $\mathfrak{q}_i \cap A = (0)$, they induce two non trivial ideals of $K[T]$: $\mathfrak{q}_1K[T] \subsetneq \mathfrak{q}_2K[T]$. But $K[T]$ is an principal ideal domain, so we must have $\mathfrak{q}_1K[T] = 0$. In particular, $\mathfrak{q}_1 = 0$. □

**Proposition 2.4.3.12.** *Let $A$ be a noetherian ring, then* $\dim(A[T]) = \dim(A) + 1$

*Proof.* Clearly, we have $\dim(A[T]) \leq \dim(A) + 1$. On the other hand, consider a chain of prime ideal of $A[T]$:

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n \subset A[T],$$

we need to show that $n \leq \dim(A[T])$. We may assume that there exists an index $i$ such that $\mathfrak{q}_i \cap A = \mathfrak{q}_{i+1} \cap A$ (otherwise, the above chain induces a chain of prime ideals of $A$, hence $n \leq \dim(A) < \dim(A) + 1$), and we choose $j$ to be the largest integer $i$ with respect to this properties. We have hence the following chain of prime ideals of $A$:

$$\mathfrak{p} := \mathfrak{q}_{j+1} \cap A \subsetneq \mathfrak{q}_{j+2} \cap A \subsetneq \cdots \subsetneq \mathfrak{q}_n \cap A \subset A.$$

As a result, we find $\dim(A) \geq n - j - 1 + \mathrm{ht}(\mathfrak{p})$. Now, on applying Lemma 2.4.3.11, we find $\mathfrak{q}_j = \mathfrak{p}A[T]$. Now Lemma 2.4.3.10 tells us $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{q}_j) \geq j$. As a result, $\dim(A) \geq n - j - 1 + \mathrm{ht}(\mathfrak{p}) \geq n - 1$, that is, $n \leq \dim(A) + 1$. This finishes the proof. □

**Corollary 2.4.3.13.** *Let $A$ be a noetherian ring. Then* $\dim(A[T_1, \cdots, T_n]) = \dim(A) + n$.

### 2.4.4 Dimension of schemes over a field

So here, we consider the case where $A$ is a $k$-algebra of finite type, with $k$ a field. Recall first that for a finite type field extension $k \subset K$, there exists an integer $d$, and elements $t_1, \cdots, t_d \in K$ such that $k(t_1, \cdots, t_d) \simeq \mathrm{Frac}(k[T_1, \cdots, T_d])$, and that $K$ is a finite algebraic extension over $k(t_1, \cdots, t_d)$. The integer $d$ is called the *transcendent degree* of $K$ over $k$, denoted by $\mathrm{tr.deg}(K/k)$. It's uniquely determined. We recall the famous

**Proposition 2.4.4.1** (Noether normalization lemma)**.** *Let $A$ be a finitely generated algebra over a field $k$. Then there exists an integer $d \geq 0$, and a finite injective homomorphism $k[T_1, \cdots, T_d] \hookrightarrow A$.*

When, moreover, $A$ is integral, the integer $d$ in the Noether normalization lemma is the transcendent degree of $K := \mathrm{Frac}(A)$ over $k$. In particular, we get the following

**Corollary 2.4.4.2.** *Let $X$ be an integral scheme of finite type over a field $k$. Then for any non empty open subset $U \subset X$, we have* $\dim(U) = \dim(X) = \mathrm{tr.deg}(K(X)/k)$.

**Lemma 2.4.4.3.** *Let $B$ be a homogeneous algebra over a field $k$, and $\mathfrak{p} \subset B$ be a prime ideal. Then the ideal $\mathfrak{p}^h$ of generated by the homogeneous elements contained in $\mathfrak{p}$ is a homogeneous prime ideal of $B$.*

*Proof.* Let $f, g \in B$ two elements, such that $f \cdot g \in \mathfrak{p}^h$. We write $f = f_0 + f_1 + \cdots + f_n$ with $f_i \in B$ homogeneous such that $\deg(f_0) < \deg(f_1) < \cdots < \deg(f_n)$, similarly, $g = g_0 + \cdots + g_m$ with $g_i \in B$ homogeneous and $\deg(g_0) < \cdots < \deg(g_m)$. We need to show that either $f \in \mathfrak{p}^h$ or $g \in \mathfrak{p}^h$. Otherwise, we may assume that $f_0, g_0 \notin \mathfrak{p}^h$. Then

$$f \cdot g = f_0 \cdot g_0 + \text{ terms of higher degree.}$$

As $\mathfrak{p}^h$ is homogeneous, hence $f_0 \cdot g_0 \in \mathfrak{p}^h \subset \mathfrak{p}$. Hence either $f_0 \in \mathfrak{p}$ or $g_0 \in \mathfrak{p}$, which implies either $f_0 \in \mathfrak{p}^h$ or $g_0 \in \mathfrak{p}^h$. This gives a contradiction, and hence the lemma. □

**Corollary 2.4.4.4.** *Let $B$ be a homogeneous algebra over a field $k$.*[13] *Then* $\dim(\mathrm{Spec}(B)) = \dim(\mathrm{Proj}(B)) + 1$.

*Proof.* According to the previous lemma, any minimal prime ideal of $B$ is homogeneous. Up to replace $B$ by the quotient of $B$ by any minimal prime ideal, we may assume that $B$ is integral. Now for any $f \in B_+$ a homogeneous element of degree 1, we have $D_+(f) = \mathrm{Spec}(B_{(f)})$ which is irreducible of dimension $\dim(B_{(f)})$. On the other hand, there exists a canonical map

$$B_{(f)}[T] \to B_f, \quad T \mapsto 1/f$$

As $B_f \subset \mathrm{Frac}(B)$ is integral, the previous homomorphism of groups is bijective. As a result, we have $\dim(B_f) = \dim(B_{(f)}) + 1$. As a result, $\dim(D_+(f)) = \dim(B_{(f)}) = \dim(B_f) - 1 = \dim(D(f)) - 1 = \dim(\mathrm{Spec}(B)) - 1$. As the opens $D_+(f)$ ($f \in B_1$) gives an open covering of $\mathrm{Proj}(B)$, we find $\dim(\mathrm{Proj}(B)) = \dim(\mathrm{Spec}(B)) - 1$. This gives the corollary. □

**Lemma 2.4.4.5.** *Let $A$ be a finitely generated integral domain over $k$, and let $\mathfrak{p}$ be a prime ideal of $A$ of height* 1. *Then* $\dim(A/\mathfrak{p}) = \dim(A) - 1$.

*Proof.* See the book of Liu. □

[13] This means that $B \simeq k[T_1, \cdots, T_n]/I$ for some homogeneous ideal, hence $B$ is generated as a $k$-algebra by $B_1$.

**Proposition 2.4.4.6.** *Let $A$ be a finitely generated integral domain over a field $k$. Let $\mathfrak{p} \subset A$ be a prime ideal.*

1. $\mathrm{ht}(\mathfrak{p}) + \dim(A/\mathfrak{p}) = \dim(A)$.

2. *If $\mathfrak{p}$ is maximal. Then* $\dim(A) = \dim(A_{\mathfrak{p}})$.

*Proof.* (1) We prove this by induction on $r = \mathrm{ht}(\mathfrak{p})$. The case when $r = 0$ is clear, while when $r = 1$, this is the previous lemma. From now on, suppose $r > 1$, and let $0 = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p}$ be a strictly ascending sequence of prime ideals contained in $\mathfrak{p}$. The prime ideal $\mathfrak{p}/\mathfrak{p}_1 \subset A/\mathfrak{p}_1$ is of height $r-1$. As a result, we have $\mathrm{ht}(\mathfrak{p}/\mathfrak{p}_1) + \dim(A/\mathfrak{p}) = \dim(A/\mathfrak{p}_1) = \dim(A) - \mathrm{ht}(\mathfrak{p}_1)$. Moreover, $\mathfrak{p}/\mathfrak{p}_1$ is of height $r-1$, hence $\dim(A/\mathfrak{p}) = \dim(A) - 1 - (r-1) = \dim(A) - r$. This gives (1). When $\mathfrak{p}$ is maximal, $A/\mathfrak{p}$ is then a finite extension of $k$, in particular, $\dim(A/\mathfrak{p}) = 0$. Hence $\dim(A) = \mathrm{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}})$. □

**Corollary 2.4.4.7.** *Let $X$ be an irreducible finite type scheme over a field $k$. Let $x \in X$ be a closed point. Then* $\dim(X) = \dim(\mathcal{O}_{X,x})$.

*Proof.* Since $\dim(X) = \dim(X_{\mathrm{red}})$, and $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X_{\mathrm{red}},x})$. Hence up to replace $X$ by $X_{\mathrm{red}}$, we may assume that $X$ is integral. Moreover, we may assume that $X$ is affine, hence the corollary follows. □

Now, for $X$ be a scheme, and $f \in \mathcal{O}_X(X)$. Let $D(f) = \{x \in X : f_x \notin \mathfrak{m}_x\mathcal{O}_{X,x}\}$, which is open in $X$. Such an open is called *principal*. We put $V(f) = X - D(f)$.

**Corollary 2.4.4.8.** *Let $X$ be an irreducible finite type scheme over a field $k$, and $f \in \mathcal{O}_X(X)$ which is not nilpotent. Then every irreducible component of $V(f)$ is of dimension* $\dim(X) - 1$.

*Proof.* As $f$ is non nilpotent, $V(f) \neq X$. Since $X$ is irreducible, any component of $X$ is dimension $\leq \dim(X) - 1$. On the other hand, for any irreducible component $Y$ of $V(f)$, let $U \subset X$ be an affine open such that $U = \mathrm{Spec}(A) \cap Y \neq \emptyset$. The image $f'$ of $f$ in $\mathcal{O}_X(U) = A$ is non nilpotent and non invertible. The prime ideal $\mathfrak{p} \subset A$ corresponds to $x$ is a minimal prime ideal containing $f'$, hence $\mathrm{ht}(\mathfrak{p}) = 1$ (as $\mathfrak{p}$ contains $(0)$), hence $\dim(Y) = \dim(U \cap Y) = \dim(A/\mathfrak{p}) = \dim(A) - 1 = \dim(X) - 1$. This gives the conclusion. □

## 2.5 Fiber products and base change

### 2.5.1 Sum of schemes

Recall that in a category $\mathcal{C}$, let $X_i$ be a family of objects of $\mathcal{C}$. The *sum* of the family $X_i$, denoted by $\coprod_i X_i$, is an objet of $\mathcal{C}$ together with morphisms $p_i : X_i \to \coprod_i X_i$ for each $i$, such that for any family of morphisms $f_i : X_i \to Z$, there exists a unique morphism $F : \coprod_i X_i \to Z$ such that $F \circ p_i = f_i$ for ech $i$. For example, in the category of sets, the sum of a family of sets is the disjoint union of these sets.

Let $(X_\imath)_{i \in I}$ be an arbitrary family of schemes, let $X$ be the sum of the topological spaces $(X_i)$. Hence $X$ is a disjoint union of its opens $X_i'$ $(i \in I)$, together a homeomorphism $\phi_i : X_i \to X_i'$. By means of $\phi_i$, we get a ringed space

$$(X, \prod_i j_{i,*}\phi_{i,*}\mathcal{O}_{X_i})$$

It's clear that this gives a scheme, called the *sum of the family of scheme* $(X_i)$, denoted by $\coprod_i X_i$. For each $X_i$, we have a then a morphism of scheme $\psi_i \colon X_i \to X$. One shows easily that the natural map is an isomorphism:

$$\operatorname{Hom}\left(\coprod_i X_i, T\right) \simeq \prod_i \operatorname{Hom}(X_i, T), \quad f \mapsto (f \circ \psi_i).$$

**Exercise 2.5.1.1.** Let $X$ be a topological space, and $\mathcal{F}_i$ a family of sheaves on $X$. Define $\prod_i \mathcal{F}_i$ the presheaf on $X$ such that for any open $U \subset X$, $\left(\prod_i \mathcal{F}_i\right)(U) = \prod_i \mathcal{F}_i(U)$. Show that the presheaf $\prod_i \mathcal{F}_i$ is a sheaf.

### 2.5.2 Fiber products of schemes

Recall that for $f \colon X \to S$ and $g \colon Y \to S$ be two maps of *sets*, the *fiber product of $f$ and $g$*, or the *fiber product of $X$ and $Y$ over $S$*, denoted by $X \times_S Y$, is defined to be the following subset of $X \times Y$;

$$X \times_S Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\} \subset X \times Y.$$

We can actually define the notion of fiber product in a category.

**Definition 2.5.2.1.** Let $\mathcal{C}$ be a category, and let $f \colon X \to S$ and $g \colon Y \to S$ be two morphisms in the category $\mathcal{C}$. The *fiber product of $f$ and $g$*, or the *fiber product of $X$ and $Y$ over $S$*, is an object of $\mathcal{C}$ which represents the following functor

$$\mathcal{C} \to \mathfrak{Set}, \quad T \mapsto \operatorname{Hom}(T, X) \times_{\operatorname{Hom}(T,S)} \operatorname{Hom}(T, Y).$$

If the fiber product exists, it will be denoted by $X \times_S Y$.

In more concrete terms, the fiber product of $X$ and $Y$ over $S$ is an object, denoted by $X \times_S Y$ of $\mathcal{C}$, together with two morphisms $p \colon X \times_S Y \to X$ and $q \colon X \times_S Y \to Y$ such that

- $f \circ p = g \circ q$;
- For any pair of morphisms $(\alpha \colon T \to X, \beta \colon T \to Y)$ such that $f \circ \alpha = g \circ \beta$, there exists a unique morphism $\gamma \colon T \to X \times_S Y$ such that $p \circ \gamma = \alpha$ and $q \circ \gamma = \beta$.

$$\begin{tikzcd}
T \arrow[dr, dotted, "\exists!\ \gamma"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& X \times_S Y \arrow[r, "q"] \arrow[d, "p"'] & X \arrow[d, "g"] \\
& Y \arrow[r, "f"'] & S.
\end{tikzcd}$$

The fiber product in a category does *not* exist in general. But if it exists, then it's unique up to a unique isomorphism since it's the solution of some universal problem.

**Proposition 2.5.2.2.** *Let $X, Y$ be two $S$-schemes. Then the fiber product of $X$ and $Y$ over $S$ exists in the category of schemes.*

*Proof.* We divide the proof into several steps.

*Case when $X, Y, S$ are affine.*

In this case, suppose $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $S = \mathrm{Spec}(R)$. We take $Z = \mathrm{Spec}(A \otimes_R B)$, and consider $p : Z \to X$ the morphism induced by $A \to A \otimes_R B, a \mapsto a \otimes 1$, $q : Z \to Y$ the morphism induced by $B \to A \otimes_R B, b \mapsto 1 \otimes b$. We claim that $(Z, p, q)$ represents the fiber product of $X$ and $Y$ over $S$: let $T$ be a scheme:

$$\begin{aligned} \mathrm{Hom}(T, Z) &\simeq \mathrm{Hom}_{\mathrm{ring}}(A \otimes_R B, \mathcal{O}_Z(Z)) \\ &\simeq \mathrm{Hom}(A, \mathcal{O}_Z(Z)) \times_{\mathrm{Hom}(R, \mathcal{O}_Z(Z))} \mathrm{Hom}(B, \mathcal{O}_Z(Z)) \\ &= \mathrm{Hom}(T, X) \times_{\mathrm{Hom}(T,S)} \mathrm{Hom}(T, Y) \end{aligned}$$

hence get a isomorphism

$$\mathrm{Hom}(T, Z) \to \mathrm{Hom}(T, X) \times_{\mathrm{Hom}(T,S)} \mathrm{Hom}(T, Y), \quad \gamma \mapsto (p \circ \gamma, q \circ \gamma).$$

As a result, $(Z, p, q)$ gives the fiber product of $X$ and $Y$ over $S$.

*Case when $X, Y$ are opens of affine schemes over $S$.*

In this case, we assume that $j_X : X \hookrightarrow X'$ and $j_Y : Y \hookrightarrow Y'$ are two open immersions over $S$ with $S, X, Y$ three affine schemes. Let $f' : X' \to S$ and $g' : Y' \to S$ be the two structural morphisms. According to the previous case, we know that the fiber product $X' \times_S Y'$ of $X'$ and $Y'$ over $S$ exists. Let $p : X' \times_S Y' \to X'$ and $q : X' \times_S Y' \to Y'$ be the canonical morphisms, we consider then the open subset $U := p^{-1}(X) \bigcap q^{-1}(Y) \subset X' \times_S Y'$, and we claim that the triple $(U, p|_U, q|_U)$ gives the fiber product of $X$ and $Y$ over $S$: indeed, let $\alpha : Z \to X$, and $\beta : Z \to Y$ be two morphisms such that $f \circ \alpha = g \circ \beta : Z \to S$. Let $\alpha' = j_X \circ \alpha$, et $\beta' = j_Y \circ \beta$. Then $f' \circ \alpha' = f' \circ j_X \circ \alpha = f \circ \alpha = g \circ \beta = g' \circ \beta'$, from where we get a morphism $\gamma' : Z \to X' \times_S Y'$ such that $p \circ \gamma' = \alpha' = j_X \circ \alpha$ and that $q \circ \gamma' = \beta' = j_Y \circ \beta$. Hence $\mathrm{Im}(\gamma) \subset U = p^{-1}(X) \cap q^{-1}(Y)$. Hence $\gamma : Z \to X' \times_S Y'$ factors through $U \hookrightarrow X' \times_S Y'$, et we get a morphism $\gamma' : Z \to U$. One verifies easily that $\gamma'$ is also unique. Hence the triple $(U, p|_U, q|_V)$ solves the universal problem, hence it gives the fiber product of $X$ and $Y$ over $S$.

*Case when $S$ is affine.*

We cover $X$ by affine opens $X = \bigcup_{i \in I} X_i$, and $Y$ by affine opens $Y = \bigcup_{j \in J} Y_j$. For each $(i, j)$, the fiber product $X_i \times_S Y_j$ exists. Moreover, according to the previous case, the fiber product $X_{ii'} \times_S Y_{jj'}$ exists also, and can be realized as an open of $X_i \times_S Y_j$ for any index $i, i' \in I$ and $j, j' \in J$. Now the fiber product $X \times_S Y$ is constructed by gluing these $X_i \times_S Y_j$ along the opens $X_{ii'} \times_S Y_{jj'} \hookrightarrow X_i \times_S Y_j$. This gives the fiber product when $S$ is *affine*. As a corollary, this gives also the case when the structural maps $f$ and $g$ can factor through some affine open $S_0 \subset S$. Indeed, in this case, one verifies easily that $X \times_S Y = X \times_{S_0} Y$, hence $X \times_S Y$ exists as the latter fiber product exists.

*General case.*

In this case, we cover $S$ by the affine opens $S = \bigcup_i S_i$ and let $X_i$ (resp. $Y_i$) be the inverse image of $S_i \hookrightarrow S$, and $X_{ij} = X_i \cap X_j$. According to the previous cases, the fiber product $X_i \times_S Y_i = X_i \times_{S_i} Y_i$ exists, and so is $X_{ij} \times_S Y_{ij} = X_{ij} \times_{S_{ij}} Y_{ij}$ (since it's the same as $X_{ij} \times_{S_i} Y_{ij}$). Moreover, $X_{ij} \times_S Y_{ij} = X_{ij} \times_{S_{ij}} Y_{ij}$ is an open of $X_i \times_S Y_i = X_i \times_{S_i} Y_i$. Hence, the fiber product $X \times_S Y$ is obtained by gluing the schemes $X_i \times_S Y_i$ along the opens $X_{ij} \times_S Y_{ij}$.

This finishes then the proof of Proposition 2.5.2.2 □

From the proof, we deduce also the following corollary:

**Corollary 2.5.2.3.** *Let $S$ be a scheme.*

1. *Let $X \hookrightarrow X'$ and $Y \hookrightarrow Y'$ be two open immersions over $S$. Let $p' \colon X' \times_S Y' \to X'$ and $q' \colon X' \times_S Y' \to Y'$ be the canonical morphisms. Then there exists a canonical isomorphism between $X' \times_S Y'$ and the open subscheme $p'^{-1}(X) \cap q'^{-1}(Y)$.*

2. *For three schemes $X, Y, Z$ over $S$, we have $(X \times_S Y) \times_S Z \simeq X \times_S (Y \times_S Z)$.*

3. *For $X$ be a scheme over $S$. Then there is a canonical isomorphism $X \times_S S \simeq X$.*

**Example 2.5.2.4.** We have $\mathbb{A}_k^n \times_{\mathrm{Spec}(k)} \mathbb{A}_k^m \simeq \mathbb{A}_k^{m+n}$, while $\mathbb{P}_k^m \times_{\mathrm{Spec}(k)} \mathbb{P}_k^n \not\simeq \mathbb{P}_k^{m+n}$.

**Exercise 2.5.2.5.** Verify the previous example.

**Remark 2.5.2.6.** In general, the underlying set of $X \times_S Y'$ is *not* the fiber product of the underlying sets of $X$ and $Y$ over $S$. For example, $\mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$ consists of two points, while the fiber product of the underlying sets has only one point.

**Definition 2.5.2.7.** Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be two morphisms over $S$, the by the universal property of fiber product, we get a morphism $f \times_S g \colon X \times_S Y \to X' \times_S Y'$ such that $p_{X'} \circ (f \times_S g) = p_X$, and $p_{Y'} \circ (f \times_S g) = p_Y$. When no confusion is possible, we often omit $S$ from the notation $f \times_S g$.

### 2.5.3 Base change; fibers

**Definition 2.5.3.1.** Let $S$ be a scheme, and $X$ be an $S$-scheme. For any $S$-scheme $S'$, the second projection $q \colon X \times_S S' \to S'$ endows $X \times_S S'$ with a structure of an $S'$-scheme. Such a process is called the *base change* of $X$ by $S' \to S$. We sometimes denote the $S'$-scheme $X \times_S S'$ by $X_{S'}$. For a morphism of $S$-schemes $f \colon X \times Y$, its *base change* by $S' \to S$ is the morphism $f \times_S S' \colon X \times_S' \to Y \times_S S'$, sometimes it's denoted by $f_{S'}$.

**Example 2.5.3.2.** Let $S = \mathrm{Spec}(A)$, and $X = \mathrm{Spec}(A[T_1, \cdots, T_n]/(P_1, \cdots, P_m))$. Then for any morphism of rings $\tau \colon A \to B$, we have $X_{\mathrm{Spec}(B)} = \mathrm{Spec}(B[T_1, \cdots, T_n]/(\widetilde{P_1}, \cdots, \widetilde{P_n}))$ where $\widetilde{P_i}$ is obtained by applying the morphism $\tau$ on the coefficients of $P_i$.

**Example 2.5.3.3.** For $A$ a ring, and $B$ a homogeneous $A$-algebra. Let $C$ be an $A$-algebra, the tensor product $D := B \otimes_A C$ is then naturally a homogeneous $C$-algebra. Moreover, we have $\mathrm{Proj}(B)_{\mathrm{Spec}(C)} \simeq \mathrm{Proj}(B \times_A C)$. In particular, for $S$ an arbitrary scheme, we can consider the *projective space over $S$ of dimension $n$*, which is given by $\mathbb{P}_S^n := \mathrm{Proj}(\mathbb{Z}[T_0, \cdots, T_n])_S$.

**Exercise 2.5.3.4.** Verify that $\mathrm{Proj}(B)_{\mathrm{Spec}(C)} \simeq \mathrm{Proj}(B \times_A C)$ in the previous example.

Now, to define the *fiber of a morphism*, recall that for a point $y \in Y$ of a scheme, there is a induced morphism of schemes $\mathrm{Spec}(k(y)) \to Y$. More precisely, let $U = \mathrm{Spec}(A) \subset Y$ be an affine open of $Y$ containing $y$, the point $y$ corresponds then a prime ideal $\mathfrak{p} \subset A$ of $A$, the canonical map $\mathrm{Spec}(k(y)) \to Y$ is then the composite of $U \hookrightarrow X$ and $\mathrm{Spec}(k(y)) \to U = \mathrm{Spec}(A)$, where the latter morphism is induced by the morphism of rings $A \to A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = k(y)$.

**Definition 2.5.3.5.** Let $f \colon X \to Y$ be a morphism of schemes. For any point $y \in Y$, we set $X_y = X \times_Y \mathrm{Spec}(k(y))$, where $\mathrm{Spec}(k(y)) \to Y$ the canonical map associated to the point $y \in Y$. We call $X_y$ be the *fiber of $f$ over $y$*. The second projection $X_y = X \times_Y \mathrm{Spec}(k(y)) \to \mathrm{Spec}(k(y))$ makes $X_y$ into a scheme over $k(y)$.

**Example 2.5.3.6.** For $y \in Y$ a point. $\mathbb{P}^n_{Y,y} \simeq \mathbb{P}^n_{k(y)}$

**Definition 2.5.3.7.** For $f\colon X \to Y$ a morphism with $Y$ irreducible of generic point $\eta \in Y$. The fiber $X_\eta$ over $\eta$ is called the *generic fiber of $f$*.

**Lemma 2.5.3.8.** *Let $B$ be a ring, and $T \subset B$ be a multiplicative subset. Then the canonical map* $\operatorname{Spec}(T^{-1}B) \to \operatorname{Spec}(B)$ *induces a homeomorphism between* $\operatorname{Spec}(T^{-1}B)$ *and the subset* $\mathfrak{T}$ *of element $\mathfrak{q} \subset B$ such that $T \cap \mathfrak{q} = \emptyset$ of* $\operatorname{Spec}(B)$

*Proof.* Let $\alpha\colon B \to T^{-1}B$ be the canonical map of rings. It's well-known in commutative algebra that, the map

$$\iota\colon \operatorname{Spec}(T^{-1}B) \to \operatorname{Spec}(B), \quad \mathfrak{p} \mapsto \alpha^{-1}(\mathfrak{p})$$

induces a one-to-one correspondence between $\operatorname{Spec}(T^{-1}B)$ and the subset $\mathfrak{T}$. So we only need to check that this correspondence respects also the topology. Indeed, it's continuous, and for any ideal $\mathfrak{a} \subset T^{-1}B$, let $\{x_i : i \in I\}$ be a family of generators of $\mathfrak{a}$. Up to multiply $x_i$ by some element in $T$ (which is invertible in $T^{-1}B$), we may assume that $x_i = b_i/1$ with $b_i \in B$. Now, consider $\mathfrak{b} \subset B$ the ideal generated by the $b_i$'s, then its clear that $T^{-1}\mathfrak{b} = \mathfrak{a}$. Now, for any prime ideal $\mathfrak{p} = T^{-1}\mathfrak{q} \subset T^{-1}B$, $\mathfrak{a} = T^{-1}(\mathfrak{b}) \subset T^{-1}\mathfrak{q} = \mathfrak{p}$ if and only if $\mathfrak{b} \subset \mathfrak{q}$. As a result, we find $\iota(V(\mathfrak{a})) = \mathfrak{T} \cap V(\mathfrak{b})$. Hence $\iota$ is a closed map. So it's a homoeomorphism, as required. □

**Proposition 2.5.3.9.** *Let $f\colon X \to Y$ be a morphism of schemes, and $y \in Y$ be a point. Then the first projection $X_y \to X$ induces a homeomorphism between the underlying topological space of $X_y$ and the subspace $f^{-1}(y)$ of $Y$ with the induced topology.*

*Proof.* Clearly, to prove our proposition, we may assume that $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are both affine. We consider $\varphi = f^\sharp\colon A \to B$ be the associated morphism between the rings. Let $\mathfrak{p} \subset A$ be the prime ideal which corresponds to $y$. Then

$$f^{-1}(y) = \{\mathfrak{q} \in \operatorname{Spec}(B) | \varphi^{-1}(\mathfrak{q}) = \mathfrak{p}\},$$

and

$$X_y = \operatorname{Spec}(B \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}) = \operatorname{Spec}\left(\frac{B \otimes_A A_\mathfrak{p}}{\varphi(\mathfrak{p})(B \otimes_A A_\mathfrak{p})}\right).$$

Moreover, the projection $X_y = \operatorname{Spec}(k(y)) \times_Y X$ to $X$ induces a map of topological spaces

$$\iota_y\colon X_y \to f^{-1}(y),$$

and we need to show that $\iota_y$ is a homeomorphism. Let $S = A - \mathfrak{p}$, and $T = \varphi(S) \subset B$. Then, we have $B \otimes_A A_\mathfrak{p} \simeq T^{-1}B$. As a result,

$$\frac{B \otimes_A A_\mathfrak{p}}{\varphi(\mathfrak{p})(B \otimes_A A_\mathfrak{p})} \simeq \frac{T^{-1}B}{\varphi(\mathfrak{p})T^{-1}B}.$$

According the the previous lemma, $\iota_y$ induces a homeomorphism between $X_y$ the set of prime ideals $\mathfrak{q} \subset B$ such that $\mathfrak{q} \cap T = \emptyset$, and that $T^{-1}\mathfrak{q} \supset \varphi(\mathfrak{p})T^{-1}B$, that is, the set of prime ideals $\mathfrak{q} \subset B$ such that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, as desired. □

Hence sometimes we write $X_y$ by $f^{-1}(y)$. Intuitively, we can consider $X$ as a family of schemes $X_y \to \operatorname{Spec}(k(y))$.

**Example 2.5.3.10.** Let $m \in \mathbb{Z}$ be an integer. The scheme $\operatorname{Spec}(\mathbb{Z}[X,Y]/(XY - m))$ gives then a family of scheme over the base scheme $\operatorname{Spec}(\mathbb{Z})$. Over each *closed* point $(p) \in \operatorname{Spec}(Z)$, the fiber over $(p)$ is $\operatorname{Spec}(\mathbb{F}_p[X,Y]/(XY - \bar{m}))$, while its generic fiber is given by $\operatorname{Spec}(\mathbb{Q}[X,Y]/(XY - m))$.

**Definition 2.5.3.11.** Let $\mathcal{P}$ be a property of morphisms of schemes $f\colon X \to Y$.

1. The property $\mathcal{P}$ is said to be *local on the base* $Y$ if the following assertions are equivalent:
   (a) $f$ verifies $\mathcal{P}$;
   (b) for any $y \in Y$, there exists an affine neighborhood $V$ of $y$ such that $f|_{f^{-1}(V)}$ verifies $\mathcal{P}$.
2. The property $\mathcal{P}$ is said to be *stable under the base change* if for any morphism $f\colon X \to Y$ verifying $\mathcal{P}$, and for any morphism $Y' \to Y$, the base change $f_{Y'}\colon X_{Y'} \to Y'$ verifies again the property $\mathcal{P}$.

**Example 2.5.3.12.** Open immersion and closed immersion are properties which are local on the base, which are also stable under base change. We will encounter more properties which are local on the base or stable under base change.

## 2.6 Some global properties of morphisms

### 2.6.1 Separatedness of schemes

We begin with a topological observation.

**Proposition 2.6.1.1.** *Let $X$ be a topological space, and $\Delta\colon X \to X \times X$ be the diagonal map. Then $X$ is separated (*i.e., *Hausdorff) iff the image $\Delta(X) \subset X \times X$ is closed.*

*Proof.* Suppose first of all $X$ is separated. Let $p = (x, y) \notin \Delta(X)$, namely $x, y \in X$ such that $x \neq y$. As $X$ is separated, there exist two opens $U, V$ of $X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. In particular, $p \in U \times V \subset X \times X - \Delta(X)$. By the definition of product topology, $U \times V$ is open in $X \times X$. Hence, $X \times X - \Delta(X)$ is open, as a result, $\Delta(X) \subset X \times X$ is closed. Conversely, suppose $\Delta(X) \subset X \times X$ is closed, and let $x, y \in X$ such that $x \neq y$. Hence $(x, y) \in X \times X - \Delta(X)$ which is open by assumption. Hence, can find $U, V \subset X$ two opens such that $(x, y) \in U \times V \subset X \times X - \Delta(X)$. Hence $U \cap V = \emptyset$. This shows that $X$ is Hausdorff. ☐

Let now $X$ be a scheme, we know that the underlying topological space of $X$ in general is not Hausdorff, nevertheless, we will define the separatedness of schemes in a similar way. Recall first that a morphism of schemes $f\colon X \to Y$ is a *closed immersion* if for any affine open $U$ of $Y$, the inverse image $f^{-1}(U) \subset X$ is again affine, moreover, the induced map $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is surjective.

**Definition 2.6.1.2.** A morphism of schemes $f\colon X \to Y$ is called *separated* if the diagonal morphism $\Delta\colon X \to X \times_Y X$ is a closed immersion. We say the $X$ is a *separated $Y$-scheme* or $X$ is *separated over $Y$*. A scheme $X$ is said to be *separated* if $X$ is separated over $\mathrm{Spec}(\mathbb{Z})$.

**Example 2.6.1.3.** Any morphism of affine schemes is separated. In particular, any affine scheme is separated.

**Lemma 2.6.1.4.** *For any morphism of schemes $f\colon X \to Y$, its diagonal morphism $\Delta\colon X \to X \times_Y X$ is an immersion.*

*Proof.* This is a local property on $Y$, hence we may assume that $Y$ is affine. To show that $\Delta$ is an immersion, we need to find an open neighborhood $V$ of the image $\Delta(X)$ such that the induced map $\Delta\colon X \to V$ is a closed immersion. So for any point $x \in X$, let $U_x \subset X$ be a affine open neighborhood of $x$. Then $U_x \times_Y U_x$ gives an open subset of $X \times_Y X$ such that $\Delta^{-1}(U_x \times_Y U_x) = U_x$. Put

$$V = \bigcup_{x \in X} U_x \times_Y U_x,$$

which gives an open subset of $X \times_Y X$ containing $\Delta(X)$. Moreover, as the $U_x$'s are affine and $Y$ is affine, the induced map $\Delta : U_x = \Delta^{-1}(U_x \times_Y U_x) \to U_x \times_Y U_x$ is a closed immersion (as it's the diagonal map of the $Y$-scheme $U_x$). As a result, the induced map $\Delta \colon X \to V$ is also a closed immersion, as desired. □

**Corollary 2.6.1.5.** *Let $f \colon X \to Y$ be a morphism of schemes. Then $f$ is separated if and only if the image $\Delta(X) \subset X \times_Y X$ is closed.*

*Proof.* Clearly, if $f$ is separated, then $\Delta(X) \subset X \times_Y X$ is closed. Conversely, if $\Delta(X)$ is closed, we need to check that $\Delta$ is then a closed immersion. As $\Delta \colon X \to X \times_Y X$ is an immersion, let $U$ be an open neighborhood of $\Delta(X)$ such that then induced map $X \to U$ is a closed immersion. As $\Delta(X) \subset X \times_Y X$ is closed, let $V = X \times_Y X - \Delta(X)$. We obtain hence an open covering $X = U \cup V$ of $X$ such that $\Delta|_{\Delta^{-1}(U)}$ and $\Delta|_{\Delta^{-1}(V)}$ are both closed immersions. As a result, $\Delta$ is a closed immersion. □

**Proposition 2.6.1.6.** *Let $f \colon Y \to X = \mathrm{Spec}(A)$ be a morphism of schemes with $X$ affine. The following three conditions are equivalent:*

1. *$f$ is separated;*
2. *For any two affine opens $U, V \subset Y$, their intersection $U \cap V \subset Y$ is again affine, moreover, the canonical map $\mathcal{O}_Y(U) \otimes_A \mathcal{O}_Y(V) \to \mathcal{O}_Y(U \cap V)$ is surjective;*
3. *There exists an open affine covering $Y = \bigcup_{i \in I} U_i$ such that $U_i \bigcap U_j$ is affine, and that the canonical map $\mathcal{O}_Y(U_i) \otimes \mathcal{O}_Y(U_j) \to \mathcal{O}_Y(U_i \cap U_j)$ is surjective for any $i, j \in I$.*

*Proof.* (1)$\Longrightarrow$(2): Suppose $f$ is separated, namely the diagonal $\Delta \colon Y \to Y \times_X Y$ is a closed immersion. Since $X$ is affine, the open $U \times_X V$ is also affine, hence its preimage, $U \bigcap V$ by $\Delta$ is also affine. Moreover, the canonical map $\mathcal{O}_{Y \times_X Y}(U \times_X V) = \mathcal{O}_Y(U) \otimes_A \mathcal{O}_Y(V) \to \mathcal{O}_Y(U \cap V)$ is surjective. This gives (2). Clearly (2) implies (3). It remains to show (3)$\Longrightarrow$(1). With the notations of (3), the family $\{U_i \times_X U_j\}_{i,j}$ forms an affine open covering of $Y \times_X Y$. Moreover the preimage of $U_i \times_X U_j$ by $\Delta$ is $U_i \bigcap U_j$ which is affine by assumption. Moreover, the canonical map

$$\mathcal{O}_{Y \times_X Y}(U_i \times_X U_j) = \mathcal{O}_Y(U_i) \otimes_A \mathcal{O}_Y(U_j) \to \mathcal{O}_Y(U_i \cap U_j)$$

is surjective, as a result, the diagonal $\Delta \colon Y \to Y \times_X Y$ is a closed immersion. This gives (1). □

**Example 2.6.1.7.** The projective space $\mathbb{P}^n_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[X_0, \cdots, X_n])$ is separated. Indeed, $\mathbb{P}^n_{\mathbb{Z}}$ can be covered by the affine opens $D_+(X_i)$. Moreover, the intersection $D_+(X_i) \bigcap D_+(X_j)$ is also affine. Hence $\mathbb{P}^n_{\mathbb{Z}}$ is separated over $\mathbb{Z}$. As a corollary, any projective space $\mathbb{P}^n_S$ is separated over $S$. More generally, a projective morphism $f \colon X \to S$ is separated.

**Example 2.6.1.8.** We consider $X_1 = X_2 = \mathrm{Spec}(\mathbb{Z})$, $X_{12} = D(p) \subset X_1$ and $X_{21} = D(p) \subset X_2$. We glue the two schemes $X_1$ and $X_2$ along the open subschmes $X_{12} \simeq X_{21}$. Now, the scheme $X$ obtained in this way is *not* separated. Indeed, $X_1$ (resp. $X_2$) can be identified with a open of $X$, still denoted by $X_1$ (resp. by $X_2$), their intersection is denoted by $U$. Clearly, $X_i$ is affine, and so is $U$. So to show that $X$ is not separated, we only need to verify that the canonical morphism

$$\mathbb{Z} \simeq \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathcal{O}_X(X_1) \otimes_Z \mathcal{O}_X(X_2) \to \mathcal{O}_X(U) \simeq \mathbb{Z}_{(p)} = \mathbb{Z}[1/p] \subset \mathbb{Q}.$$

But clearly, this map is not surjective. This gives the statement.

**Proposition 2.6.1.9.** *1. Open and closed immersions are separated.*

*2. Separated morphisms are stable under base change.*

*3. The composition of two separated morphisms is again separated. In particular, immersions are separated.*

*4. Let $f\colon X \to Y$, and $g\colon Y \to Z$ be two morphisms such that $g \circ f$ is separated. Then $f$ is separated.*

*Proof.* (1) is clear since in this case, the diagonal is an isomorphism. (2) is also clear. (3) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two separated morphisms. Then the diagonal $\Delta_{g\circ f}\colon X \to X \times_Z X$ can factor as

$$X \to X \times_Y X \to X \times_Z X.$$

Hence we only need to verify that the second morphism is a closed immersion. Indeed, this second morphism can be obtained by the following cartesian diagram

$$\begin{tikzcd} X \times_Y X \arrow[r] \arrow[d] & X \times_Z X \arrow[d] \\ Y \arrow[r, "\Delta_g"] & Y \times_Z Y. \end{tikzcd}$$

Now since $g$ is separated, the diagonal $\Delta_g$ is closed immersion. As a result, the canonical morphism $X \times_Y X \to X \times_Z X$ is a closed immersion. Hence $\Delta_{g\circ f}$ is a closed immersion, namely $g \circ f$ is separated. (4) The morphism $f$ can factor as

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y \ .$$

The second morphism $p_2$ is separated since it is the base change of $g \circ f$ by $Y \to Z$. Moreover, $\Gamma_f$ is an immersion (see the exercise below), in particular is separated. As a result, $f = p_2 \circ \Gamma_f$ is separated. □

**Exercise 2.6.1.10.** Let $f\colon X \to Y$ be a morphism of $S$-schemes, then the graph

$$\Gamma_f\colon X \to X \times_S Y, \quad x \mapsto (x, f(x)).$$

Show that the following diagram is cartesian

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_S Y \arrow[d, "f\times \mathrm{id}_Y"] \\ Y \arrow[r, "\Delta"] & Y \times_S Y \end{tikzcd} \ .$$

Deduce then $\Gamma_f$ is always an immersion, and is a closed immersion if $Y/S$ is separated.

**Proposition 2.6.1.11.** *Let $\mathcal{P}$ be a property about the morphism of schemes. Suppose that*

- *the property $\mathcal{P}$ is stable under base change.*
- *the property $\mathcal{P}$ is stable under composition.*

- *closed immersions satisfy the property $\mathcal{P}$.*

*Let $f : X \to Y$ and $g\colon Y \to Z$ be two morphisms of schemes such that $g$ is separated. Suppose that the composition $g \circ f$ satisfies $\mathcal{P}$, then so is $f$.*

*Proof.* The morphism $f$ can factor as

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y .$$

As $p_2$ is obtained as the base change of $g \circ f\colon X \to Z$ by $Y \to Z$, it satisfies also the property $\mathcal{P}$. By the exercise above, since $Y/S$ is separated, the graph $\Gamma_f$ is a closed immersion. As a result, it verifies the property $\mathcal{P}$. Thus, the morphism $f$, being the composition of $\Gamma_f$ with $p_2$, verifies the property $\mathcal{P}$. □

**Exercise 2.6.1.12.** Let $X, Y$ be two $S$-schemes such that $Y/S$ is separated, and that $X$ is reduced. Let $f, g\colon X \to Y$ be two morphism of $S$-schemes which coincide over a dense open subset of $X$. Show that $f = g$.

### 2.6.2 Proper morphisms

The properness of a morphism is essentially a topological property. Recall that a map of topological spaces $f\colon X \to Y$ is said to be *closed* if for any closed subset of $X$, its image $f(F) \subset Y$ is closed.

**Definition 2.6.2.1.** Let $f\colon X \to Y$ be a morphism of locally noetherian schemes.

1. $f$ is said to be *universally closed* if for any morphism $Y' \to Y$, the base change

$$f_{Y'} : X' = X \times_Y Y' \to Y'$$

is a closed map between the underlying topological spaces.

2. $f$ is said to be *proper* if $f$ is separated, of finite type,[14] and universally closed.

**Proposition 2.6.2.2.** *We have the following properties:*

1. *Closed immersions are proper.*

2. *The composition of two proper morphisms is proper.*

3. *The base change of a proper morphism is still proper.*

4. *If the composition of $X \to Y$ and $Y \to Z$ is proper, and if the second morphism $Y \to Z$ is separated. Then the first morphism $X \to Y$ is proper.*

5. *Let $f\colon X \to Y$ be a surjective morphism of $S$-schemes. Let us suppose that $Y$ is separated and of finite type over $S$, and that $X$ is proper over $S$, then $Y$ is proper over $S$.*

A fundamental property about proper morphisms is the finiteness result. Here we will show a very special case, and we will refer to Liu's book for the proof.

**Proposition 2.6.2.3.** *Let $X \to \operatorname{Spec}(A) = S$ be a proper morphism. Then $\mathcal{O}_X(X)$ is finite over $A$.*

[14] Over a non noetherian base $Y$, the condition "of finite type" should be replaced by "of finite presentation", but here for simplicity, we will not use this definition

### 2.6.3 Projective morphisms

Recall that, a morphism of schemes $f\colon X \to Y$ is called projective if there exists an open covering $Y = \bigcup_i Y_i$, such that $f|_{f^{-1}(Y_i)}\colon f^{-1}(Y_i) \to Y_i$ can be factored as

$$f^{-1}(Y_i) \hookrightarrow \mathbb{P}^{m_i}_{Y_i} \to Y_i$$

with the first morphism a closed immersion.

**Theorem 2.6.3.1.** *A projective morphism is proper.*

*Proof.* Let $f\colon X \to Y$ be a projective morphism. Then we have seen that $f$ is of finite type and separated. Hence it remains to show that $f$ is universally closed. Since a base change of $f$ is still projective, we only need to show that $f$ is a closed map. This is a local question on $Y$, hence we may assume that $Y = \mathrm{Spec}(R)$, and that $f$ can be factored as $X \hookrightarrow \mathbb{P}^n_Y \to Y$ with first map a closed immersion. Since any closed subset of $X$ gives a closed subset of $\mathbb{P}^n_Y$, we are reduced to the case where $X = \mathbb{P}^n_Y = \mathrm{Proj}(R[X_0, \cdots, X_n])$. We denote in the following $A = R[X_0, \cdots, X_n]$ with the natural gradation. Let $F = V_+(I)$ be a closed subset with $I \subset A$ a homogeneous ideal. Let $B = A/I = \oplus_{n\geq 0} B_n$ be the quotient with the natural gradation. We need to show that $f(F) \subset Y$ is closed. Now, for any $y \in Y$, $y \in f(F)$ if and only if $F \cap \mathbb{P}^n_{k(y)} \neq \emptyset$. Suppose $I$ can be generated by the homogeneous polynomials $g_1, \cdots, g_r \in A$, then $F \cap \mathbb{P}^n_{k(y)} = V_+(\overline{g_1}, \cdots, \overline{g_r})$ with $\overline{g_i}$ the image of $g_i$ by the canonical map $R[X_0, \cdots, X_n] \to k(y)[X_0, \cdots, X_n]$. Hence $F \cap \mathbb{P}^n_{k(y)} = \emptyset$ if and only if $A_+ \otimes_R k(y) \subset \sqrt{(\overline{g_1}, \cdots, \overline{g_r})}$. In an equivalent way, this means that $B_n \otimes_R k(y) = 0$ for some integer $n$. Now for each $n$, as a $R$-module, $B_n$ is of finite type, hence by Nakayama's lemma, $B_n \otimes_R k(y) = 0$ is equivalent to $B_{n,y} = 0$. Hence $y \in f(F)$ if and only if $y \in \mathrm{Supp}(B_n)$ for any $n$. Now, to complete the proof, it suffices to remark that the support $\mathrm{Supp}(B_n)$ is closed as it's give by $V(\mathrm{Ann}(B_n)) \subset \mathrm{Spec}(R)$. □

**Lemma 2.6.3.2.** *Let $S$ be a scheme. Then there exists a closed immersion $\mathbb{P}^n_S \times \mathbb{P}^m_S \to \mathbb{P}^{(n+1)(m+1)-1}_S$, called the* Segre embedding.

*Proof.* We will just give a set theoretical description of this morphism: let $N = (n+1)(m+1)-1$, then this map is given by

$$\mathbb{P}^n_S \times_S \mathbb{P}^m_S \to \mathbb{P}^N_S, \quad ((x_0 : \cdots : x_n), (y_0 : \cdots : y_m)) \to (x_0 y_: x_0 y_1 : \cdots : x_i y_j : \cdots : x_n y_m).$$

□

**Corollary 2.6.3.3.** *The following statements are true:*

1. *Closed immersions are projective morphisms.*
2. *The composition of two projective morphisms is projective morphisms.*
3. *Projective morphisms are stable under base change.*
4. *Let $X \to S$ and $Y \to S$ be projective morphisms, then so is $X \times_S Y \to S$.*
5. *If the composition of $X \to Y$, $Y \to Z$ is projective, and if $Y \to Z$ is projective, then so is $X \to Y$.*

*Proof.* See Liu's book. □

**Remark 2.6.3.4.** Let $k$ be a field. Let $X$ a proper scheme over $k$. We can show that (1) if $\dim(X) \leq 1$ then $X$ is projective; (2) if $\dim(X) = 2$ and $X$ is smooth (see below for the definition of smoothness), then $X$ is projective; (3) there exist proper scheme over $k$ of dimension 2 and smooth proper scheme over $k$ of dimension $\geq 3$ which are not projective.

**Theorem 2.6.3.5** (Chow's lemma)**.** *Let $Y$ be a noetherian scheme, and $f\colon X \to Y$ be a proper morphism. Then there exists a* projective *$Y$-scheme $g\colon X' \to Y$, and a $Y$-morphism $f\colon X' \to X$ such that there exists an everywhere dense open subset $U \subset X$ such that $f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$ is an isomorphism.*

*Proof.* Omitted. □

## 2.7 Some local properties of schemes and morphisms of schemes

### 2.7.1 Zariski tangent spaces

**Definition 2.7.1.1.**

1. Let $X$ be a scheme, and $x \in X$. Let $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ be the maximal ideal and $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ the residue field. We define the *tangent space* to $X$ at $x$ to the $k(x)$-vector space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee := \mathrm{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x))$.

2. For $f\colon X \to Y$ be a morphism of schemes. Let $x \in X$ and $y = f(x)$. Then $f_x^\sharp\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ canonically induces a $k(x)$-linear map $T_{X,x} \to T_{Y,y} \otimes_{k(y)} k(x)$, which will be denoted by $T_{f,x}$, called the *tangent map* of $f$ at $x$.

The following lemma is very useful.

**Lemma 2.7.1.2.** *Let $A$ be a ring with $\mathfrak{m} \subset A$ a maximal ideal. Then the canonical morphism of $A$-modules $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}A_\mathfrak{m}/\mathfrak{m}^2A_\mathfrak{m}$ is an isomorphism.*

*Proof.* We remark first that the canonical map $A/\mathfrak{m} \to A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}$ is an isomorphism of $A$-modules, hence so is the canonical maps

$$\mathfrak{m}/\mathfrak{m}^2 \simeq \mathfrak{m} \otimes_A A/\mathfrak{m} \simeq \mathfrak{m} \otimes_A (A_\mathfrak{m}/\mathfrak{m}^2A_\mathfrak{m}) \simeq \mathfrak{m}A_\mathfrak{m}/\mathfrak{m}^2A_\mathfrak{m},$$

which is exactly the canonical morphism considered in this lemma. □

In the following, we will consider an explicit example. Let

- $k$ be a field;
- $Y = \mathbb{A}_k^n = \mathrm{Spec}(k[T_1, \cdots, T_n])$, with $y = (\lambda_1, \cdots, \lambda_n) \in \mathbb{A}_k^n(k)$ a rational point;
- $E = k^n$ be the $k$-vector spaces of dimension $n$.

Associate with the point $y$, we consider the following $k$-linear map

$$D_y\colon k[T_1, \cdots, T_n] \to E^\vee, \quad f \mapsto \left((t_1, \cdots, t_n) \mapsto \sum_{i=1}^n \frac{\partial f}{\partial T_i}(y)t_i\right).$$

**Lemma 2.7.1.3.** *Let $\mathfrak{m} = (T_1 - \lambda_1, \cdots, T_n - \lambda_n) \subset k[T_1, \cdots, T_n]$ be the maximal ideal corresponding to $y$.*

1. *The restriction of $D_y$ to $\mathfrak{m}$ induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \simeq E^\vee$.*

2. $T_{Y,y} \simeq E$.

*Proof.* Direct computation. □

In particular, we obtain the following perfect pairing

$$\mathfrak{m}/\mathfrak{m}^2 \times E \to k, \quad (f, (t_1, \cdots, t_n)) \mapsto \sum_{i=1}^{n} \frac{\partial f}{\partial T_i}(y) t_i.$$

Consider a closed subscheme $X = \mathrm{Spec}(k[T_1, \cdots, T_n]/I) \hookrightarrow Y$ with $I \subset k[T_1, \cdots, T_n]$ an ideal. Let $x = (\lambda_1, \cdots, \lambda_n) \in X(k) \subset Y(k)$ be a rational point of $X$. Let $\mathfrak{m} \subset k[T_1, \cdots, T_n]$ (resp. $\mathfrak{n} \subset A := k[T_1, \cdots, T_n]/I$) be the maximal ideal corresponding to $x \in Y$ (resp. to $x \in X$). Then $I \subset \mathfrak{m}$, and $\mathfrak{n} = \mathfrak{m}/I$. In particular, $\mathfrak{n}/\mathfrak{n}^2 = \mathfrak{m}/(\mathfrak{m}^2 + I)$, and we obtain the following short exact sequence:

$$0 \to (\mathfrak{m}^2 + I)/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

As a result, the isomorphism $T_{Y,x} = (\mathfrak{m}/\mathfrak{m}^2)^\vee \simeq E$ induces an identification between $T_{X,x}$ and the subspace

$$(D_y(I))^\perp := \left\{ (t_1, \cdots, t_n) \in E \colon \sum_{i=1}^{n} \frac{\partial P}{\partial T_i}(x) t_i = 0 \ \ \forall P \in I \right\}.$$

If $I = (P_1, \cdots, P_m)$, the latter space can also be written as:

$$(D_y(I))^\perp := \left\{ (t_1, \cdots, t_n) \in E \colon \sum_{i=1}^{n} \frac{\partial P_j}{\partial T_i}(x) t_i = 0 \ \ \forall j = 1, \cdots, m \right\}.$$

**Example 2.7.1.4.** Let $k$ be an algebraically closed field. Consider $X = \mathrm{Spec}(k[T, S]/(T^2 - S^3)) \hookrightarrow \mathbb{A}_k^2$. Let $P = (a, b) \in X$ be a closed point (hence rational as $k$ is algebraically closed). Then according to the previous calculations, we have

$$T_{X,P} \simeq \{(t, s) \in k^2 : 2at - 3b^2 s = 0\}.$$

Hence $\dim_k T_{X,P} = 1$ if $P$ is not equal to the origin of $\mathbb{A}_k^2$, otherwise it's of dimension 2.

### 2.7.2 Regular schemes

Let $(A, \mathfrak{m})$ be a neotherian local ring with residue field $k$. Recall that $\dim(A) \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$. We consider also the following $k$-graded algebra associated with $A$:

$$\mathrm{gr}_{\mathfrak{m}}(A) = \bigoplus_{n=0}^{\infty} \mathfrak{m}^n/\mathfrak{m}^{n+1}.$$

**Definition 2.7.2.1.** A neotherian local ring $(A, \mathfrak{m})$ is called *regular* if $\dim(A) = \dim_k \mathfrak{m}/\mathfrak{m}^2$

**Theorem 2.7.2.2.** *Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension $d$. The following three statements are equivalent.*

1. *$A$ is regular.*
2. *$\mathfrak{m} \subset A$ can be generated by $d$ elements.*
3. *$\mathrm{gr}_{\mathfrak{m}}(A)$ is isomorphic, as a $k$-graded algebra, to the ring of polynomials in $d$ variables.*

*Proof.* Cf. Atiyah-MacDonald Theorem 11.22. □

**Proposition 2.7.2.3.** *Let $(A, \mathfrak{m})$ be a regular neotherian local ring. Then $A$ is an integral domain.*

*Proof.* This proposition can be proved by using the theorem above, see Atiyah-MacDonald 11.23. Here is a direct proof of this result. We will prove by induction on $d := \dim(A)$. If $d = 0$, as $A$ is regular, we find $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 0$. Hence $\mathfrak{m} = \mathfrak{m}^2$, which implies $\mathfrak{m} = 0$ by Nakayama's lemma. So $A$ is a field in this case. In particular, it's an integral domain. Suppose now $d \geq 1$, and the statement has been verified for the local rings of dimension $\leq d-1$. Let $f \in \mathfrak{m} - \mathfrak{m}^2$, and consider $A/(f)$. We claim first that $\dim(A/(f))$ is regular of dimension $d-1$. Indeed, let $\mathfrak{n} = \mathfrak{m}/(f) \subset A/(f)$ be its maximal ideal. Then $\mathfrak{n}/\mathfrak{n}^2 = \mathfrak{m}/(\mathfrak{m}^2 + (f))$ which is of dimension $d-1$ (as $f \notin \mathfrak{m}^2$). On the other hand, as $A$ is noetherian, $\dim(A/(f)) \geq \dim(A) - 1 = d-1$, hence we find the following inequalities

$$d - 1 = \dim(A) - 1 \leq \dim(A/f) \leq \dim_k \mathfrak{n}/\mathfrak{n}^2 = d - 1.$$

On the other hand, let $\mathfrak{p} \subset A$ be a (minimal) prime ideal of depth $d$, then $A/\mathfrak{p}$ is again regular of dimension $d$, and the quotient $A/(f + \mathfrak{p})$ is a local ring of dimension $\geq \dim(A/\mathfrak{p}) - 1 = d-1$ (being the quotient of $A/\mathfrak{p}$ by a single element) whose maximal ideal can be generated by $d-1$ elements (being a quotient of $A/(f)$ which is regular local of dimension $d-1$), as a result, $A/(f + \mathfrak{p})$ is equally regular of dimension $d-1$. Now, as $A/(f + \mathfrak{p})$ is a quotient of $A/f$, which are both integral by induction hypothesis, we have $(f) + \mathfrak{p} = (f)$. Therefore, $\mathfrak{p} \in (f)$. So for any $u \in \mathfrak{p}$, there exists $v \in A$ such that $u = fv$. But note that $f \notin \mathfrak{p}$ (otherwise $\dim(A/f) \geq \dim(A/\mathfrak{p}) = d$), we must have $v \in \mathfrak{p}$. This shows $\mathfrak{p} \subset (f)\mathfrak{p} \subset \mathfrak{m}\mathfrak{p}$. So the Nakayama lemma implies $\mathfrak{p} = 0$. So $A$ is integral. This finishes the proof. □

From the proof, we obtain also the following

**Corollary 2.7.2.4.** *Let $(A, \mathfrak{m})$ be a noetherian regular local ring of dimension $d$. Let $f \in \mathfrak{m}$. Then $A/(f)$ is regular of dimension $d-1$ if and only if $f \notin \mathfrak{m}^2$.*

*Proof.* From the proof, we know that if $f \in \mathfrak{m} - \mathfrak{m}^2$, then $A/(f)$ is regular of dimension $d-1$. Conversely, suppose $A/(f)$ is regular of dimension $d-1$. In particular, $f \neq 0$. Moreover, to require that $A/(f)$ is regular of dimension $d-1$ is the same to require that $\dim(\mathfrak{m}/(\mathfrak{m}^2 + (f))) = \dim(\mathfrak{m}/\mathfrak{m}^2) - 1$. Hence we must have $f \notin \mathfrak{m}^2$. □

By induction, we obtain also the following

**Corollary 2.7.2.5.** *Let $(A, \mathfrak{m})$ be a regular noetherian local ring of dimension $d$. Let $I \subset A$ be a proper ideal of $A$. Then $A/I$ is regular if and only if $I$ can be generated by $r$ elements $f_1, \cdots, f_r$ with $r = \dim(A) - \dim(A/I)$, such that the $f_i$'s can be generated to a family $\{f_1, \cdots, f_r, f_{r+1}, \cdots, f_d\}$ of generators of $\mathfrak{m}$.*

**Theorem 2.7.2.6.** *Let $A$ be a regular neotherian local ring. Then the following properties hold.*

1. *For any prime ideal $\mathfrak{p} \subset A$, its localization $A_{\mathfrak{p}}$ is also regular.*

2. *The ring $A$ is factorial.*

**Definition 2.7.2.7.** Let $X$ be a locally neotherian scheme, and let $x \in X$ be a point. We say that $X$ is *regular* at $x \in X$, or $x$ is a regular point of $X$ if $\mathcal{O}_{X,x}$ is regular. We say that $X$ is regular if $X$ is regular at all its points. A point $x \in X$ which is not regular is called a *singular point* of $X$. A scheme that is not regular is said to be *singular*.

**Corollary 2.7.2.8.** *Let $X$ be a noetherian scheme. Then $X$ is regular if and only if $X$ is regular at all its closed points.*

*Proof.* Note that, as $X$ is noetherian, any closed subset of $X$ admits a closed point. □

**Example 2.7.2.9.** Let $k$ be an algebraically closed field.[15] The affine space $\mathbb{A}_k^n$ is regular. Indeed, as $\mathbb{A}_k^n$ is one quasi-compact, we only need to check that $\mathbb{A}_k^n$ is regular at all its closed points. Let $x = (x_1, \cdots, x_n) \in \mathbb{A}_k^n$ be a closed point, and let $\mathfrak{m} = (X_1 - x_1, \cdots, X_n - x_n) \subset k[X_1, \cdots, X_n]$ be the corresponding maximal idea. Then $\mathfrak{m}/\mathfrak{m}^2$ is of dimension $n$ over $k$ with a base given by the images of $X_i - x_i$ in $\mathfrak{m}/\mathfrak{m}^2$. On the other hand, we know $\mathbb{A}_k^n$ is of dimension $n$, the local ring $\mathcal{O}_{\mathbb{A}_k^n,x}$ is of dimension $n$. Thus, by Lemma 2.7.1.2, $\mathcal{O}_{\mathbb{A}_k^n,x}$ is regular of dimension $n$. In this way, one shows that $\mathbb{A}_k^n$ is regular. As a corollary, the projective space $\mathbb{P}_k^n$ is also regular.

**Theorem 2.7.2.10** (Jacobian criterion)**.** *Let $k$ be a field. Let $X = V(I) \hookrightarrow \mathbb{A}_k^n$ be a subscheme defined by the ideal $I \subset k[T_1, \cdots, T_n]$. Suppose $I = (F_1, \cdots, F_r)$, and let $x \in X(k)$ be a rational point. Then $X$ is regular at $x$ if and only if the following matrix*

$$J_x = \left(\frac{\partial F_i}{\partial T_j}(x)\right)_{1\leq i\leq r, 1\leq j\leq n}$$

*is of rank $n - \dim(\mathcal{O}_{X,x})$.*

**Example 2.7.2.11.** Consider $X = V(T^2 - S^3) \subset \mathbb{A}_k^2 = \mathrm{Spec}(k[T, S])$. Let $P = (a, b)$ be a rational point of $X$. The localization $\mathcal{O}_{X,P}$ is always of dimension 1. While for the corresponding matrix $J_P$, it's of rank 1 if and only if $x$ it not the origin of $\mathbb{A}_k^2$. Hence $P \in X$ is a regular point if and only if $P$ is not the origin.

For $X$ a noetherian scheme, we will denote by $\mathrm{Reg}(X)$ the set of regular points of $X$, and $X_{\text{sing}} := X - \mathrm{Reg}(X)$ the singular locus of $X$.

**Proposition 2.7.2.12.** *Let $k$ be an algebraically closed field, and $X/k$ be a scheme of finite type. Then*

1. *If moreover $X$ is reduced. Then $\mathrm{Reg}(X)$ contains a closed point of $X$.*
2. *In general, $\mathrm{Reg}(X) \subset X$ is an open subset.*

*Proof.* (1) See Liu's book.

(2) This is a local question on $X$, hence we may assume that $X = \mathrm{Spec}(A)$ is affine. Moreover, we may assume that $X$is irreducible. Hence for each closed point $x \in X(k)$, $\mathcal{O}_{X,x}$ is of the same dimension $d = \dim(X)$. As $X$ is of finite type over $k$, $A \simeq k[T_1, \cdots, T_n]/I$ for some ideal $I \subset k[T_1, \cdots, T_n]$. Suppose $I = (F_1, \cdots, F_r)$, and consider the matrix

$$M = \left(\frac{\partial F_i}{\partial T_j}\right)$$

with coefficients in $k[T_1, \cdots, T_n]$. For each closed point $x \in X$, the matrix $M(x)$ is then of rank $\leq n - d$. Let $J \subset A$ be image of the ideal of $k[T_1, \cdots, T_n]$ generated by the minors of $M$ of order $n - d$. Then a closed point $x \in X$ is singular if and only if all the minors of order $n - d$ of $M(x)$ vanishes, in other words, if and only if $x \in V(J) \subset X$. Now we claim that $\mathrm{Reg}(X) = X - V(J)$. Indeed, for a point $y \in X - V(J)$, it can specialize to a

[15] In fact, this assumption is unnecessary. Here we make this assumption just for simplicity.

closed point $x$ of $X$ contained in $X - V(J)$. Hence $y$ is a regular point of $X$ since $\mathcal{O}_{X,y}$ is a localization of a regular local ring $\mathcal{O}_{X,x}$. In this way, we see $X - V(J) \subset \mathrm{Reg}(X)$. Conversely, consider a regular point $x \in X$, and let $Z = \overline{\{x\}} \subset X$ with the reduced subscheme structure. Let $I \subset A$ be the ideal defining $Z$. As $Z$ is integral with $x \in Z$ its generic point, the local ring $\mathcal{O}_{Z,x} = \mathcal{O}_{X,x}/I\mathcal{O}_{X,x}$ is a field. Since $\mathcal{O}_{X,x}$ is a local ring, one can find $r := \dim(\mathcal{O}_{X,x})$ elements $\{f_1, \cdots, f_r\} \subset I$ such that $\mathfrak{m}_{X,x} = I\mathcal{O}_{X,x} = (f_1, \cdots, f_r)\mathcal{O}_{X,x}$. As a result, we find $\mathfrak{m}_{X,x}\mathcal{O}_{X,y} = (f_1, \cdots, f_r)\mathcal{O}_{X,y}$. According to (1), there exists a regular closed point $y_0 \in Z$. Hence $\mathcal{O}_{Z,y_0} = \mathcal{O}_{X,y_0}/(f_1, \cdots, f_r) = \mathcal{O}_{X,y_0}/\mathfrak{m}_{X,x}\mathcal{O}_{X,y_0}$ is regular. As a result, $\mathcal{O}_{X,y_0}$ is regular by the lemma below. Since $y_0 \in X$ is closed, by the jacobian criterion, $y_0 \in X - V(J)$. This implies $x \in X - V(J)$ since $y \in \overline{\{x\}}$. □

**Lemma 2.7.2.13.** *Let $(A, \mathfrak{m})$ be a noetherian local ring. Let $\mathfrak{p} \subset A$ be a prime ideal such that $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$ are both regular. Moreover, suppose that $\mathfrak{p}$ can be generated by $e$ elements with $e = \dim(A_{\mathfrak{p}})$. Then $A$ is also regular.*

*Proof.* Write $d = \dim(A)$. As $\mathfrak{p}$ can be generated by $e$ elements $\{f_1, \cdots, f_e\}$, we find $\dim(A/\mathfrak{p}) \geq d - e$. On the other hand, as $\dim(A_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) \leq \dim(A)$, we find exactly $\dim(A/\mathfrak{p}) = d - e$. Let $\{g_1, \cdots, g_{d-e}\} \subset \mathfrak{m}$ whose images in $A/\mathfrak{p}$ generate $\mathfrak{m}/\mathfrak{p}$, then we obtain $\mathfrak{m} = (f_1, \cdots, f_e, g_1, \cdots, g_{d-e})$ which hence can be generated by $d$ elements. As a result, $A$ is regular. □

**Remark 2.7.2.14.** With the notations of the proof of Proposition 2.7.2.12 (2), the singular locus $X_{\mathrm{sing}}$ of an irreducible affine scheme $X = \mathrm{Spec} k[T_1, \cdots, T_n]/(F_1, \cdots, F_r)$ is given by $V(J)$, where $J \subset k[T_1, \cdots, T_n]/(F_1, \cdots, F_r)$ is the ideal generated by the images of the minors of order $n - \dim(X)$.

### 2.7.3 Flatness and smoothness

As we mentioned before, intuitively, a morphism of schemes $f\colon X \to Y$ can be thought as a family of schemes defined over fields $\{X_y\}_{y \in Y}$ indexed by the scheme $Y$. As in the analytic case, to have some good properties on $f$, we usually need to suppose that the family moves *continuously* along the scheme $Y$. The right notion for the continuity in algebraic geometry is the flatness. Recall first the following definition in commutative algebra.

**Definition 2.7.3.1.** Let $A$ be a ring.

1. An $A$-module $M$ is said to be flat if for any injective morphism $N' \hookrightarrow N$, the induced morphism $M \otimes_A N' \to M \otimes_A N$ is still injective.

2. A morphism of ring $A \to B$ is said to be flat, if $B$ is flat as an $A$-module.

**Definition 2.7.3.2.** Let $f\colon X \to Y$ be a morphism of schemes. We say that $f$ is flat at the point $x \in X$ if the homomorphism $f_x^\sharp\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a flat. We say that $f$ is flat if it's flat at all the points of $X$.

**Example 2.7.3.3.** If $Y = \mathrm{Spec}(k)$ is the spectrum of a field, then every morphism $f\colon X \to Y$ is flat.

**Example 2.7.3.4.** For $A$ a *discrete valuation ring*, let $\pi \in A$ be a uniformizing element. Then an $A$-module $M$ is flat if and only if $M$ has no non-trivial $\pi$-torsion. As a result, a morphism $f\colon X \to Y$ is flat if and only if every irreducible component of $X$ dominates $Y$. See 4.3.9 of Liu's book for a proof of this statement.

**Proposition 2.7.3.5.** *The following properties are true.*

1. *Open immersions are flat morphisms.*
2. *Flat morphisms are stable under composition and base change.*
3. *Let $A \to B$ be a morphism of rings. Then $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat if and only if $B$ is flat over $A$.*

For a flat morphism $f\colon X \to Y$, the family $\{X_y\}_{y\in Y}$ moves continuously along the scheme $Y$, which allows to deduce many interesting properties about the morphism $f$. For example, one have the following theorem, whose proof can be found in the book of Liu.

**Theorem 2.7.3.6.** *Let $f\colon X \to Y$ be a morphism of locally neotherian schemes. Let $x \in X$ and $y = f(x)$. Then*

$$\dim(\mathcal{O}_{X_y,x}) \le \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{Y,y}).$$

*If, moreover, $f$ is flat, then we have equality.*

**Corollary 2.7.3.7.** *Let $X, Y$ be two irreducible schemes of finite type over a field $k$, and $f\colon X \to Y$ be a surjective flat morphism of $k$-schemes. Then for every $y \in Y$, every irreducible component of $X_y$ is of dimension $\dim(X) - \dim(Y)$.*

Moreover, we have

**Theorem 2.7.3.8.** *Let $f\colon X \to Y$ be a morphism of schemes between locally noetherian schemes. Assume that $f$ is flat. Then $f$ is an open map between the underlying topological spaces of $X$ and $Y$.*

**Definition 2.7.3.9.**
1. Let $f\colon X \to Y$ be a morphism of finite type of locally noetherian schemes. Let $x \in X$ and $y = f(x)$. We say that $f$ is *unramified* at $x$ if the homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ verifies $\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_x$, and the (finite) extension of residue fields $k(y) \to k(x)$ is separable. We say that $f$ is *étale* at $x$ if it's unramified and flat at $x$.
2. A morphism of noetherian local rings $A \to B$ is called *étale* is it's unramified, flat and if $B$ is a localization of a finitely generated $A$-algebra.

**Lemma 2.7.3.10.** *Let $f\colon X \to Y$ be a morphism of finite type of locally noetherian schemes.*

1. *Then $f$ is unramified if and only if for each $y \in Y$, the fiber $X_y$ is unramified (hence automatically étale over $k(y)$).*
2. *If $Y = \mathrm{Spec}(k)$ is the spectrum of a field, then $X$ is unramified over $k$ if and only if $X$ affine with $\mathcal{O}_X(X)$ a finite product of separable extensions of $k$.*

*Proof.* (1) For any $x \in X$ over $y \in Y$, the quotient $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ remains unchange after we replace $X$ by its fiber $X_y$ over $y$. As a result, $f$ is unramified at $x$ if and only if $X_y$ is unramified at $x$ over $k(y)$, which shows (1). For (2), let $x \in X$ be an arbitrary point, as the maximal ideal of $\mathcal{O}_{X,x}$ is 0, $\mathcal{O}_{X,x}$ is a field. As a result, $\dim(\mathcal{O}_{X,x}) = 0$ for any $x \in X$. Hence $\dim(X) = 0$ and $X$ is a disjoint union of finitely many points. Moreover, for each point $x \in X$, $\mathcal{O}_{X,x} = k(x)$ is a separable extension of $k$, as a result, $\mathcal{O}_X(X)$ is a finite product of separable extension of $k$. □

**Proposition 2.7.3.11.** *The following properties are true.*

1. *All closed immersion of locally noetherian scheme is an unramified morphism.*

2. *All open immersion of locally noetherian scheme is an étale morphism.*

3. *Unramified morphisms and étale morphisms are both stable under composition and base change.*

**Example 2.7.3.12** (Standard étale ring homomorphism)**.** Let $A$ be a noetherian ring, $f \in A[X]$ be a *monic* polynomial with coefficients in $A$, and $g \in A[X]$. Let $B = A[X]/(f)$, and suppose that the image of the derivative $f'$ of $f$ in $B_g$ is invertible. Then $B_g$ is étale over $A$. Such a ring homomorphism $A \to (A[X]/f)_g$ is called a *standard étale ring homomorphism*. To justify the terminology, we prove first that the $A$-algebra $B$ is flat. Indeed, write

$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0,$$

then the qotient $B = A[X]/(f)$ is a free $A$-module free of rank $n$. In particular, its localization $B_g$ is flat over $A$. Hence, to show that $B_g$ is étale over $A$, we may assume that $A = k$ is a field. Let

$$f(X) = \prod_{i=1}^{s} P_i(X)^{m_i}$$

be the decomposition of $f$ into product of irreducible polynomials such that $P_i \nmid P_j$ for any $i \neq j$. Then we know

$$k[X]/(f(X)) = \prod_{i=1}^{s} k[X]/(P_i(X)^{m_i}),$$

and hence $X = \coprod_{i=1}^{s} \mathrm{Spec}(k[X]/(P_i(X)^{m_i}) = \{x_1, \cdots, x_s\}$. So we need to show that, if the image of $f'$ in $k[X]/(P_i(X)^{m_i})$ is invertible, then $m_i = 1$, and $P_i(X)$ is a separable polynomial. By definition, we have

$$f'(X) = \sum_{i=1}^{s} m_i P_i(X)^{m_i - 1} P_i'(X) \cdot \prod_{j \neq i} P_j(X)^{m_j}.$$

Hence $f'(X)$ is invertible in $B_i := k[X]/(P_i(X)^{m_i})$ if and only if $P_i(X)^{m_i-1}P_i'(X)$ is invertible in $B_i$. So we must have $m_i = 1$, and $P_i'(X)$ is non zero, which implies that $P_i$ is separable. This prove the assertion.

The importance of the previous example is that, any étale morphism of rings locally looks like a standard étale ring homomorphism. More precisely, we have

**Theorem 2.7.3.13** (local structure of étale and unramified morphisms)**.** *Let $B$ be an $A$-algebra and $\mathfrak{q} \subset B$ be a prime ideal over a prime ideal $\mathfrak{p} \subset A$.*

1. *The following two conditions are equivalent:*

   (a) $\mathrm{Spec}(B)$ *is étale over* $\mathrm{Spec}(A)$ *over some open neighborhood of* $\mathfrak{q} \in \mathrm{Spec}(B)$.

   (b) *There exist $f \in B - \mathfrak{q}$ and $h \in A - \mathfrak{p}$ such that $B_f$ is $A$-isomorphic to a standard étale $A_h$-algebra $C = (A_h[X]/P)_g$.*

2. *The following two conditions are equivalent:*

   (a) $\mathrm{Spec}(B)$ *is unramifield over* $\mathrm{Spec}(A)$ *over some open neighborhood of* $\mathfrak{q} \in \mathrm{Spec}(B)$.

   (b) *There exist $f \in B - \mathfrak{q}$, $h \in A - \mathfrak{p}$ and a standard étale $A_h$-algebra $C = (A_h[X]/P)_g$, and a surjective morphism of $A$-algebras $C \to B_f$.*

*Proof.* We refer to Raynaud's book *Anneaux locaux henséliens* Chapitre V for a proof of this important theorem. □

**Definition 2.7.3.14.** Let $X/k$ be a scheme of finite type over a field $k$. Let $\overline{k}$ be an algebraic closure of $k$. We say that $X$ is smooth at a point $x \in X$ if the points of $X_{\overline{k}}$ over $x$ are regular points of $X_{\overline{k}}$. We say $X$ is smooth if $X$ is smooth at any points of $X$, or equivalently, if $X_{\overline{k}}$ is regular.

**Example 2.7.3.15.** Let $k$ be a field.

1. The affine space $\mathbb{A}^n_k$ and projective space $\mathbb{P}^n_k$ are all smooth over $k$. Indeed, by Example 2.7.2.9, the base changes $\mathbb{A}^n_{\overline{k}}$ and $\mathbb{P}^n_{\overline{k}}$ are all regular.

2. Let $L/k$ be a *finite separable* extension of $k$, then $X := \mathrm{Spec}(L) \to \mathrm{Spec}(k)$ is smooth. In fact, its base change $X_{\bar{k}} = \mathrm{Spec}(L \otimes_k \bar{k})$. As $L/k$ is finite separable, it can be generated by just one element $x \in L$. Let $P(X) \in k[X]$ be the minimal polynomial of $x$ over $k$, then $L \simeq k[X]/(f(X))$. Hence $X_{\bar{k}} = \mathrm{Spec}(\bar{k}[X]/(f(X)))$. As $f(X)$ is separable, $f(X)$ can be decomposed in $\bar{k}[X]$ as follows:

$$f(X) = (X - a_1)(X - a_2) \cdots (X - a_n)$$

where $a_i \in k$ such that $a_i \neq a_j$ $(i \neq j)$. Now, the Chinese remainder theorem implies that

$$\bar{k}[X]/(f(X)) \simeq \bar{k}[X]/(X - a_1) \times \cdots \times \bar{k}[X]/(X - a_n) \simeq \bar{k}^n.$$

So $X_{\bar{k}} = \mathrm{Spec}(\bar{k}^n)$ is a disjoint union of finitely many closed points $x_i$ $(1 \leq i \leq n)$, and the corresponding local ring $\mathcal{O}_{X_{\bar{k}},x_i}$ is isomorphic to $k$. Hence $X_{\bar{k}}$ is regular of dimension 0. This shows that $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ is smooth.

3. Suppose now $k$ is *imperfect* of characteristic $p$. Consider $a \in k - k^p$, and the finite extension $L = k[X]/(X^p - a)$. Then $X = \mathrm{Spec}(L)$ is regular, but $X$ is *not* smooth over $\mathrm{Spec}(k)$. Indeed, its base change to $\bar{k}$ is given by $\mathrm{Spec}(L \otimes_k \bar{k}) = \mathrm{Spec}(\bar{k}[X]/(X - \alpha)^p)$ where $\alpha \in \bar{k}$ such that $\alpha^p = a$. So $X_{\bar{k}}$ is a non-reduced scheme. In particular, $X_{\bar{k}}$ is not regular. As a result, $X$ is not smooth over $\mathrm{Spec}(k)$.

**Exercise 2.7.3.16.** Let $L/k$ be a finite field extension. Then $\mathrm{Spec}(L) \to \mathrm{Spec}(k)$ is smooth if and only if $L/k$ is separable. Deduce that for a scheme $X$ of finite type over a field $k$ such that $\dim(X) = 0$, $X$ is smooth over $k$ if and only if $X$ is étale over $k$.

**Lemma 2.7.3.17.** *Let $k$ be an algebraically closed field, and let $\Omega$ be an algebraically closed field containing $k$. Let $X$ be a scheme of finite type over $k$ with is irreducible of dimension $d$. Then so is the base change $X_\Omega$.*

*Proof.* To be added later. The key point here is the following: for $k$ an algebraic closed field, and for $K, L$ two field extension of $k$, the tensor product $K \otimes_k L$ is an integral domain. □

**Lemma 2.7.3.18.** *Suppose $k = \overline{k}$ is algebraically closed, and let $\Omega/k$ be an extension of $k$ which is also algebraically closed. Let $X/k$ be a scheme of finite type. Then $X/k$ is smooth, if and only if the base change $X_\Omega$ over $\Omega$ is smooth.*

*Proof.* Up to replace $X$ by a connected component of $X$, we may assume that $X$ is connected. As $X$ is smooth over $k$, with $k$ algebraically closed, each local ring of $X$ is regular, and hence is an integral domain. As a result, $X$ is irreducible (see the exercise below), hence $X$ is integral. As a result, $X_\Omega$ is also irreducible, and $\dim(X_\Omega) = \dim(X) =: d$ (see the lemma above). We may also assume that $X = \mathrm{Spec}(A)$ is affine, with $A = k[T_1, \cdots, T_n]/(F_1, \cdots, F_r)$, as a result, $X_\Omega = \mathrm{Spec}(\Omega[T_1, \cdots, T_n]/(F_1, \cdots, F_r))$. Let $J$ be the ideal generated by the images of the minors of order $n - d$ of the following matrix $M$

$$M := \left(\frac{\partial F_i}{\partial T_j}\right)_{1 \leq i \leq r, 1 \leq j \leq n}$$

Then as we see in the proof of Proposition 2.7.2.12 (2) (or Remark 2.7.2.14), $V(J) \subset X$ is the singular locus $X_{\mathrm{sing}}$ of $X$. Similarly, the ideal

$$J_\Omega := J \otimes_k \Omega \subset A_\Omega = A \otimes_k \Omega$$

defines also the singular locus $X_{\Omega,\mathrm{sing}}$ of $X_\Omega$. Hence, if $X$ is smooth over $k$, then $V(J) = \emptyset$. This implies that the ideal $J \subset A$ is equal to $A$. Hence so is the ideal $J_\Omega \subset A_\Omega$. As a result, $V(J_\Omega) = \emptyset$. In other words, $X_\Omega$ is smooth. □

**Exercise 2.7.3.19.** Let $X$ be a noetherian scheme whose local rings are all integral domains. Prove that any connected component of $X$ is also irreducible.

**Corollary 2.7.3.20.** *Let $X/k$ be a scheme over $k$, and let $k \subset k'$ be a field extension. If $X$ is smooth over $k$, then so is the base change $X_{k'}$.*

**Definition 2.7.3.21.** Let $f: X \to Y$ be a morphism of schemes. We say that $f$ is *smooth* at a point $x \in X$ if $f$ is flat at $x$, and $X_y$ is smooth at $x$ (where $y = f(x)$). If $f$ is smooth at any point of $X$, then we say that $f$ is smooth.

For a flat morphism $f: X \to Y$, we define the *relative dimension* of $f$ at a point $y \in Y$ to be the dimension of the fiber $X_y$. With this terminology, a étale morphism is just a smooth morphism of relative dimension 0. More precisely, we have

**Proposition 2.7.3.22.** *Let $f: X \to Y$ be a finite type morphism between locally noetherian schemes. Suppose moreover that $f$ is a smooth morphism. Then $f$ is étale if and only if for any point $y \in f(X) \subset Y$, the relative dimension of $f$ at $y$ is equal to zero.*

*Proof.* Use Exercise 2.7.3.16. □

**Proposition 2.7.3.23.** *Let $Y$ be a regular locally noetherian scheme, and $f: X \to Y$ be a smooth morphism. Then $X$ is also regular.*

*Proof.* Omitted, see Liu's book, theorem 4.3.36. □

**Proposition 2.7.3.24.** *Smooth morphisms are stable under base change and composition.*

**Proposition 2.7.3.25.** *Let $f: X \to Y$ be a morphism of finite type between locally noetherian schemes. Let $x \in X$ and $y = f(x)$. Then we have an exact sequence of $k(x)$-vector spaces*

$$0 \to T_{X_y,x} \to T_{X,x} \to T_{Y,y} \otimes_{k(y)} k(x)$$

*If moreover $f$ is smooth at $x$, then the sequence above is also exact at right.*

### 2.7.4 Normal schemes

Recall that an integral domain $R$ is called *normal* if $R$ is integrally closed in its fraction fields. Namely, for any element $a \in \mathrm{Frac}(R)$, if there exists a *monic* polynomial $P(X) \in A[X]$ such that $P(a) = 0$, then $a \in R$.

**Exercise 2.7.4.1.** Let $A$ be a ring.

1. If $A$ is a unique factorization domain. Then $A$ is normal. Deduce that any regular noetherian local ring is normal (use Theorem 2.7.2.6 (2)).

2. If $A$ is a normal integral domain. Then $S^{-1}A$ is normal for any multiplicative subset $S$ of $A$.

**Remark 2.7.4.2.** Let $(A, \mathfrak{m})$ be a noetherian local ring.

1. If $\dim(A) = 0$. Then $A$ is normal iff $A$ is regular iff $A$ is an integral domain iff $A$ is a field.

2. If $\dim(A) = 1$. The we claim that the following three statements are all equivalents: (a) $A$ is a principal ideal domain; (b) $A$ is a regular; (c) $A$ is normal.

   - Clearly (a) implies (b) and (c).
   - Now assuming $A$ normal, we will show that $\mathfrak{m}$ can be generated by one element, that is $A$ is regular. Let $x \in \mathfrak{m} - \mathfrak{m}^2$. As $A$ is a domain of dimension 1, the quotient $A/(x)$ is of dimension 0. In particular, its maximal ideal $\mathfrak{m}/(x) \subset A/(x)$ is nilpotent, which implies that there exists an integer $r \gg 1$ such that $\mathfrak{m}^r \subset xA$. If $r = 1$, we can then conclude $\mathfrak{m} = (x)$ is principal. Otherwise, assume $r \geq 2$, and we claim that claim that $\mathfrak{m}^{r-1} \subset xA$. Indeed, let $y \in \mathfrak{m}^{r-1}$, then $yx^{-1}\mathfrak{m} \subset x^{-1}\mathfrak{m}^r \subset A$. Thus $yx^{-1}\mathfrak{m}$ is an ideal of $A$. It cannot be equal to $A$ since otherwise one would have $x \in y\mathfrak{m} \subset \mathfrak{m}^2$, which is a contradiction. Hence $yx^{-1}\mathfrak{m} \subset \mathfrak{m}$. Now we claim $yx^{-1} \in \mathrm{Frac}(A)$ is integral over $A$: indeed, suppose $\mathfrak{m} = (x_1, \cdots, x_n)$, there exist $a_{ij} \in A$ such that

     $$yx^{-1}x_i = \sum_j a_{ij}x_j.$$

     In other words, we have the following equality in terms of matrix

     $$(yx^{-1}\mathrm{I}_n - (a_{ij})) \cdot (x_1, \cdots, x_n)' = 0.$$

     As a result, $\det\big(yx^{-1}\mathrm{I}_n - (a_{ij})\big) \cdot x_i = 0$. Since $A$ is a domain, $\mathrm{Frac}(A)$ is a field. Moreover $\mathfrak{m}$ is non-zero, we must have then $\det\big(yx^{-1}\mathrm{I}_n - (a_{ij})\big) = 0$. So the element $yx^{-1} \in \mathrm{Frac}(A)$ verifies an integral relation over $A$. By consequent, $yx^{-1}$ is integral over $A$. As $A$ is normal, we must have $yx^{-1} \in A$. Hence $y \in xA$, and we obtain $\mathfrak{m}^{r-1} \subset A$. If we continue this proof in finitely many times, we find $\mathfrak{m} \subset xA$, hence $\mathfrak{m} = xA$ is principal. So this implies (b).
   - To complete the proof, it remains to show (b) implies (a). Let $x \in A$ such that $\mathfrak{m} = (x)$. We claim first that $I := \bigcap_{n\geq} \mathfrak{m}^n$ is equal to $(0)$. Clearly $\mathfrak{m}I \subset I$, so by Nakayama's lemma, it remains to show $I \subset \mathfrak{m}I$. If $\mathfrak{m}I = 0$, as $A$ is an integral domain, we must have $I = 0$, and we have trivially $I \subset \mathfrak{m}I$. If $\mathfrak{m}I \neq 0$. Then $A/\mathfrak{m}I$ is a noetherian local ring of dimension 0, as a result, there exist some integer $r \gg 0$ such that $\mathfrak{m}^r \subset \mathfrak{m}I$. Hence $I \subset \mathfrak{m}^r \subset \mathfrak{m}I$. So in both case, we have $I \subset \mathfrak{m}I$. This gives finally $\mathfrak{m}I = I$, as a result, $I = 0$ by Nakayama's lemma. Next, let $J \subset A$

be a *non-zero* ideal contained in $\mathfrak{m}$. As $\bigcap_{n\geq 0}\mathfrak{m}^n = 0$, there exists an integer $n \geq 1$ such that $J \subset \mathfrak{m}^n = (x^n)$ but $J \not\subseteq \mathfrak{m}^{n+1}$. We claim that $J = x^n A$. Indeed, choose $y \in J - \mathfrak{m}^{n+1}$, then the image of $y$ in the quotient $\mathfrak{m}^n/\mathfrak{m}^{n+1} = \mathfrak{m}^n \otimes_A A/\mathfrak{m}$ is non-zero. But the latter quotient is of one dimensional over $A/\mathfrak{m}$, as a result, the image of $y$ gives a base of this quotient. So by Nakayama's lemma, $y$ generates also the ideal $\mathfrak{m}^n$. Hence $\mathfrak{m}^n = (y) \subset J$. This shows $J = \mathfrak{m}^n = (x^n)$, and hence gives (1).

3. If $\dim(A) \geq 2$. Then there exits example of normal local ring which is not regular. For example, consider the ring $A = k[X,Y,Z]/(Z^2 - XY)$. The scheme $X = \mathrm{Spec}(A)$ is not regular (for example, one can use jacobian criterion to see that $X$ is not regular at the point $o = (0,0,0)$), but the ring $A$ is however normal. As a result, $\mathcal{O}_{X,o}$ is a normal noetherian local ring which is not regular.

**Example 2.7.4.3.** For $k$ a field, show that $A = k[X,Y,Z]/(Z^2 - XY)$ is a normal ring (**Hint:** show firstly the following morphism is injective

$$A \to k[U,V], \quad X \mapsto U^2, Y \mapsto V^2, Z \mapsto UV.$$

Then use this injection to identify $A$ with the subring $k[U^2,V^2,UV]$. Finally verify the equality $k[U^2,V^2,UV] = k[U,V] \cap k(U^2,V/U)$, where the intersection is taken in $k(U,V)$, and conclude that $k[U^2,V^2,UV]$ is normal).

**Definition 2.7.4.4.** Let $X$ be a scheme. We say that $X$ is *normal at* $x \in X$ or that $x$ is a *normal point of* $X$ if $\mathcal{O}_{X,x}$ is normal. We say that $X$ is *normal* if it is normal at all of its points.[16] Similarly, we define the notion of *factorial scheme*.

**Proposition 2.7.4.5.** *Let $X$ be an irreducible scheme. The following properties are equivalent:*

1. *The scheme $X$ is normal.*
2. *For every open subset $U$ of $X$, $\mathcal{O}_X(U)$ is a normal integral domain.*

*If, moreover, $X$ is quasi-compact, these properties are equivalent to*

*3 The scheme $X$ is normal at its closed points.*

*Proof.* (1)$\Longrightarrow$(2). As $X$ is normal, the scheme $X$ is integral. In particular, $\mathcal{O}_X(U)$ is integral domain. Let $\alpha \in \mathrm{Frac}(\mathcal{O}_X(U))$ which is integral over $\mathcal{O}_X(U)$. We may assume $U$ is affine. Write $U = \mathrm{Spec}(A)$. As $X$ is normal, for any prime ideal $\mathfrak{p}$ of $A$, the localization $A_{\mathfrak{p}}$ is normal. In particular, we have $x \in A_{\mathfrak{p}}$. Hence $\alpha = a_{\mathfrak{p}}/b_{\mathfrak{p}}$ with $a_{\mathfrak{p}} \in A$, and $b_{\mathfrak{p}} \in A - \mathfrak{p}$. In particular, we find in the field $\mathrm{frac}(A)$ the equality $xb_{\mathfrak{p}} = a_{\mathfrak{p}}$. Now we consider $I \subset A$ the ideal generated by $b_{\mathfrak{p}}$, then $I = A$. In particular, one can find the equality $1 = \sum_{\mathfrak{p}} \lambda_{\mathfrak{p}} b_{\mathfrak{p}}$. Hence $x = \sum_{\mathfrak{p}} \lambda_{\mathfrak{p}} b_{\mathfrak{p}} x = \sum_{\mathfrak{p}} \lambda_{\mathfrak{p}} a_{\mathfrak{p}} \in A$. This gives (2). The inverse direction is clear. Suppose now $X$ quasi-compact.

It remains to show that (3) implies (1). In fact, for any point $x \in X$, we claim that there exists a closed point $y \in X$ such that $y \in \overline{\{x\}}$.[17] Indeed, as $X$ is quasi-compact, and $\overline{\{x\}} \subset X$ is a closed subset, we can find finitely many affine open subschemes $U_i$ $(i = 1, \cdots, n)$ such that $\overline{\{x\}} \subset \bigcup_{i=1}^n U_i$. Now suppose $x \in U_1$, then $x$ corresponds to a prime ideal $\mathfrak{p}_1$ of the ring $\mathcal{O}_X(U_1)$, hence there exists a maximal ideal containing $\mathfrak{p}_1$, which corresponds then a closed point $y_1$ of $U_1$. In other words, $y_1 \in U_1$ is a closed point of $U_1$ such that $y_1 \in \overline{\{x\}}$. Then $y_1 \in X$ is closed

[16] This definition differs from the definition in the book of Liu where we require for simplicity that $X$ is irreducible.

[17] In this proof, the closure is always taken in $X$

if and only if $\{y_1\}\bigcap U_i$ is a closed subset of $U_i$ for each $i$. If it is not the case, there exists $i\in\{2,\cdots,n\}$, say $i=2$ such that $\overline{\{y_1\}}\bigcap U_2$ is not closed. Hence one can find $y_2\in U_2$ which is closed in $U_2$ such that $y_2\in\overline{\{y_1\}}$. Moreover, since $y_1\in U_2$ is not closed, $y_2\neq y_1$. Hence $y_2\notin U_1$ since $y_1\in U_1$ is already closed. Now we continue with this processus, and since the covering is finite, after at most $n$ steps, one find a point $y\in X$ such that $y\in\overline{\{x\}}$ such that $\{y\}\cap U_i$ is closed in $U_i$ for each $i$. In other words, $y\in X$ is a closed point. This gives the claim. Now by the assumption, $\mathcal{O}_{X,y}$ is a normal ring. Since $\mathcal{O}_{X,x}$ is a localization of $\mathcal{O}_{X,y}$, it's hence still normal. In this way, we show that $X$ is normal. This proves (1), and hence finishes the proof of this proposition. □

**Theorem 2.7.4.6** (Krull's structure theorem)**.** *Let $A$ be a noetherian integral domain. Then $A$ is normal if and only if the following two conditions are satisfied:*

*(i) For any prime ideal $\mathfrak{p}\subset A$ of height 1, $A_{\mathfrak{p}}$ is a principal ideal domain.*

*(ii) We have the following equality in $\mathrm{Frac}(A)=:K$:*

$$A=\bigcap_{\mathfrak{p}\in\mathrm{Spec}(A),\mathrm{ht}(\mathfrak{p})=1}A_{\mathfrak{p}}.$$

*Proof.* Clearly, if $A$ is an integral domain verifying the two properties (i) and (ii) above, then $A$ is normal as it's the intersection of normals subrings. Now let $A$ be a normal ring, we need to show that the two properties (i) and (ii) are true. In fact, for any prime ideal $\mathfrak{p}\subset A$ of height 1, $A_{\mathfrak{p}}$ is a normal local ring of dimension 1, hence $A_{\mathfrak{p}}$ is a principal ideal domain. It remains to verify the second property. In fact, let $A'=\bigcap_{\mathfrak{p}\in\mathrm{Spec}(A),\mathrm{ht}(\mathfrak{p})=1}A_{\mathfrak{p}}\subset\mathrm{Frac}(A)$, we need to show that $A=A'$. Clearly $A\subset A'$. If $A\subsetneq A'$, for any element $f\in A'-A$, let $I_f:=\{a\in A|af\in A\}\subset A$, which is clearly non-zero. As $A$ is noetherian, the set $\{I_f:f\in A'-A\}$ contains a maximal element, say $\mathfrak{q}:=I_g$. We claim that $\mathfrak{q}\subset A$ is a prime ideal. Indeed, let $a_1,a_2\in A$ such that $a_1a_2\in\mathfrak{q}$ but $a_2\notin\mathfrak{q}$. Then $a_2g\in A'-A$, and $a_1\in I_{a_2g}$. Furthermore, by definition, we have $\mathfrak{q}=I_g\subset I_{a_2g}$. By the choice of $\mathfrak{q}=I_g$, we must have $\mathfrak{q}=I_{a_2g}$. Hence $a_1\in\mathfrak{q}$. This shows that $\mathfrak{q}$ is a prime ideal. Moreover, we remark that $\mathrm{ht}(\mathfrak{q})\geq 1$ since $0\subsetneq\mathfrak{q}$.

Consider now the ideal $g\mathfrak{q}A_{\mathfrak{q}}\subset A_{\mathfrak{q}}$ (note that $g\mathfrak{q}\subset A$). If $g\mathfrak{q}A_{\mathfrak{q}}=A_{\mathfrak{q}}$, then we obtain $\mathfrak{q}A_{\mathfrak{q}}=g^{-1}A_{\mathfrak{q}}$ (equality in $\mathrm{Frac}(A)$). As a result, $A_{\mathfrak{q}}$ is a local ring of dimension $\leq 1$ (as its maximal ideal can be generated by just one element). So this implies that $\mathfrak{q}$ is of height 1. But

$$g\in A'=\bigcap_{\mathfrak{p}\in\mathrm{Spec}(A),\mathrm{ht}(\mathfrak{p})=1}A_{\mathfrak{p}},$$

we have $g\in A_{\mathfrak{q}}$, so we obtain $1=g^{-1}g\in g^{-1}A_{\mathfrak{q}}=\mathfrak{q}A_{\mathfrak{q}}$, a contradiction. Thus, we must have $g\mathfrak{q}A_{\mathfrak{q}}\subset\mathfrak{q}A_{\mathfrak{q}}$. As $A_{\mathfrak{q}}$ is noetherian, the ideal $\mathfrak{q}A_{\mathfrak{q}}$, is of finite type. By the same argument as we see in Remark 2.7.4.2 (2) ((c)$\Longrightarrow$(b)), $g\in\mathrm{Frac}(A)$ is integral over $A_{\mathfrak{q}}$. As $A_{\mathfrak{q}}$, being a localization of the normal ring $A$, is normal, we must have $g\in A_{\mathfrak{q}}$. As a result, $g=a/s$ with $a\in A$ and $s\in A-\mathfrak{q}$. So $sg=a\in A$, which implies that $s\in\mathfrak{q}=I_g$. A contradiction. As a result, one cannot find such ideal $\mathfrak{q}$, hence the set $\{I_f|f\in A'-A\}$ must be empty, which means $A'=A$. This finishes the proof of the theorem. □

As a geometric application of Krull's theorem, we have the following

**Proposition 2.7.4.7.** *Let $X$ be a normal locally noetherian scheme. Let $F\subset X$ be a closed subset of codimension $\geq 2$ in $X$. Then the restriction map $\mathcal{O}_X(X)\to\mathcal{O}_X(X-F)$ is bijective.*

*Proof.* We may assume that $X = \mathrm{Spec}(A)$ is affine and irreducible. In particular, $X$ is integral. Let $\xi \in X$ be its generic point. According to Proposition 2.3.3.5, for any open $U \subset X$ (resp. any point $x \in X$), $\mathcal{O}_X(U)$ (resp. $\mathcal{O}_{X,x}$) is naturally a subring of $\mathcal{O}_{X,\xi}$. Moreover, under these identifications, if $x \in U$, then $\mathcal{O}_X(U) \subset \mathcal{O}_{X,x} \subset \mathcal{O}_{X,\xi} = \mathrm{Frac}(A)$. Now, as $F \subset X$ is of codimension $\geq 2$, its complement $X - F$ contains all the primes of $A$ of height 1. In particular,

$$A = \mathcal{O}_X(X) \subset \mathcal{O}_X(X-F) \subset \bigcap_{\mathfrak{p}\in\mathrm{Spec}(A),\mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}} = A.$$

As a result, $\mathcal{O}_X(X)\mathcal{O}_X(X-F) = A$. This finishes the proof. □

The following properties is also very useful.

**Proposition 2.7.4.8.** *Let $S$ be a scheme, $X, Y$ be two locally noetherian $S$-schemes, and $U \subset X$ is an non-empty open subset. We assume*

- *$X$ is normal.*
- *$Y/S$ is proper.*

*Let $f\colon U \to X$ be an $S$-morphism. There there exists a open subset $V \subset X$ containing $U$ such that its complement $X - V$ is of codimension $\geq 2$ in $X$, and a unique morphism of $S$-schemes $g\colon V \to Y$ such that $g|_U = f$. In particular, if $\dim(X) = 1$, then $f$ can be extended to a unique morphism of $S$-schemes $X \to Y$.*

*Proof.* The proof requires the valuative criterion for proper morphisms, so we omit the proof here. □

For a integral scheme, there is a canonical normal scheme attached to it, called its *normalization*. More precisely, we have

**Proposition 2.7.4.9.** *For any integral scheme $X$, there exists a unique normal scheme together with a morphism $\pi\colon X' \to X$ verifying the following universal property: for any dominant morphism $f\colon Y \to X$ with $Y$ a normal scheme, there exists a unique morphism $f'\colon Y \to X'$ such that $\pi \circ f' = f$.*

*Proof.* We will just give the construction of $X'$ when $X = \mathrm{Spec}(A)$ is affine. Let $K$ be the fraction field of $A$, and $A'$ be the set of elements which are integral over $A$. Then $A' \subset K$ is a subring containing $A$. Hence we get a morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ which satisfies the universal property required by the proposition. □

**Exercise 2.7.4.10.** Let $m$ be an integer, and consider $X = \mathrm{Spec}(\mathbb{Z}[S_0, S_1, S_2]/(S_2^2 - mS_1S_2))$. Is $X$ normal? If so, prove your conclusion, otherwise find the normalization of $X$ (one can refer to Liu's book for a detail answer is this exercise).

# Chapter 3

# An introduction to algebraic curves

For simplicity, in this chapter, $k$ denotes an algebraically closed field of characteristic $p \geq 0$. An *algebraic curve* defined over $k$ is a $k$-scheme of finite type of dimension 1. We will mostly suppose in this chapter that $X$ is irreducible smooth and proper over $k$.

## 3.1 Divisors and invertible sheaves

### 3.1.1 Divisors

**Definition 3.1.1.1.** Let $X/k$ be a smooth irreducible curve.

1. A *divisor* $D$ on $X$ is a formal sum $D = \sum_x n_x x$, where the $x$'s are closed points of $X$, $n_x \in \mathbb{Z}$ and $n_x = 0$ for all but finitely many $x$. The divisor $D$ is called *effective* if $n_x \geq 0$ for all $x$. For two divisors $D = \sum_x n_x x$ and $D' = \sum_x m_x x$, we say that $D \geq D'$ if $n_x \geq m_x$ for all $x$.

2. For $D = \sum_x n_x x$ a divisor, we define its degree $\deg(D) = \sum_x n_x \in \mathbb{Z}$.

3. We will denote by $\mathrm{Div}(X)$ be the set of divisors on $X$, which is then the free abelian group generated by the closed points of $X$, and $\mathrm{Div}^0(X)$ the set of divisors on $X$ of degree 0.

Let $X$ be a smooth curve over $k$ with function field $K(X)$. In particular, $X$ is a regular scheme, thus, for any closed point $x \in X$, its local ring $\mathcal{O}_{X,x}$ is a discrete valuation ring. As a result, there exists a *valuation* on $K$ defined by: for any $f \in K^*$, we define $v_x(f)$ to be the integer $n \in \mathbb{Z}$ such that $f \in \mathfrak{m}_x^n - \mathfrak{m}_x^{n+1}$. Consider the following formal sum

$$\mathrm{div}(f) := \sum_{x \in X \text{ closed}} v_x(f) \cdot x.$$

**Lemma 3.1.1.2.** *The formal sum* $\mathrm{div}(f)$ *is a divisor, and* $\mathrm{div}(f) \geq 0$ *if and only if* $f \in \mathcal{O}_X(X)$. *Moreover, if $X$ is proper, then* $\deg(\mathrm{div}(f)) = 0$. *Such a divisor is called* principal.

*Proof.* Since $k(X) = \mathcal{O}_{X,\eta}$ with $\eta \in X$ the generic point, for any $f \in k(X)$, there exists some open subset $U \subset X$ such that $f$ provides from an element of $\mathcal{O}_X(U)$. Similarly, as $f \in k(X)^*$, up to replace $U$ by some smaller open subset, we may assume that $f \in \mathcal{O}_X(U)^*$. Thus $v_x(f) = 0$ for any $x \in U$. Since the complement $X - U$ of $U$ is a finite set, we find that $v_x(f) = 0$ for all but finitely many points of $X$, which gives the first statement. The proof of the second statement will be omitted for the moment. $\square$

**Definition 3.1.1.3.** Let $X$ be a connected smooth curve defined over $k$.

1. We define $\mathrm{P}(X) \subset \mathrm{Div}(X)$ to be the set of principal divisors on $X$.
2. Two divisors $D, D'$ on $X$ are said to be *linearly equivalent* if $D - D'$ is a principal divisor on $X$.

### 3.1.2 Invertible sheaves

**Definition 3.1.2.1.** For $X$ be a scheme.

1. An *invertible sheaf* on $X$ is an $\mathcal{O}_X$-module $\mathcal{L}$ such that for each point $x$ of $X$, there exists an open neighborhood $U$ of $x$ such that $\mathcal{L}|_U \simeq \mathcal{O}_X|_U$ as an $\mathcal{O}_X|_U$-module. We will denote by $\mathrm{Pic}(X)$ the set of isomorphism classes of invertible sheaves on $X$.
2. For two invertible sheaves $\mathcal{L}$ and $\mathcal{L}'$ on $X$, we can consider the *tensor product*: $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$; and the *dual* of $\mathcal{L}$: $\mathcal{L}^\vee := \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$. With these two operations, the set $\mathrm{Pic}(X)$ becomes an abelian group, called the *Picard group* of $X$.

Let $X$ be a smooth curve over $k$, and $D$ a divisor on $X$. We define in the following way a sheaf $\mathcal{O}_X(D)$ on $X$: for any open subset $U \subset X$, we define

$$\mathcal{O}_X(D)(U) = \{f \in K^* | (\mathrm{div}(f) + D)|_U \geq 0\} \cup \{0\} \subset K,$$

which is naturally an $\mathcal{O}_X(U)$-module. Together with the usual restriction maps, we obtain in this way a sheaf $\mathcal{O}_X(D)$ such that for each open $U \subset X$, $\mathcal{O}_X(D)(U)$ is naturally a $\mathcal{O}_X(U)$-modules. Hence we get an $\mathcal{O}_X$-module.

**Lemma 3.1.2.2.** *$\mathcal{O}_X(D)$ is locally free of rank one over $\mathcal{O}_X$.*

*Proof.* Write $D = \sum_x n_x x$, and let $\mathrm{supp}(D) = \{x : n_x \neq 0\}$. Hence $\mathrm{Supp}(D) \subset X$ is a finite set, and its complement $U = X - \mathrm{Supp}(D) \subset X$ is an open subset of $X$. Moreover, for any open $V \subset U$, we have

$$\begin{aligned} \mathcal{O}_X(D)(V) &= \{f \in K^* | (\mathrm{div}(f) + D)|_V \geq 0\} \cup \{0\} \\ &= \{f \in K^* | \mathrm{div}(f) \geq 0\} \cup \{0\} \\ &= \mathcal{O}_X(V). \end{aligned}$$

Hence, we find $\mathcal{O}_X(D)|_U \simeq \mathcal{O}_X|_U$. In particular, $\mathcal{O}_X(D)|_U$ is free over $\mathcal{O}_X|_U$. It remains to show that $\mathcal{O}_X(D)$ is locally free around an open neighborhood of $x \in \mathrm{Supp}(D)$. Since the local ring $\mathcal{O}_{X,x} \subset K$ is a discrete valuation ring, let $f \in \mathcal{O}_{X,x}$ be a uniformizing element. Then $g$ is an element of $\mathcal{O}_X(W)$ for some small open neighborhood of $x$. As a result, we can write

$$\mathrm{div}(g) = x + \sum_{y \neq x} n_y y.$$

So we find $D|_{W_0} = \mathrm{div}(g^{n_x})|_{W_0} = n_x x$, with

$$W_0 = W - (\mathrm{Supp}(D) - \{x\}) \cup (\mathrm{div}(g) - \{x\}),$$

which is an open neighborhood of $x$. Now for any $V_0 \subset W_0$ open subset,

$$\begin{aligned} \mathcal{O}_X(D)(V_0) &= \{f \in K^* | (\mathrm{div}(f) + D)|_{V_0} \geq 0\} \cup \{0\} \\ &= \{f \in K^* | \mathrm{div}(f g^{n_x})|_{V_0} \geq 0\} \cup \{0\} \\ &= \{f \in K^* | f g^{n_x} \in \mathcal{O}_X(V_0)\} \cup \{0\} \\ &\simeq \mathcal{O}_X(V_0), \end{aligned}$$

where the last isomorphism is given by $f \mapsto fg^{n_x}$. In this way, we find $\mathcal{O}_X(D)|_{V_0} \simeq \mathcal{O}_X|_{V_0}$, which is again free of rank one over $\mathcal{O}_X|_{V_0}$. This finishes then the proof of the lemma. $\square$

In this way, we obtain the following map

$$\varphi\colon \mathrm{Div}(X) \to \mathrm{Pic}(X), \quad D \mapsto \mathcal{O}_X(D).$$

One can show that the following two formulas hold: for any two divisors $D, D' \in \mathrm{Div}(X)$, we have

$$\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D') \simeq \mathcal{O}_X(D + D'), \quad \mathcal{O}_X(D)^\vee \simeq \mathcal{O}_X(-D).$$

In particular, the map above is a morphism of groups.

**Proposition 3.1.2.3.** *$\varphi$ is surjective with kernel given by* $\mathrm{P}(X)$. *As a result, we obtain an isomorphism of groups* $\mathrm{Div}(X)/\mathrm{P}(X) \simeq \mathrm{Pic}(X)$.

**Definition 3.1.2.4.** Let $X/k$ be a proper connected smooth curve, and $\mathcal{L}$ an invertible sheaf on $X$. We define the degree of $\mathcal{L}$, denoted by $\deg(\mathcal{L})$, to be $\deg(D)$ where $D$ is an divisor on $X$ such that $\mathcal{L} = \mathcal{O}_X(D)$. Note that by our assumption, any principal divisor is of degree 0, hence $\deg(\mathcal{L})$ is independent of the choice of $D$.

### 3.1.3 Čech cohomology of a topological space

We will discuss here the Čech cohomology. Consider first of all $Y$ a topological space, and $\mathcal{U} := \{U_i | i \in I\}$ an open covering of $Y$. Let $\mathcal{F}$ be an abelian sheaf on $Y$. For any integer $n \geq 0$, and for any sequence of indices $i_0, \cdots, i_n \in I$, let

$$U_{i_0 \cdots i_n} = U_{i_0} \cap \cdots \cap U_{i_n}.$$

We set

$$C^n(\mathcal{U}, \mathcal{F}) = \prod_{(i_1, \cdots, i_n) \in I^{n+1}} \mathcal{F}(U_{i_0 \cdots i_n}).$$

Now for $f \in C^n(\mathcal{U}, \mathcal{F})$, we define $df \in C^{n+1}(\mathcal{U}, \mathcal{F})$ to be the element given by

$$(df)_{i_0 \cdots i_{n+1}} = \sum_{j=0}^{n+1} (-1)^j f_{i_0 \cdots \hat{i}_j \cdots i_{n+1}}|_{U_{i_0 \cdots i_{n+1}}},$$

where the symbol $\hat{i}_j$ means to remove the index $i_j$. For example, if $n = 1$, then

$$(df)_{i_0 i_1 i_2} = f_{i_1 i_2}|_{U_{i_0 i_1 i_2}} - f_{i_0 i_2}|_{U_{i_0 i_1 i_2}} + f_{i_0 i_1}|_{U_{i_0 i_1 i_2}}.$$

In this way, we obtain a complex of abelian groups

$$C^\bullet(\mathcal{U}, \mathcal{F}) = (\cdots \longrightarrow 0 \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d} \cdots).$$

Now for any integer $n \geq 0$, we set

$$\mathrm{H}^n(\mathcal{U}, \mathcal{F}) := \mathrm{H}^n(C^\bullet(\mathcal{U}, \mathcal{F})) = \frac{\ker(d : C^n(\mathcal{U}, \mathcal{F}) \to C^{n+1}(\mathcal{U}, \mathcal{F}))}{\mathrm{im}(d : C^{n-1}(\mathcal{U}, \mathcal{F}) \to C^n(\mathcal{U}, \mathcal{F}))},$$

by convention, $C^{-1}(\mathcal{U}, \mathcal{F}) = 0$.

**Proposition 3.1.3.1.** *The canonical map $\mathcal{F}(X) \to \mathrm{H}^0(\mathcal{U}, \mathcal{F})$ is an isomorphism.*

Let $\mathcal{V} = \{V_j | j \in I\}$ be a second open covering of $Y$. We say that $\mathcal{V}$ is a refinement of $\mathcal{U}$ if there exists a map $\sigma\colon J \to I$ such that $V_j \subset U_{\sigma(j)}$ for every index $j$. We have then a homomorphism, which will also denoted by $\sigma$:

$$\sigma\colon C^n(\mathcal{U}, \mathcal{F}) \to C^n(\mathcal{V}, \mathcal{F})$$

given by

$$\sigma(f)_{j_0 \cdots j_n} = f_{\sigma(j_0) \cdots \sigma(j_n)}|_{V_{j_0 \cdots j_n}}.$$

This homomorphism commutes with the differential operators, we obtain a morphism of complexes

$$C^\bullet(\mathcal{U}, \mathcal{F}) \to \mathbb{C}^\bullet(\mathcal{V}, \mathcal{F}),$$

and hence a morphism between the cohomology groups

$$\mathrm{H}^n(\mathcal{U}, \mathcal{F}) \to \mathrm{H}^n(\mathcal{V}, \mathcal{F}).$$

We have now the following important observation:

**Lemma 3.1.3.2.** *The morphism above is independent of the choice of $\sigma$ for all $n \geq 0$.*

**Definition 3.1.3.3.** Let $Y$ be a topological space, and $\mathcal{F}$ be an abelian sheaf on $Y$. We set

$$\check{\mathrm{H}}^n(Y, \mathcal{F}) \to \varinjlim_{\mathcal{U}} \mathrm{H}^n(\mathcal{U}, \mathcal{F}),$$

where $\mathcal{U}$ runs through the classes of open coverings of $Y$. The group $\check{\mathrm{H}}^n(Y, \mathcal{F})$ is called the $n$-th Čech cohomology group of $\mathcal{F}$.

On applying this construction to a scheme $X$, we obtain the notion of *Čech cohomology* of the scheme $X$. For separated schemes, the direct limit in the definition of Čech cohomology above is even unnecessary. More precisely, we have the following result:

**Proposition 3.1.3.4.** *Let $X$ be a* separated *scheme, $\mathcal{U} = \{U_i | i \in I\}$ an open covering of $X$ by* affine *schemes. Let $\mathcal{L}$ be a line bundle on $X$.*[1] *Then the canonical map*

$$\mathrm{H}^n(\mathcal{U}, \mathcal{L}) \to \check{\mathrm{H}}^n(X, \mathcal{L})$$

*is an isomorphism for any $n$. Moreover, we have*

1. *If $X$ is projective of dimension $d$, then $\check{\mathrm{H}}^n(X, \mathcal{L}) = 0$ for $n > d$.*

2. *If $X$ is affine, then $\check{\mathrm{H}}^n(X, \mathcal{L}) = 0$ for any $n > 0$.*

**Remark 3.1.3.5.** The more correct notion of cohomology is defined by using derived functors. But for quasi-coherent sheaf $\mathcal{F}$ on a separated scheme (for example, when $\mathcal{F} = \mathcal{L}$ is an invertible sheaf), the Čech cohomology is the same as the cohomology group defined by using derived functors. Hence, for this reason, for the following, we will also use the notation $\mathrm{H}^n(X, \mathcal{L})$ to denote $\check{\mathrm{H}}^n(X, \mathcal{L})$ for an invertible sheaf on $X$. As a result, by a theorem of Grothendieck, we have $\mathrm{H}^n(X, \mathcal{L}) = 0$ for $n \geq \dim(X) + 1$. Moreover, if $X/k$ is proper over a field $k$, $\mathrm{H}^n(X, \mathcal{L})$ is a $k$-vector space of finite dimension.

[1] or more generally, a *quasi-coherent* sheaf on $X$.

### 3.1.4 Riemann-Roch theorem for a curve (first form)

In this subsection, let $X/k$ be *proper* a smooth connected curve. In particular, for an invertible sheaf $\mathcal{L}$ on $X$, for each $n$, the cohomology group $\mathrm{H}^n(X, \mathcal{L})$ is a $k$-vector space of finite dimension. Moreover, $\mathrm{H}^n(X, \mathcal{L}) = 0$ for $n \neq 0, 1$.

**Definition 3.1.4.1.** Let $\mathcal{L}$ be an invertible sheaf on $X$. Its *Euler-Poincaré characteristic* is defined to be the following alternative sum

$$\chi(\mathcal{L}) = \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{dim}_k(\mathrm{H}^i(X, \mathcal{L})) = \mathrm{dim}_k(\mathrm{H}^0(X, \mathcal{L})) - \mathrm{dim}_k(\mathrm{H}^1(X, \mathcal{L})).$$

**Lemma 3.1.4.2.** *If $X$ is a connected smooth proper curve over $k$. Then the canonical map $k \to \mathcal{O}_X(X)$ is an isomorphism of rings.*

*Proof.* As $X$ is connected and smooth, $\mathcal{O}_X(X)$ is a integral domain. Since $X$ is proper over $k$, $\mathcal{O}_X(X)$ is finite as a $k$-module. In particular, $\mathcal{O}_X(X)$ is a field, hence is a finite extension of $k$. As we assume $k$ algebraically closed, we must have $k = \mathcal{O}_X(X)$. $\square$

**Example 3.1.4.3.** We take $\mathcal{L} = \mathcal{O}_X$ the trivial invertible sheaf. According to $\mathrm{H}^0(X, \mathcal{O}_X) \simeq k$. We set $g = \dim(\mathrm{H}^1(X, \mathcal{O}_X))$, called the *genus* of the curve $X/k$. So we obtain $\chi(\mathcal{O}_X) = 1 - g$, which is sometimes referred to be the *Euler-Poincaré characteristic of $X$*.

**Theorem 3.1.4.4** (Riemann-Roch theorem–first form)**.** *Let $\mathcal{L}$ be an invertible sheaf of degree $d$. Then*

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \deg(\mathcal{L}) = 1 - g + \deg(\mathcal{L}).$$

But this theorem is not enough "compute" $\mathrm{dim}_k(\mathrm{H}^0(X, \mathcal{L}))$. We need some information about $\mathrm{dim}_k(\mathrm{H}^1(X, \mathcal{L}))$, which will be furnished by the Serre's duality theorem.

## 3.2 Serre's duality theorem

### 3.2.1 Differentials

Let $k$ be a field, $A$ be an $k$-algebra and $M$ an $A$-module. A $k$-derivation of $A$ into $M$ is a $k$-linear map $d\colon A \to M$ such that the Leibniz rule

$$d(a_1 a_2) = a_1 d(a_2) + a_2 d(a_1), \quad \forall a_i \in A$$

is verified, and that $d(\lambda) = 0$ for any $\lambda \in k$. Sometimes the set of $k$-derivations of $A$ into $M$ is denoted by $\mathrm{Der}_k(A, M)$.

**Definition 3.2.1.1.** With the notations above. The *module of relative differential forms of $A$ over $k$* is an $A$-module $\Omega^1_{A/k}$ endowed with a $k$-derivation $d\colon A \to \Omega^1_{A/k}$ having the following universal property: for any $A$-module $M$, and for any $k$-derivation $D\colon A \to M$, there exists a unique homomorphism of $A$-modules $\phi\colon \Omega^1_{A/k} \to M$ such that $D = d \circ \phi$. In other words, the map below is an bijection.

$$\mathrm{Hom}_A(\Omega^1_{A/k}, M) \to \mathrm{Der}_k(A, M), \quad \phi \mapsto d \circ \phi.$$

**Proposition 3.2.1.2.** *The module of relative differential forms $(\Omega^1_{A/k}, d)$ exists and is unique up to unique isomorphism.*

*Proof.* Let $F$ be the free $A$-module generated by the symbols $da$ for $a \in A$. Let $E \subset F$ be the submodule of $F$ generated by the elements of the form $d(\lambda)$, $\lambda \in k$; $d(a_1 + a_2) - da_1 - da_2$, and $d(a_1a_2) - a_1da_2 - a_2da_1$ with $a_i \in A$. Now we define $\Omega^1_{A/k} = F/E$, and $d\colon A \to \Omega^1_{A/k}$ sending $a$ to the image of $da$ in $\Omega^1_{A/k}$. Then it's clear that $(\Omega^1_{A/k}, d)$ has the required properties. □

Let now $X$ be a smooth connected curve defined over $k$, and let $K$ be its function field. Let $x \in X$. Consider now $\Omega^1_{K/k}$ (resp. $\Omega^1_{\mathcal{O}_{X,x}/k}$) the $K$-space (resp. the $\mathcal{O}_{X,x}$-module) of relative differential forms of $K$ over $k$ (resp. of $\mathcal{O}_{X,x}$ over $k$).

**Remark 3.2.1.3.** Let $k$ be a field, and $A$ a $k$-algebra.

1. Let $\alpha\colon A \to B$ be a morphism of $k$-algebras. There exists then a canonical morphism of $B$-modules:
$$B \otimes_A \Omega^1_{A/k} \to \Omega^1_{B/k}, \quad b \otimes da \mapsto b \cdot d(\alpha(a)).$$

2. Let $S \subset A$ be a multiplicative subset. The canonical map $A \to S^{-1}A$ induces then an isomorphism of $S^{-1}A$-modules
$$S^{-1}\Omega^1_{A/k} = S^{-1}A \otimes_A \Omega^1_{A/k} \to \Omega^1_{S^{-1}A/k}.$$
Indeed, we have $\mathrm{Der}_k(S^{-1}A, S^{-1}M) \simeq \mathrm{Der}_k(A, S^{-1}M)$, and
$$\mathrm{Hom}_A(\Omega^1_{A/k}, S^{-1}M) \simeq \mathrm{Hom}_{S^{-1}A}(S^{-1}\Omega^1_{A/k}, S^{-1}M).$$
As a result, $S^{-1}\Omega^1_{A/k} \simeq \Omega^1_{S^{-1}A/k}$.

3. Let $I \subset A$ be an ideal, and put $B = A/I$. We have then the following exact sequence
$$I/I^2 \to B \otimes_A \Omega^1_{A/k} \to \Omega^1_{B/k} \to 0.$$

4. Now, we take $A = k[T_1, \cdots, T_n]$. Then $\Omega^1_{A/k}$ is the free $A$-module with a base given by $\{dT_i | i = 1 \cdots, n\}$. For $B = k[T_1, \cdots, T_n]/(F_1, \cdots, F_m)$ with $F_i \in k[T_1, \cdots, T_n]$, by (3), $\Omega^1_{B/k}$ is given by
$$\frac{\oplus_i BdT_i}{< \text{image of } dF_j \text{ in } \oplus_i BdT_i >} \simeq \mathrm{coker}(\beta\colon B^m \to B^n)$$
where the matrix in the canonical basis of the $A$-linear morphism $\beta$ above is given by
$$\left(\overline{\frac{\partial F_j}{\partial T_i}}\right)_{1\leq i\leq n, 1\leq j\leq m} \in \mathrm{M}_{n\times m}(B).$$

**Lemma 3.2.1.4.** *$\Omega^1_{\mathcal{O}_{X,x}/k}$ is free of rank one over $\mathcal{O}_{X,x}$. In particular, $\Omega^1_{k(X)/k}$ is a $k(X)$-vector space of dimension one.*

*Proof.* Let $U = \mathrm{Spec}(B)$ be an affine neighborhood of $x$ with $B = k[T_1, \cdots, T_n]/(F_1, \cdots, F_m)$. Then $B$ is an integral domain. Consider moreover the following matrix with coefficient in $B$:
$$M = \left(\overline{\frac{\partial F_j}{\partial T_i}}\right)_{1\leq i\leq n, 1\leq j\leq m} \in \mathrm{M}_{n\times m}(B).$$

As $X$ is regular of dimension one at $x$, by the jacobian criterion, the matrix

$$M(x) = \left(\frac{\partial F_j}{\partial T_i}(x)\right)_{1\le i\le n, 1\le j\le m} \in \mathrm{M}_{n\times m}(k(x))$$

is of rank $n-1$. As a result, there exists a minor of order $n-1$ of $M$ which is invertible in $B_{\mathfrak{p}}$ with $\mathfrak{p} \subset B$ the prime ideal corresponding to $x$. Hence there is a submatrix $N$ of order $n-1$ which is invertible as a matrix with coefficients in $B_{\mathfrak{p}}$. Hence, there exists an element $b \in B - \mathfrak{p}$ such that, viewed as a matrix with coefficients in $B_b$, the matrix $N$ is invertible. Hence, we can find invertible matrix $P \in \mathrm{GL}_n(B_b)$ and $Q \in \mathrm{GL}_m(B_b)$ such that the matrix $PMQ$ has the form

$$\begin{pmatrix} N & 0 \\ 0 & N' \end{pmatrix}$$

with $N' \in \mathrm{M}_{1\times(m-n)}(B_b)$. But since $X/k$ is smooth of dimension 1, we must have $N'(y) = 0$ for any closed point $y \in \mathrm{Spec}(B_b) \subset \mathrm{Spec}(B)$. Hence, we find that the coefficients of $N'$ are all contained in

$$\bigcap_{\mathfrak{m}\in \mathrm{Max}(B_b)} \mathfrak{m}$$

which, by Hilbert's zero theorem, is the nilpotent radical of $B_b$. But $B_b$ is integral, so we find finally $N' = 0$. In this way, we find that $\Omega^1_{B_b/k}$ is a free $B_b$-module of rank 1. As a result, the $B_{\mathfrak{p}}$-module $\Omega^1_{B_{\mathfrak{p}}/k}$ is equally free of rank 1. □

**Remark 3.2.1.5.** According to (2) of the previous remark, the notion of module of differential forms can be globalized. So we can define the sheaf of differential forms $\Omega^1_{X/k}$, and by the last lemma, this is an $\mathcal{O}_X$-module locally free of rank 1. In other words, $\Omega^1_{X/k}$ is an invertible module on $X$.

For $x \in X$ a closed point, let $t \in \mathcal{O}_{X,x} \subset k(X)$ be the uniformizing element of the discrete valuation ring $\mathcal{O}_{X,x}$. The differential $dt$ gives a base of the one dimension $k(X)$-space $\Omega^1_{k(X)/k}$. Hence, for any element $\omega \in \Omega^1_{k(X)/k}$, it can be written as $\omega = f \cdot dt$ with $f \in k(X)$. When $\omega \neq 0$, we define

$$v_x(\omega) = v_x(f) \in \mathbb{Z}, \quad \text{and} \quad \mathrm{div}(\omega) = \sum_x v_x(\omega)x.$$

**Definition 3.2.1.6.** The $k(X)$-space $\Omega^1_{k(X)/k}$ is sometimes called the space of *meromorphic differential forms*. An element $\omega \in \Omega^1_{k(X)/k}$ is called *holomorphic* if it's either zero, or $v_x(\omega) \geq 0$ for all closed point $x \in X$.

**Lemma 3.2.1.7.** *The formal sum* $\mathrm{div}(\omega)$ *above is a divisor.*

*Proof.* Indeed, take a closed point $x \in X$, and $U = \mathrm{Spec}(A)$ an open neighborhood of $x$. Let $t \in \mathcal{O}_{X,x}$ a uniformizing element. We write $\omega = fdt$ with $f \in \mathcal{O}_{X,x}$. Up to replace $U$ by a smaller open neighborhood, we may assume that $f, t \in \mathcal{O}_X(U)$. Moreover, we may assume $\Omega^1_{A/k}$ is free of rank 1 with a base given by $dt$. As a result, $t - t(y)$ is a uniformizing element of $\mathcal{O}_{X,y}$ for any $y \in U$, and $\omega = fd(t - t(y))$. Hence $v_y(\omega) = v_y(f)$. Since $v_y(f) = 0$ for almost all $y \in U$, we find $v_y(\omega) = 0$ for almost all $y \in X$. This finishes the proof. □

**Definition 3.2.1.8.** For $D$ a divisor on $X$, we define $\Omega(D)$ to be

$$\Omega(D) := \{\omega \in \Omega^1_{k(X)/k} | \mathrm{div}(\omega) \geq D\} \cup \{0\}.$$

When $D = 0$, $\Omega := \Omega(0)$ is then the space of holomorphic differentials on $X$.

### 3.2.2 Residues

Recall that, for a noetherian local ring $(A, \mathfrak{m})$, its ($\mathfrak{m}$-adic) completion is defined to be

$$\widehat{A} := \varprojlim_n A/\mathfrak{m}^n.$$

Moreover, the canonical map $A \to \widehat{A}$ is injective. Hence we can identify $A$ to a subring of $\widehat{A}$.

**Lemma 3.2.2.1.** *Let $X/k$ be a smooth curve, and $x \in X$ a closed point. Let $t \in \mathcal{O}_{X,x}$ be a uniformizing element. Then the canonical map $k[[X]] \to \widehat{\mathcal{O}_{X,x}}$ sending $X$ to the image of $t$ is an isomorphism.*

*Proof.* To be added later. □

By using this lemma, we can deduce another local invariant of $\omega$, its *residue*: let $t$ be a uniformizing element of $\mathcal{O}_{X,x}$, we write $\omega = f dt$. As $f \in K$, there exists an integer $n$ such that $t^n f \in \mathcal{O}_{X,x}$. Hence viewed as an element in $\widehat{\mathcal{O}_{X,x}}$, we find $t^n f \in k[[t]]$. As a result, $f \in k((t))$. Note that this description is independent of the choice of $t$. So we can write $f \in k((t))$ as

$$f(t) = \sum_{n \gg -\infty} a_n t^n.$$

Hence the *residue* of the differential form $\omega$ at the point $x$ is defined to be

$$\mathrm{Res}_x(\omega) := a_{-1} \in k.$$

We have the following two fundamental propositions.

**Proposition 3.2.2.2** (Invariance of the residue)**.** *The preceding definition is independent of the choice of the local uniformizing element $t$.*

*Proof.* To be added later. □

**Proposition 3.2.2.3** (Residue formula)**.** *For every meromorphic differential forms $\omega \in \Omega^1_{K/k}$, we have $\sum_{x \in X(k)} \mathrm{Res}_x(\omega) = 0$.*

*Proof.* To be added later. □

### 3.2.3 Classes of répartitions

Following Weil, introduce the following definition of *répartitions* (or *adèles*)

**Definition 3.2.3.1.**

1. A *répartition* $r$ is a family $\{r_x\}_{x \in X(k)}$ of elements of $k(X)$ such that $r_x \in \mathcal{O}_{X,x}$ for almost all $x \in X$. The répartitions form an algebra $R$ over the field $k$.
2. For $D$ a divisor, we write $R(D)$ for the vector subspace of $R$ formed by the $r = \{r_x\}$ such that $v_x(r_x) \geq -v_x(D)$.

As $D$ runs through the ordered set of divisors of $X$, the $R(D)$ form an increasing filtered family of subspaces of $R$ whose union is $R$ itself. Moreover, we have a natural injection from the function field $k(X)$ to the algebra of répartitions over $k$.

**Proposition 3.2.3.2.** *For a divisor $D$ on $X$. There is a canonical isomorphism between $\mathrm{H}^1(X, \mathcal{O}_X(D))$ and $R/(R(D) + k(X))$.*

*Proof.* We consider the constant presheaf sheaf $\mathcal{K}$ defined as follows: for any non empty subset $U \subset X$, $\mathcal{K}(U) = k(X)$. As our scheme $X$ is irreducible, $\mathcal{K}$ is a sheaf on $X$. Consider now the following short exact sequence of abelian sheaves on $X$:

$$0 \to \mathcal{O}_X(D) \to \mathcal{K} \to \mathcal{K}/\mathcal{O}_X(D) \to 0.$$

The Čech cohomology gives then a long exact sequence

$$0 \to \mathrm{H}^0(X, \mathcal{O}_X(D)) \to K \to \mathrm{H}^0(X, \mathcal{K}/\mathcal{O}_X(D)) \to \mathrm{H}^1(X, \mathcal{O}_X(D)) \to \mathrm{H}^1(X, \mathcal{K}).$$

As $X$ is irreducible, and $\mathcal{K}$ is constant, we find $\mathrm{H}^1(X, \mathcal{K}) = 0$.[2] Hence $\mathrm{H}^1(X, \mathcal{O}_X(D))$ is the cokernel of the morphism $K \to \mathrm{H}^0(X, \mathcal{K}/\mathcal{O}_X(D))$. On the other hand, for any element $s \in \mathrm{H}^0(X, \mathcal{K}/\mathcal{O}_X(D))$, let $\tilde{s}_x \in \mathcal{K}_x$ be a lifting of the stalk $s_x \in (\mathcal{K}/\mathcal{O}_X(D))_x$. We obtain hence a family $\{\tilde{s}_x\}_{x \in X(k)} \in R$ whose image in the quotient $R/R(D)$ is independent of the choice of $\tilde{s}_x$. So we get in this way a morphism

$$\alpha \colon \mathrm{H}^0(X, \mathcal{K}/\mathcal{O}_X(D)) \to R/R(D).$$

This map is bijective. Indeed, for any $r = \{r_x\} \in R$, let

$$Y = \{x \in X(k) | r_x \in \mathcal{O}_{X,x}\} \cup \mathrm{Supp}(D), \quad \text{and} \quad U = X - Y.$$

For any element $y \in Y$, let $V_y \subset X$ be an open such that $r_y|_{V_y - \{y\}} \in \mathcal{O}_X(D)(V_y)$. As a result, the class $\{(r_y, V_y), (0, U)$ defines a global section of $\mathcal{K}/\mathcal{O}_X(D)$. As a result, the morphism $\alpha$ above is surjective. It's also clear that $\alpha$ is injective. Hence $\alpha$ gives an isomorphism. As a result, the cokernel of the morphism $K \to \mathrm{H}^0(X, \mathcal{K}/\mathcal{O}_X(D))$ is the same as the cokernel of the morphism $K \to R/R(D)$, that is, $R/(R(D) + K)$, as desired. □

### 3.2.4 Duality theorem

Let $D$ be a divisor, recall that we write $\Omega(D)$ the set of meromorphic differential forms formed by 0 and the differentials $\omega \neq 0$ such that $\mathrm{div}(\omega) \geq D$. Now for any $\omega \in \Omega^1_{K/k}$, and any $r \in R$, we define

$$< \omega, r > := \sum_{x \in X(k)} \mathrm{Res}_x(r_x \omega).$$

Since $r_x \omega \in \Omega^1_{\mathcal{O}_{X,x}/k}$ for almost all $r$, the previous sum is a finite sum. We obtain in this way the following pairing

$$< \cdot, \cdot > \colon \Omega^1_{K/k} \times R \to k.$$

Moreover, one can verify the following properties:

(a) $< \omega, r > = 0$ if $r \in K$ by the residue formula.

(b) $< \omega, r > = 0$ if $r \in R(D)$ and $\omega \in \Omega(D)$, since $r_P \omega \in \Omega^1_{\mathcal{O}_{X,x}/k}$ for all $x \in X(k)$.

(c) If $f \in K$, then $< f\omega, r > = < \omega, fr >$

As a result of (a) and (b), we obtain the following pairing

$$< \cdot, \cdot > \colon \Omega(D) \times R/(R(D) + K) \to k.$$

[2] Details to be added later.

**Theorem 3.2.4.1** (Serre's duality theorem)**.** *For every divisor $D$, the previous pairing is perfect.*

Now, for $\omega \in \Omega^1_{K/k}$, we denote by

$$\theta(\omega)\colon R \to k$$

the map induced by the pairing above.

**Lemma 3.2.4.2.** *If $\omega$ is a differential such that $\theta|_{R(D)+K} = 0$, then $\omega \in \Omega(D)$.*

*Proof.* Indeed, otherwise there would be a point $x \in X(k)$ such that $v_x(\omega) < v_x(D)$. Put $n = v_x(\omega) + 1$, and let $r$ be the répartition whose components are

$$r_y = \begin{cases} 0 & \text{if } y \neq x, \\ 1/t^n & y = x \end{cases}$$

where $t$ is a uniformizing element of $\mathcal{O}_{X,x}$. As a result, $v_x(r_x\omega) = -1$, whence $\mathrm{Res}_x(r_x\omega) \neq 0$ and $< \omega, r > \neq 0$; but since $n \leq v_x(D)$, $r \in R(D)$, and we arrive at a contradiction since $\theta(\omega)$ is assumed to vanish on $R(D)$. □

*Proof of duality theorem.* To be added later. □

## 3.3 Riemann-Roch theorem – Definitive form

### 3.3.1 Riemann-Roch theorem

Let $X/k$ be a connected proper smooth curve. Recall that for a meromorphic differential form $\omega \in \Omega^1_{k(X)/k}$, $\mathrm{div}(\omega)$ is a divisor on $X$. We will denote this divisor by $K_\omega$, called a *canonical divisor* of $X$. If $\omega'$ is another meromorphic differential form, the corresponding divisor $K_{\omega'}$ is different from $K_\omega$ by a principal divisor, hence the class of $K_\omega$ in $\mathrm{Div}(X)/\mathrm{P}(X)$ is independent of the choice of $\omega$, as a result, $\mathcal{O}_X(K_\omega)$ is independent of the choice of $\omega$. Because of this reason, by abuse of notation, we will denote by $K$ a canonical divisor, called *the* canonical divisor of $X$. With this terminology, we have trivially the following

**Lemma 3.3.1.1.** *For each divisor $D$ on $X$, there is a $k$-linear bijection between $\mathrm{H}^0(X, \mathcal{O}_X(K - D))$ and $\Omega(D)$.*

Hence, on combining 3.1.4.4, 3.2.3.2 and 3.2.4.1, we obtain the following

**Theorem 3.3.1.2** (Riemann-Roch theorem – Definitive form)**.** *Let $X/k$ be a proper connected smooth curve, and $D$ a divisor on $X$. Then*

$$\dim_k(\mathrm{H}^0(X, \mathcal{O}_X(D))) - \dim_k(\mathrm{H}^0(X, \mathcal{O}_X(K - D))) = 1 - g + \deg(D).$$

### 3.3.2 Some applications

**Lemma 3.3.2.1.** *Let $X/k$ be a proper connected smooth curve, and $D$ a divisor of degree $< 0$. Then $\mathrm{H}^0(X, \mathcal{O}_X(D)) = 0$.*

*Proof.* Otherwise, let $f \in k(X)^*$ such that $\mathrm{div}(f) + D \geq 0$. As a result, we have

$$0 \leq \deg(\mathrm{div}(f) + D) = \deg(D),$$

whence a contradiction. □

**Lemma 3.3.2.2.** *Let $K$ be the canonical divisor of $X/k$, then* $\deg(K) = 2g - 2$.

*Proof.* Recall that $\mathrm{H}^0(X, \mathcal{O}_X) = k$. As a result, we obtain by Riemann-Roch theorem that

$$1 - \dim_k \mathrm{H}^0(X, \mathcal{O}_X(K)) = 1 - g + 0.$$

Similarly, on applying the Riemann-Roch theorem to the divisor $K$, we obtain

$$\dim_k \mathrm{H}^0(X, \mathcal{O}_X(K)) - 1 = 1 - g + \deg(K).$$

As a result, $\deg(K) = 2g - 2$. □

**Corollary 3.3.2.3.** *Let $X/k$ a proper smooth connected curve, and $D$ a divisor of $X$. If* $\deg(D) > 2g - 2$, *then* $\dim_k\ \mathrm{H}^0(X, \mathcal{O}_X(D)) = 1 - g + \deg(D)$.

Now, we give the following characterization of projective line.

**Proposition 3.3.2.4.** *The genus of $\mathbb{P}^1_k$ is equal to 0. Conversely, for any proper smooth connected curve of genus 0 is $k$-isomorphic to $\mathbb{P}^1_k$.*

*Proof.* Indeed, consider let $t$ be the uniformizing element of $\mathcal{O}_{\mathbb{P}^1_k, 0}$, then $k(\mathbb{P}^1_k) = k(t)$. We consider the differential form $dt$, then we find

$$v_x(dt) = \begin{cases} 0 & \text{if } x \neq \infty; \\ -2 & \text{if } x = \infty. \end{cases}$$

Hence $\mathrm{div}(dt) = -2 \cdot \infty$. As a result, $2g - 2 = \deg(\mathrm{div}(dt)) = -2$, which implies $g = 0$. Conversely, $X$ be a smooth proper connected curve of genus 0. In particular, $2g - 2 < 0$. Let $x \in \mathbb{P}^1_k$ be a closed point. By Riemann-Roch theorem, the $k$-space $\mathrm{H}^0(X, \mathcal{O}_X(x))$ is of dimension 2. Hence there exists a element $f \in k(X) - k$ such that

$$\mathrm{div}(f) + x \geq 0.$$

As $f \notin k$, $\mathrm{div}(f) \neq 0$. Hence we must have $v_x(f) = -1$. Now using $f$, we can define a morphism of $k$-schemes $\alpha \colon X \to \mathbb{P}^1_k$. Now the fact that $v_x(f) = -1$ implies that $\alpha$ is an isomorphism. □

We now comes to curves of genus 1. Such a curve (with a marked point) is called an *elliptic curve*. For elliptic curves, we have the following

**Proposition 3.3.2.5.** *Let $X = E/k$ be an elliptic curve. Then one can realize $X$ as a closed subscheme $\mathbb{P}^2_k$ whose (inhomogeneous) equation can be given by the following Weierstrass equation:*

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

*Proof.* To be added later. □

**Remark 3.3.2.6.** When the field $k$ is of characteristic $\neq 2, 3$, the Weierstrass equation can be further simplified. The details will be added later..The interested readers can refer to the book of Silverman *Arithmetic of elliptic curves.*

More should be discussed..But we don't have time......