Feuille 2

ex1

1)

* $\ln(1+x)$

alors $\left(\ln(1+x)\right)' = \dfrac{1}{1+x}$

$\left(\ln(1+x)\right)'' = \left(\dfrac{1}{1+x}\right)' = -\dfrac{1}{(1+x)^2}$

donc $\ln(1+x) = \ln(1+0) + \dfrac{1}{1+0}\cdot x + \left(-\dfrac{1}{(1+0)^2}\right)\dfrac{x^2}{2} + o(x^2)$

$= x - \dfrac{1}{2}x^2 + o(x^2)$

* $\dfrac{1}{1+x}$

$\left(\dfrac{1}{1+x}\right)' = -\dfrac{1}{(1+x)^2}$

$\left(\dfrac{1}{1+x}\right)'' = \left(-\dfrac{1}{(1+x)^2}\right)' = 2\dfrac{1}{(1+x)^3}$

donc $\dfrac{1}{1+x} = 1 - x + x^2 + o(x^2)$

* $\tan x$

$(\tan x)' = \left(\dfrac{\sin x}{\cos x}\right)' = \dfrac{\cos x \cos x - \sin x\,(\cos x)'}{\cos^2 x} = \dfrac{1}{\cos^2 x}$

$(\tan x)'' = \left(\dfrac{1}{\cos^2 x}\right)' = -2\dfrac{1}{\cos^3 x}\cdot(\cos x)'$

$= \dfrac{2\sin x}{\cos^3 x}$

donc $(\tan x)'\big|_{x=0} = 1$

$(\tan x)''\big|_{x=0} = \dfrac{2\sin 0}{\cos^3 0} = 0$

D'où
$$\tan(x) = 0 + x + 0 \cdot \frac{x^2}{2} + o(x^2)$$
$$= x + o(x^2)$$

2) posons $f(x) = \cos \pi x$

on a alors
$$f(x) = f(1) + f'(1)(x-1) + f''(1) \cdot \frac{(x-1)^2}{2} + o((x-1)^2)$$

or $f(x) = \cos \pi x \Rightarrow f'(x) = \cos'(\pi x) \cdot (\pi x)' = -\pi \sin \pi x$

donc $f''(x) = -\pi \sin' \pi x \cdot (\pi x)' = -\pi^2 \cos(\pi x)$

d'où la formule de Taylor d'ordre 2 de f en $x_0 = 1$:
$$\cos \pi x = f(x) = f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2} + o((x-1)^2)$$
$$= -1 + 0 + \pi^2 \cdot \frac{(x-1)^2}{2} + o((x-1)^2)$$

d'où
$$\lim_{x \to 1} \frac{\cos \pi x + 1}{(x-1)^2} = \lim_{x \to 1} \frac{\pi^2 \cdot \frac{(x-1)^2}{2} + o((x-1)^2)}{(x-1)^2}$$
$$= \frac{\pi^2}{2} + \lim_{x \to 1} \frac{o((x-1)^2)}{(x-1)^2} = \frac{\pi^2}{2}$$

ex2 * $\ln(1-xy)$
$$\frac{\partial}{\partial x}(\ln(1-xy)) = \frac{1}{1-xy} \cdot \frac{\partial}{\partial x}(1-xy) = -\frac{y}{1-xy}$$

* $\dfrac{1}{\sqrt{1+x^2+y^2}}$
$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1+x^2+y^2}}\right) = \frac{\partial}{\partial x}\left((1+x^2+y^2)^{-\frac{1}{2}}\right) = -\frac{1}{2}(1+x^2+y^2)^{-\frac{3}{2}} \cdot \frac{\partial}{\partial x}(1+x^2+y^2)$$
$$= -\frac{x}{(\sqrt{1+x^2+y^2})^3}$$

* $y e^x \sin(x+y)$
$$\frac{\partial}{\partial x}(y e^x \sin(x+y)) = \frac{\partial}{\partial x}(y e^x) \sin(x+y) + y e^x \frac{\partial}{\partial x} \sin(x+y)$$
$$= y e^x \sin(x+y) + y e^x \cos(x+y)$$

ex3

si $(x,y) \neq (0,0)$

on a $\displaystyle \frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(\frac{xy(x^2-y^2)}{(x^2+y^2)}\right)$

$$= \frac{\left(\frac{\partial}{\partial x}(xy(x^2-y^2))\right)\cdot(x^2+y^2) - (xy(x^2-y^2))\cdot\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(3x^2y-y^3)(x^2+y^2) - (x^3y-xy^3)\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{3x^4y+3x^2y^3-x^2y^3-y^5-2x^4y+2x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$$

et $\displaystyle \frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{0-0}{t} = 0$

d'où $\displaystyle \frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} & \text{si } (x,y)\neq(0,0) \\ 0 & \text{sinon.} \end{cases}$

en particulier, si $t\neq 0$, $\displaystyle \frac{\partial f}{\partial x}(0,t) = -t$

De la même manière, on a

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2} & \text{si } (x,y)\neq(0,0) \\ 0 & \text{sinon} \end{cases}$$

d'où $\displaystyle \frac{\partial f}{\partial y}(t,0) = t$

par suit, $\displaystyle \frac{\partial^2 f}{\partial y\partial x}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0)$

$$= \lim_{t\to 0}\frac{\frac{\partial f}{\partial x}(0,t)-\frac{\partial f}{\partial x}(0,0)}{t} = \lim_{t\to 0}\frac{-t-0}{t} = -1$$

$$\frac{\partial^2 f}{\partial x\partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0)$$

$$= \lim_{t\to 0}\left(\frac{\frac{\partial f}{\partial y}(t,0)-\frac{\partial f}{\partial y}(0,0)}{t}\right) = \lim_{t\to 0}\frac{t-0}{t} = 1$$

donc $\displaystyle \frac{\partial^2 f}{\partial x\partial y}(0,0) \neq \frac{\partial^2 f}{\partial y\partial x}(0,0)$!

ex 4 formule de Taylor d'ordre 2 de

$f(x,y) = \sqrt{1+x+y}$ en $(0,0)$

$f(0,0) = \sqrt{0+0+1} = 1$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{1+x+y}\right) = \frac{1}{2}\frac{1}{\sqrt{1+x+y}}$.

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{1+x+y}\right) = \frac{1}{2\sqrt{1+x+y}}$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{1+x+y}}\right)$

$= -\frac{1}{4}\cdot\frac{1}{\left(\sqrt{1+x+y}\right)^3}$

$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{1+x+y}}\right)$

$= -\frac{1}{4}\frac{1}{\left(\sqrt{1+x+y}\right)^3}$

De la m̂ manière, on a

$\frac{\partial^2 f}{\partial y^2} = -\frac{1}{4}\cdot\frac{1}{\left(\sqrt{1+x+y}\right)^3}$

$\frac{\partial^2 f}{\partial y\partial x} = -\frac{1}{4}\cdot\frac{1}{\left(\sqrt{1+x+y}\right)^3}$

donc $f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y + \frac{\partial^2 f}{\partial x^2}(0,0)\frac{x^2}{2}$

$+ \frac{\partial^2 f}{\partial x\partial y}(0,0)\,xy + \frac{\partial^2 f}{\partial y^2}(0,0)\frac{y^2}{2} + o(x^2+y^2)$

$= 1 + \frac{1}{2}x + \frac{1}{2}y - \frac{1}{8}x^2 - \frac{1}{4}xy - \frac{1}{8}y^2 + o(x^2+y^2)$

ex 5.

$$* \quad \int_{[0,1]\times[0,1]} e^{x+y}\, d(x,y) = \int_{[0,1]\times[0,1]} e^{x}\cdot e^{y}\, d(x,y)$$

$$\overset{\text{Fubini allégé}}{=} \left(\int_0^1 e^x\,dx\right)\cdot\left(\int_0^1 e^y\,dy\right)$$

$$= \left[e^x\right]_0^1 \cdot \left[e^y\right]_0^1$$

$$= (e-1)^2$$

$$* \quad \int_{\{x^2+y^2\le 1\}} e^{x^2+y^2}\, d(x,y)$$

$$= \int_{[0,1]\times[0,2\pi]} e^{(\rho\cos x)^2+\rho^2(\sin x)^2}\cdot \rho\, d(\rho,\theta)$$

$$= \int_{[0,1]\times[0,2\pi]} \rho e^{\rho^2}\, d(\rho,\theta)$$

$$= \left(\int_0^1 \rho e^{\rho^2} d\rho\right)\cdot \int_0^{2\pi} 1\, d\theta$$

$$= \int_0^1 \left(\frac{1}{2}e^{\rho^2}\right)' d\rho \cdot 2\pi$$

$$= \left[\frac{1}{2}e^{\rho^2}\right]_0^1 \cdot 2\pi = \left(\frac{1}{2}e - \frac{1}{2}\right)\cdot 2\pi$$

$$= (e-1)\cdot\pi$$

$$\int_{[0,1]^3} \sqrt{1+x+y}\, d(x,y,z)$$

$$= \int_0^1 dx \int_0^1 dy \cdot \int_0^1 \sqrt{1+y+x}\, dz$$

$$= \int_0^1 dx \int_0^1 \sqrt{1+x+y}\, dy$$

$$= \int_0^1 dx \int_0^1 \sqrt{1+x+y} \cdot (1+x+y)_y'\, dy$$

$$= \int_0^1 dx \int_{1+x}^{2+x} \sqrt{u}\, du$$

$$= \int_0^1 \left[\frac{2}{3} u^{\frac{3}{2}}\right]_{1+x}^{2+x} dx$$

$$= \frac{2}{3} \int_0^1 (2+x)^{\frac{3}{2}}\, dx - \frac{2}{3} \int_0^1 (1+x)^{\frac{3}{2}}\, dx$$

$$= \frac{2}{3} \int_0^1 (2+x)^{\frac{3}{2}} \cdot (2+x)'\, dx - \frac{2}{3} \int_0^1 (1+x)^{\frac{3}{2}} (1+x)'\, dx$$

$$= \frac{2}{3} \int_2^3 u^{\frac{3}{2}}\, du - \frac{2}{3} \int_1^2 u^{\frac{3}{2}}\, du$$

$$= \frac{2}{3} \left(\left[\frac{2}{5} u^{\frac{5}{2}}\right]_2^3 - \left[\frac{2}{5} u^{\frac{5}{2}}\right]_1^2 \right)$$

$$= \frac{4}{15} \left(3^{\frac{5}{2}} - 2^{\frac{5}{2}} - \left(2^{\frac{5}{2}} - 1^{\frac{5}{2}} \right) \right)$$

$$= \frac{4}{15} \left(9\sqrt{3} - 4\sqrt{2} - 4\sqrt{2} + 1 \right)$$

$$= \frac{4}{15} \left(9\sqrt{3} - 8\sqrt{2} + 1 \right)$$

ex 6

1. $\mathrm{Vol}(C) = \int_C 1 \cdot d(x,y,z)$

$= \int_0^1 1\, dx \cdot \int_2^4 1 \cdot dy \cdot \int_1^3 1\, dz$

$= 1 \times 2 \times 2 = 4$

le barycentre de C, noté par (x_0, y_0, z_0) est alors donné par

$x_0 = \int_C x\, d(x,y,z) \Big/ \mathrm{Vol}(C)$

$= \frac{1}{4} \int_0^1 x\, dx \cdot \int_2^4 dy \cdot \int_1^3 1\, dz$

$= \frac{1}{4}\left(\frac{1}{2} \times 2 \times 2\right) = \frac{1}{2}$

$y_0 = \int_C y\, d(x,y,z) \Big/ \mathrm{Vol}(C)$

$= \frac{1}{4} \int_0^1 dx \int_2^4 y\, dy \int_1^3 1\, dz$

$= \frac{1}{4} \times 1 \times \left(\frac{1}{2} \cdot 4^2 - \frac{1}{2} \times 2^2\right) \times 2$

$= \frac{1}{4} \times 1 \times \frac{1}{2} \times 12 \times 2$

$= 3$

$z_0 = \int_C z\, d(x,y,z) \Big/ \mathrm{vol}(C)$

$= \frac{1}{4} \int_0^1 dx \int_2^4 dy \int_1^3 z\, dz$

$= \frac{1}{4} \times 1 \times 2 \times \left(\frac{1}{2} \times 3^2 - \frac{1}{2} \times 1^2\right)$

$= \frac{1}{4} \times 1 \times 2 \times \frac{1}{2} \times 8 = 2$

$\Rightarrow$ barycentre $= \left(\frac{1}{2}, 3, 2\right)$.

2. $C = \{ (x,y) \in \mathbb{R}^2 / (x^2+y^2) \leq 1 \} \subset \mathbb{R}^2$

$$\mathrm{Vol}(C) = \int_C d(x,y) = \int_{[0,1]\times[0,2\pi]} 1 \cdot \rho \, d(\rho,\theta)$$

$$= \int_0^1 \rho \, d\rho \int_0^{2\pi} d\theta$$

$$= \frac{1}{2} \cdot 2\pi = \pi$$

le barycentre de C est alors donné par (x_0, y_0) avec

$$x_0 = \int_C x \, d(x,y) = \int_{[0,1]\times[0,2\pi]} \rho\cos\theta \cdot \rho \, d(\rho,\theta)$$

$$= \int_0^1 \rho^2 d\rho \cdot \int_0^{2\pi} \cos\theta \, d\theta$$

$$= \left[\frac{1}{3}\rho^3\right]_0^1 \cdot \left[\sin\theta\right]_0^{2\pi} = 0$$

$$y_0 = \int_C y \, d(x,y) = \int_{[0,1]\times[0,2\pi]} \rho\sin\theta \; \rho \, d(\rho,\theta)$$

$$= \int_0^1 \rho^2 d\rho \cdot \int_0^{2\pi} \sin\theta \, d\theta$$

$$= \left[\frac{1}{3}\rho^3\right]_0^1 \times \left[-\cos\theta\right]_0^{2\pi}$$

$$= 0$$

donc barycentre de C $= (x_0, y_0) = (0,0)$

#!