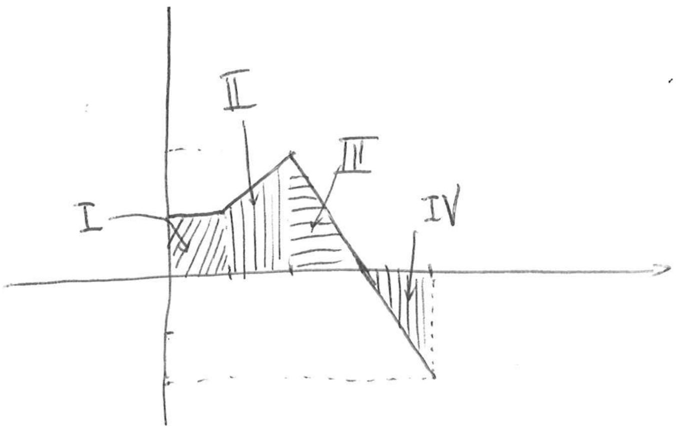
Feuille d'exercices 1

ex 3



Donc $\int_0^4 f(x)\,dx =$ aire de I + aire de II + aire de III − aire de IV

$= 1 + \frac{1}{2}(1+2) + \frac{1}{2}(2+0) - \frac{1}{2}(2+0)$

$= \frac{5}{2}$

ex 4

$\int_e^{1/e} |\ln x|\,dx \qquad e = 2.71828 < 1 < \frac{1}{e}$

$= \int_1^{1/e} |\ln x|\,dx + \int_e^1 |\ln(x)|\,dx$

$= -\int_1^{1/e} \ln x\,dx + \int_e^1 \ln x\,dx$

$= -\left[x\ln x - x\right]_1^{\frac{1}{e}} + \left[x\ln x - x\right]_e^1$

$= -\left(\frac{1}{e}\ln\frac{1}{e} - \frac{1}{e}\right) + (\ln 1 - 1) + (\ln 1 - 1) - (e\ln e - e)$

$= \frac{1}{e} + \frac{1}{e} - 1 - 1 - (e - e)$

$= \frac{2}{e} - 2$

<u>ex6</u> 1) $\int_0^3 \sqrt{9-x^2}\,dx = \frac{9\pi}{4}$

<u>Donc</u> $A = \int_0^3 (\sqrt{9-x^2} - 3)\,dx = \int_0^3 \sqrt{9-x^2}\,dx - 3\int_0^3 dx$

$= \frac{9\pi}{4} - 9$

$A+B = \int_0^3 [(\sqrt{9-x^2} - 3) + \frac{x^2}{\sqrt{9-x^2} + 3})\,dx$

$= \int_0^3 \frac{9-x^2-9 + x^2}{\sqrt{9-x^2} + 3}\,dx = 0$

par suite $B = -A = 9 - 9\pi/4$

2) $\int_0^3 \sqrt{9-x^2}\,dx$

posons $x = 3\sin t$ $\quad t \in [0, \frac{\pi}{2}] \Rightarrow \sin 0 = 0, \quad \sin\frac{\pi}{2} = 1$

$dx = 3\cos t\,dt$

$\Rightarrow \int_0^3 \sqrt{9-x^2}\,dx = \int_0^{\pi/2} \sqrt{9 - 9\sin^2 t}\,(3\cos t)\,dt$

$= 9\int_0^{\pi/2} \sqrt{1-\sin^2 t}\,\cos t\,dt$

$= 9\int_0^{\pi/2} \cos^2 t\,dt = 9\int_0^{\pi/2} \frac{\cos 2t + 1}{2}\,dt$

Or $\begin{cases} \cos 2t = 2\cos^2 t - 1 \\ \sin 2t = 2\cos t \sin t \end{cases}$

$= 9\int_0^{\pi/2} \frac{\cos 2t}{2}\,dt + 9\int_0^{\pi/2} \frac{1}{2}\,dt$

$= 0 + \frac{9\pi}{4} = \frac{9\pi}{4}.$

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2) * $\int_0^1 (x+5)e^{x+1}\,dx$

$= \int_0^1 (x+5)\, d(e^{x+1}) = \left[(x+5)e^{x+1}\right]_0^1 - \int_0^1 e^{x+1}\,dx$

$= 6e^2 - 5e - \left[e^{x+1}\right]_0^1$

$= 6e^2 - 5e - (e^2 - e)$

$= 5e^2 - 4e$

* $\int_0^1 (x+5)^2 e^{x+1}\,dx$

$= \int_0^1 (x+5)^2\, d(e^{x+1}) = \left[(x+5)^2 e^{x+1}\right]_0^1 - 2\int (x+5)e^{x+1}\,dx$

$= 36e^2 - 25e - 2(5e^2 - 4e)$

$= 26e^2 - 17e$

ex 8. IPP

* $\int_0^{\pi/2} x\sin 2x\,dx = \int_0^{\pi/2} x\left(-\frac{1}{2}\cos 2x\right)'dx$

$= \left[-\frac{x}{2}\cos 2x\right]_0^{\pi/2} + \int_0^{\pi/2}\left(-\frac{1}{2}\cos(2x)\right)dx$

$= -\frac{\pi}{4}\cos\pi - \frac{1}{4}\left[\sin 2x\right]_0^{\pi/2}$

$= \frac{\pi}{4}$

* $\int_1^2 t^n \ln t\,dt$

si $n = -1$ $\int_1^2 \frac{\ln t}{t}\,dt = \int_1^2 \ln t\,(d\ln t)) = \cancel{\int_0^{\ln 2} \ln x\,dx}$

$= \left[\ln^2 t\right]_1^2 - \int_1^2 \frac{\ln t}{t}\,dt$

$\Rightarrow \int_1^2 \frac{\ln t}{t}\,dt = \frac{1}{2}\ln^2 2.$

Si $n \neq -1$

$$\int_1^2 t^n \ln t \, dt = \int_1^2 \ln t \cdot \left(\frac{1}{n+1} t^{n+1}\right)' dt$$

$$= \left[\frac{1}{n+1} t^{n+1} \ln t\right]_1^2 - \int_1^2 \frac{1}{(n+1)} t^{n+1} \frac{1}{t} dt$$

$$= \frac{1}{n+1} 2^{n+1} \ln 2 - \frac{1}{n+1} \int_1^2 t^n dt$$

$$= \frac{1}{n+1} 2^{n+1} \ln 2 - \frac{1}{(n+1)^2} \left[t^{n+1}\right]_1^2$$

$$= \frac{1}{n+1} 2^{n+1} \ln 2 - \frac{1}{(n+1)^2} 2^{n+1} + \frac{1}{(n+1)^2}$$

* $A := \int_0^{\pi/2} e^x \sin(2x) \, dx$

$$= \int_0^{\pi/2} \sin 2x \, (e^x)' dx = \left[e^x \sin 2x\right]_0^{\pi/2} - 2\int_0^{\pi/2} e^x \cos 2x \, dx$$

$$= -2 \int_0^{\pi/2} \cos 2x \, (e^x)' dx$$

$$= -2 \left[e^x \cos 2x\right]_0^{\pi/2} + 2\int_0^{\pi/2} e^x (\cos 2x)' dx$$

$$= 2(e^{\pi/2}+1) - 4 \underbrace{\int_0^{\pi/2} e^x \sin 2x \, dx}_{A}$$

$\Rightarrow$

$$5A = 2(e^{\pi/2}+1) \Rightarrow A = \frac{2}{5}(e^{\pi/2}+1)$$

car $A = \int_0^{\pi/2} e^x \sin(2x) dx \in \mathbb{R}$

est un <u>réel</u> bien défini.

ex 9

$$\int_{\pi/4}^{\pi/3} \tan x \, dx = \int_{\pi/4}^{\pi/3} \frac{\sin x}{\cos x} dx$$

$$= -\int_{\pi/4}^{\pi/3} \frac{1}{\cos x} (\cos x)' dx = -\int_{\sqrt{2}/2}^{\frac{1}{2}} \frac{1}{t} dt$$

$$= -\left[\ln t\right]_{\sqrt{2}/2}^{\frac{1}{2}} = -\left(\ln\left(\tfrac{1}{2}\right) - \ln\tfrac{\sqrt{2}}{2}\right)$$

$$= \frac{1}{2} \ln 2.$$

ex 10

$$\int_0^1 \arctan x \, dx = \int_0^1 (\arctan x).(x)' dx = \left[x \arctan x\right]_0^1 - \int_0^1 x.(\arctan x)' dx$$

$$= -\int_0^1 \frac{x}{1+x^2} dx + \arctan 1$$

$$= \frac{\pi}{4} - \frac{1}{2} \int_0^1 \frac{2x}{1+x^2} dx$$

$$= \frac{\pi}{4} - \frac{1}{2} \left[\ln(1+x^2)\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

ex 11

$$\int e^x \cos x \, dx = \int \cos x \, (e^x)' dx = e^x \cos x - \int e^x (\cos x)' dx$$

$$= e^x \cos x + \int e^x \sin x \, dx$$

$$= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

d'où $\displaystyle\int e^x \cos x \, dx = \frac{1}{2}(e^x \cos x + e^x \sin x) + C$

$$\int \arcsin x \, dx = \int (\arcsin x) \cdot (x)' dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= x \arcsin x - \frac{1}{2} \int \frac{d(x^2)}{\sqrt{1-x^2}}$$

$$= x \arcsin x + \frac{1}{2} \int \frac{d(1-x^2)}{\sqrt{1-x^2}}$$

$$= x \arcsin x + \sqrt{1-x^2} + C$$

<u>ex 12.</u>

$$\frac{1}{x^2-4x-5} = \frac{1}{(x-5)(x+1)}$$

$$\begin{array}{c|cc} 1 & -5 \\ 1 & +1 \end{array}$$

$$= \frac{(x+1)-(x-5)}{6(x+1)(x-5)} = \frac{1}{6}\left(\frac{1}{x-5} - \frac{1}{x+1} \right)$$

$$= \frac{-1/6}{x+1} + \frac{1/6}{x-5}$$

<u>dom</u> $$\int_0^2 \frac{1}{x^2-4x-5} dx = \frac{1}{6} \int_0^2 \left(\frac{1}{x-5} - \frac{1}{x+1} \right) dx$$

$$= \frac{1}{6} \left[\ln|x-5| - \ln|x+1| \right]_0^2$$

$$= \frac{1}{6} (\ln 3 - \ln 5 - \ln 3 + \ln 1)$$

$$= -\frac{\ln 5}{6}$$

ex 13

$$\frac{x}{x^2-x-2} = \frac{x}{(x-2)(x+1)} = \frac{2(x+1)+(x-2)}{3(x-2)(x+1)}$$

$$= \frac{2}{3(x-2)} + \frac{1}{3(x+1)}.$$

$$\Rightarrow \quad \int \frac{x}{x^2-x-2}\,dx = \frac{2}{3}\ln|x-2| + \frac{1}{3}\ln|x+1| + C$$

défini sur $\mathbb{R}-\{-1,2\}$

Donc sur $]2,+\infty[$, on a

$$\int \frac{x}{x^2-x-2}\,dx = \frac{2}{3}\ln(x-2) + \frac{1}{3}\ln(x+1) + C$$

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ex 7 $\quad n \geq 2$

4). $I_n = \int_0^1 x^n e^{x^2}\,dx$

$$= \frac{1}{2}\int_0^1 x^{n-1}\, d(e^{x^2})$$

$$= \frac{1}{2}\left[x^{n-1}e^{x^2}\right]_0^1 - \frac{1}{2}\int_0^1 e^{x^2}\, x^{n-2}\cdot(n-1)\,dx$$

$$= \frac{1}{2}e - \frac{n-1}{2} I_{n-2}$$

or $\quad I_n = \int_0^1 x^n e^{x^2}\,dx = \int_0^1 e^{x^2}\left(\frac{1}{n+1}x^{n+1}\right)'dx$

$$= \left[e^{x^2}\cdot\frac{1}{n+1}x^{n+1}\right]_0^1 - \frac{2}{n+1}\int_0^1 x^{n+2}e^{x^2}\,dx$$

$$= \frac{e}{n+1} - \frac{2}{n+1} I_{n+2}.$$

$\Rightarrow \quad I_n = \frac{1}{n+1}e - \frac{2}{n+1}I_{n+2} \quad$ or $\quad I_{n+2} = \frac{1}{2}e - \frac{n+1}{2}I_n$

I_1 facile à calculer $\Rightarrow \dots$

ex13

Comme $\dfrac{x}{x^2-x-2} = \dfrac{x}{(x-2)(x+1)}$

cherchons $a, b \in \mathbb{R}$ tq

$$\frac{x}{(x-2)(x+1)} = \frac{a}{x-2} + \frac{b}{x+1} = \frac{(a+b)x+(a-2b)}{(x-2)(x+1)}$$

$$\Leftrightarrow \begin{cases} a+b=1 \\ a-2b=0 \end{cases} \Leftrightarrow \begin{cases} a=2/3 \\ b=1/3 \end{cases}$$

donc $\dfrac{x}{x^2-x-2} = \dfrac{2/3}{x-2} + \dfrac{1/3}{x+1}$

$$\Rightarrow \int \frac{x}{x^2-x-2}\,dx = \int \frac{2/3}{x-2}\,dx + \int \frac{1/3}{x+1}\,dx$$

$$= \frac{2}{3}\ln|x-2| + \frac{1}{3}\ln|x+1| + c \quad \text{pour } x \notin \{-1,2\}$$

Donc sur l'intervalle $]2,+\infty[$

les primitives de $\dfrac{x}{x^2-x-2}$ sont les fcts

$$\frac{2}{3}\ln(x-2) + \frac{1}{3}\ln(x+1) + c \qquad \forall c \in \mathbb{R}.$$