CRYSTALLINE COMPARISON ISOMORPHISMS IN p-ADIC HODGE THEORY: THE ABSOLUTELY UNRAMIFIED CASE

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Abstract. We construct the crystalline comparison isomorphisms for proper smooth formal schemes over an absolutely unramified base. Such isomorphisms hold for étale cohomology with nontrivial coefficients, as well as in the relative setting, i.e. for proper smooth morphisms of smooth formal schemes. The proof is formulated in terms of the pro-étale topos introduced by Scholze, and uses his primitive comparison theorem for the structure sheaf on the pro-étale site. Moreover, we need to prove the Poincaré lemma for crystalline period sheaves, for which we adapt the idea of Andreatta and Iovita. Another ingredient for the proof is the geometric acyclicity of crystalline period sheaves, whose computation is due to Andreatta and Brinon.

Contents

Notation

- Let p be a prime number.

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- Let k be a p-adic field, i.e., a discretely valued complete nonarchimedean extension of $\mathbb{Q}_p$, whose residue field κ is a perfect field of characteristic p. (We often assume k to be absolutely unramified in this paper.)
- Let $\overline{k}$ be a fixed algebraic closure of k. Set $\mathbb{C}_p := \widehat{\overline{k}}$ the p-adic completion of $\overline{k}$. The p-adic valuation v on $\mathbb{C}_p$ is normalized so that $v(p) = 1$. Write the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ as G_k.
- For a topological ring A which is complete with respect to p-adic topology, let $A\langle T_1, \ldots, T_d\rangle$ be the PD-envelope of the polynomial ring $A[T_1, \ldots, T_d]$ with respect to the ideal $(T_1, \ldots, T_d) \subset A[T_1, \ldots, T_d]$ (with the requirement that the PD-structure be compatible with the one on the ideal (p)) and then let $A\{\langle T_1, \ldots, T_d\rangle\}$ be its p-adic completion.
- We use the symbol $\simeq$ to denote canonical isomorphisms and sometimes use $\approx$ for almost isomorphisms (often with respect to the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$).

1. Introduction

Let k be a discretely valued complete nonarchimedean field over $\mathbb{Q}_p$ which is absolutely unramified.

Consider a rigid analytic variety over k, or more generally an adic space X over $\mathrm{Spa}(k, \mathcal{O}_k)$ which admits a proper smooth formal model $\mathcal{X}$ over $\mathrm{Spf}\,\mathcal{O}_k$, whose special fiber is denoted by $\mathcal{X}_0$. Let $\mathbb{L}$ be a lisse $\mathbb{Z}_p$-sheaf on $X_{\text{ét}}$. On one hand, we have the p-adic étale cohomology $H^i(X_{\overline{k}}, \mathbb{L})$ which is a finitely generated $\mathbb{Z}_p$-module carrying a continuous $G_k = \mathrm{Gal}(\overline{k}/k)$-action. On the other hand, one may consider the crystalline cohomology $H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{E})$ with the coefficient $\mathcal{E}$ being a filtered (convergent) F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$. At least in the case that X comes from a scheme and the coefficients $\mathbb{L}$ and $\mathcal{E}$ are trivial, it was Grothendieck's problem of *mysterious functor* to find a comparison between the two cohomology theories. This problem was later formulated as the *crystalline conjecture* by Fontaine [Fon82].

In the past decades, the crystalline conjecture was proved in various generalities, by Fontaine-Messing, Kato, Tsuji, Niziol, Faltings, Andreatta-Iovita, Beilinson and Bhatt. Among them, the first proof for the whole conjecture was given by Faltings [Fal]. Along this line, Andreatta-Iovita introduced the Poincaré lemma for the crystalline period sheaf $\mathbb{B}_{\text{cris}}$ on the Faltings site, a sheaf-theoretic generalization of Fontaine's period ring B_{cris}. Both the approach of Fontaine-Messing and that of Faltings-Andreatta-Iovita use an intermediate topology, namely the syntomic topology and the Faltings topology, respectively. The approach of Faltings-Andreatta-Iovita, however, has the advantage that it works for nontrivial coefficients $\mathbb{L}$ and $\mathcal{E}$.

More recently, Scholze [Sch13] introduced the pro-étale site $X_{\text{proét}}$, which allows him to construct the de Rham comparison isomorphism for any proper smooth adic space over a discretely valued complete nonarchimedean field over $\mathbb{Q}_p$, with coefficients being lisse $\mathbb{Z}_p$-sheaves on $X_{\text{proét}}$. (The notion of lisse $\mathbb{Z}_p$-sheaf on $X_{\text{ét}}$ and that on $X_{\text{proét}}$ are equivalent.) Moreover, his approach is direct and flexible enough to attack the relative version of the de Rham comparison isomorphism, i.e. the comparison for a proper smooth morphism between two smooth adic spaces.

It seems that to deal with nontrivial coefficients in a comparison isomorphism, one is forced to work over analytic bases. For the generality and some technical advantages provided by the pro-étale topology, we adapt Scholze's approach to give

a proof of the crystalline conjecture for proper smooth formal schemes over Spf $\mathcal{O}_k$, with nontrivial coefficients, in both absolute and relative settings. In a sequel paper we shall construct crystalline comparison isomorphisms for arbitrary varieties over p-adic fields.

Let us explain our construction of crystalline comparison isomorphism (in the absolutely unramified case) in more details. First of all, Scholze is able to prove the finiteness of the étale cohomology of a proper smooth adic space X over $K = \widehat{\overline{k}}$ with coefficient $\mathbb{L}'$ being an $\mathbb{F}_p$-local system. Consequently, he shows the following "primitive comparison", an almost (with respect to the maximal ideal of $\mathcal{O}_K$) isomorphism

$$H^i(X_{\overline{k},\text{ét}}, \mathbb{L}') \otimes_{\mathbb{F}_p} \mathcal{O}_K/p \xrightarrow{\approx} H^i(X_{\overline{k},\text{ét}}, \mathbb{L}' \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p).$$

With some more efforts, one can produce the primitive comparison isomorphism in the crystalline case:

Theorem 1.1 (See Theorem 4.3)**.** *For $\mathbb{L}$ a lisse $\mathbb{Z}_p$-sheaf on $X_{\text{ét}}$, we have a canonical isomorphism of B_{cris}-modules*

$$H^i(X_{\overline{k},\text{ét}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} B_{\text{cris}} \xrightarrow{\sim} H^i(X_{\overline{k},\text{proét}}, \mathbb{L} \otimes \mathbb{B}_{\text{cris}}). \tag{1.0.1}$$

compatible with G_k-action, filtration, and Frobenius.

It seems to us that such a result alone may have interesting arithmetic applications, since it works for any lisse $\mathbb{Z}_p$-sheaves, without the crystalline condition needed for comparison theorems.

Following Faltings, we say a lisse $\mathbb{Z}_p$-sheaf $\mathbb{L}$ on the pro-étale site $X_{\text{proét}}$ is *crystalline* if there exists a filtered F-isocrystal $\mathcal{E}$ on $\mathcal{X}_0/\mathcal{O}_k$ together with an isomorphism of $\mathcal{O}\mathbb{B}_{\text{cris}}$-modules

$$\mathcal{E} \otimes_{\mathcal{O}_X^{\text{ur}}} \mathcal{O}\mathbb{B}_{\text{cris}} \simeq \mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\text{cris}} \tag{1.0.2}$$

which is compatible with connection, filtration and Frobenius. Here, $\mathcal{O}_X^{\text{ur}}$ is the pullback to $X_{\text{proét}}$ of $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$ and $\mathcal{O}\mathbb{B}_{\text{cris}}$ is the crystalline period sheaf of $\mathcal{O}_X^{\text{ur}}$-module with connection ∇ such that $\mathcal{O}\mathbb{B}_{\text{cris}}^{\nabla=0} = \mathbb{B}_{\text{cris}}$. When this holds, we say the lisse $\mathbb{Z}_p$-sheaf $\mathbb{L}$ and the filtered F-isocrystal $\mathcal{E}$ are *associated*.

We illustrate the construction of the crystalline comparison isomorphism briefly. Firstly, we prove a Poincaré lemma for the crystalline period sheaf $\mathbb{B}_{\text{cris}}$ on $X_{\text{proét}}$. It follows from the Poincaré lemma (Proposition 2.16) that the natural morphism from $\mathbb{B}_{\text{cris}}$ to the de Rham complex $DR(\mathcal{O}\mathbb{B}_{\text{cris}})$ of $\mathcal{O}\mathbb{B}_{\text{cris}}$ is a quasi-isomorphism, which is compatible with filtration and Frobenius. When $\mathbb{L}$ and $\mathcal{E}$ are associated, the natural morphism

$$\mathbb{L} \otimes_{\mathbb{Z}_p} DR(\mathcal{O}\mathbb{B}_{\text{cris}}) \to DR(\mathcal{E}) \otimes \mathcal{O}\mathbb{B}_{\text{cris}}$$

is an isomorphism compatible with Frobenius and filtration. Therefore we find a quasi-isomorphism

$$\mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{cris}} \simeq DR(\mathcal{E}) \otimes \mathcal{O}\mathbb{B}_{\text{cris}}.$$

From this we deduce

$$R\Gamma(X_{\overline{k},\text{proét}}, \mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{cris}}) \xrightarrow{\sim} R\Gamma(X_{\overline{k},\text{proét}}, DR(\mathcal{E}) \otimes \mathcal{O}\mathbb{B}_{\text{cris}}). \tag{1.0.3}$$

Via the natural morphism of topoi $\overline{w} : X_{\overline{k},\text{proét}}^{\sim} \to \mathcal{X}_{\text{ét}}^{\sim}$, one has

$$R\Gamma(X_{\overline{k},\text{proét}}, DR(\mathcal{E}) \otimes \mathcal{O}\mathbb{B}_{\text{cris}}) \simeq R\Gamma(\mathcal{X}_{\text{ét}}, DR(\mathcal{E}) \widehat{\otimes}_{\mathcal{O}_k} B_{\text{cris}})) \tag{1.0.4}$$

for which we have used the fact that the natural morphism

$$\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_{\mathcal{O}_k}B_{\mathrm{cris}}\to R\overline{w}_*\mathcal{O}\mathbb{B}_{\mathrm{cris}}$$

is an isomorphism (compatible with extra structures), which is a result of Andreatta-Brinon.

Combining the isomorphisms above, we obtain the desired crystalline comparison isomorphism.

Theorem 1.2 (See Theorem 4.5)**.** *Let $\mathbb{L}$ be a lisse $\mathbb{Z}_p$-sheaf on X and $\mathcal{E}$ be a filtered F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$ which are associated as in (1.0.2). Then there is a natural isomorphism of B_{cris}-modules*

$$H^i(X_{\overline{k},\text{ét}},\mathbb{L})\otimes B_{\mathrm{cris}}\xrightarrow{\sim}H^i_{\mathrm{cris}}(\mathcal{X}_0/\mathcal{O}_k,\mathcal{E})\otimes_k B_{\mathrm{cris}}$$

which is compatible with G_k-action, filtration and Frobenius.

After obtaining a refined version of the acyclicity of crystalline period sheaf $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$ in §6, we achieve the crystalline comparison in the relative setting, which reduces to Theorem 1.2 when $\mathcal{Y}=\mathrm{Spf}\,\mathcal{O}_k$:

Theorem 1.3 (See Theorem 5.6)**.** *Let $f\colon\mathcal{X}\to\mathcal{Y}$ be a proper smooth morphism of smooth formal schemes over* $\mathrm{Spf}\,\mathcal{O}_k$*, with $f_k\colon X\to Y$ the generic fiber and f_{cris} the morphism between the crystalline topoi. Let $\mathbb{L},\mathcal{E}$ be as in Theorem 1.2. Suppose that $R^if_{k*}\mathbb{L}$ is a lisse $\mathbb{Z}_p$-sheaf on Y. Then it is crystalline and is associated to the filtered F-isocrystal $R^if_{\mathrm{cris}*}\mathcal{E}$.*

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2. Crystalline period sheaves

Let k be a discretely valued nonarchimedean extension of $\mathbb{Q}_p$, with κ its residue field. Let X be a locally noetherian adic space over $\mathrm{Spa}(k,\mathcal{O}_k)$. For the fundamentals on the pro-étale site $X_{\text{proét}}$, we refer to [Sch13].

The following terminology and notation will be used frequently throughout the paper. We shall fix once for all an algebraic closure $\overline{k}$ of k, and consider $X_{\overline{k}}:=X\times_{\mathrm{Spa}(k,\mathcal{O}_k)}\mathrm{Spa}(\overline{k},\mathcal{O}_{\overline{k}})$ as an object of $X_{\text{proét}}$. As in [Sch13, Definition 4.3], an object $U\in X_{\text{proét}}$ lying above $X_{\overline{k}}$ is called an *affinoid perfectoid* (lying above $X_{\overline{k}}$) if U has a pro-étale presentation $U=\varprojlim U_i\to X$ by affinoids $U_i=\mathrm{Spa}(R_i,R_i^+)$ above $X_{\overline{k}}$ such that, with R^+ the p-adic completion of $\varinjlim R_i^+$ and $R=R^+[1/p]$, the pair (R,R^+) is a perfectoid affinoid $(\widehat{\overline{k}},\mathcal{O}_{\widehat{\overline{k}}})$-algebra. Write $\widehat{U}=\mathrm{Spa}(R,R^+)$. By [Sch13, Proposition 4.8, Lemma 4.6], the set of affinoid perfectoids lying above $X_{\overline{k}}$ of $X_{\text{proét}}$ forms a basis for the topology.

2.1. Period sheaves and their acyclicities.

Following [Sch13], let

$$\nu\colon X^{\sim}_{\text{proét}}\longrightarrow X^{\sim}_{\text{ét}}$$

be the morphism of topoi, which, on the underlying sites, sends an étale morphism $U\to X$ to the pro-étale morphism from U (viewed as a constant projective system) to X. Consider $\mathcal{O}_X^+=\nu^{-1}\mathcal{O}^+_{X_{\text{ét}}}$ and $\mathcal{O}_X=\nu^{-1}\mathcal{O}_{X_{\text{ét}}}$, the (uncompleted) structural sheaves on $X_{\text{proét}}$. More concretely, for $U=\varprojlim U_i$ a qcqs (quasi-compact and

quasi-separated) object of $X_{\text{proét}}$, one has $\mathcal{O}_X(U) = \varinjlim \mathcal{O}_X(U_i) = \varinjlim \mathcal{O}_{X_{\text{ét}}}(U_i)$ ([Sch13, Lemma 3.16]). Set

$$\widehat{\mathcal{O}_X^+} := \varprojlim_n \mathcal{O}_X^+/p^n, \quad \mathcal{O}_X^{\flat+} = \varprojlim_{x\mapsto x^p} \mathcal{O}_X^+/p\mathcal{O}_X^+.$$

For $U = \varprojlim U_i \in X_{\text{proét}}$ an affinoid perfectoid lying above $X_{\overline{k}}$ with $\widehat{U} = \text{Spa}(R, R^+)$, by [Sch13, Lemmas 4.10, 5.10], we have

$$\widehat{\mathcal{O}_X^+}(U) = R^+, \quad \text{and} \quad \mathcal{O}_X^{\flat+}(U) = R^{\flat+} := \varprojlim_{x\mapsto x^p} R^+/pR^+. \tag{2.1.1}$$

Denote

$$R^{\flat+} \to R^+, \quad x = (x_0, x_1, \cdots) \mapsto x^\sharp := \lim_{n\to\infty} \widehat{x}_n^{p^n},$$

for $\widehat{x}_n$ any lifting from R^+/p to R^+. We have the multiplicative homeomorphism (induced by projection):

$$\varprojlim_{x\mapsto x^p} R^+ \xrightarrow{\sim} R^{\flat+}, \quad (x^\sharp, (x^{1/p})^\sharp, \cdots) \hookleftarrow x$$

which can be extended to a ring homomorphism.

Put $\mathbb{A}_{\text{inf}} := W(\mathcal{O}_X^{\flat+})$ and $\mathbb{B}_{\text{inf}} = \mathbb{A}_{\text{inf}}[\frac{1}{p}]$. As $R^{\flat+}$ is a perfect ring, $\mathbb{A}_{\text{inf}}(U) = W(R^{\flat+})$ has no p-torsions. In particular, $\mathbb{A}_{\text{inf}}$ has no p-torsions and it is a subsheaf of $\mathbb{B}_{\text{inf}}$. Following Fontaine, define as in [Sch13, Definition 6.1] a natural morphism

$$\theta\colon \mathbb{A}_{\text{inf}} \to \widehat{\mathcal{O}_X^+} \tag{2.1.2}$$

which, on an affinoid perfectoid U with $\widehat{U} = \text{Spa}(R, R^+)$, is given by

$$\theta(U)\colon \mathbb{A}_{\text{inf}}(U) = W(R^{\flat+}) \longrightarrow \widehat{\mathcal{O}_X^+}(U) = R^+, \quad (x_0, x_1, \cdots) \mapsto \sum_{n=0}^{\infty} p^n x_{n,n}^\sharp \tag{2.1.3}$$

with $x_n = (x_{n,i})_{i\in\mathbb{N}} \in R^{\flat+} = \varprojlim_{x\mapsto x^p} R^+/p$. As (R, R^+) is a perfectoid affinoid algebra, $\theta(U)$ is known to be surjective (cf. [Bri, 5.1.2]). Therefore, θ is also surjective.

Definition 2.1. Let X be a locally noetherian adic space over $\text{Spa}(k, \mathcal{O}_k)$ as above. Consider the following sheaves on $X_{\text{proét}}$.

1. Define $\mathbb{A}_{\text{cris}}$ to be the p-adic completion of the PD-envelope $\mathbb{A}_{\text{cris}}^0$ of $\mathbb{A}_{\text{inf}}$ with respect to the ideal sheaf $\ker(\theta) \subset \mathbb{A}_{\text{inf}}$, and define $\mathbb{B}_{\text{cris}}^+ := \mathbb{A}_{\text{cris}}[1/p]$.
2. For $r \in \mathbb{Z}_{\geq 0}$, set $\text{Fil}^r \mathbb{A}_{\text{cris}}^0 := \ker(\theta)^{[r]}\mathbb{A}_{\text{cris}}^0 \subset \mathbb{A}_{\text{cris}}^0$ to be the r-th divided power ideal, and $\text{Fil}^{-r} \mathbb{A}_{\text{cris}}^0 = \mathbb{A}_{\text{cris}}^0$. So the family $\{\text{Fil}^r \mathbb{A}_{\text{cris}}^0 : r \in \mathbb{Z}\}$ gives a descending filtration of $\mathbb{A}_{\text{cris}}^0$.
3. For $r \in \mathbb{Z}$, define $\text{Fil}^r \mathbb{A}_{\text{cris}} \subset \mathbb{A}_{\text{cris}}$ to be the image of the following morphism of sheaves (we shall see below that this map is actually injective):

$$\varprojlim_n (\text{Fil}^r \mathbb{A}_{\text{cris}}^0)/p^n \longrightarrow \varprojlim_n \mathbb{A}_{\text{cris}}^0/p^n = \mathbb{A}_{\text{cris}}, \tag{2.1.4}$$

and define $\text{Fil}^r \mathbb{B}_{\text{cris}}^+ = \text{Fil}^r \mathbb{A}_{\text{cris}}[1/p]$.

Let $p^\flat = (p_i)_{i\geq 0}$ be a family of elements of $\overline{k}$ such that $p_0 = p$ and that $p_{i+1}^p = p_i$ for any $i \geq 0$. Set

$$\xi = [p^\flat] - p \in \mathbb{A}_{\text{inf}}|_{X_{\overline{k}}}.$$

Lemma 2.2. *We have* $\ker(\theta)|_{X_{\overline{k}}} = (\xi) \subset \mathbb{A}_{\inf}|_{X_{\overline{k}}}$. *Furthermore,* $\xi \in \mathbb{A}_{\inf}|_{X_{\overline{k}}}$ *is not a zero-divisor.*

Proof. As the set of affinoids perfectoids U lying above $X_{\overline{k}}$ forms a basis for the topology of $X_{\text{pro\'et}}/X_{\overline{k}}$, we only need to check that, for any such U, $\xi \in \mathbb{A}_{\inf}(U)$ is not a zero-divisor and that the kernel of $\theta(U)\colon \mathbb{A}_{\inf}(U) \to \widehat{\mathcal{O}_X^+}(U)$ is generated by ξ. Write $\widehat{U} = \mathrm{Spa}(R, R^+)$. Then $\mathbb{A}_{\inf}(U) = W(R^{\flat +})$ and $\widehat{O_X^+}(U) = R^+$, hence we reduce our statement to (the proof of) [Sch13, Lemma 6.3]. □

Corollary 2.3. *(1) We have* $\mathbb{A}^0_{\mathrm{cris}}|_{X_{\overline{k}}} = \mathbb{A}_{\inf}|_{X_{\overline{k}}}[\xi^n/n! : n \in \mathbb{N}] \subset \mathbb{B}_{\inf}|_{X_{\overline{k}}}$. *Moreover, for* $r \geq 0$, $\mathrm{Fil}^r \mathbb{A}^0_{\mathrm{cris}}|_{X_{\overline{k}}} = \mathbb{A}_{\inf}|_{X_{\overline{k}}}[\xi^n/n! : n \geq r]$ *and* $\mathrm{gr}^r \mathbb{A}^0_{\mathrm{cris}}|_{X_{\overline{k}}} \xrightarrow{\sim} \widehat{\mathcal{O}_X^+}|_{X_{\overline{k}}}$.

(2) The morphism (2.1.4) *is injective, hence* $\varprojlim_n \mathrm{Fil}^r \mathbb{A}^0_{\mathrm{cris}}/p^n \xrightarrow{\sim} \mathrm{Fil}^r \mathbb{A}_{\mathrm{cris}}$. *Moreover, for* $r \geq 0$, $\mathrm{gr}^r \mathbb{A}_{\mathrm{cris}}|_{X_{\overline{k}}} \xrightarrow{\sim} \widehat{\mathcal{O}_X^+}|_{X_{\overline{k}}}$.

Proof. The first two statements in (1) are clear. In particular, for $r \geq 0$ we have the following exact sequence

$$0 \longrightarrow \mathrm{Fil}^{r+1} \mathbb{A}^0_{\mathrm{cris}}|_{X_{\overline{k}}} \longrightarrow \mathrm{Fil}^r \mathbb{A}^0_{\mathrm{cris}}|_{X_{\overline{k}}} \longrightarrow \widehat{\mathcal{O}_X^+}|_{X_{\overline{k}}} \longrightarrow 0\ , \tag{2.1.5}$$

where the second map sends $a\xi^r/r!$ to $\theta(a)$. This gives the last assertion of (1).

As $\widehat{\mathcal{O}_X^+}$ has no p-torsions, an induction on r shows that the cokernel of the inclusion $\mathrm{Fil}^r \mathbb{A}^0_{\mathrm{cris}} \subset \mathbb{A}^0_{\mathrm{cris}}$ has no p-torsions. As a result, the morphism (2.1.4) is injective and $\mathrm{Fil}^r \mathbb{A}_{\mathrm{cris}}$ is the p-adic completion of $\mathrm{Fil}^r \mathbb{A}^0_{\mathrm{cris}}$. Since $\widehat{\mathcal{O}_X^+}$ is p-adically complete, we deduce from (2.1.5) also the following short exact sequence after passing to p-adic completions:

$$0 \longrightarrow \mathrm{Fil}^{r+1} \mathbb{A}_{\mathrm{cris}}|_{X_{\overline{k}}} \longrightarrow \mathrm{Fil}^r \mathbb{A}_{\mathrm{cris}}|_{X_{\overline{k}}} \longrightarrow \widehat{\mathcal{O}_X^+}|_{X_{\overline{k}}} \longrightarrow 0 \tag{2.1.6}$$

giving the last part of (2). □

Let $\epsilon = (\epsilon^{(i)})_{i \geq 0}$ be a sequence of elements of $\overline{k}$ such that $\epsilon^{(0)} = 1$, $\epsilon^{(1)} \neq 1$ and $(\epsilon^{(i+1)})^p = \epsilon^{(i)}$ for all $i \geq 0$. Then $1 - [\epsilon]$ is a well-defined element of the restriction $\mathbb{A}_{\inf}|_{X_{\overline{k}}}$ to $X_{\text{pro\'et}}/X_{\overline{k}}$ of $\mathbb{A}_{\inf}$. Moreover $1 - [\epsilon] \in \ker(\theta)|_{X_{\overline{k}}} = \mathrm{Fil}^1 \mathbb{A}_{\mathrm{cris}}|_{X_{\overline{k}}}$. Let

$$t := \log([\epsilon]) = -\sum_{n=1}^{\infty} \frac{(1-[\epsilon])^n}{n}, \tag{2.1.7}$$

which is well-defined in $\mathbb{A}_{\mathrm{cris}}|_{X_{\overline{k}}}$ since $\mathrm{Fil}^1 \mathbb{A}_{\mathrm{cris}}$ is a PD-ideal.

Definition 2.4. Let X be a locally noetherian adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$. Define $\mathbb{B}_{\mathrm{cris}} = \mathbb{B}^+_{\mathrm{cris}}[1/t]$. For $r \in \mathbb{Z}$, set $\mathrm{Fil}^r \mathbb{B}_{\mathrm{cris}} = \sum_{s \in \mathbb{Z}} t^{-s} \mathrm{Fil}^{r+s} \mathbb{B}^+_{\mathrm{cris}} \subset \mathbb{B}_{\mathrm{cris}}$. (As the canonical element $t \in \mathbb{A}_{\mathrm{cris}}$ exists locally, these definitions do make sense.)

Before investigating these period sheaves in details, we first study them over a perfectoid affinoid $(\widehat{\overline{k}}, \mathcal{O}_{\widehat{\overline{k}}})$-algebra (R, R^+). Consider

$$\mathbb{A}_{\inf}(R, R^+) := W(R^{\flat +}), \quad \mathbb{B}_{\inf}(R, R^+) := \mathbb{A}_{\inf}(R, R^+)[1/p],$$

and define the morphism

$$\theta_{(R,R^+)}\colon \mathbb{A}_{\inf}(R, R^+) \longrightarrow R^+ \tag{2.1.8}$$

in the same way as in (2.1.3). It is known to be surjective as (R, R^+) is perfectoid. Furthermore, we have seen that the element $\xi \in \mathbb{A}_{\mathrm{inf}}(\widehat{\overline{k}}, \mathcal{O}_{\widehat{\overline{k}}})$ generates $\ker(\theta_{(R,R^+)})$ and is not a zero-divisor in $\mathbb{A}_{\mathrm{inf}}(R, R^+)$. Let $\mathbb{A}_{\mathrm{cris}}(R, R^+)$ be the p-adic completion of the PD-envelope of $\mathbb{A}_{\mathrm{inf}}(R, R^+)$ with respect to the ideal $\ker(\theta_{(R,R^+)})$. So $\mathbb{A}_{\mathrm{cris}}(R, R^+)$ is the p-adic completion of $\mathbb{A}^0_{\mathrm{cris}}(R, R^+)$ with

$$\mathbb{A}^0_{\mathrm{cris}}(R, R^+) := \mathbb{A}_{\mathrm{inf}}(R, R^+)\left[\frac{\xi^n}{n!} : n \in \mathbb{N}\right] \subset \mathbb{B}_{\mathrm{inf}}(R, R^+).$$

For r an integer, let $\mathrm{Fil}^r \mathbb{A}^0_{\mathrm{cris}}(R, R^+) \subset \mathbb{A}^0_{\mathrm{cris}}(R, R^+)$ be the r-th PD-ideal, *i.e.,* the ideal generated by $\xi^n/n!$ for $n \geq \max\{r, 0\}$. Let $\mathrm{Fil}^r \mathbb{A}_{\mathrm{cris}}(R, R^+) \subset \mathbb{A}_{\mathrm{cris}}(R, R^+)$ be the closure (for the p-adic topology) of $\mathrm{Fil}^r \mathbb{A}^0_{\mathrm{cris}}(R, R^+)$ inside $\mathbb{A}_{\mathrm{cris}}(R, R^+)$. Finally, put $\mathbb{B}^+_{\mathrm{cris}}(R, R^+) := \mathbb{A}_{\mathrm{cris}}(R, R^+)[1/p]$, $\mathbb{B}_{\mathrm{cris}}(R, R^+) := \mathbb{B}^+_{\mathrm{cris}}(R, R^+)[1/t]$, and for $r \in \mathbb{Z}$, set

$$\mathrm{Fil}^r \mathbb{B}^+_{\mathrm{cris}}(R, R^+) := \mathrm{Fil}^r \mathbb{A}_{\mathrm{cris}}(R, R^+)[1/p] \text{ and}$$
$$\mathrm{Fil}^r \mathbb{B}_{\mathrm{cris}}(R, R^+) := \sum_{s \in \mathbb{Z}} t^{-s} \mathrm{Fil}^{r+s} \mathbb{B}^+_{\mathrm{cris}}(R, R^+).$$

In particular, taking $R^+ = \mathcal{O}_{\mathbb{C}_p}$ with $\mathbb{C}_p$ the p-adic completion of the fixed algebraic closure $\overline{k}$ of k, we get Fontaine's rings A_{cris}, B^+_{cris}, B_{cris} as in [Fon94].

Lemma 2.5. *Let X be a locally noetherian adic space over $(k, \mathcal{O}_k)$. Let $U \in X_{\mathrm{pro\acute{e}t}}$ be an affinoid perfectoid above $X_{\overline{k}}$ with $\widehat{U} = \mathrm{Spa}(R, R^+)$. Let $\mathcal{F} \in \{\mathbb{A}^0_{\mathrm{cris}}, \mathbb{A}_{\mathrm{cris}}\}$. Then for any $r \geq \mathbb{Z}$, we have a natural almost isomorphism $\mathrm{Fil}^r \mathcal{F}(R, R^+)^a \xrightarrow{\approx} \mathrm{Fil}^r \mathcal{F}(U)^a$, and have $H^i(U, \mathrm{Fil}^r \mathcal{F})^a = 0$ for any $i > 0$.*

Proof. As U is affinoid perfectoid, we have $\widehat{\mathcal{O}^+_X}(U) = R^+$, $\mathcal{O}^{\flat+}_X(U) = R^{\flat+}$ and $\theta(U) = \theta_{(R,R^+)}$. In particular, $\mathbb{A}_{\mathrm{inf}}(U) = \mathbb{A}_{\mathrm{inf}}(R, R^+)$, and the natural morphism $\alpha\colon \mathbb{A}_{\mathrm{inf}}(R, R^+) = \mathbb{A}_{\mathrm{inf}}(U) \longrightarrow \mathbb{A}^0_{\mathrm{cris}}(U)$ sends $\ker(\theta_{(R,R^+)})$ into $\mathrm{Fil}^1 \mathbb{A}^0_{\mathrm{cris}}(U)$. As a consequence, we get a natural morphism $\mathbb{A}^0_{\mathrm{cris}}(R, R^+) \to \mathbb{A}^0_{\mathrm{cris}}(U)$, inducing morphisms $\mathrm{Fil}^r \mathbb{A}^0_{\mathrm{cris}}(R, R^+) \to \mathrm{Fil}^r \mathbb{A}^0_{\mathrm{cris}}(U)$ between the filtrations. Passing to p-adic completions, we obtain the natural maps $\mathrm{Fil}^r \mathbb{A}_{\mathrm{cris}}(R, R^+) \to \mathrm{Fil}^r \mathbb{A}_{\mathrm{cris}}(U)$ for all $r \in \mathbb{Z}$.

We need to show that the morphisms constructed above are almost isomorphisms. Recall that, as U is affinoid perfectoid, $H^i(U, \mathbb{A}_{\mathrm{inf}})^a = 0$ for $i > 0$ ([Sch13, Theorem 6.5]). Consider $\widetilde{\mathbb{A}^0_{\mathrm{cris}}} := \frac{\mathbb{A}_{\mathrm{inf}}[X_i : i \in \mathbb{N}]}{(X_i^p - a_i X_{i+1} : i \in \mathbb{N})}$ with $a_i = \frac{p^{i+1}!}{(p^i!)^p}$, then one has the following short exact sequence

$$0 \longrightarrow \widetilde{\mathbb{A}^0_{\mathrm{cris}}} \xrightarrow{\cdot(X_0 - \xi)} \widetilde{\mathbb{A}^0_{\mathrm{cris}}} \longrightarrow \mathbb{A}^0_{\mathrm{cris}} \longrightarrow 0\ .$$

Since $\widetilde{\mathbb{A}^0_{\mathrm{cris}}}$ is a direct sum of $\mathbb{A}_{\mathrm{inf}}$'s as an abelian sheaf and U is qcqs, we get $H^i\left(U, \widetilde{\mathbb{A}^0_{\mathrm{cris}}}\right)^a = 0$ for $i > 0$. Hence $H^i(U, \mathbb{A}^0_{\mathrm{cris}})^a = 0$ for $i > 0$ and the short exact sequence above stays almost exact after taking sections over U:

$$0 \longrightarrow \widetilde{\mathbb{A}^0_{\mathrm{cris}}}(U)^a \xrightarrow{\cdot(X_0 - \xi)} \widetilde{\mathbb{A}^0_{\mathrm{cris}}}(U)^a \longrightarrow \mathbb{A}^0_{\mathrm{cris}}(U)^a \longrightarrow 0\ . \tag{2.1.9}$$

On the other hand, set $\widetilde{\mathbb{A}^0_{\mathrm{cris}}}(R,R^+) := \frac{\mathbb{A}_{\mathrm{inf}}(R,R^+)[X_i:i\in\mathbb{N}]}{(X_i^p - a_i X_{i+1}:i\in\mathbb{N})}$ with $a_i = \frac{p^{i+1}!}{(p^i!)^p}$ as above, then the following similar sequence is exact:

$$0 \longrightarrow \widetilde{\mathbb{A}^0_{\mathrm{cris}}}(R,R^+)^a \xrightarrow{\cdot(X_0-\xi)} \widetilde{\mathbb{A}^0_{\mathrm{cris}}}(R,R^+)^a \longrightarrow \mathbb{A}^0_{\mathrm{cris}}(R,R^+)^a \longrightarrow 0\,. \tag{2.1.10}$$

Since U is qcqs, $\widetilde{\mathbb{A}^0_{\mathrm{cris}}}(U) = \widetilde{\mathbb{A}^0_{\mathrm{cris}}}(R,R^+)$. Combining (2.1.9) and (2.1.10), we find $\mathbb{A}^0_{\mathrm{cris}}(R,R^+)^a \xrightarrow{\sim} \mathbb{A}^0_{\mathrm{cris}}(U)^a$. This proves the statement for $\mathbb{A}^0_{\mathrm{cris}}$.

Next, as $\mathbb{A}^0_{\mathrm{cris}} \subset \mathbb{B}_{\mathrm{inf}}$ (Corollary 2.3), the sheaf $\mathbb{A}^0_{\mathrm{cris}}$ has no p-torsions. So we get the following tautological exact sequence

$$0 \longrightarrow \mathbb{A}^0_{\mathrm{cris}} \xrightarrow{p^n} \mathbb{A}^0_{\mathrm{cris}} \longrightarrow \mathbb{A}^0_{\mathrm{cris}}/p^n \longrightarrow 0.$$

By what we have shown for $\mathbb{A}^0_{\mathrm{cris}}$, we get from the exact sequence above that $\left(\mathbb{A}^0_{\mathrm{cris}}(R,R^+)/p^n\right)^a \xrightarrow{\sim} \left(\mathbb{A}^0_{\mathrm{cris}}(U)/p^n\right)^a \xrightarrow{\sim} \left(\mathbb{A}^0_{\mathrm{cris}}/p^n\right)(U)^a$, and $H^i(U,\mathbb{A}^0_{\mathrm{cris}}/p^n)^a = 0$ for $i>0$. Therefore the transition maps of the projective system $\{(\mathbb{A}^0_{\mathrm{cris}}/p^n)(U)\}_{n\geq 0}$ are almost surjective, thus $R^1\varprojlim \left((\mathbb{A}^0_{\mathrm{cris}}/p^n)(U)\right)^a = 0$. So the projective system $\{\mathbb{A}^0_{\mathrm{cris}}/p^n\}$ verifies the assumptions of (the almost version of)[Sch13, Lemma 3.18]. As a result, we find

$$R^j \varprojlim A^0_{\mathrm{cris}}/p^n = 0, \quad \text{for any } j>0, \tag{2.1.11}$$

$$\mathbb{A}_{\mathrm{cris}}(R,R^+)^a = \varprojlim (\mathbb{A}^0_{\mathrm{cris}}(R,R^+)/p^n)^a \xrightarrow{\sim} \varprojlim \left(\left(\mathbb{A}^0_{\mathrm{cris}}/p^n\right)(U)\right)^a = \mathbb{A}_{\mathrm{cris}}(U)^a,$$

and

$$H^i(U,\mathbb{A}_{\mathrm{cris}})^a = H^i(U, \varprojlim_n \mathbb{A}^0_{\mathrm{cris}}/p^n)^a = 0$$

for $i>0$.

To prove the statements for $\mathrm{Fil}^r\,\mathbb{A}^0_{\mathrm{cris}}$ for $r\geq 0$, we shall use the exact sequence (2.1.5). As $H^i(U,\widehat{\mathcal{O}^+_X})^a = 0$ for $i>0$ ([Sch13, Lemma 4.10]), we find $H^i(U,\mathrm{Fil}^r\,\mathbb{A}^0_{\mathrm{cris}})^a = 0$ for all $i\geq 2$, and also the induced long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Fil}^{r+1}\,\mathbb{A}^0_{\mathrm{cris}}(U)^a \longrightarrow \mathrm{Fil}^r\,\mathbb{A}^0_{\mathrm{cris}}(U)^a \longrightarrow \widehat{\mathcal{O}^+_X}(U)^a \longrightarrow \\ H^1(U,\mathrm{Fil}^{r+1}\,\mathbb{A}^0_{\mathrm{cris}})^a \longrightarrow H^1(U,\mathrm{Fil}^r\,\mathbb{A}^0_{\mathrm{cris}})^a \longrightarrow 0. \end{aligned}$$

On the other hand, we have the analogous exact sequence for $\mathrm{Fil}^r\,\mathbb{A}^0_{\mathrm{cris}}(R,R^+)$:

$$0 \longrightarrow \mathrm{Fil}^{r+1}\,\mathbb{A}^0_{\mathrm{cris}}(R,R^+) \longrightarrow \mathrm{Fil}^r\,\mathbb{A}^0_{\mathrm{cris}}(R,R^+) \longrightarrow R^+ \longrightarrow 0,$$

where the second morphism sends $a\xi^r/r!$ to $\theta_{(R,R^+)}(a)$. Together with the two exact sequences above and the vanishing $H^1(U,\mathbb{A}^0_{\mathrm{cris}})^a = 0$, an induction on $r\geq 0$ shows $\mathrm{Fil}^r\,\mathbb{A}^0_{\mathrm{cris}}(R,R^+)^a \xrightarrow{\sim} \mathrm{Fil}^r\,\mathbb{A}^0_{\mathrm{cris}}(U)^a$ and $H^1(U,\mathrm{Fil}^r\,\mathbb{A}^0_{\mathrm{cris}})^a = 0$. This gives the statement for $\mathrm{Fil}^r\,\mathbb{A}^0_{\mathrm{cris}}$ for $r\in\mathbb{Z}$ (note that $\mathrm{Fil}^r\,\mathbb{A}^0_{\mathrm{cris}} = \mathbb{A}^0_{\mathrm{cris}}$ when $r\leq 0$). The statement for $\mathrm{Fil}^r\,\mathbb{A}_{\mathrm{cris}}$ can be done in the same way; one starts with (2.1.6) and uses the vanishing of $H^1(U,\mathbb{A}_{\mathrm{cris}})^a$. □

Corollary 2.6. *Keep the notation of Lemma 2.5.*

1. *The morphism multiplication by t on $\mathbb{A}_{\mathrm{cris}}$ is almost injective. In particular, $t\in\mathbb{B}^+_{\mathrm{cris}}|_{X_{\overline{k}}}$ is not a zero-divisor.*
2. *Let $\mathcal{F}\in\{\mathbb{B}^+_{\mathrm{cris}},\mathbb{B}_{\mathrm{cris}}\}$. Then for any $r\in\mathbb{Z}$, there is a natural isomorphism $\mathrm{Fil}^r\,\mathcal{F}(R,R^+) \xrightarrow{\sim} \mathrm{Fil}^r\mathcal{F}(U)$, and $H^i(U,\mathrm{Fil}^r\,\mathcal{F}) = 0$ for $i\geq 1$.*

Proof. (1) By Lemma 2.5, we are reduced to showing that $t \in \mathbb{A}_{\mathrm{cris}}(R, R^+)$ is not a zero-divisor. So we just need to apply [Bri, Corollaire 6.2.2], whose proof works for any such perfectoid (R, R^+).

(2) As U is qcqs, we deduce from Lemma 2.5 the statement for $\mathbb{B}^+_{\mathrm{cris}}$ and for $\mathrm{Fil}^r \mathbb{B}^+_{\mathrm{cris}}$ on inverting p. Similarly, inverting t we get the statement for $\mathbb{B}_{\mathrm{cris}}$. Finally, as $\mathrm{Fil}^r \mathbb{B}_{\mathrm{cris}}|_{X_{\overline{k}}} = \varinjlim_s t^{-s} \mathrm{Fil}^{r+s} \mathbb{B}^+_{\mathrm{cris}}|_{X_{\overline{k}}}$ and as $t^{-s} \mathrm{Fil}^{r+s} \mathbb{B}^+_{\mathrm{cris}}|_{X_{\overline{k}}} \simeq \mathrm{Fil}^{r+s} \mathbb{B}^+_{\mathrm{cris}}|_{X_{\overline{k}}}$, passing to limits we get the statement for $\mathrm{Fil}^r \mathbb{B}_{\mathrm{cris}}$. □

2.2. **Period sheaves with connections.** In this section, assume that the p-adic field k is *absolutely unramified*. Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$. Set $X := \mathcal{X}_k$ the generic fiber of $\mathcal{X}$, viewed as an adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$. For any étale morphism $\mathcal{Y} \to \mathcal{X}$, by taking the generic fibers, we obtain an étale morphism $\mathcal{Y}_k \to X$ of adic spaces, hence an object of the pro-étale site $X_{\mathrm{proét}}$. In this way, we get a morphism of sites $\mathcal{X}_{\text{ét}} \to X_{\mathrm{proét}}$, with the induced morphism of topoi

$$w \colon X^{\sim}_{\mathrm{proét}} \longrightarrow \mathcal{X}^{\sim}_{\text{ét}}.$$

Let $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$ denote the structure sheaf of the étale site $\mathcal{X}_{\text{ét}}$: for any étale morphism $\mathcal{Y} \to \mathcal{X}$ of formal schemes over $\mathcal{O}_k$, $\mathcal{O}_{\mathcal{X}_{\text{ét}}}(\mathcal{Y}) = \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. Define $\mathcal{O}_X^{\mathrm{ur}+} := w^{-1}\mathcal{O}_{\mathcal{X}_{\text{ét}}}$ and $\mathcal{O}_X^{\mathrm{ur}} := w^{-1}\mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p]$. Thus $\mathcal{O}_X^{\mathrm{ur}+}$ is the associated sheaf of the presheaf $\widetilde{\mathcal{O}_X^{\mathrm{ur}+}}$:

$$X_{\mathrm{proét}} \ni U \mapsto \varinjlim_{(\mathcal{Y}, a)} \mathcal{O}_{\mathcal{X}_{\text{ét}}}(\mathcal{Y}) =: \widetilde{\mathcal{O}_X^{\mathrm{ur}+}}(U),$$

where the limit runs through all pairs $(\mathcal{Y}, a)$ with $\mathcal{Y} \in \mathcal{X}_{\text{ét}}$ and $a \colon U \to \mathcal{Y}_k$ a morphism making the following diagram commutative

(2.2.1)
$$\begin{array}{ccc} U & \longrightarrow & X = \mathcal{X}_k \\ & \overset{a}{\searrow} & \uparrow \\ & & \mathcal{Y}_k. \end{array}$$

The morphism $a \colon U \to \mathcal{Y}_k$ induces a map $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to \mathcal{O}_X(U)$. There is then a morphism of presheaves $\widetilde{\mathcal{O}_X^{\mathrm{ur}+}} \to \mathcal{O}_X^+$, whence a morphism of sheaves

(2.2.2)
$$\mathcal{O}_X^{\mathrm{ur}+} \to \mathcal{O}_X^+.$$

Recall $\mathbb{A}_{\mathrm{inf}} := W(\mathcal{O}_X^{\flat+})$. Set $\mathcal{O}\mathbb{A}_{\mathrm{inf}} := \mathcal{O}_X^{\mathrm{ur}+} \otimes_{\mathcal{O}_k} \mathbb{A}_{\mathrm{inf}}$ and

(2.2.3)
$$\theta_X \colon \mathcal{O}\mathbb{A}_{\mathrm{inf}} \longrightarrow \widehat{\mathcal{O}_X^+}$$

to be the map induced from $\theta \colon \mathbb{A}_{\mathrm{inf}} \to \widehat{\mathcal{O}_X^+}$ of (2.1.2) by extension of scalars.

Definition 2.7. Consider the following sheaves on $X_{\mathrm{proét}}$.

(1) Let $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ be the p-adic completion of the PD-envelope $\mathcal{O}\mathbb{A}^0_{\mathrm{cris}}$ of $\mathcal{O}\mathbb{A}_{\mathrm{inf}}$ with respect to the ideal sheaf $\ker(\theta_X) \subset \mathcal{O}\mathbb{A}_{\mathrm{inf}}$, $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}} := \mathcal{O}\mathbb{A}_{\mathrm{cris}}[1/p]$, and $\mathcal{O}\mathbb{B}_{\mathrm{cris}} := \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}[1/t]$ with $t = \log([\epsilon])$ defined in (2.1.7).

(2) For $r \in \mathbb{Z}_{\geq 0}$, define $\mathrm{Fil}^r \mathcal{O}\mathbb{A}^0_{\mathrm{cris}} \subset \mathcal{O}\mathbb{A}^0_{\mathrm{cris}}$ to be the r-th PD-ideal $\ker(\theta_X)^{[r]}$, and $\mathrm{Fil}^r \mathcal{O}\mathbb{A}_{\mathrm{cris}}$ the image of the canonical map

$$\varprojlim \mathrm{Fil}^r \mathcal{O}\mathbb{A}^0_{\mathrm{cris}}/p^n \longrightarrow \varprojlim \mathcal{O}\mathbb{A}^0_{\mathrm{cris}}/p^n = \mathcal{O}\mathbb{A}_{\mathrm{cris}}.$$

Also set $\mathrm{Fil}^{-r} \mathcal{O}\mathbb{A}_{\mathrm{cris}} = \mathcal{O}\mathbb{A}_{\mathrm{cris}}$ for $r > 0$.

(3) For $r \in \mathbb{Z}$, set

$$\mathrm{Fil}^r \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} := \mathrm{Fil}^r \mathcal{O}\mathbb{A}_{\mathrm{cris}}[1/p] \quad \text{and} \quad \mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris}} := \sum_{s\in\mathbb{Z}} t^{-s} \mathrm{Fil}^{r+s} \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}.$$

Remark 2.8. As $t^p = p!\cdot t^{[p]}$ in $A_{\mathrm{cris}} := \mathbb{A}_{\mathrm{cris}}(\widehat{\overline{k}}, \mathcal{O}_{\widehat{\overline{k}}})$, one can also define $\mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris}}$ as $\sum_{s\in\mathbb{N}} t^{-s} \mathrm{Fil}^{r+s} \mathcal{O}\mathbb{A}_{\mathrm{cris}}$. A similar observation holds equally for $\mathrm{Fil}^r \mathbb{B}_{\mathrm{cris}}$.

Remark 2.9. (1) As $\mathcal{X}$ is flat over $\mathcal{O}_k$, the structure sheaf $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$ has no p-torsions. It follows that $\mathcal{O}\mathbb{A}_{\mathrm{inf}}$, $\mathcal{O}\mathbb{A}^0_{\mathrm{cris}}$ and $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ have no p-torsions. So $\mathcal{O}\mathbb{A}_{\mathrm{cris}} \subset \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}$.

(2) The morphism θ_X of (2.2.3) extends to a surjective morphism from $\mathcal{O}\mathbb{A}^0_{\mathrm{cris}}$ to $\widehat{\mathcal{O}^+_X}$ with kernel $\mathrm{Fil}^1 \mathcal{O}\mathbb{A}^0_{\mathrm{cris}}$, hence also a morphism from $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ to $\widehat{\mathcal{O}^+_X}$. Let us denote these two morphisms again by θ_X. As $\mathrm{coker}(\mathrm{Fil}^1 \mathcal{O}\mathbb{A}^0_{\mathrm{cris}} \hookrightarrow \mathcal{O}\mathbb{A}^0_{\mathrm{cris}}) \simeq \widehat{\mathcal{O}^+_X}$ is p-adically complete and has no p-torsions, using the snake lemma and passing to limits one can deduce the following short exact sequence

$$0 \longrightarrow \varprojlim_n (\mathrm{Fil}^1 \mathcal{O}\mathbb{A}^0_{\mathrm{cris}}/p^n) \longrightarrow \mathcal{O}\mathbb{A}_{\mathrm{cris}} \xrightarrow{\theta_X} \widehat{\mathcal{O}^+_X} \longrightarrow 0.$$

In particular, $\mathrm{Fil}^1 \mathcal{O}\mathbb{A}_{\mathrm{cris}} = \ker(\theta_X)$.

In order to describe explicitly the sheaf $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, we shall need some auxiliary sheaves on $X_{\mathrm{proét}}$.

Definition 2.10. Consider the following sheaves on $X_{\mathrm{proét}}$.

(1) Let $\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}$ be the p-adic completion of the sheaf of PD polynomial rings $\mathbb{A}^0_{\mathrm{cris}}\langle u_1, \ldots, u_d\rangle \subset \mathbb{B}_{\mathrm{inf}}[u_1, \ldots, u_d]$. Set $\mathbb{B}^+_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\} := \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}[1/p]$ and $\mathbb{B}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\} := \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}[1/t]$.

(2) For $r \in \mathbb{Z}$, let $\mathrm{Fil}^r \mathbb{A}^0_{\mathrm{cris}}\langle u_1, \ldots, u_d\rangle \subset \mathbb{A}^0_{\mathrm{cris}}\langle u_1, \ldots, u_d\rangle$ be the ideal sheaf given by

$$\mathrm{Fil}^r(\mathbb{A}^0_{\mathrm{cris}}\langle u_1, \ldots, u_d\rangle) := \sum_{i_1,\ldots,i_d\geq 0} \mathrm{Fil}^{r-(i_1+\ldots+i_d)} \mathbb{A}^0_{\mathrm{cris}} \cdot u_1^{[i_1]} \cdots u_d^{[i_d]},$$

and $\mathrm{Fil}^r(\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}) \subset \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}$ the image of the morphism

$$\varprojlim_n \left(\mathrm{Fil}^r \mathbb{A}^0_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}/p^n\right) \longrightarrow \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}.$$

The family $\{\mathrm{Fil}^r(\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}) : r \in \mathbb{Z}\}$ gives a descending filtration of $\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}$. Inverting p, we obtain $\mathrm{Fil}^r(\mathbb{B}^+_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\})$. Set finally

$$\mathrm{Fil}^r(\mathbb{B}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}) := \sum_{s\in\mathbb{Z}} t^{-s} \mathrm{Fil}^{r+s}(\mathbb{B}^+_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}).$$

The proof of the following lemma is similar to that of Lemma 2.5. We omit the details here.

Lemma 2.11. *Let $V \in X_{\mathrm{proét}}$ be an affinoid perfectoid lying above $X_{\overline{k}}$ with $\widehat{V} = \mathrm{Spa}(R, R^+)$. Then the following natural map is an almost isomorhpism*

$$\mathbb{A}_{\mathrm{cris}}(R, R^+)\{\langle u_1, \ldots, u_d\rangle\} \xrightarrow{\approx} (\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\})(V).$$

Under this identification, $\mathrm{Fil}^r(\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\})(V)^a$ *consists of series*

$$\sum_{i_1,\dots,i_d\geq 0} a_{i_1,\dots,i_d} u_1^{[i_1]}\cdots u_d^{[i_d]} \in \mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\dots,u_d\rangle\}$$

such that $a_{i_1,\dots,i_d}\in \mathrm{Fil}^{r-(i_1+\dots+i_d)}\,\mathbb{A}_{\mathrm{cris}}(R,R^+)$ *and* $a_{i_1,\dots,i_d}$ *tends to* 0 *for the p-adic topology when the sum* $i_1+\dots+i_d$ *tends to infinity. Furthermore*

$$H^i(V,\mathrm{Fil}^r(\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\}))^a=0,\quad \text{for } i>0.$$

We want to describe $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ more explicitly . For this, assume there is an étale morphism $\mathcal{X}\to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1},\dots,T_d^{\pm 1}\})=:\mathcal{T}^d$ of formal schemes over $\mathcal{O}_k$. Let $\mathbb{T}^d$ denote the generic fiber of $\mathcal{T}^d$ and $\widetilde{\mathbb{T}^d}$ be obtained from $\mathbb{T}^d$ by adding a compatible system of p^n-th root of T_i for $1\leq i\leq d$ and $n\geq 1$:

$$\widetilde{\mathbb{T}^d}:=\mathrm{Spa}(k\{T_1^{\pm 1/p^\infty},\dots,T_d^{\pm 1/p^\infty}\},\mathcal{O}_k\{T_1^{\pm 1/p^\infty},\dots,T_d^{\pm 1/p^\infty}\}).$$

Set $\widetilde{X}:=X\times_{\mathbb{T}^d}\widetilde{\mathbb{T}^d}$. Let $T_i^\flat\in\mathcal{O}_X^{\flat+}|_{\widetilde{X}}$ be the element $(T_i,T_i^{1/p},\dots,T_i^{1/p^n},\dots)$. Then $\theta_X(T_i\otimes 1-1\otimes[T_i^\flat])=0$, which allows us to define an $\mathbb{A}_{\mathrm{cris}}$-linear morphism

$$\alpha\colon \mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\}|_{\widetilde{X}}\longrightarrow \mathcal{O}\mathbb{A}_{\mathrm{cris}}|_{\widetilde{X}},\quad u_i\mapsto T_i\otimes 1-1\otimes[T_i^\flat]. \tag{2.2.4}$$

Proposition 2.12. *The morphism* α *of* (2.2.4) *is an almost isomorphism. Furthermore, it respects the filtrations on both sides.*

Lemma 2.13. *Let* $\overline{k}$ *be an algebraic closure of* k*. Then* $\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\}|_{\widetilde{X}_{\overline{k}}}$ *has a natural* $\mathcal{O}_X^{\mathrm{ur}+}|_{\widetilde{X}_{\overline{k}}}$*-algebra structure, sending* T_i *to* $u_i+[T_i^\flat]$*, such that the composition*

$$\mathcal{O}_X^{\mathrm{ur}+}|_{\widetilde{X}_{\overline{k}}}\longrightarrow \mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\}|_{\widetilde{X}_{\overline{k}}}\xrightarrow{\theta'|_{\widetilde{X}_{\overline{k}}}}\widehat{\mathcal{O}_X^+}|_{\widetilde{X}_{\overline{k}}}$$

is the map (2.2.2) *composed with* $\mathcal{O}_X^+\to\widehat{\mathcal{O}_X^+}$*. Here* $\theta'\colon\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\}\to\widehat{\mathcal{O}_X^+}$ *is induced from the map* $\mathbb{A}_{\mathrm{cris}}\xrightarrow{\theta}\widehat{\mathcal{O}_X^+}$ *by sending* U_i*'s to* 0*.*

Proof. As $\mathcal{O}_X^{\mathrm{ur}+}$ is the associated sheaf of $\widetilde{\mathcal{O}_X^{\mathrm{ur}+}}$, and as the affinoids perfectoids lying above $\widetilde{X}_{\overline{k}}$ form a basis for the topology of $X_{\mathrm{proét}}/\widetilde{X}_{\overline{k}}$, we only need to define naturally, for any affinoid perfectoid $U\in X_{\mathrm{proét}}$ lying above $\widetilde{X}_{\overline{k}}$, a morphism of rings:

$$\widetilde{\mathcal{O}_X^{\mathrm{ur}+}}(U)\longrightarrow(\mathbb{A}_{\mathrm{cris}}^a\{\langle u_1,\dots,u_d\rangle\})(U),$$

sending T_i to $u_i+[T_i^\flat]$.

Write $\widehat{U}=\mathrm{Spa}(R,R^+)$. Let $p^\flat\in\mathcal{O}_{\widehat{\overline{k}}}^\flat\subset R^{\flat+}$ be the element given by a compatible system of p^n-th roots of p. Then we have (see [Bri, Proposition 6.1.2])

$$\frac{\mathbb{A}_{\mathrm{cris}}(R,R^+)}{(p)}\simeq\frac{(R^{\flat+}/(p^\flat)^p)[\delta_0,\delta_1,\dots]}{(\delta_0^p,\delta_1^p,\dots)},$$

where δ_i is the image of $\gamma^{i+1}(\xi)$ with $\gamma: x\mapsto x^p/p$. So

$$\begin{aligned}\frac{\mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\dots,u_d\rangle\}}{(p)} &\simeq \frac{(\mathbb{A}_{\mathrm{cris}}(R,R^+)/p)[u_i,u_{i,j}:1\leq i\leq d,j=0,1,\dots]}{(u_i^p,u_{i,j}^p)}\\ &\simeq \frac{\frac{R^{\flat+}[u_1,\dots,u_d]}{((p^\flat)^p,u_1^p,\dots,u_d^p)}[\delta_j,u_{i,j}:1\leq i\leq d,j=0,1,\dots]}{(\delta_j^p,u_{i,j}^p)}.\end{aligned}$$

Set $I := (p^\flat, u_1, \ldots, u_d) \subset R^{\flat+}[u_1, \ldots, u_d]/((p^\flat)^p, u_1^p, \ldots, u_d^p)$: this is an nilpotent ideal with cokernel $R^{\flat+}/(p^\flat) \simeq R^+/p$. Furthermore, there is a canonical morphism of schemes

$$\mathrm{Spec}\left(\frac{\mathbb{A}_{\mathrm{cris}}(R, R^+)\{\langle u_1, \ldots, u_d\rangle\}}{(p)}\right) \longrightarrow \mathrm{Spec}\left(\frac{R^{\flat+}[u_1, \ldots, u_d]}{((p^\flat)^p, u_1^p, \ldots, u_d^p)}\right)$$

induced from the natural inclusion $\frac{R^{\flat+}[u_1,\ldots,u_d]}{((p^\flat)^p,u_1^p,\ldots,u_d^p)} \subset \frac{\mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\ldots,u_d\rangle\}}{(p)}$.

Let $\mathcal{Y}$ be a formal scheme étale over $\mathcal{X}$ together with a factorization $a\colon U \to \mathcal{Y}_k$ as in (2.2.1). We shall construct a natural morphism of $\mathcal{O}_k$-algebras

$$\mathcal{O}_{\mathcal{Y}}(\mathcal{Y}) \longrightarrow \left(\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}\right)(U) \tag{2.2.5}$$

sending T_i to $u_i + [T_i^\flat]$. Assume firstly that the image $a(U)$ is contained in the generic fiber $\mathrm{Spa}(A[1/p], A)$ of some affine open subset $\mathrm{Spf}(A) \subset \mathcal{Y}$. In particular, there exists a morphism of $\mathcal{O}_k$-algebras $A \to R^+$, whence a morphism of κ-schemes

$$\mathrm{Spec}(R^+/p) \longrightarrow \mathrm{Spec}(A/p) \longrightarrow \mathcal{Y} \otimes_{\mathcal{O}_k} \kappa =: \mathcal{Y}_1. \tag{2.2.6}$$

Composing $\mathcal{Y} \to \mathcal{X}$ with the étale morphism $\mathcal{X} \to \mathcal{T}^d$, we obtain an étale morphism $\mathcal{Y} \to \mathcal{T}^d$, hence an étale morphism $\mathcal{Y}_n := \mathcal{Y} \otimes \mathcal{O}_k/p^n \to \mathcal{T}^d \otimes \mathcal{O}_k/p^n =: \mathcal{T}^{d,n}$ for each n. As $T_i^\flat + u_i \in R^{\flat+}[u_1, \ldots, u_d]/((p^\flat)^p, u_1^p, \ldots, u_d^p)$ is invertible, one deduces a map

$$\mathrm{Spec}(R^{\flat+}[u_1, \ldots, u_d]/((p^\flat)^p, u_1^p, \ldots, u_d^p)) \longrightarrow \mathcal{T}^{d,1}, \quad T_i \mapsto T_i^\flat + u_i.$$

But we have the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Spec}(R^+/p) & = & \mathrm{Spec}(R^{\flat+}/(p^\flat)) & \xrightarrow{(2.2.6)} & \mathcal{Y}_1 \\
 & & \Big\downarrow{\scriptstyle \text{nil-immersion}} & \overset{\exists\, g}{\nearrow} & \Big\downarrow{\scriptstyle \text{étale morphism}} \\
 & & \mathrm{Spec}\left(\frac{R^{\flat+}[u_1,\ldots,u_d]}{((p^\flat)^p,u_1^p,\ldots,u_d^p)}\right) & \longrightarrow & \mathcal{T}^{d,1},
\end{array}$$

from which we deduce a morphism, denoted by g_1:

$$g_1 : \mathrm{Spec}\left(\frac{\mathbb{A}_{\mathrm{cris}}(R, R^+)\{\langle u_1, \ldots, u_d\rangle\}}{(p)}\right) \longrightarrow \mathrm{Spec}\left(\frac{R^{\flat+}[u_1, \ldots, u_d]}{((p^\flat)^p, u_1^p, \ldots, u_d^p)}\right) \xrightarrow{g} \mathcal{Y}_1.$$

Then we have the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Spec}\left(\frac{\mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\ldots,u_d\rangle\}}{(p)}\right) & \xrightarrow{g_1} & \mathcal{Y}_1 & \hookrightarrow & \mathcal{Y}_n \\
\Big\downarrow{\scriptstyle \text{nil-immersion}} & & & \overset{\exists\, g_n}{\nearrow} & \Big\downarrow{\scriptstyle \text{étale morphism}} \\
\mathrm{Spec}\left(\frac{\mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\ldots,u_d\rangle\}}{(p^n)}\right) & & \longrightarrow & & \mathcal{T}^{d,n}.
\end{array}$$

with the bottom map $T_i \mapsto T_i^\flat + u_i$. These g_n's are compatible with each other, so that they give rise to a morphism $\mathrm{Spf}(\mathbb{A}_{\mathrm{cris}}(R, R^+)\{\langle u_1, \ldots, u_d\rangle\}) \to \mathcal{Y}$ of formal schemes over $\mathcal{O}_k$, inducing a morphism of $\mathcal{O}_k$-algebras

$$\mathcal{O}_{\mathcal{Y}}(\mathcal{Y}) \to \mathbb{A}_{\mathrm{cris}}(R, R^+)\{\langle u_1, \ldots, u_d\rangle\}$$

sending T_i to $u_i + [T_i^\flat]$. Combining it with the natural morphism

$$\mathbb{A}_{\mathrm{cris}}(R, R^+)\{\langle u_1, \ldots, u_d\rangle\} \to \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}(U)$$

we obtain the desired map (2.2.5). For the general case, *i.e.,* without assuming that $a(U)$ is contained in the generic fiber of some affine open of $\mathcal{Y}$, cover U by affinoids perfectoids V_j such that each $a(V_j)$ is contained in the generic fiber of some affine open subset of $\mathcal{Y}$. The construction above gives, for each j, a morphism of $\mathcal{O}_k$-algebras

$$\mathcal{O}_{\mathcal{Y}}(\mathcal{Y}) \longrightarrow \left(\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}\right)(V_j), \quad T_i \mapsto u_i + [T_i^\flat].$$

As the construction is functorial on the affinoid perfectoid V_j, we deduce the morphism (2.2.5) in the general case using the sheaf property of $\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}$.

Finally, since $\widetilde{\mathcal{O}_X^{\mathrm{ur}+}}(U) = \varinjlim \mathcal{O}_{\mathcal{Y}}(\mathcal{Y})$ with the limit runs through the diagrams (2.2.1), we get a morphism of $\mathcal{O}_k$-algebras

$$\widetilde{\mathcal{O}_X^{\mathrm{ur}+}}(U) \longrightarrow \mathbb{A}_{\mathrm{cris}}(U)\{\langle u_1, \ldots, u_d\rangle\}, \quad T_i \mapsto u_i + [T_i^\flat]$$

which is functorial with respect to affinoid perfectoid $U \in X_{\mathrm{pro\acute{e}t}}$ lying above $\widetilde{X}_{\overline{k}}$. Passing to the associated sheaf, we obtain finally a natural morphism of sheaves of $\mathcal{O}_k$-algebras $\mathcal{O}_X^{\mathrm{ur}}|_{\widetilde{X}_{\overline{k}}} \to \mathbb{A}_{\mathrm{cris}}|_{\widetilde{X}_{\overline{k}}}\{\langle u_1, \ldots, u_d\rangle\}$ sending T_i to $u_i + [T_i^\flat]$. The last statement follows from the assignment $\theta'(U_i) = 0$ and the fact that $\theta([T_i^\flat]) = T_i$. □

Proof of Proposition 2.12. As $\widetilde{X}_{\overline{k}} \to \widetilde{X}$ is a covering in the pro-étale site $X_{\mathrm{pro\acute{e}t}}$, we only need to show that $\alpha|_{\widetilde{X}_{\overline{k}}}$ is an almost isomorphism. By the lemma above, there exists a morphism of sheaves of $\mathcal{O}_k$-algebras $\mathcal{O}_X^{\mathrm{ur}+}|_{\widetilde{X}_{\overline{k}}} \to \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}|_{\widetilde{X}_{\overline{k}}}$ sending T_i to $u_i + [T_i^\flat]$. By extension of scalars, we find the following morphism

$$\beta : \mathcal{O}\mathbb{A}_{\mathrm{inf}}|_{\widetilde{X}_{\overline{k}}} = \left(\mathcal{O}_X^{\mathrm{ur}+} \otimes_{\mathcal{O}_k} \mathbb{A}_{\mathrm{inf}}\right)|_{\widetilde{X}_{\overline{k}}} \longrightarrow \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}|_{\widetilde{X}_{\overline{k}}}$$

which maps $T_i \otimes 1$ to $u_i + [T_i^\flat]$. Consider the composite (with θ' as in Lemma 2.13)

$$\theta'|_{\widetilde{X}_{\overline{k}}} \circ \beta \colon \mathcal{O}\mathbb{A}_{\mathrm{inf}}|_{\widetilde{X}_{\overline{k}}} \longrightarrow \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}|_{\widetilde{X}_{\overline{k}}} \longrightarrow \widehat{\mathcal{O}_X^+}|_{\widetilde{X}_{\overline{k}}},$$

which is simply $\theta_X|_{\widetilde{X}_{\overline{k}}}$ by Lemma 2.13. Therefore, $\beta\left(\ker(\theta_X|_{\widetilde{X}_{\overline{k}}})\right) \subset \ker(\theta'|_{\widetilde{X}_{\overline{k}}})$. Since $\ker(\theta')$ has a PD-structure, the map β extends to the PD-envelope $\mathcal{O}\mathbb{A}^0_{\mathrm{cris}}|_{\widetilde{X}_{\overline{k}}}$ of the source. Furthermore, $\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}^a$ is by definition p-adically complete: $\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}^a \xrightarrow{\sim} \varprojlim_n (\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}^a/p^n)$. The map β in turn extends to the p-adic completion of $\mathcal{O}\mathbb{A}^{0,a}_{\mathrm{cris}}|_{\widetilde{X}_{\overline{k}}}$. So we obtain the following morphism, still denoted by β:

$$\beta \colon \mathcal{O}\mathbb{A}^a_{\mathrm{cris}}|_{\widetilde{X}_{\overline{k}}} \longrightarrow \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\}^a|_{\widetilde{X}_{\overline{k}}}, \quad T_i \otimes 1 \mapsto u_i + [T_i^\flat].$$

Then one shows that β and α are inverse to each other, giving the first part of our proposition.

It remains to check that α respects the filtrations. As $\theta_X|_{\widetilde{X}} \circ \alpha = \theta'|_{\widetilde{X}}$ and as α is an almost isomorphism, α induces an almost isomorphism $\alpha \colon \ker(\theta'|_{\widetilde{X}}) \xrightarrow{\approx} \ker(\theta_X|_{\widetilde{X}})$. Therefore, since $\ker(\theta') = \mathrm{Fil}^1(\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\})$ and $\mathrm{Fil}^1 \mathcal{O}\mathbb{A}_{\mathrm{cris}} = \ker(\theta_X)$ (Remark 2.9), α gives an almost isomorphism

$$\alpha \colon \mathrm{Fil}^1(\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \ldots, u_d\rangle\})^a|_{\widetilde{X}} \xrightarrow{\sim} \mathrm{Fil}^1 \mathcal{O}\mathbb{A}^a_{\mathrm{cris}}|_{\widetilde{X}}.$$

As $\mathrm{Fil}^1 \mathcal{O}\mathbb{A}_{\mathrm{cris}}|_{\widetilde{X}} \subset \mathcal{O}\mathbb{A}_{\mathrm{cris}}|_{\widetilde{X}}$ is a PD-ideal, we can consider its i-th PD ideal subsheaf $\mathcal{I}^{[i]} \subset \mathcal{O}\mathbb{A}_{\mathrm{cris}}|_{\widetilde{X}}$. Using the almost isomorphism above and the explicit description in Lemma 2.11, one checks that the p-adic completion of $\mathcal{I}^{[i]}$ is (almost)

equal to $\alpha(\mathrm{Fil}^i(\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\})|_{\widetilde{X}})$. As the image of the morphism

$$\mathrm{Fil}^i\,\mathcal{O}\mathbb{A}^0_{\mathrm{cris}}|_{\widetilde{X}} \longrightarrow \mathcal{O}\mathbb{A}_{\mathrm{cris}}|_{\widetilde{X}}$$

is naturally contained in $\mathcal{I}^{[i]}$, on passing to p-adic completion, we obtain

$$\mathrm{Fil}^i\,\mathcal{O}\mathbb{A}^a_{\mathrm{cris}}|_{\widetilde{X}} \subset \alpha(\mathrm{Fil}^i\,\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\}^a|_{\widetilde{X}}).$$

On the other hand, we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Fil}^i(\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\})|_{\widetilde{X}} & \xrightarrow{\alpha} & \mathcal{O}\mathbb{A}_{\mathrm{cris}}|_{\widetilde{X}} \\
\cong\uparrow & & \uparrow \\
\varprojlim \frac{\mathrm{Fil}^i(\mathbb{A}^0_{\mathrm{cris}}\langle u_1,\dots,u_d\rangle)|_{\widetilde{X}}}{p^n} & \xrightarrow{\alpha} & \varprojlim \frac{\mathrm{Fil}^i\,\mathcal{O}\mathbb{A}^0_{\mathrm{cris}}|_{\widetilde{X}}}{p^n} = \mathrm{Fil}^i\,\mathcal{O}\mathbb{A}^a_{\mathrm{cris}}|_{\widetilde{X}}.
\end{array}$$

Therefore $\alpha(\mathrm{Fil}^i\,\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\}|_{\widetilde{X}}) \subset \mathrm{Fil}^i\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}|_{\widetilde{X}}$, whence the equality

$$\alpha(\mathrm{Fil}^i(\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\})^a|_{\widetilde{X}}) = \mathrm{Fil}^i\,\mathcal{O}\mathbb{A}^a_{\mathrm{cris}}|_{\widetilde{X}}.$$

□

Corollary 2.14. *Keep the notation above. There are natural filtered isomorphisms*

$$\mathbb{B}^+_{\mathrm{cris}}|_{\widetilde{X}}\{\langle u_1,\dots,u_d\rangle\} \xrightarrow{\sim} \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}|_{\widetilde{X}}, \quad \textit{and} \quad \mathbb{B}_{\mathrm{cris}}|_{\widetilde{X}}\{\langle u_1,\dots,u_d\rangle\} \xrightarrow{\sim} \mathcal{O}\mathbb{B}_{\mathrm{cris}}|_{\widetilde{X}}$$

both sending u_i to $T_i \otimes 1 - 1 \otimes [T_i^\flat]$.

Corollary 2.15. *Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$. Then the morphism of multiplication by t on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}|_{X_{\overline{k}}}$ is almost injective. In particular $t \in \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}|_{X_{\overline{k}}}$ is not a zero-divisor and $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \subset \mathcal{O}\mathbb{B}_{\mathrm{cris}}$.*

Proof. This is a local question on X. Hence we may and do assume there is an étale morphism $X \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1},\dots,T_d^{\pm 1}\})$. Thus our corollary results from Proposition 2.12 and Corollary 2.6 (1). □

An important feature of $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ is that it has an $\mathbb{A}_{\mathrm{cris}}$-linear connection on it. To see this, set $\Omega^{1,\mathrm{ur}+}_{X/k} := w^{-1}\Omega^1_{\mathcal{X}_{\text{ét}}/\mathcal{O}_k}$, which is locally free of finite rank over $\mathcal{O}_X^{\mathrm{ur}+}$. Let

$$\Omega^{i,\mathrm{ur}+}_{X/k} := \wedge^i_{\mathcal{O}_X^{\mathrm{ur}+}}\Omega^{1,\mathrm{ur}+}_{X/k}, \quad \text{and} \quad \Omega^{i,\mathrm{ur}}_{X/k} := \Omega^{1,\mathrm{ur}+}_{X/k}[1/p] \quad \forall i \geq 0.$$

Then $\mathcal{O}\mathbb{A}_{\mathrm{inf}}$ admits a unique $\mathbb{A}_{\mathrm{inf}}$-linear connection

$$\nabla\colon \mathcal{O}\mathbb{A}_{\mathrm{inf}} \longrightarrow \mathcal{O}\mathbb{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_X^{\mathrm{ur}+}} \Omega^{1,\mathrm{ur}+}_{X/k}$$

induced from the usual one on $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$. This connection extends uniquely to $\mathcal{O}\mathbb{A}^0_{\mathrm{cris}}$ and to its completion

$$\nabla\colon \mathcal{O}\mathbb{A}_{\mathrm{cris}} \longrightarrow \mathcal{O}\mathbb{A}_{\mathrm{cris}} \otimes_{\mathcal{O}_X^{\mathrm{ur}+}} \Omega^{1,\mathrm{ur}+}_{X/k}$$

This extension is $\mathbb{A}_{\mathrm{cris}}$-linear. Inverting p (resp. t), we get also a $\mathbb{B}^+_{\mathrm{cris}}$-linear (resp. $\mathbb{B}_{\mathrm{cris}}$-linear) connection on $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}$ (resp. on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$):

$$\nabla\colon \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \longrightarrow \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \Omega^{1,\mathrm{ur}}_{X/k}, \quad \text{and} \quad \nabla\colon \mathcal{O}\mathbb{B}_{\mathrm{cris}} \longrightarrow \mathcal{O}\mathbb{B}_{\mathrm{cris}} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \Omega^{1,\mathrm{ur}}_{X/k}.$$

From Proposition 2.12, we obtain

Corollary 2.16 (Crystalline Poincaré lemma)**.** *Let $\mathcal{X}$ be a smooth formal scheme of dimension d over $\mathcal{O}_k$. Then there is an exact sequence of pro-étale sheaves:*

$$0 \to \mathbb{B}^+_{\mathrm{cris}} \to \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \xrightarrow{\nabla} \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \otimes_{\mathcal{O}^{\mathrm{ur}}_X} \Omega^{1,\mathrm{ur}}_{X/k} \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \otimes_{\mathcal{O}^{\mathrm{ur}}_X} \Omega^{d,\mathrm{ur}}_{X/k} \to 0,$$

which is strictly exact with respect to the filtration giving $\Omega^{i,\mathrm{ur}}_{X/k}$ degree i. In particular, the connection ∇ is integrable and satisfies Griffiths transversality with respect to the filtration on $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}$, i.e. $\nabla(\mathrm{Fil}^i \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}) \subset \mathrm{Fil}^{i-1} \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \otimes_{\mathcal{O}^{\mathrm{ur}}_X} \Omega^{1,\mathrm{ur}}_{X/k}$.

Proof. We just need to establish the almost version of our corollary for $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. This is a local question on X, hence we may and do assume there is an étale morphism from $\mathcal{X}$ to $\mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$. Then under the almost isomorphism (2.2.4) of Proposition 2.12, $\mathrm{Fil}^i \mathcal{O}\mathbb{A}^a_{\mathrm{cris}}|_{\widetilde{X}}$ is the p-adic completion of

$$\sum_{i_1, \dots i_d \geq 0} \mathrm{Fil}^{i-(i_0+\dots+i_d)} \mathbb{A}^a_{\mathrm{cris}}|_{\widetilde{X}} u_1^{[i_1]} \cdots u_d^{[i_d]}$$

with $T_i \otimes 1 - 1 \otimes [T_i^\flat]$ sent to u_i. Moreover $\nabla(u_i^{[n]}) = u_i^{[n-1]} \otimes dT_i$ for any $i, n \geq 1$, since the connection ∇ on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ is $\mathbb{A}_{\mathrm{cris}}$-linear. The strict exactness and Griffiths transversality then follow. □

Using Proposition 2.12, we can also establish an analogous acyclicity result for $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ as in Lemma 2.5. Let $\mathcal{U} = \mathrm{Spf}(R^+)$ be an affine subset of $\mathcal{X}$ admitting an étale morphism to $\mathcal{T}^d = \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$. Let U be the generic fiber, and set $\widetilde{U} := U \times_{\mathbb{T}^d} \widetilde{\mathbb{T}}^d$. Let V be an affinoid perfectoid of $X_{\mathrm{pro\acute{e}t}}$ lying above $\widetilde{U}_{\overline{k}}$. Write $\widehat{V} = \mathrm{Spa}(S, S^+)$. Let $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+)$ be the p-adic completion of the PD-envelope $\mathcal{O}\mathbb{A}^0_{\mathrm{cris}}(S, S^+)$ of $R^+ \otimes_{\mathcal{O}_k} W(S^{\flat +})$ with respect to the kernel of the following morphism induced from $\theta_{(S,S^+)}$ by extending scalars to R^+:

$$\theta_{R^+} \colon R^+ \otimes_{\mathcal{O}_k} W(S^{\flat +}) \longrightarrow S^+.$$

Set $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}(S, S^+) := \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+)[1/p]$, $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S^+) := \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}(S, S^+)[1/t]$. For $r \in \mathbb{Z}$, define $\mathrm{Fil}^r \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+)$ to be the closure inside $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+)$ for the p-adic topology of the r-th PD-ideal of $\mathcal{O}\mathbb{A}^0_{\mathrm{cris}}(S, S^+)$. Finally, set $\mathrm{Fil}^r \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}(S, S^+) := \mathrm{Fil}^r \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+)[1/p]$ and $\mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S^+) := \sum_{s \in \mathbb{Z}} t^{-s} \mathrm{Fil}^{r+s} \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}(S, S^+)$.

Lemma 2.17. *Keep the notation above. For any $\mathcal{F} \in \{\mathcal{O}\mathbb{A}_{\mathrm{cris}}, \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}, \mathcal{O}\mathbb{B}_{\mathrm{cris}}\}$ and any $r \in \mathbb{Z}$, there exists a natural almost isomorphism $\mathrm{Fil}^r \mathcal{F}(S, S^+) \xrightarrow{\approx} \mathrm{Fil}^r \mathcal{F}(V)$. Moreover, $H^i(V, \mathrm{Fil}^r \mathcal{F})^a = 0$ whenever $i > 0$.*

Proof. Consider the following morphism, again denoted by α

$$\alpha \colon \mathbb{A}_{\mathrm{cris}}(S, S^+)\{\langle u_1, \dots, u_d \rangle\} \longrightarrow \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+), \quad u_i \mapsto T_i \otimes 1 - 1 \otimes [T_i^\flat].$$

One checks similarly as in Proposition 2.12 that this morphism is an isomorphism. Moreover, if we define $\mathrm{Fil}^r(\mathbb{A}_{\mathrm{cris}}(S, S^+)\{\langle u_1, \dots, u_d \rangle\})$ to be the p-adic completion of

$$\sum_{i_1, \dots, i_d \geq 0} \mathrm{Fil}^{r-(i_1+\dots+i_d)} \mathbb{A}^0_{\mathrm{cris}}(S, S^+) u_1^{[i_1]} \cdots u_d^{[i_d]} \subset \mathbb{A}_{\mathrm{cris}}(S, S^+)\{\langle u_1, \dots, u_d \rangle\},$$

then α respects the filtrations of both sides. The first part of our lemma can be deduced from the following commutative diagram

$$\begin{array}{ccc} \mathrm{Fil}^r(\mathbb{A}_{\mathrm{cris}}(S,S^+)\{\langle u_1,\dots,u_d\rangle\}) & \xrightarrow[\alpha]{\approx} & \mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S,S^+) \\ \approx\downarrow & & \downarrow \\ \mathrm{Fil}^r(\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\})(V) & \xrightarrow[\alpha]{\approx} & \mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}(V) \end{array}$$

where the left vertical almost isomorphism comes from Lemma 2.11. Using the last part of Lemma 2.11 and the almost isomorphism $\mathrm{Fil}^r(\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\})|_{\widetilde{X}} \xrightarrow{\approx} \mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}|_{\widetilde{X}}$ of Proposition 2.12, we deduce $H^i(V,\mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}})^a = 0$ for $i > 0$. Inverting respectively p and t, we obtain the statements for $\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}$ and for $\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris}}$. □

2.3. **Frobenius on crystalline period sheaves.** We keep the notations in the previous §. So k is *absolutely unramified* and $\mathcal{X}$ is a smooth formal scheme of dimension d over $\mathcal{O}_k$. We want to endow Frobenius endomorphisms on the crystalline period sheaves.

On $\mathbb{A}_{\mathrm{inf}} = W(\mathcal{O}_X^{\flat+})$, we have the Frobenius map

$$\varphi\colon \mathbb{A}_{\mathrm{inf}} \longrightarrow \mathbb{A}_{\mathrm{inf}}, \quad (a_0,a_1,\dots,a_n,\dots) \mapsto (a_0^p,a_1^p,\dots,a_n^p,\dots).$$

Then for any $a \in \mathbb{A}_{\mathrm{inf}}$, we have $\varphi(a) \equiv a^p \bmod p$. Thus, $\varphi(\xi) = \xi^p + p\cdot b$ with $b \in \mathbb{A}_{\mathrm{inf}}|_{X_{\overline{k}}}$. In particular $\varphi(\xi) \in \mathbb{A}^0_{\mathrm{cris}}|_{X_{\overline{k}}}$ has all divided powers. As a consequence we obtain a Frobenius φ on $\mathbb{A}^0_{\mathrm{cris}}$ extending that on $\mathbb{A}_{\mathrm{inf}}$. By continuity, φ extends to $\mathbb{A}_{\mathrm{cris}}$ and $\mathbb{B}^+_{\mathrm{cris}}$. Note that $\varphi(t) = \log([\epsilon^p]) = pt$. Consequently φ is extended to $\mathbb{B}_{\mathrm{cris}}$ by setting $\varphi(\frac{1}{t}) = \frac{1}{pt}$.

To endow a Frobenius on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, we first assume that the Frobenius of $\mathcal{X}_0 = \mathcal{X}\otimes_{\mathcal{O}_k}\kappa$ lifts to a morphism σ on $\mathcal{X}$, which is compatible with the Frobenius on $\mathcal{O}_k$. Then for $\mathcal{Y} \in \mathcal{X}_{\text{ét}}$, consider the following diagram:

$$\begin{array}{ccccccc} \mathcal{Y}_\kappa & \hookrightarrow & & & \mathcal{Y} & & \\ \uparrow & & \nearrow_{\exists\sigma_{\mathcal{Y}}} & & & & \\ \mathcal{Y}_\kappa & \hookrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{X} & \xrightarrow{\sigma} & \mathcal{X}. \end{array} \qquad \begin{array}{l} \text{absolute Frobenius} \\ \\ \\ \end{array} \qquad \Big\downarrow \text{étale}$$

As the right vertical map is étale, there is a unique dotted morphism above making the diagram commute. When $\mathcal{Y}$ varies in $\mathcal{X}_{\text{ét}}$, the $\sigma_{\mathcal{Y}}$'s give rise to a σ-semilinear endomorphism on $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$ whence a σ-semilinear endomorphism φ on $\mathcal{O}_X^{\mathrm{ur}+}$.

Remark 2.18. In general $\mathcal{X}$ does not admit a lifting of Frobenius. But as $\mathcal{X}$ is smooth over $\mathcal{O}_k$, for each open subset $\mathcal{U} \subset \mathcal{X}$ admitting an étale morphism $\mathcal{U} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm1},\dots,T_d^{\pm1}\})$, a similar argument as above shows that there exists a unique lifting of Frobenius on $\mathcal{U}$ mapping T_i to T_i^p.

We deduce from above a Frobenius on $\mathcal{O}\mathbb{A}_{\mathrm{inf}} = \mathcal{O}_X^{\mathrm{ur}+}\otimes_{\mathcal{O}_k}\mathbb{A}_{\mathrm{inf}}$ given by $\varphi\otimes\varphi$. Abusing notation, we will denote it again by φ. A similar argument as in the previous paragraphs shows that φ extends to $\mathcal{O}\mathbb{A}^0_{\mathrm{cris}}$, hence to $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ by continuity, and finally to $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}$ and $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$. Moreover, under the almost isomorphism (2.2.4), the Frobenius on $\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\} \xrightarrow{\approx} \mathcal{O}\mathbb{A}_{\mathrm{cris}}$ sends u_i to $\varphi(u_i) = \sigma(T_i) - [T_i^\flat]^p$.

Lemma 2.19. *Assume as above that the Frobenius of $\mathcal{X}_0 = \mathcal{X} \otimes_{\mathcal{O}_k} \kappa$ lifts to a morphism σ on $\mathcal{X}$ compatible with the Frobenius on $\mathcal{O}_k$. The Frobenius φ on $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}$ is horizontal with respect to the connection $\nabla\colon \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \to \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \otimes \Omega^{1,\mathrm{ur}}_{X/k}$.*

Proof. We just need to check $\nabla \circ \varphi = (\varphi \otimes d\sigma) \circ \nabla$ on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ in the almost sense. It is enough to do this locally. Thus we may assume there exists an étale morphism $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\})$. Recall the almost isomorphism (2.2.4). By $\mathbb{A}_{\mathrm{cris}}$-linearity it reduces to check the equality on the $U_i^{[n]}$. We have

$$(\nabla \circ \varphi)(u_i^{[n]}) \quad = \quad \nabla(\varphi(u_i)^{[n]}) = \varphi(u_i)^{[n-1]}\nabla(\varphi(u_i))$$

Meanwhile, note that $\varphi(u_i) - \sigma(T_i) = -[T_i^\flat]^p \in \mathbb{A}_{\mathrm{cris}}$, hence $\nabla(\varphi(u_i)) = d\sigma(T_i)$. Thus

$$\begin{aligned} \left((\varphi \otimes d\sigma) \circ \nabla\right)(u_i^{[n]}) &= (\varphi \otimes d\sigma)(u_i^{[n-1]} \otimes dT_i) \\ &= \varphi(u_i^{[n-1]}) \otimes d\sigma(T_i) \\ &= (\nabla \circ \varphi)(u_i^{[n]}), \end{aligned}$$

as desired. □

Clearly, the Frobenius on $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}$ above depends on the initial lifting of Frobenius on $\mathcal{X}$. For different choices of liftings of Frobenius on $\mathcal{X}$, it is possible to compare explicitly the resulting Frobenius endomorphisms on $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}$ with the help of the connection on it, at least when the formal scheme $\mathcal{X}$ is *small*, i.e. when it admits an étale morphism to $\mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\})$.

Lemma 2.20. *Assume there is an étale morphism $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\})$. Let σ_1, σ_2 be two Frobenius liftings on $\mathcal{X}$, and let φ_1 and φ_2 be the induced Frobenius maps on $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}$, respectively. Then for any quasi-compact $U \in X_{\mathrm{pro\acute{e}t}}$, we have the following relation on $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}(U)$:*

$$\varphi_2 = \sum_{(n_1,\ldots,n_d)\in\mathbb{N}^d} \Big(\prod_{i=1}^d (\sigma_2(T_i) - \sigma_1(T_i))^{[n_i]}\Big)(\varphi_1 \circ \Big(\prod_{i=1}^d N_i^{n_i}\Big)) \tag{2.3.1}$$

where the N_i's are the endomorphisms of $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}$ such that $\nabla = \sum_{i=1}^d N_i \otimes dT_i$.

Proof. We only need to check the almost analogue for $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. To simplify the notations, we shall use the multi-index: for $\underline{m} = (m_1, \ldots, u_d) \in \mathbb{N}^d$, set $\underline{N}^{\underline{m}} := \prod_{i=1}^d N_i^{m_i}$ and $|\underline{m}| := \sum_i m_i$. Let us remark first that for any $a \in \mathcal{O}\mathbb{A}_{\mathrm{cris}}(U)$ and any $r \in \mathbb{N}$, $\underline{N}^{\underline{m}}(a) \in p^r \cdot \mathcal{O}\mathbb{A}_{\mathrm{cris}}(U)$ when $|m|$ is sufficiently large. As U is quasi-compact, we may and do assume U is affinoid perfectoid with $\widehat{U} = \mathrm{Spa}(R, R^+)$. As $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ has no p-torsions, up to replacing a by $p \cdot a$, we may and do assume that a is of the form

$$a = \sum_{\underline{m}\in\mathbb{N}^d} b_{\underline{m}} \cdot \underline{u}^{[\underline{m}]}, \quad b_{\underline{m}} \in \mathbb{A}_{\mathrm{cris}}(R, R^+) \text{ and } \lim_{|m|\to\infty} b_{\underline{m}} = 0.$$

Here we have again used the almost isomorphism (2.2.4). An easy calculation shows

$$\underline{N}^{\underline{n}}(a) = \sum_{\underline{m}\geq\underline{n}} b_{\underline{m}}\underline{u}^{[\underline{m}-\underline{n}]} = \sum_{\underline{m}\in\mathbb{N}^d} b_{\underline{m}+\underline{n}}\underline{u}^{[\underline{m}]}.$$

As the coefficient $b_{\underline{m}}$ tends to 0 for the p-adic topology when $|m|$ goes to infinity, it follows that $\underline{N}^{\underline{n}}(a) \in p^r \cdot \mathcal{O}\mathbb{A}_{\mathrm{cris}}(U)$ when $|n| \gg 0$, as desired. Meanwhile, note that

$\sigma_2(T_i)-\sigma_1(T_i)\in p\mathcal{O}_X^{\mathrm{ur}+}(U)$, hence their divided powers lie in $\mathcal{O}_X^{\mathrm{ur}+}(U)$. Therefore the series of the righthand side of (2.3.1) applied to a converges.

It remains to verify the formula (2.3.1) for any $a\in\mathcal{O}\mathbb{A}_{\mathrm{cris}}(U)$. Assume again U is affinoid perfectoid. Since both sides of (2.3.1) are semilinear with respect to the Frobenius of $\mathbb{A}_{\mathrm{cris}}$, it suffices to check the equality for $a=\underline{u}^{[\underline{m}]}$. In fact, we have

$$\begin{aligned}
&\Big(\textstyle\sum_{(n_1,\dots,n_d)\in\mathbb{N}^d}(\prod_{i=1}^d(\sigma_2(T_i)-\sigma_1(T_i))^{[n_i]})(\varphi_1\circ(\prod_{i=1}^d N_i^{n_i}))\Big)\,(u^{[\underline{m}]})\\
=\;&\textstyle\sum_{\underline{n}\in\mathbb{N}^d}(\sigma_2(\underline{T})-\sigma_1(\underline{T}))^{[\underline{n}]}(\varphi_1(\underline{N}^{\underline{n}}(\underline{u}^{[m]})))\\
=\;&\textstyle\sum_{\underline{n}\in\mathbb{N}^d}(\varphi_2(\underline{u})-\varphi_1(\underline{u}))^{[\underline{n}]}(\varphi_1(\underline{N}^{\underline{n}}(\underline{u}^{[m]})))\\
=\;&\textstyle\sum_{\underline{n}\in\mathbb{N}^d\text{ s.t. }\underline{n}\le\underline{m}}(\varphi_2(\underline{u})-\varphi_1(\underline{u}))^{[\underline{n}]}\cdot\varphi_1(\underline{u})^{[\underline{m}-\underline{n}]}\\
=\;&(\varphi_2(\underline{u})-\varphi_1(\underline{u})+\varphi_1(\underline{u}))^{[\underline{m}]}\\
=\;&\varphi_2(u^{[\underline{m}]}).
\end{aligned}$$

This finishes the proof. □

2.4. **Comparison with de Rham period sheaves.** Let X be a locally noetherian adic space over $\mathrm{Spa}(k,\mathcal{O}_k)$ and recall the map (2.1.2). Set $\mathbb{B}_{\mathrm{dR}}^+=\varprojlim\mathbb{B}_{\mathrm{inf}}/(\ker\theta)^n$, $\mathbb{B}_{\mathrm{dR}}=\mathbb{B}_{\mathrm{dR}}^+[1/t]$. For $r\in\mathbb{Z}$, let $\mathrm{Fil}^r\,\mathbb{B}_{\mathrm{dR}}=(\ker\theta)^r\mathbb{B}_{\mathrm{dR}}^+$. By its very definition, the filtration on $\mathbb{B}_{\mathrm{dR}}$ is decreasing, separated and exhaustive. Similarly, with $\mathcal{O}\mathbb{B}_{\mathrm{inf}}$ in place of $\mathbb{B}_{\mathrm{inf}}$, we define $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$. Define $\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+=(\ker\theta_X)^r\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ and $\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}=\sum_{s\in\mathbb{Z}}t^{-s}\,\mathrm{Fil}^{r+s}\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$. The filtration on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ is decreasing, separated and exhaustive. Moreover, as in [Bri, 5.2.8, 5.2.9], one can also show that

$$\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+\cap\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}=\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+,$$

which in particular implies that the filtration on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ is also (decreasing and) separated and exhaustive.

In the rest of this subsection, assume $k/\mathbb{Q}_p$ is absolutely unramified.

Lemma 2.21. *Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$.*

(1) *There are natural injective morphisms*

$$\mathbb{B}_{\mathrm{cris}}^+\hookrightarrow\mathbb{B}_{\mathrm{dR}}^+,\quad \mathcal{O}\mathbb{B}_{\mathrm{cris}}^+\hookrightarrow\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+.$$

In the following, we will view $\mathbb{B}_{\mathrm{cris}}^+$ (resp. $\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+$) as a subring of $\mathbb{B}_{\mathrm{dR}}^+$ (resp. of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$).

(2) *For any integer $i\ge 0$, one has*

$$\mathrm{Fil}^i\,\mathbb{B}_{\mathrm{cris}}^+=\mathrm{Fil}^i\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+\cap\mathbb{B}_{\mathrm{cris}}^+\quad \mathrm{Fil}^i\,\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+=\mathrm{Fil}^i\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+\cap\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+.$$

In particular, the filtration $\{\mathrm{Fil}^i\,\mathbb{B}_{\mathrm{cris}}^+ : i\in\mathbb{Z}\}$ (resp. $\{\mathrm{Fil}^i\,\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+ : i\in\mathbb{Z}\}$) on $\mathbb{B}_{\mathrm{cris}}^+$ (resp. on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+$) is decreasing, separated and exhaustive.

(3) *For $i\ge 0$, the following canonical morphisms are isomorphisms:*

$$\mathrm{gr}^i\mathbb{B}_{\mathrm{cris}}^+\xrightarrow{\sim}\mathrm{gr}^i\mathbb{B}_{\mathrm{dR}}^+,\quad \mathrm{gr}^i\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+\xrightarrow{\sim}\mathrm{gr}^i\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+.$$

Proof. (1) We will first construct the two natural morphisms claimed in our lemma. Recall that we have the natural mophism $\theta\colon W(\mathcal{O}_X^{\flat+})\to\widehat{\mathcal{O}_X^+}$, and that $\mathbb{B}_{\mathrm{dR}}^+$ is a sheaf of $\mathbb{Q}_p$-algebras. In particular, under the natural morphism

$$\mathbb{A}_{\mathrm{inf}}=W(\mathcal{O}_X^{\flat+})\longrightarrow W(\mathcal{O}_X^{\flat+})[1/p]\longrightarrow\mathbb{B}_{\mathrm{dR}}^+,$$

the ideal of $\mathbb{B}_{\mathrm{dR}}^+$ generated by the image of $\ker(\theta)$ has a PD-structure. Therefore, the composed morphism above extends to a unique morphism $\mathbb{A}_{\mathrm{cris}}^0\to\mathbb{B}_{\mathrm{dR}}^+$. On

the other hand, for each n, the quotient $\mathbb{B}^+_{\mathrm{dR}}/\operatorname{Fil}^n \mathbb{B}^+_{\mathrm{dR}}$ is p-adically complete, the composite

$$\mathbb{A}^0_{\mathrm{cris}} \longrightarrow \mathbb{B}^+_{\mathrm{dR}} \longrightarrow \mathbb{B}^+_{\mathrm{dR}}/\operatorname{Fil}^n \mathbb{B}^+_{\mathrm{dR}}$$

factors through the p-adic completion $\mathbb{A}_{\mathrm{cris}}$ of $\mathbb{A}^0_{\mathrm{cris}}$, giving a morphism $\mathbb{A}_{\mathrm{cris}} \to \mathbb{B}^+_{\mathrm{dR}}/\operatorname{Fil}^n \mathbb{B}^+_{\mathrm{dR}}$. On passing to limit with respect to n, we get a morphism $\mathbb{A}_{\mathrm{cris}} \to \mathbb{B}^+_{\mathrm{dR}}$, whence the required natural morphism $\mathbb{B}^+_{\mathrm{cris}} \to \mathbb{B}^+_{\mathrm{dR}}$ by inverting $p \in \mathbb{A}_{\mathrm{cris}}$. The natural morphism from $\mathcal{O}\mathbb{B}_{\mathrm{cris}} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$ is constructed in a similar way.

The two morphisms constructed above are compatible with the isomorphisms in Corollary 2.14 and its de Rham analogue [Sch13, Proposition 6.10]. To finish the proof of (1), we only need to show the morphism $\mathbb{B}^+_{\mathrm{cris}} \to \mathbb{B}^+_{\mathrm{dR}}$ constructed above is injective. Let $\overline{k}$ be an algebraic closure of k. We only need to check that for any affinoid perfectoid U lying above $X_{\overline{k}}$, the induced map $\mathbb{B}^+_{\mathrm{cris}}(U) \to \mathbb{B}^+_{\mathrm{dR}}(U)$ is injective. Write $\widehat{U} = \operatorname{Spa}(R, R^+)$. Using the identification in Lemma 2.5 together with its de Rham analogue ([Sch13, Theorem 6.5]), we are reduced to showing that the map $h\colon \mathbb{B}^+_{\mathrm{cris}}(R, R^+) \to \mathbb{B}^+_{\mathrm{dR}}(R, R^+)$ is injective, where h is constructed analogously as the natural map $\mathbb{B}^+_{\mathrm{cris}} \to \mathbb{B}^+_{\mathrm{dR}}$ above. This is proved in [Bri, Proposition 6.2.1], and we reproduce the proof here for the sake of completeness.

As $\mathbb{B}^+_{\mathrm{cris}}(R, R^+) = \mathbb{A}_{\mathrm{cris}}(R, R^+)[1/p]$, we only need to show the following composite h' is injective:

$$h'\colon\ \mathbb{A}_{\mathrm{cris}}(R, R^+) \lhook\joinrel\longrightarrow \mathbb{B}^+_{\mathrm{cris}}(R, R^+) \xrightarrow{h} \mathbb{B}^+_{\mathrm{dR}}(R, R^+)\ .$$

For $n \geq 0$, let $J^{[n]} \subset \mathbb{A}_{\mathrm{cris}}(R, R^+)$ denote the closure for the p-adic topology of the ideal generated by $\xi^{[i]} = \xi^i/i!$ $(i \geq n)$, with $\xi := [p^\flat] - p \in \mathbb{A}_{\mathrm{cris}}$ generating the kernel of θ. We claim first that ${h'}^{-1}(\operatorname{Fil}^n \mathbb{B}^+_{\mathrm{dR}}(R, R^+)) = J^{[n]}$. Clearly only the inclusion ${h'}^{-1}(\operatorname{Fil}^n \mathbb{B}^+_{\mathrm{dR}}(R, R^+)) \subset J^{[n]}$ requires verification. The case $n = 1$ is obvious from the definition. For general $n \geq 2$, we will proceed by induction. Let $x \in {h'}^{-1}(\operatorname{Fil}^n \mathbb{B}^+_{\mathrm{dR}}(R, R^+))$. By induction hypothesis, $x \in J^{[n-1]}$. Write $x = x_0\xi^{[n-1]} + x_1$ with $x_0 \in W(R^{\flat+})$ and $x_1 \in J^{[n]}$. Under the identification

$$\frac{\operatorname{Fil}^{n-1} \mathbb{B}^+_{\mathrm{dR}}(R, R^+)}{\operatorname{Fil}^n \mathbb{B}^+_{\mathrm{dR}}(R, R^+)} \xrightarrow{\sim} R \cdot \xi^{n-1}, \quad \overline{a\xi^{n-1}} \mapsto \theta(a)\xi^{n-1},$$

the class of $h'(x) \in \operatorname{Fil}^{n-1} \mathbb{B}^+_{\mathrm{dR}}(R, R^+)$ corresponds to $\theta(x_0)\xi^{n-1}/(n-1)!$. As $h'(x)$ lies in $\operatorname{Fil}^n \mathbb{B}^+_{\mathrm{dR}}(R, R^+)$, it follows that $\theta(x_0) = 0$. So $x_0 \in J$ and thus $x = x_0\xi^{[n-1]} + x_1 \in J^{[n]}$. Consequently $J^{[n]} = h^{-1}(\operatorname{Fil}^n \mathbb{B}^+_{\mathrm{dR}}(R, R^+))$.

To conclude the proof of injectivity of h', it remains to show $\bigcap_n J^{[n]} = 0$ in $\mathbb{A}_{\mathrm{cris}}(R, R^+)$. Recall from the proof of Lemma 2.13

$$\mathbb{A}_{\mathrm{cris}}(R, R^+)/(p) \simeq \frac{(R^{\flat+}/(p^\flat)^p)[\delta_0, \delta_1, \ldots]}{(\delta_0^p, \delta_1^p, \ldots)}$$

for δ_i the image of $\gamma^{i+1}(\xi)$, where $\gamma(x) = x^p/p$. Under this isomorphism, for any $n \geq 1$, the image of $J^{[p^n]}$ in $\mathbb{A}_{\mathrm{cris}}(R, R^+)/(p)$ is generated by $(\delta_i)_{i \geq n-1}$. This implies that $\bigcap_{i \in \mathbb{Z}_{\geq 1}} J^{[i]} \subset p\mathbb{A}_{\mathrm{cris}}(R, R^+)$. Take $x = px' \in \bigcap_{i \geq 1} J^{[i]}$ with $x' \in \mathbb{A}_{\mathrm{cris}}(R, R^+)$. Then $h'(x') \in \bigcap_i \operatorname{Fil}^i \mathbb{B}^+_{\mathrm{dR}}(R, R^+)$ as $p \in \mathbb{B}^+_{\mathrm{cris}}$ is invertible. Thus $x' \in \bigcap_i {h'}^{-1}(\operatorname{Fil}^i \mathbb{B}_{\mathrm{dR}}(R, R^+)) = \bigcap_i J^{[i]} \subset p\mathbb{A}_{\mathrm{cris}}(R, R^+)$. Hence $x = px' \in p^2\mathbb{A}_{\mathrm{cris}}(R, R^+)$. Repeating this argument, one sees that $\bigcap_{i \in \mathbb{Z}_{\geq 1}} J^{[i]} \subset p^n\mathbb{A}_{\mathrm{cris}}$ for any $n \geq 1$. So $\bigcap_{i \in \mathbb{Z}_{\geq 1}} J^{[i]} = \{0\}$ as $\mathbb{A}_{\mathrm{cris}}$ is p-adically separated. Thus we have

constructed a natural injection $\mathbb{B}^+_{\mathrm{cris}} \hookrightarrow \mathbb{B}^+_{\mathrm{dR}}$. The claim $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \hookrightarrow \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$ follows from this: this is a local question on $X_{\mathrm{pro\acute{e}t}}$, thus we may assume that our formal scheme $\mathcal{X}$ admits an étale map to $\mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$, and then conclude by Corollary 2.14 and its de Rham analogue [Sch13, Proposition 6.10].

(2) To check $\mathrm{Fil}^i \mathbb{B}^+_{\mathrm{cris}} = \mathbb{B}^+_{\mathrm{cris}} \bigcap \mathrm{Fil}^i \mathbb{B}^+_{\mathrm{dR}}$, it suffices to show $\mathrm{Fil}^i \mathbb{B}^+_{\mathrm{cris}}(U) = \mathbb{B}^+_{\mathrm{cris}}(U) \bigcap \mathrm{Fil}^i \mathbb{B}^+_{\mathrm{dR}}(U)$ for any affinoid perfectoid U above $X_{\overline{k}}$. Write $\widehat{U} = \mathrm{Spa}(R, R^+)$. Under the identifications $\mathbb{B}^+_{\mathrm{dR}}(R, R^+) \simeq \mathbb{B}^+_{\mathrm{dR}}(U)$ and $\mathbb{B}^+_{\mathrm{cris}}(R, R^+) \simeq \mathbb{B}^+_{\mathrm{cris}}(U)$, we have $\mathrm{Fil}^i \mathbb{B}^+_{\mathrm{cris}}(U) = \mathrm{Fil}^i \mathbb{B}^+_{\mathrm{cris}}(R, R^+)$, and $\mathrm{Fil}^i \mathbb{B}^+_{\mathrm{dR}}(U) = \mathrm{Fil}^i \mathbb{B}^+_{\mathrm{dR}}(R, R^+)$ (Lemma 2.5). On the other hand, from the proof of (1), we have $J^{[i]} = h^{-1}(\mathrm{Fil}^i \mathbb{B}_{\mathrm{dR}}(R, R^+)) = \mathbb{A}_{\mathrm{cris}}(R, R^+) \bigcap \mathrm{Fil}^i \mathbb{B}_{\mathrm{dR}}(R, R^+)$, from which we deduce

$$\mathrm{Fil}^i \mathbb{B}^+_{\mathrm{cris}}(R, R^+) = \mathbb{B}^+_{\mathrm{cris}}(R, R^+) \bigcap \mathrm{Fil}^i \mathbb{B}^+_{\mathrm{dR}}(R, R^+),$$

as desired.

To show $\mathrm{Fil}^i \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} = \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \bigcap \mathrm{Fil}^i \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$, we only need to check $\mathrm{Fil}^i \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \supset \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \bigcap \mathrm{Fil}^i \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$. For this we may again assume the formal scheme $\mathcal{X}$ admits an étale map to $\mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$. Then we can conclude by Corollary 2.14 and its de Rham analogue.

(3) From (2), we know that the canonical morphism $\mathrm{gr}^i \mathbb{B}^+_{\mathrm{cris}} \to \mathrm{gr}^i \mathbb{B}^+_{\mathrm{dR}}$ is injective. Furthermore we have the identification

$$\mathrm{gr}^i \mathbb{B}^+_{\mathrm{dR}}|_{X_{\overline{k}}} = \frac{\mathrm{Fil}^i \mathbb{B}^+_{\mathrm{dR}}|_{X_{\overline{k}}}}{\mathrm{Fil}^{i+1} \mathbb{B}^+_{\mathrm{dR}}|_{X_{\overline{k}}}} \xrightarrow{\sim} \widehat{\mathcal{O}_X}|_{X_{\overline{k}}} \cdot \xi^i, \quad \overline{a\xi^i} \mapsto \theta(a)\xi^i.$$

As $\xi^i = i!\xi^{[i]} \in \mathrm{Fil}^i \mathbb{B}^+_{\mathrm{cris}}$, the injection $\mathrm{gr}^i \mathbb{B}^+_{\mathrm{cris}} \hookrightarrow \mathrm{gr}^i \mathbb{B}^+_{\mathrm{dR}}$ is also surjective. Thus $\mathrm{gr}^i \mathbb{B}^+_{\mathrm{cris}} \xrightarrow{\sim} \mathrm{gr}^i \mathbb{B}^+_{\mathrm{dR}}$. Using Corollary 2.14 and its de Rham analogue, we see that the latter isomorphism also implies that $\mathrm{gr}^i \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \xrightarrow{\sim} \mathrm{gr}^i \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$. □

Corollary 2.22. *Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$.*

(1) *There are two natural injections*

$$\mathbb{B}_{\mathrm{cris}} \hookrightarrow \mathbb{B}_{\mathrm{dR}}, \quad \mathcal{O}\mathbb{B}_{\mathrm{cris}} \hookrightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}}.$$

(2) *For any $i \in \mathbb{Z}$, we have $\mathrm{Fil}^i \mathbb{B}_{\mathrm{cris}} = \mathbb{B}_{\mathrm{cris}} \bigcap \mathrm{Fil}^i \mathbb{B}_{\mathrm{dR}}$ and $\mathrm{Fil}^i \mathcal{O}\mathbb{B}_{\mathrm{cris}} = \mathcal{O}\mathbb{B}_{\mathrm{cris}} \bigcap \mathrm{Fil}^i \mathcal{O}\mathbb{B}_{\mathrm{dR}}$.*

Furthermore, $\mathrm{gr}^i \mathbb{B}_{\mathrm{cris}} \xrightarrow{\sim} \mathrm{gr}^i \mathbb{B}_{\mathrm{dR}}$ and $\mathrm{gr}^i \mathcal{O}\mathbb{B}_{\mathrm{cris}} \xrightarrow{\sim} \mathrm{gr}^i \mathcal{O}\mathbb{B}_{\mathrm{dR}}$. In particular, the filtration $\{\mathrm{Fil}^i \mathbb{B}_{\mathrm{cris}}\}_{i \in \mathbb{Z}}$ (resp. $\{\mathrm{Fil}^i \mathcal{O}\mathbb{B}_{\mathrm{cris}}\}_{i \in \mathbb{Z}}$) is decreasing, separated and exhaustive.

Proof. These follow from the previous lemma, by inverting t. □

As a consequence, we can compute the cohomology of the graded quotients $\mathrm{gr}^i \mathcal{F}$ for $\mathcal{F} \in \{\mathbb{B}^+_{\mathrm{cris}}, \mathbb{B}_{\mathrm{cris}}, \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}, \mathcal{O}\mathbb{B}_{\mathrm{cris}}\}$: one just reduces to its de Rham analogue such as [Sch13, Proposition 6.16] etc.

Corollary 2.23. *Let X be a smooth adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$ which admits a smooth formal model $\mathcal{X}$ over $\mathcal{O}_k$ (so that we can define $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$), then*

$$w_* \mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq \mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p].$$

Proof. Let $\nu : X^{\sim}_{\mathrm{pro\acute{e}t}} \to X^{\sim}_{\text{ét}}$ and $\nu' \colon X^{\sim}_{\text{ét}} \to \mathcal{X}^{\sim}_{\text{ét}}$ the natural morphisms of topoi. Then $w = \nu' \circ \nu$. Therefore

$$\mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p] \xrightarrow{\sim} \nu'_* \mathcal{O}_{X_{\text{ét}}} \xrightarrow{\sim} \nu'_* \nu_* \mathcal{O}_X = w_* \mathcal{O}_X.$$

By [Sch13, Corollary 6.19], the natural morphism $\mathcal{O}_{X_{\text{ét}}} \to \nu_*\mathcal{O}\mathbb{B}_{\text{dR}}$ is an isomorphism. Thus, $w_*\mathcal{O}\mathbb{B}_{\text{dR}} = \nu'_*(\nu_*\mathcal{O}\mathbb{B}_{\text{dR}}) \simeq \nu'_*\mathcal{O}_{X_{\text{ét}}} \simeq \mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p]$. On the other hand, we have the injection of $\mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p]$-algebras $w_*\mathcal{O}\mathbb{B}_{\text{cris}} \hookrightarrow w_*\mathcal{O}\mathbb{B}_{\text{dR}}$. Thereby $\mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p] \xrightarrow{\sim} w_*\mathcal{O}\mathbb{B}_{\text{cris}}$. □

3. Crystalline cohomology and pro-étale cohomology

In this section, we assume k is *absolutely unramified*. Let σ denote the Frobenius on $\mathcal{O}_k$ and on k, lifting the Frobenius of the residue field κ. Note that the ideal $(p) \subset \mathcal{O}_k$ is endowed naturally with a PD-structure and $\mathcal{O}_k$ becomes a PD-ring in this way.

3.1. **A reminder on convergent F-isocrystals.** Let $\mathcal{X}_0$ be a κ-scheme of finite type. Let us begin with some general definitions about crystals on the small crystalline site $(\mathcal{X}_0/\mathcal{O}_k)_{\text{cris}}$ endowed with étale topology. For basics of crystals, we refer to [Ber96], [BO]. Recall that a *crystal of $\mathcal{O}_{\mathcal{X}_0/\mathcal{O}_k}$-modules* is an $\mathcal{O}_{\mathcal{X}_0/\mathcal{O}_k}$-module $\mathbb{E}$ on $(\mathcal{X}_0/\mathcal{O}_k)_{\text{cris}}$ such that (i) for any object $(U,T) \in (\mathcal{X}_0/\mathcal{O}_k)_{\text{cris}}$, the restriction $\mathbb{E}_T$ of $\mathbb{E}$ to the étale site of T is a coherent $\mathcal{O}_T$-module; and (ii) for any morphism $u : (U',T') \to (U,T)$ in $(\mathcal{X}_0/\mathcal{O}_k)_{\text{cris}}$, the canonical morphism $u^*\mathbb{E}_T \xrightarrow{\sim} \mathbb{E}_{T'}$ is an isomorphism.

Remark 3.1. Write $\mathcal{X}_0$ the closed fiber of a smooth formal scheme $\mathcal{X}$ over $\mathcal{O}_k$. Then the category of crystals on $(\mathcal{X}_0/\mathcal{O}_k)_{\text{cris}}$ is equivalent to that of coherent $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{M}$ equipped with an integrable and quasi-nilpotent connection $\nabla\colon \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^1_{\mathcal{X}/\mathcal{O}_k}$. Here the connection ∇ is said to be *quasi-nilpotent* if its reduction modulo p is quasi-nilpotent in the sense of [BO, Definition 4.10]. The correspondence between these two categories is given as follows: for $\mathbb{E}$ a crystal on $\mathcal{X}_0/\mathcal{O}_k$, as $\mathcal{X}_0 \hookrightarrow \mathcal{X}$ is a p-adic PD-thickening, we can evalue $\mathbb{E}$ at it: set $\mathbb{E}_{\mathcal{X}} := \varprojlim_n \mathbb{E}_{\mathcal{X}\otimes\mathcal{O}_k/p^n}$. Let $\Delta_1 \hookrightarrow \mathcal{X}\times\mathcal{X}$ be the PD-thickening of order 1 of the diagonal embedding $\mathcal{X} \hookrightarrow \mathcal{X}\times\mathcal{X}$. The two projections $p_i\colon \Delta_1 \to \mathcal{X}$ are PD-morphisms. So we have two isomorphisms $p_i^*\mathbb{E}_{\mathcal{X}} \xrightarrow{\sim} \mathbb{E}_{\Delta_1} := \varprojlim_n \mathbb{E}_{\Delta_1\otimes\mathcal{O}_k/p^n}$, whence a natural isomorphism $p_2^*\mathbb{E}_{\mathcal{X}} \xrightarrow{\sim} p_1^*\mathbb{E}_{\mathcal{X}}$. The latter isomorphism gives a connection $\nabla\colon \mathbb{E}_{\mathcal{X}} \to \mathbb{E}_{\mathcal{X}} \otimes \Omega^1_{\mathcal{X}/\mathcal{O}_k}$ on $\mathbb{E}_{\mathcal{X}}$. Together with a limit argument, that ∇ is integrable and quasi-nilpotent is due to [BO, Theorem 6.6].

The absolute Frobenius $F\colon \mathcal{X}_0 \to \mathcal{X}_0$ is a morphism over the Frobenius σ on $\mathcal{O}_k$, hence it induces a morphism of topoi, still denoted by F:

$$F\colon (\mathcal{X}_0/\mathcal{O}_k)^{\sim}_{\text{cris}} \longrightarrow (\mathcal{X}_0/\mathcal{O}_k)^{\sim}_{\text{cris}}\,.$$

An *F-crystal* on $(\mathcal{X}_0/\mathcal{O}_k)_{\text{cris}}$ is a crystal $\mathbb{E}$ equipped with a morphism $\varphi\colon F^*\mathbb{E} \to \mathbb{E}$ of $\mathcal{O}_{\mathcal{X}_0/\mathcal{O}_k}$-modules, which is nondegenerate, i.e. there exists a map $V : \mathbb{E} \to F^*\mathbb{E}$ of $\mathcal{O}_{\mathcal{X}_0/\mathcal{O}_k}$-modules such that $\varphi V = V\varphi = p^m$ for some $m \in \mathbb{N}$. In the following, we will denote by F-Cris$(\mathcal{X}_0, \mathcal{O}_k)$ the category of F-crystals on $\mathcal{X}_0/\mathcal{O}_k$.

Before discussing isocrystals, let us first observe the following facts.

Remark 3.2. Let X^{rig} be a classical rigid analytic space over k, with associated adic space X. Using [Sch13, Theorem 9.1], one sees that the notion of coherent $\mathcal{O}_{X^{\text{rig}}}$-modules on X^{rig} coincides with that of coherent $\mathcal{O}_{X_{\text{an}}}$-modules on X_{an}, where X_{an} denote the site of open subsets of the adic space X.

Remark 3.3. Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$, with X its generic fiber in the sense of Huber. Let $\mathrm{Coh}(\mathcal{O}_\mathcal{X}[1/p])$ denote the category of coherent $\mathcal{O}_\mathcal{X}[1/p]$-modules on $\mathcal{X}$, or equivalently, the full subcategory of the category of $\mathcal{O}_\mathcal{X}[1/p]$-modules on $\mathcal{X}$ consisting of $\mathcal{O}_\mathcal{X}[1/p]$-modules which are isomorphic to $\mathcal{M}^+[1/p]$ for some coherent sheaf $\mathcal{M}^+$ on $\mathcal{X}$. Denote also $\mathrm{Coh}(\mathcal{O}_{X_{\mathrm{an}}})$ the category of coherent $\mathcal{O}_{X_{\mathrm{an}}}$-modules on X. The analytification functor gives a fully faithful embedding $\mathrm{Coh}(\mathcal{O}_\mathcal{X}[1/p]) \hookrightarrow \mathrm{Coh}(\mathcal{O}_{X_{\mathrm{an}}})$. Moreover, the essential image is stable under taking direct summands (as $\mathcal{O}_{X_{\mathrm{an}}}$-modules). Indeed, let $\mathcal{E}$ be a coherent $\mathcal{O}_{X_{\mathrm{an}}}$-module admitting a coherent formal model $\mathcal{E}^+$ over $\mathcal{X}$, and $\mathcal{E}' \subset \mathcal{E}$ a direct summand. Let $f\colon \mathcal{E} \to \mathcal{E}$ be the idempotent corresponding to $\mathcal{E}'$: write $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$, then f is the composite of the projection from $\mathcal{E}$ to $\mathcal{E}'$ followed by the inclusion $\mathcal{E}' \subset \mathcal{E}$. Therefore, there exists some $n \gg 0$ such that $p^n f$ comes from a morphism $f^+\colon \mathcal{E}^+ \to \mathcal{E}^+$ of $\mathcal{O}_\mathcal{X}$-modules. Then the image of f^+ gives a formal model of $\mathcal{E}'$ over $\mathcal{X}$, as desired.

Let $\mathcal{X}_0$ be a κ-scheme of finite type. Asumme that it can be embedded as a closed subscheme into a smooth formal scheme $\mathcal{P}$. Let P be the associated adic space of $\mathcal{P}$ and $]\mathcal{X}_0[_\mathcal{P} \subset P$ the pre-image of the closed subset $\mathcal{X}_0 \subset \mathcal{P}$ under the specialization map. Following [Ber96, 2.3.2 (i)] (with Remark 3.2 in mind), the *realization on $\mathcal{P}$ of a convergent isocrystal* on $\mathcal{X}_0/\mathcal{O}_k$ is a coherent $\mathcal{O}_{]\mathcal{X}_0[_\mathcal{P}}$-module $\mathcal{E}$ equipped with an integrable and *convergent* connection $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_{]\mathcal{X}_0[_\mathcal{P}}} \Omega^1_{]\mathcal{X}_0[_\mathcal{P}/k}$ (we refer to [Ber96, 2.2.5] for the definition of convergent connections). Being a coherent $\mathcal{O}_{]\mathcal{X}_0[_\mathcal{P}}$-module with integrable connection, $\mathcal{E}$ is locally free of finite rank by [Ber96, 2.2.3 (ii)]. The category of realizations on $\mathcal{P}$ of convergent isocrystals on $\mathcal{X}_0/\mathcal{O}_k$ is denoted by $\mathrm{Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k, \mathcal{P})$, where the morphisms are morphisms of $\mathcal{O}_{]\mathcal{X}_0[_\mathcal{P}}$-modules which commute with connections.

Let $\mathcal{X}_0 \hookrightarrow \mathcal{P}'$ be a second embedding of $\mathcal{X}_0$ into a smooth formal scheme $\mathcal{P}'$ over $\mathcal{O}_k$, and assume there exists a morphism $u\colon \mathcal{P}' \to \mathcal{P}$ of formal schemes inducing identity on $\mathcal{X}_0$. The generic fiber of u gives a morphism of adic spaces $u_k\colon]\mathcal{X}_0[_{\mathcal{P}'} \to]\mathcal{X}_0[_\mathcal{P}$, hence a natural functor

$$u_k^*\colon \mathrm{Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k, \mathcal{P}) \longrightarrow \mathrm{Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k, \mathcal{P}'), \quad (\mathcal{E}, \nabla) \mapsto (u_k^*\mathcal{E}, u_k^*\nabla).$$

By [Ber96, 2.3.2 (i)], the functor u_k^* is an equivalence of categories. Furthermore, for a second morphism $v\colon \mathcal{P}' \to \mathcal{P}$ of formal schemes inducing identity on $\mathcal{X}_0$, the two equivalence u_k^*, v_k^* are canonically isomorphic ([Ber96, 2.2.17 (i)]). Now the category of *convergent isocrystal on* $\mathcal{X}_0/\mathcal{O}_k$, denoted by $\mathrm{Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k)$, is defined as

$$\mathrm{Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k) := 2\text{-}\varinjlim_{\mathcal{P}} \mathrm{Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k, \mathcal{P}),$$

where the limit runs through all smooth formal embedding $\mathcal{X}_0 \hookrightarrow \mathcal{P}$ of $\mathcal{X}_0$.

Remark 3.4. In general, $\mathcal{X}_0$ does not necessarily admit a global formal embedding. In this case, the category of convergent isocrystals on $\mathcal{X}_0/\mathcal{O}_k$ can still be defined by a gluing argument (see [Ber96, 2.3.2(iii)]). But the definition recalled above will be enough for our purpose.

As for the category of crystals on $\mathcal{X}_0/\mathcal{O}_k$, the Frobenius morphism $F\colon \mathcal{X}_0 \to \mathcal{X}_0$ induces a natural functor (see [Ber96, 2.3.7] for the construction):

$$F^*\colon \mathrm{Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k) \longrightarrow \mathrm{Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k).$$

A *convergent F-isocrystal* on $\mathcal{X}_0/\mathcal{O}_k$ is a convergent isocrystal $\mathcal{E}$ on $\mathcal{X}_0/\mathcal{O}_k$ equipped with an isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ in $\mathrm{Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k)$. The category of convergent F-isocrystals on $\mathcal{X}_0/\mathcal{O}_k$ will be denoted in the following by $F\text{-Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k)$.

Remark 3.5. The category $F\text{-Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k)$ has as a full subcategory the isogeny category $F\text{-Cris}(\mathcal{X}_0/\mathcal{O}_k)\otimes\mathbb{Q}$ of F-crystals $\mathbb{E}$ on $(\mathcal{X}_0/\mathcal{O}_k)_{\mathrm{cris}}$. To explain this, assume for simplicity that $\mathcal{X}_0$ is the closed fiber of a smooth formal scheme $\mathcal{X}$ over $\mathcal{O}_k$. So $]\mathcal{X}_0[_{\mathcal{X}} = X$, the generic fiber of $\mathcal{X}$. Let $(\mathcal{M},\nabla)$ be the $\mathcal{O}_{\mathcal{X}}$-module with integrable and quasi-nilpotent connection associated to the F-crystal $\mathbb{E}$ (Remark 3.1). Let $\mathbb{E}^{\mathrm{an}} := \mathcal{M}_\nu$ denote the generic fiber of $\mathcal{M}$, which is a coherent (hence locally free by [Ber96, 2.3.2 (ii)]) $\mathcal{O}_{X_{\mathrm{an}}}$-module equipped with an integrable connection $\nabla^{\mathrm{an}}\colon \mathbb{E}^{\mathrm{an}} \longrightarrow \mathbb{E}^{\mathrm{an}}\otimes\Omega^1_{X_{\mathrm{an}}/k}$, which is nothing but the generic fiber of ∇. Because of the F-crystal structure on $\mathbb{E}$, the connection ∇^{an} is necessarily convergent ([Ber96, 2.4.1]). In this way we obtain an F-isocrystal $\mathbb{E}^{\mathrm{an}}$ on $\mathcal{X}_0/\mathcal{O}_k$, whence a natural functor

$$(-)^{\mathrm{an}}\colon F\text{-Cris}(\mathcal{X}_0/\mathcal{O}_k)\otimes\mathbb{Q} \longrightarrow F\text{-Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k), \quad \mathbb{E}\mapsto\mathbb{E}^{\mathrm{an}}. \tag{3.1.1}$$

By [Ber96, 2.4.2], this analytification functor is fully faithful, and for $\mathcal{E}$ a convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$, there exists an integer $n\geq 0$ and an F-crystal $\mathbb{E}$ such that $\mathcal{E}\xrightarrow{\sim}\mathbb{E}^{\mathrm{an}}(n)$, where for $\mathcal{F}=(\mathcal{F},\nabla,\varphi\colon F^*\mathcal{F}\xrightarrow{\sim}\mathcal{F})$ an F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$, $\mathcal{F}(n)$ denotes the Tate twist of $\mathcal{F}$, given by $(\mathcal{F},\nabla,\frac{\varphi}{p^n}\colon F^*\mathcal{F}\xrightarrow{\sim}\mathcal{F})$ ([Ber96, 2.3.8 (i)]).

Our next goal is to give a more explicit description of the Frobenii on convergent F-isocrystals on $\mathcal{X}_0/\mathcal{O}_k$. From now on, assume for simplicity that $\mathcal{X}_0$ *is the closed fiber of a smooth formal scheme* $\mathcal{X}$ and we shall identify the notion of convergent iscrystals on $\mathcal{X}_0/\mathcal{O}_k$ with its realizations on $\mathcal{X}$. Let X be the generic fiber of $\mathcal{X}$. The proof of the following lemma is obvious.

Lemma 3.6. *Assume that the Frobenius $F\colon\mathcal{X}_0\to\mathcal{X}_0$ can be lifted to a morphism $\sigma\colon\mathcal{X}\to\mathcal{X}$ compatible with the Frobenius on $\mathcal{O}_k$. Still denote by σ the endomorphism on X induced by σ. Then there is an equivalence of categories between*

1. *the category $F\text{-Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k)$ of convergent F-isocrystals on $\mathcal{X}_0/\mathcal{O}_k$; and*
2. *the category $\mathbf{Mod}^{\sigma,\nabla}_{\mathcal{O}_X}$ of $\mathcal{O}_{X_{\mathrm{an}}}$-vector bundles $\mathcal{E}$ equipped with an integrable and convergent connection ∇ and an $\mathcal{O}_{X_{\mathrm{an}}}$-linear horizontal isomorphism $\varphi\colon\sigma^*\mathcal{E}\to\mathcal{E}$.*

Consider two liftings of Froebnius σ_i $(i=1,2)$ on $\mathcal{X}$. By the lemma above, for $i=1,2$, both categories $\mathbf{Mod}^{\sigma_i,\nabla}_{\mathcal{O}_X}$ are naturally equivalent to the category of convergent F-iscrystals on $\mathcal{X}_0/\mathcal{O}_k$:

$$\mathbf{Mod}^{\sigma_1,\nabla}_{\mathcal{O}_X} \xleftarrow{\sim} F\text{-Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k) \xrightarrow{\sim} \mathbf{Mod}^{\sigma_2,\nabla}_{\mathcal{O}_X}\ .$$

Therefore we deduce a natural equivalence of categories

$$F_{\sigma_1,\sigma_2}\colon \mathbf{Mod}^{\sigma_1,\nabla}_{\mathcal{O}_X} \longrightarrow \mathbf{Mod}^{\sigma_2,\nabla}_{\mathcal{O}_X}. \tag{3.1.2}$$

When our formal scheme $\mathcal{X}$ is small, we can explicitly describe this equivalence as follows. Assume there is an étale morphism $\mathcal{X}\to\mathcal{T}^d=\mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm1},\dots,T_d^{\pm1}\})$. So $\Omega^1_{X_{\mathrm{an}}/k}$ is a free $\mathcal{O}_{X_{\mathrm{an}}}$-module with a basis given by dT_i $(i=1,\dots,d)$. In the following, for ∇ a connection on an $\mathcal{O}_{X_{\mathrm{an}}}$-module $\mathcal{E}$, let N_i be the endomorphism of $\mathcal{E}$ (as an abelian sheaf) such that $\nabla=\sum_{i=1}^d N_i\otimes dT_i$.

Lemma 3.7 (see also [Bri] 7.2.3)**.** *Assume $\mathcal{X} = \mathrm{Spf}(A)$ is affine and that there exists an étale morphism $\mathcal{X} \to \mathcal{T}^d$ as above. For $(\mathcal{E}, \nabla, \varphi_1) \in \mathbf{Mod}^{\sigma_1,\nabla}_{\mathcal{O}_X}$, with $(\mathcal{E}, \nabla, \varphi_2)$ the corresponding object of $\mathbf{Mod}^{\sigma_2,\nabla}_{\mathcal{O}_X}$ under the functor F_{σ_1,σ_2}. Then on $\mathcal{E}(X)$ we have*

$$\varphi_2 = \sum_{(n_1,\ldots,n_d)\in\mathbb{N}^d} \left(\prod_{i=1}^{d} (\sigma_2(T_i) - \sigma_1(T_i))^{[n_i]}\right) \left(\varphi_1 \circ \left(\prod_{i=1}^{d} N_i^{n_i}\right)\right). \tag{3.1.3}$$

Furthermore, φ_1 and φ_2 coincide on $\mathcal{E}(X)^{\nabla=0}$.

Proof. To simplify the notations, we shall use the multi-index: so $\underline{N}^{\underline{n}} := \prod_n N_i^{n_i}$ for $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ etc. We observe first that the right hand side of (3.1.3) converges. Indeed, by [Ber96, 2.4.2], there exists some $n \in \mathbb{N}$ such that $\mathcal{E}(-n)$ lies in the essential image of the functor (3.1.1). In particular, this implies that there exists a coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}^+$ equipped with a quasi-nilpotent connection ∇^+ such that $(\mathcal{E}, \nabla)$ is the generic fiber of $(\mathcal{E}^+, \nabla^+)$. In particular, for any $e \in E^+ := \mathcal{E}^+(\mathcal{X})$, $\underline{N}^{\underline{n}}(e) \in p \cdot E^+$ for all but finitely many $\underline{n} \in \mathbb{N}^d$. Furthermore, as σ_1, σ_2 are liftings of Frobenius, $\sigma_2(T_i) - \sigma_1(T_i) \in p \cdot A$. So the divided power $(\sigma_2(T_i) - \sigma_1(T_i))^{[n_i]} \in A$. Thereby the right hand side of (3.1.3) applied to an element of $\mathcal{E}(X) = E^+[1/p]$ converges for the cofinite filter on $\mathbb{N}^d$.

Next we claim that the equality (3.1.3) holds when $\mathcal{E} = \mathcal{O}_{X_{\mathrm{an}}}$ endowed with the natural structure of F-isocrystal. Indeed, necessarily $\varphi_i = \sigma_i$ in this case. Let σ_2' be the endomorphism of A defined by the right hand side of (3.1.3). Then both σ_2, σ_2' are liftings of Frobenius on A, and it is elementary to check that they coincide on the $\mathcal{O}_k$-subalgbra $\mathcal{O}_k\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\}$. As A is étale over $\mathcal{O}_k\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\}$, we deduce $\sigma_2 = \sigma_2'$, giving our claim. By consequence, in the general case, the right hand side of (3.1.3) defines a σ_2-semilinear endomorphism φ_2' on $\mathcal{E}(X)$. In particular, we obtain a morphism of $\mathcal{O}_{X_{\mathrm{an}}}$-modules $\sigma_2^*\mathcal{E} \to \mathcal{E}$, still denoted by φ_2'. One checks that $\varphi_2' \colon \sigma_2^*\mathcal{E} \to \mathcal{E}$ is horizontal, hence it is a morphism of convergent isocrystals.

Now one needs to verify the equality (3.1.3). Consider the fiber product $\mathcal{X} \times \mathcal{X}$ and its generic fiber $X \times X$. Let $U =]\mathcal{X}_0[_{\mathcal{X}\times\mathcal{X}} \subset X \times X$ denote the pre-image of the closed subset $\delta(\mathcal{X}_0) \subset \mathcal{X}_0 \times \mathcal{X}_0$ under the specialization map, and $q_i \colon U \to X$ $(i = 1, 2)$ the two projections. If we endow $\mathcal{X} \times \mathcal{X}$ with the Frobenius $\sigma := \sigma_1 \times \sigma_2$, then $\sigma_i \circ q_i = q_i \circ \sigma$. In particular, the pull-back $q_1^*\mathcal{E}$ is endowed with a morphism

$$q_1^*\varphi_1 \colon \sigma^* q_1^*\mathcal{E} = q_1^*\sigma_1^*\mathcal{E} \longrightarrow q_1^*\mathcal{E}.$$

Similarly, $q_2^*\mathcal{E}$ is a convergent isocrystal on U endowed with a horizontal morphism $q_2^*\varphi_2'$. Consider the following $\mathcal{O}_{U_{\mathrm{an}}}$-linear isomorphism induced by the connection on $\mathcal{E}$:

$$\eta \colon \mathcal{O}_{U_{\mathrm{an}}} \otimes_{\mathcal{O}_{X_{\mathrm{an}}}} \mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{O}_{X_{\mathrm{an}}}} \mathcal{O}_{U_{\mathrm{an}}}, \quad 1 \otimes m \mapsto \sum_{\underline{n}\in\mathbb{N}^d} \frac{\underline{N}^{\underline{n}}(m) \otimes \underline{\tau}^{\underline{n}}}{\underline{n}!},$$

where $\underline{\tau} = (\tau_1, \ldots, \tau_d)$ with $\tau_i = 1 \otimes T_i - T_i \otimes 1$ (note that this formula makes sense as the connection on $\mathcal{E}$ is convergent (see [Ber96, 2.2.5])). By a direct calculation,

one checks the commutativity of the following diagram:

$$\begin{array}{ccc} \sigma^*(\mathcal{O}_{U_{\mathrm{an}}}\otimes\mathcal{E}) & \xrightarrow{q_2^*\varphi_2'} & \mathcal{O}_{U_{\mathrm{an}}}\otimes\mathcal{E} \\ \downarrow{\scriptstyle \sigma^*(\eta)} & & \downarrow{\scriptstyle \eta} \\ \sigma^*(\mathcal{E}\otimes\mathcal{O}_{U_{\mathrm{an}}}) & \xrightarrow[q_1^*\varphi_1]{} & \mathcal{E}\otimes\mathcal{O}_{U_{\mathrm{an}}}. \end{array}$$

Indeed, taking $m\in\mathcal{E}$, we find

$$\sum_{\underline{n}\in\mathbb{N}^d}\frac{\underline{N}^{\underline{n}}(\varphi_2'(m))\otimes\underline{\tau}^{\underline{n}}}{\underline{n}!}=\sum_{\underline{n}\in\mathbb{N}^d}\frac{\varphi_1(\underline{N}^{\underline{n}}(m))\otimes\widehat{\sigma}(\underline{\tau}^{\underline{n}})}{\underline{n}!}.$$

Finally, let $\mathcal{I}=(\tau_1,\dots,\tau_d)$ be the ideal defining the closed immersion $X\hookrightarrow U$. Then modulo the ideal $\mathcal{I}$ in the above equality, the left hand side becomes just $\varphi_2'(m)$. Moreover, as $\sigma(\tau_i)=\sigma_2(T_i)-\sigma_1(T_i)$, we find the following equality in $\mathcal{E}$:

$$\varphi_2(m)=\sum_{\underline{n}\in\mathbb{N}^d}\frac{\varphi_1(\underline{N}^{\underline{n}}(m))\cdot\prod_i(\sigma_2(T_i)-\sigma_1(T_i))^{n_i}}{\underline{n}!},$$

as desired.

In particular, $\varphi_2'\colon\sigma_2^*\mathcal{E}\to\mathcal{E}$ is a horizontal isomorphism, hence $(\mathcal{E},\nabla,\varphi_2')$ is an F-isocrystal. Moreover $q_1^*(\mathcal{E},\nabla,\varphi_1)$ and $q_2^*(\mathcal{E},\nabla,\varphi_2')$ are isomorphic as realizations of F-isocrystals on the embedding $\mathcal{X}_0\hookrightarrow\mathcal{X}\times\mathcal{X}$. Therefore, by definition [Ber96, 2.3.2], $(\mathcal{E},\nabla,\varphi_1)$ and $(\mathcal{E},\nabla,\varphi_2')$ realize the same F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$.

The last statement of our lemma is clear from the formula just proved as $\mathcal{E}(X)^{\nabla=0}=\bigcap_{i=1}^d\ker(N_i)$. □

More generally, i.e. without assuming the existence of Frobenius lifts to $\mathcal{X}$, for $(\mathcal{E},\nabla)$ an $\mathcal{O}_{X_{\mathrm{an}}}$-module with integrable and convergent connection, *a compatible system of Frobenii on* $\mathcal{E}$ consists of, for any open subset $\mathcal{U}\subset\mathcal{X}$ equipped with a lifting of Frobenius $\sigma_{\mathcal{U}}$, a horizontal isomorphism $\varphi_{(\mathcal{U},\sigma_{\mathcal{U}})}\colon\sigma_{\mathcal{U}}^*\mathcal{E}|_{\mathcal{U}_k}\to\mathcal{E}|_{\mathcal{U}_k}$ satisfying the following condition: for $\mathcal{V}\subset\mathcal{X}$ another open subset equipped with a lifting of Frobenius $\sigma_{\mathcal{V}}$, the functor

$$F_{\sigma_{\mathcal{U}},\sigma_{\mathcal{V}}}\colon\mathbf{Mod}^{\sigma_{\mathcal{U}},\nabla}_{\mathcal{O}_{\mathcal{U}_k\cap\mathcal{V}_k}}\longrightarrow\mathbf{Mod}^{\sigma_{\mathcal{V}},\nabla}_{\mathcal{O}_{\mathcal{U}_k\cap\mathcal{V}_k}}$$

sends $(\mathcal{E}|_{\mathcal{U}_k\cap\mathcal{V}_k},\nabla,\varphi_{(\mathcal{U},\sigma_{\mathcal{U}})}|_{\mathcal{U}_k\cap\mathcal{V}_k})$ to $(\mathcal{E}|_{\mathcal{U}_k\cap\mathcal{V}_k},\nabla,\varphi_{(\mathcal{V},\sigma_{\mathcal{V}})}|_{\mathcal{U}_k\cap\mathcal{V}_k})$. We denote a compatible system of Frobenii on $\mathcal{E}$ by the symbol φ, when no confusion arises. Let $\mathbf{Mod}^{\sigma,\nabla}_{\mathcal{O}_X}$ be the category of $\mathcal{O}_{X_{\mathrm{an}}}$-vector bundles equipped with an integrable and convergent connection, and with a compatible system of Frobenii. The morphism in $\mathbf{Mod}^{\sigma,\nabla}_{\mathcal{O}_X}$ are the morphisms of $\mathcal{O}_{X_{\mathrm{an}}}$-modules which commute with the connections, and with the Frobenius morphisms on any open subset $\mathcal{U}\subset\mathcal{X}$ equipped with a lifting of Frobenius.

Remark 3.8. Let $\mathcal{E}$ be a convergent isocrystal on $\mathcal{X}_0/\mathcal{O}_k$. To define a compatible system of Frobenii on $\mathcal{E}$, we only need to give, for a cover $\mathcal{X}=\bigcup_i\mathcal{U}_i$ of $\mathcal{X}$ by open subsets $\mathcal{U}_i$ equipped with a lifting of Frobenius σ_i, a family of Frobenius morphisms $\varphi_i\colon\sigma_i^*\mathcal{E}|_{U_i}\xrightarrow{\sim}\mathcal{E}|_{U_i}$ such that $\varphi_i|_{U_i\bigcap U_j}$ corresponds to $\varphi_j|_{U_i\bigcap U_j}$ under the functor $F_{\sigma_i,\sigma_j}\colon\mathbf{Mod}^{\sigma_i,\nabla}_{\mathcal{O}_{U_i\bigcap U_j}}\to\mathbf{Mod}^{\sigma_j,\nabla}_{\mathcal{O}_{U_i\bigcap U_j}}$ (Here $U_\bullet:=\mathcal{U}_{\bullet,k}$). Indeed, for $\mathcal{U}$ any open subset equipped with a lifting of Frobenius $\sigma_{\mathcal{U}}$, one can first use the functor $F_{\sigma_i,\sigma_{\mathcal{U}}}$ of (3.1.2) applied to $(\mathcal{E}|_{U_i},\nabla|_{U_i},\varphi_i)|_{U_i\bigcap U}$ to obtain a horizontal

isomorphism $\varphi_{\mathcal{U},i}\colon (\sigma_{\mathcal{U}}^*(\mathcal{E}|_{\mathcal{U}}))|_{U_i\bigcap U}\to \mathcal{E}|_{U_i\bigcap U}$. From the compatibility of the φ_i's, we deduce $\varphi_{\mathcal{U},i}|_{U\bigcap U_i\bigcap U_j}=\varphi_{\mathcal{U},j}|_{U\bigcap U_i\bigcap U_j}$. Consequently we can glue the $\varphi_{\mathcal{U},i}$'s $(i\in I)$ to get a horizontal isomorphism $\varphi_{\mathcal{U}}\colon \sigma_{\mathcal{U}}^*(\mathcal{E}|_U)\to\mathcal{E}|_U$. One checks that these $\varphi_{\mathcal{U}}$'s give the desired compatible system of Frobenii on $\mathcal{E}$.

Let $\mathcal{E}$ be a convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$. For $\mathcal{U}\subset\mathcal{X}$ an open subset equipped with a lifting of Frobenius $\sigma_{\mathcal{U}}$, the restriction $\mathcal{E}|_{\mathcal{U}_k}$ gives rise to a convergent F-isocrystal on $\mathcal{U}_0/k$. Thus there exists a ∇-horizontal isomorphism $\varphi_{(\mathcal{U},\sigma_{\mathcal{U}})}\colon \sigma_{\mathcal{U}}^*\mathcal{E}|_{\mathcal{U}_k}\to\mathcal{E}|_{\mathcal{U}_k}$. Varying $(\mathcal{U},\sigma_{\mathcal{U}})$ we obtain a compatible system of Frobenii φ on $\mathcal{E}$. In this way, $(\mathcal{E},\nabla,\varphi)$ becomes an object of $\mathbf{Mod}_{\mathcal{O}_X}^{\sigma,\nabla}$. Directly from the definition, we have the following

Corollary 3.9. *The natural functor F-$\mathrm{Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k)\to\mathbf{Mod}_{\mathcal{O}_X}^{\sigma,\nabla}$ is an equivalence of categories.*

In the following, denote by $\mathbf{FMod}_{\mathcal{O}_X}^{\sigma,\nabla}$ the category of quadruples $(\mathcal{E},\nabla,\varphi,\mathrm{Fil}^\bullet(\mathcal{E}))$ with $(\mathcal{E},\nabla,\varphi)\in\mathbf{Mod}_{\mathcal{O}_X}^{\sigma,\nabla}$ and a decreasing, separated and exhaustive filtration $\mathrm{Fil}^\bullet(\mathcal{E})$ on $\mathcal{E}$ by locally free direct summands, such that ∇ satisfies Griffiths transversality with respect to $\mathrm{Fil}^\bullet(\mathcal{E})$, *i.e.,* $\nabla(\mathrm{Fil}^i(\mathcal{E}))\subset\mathrm{Fil}^{i-1}(\mathcal{E})\otimes_{\mathcal{O}_{X_{\mathrm{an}}}}\Omega^1_{X_{\mathrm{an}}/k}$. The morphisms are the morphisms in $\mathbf{Mod}_{\mathcal{O}_X}^{\sigma,\nabla}$ which respect the filtrations. We call the objects in $\mathbf{FMod}_{\mathcal{O}_X}^{\sigma,\nabla}$ *filtered (convergent) F-isocrystals* on $\mathcal{X}_0/\mathcal{O}_k$. By analogy with the category F-$\mathrm{Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k)$ of F-isocrystals, we also denote the category of filtered F-isocrystals on $\mathcal{X}_0/\mathcal{O}_k$ by FF-$\mathrm{Isoc}^\dagger(\mathcal{X}_0/\mathcal{O}_k)$.

3.2. **Lisse $\widehat{\mathbb{Z}}_p$-sheaves and filtered F-isocystals.** Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$ with X its generic fiber in the sense of Huber. Define $\widehat{\mathbb{Z}}_p:=\varprojlim\mathbb{Z}/p^n$ and $\widehat{\mathbb{Q}}_p:=\widehat{\mathbb{Z}}_p[1/p]$ as sheaves on $X_{\mathrm{pro\acute{e}t}}$. Recall that a *lisse $\mathbb{Z}_p$-sheaf* on $X_{\mathrm{\acute{e}t}}$ is an inverse system of sheaves of $\mathbb{Z}/p^n$-modules $\mathbb{L}_\bullet=(\mathbb{L}_n)_{n\in\mathbb{N}}$ on $X_{\mathrm{\acute{e}t}}$ such that each $\mathbb{L}_n$ is locally a constant sheaf associated to a finitely generated $\mathbb{Z}/p^n$-modules, and such that the inverse system is isomorphic in the pro-category to an inverse system for which $\mathbb{L}_{n+1}/p^n\simeq\mathbb{L}_n$. A *lisse $\widehat{\mathbb{Z}}_p$-sheaf* on $X_{\mathrm{pro\acute{e}t}}$ is a sheaf of $\widehat{\mathbb{Z}}_p$-modules on $X_{\mathrm{pro\acute{e}t}}$, which is locally isomorphic to $\widehat{\mathbb{Z}}_p\otimes_{\mathbb{Z}_p}M$ where M is a finitely generated $\mathbb{Z}_p$-module. By [Sch13, Proposition 8.2] these two notions are equivalent via the functor $\nu^*\colon X_{\mathrm{\acute{e}t}}^\sim\to X_{\mathrm{pro\acute{e}t}}^\sim$. In the following, we use the natural morphism of topoi $w\colon X_{\mathrm{pro\acute{e}t}}^\sim\xrightarrow{\nu}X_{\mathrm{\acute{e}t}}^\sim\to\mathcal{X}_{\mathrm{\acute{e}t}}^\sim$ frequently. Before defining crystalline sheaves let us first make the following observation.

Remark 3.10. (1) Let $\mathcal{M}$ be a crystal on $\mathcal{X}_0/\mathcal{O}_k$, viewed as a coherent $\mathcal{O}_{\mathcal{X}}$-module admitting an integrable connection. Then $w^{-1}\mathcal{M}$ is a coherent $\mathcal{O}_X^{\mathrm{ur}+}$-module with an integrable connection $w^{-1}\mathcal{M}\to w^{-1}\mathcal{M}\otimes_{\mathcal{O}_X^{\mathrm{ur}+}}\Omega^{1,\mathrm{ur}+}_{X/k}$. If furthermore $\mathcal{M}$ is an F-crystal, then $w^{-1}\mathcal{M}$ inherits a system of Frobenii: for any open subset $\mathcal{U}\subset\mathcal{X}$ equipped with a lifting of Frobenius $\sigma_{\mathcal{U}}$, there is naturally an endomorphism of $w^{-1}\mathcal{M}|_U$ which is semilinear with respect to the Frobenius $w^{-1}\sigma_{\mathcal{U}}$ on $\mathcal{O}_X^{\mathrm{ur}+}|_U$ (here $U:=\mathcal{U}_k$). Indeed, the Frobenius structure on $\mathcal{M}$ gives a horizontal $\mathcal{O}_{\mathcal{U}}$-linear morphism $\sigma_{\mathcal{U}}^*\mathcal{M}|_{\mathcal{U}}\to\mathcal{M}|_{\mathcal{U}}$, or equivalently, a $\sigma_{\mathcal{U}}$-semilinear morphism $\varphi_{\mathcal{U}}\colon\mathcal{M}|_{\mathcal{U}}\to\mathcal{M}|_{\mathcal{U}}$ (as $\sigma_{\mathcal{U}}$ is the identity map on the underlying topological space). So we obtain a natural endomorphism $w^{-1}\varphi_{\mathcal{U}}$ of $w^{-1}\mathcal{M}|_U$, which is $w^{-1}\sigma_{\mathcal{U}}$-semilinear.

(2) Let $\mathcal{E}$ be a convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$. By Remark 3.5, there exists an F-crystal $\mathcal{M}$ on $\mathcal{X}_0/\mathcal{O}_k$ and $n \in \mathbb{N}$ such that $\mathcal{E} \simeq \mathcal{M}^{\mathrm{an}}(n)$. By (1), $w^{-1}\mathcal{M}$ is a coherent $\mathcal{O}_X^{\mathrm{ur}+}$-module equipped with an integrable connection and a compatible system of Frobenii φ. Inverting p, we get an $\mathcal{O}_X^{\mathrm{ur}}$-module $w^{-1}\mathcal{M}[1/p]$ equipped with an integrable connection and a system of Frobenii φ/p^n, which does not depend on the choice of the formal model $\mathcal{M}$ or the integer n. For this reason, abusing notation, let us denote $w^{-1}\mathcal{M}[1/p]$ by $w^{-1}\mathcal{E}$, which is equipped with an integrable connection and a system of Frobenii inherited from $\mathcal{E}$. If furthermore $\mathcal{E}$ has a descending filtration $\{\mathrm{Fil}^i \mathcal{E}\}$ by locally direct summands, by Remark 3.3, each $\mathrm{Fil}^i \mathcal{E}$ has a coherent formal model $\mathcal{E}_i^+$ on $\mathcal{X}$. Then $\{w^{-1}\mathcal{E}_i^+[1/p]\}$ gives a descending filtration (by locally direct summands) on $w^{-1}\mathcal{E}$.

Definition 3.11. We say a lisse $\widehat{\mathbb{Z}}_p$-sheaf $\mathbb{L}$ on $X_{\mathrm{pro\acute{e}t}}$ is *crystalline* if there exists a filtered F-isocrystal $\mathcal{E}$ together with an isomorphism of $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$-modules

$$w^{-1}\mathcal{E} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq \mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}} \tag{3.2.1}$$

which is compatible with connection, filtration and Frobenius. In this case, we say that the lisse $\widehat{\mathbb{Z}}_p$-sheaf $\mathbb{L}$ and the filtered F-isocrystal $\mathcal{E}$ are *associated*.

Remark 3.12. The Frobenius compatibility of the isomorphism (3.2.1) means the following. Take any open subset $\mathcal{U} \subset \mathcal{X}$ equipped with a lifting of Frobenius $\sigma\colon \mathcal{U} \to \mathcal{U}$. By the discussion in §2.3, we know that $\mathcal{O}\mathbb{B}_{\mathrm{cris}}|_{\mathcal{U}_k}$ is naturally endowed with a Frobenius φ. Meanwhile, as $\mathcal{E}$ is an F-isocrystal, by Remark 3.10 $w^{-1}\mathcal{E}|_{\mathcal{U}_k}$ is endowed with a $w^{-1}\sigma$-semilinear Frobenius, still denoted by φ. Now the required Frobenius compatibility means that when restricted to any such $\mathcal{U}_k$, we have $\varphi \otimes \varphi = \mathrm{id} \otimes \varphi$ via the isomorphism (3.2.1).

Definition 3.13. For $\mathbb{L}$ a lisse $\widehat{\mathbb{Z}}_p$-sheaf and $i \in \mathbb{Z}$, set

$$\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) := w_*(\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}), \quad \text{and} \quad \mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathbb{L}) := w_*(\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathrm{Fil}^i \mathcal{O}\mathbb{B}_{\mathrm{cris}}).$$

All of them are $\mathcal{O}_{\mathcal{X}}[1/p]$-modules, and the $\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ give a separated exhaustive decreasing filtration on $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ (as the same holds for the filtration on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$; see Corollary 2.22).

Next we shall compare the notion of crystalline sheaves with other related notions considered in [Bri, Chapitre 8], [Fal] and [Sch13]. We begin with the following characterization of crystalline sheaves, which is more closely related to the classical definition of crystalline representations by Fontaine (see also [Bri, Chapitre 8]).

Proposition 3.14. *Let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf on $X_{\mathrm{pro\acute{e}t}}$. Then $\mathbb{L}$ is crystalline if and only if the following two conditions are verified:*

1. *the $\mathcal{O}_{\mathcal{X}}[1/p]$-modules $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ and $\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ ($i \in \mathbb{Z}$) are all coherent.*
2. *the adjunction morphism $w^{-1}\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}} \to \mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}$ is an isomorphism of $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$-modules.*

Before proving this proposition, let us express locally the sheaf $\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) = w_*(\mathbb{L} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}})$ as the Galois invariants of some Galois module. Consider $\mathcal{U} = \mathrm{Spf}(R^+) \subset \mathcal{X}$ a connected affine open subset admitting an étale map $\mathcal{U} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\})$. Write $R = R^+[1/p]$ and denote U the generic fiber of $\mathcal{U}$. As $\mathcal{U}$ is smooth and connected, R^+ is an integral domain. Fix an algebraic closure Ω of $\mathrm{Frac}(R)$, and let $\overline{R}^+$ be the union of finite and normal R^+-algebras Q^+ contained in Ω such that

$Q^+[1/p]$ is étale over R. Write $\overline{R} = \overline{R}^+[1/p]$. Write $G_U := \mathrm{Gal}(\overline{R}/R)$, which is nothing but the fundamental group of $U = \mathcal{U}_k$. Let U^{univ} be the profinite étale cover of U corresponding to $(\overline{R}, \overline{R}^+)$. One checks that U^{univ} is affinoid perfectoid (over the completion of $\overline{k}$). As $\mathbb{L}$ is a lisse $\widehat{\mathbb{Z}}_p$-sheaf on X, its restriction to U corresponds to a continuous $\mathbb{Z}_p$-representation $V_U(\mathbb{L}) := \mathbb{L}(U^{\mathrm{univ}})$ of G_U. Write $\widehat{U^{\mathrm{univ}}} = \mathrm{Spa}(S, S^+)$, where (S, S^+) is the p-adic completion of $(\overline{R}, \overline{R}^+)$.

Lemma 3.15. *Keep the notation above. Let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf on X. Then there exist natural isomorphisms of R-modules*

$$\mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{U}) \xrightarrow{\sim} \left(V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S^+)\right)^{G_U} =: D_{\mathrm{cris}}(V_U(\mathbb{L}))$$

and, for any $r \in \mathbb{Z}$,

$$\left(\mathrm{Fil}^r \mathbb{D}_{\mathrm{cris}}(\mathbb{L})\right)(\mathcal{U}) \xrightarrow{\sim} \left(V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S^+)\right)^{G_U}.$$

Moreover, the R-module $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{U})$ is projective of rank at most that of $V_U(\mathbb{L}) \otimes \mathbb{Q}_p$.

Proof. As $\mathbb{L}$ is a lisse $\widehat{\mathbb{Z}}_p$-sheaf, it becomes constant restricted to U^{univ}. In other words, we have $\mathbb{L}|_{U^{\mathrm{univ}}} \simeq V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \widehat{\mathbb{Z}}_p|_{U^{\mathrm{univ}}}$. For $i \geq 0$ an integer, we denote by $U^{\mathrm{univ},i}$ the (i+1)-fold product of U^{univ} over U. Then $U^{\mathrm{univ},i} \simeq U^{\mathrm{univ}} \times G_U^i$, and it is again an affinoid perfectoid. By the use of Lemma 2.17, we find $H^j(U^{\mathrm{univ},i}, \mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}) = 0$ for $j > 0$. Moreover

$$H^0(U^{\mathrm{univ},i}, \mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}) = \mathrm{Hom}_{\mathrm{cont}}\left(G_U^i, V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(U^{\mathrm{univ}})\right)[1/t].$$

Again, as U^{univ} is affinoid perfectoid, $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(U^{\mathrm{univ}}) \simeq \mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S^+)$ by Lemma 2.17. Consider the Cartan-Leray spectral sequence (cf. [SGA4, V.3]) associated to the cover $U^{\mathrm{univ}} \to U$:

$$E_1^{i,j} = H^j(U^{\mathrm{univ},i}, \mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}) \Longrightarrow H^{i+j}(U, \mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}).$$

As $E_1^{i,j} = 0$ for $j \geq 1$, we have $E_2^{n,0} = H^n(U, \mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}})$. Thus, we deduce a natural isomorphism

$$H^j(U, \mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}) \xrightarrow{\sim} H^j_{\mathrm{cont}}(G_U, V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+))[1/t]$$

where the right hand side is the continuous group cohomology. Taking $j = 0$, we obtain our first assertion. The isomorphism concerning $\mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris}}$ can be proved exactly in the same way. The last assertion follows from the first isomorphism and [Bri, Proposition 8.3.1], which gives the assertion for the right hand side. □

The lemma above has the following two consequences.

Corollary 3.16. *Let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf on $X_{\mathrm{pro\acute{e}t}}$, which satisfies the condition (1) of Proposition 3.14. Let $\mathcal{U} = \mathrm{Spf}(R^+)$ be a small connected affine open subset of $\mathcal{X}$. Write $R = R^+[1/p]$ and $U = \mathcal{U}_k$. Then for any $V \in X_{\mathrm{pro\acute{e}t}}/U$, we have*

$$\mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{U}) \otimes_R \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V) \xrightarrow{\sim} (w^{-1}\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}})(V).$$

Proof. By Lemma 3.15, the R-module $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{U})$ is projective of finite type over R, hence it is a direct summand of a finite free R-module. As $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is coherent over $\mathcal{O}_{\mathcal{X}}[1/p]$ and as $\mathcal{U}$ is affine, $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})|_{\mathcal{U}}$ is then a direct summand of a finite free $\mathcal{O}_{\mathcal{X}}[1/p]|_{\mathcal{U}}$-module. The isomorphism in our corollary then follows, since we have similar isomorphism when $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})|_{\mathcal{U}}$ is replace by a free $\mathcal{O}_{\mathcal{X}}[1/p]|_{\mathcal{U}}$-module. □

Corollary 3.17. *Let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf verifying the condition (1) of Proposition 3.14. Then the condition (2) of Proposition 3.14 holds for $\mathbb{L}$ if and only if for any small affine connected open subset $\mathcal{U} \subset \mathcal{X}$ (with $U := \mathcal{U}_k$), the G_U-representation $V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline in the sense that the following natural morphism is an isomorphism (*[Bri, Chapitre 8]*)*

$$D_{\mathrm{cris}}(V_U(\mathbb{L})) \otimes_R \mathcal{OB}_{\mathrm{cris}}(S, S^+) \xrightarrow{\sim} V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{OB}_{\mathrm{cris}}(S, S^+),$$

where $G_U, U^{\mathrm{univ}}, \widehat{U^{\mathrm{univ}}} = \mathrm{Spa}(S, S^+)$ are as in the paragraph before Lemma 3.15.

Proof. If $\mathbb{L}$ satisfies in addition the condition (2) of Proposition 3.14, combining with Corollary 3.16, we find

$$\begin{aligned} \mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{U}) \otimes_R \mathcal{OB}_{\mathrm{cris}}\left(U^{\mathrm{univ}}\right) &\xrightarrow{\sim} (w^{-1}\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{OB}_{\mathrm{cris}})(U^{\mathrm{univ}}) \\ &\xrightarrow{\sim} (\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{OB}_{\mathrm{cris}})(U^{\mathrm{univ}}) \\ &= V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{OB}_{\mathrm{cris}}(U^{\mathrm{univ}}). \end{aligned}$$

So, by Lemmas 2.17 and 3.15, the G_U-representation $V_U(\mathbb{L}) \otimes \mathbb{Q}_p$ is crystalline.

Conversely, assume that for any small connected affine open subset $\mathcal{U} = \mathrm{Spf}(R^+)$ of $\mathcal{X}$, the G_U-representation $V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline. Together with Lemmas 2.17 and 3.15, we get $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{U}) \otimes_R \mathcal{OB}_{\mathrm{cris}}(U^{\mathrm{univ}}) \xrightarrow{\sim} V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{OB}_{\mathrm{cris}}(U^{\mathrm{univ}})$ and the similar isomorphism after replacing U^{univ} by any $V \in X_{\mathrm{pro\acute{e}t}}/U^{\mathrm{univ}}$. Using Corollary 3.16, we deduce $(w^{-1}\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{OB}_{\mathrm{cris}})(V) \xrightarrow{\sim} (\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{OB}_{\mathrm{cris}})(V)$ for any $V \in X_{\mathrm{pro\acute{e}t}}/U^{\mathrm{univ}}$, *i.e.*, $(w^{-1}\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{OB}_{\mathrm{cris}})|_{U^{\mathrm{univ}}} \xrightarrow{\sim} (\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{OB}_{\mathrm{cris}})|_{U^{\mathrm{univ}}}$. When the small opens $\mathcal{U}$'s run through a cover of $\mathcal{X}$, the U^{univ}'s form a cover of X for the pro-étale topology. Therefore, $w^{-1}\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \otimes \mathcal{OB}_{\mathrm{cris}} \xrightarrow{\sim} \mathbb{L} \otimes \mathcal{OB}_{\mathrm{cris}}$, as desired. □

Lemma 3.18. *Let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf on X satisfying the two conditions of Proposition 3.14. Then (the analytification of) $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ has a natural structure of filtered convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$.*

Proof. First of all, the $\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$'s ($i \in \mathbb{Z}$) endow a separated exhaustive decreasing filtration on $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ by Corollary 2.22, and the connection on $\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) = w_*(\mathbb{L} \otimes \mathcal{OB}_{\mathrm{cris}})$ can be given by the composite of

$$\begin{aligned} w_*(\mathbb{L} \otimes \mathcal{OB}_{\mathrm{cris}}) &\xrightarrow{w_*(\mathrm{id}\otimes\nabla)} w_*(\mathbb{L} \otimes \mathcal{OB}_{\mathrm{cris}} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \Omega^{1,\mathrm{ur}}_{X/k}) \\ &\xrightarrow{\sim} w_*(\mathbb{L} \otimes \mathcal{OB}_{\mathrm{cris}}) \otimes_{\mathcal{O}_{\mathcal{X}}[1/p]} \Omega^1_{\mathcal{X}/\mathcal{O}_k}[1/p] \end{aligned}$$

where the last isomorphism is the projection formula. That the connection satisfies the Griffiths transversality with respect to the filtration $\mathrm{Fil}^\bullet \mathbb{D}_{\mathrm{cris}}$ follows from the analogous assertion for $\mathcal{OB}_{\mathrm{cris}}$ (Proposition 2.16).

Now consider the special case where $\mathcal{X} = \mathrm{Spf}(R^+)$ is affine connected admitting an étale map $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\})$, such that $\mathcal{X}$ is equipped with a lifting of Frobenius σ. As in the paragraph before Lemma 3.15, let X^{univ} be the univeral profinite étale cover of X (which is an affinoid perfectoid). Write $\widehat{X^{\mathrm{univ}}} = \mathrm{Spa}(S, S^+)$ and G_X the fundamental group of X. As $\mathcal{X}$ is affine, the category $\mathrm{Coh}(\mathcal{O}_{\mathcal{X}}[1/p])$ is equivalent to the category of finite type R-modules (here $R := R^+[1/p]$). Under this equivalence, $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ corresponds to $D_{\mathrm{cris}}(V_X(\mathbb{L})) := (V_X(\mathbb{L}) \otimes \mathcal{OB}_{\mathrm{cris}}(S, S^+))^{G_X}$, denoted by D for simplicity. So D is a projective

R-module of finite type (Lemma 3.15) equipped with a connection $\nabla\colon D \to D \otimes \Omega^1_{R/k}$. Under the same equivalence, $\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ corresponds to $\mathrm{Fil}^i D := (V_X(\mathbb{L}) \otimes \mathrm{Fil}^i \mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S^+))^{G_X}$, by Lemma 3.15 again. By the same proof as in [Bri, 8.3.2], the graded quotient $\mathrm{gr}^i(D)$ is a projective module. In particular, $\mathrm{Fil}^i D \subset D$ is a direct summand. Therefore, each $\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is a direct summand of $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$. Furthermore, since $\mathcal{X}$ admits a lifting of Frobenius σ, we get from § 2.3 a σ-semilinear endomorphism φ on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(X^{\mathrm{univ}}) \simeq \mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S^+)$, whence a σ-semilinear endomorphism on D, still denoted by φ. From Lemma 2.19, One checks that the Frobenius φ on D is horizontal with respect to its connection. Thus $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is endowed with a horizontal σ-semilinear morphism $\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \to \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$, always denoted by φ in the following.

To finish the proof in the special case, one still needs to show that the triple $(\mathbb{D}_{\mathrm{cris}}(\mathbb{L}), \nabla, \varphi)$ gives an F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$. As D is of finite type over R, there exists some $n \in \mathbb{N}$ such that $D = D^+[1/p]$ with $D^+ := (V_X(\mathbb{L}) \otimes_{\mathbb{Z}_p} t^{-n}\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+))^{G_X}$. The connection on $t^{-n}\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+)$ induces a connection $\nabla^+\colon D^+ \to D^+ \otimes_{R^+} \Omega^1_{R^+/\mathcal{O}_k}$ on D^+, compatible with that of $D_{\mathrm{cris}}(V_X(\mathbb{L}))$. Moreover, if we take N_i to be the endomorphism of D^+ so that $\nabla^+ = \sum_{i=1}^d N_i \otimes dT_i$, then for any $a \in D^+$, $\underline{N}^{\underline{m}}(a) \in p \cdot D^+$ for all but finitely many $\underline{m} \in \mathbb{N}^d$ (as this holds for the connection on $t^{-n}\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, seen in the proof of Lemma 2.20). Similarly, the Frobenius on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S^+)$ induces a map (note that the Frobenius on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S^+)$ sends t to $p \cdot t$)

$$\varphi\colon D^+ \longrightarrow (V_X(\mathbb{L}) \otimes p^{-n}t^{-n}\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+))^{G_X}.$$

Thus $\psi := p^n\varphi$ gives a well-defined σ-semilinear morphism on D^+. One checks that ψ is horizontal with respect to the connection ∇^+ on D^+ and it induces an R^+-linear isomorphism $\sigma^* D^+ \xrightarrow{\sim} D^+$. As a result, the triple (D^+, ∇, ψ) will define an F-crystal on $\mathcal{U}_0/\mathcal{O}_k$, once we know D^+ is of finite type over R^+. The required finiteness of D^+ is explained in [AI, Proposition 3.6], and for the sake of completeness we recall briefly their proof here. As D is projective of finite type (Lemma 3.15), it is a direct summand of a finite free R-module T. Let $T^+ \subset T$ be a finite free R^+-submodule of T such that $T^+[1/p] = T$. Then we have the inclusion $D \otimes_R \mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S^+) \hookrightarrow T^+ \otimes_{R^+} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S^+)$. As $V_X(\mathbb{L})$ is of finite type over $\mathbb{Z}_p$ and $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S^+) = \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+)[1/t]$, there exists $m \in \mathbb{N}$ such that the $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+)$-submodule $V_X(\mathbb{L}) \otimes t^{-n}\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+)$ of $V_X(\mathbb{L}) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S+) \simeq D \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S^+)$ is contained in $T^+ \otimes_{R^+} t^{-m}\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+)$. By taking G_U-invariants and using the fact that R^+ is noetherian, we are reduced to showing that $R' := (t^{-m}\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S, S^+))^{G_X}$ is of finite type over R^+. From the construction, R' is p-adically separated and $R^+ \subset R' \subset R = (\mathcal{O}\mathbb{B}_{\mathrm{cris}}(S, S^+))^{G_X}$. As R^+ is normal, we deduce $p^N R' \subset R^+$ for some $N \in \mathbb{N}$. Thus $p^N R'$ and hence R' are of finite type over R^+. As a result, (D^+, ∇, ψ) defines an F-crystal $\mathcal{D}^+$ on $\mathcal{U}_0/\mathcal{O}_k$. As $D = D^+[1/p]$ and $\nabla = \nabla^+[1/p]$, the connection ∇ on $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is convergent; this is standard and we refer to [Ber96, 2.4.1] for detail. Consequently, the triple $(\mathbb{D}_{\mathrm{cris}}(\mathbb{L}), \nabla, \varphi)$ is an F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$, which is isomorphic to $\mathcal{D}^{+,\mathrm{an}}(n)$. This finishes the proof in the special case.

In the general case, consider a covering $\mathcal{X} = \bigcup_i \mathcal{U}_i$ of $\mathcal{X}$ by connected small affine open subsets such that each $\mathcal{U}_i$ admits a lifting of Frobenius σ_i and an étale morphism to some torus over $\mathcal{O}_k$. By the special case, we have seen that each $\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \subset \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is locally a direct summand, and that the connection on

$\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is convergent ([Ber96, 2.2.8]). Furthermore, each $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})|_{\mathcal{U}_i}$ is equipped with a Frobenius φ_i, and over $\mathcal{U}_i \cap \mathcal{U}_j$, the two Frobenii φ_i, φ_j on $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})|_{\mathcal{U}_i \bigcap \mathcal{U}_j}$ are related by the formula in Lemma 3.7 as it is the case for φ_i, φ_j on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}|_{U_i \bigcap U_j}$ (Lemma 2.20). So these local Frobenii glue together to give a compatible system of Frobenii φ on $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ and the analytification of the quadruple $(\mathbb{D}_{\mathrm{cris}}(\mathbb{L}), \mathrm{Fil}^\bullet \mathbb{D}_{\mathrm{cris}}(\mathbb{L}), \nabla, \varphi)$ is a filtered F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$, as wanted. □

Proof of Proposition 3.14. If a lisse $\widehat{\mathbb{Z}}_p$-sheaf $\mathbb{L}$ on X is associated to a filtered F-isocrystal $\mathcal{E}$ on X, then we just have to show $\mathcal{E} \simeq \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$. By assumption, we have $\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq w^{-1}\mathcal{E} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}}$. Then

$$w_*(\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}) \simeq w_*(w^{-1}\mathcal{E} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}}) \simeq \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p]} w_*\mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq \mathcal{E}$$

where the second isomorphism has used Remark 3.12, and the last isomorphism is by the isomorphism $w_*\mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq \mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p]$ from Corollary 2.23.

Conversely, let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf verifying the two conditions of our proposition. By Lemma 3.18, $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is naturally a filtered F-isocrystal. To finish the proof, we need to show that the isomorphism in (2) is compatible with the extra structures. Only the compatibility with filtrations needs verification. This is a local question, hence we shall assume $\mathcal{X} = \mathrm{Spf}(R^+)$ is a small connected affine formal scheme. As $\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is coherent over $\mathcal{O}_{\mathcal{X}}[1/p]$ and is a direct summand of $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$, the same proof as that of Corollary 3.16 gives

$$\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{X}) \otimes_R \mathrm{Fil}^j \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V) \xrightarrow{\sim} (w^{-1}\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathrm{Fil}^j \mathcal{O}\mathbb{B}_{\mathrm{cris}})(V)$$

for any $V \in X_{\mathrm{proét}}$. Consequently, the isomorphism in Corollary 3.16 is strictly compatible with filtrations on both sides. Thus, we reduce to show that, for any affinoid perfectoid $V \in X_{\mathrm{proét}}/X^{\mathrm{univ}}$, the isomorphism $D_{\mathrm{cris}}(V_X(\mathbb{L})) \otimes_R \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V) \xrightarrow{\sim} V_X(\mathbb{L}) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V)$ is strictly compatible with the filtrations, or equivalently, the induced morphisms between the graded quotients are isomorphisms:

$$\oplus_{i+j=n} \left(\mathrm{gr}^i D_{\mathrm{cris}}(V_X(\mathbb{L})) \otimes_R \mathrm{gr}^j \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V)\right) \longrightarrow \mathbb{L} \otimes \mathrm{gr}^n \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V). \tag{3.2.2}$$

When $V = X^{\mathrm{univ}}$, this follows from [Bri, 8.4.3]. For the general case, write $\widehat{X^{\mathrm{univ}}} = \mathrm{Spa}(S, S^+)$ and $\widehat{V} = \mathrm{Spa}(S_1, S_1^+)$. Then by [Sch13, Corollary 6.15] and Corollary 2.22, we have $\mathrm{gr}^j \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V) \simeq S_1 \xi^j [U_1/\xi, \ldots, U_d/\xi]$. So the natural morphism $\mathrm{gr}^j \mathcal{O}\mathbb{B}_{\mathrm{cris}}(X^{\mathrm{univ}}) \otimes_S S_1 \xrightarrow{\sim} \mathrm{gr}^j \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V)$ is an isomorphism. The required isomorphism (3.2.2) for general V then follows from the special case for X^{univ}. □

Let $\mathrm{Lis}^{\mathrm{cris}}_{\widehat{\mathbb{Z}}_p}(X)$ denote the category of lisse crystalline $\widehat{\mathbb{Z}}_p$-sheaves on X, and $\mathrm{Lis}^{\mathrm{cris}}_{\widehat{\mathbb{Q}}_p}(X)$ the corresponding isogeny category. The functor

$$\mathbb{D}_{\mathrm{cris}} \colon \mathrm{Lis}^{\mathrm{cris}}_{\widehat{\mathbb{Q}}_p}(X) \longrightarrow FF\text{-}\mathrm{Iso}^\dagger(\mathcal{X}_0/\mathcal{O}_k), \quad \mathbb{L} \mapsto \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$$

allows us to relate $\mathrm{Lis}^{\mathrm{cris}}_{\widehat{\mathbb{Q}}_p}(X)$ to the category $FF\text{-}\mathrm{Iso}^\dagger(\mathcal{X}_0/\mathcal{O}_k)$ of filtered convergent F-isocrystals on $\mathcal{X}_0/\mathcal{O}_k$, thanks to Proposition 3.14. A filtered F-isocrystal $\mathcal{E}$ on $\mathcal{X}_0/\mathcal{O}_k$ is called *admissible* if it lies in the essential image of the functor above. The full subcategory of admissible filtered F-isocrystals on $\mathcal{X}_0/\mathcal{O}_k$ will be denoted by $FF\text{-}\mathrm{Iso}^\dagger(\mathcal{X}_0/\mathcal{O}_k)^{\mathrm{adm}}$.

Theorem 3.19. *The functor $\mathbb{D}_{\mathrm{cris}}$ above induces an equivalence of categories*

$$\mathbb{D}_{\mathrm{cris}} \colon \mathrm{Lis}^{\mathrm{cris}}_{\widehat{\mathbb{Q}}_p}(X) \xrightarrow{\sim} FF\text{-}\mathrm{Iso}^\dagger(\mathcal{X}_0/\mathcal{O}_k)^{\mathrm{adm}}.$$

A quasi-inverse of $\mathbb{D}_{\mathrm{cris}}$ is given by

$$\mathbb{V}_{\mathrm{cris}} : \mathcal{E} \mapsto \mathrm{Fil}^0(w^{-1}\mathcal{E} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}})^{\nabla=0,\varphi=1}$$

where φ denotes the compatible system of Frobenii on $\mathcal{E}$ as before.

Proof. Observe first that, for $\mathcal{E}$ a filtered convergent F-isocrystal, the local Frobenii on $\mathcal{E}^{\nabla=0}$ glue to give a unique σ-semilinear morphism on $\mathcal{E}^{\nabla=0}$ (Lemma 3.7). In particular, the abelian sheaf $\mathbb{V}_{\mathrm{cris}}(\mathcal{E})$ is well-defined. Assume moreover $\mathcal{E}$ is admissible, and let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf such that $\mathcal{E} \simeq \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$. So $\mathbb{L}$ and $\mathcal{E}$ are associated by Proposition 3.14. Hence $\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq w^{-1}\mathcal{E} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}}$, and we find

$$\begin{aligned}
\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathbb{Q}}_p &\xrightarrow{\sim} \mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathrm{Fil}^0(\mathcal{O}\mathbb{B}_{\mathrm{cris}})^{\nabla=0,\varphi=1} \\
&\xrightarrow{\sim} \mathrm{Fil}^0(\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}})^{\nabla=0,\varphi=1} \\
&\xrightarrow{\sim} \mathrm{Fil}^0(w^{-1}\mathcal{E} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}})^{\nabla=0,\varphi=1} \\
&= \mathbb{V}_{\mathrm{cris}}(\mathcal{E}),
\end{aligned}$$

where the first isomorphism following from the the fundamental exact sequence (by Lemma 2.5 and [Bri, Corollary 6.2.19])

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \mathrm{Fil}^0 \mathbb{B}_{\mathrm{cris}} \xrightarrow{1-\varphi} \mathbb{B}_{\mathrm{cris}} \longrightarrow 0.$$

In particular, $\mathbb{V}_{\mathrm{cris}}(\mathcal{E})$ is the associated $\widehat{\mathbb{Q}}_p$-sheaf of a lisse $\widehat{\mathbb{Z}}_p$-sheaf. Thus $\mathbb{V}_{\mathrm{cris}}(\mathcal{E}) \in \mathrm{Lis}^{\mathrm{cris}}_{\widehat{\mathbb{Q}}_p}(X)$ and the functor $\mathbb{V}_{\mathrm{cris}}$ is well-defined. Furthermore, as we can recover the the lisse $\widehat{\mathbb{Z}}_p$-sheaf up to isogeny, it follows that $\mathbb{D}_{\mathrm{cris}}$ is fully faithful, and a quasi-inverse on its essential image is given by $\mathbb{V}_{\mathrm{cris}}$. □

Remark 3.20. Using [Bri, Theorem 8.5.2], one can show that the equivalence above is an equivalence of tannakian category.

Next we compare Definition 3.11 with the "associatedness" defined in [Fal]. Let $\mathcal{E}$ be a filtered convergent F-isocrystal $\mathcal{E}$ on $\mathcal{X}_0/\mathcal{O}_k$, and $\mathcal{M}$ an F-crystal on $\mathcal{X}_0/\mathcal{O}_k$ such that $\mathcal{M}^{\mathrm{an}} = \mathcal{E}(-n)$ for some $n \in \mathbb{N}$ (see Remark 3.5 for the notations). Let $\mathcal{U} = \mathrm{Spf}(R^+)$ be a small connected affine open subset of $\mathcal{X}$, equipped with a lifting of Frobenius σ. Write $U = \mathrm{Spa}(R, R^+)$ the generic fiber of $\mathcal{U}$. As before, let $\overline{R}^+$ be the union of all finite normal R^+-algebras (contained in some fixed algebraic closure of $\mathrm{Frac}(R^+)$) which are étale over $R := R^+[1/p]$, and $\overline{R} := \overline{R}^+[1/p]$. Let $G_U := \mathrm{Gal}(\overline{R}/R)$ and (S, S^+) the p-adic completion of $(\overline{R}, \overline{R}^+)$. Then (S, S^+) is an perfectoid affinoid algebra over the p-adic completion of $\overline{k}$. So we can consider the period sheaf $\mathbb{A}_{\mathrm{cris}}(S, S^+)$. Moreover the composite of the following two natural morphisms

$$\mathbb{A}_{\mathrm{cris}}\left(S, S^+\right) \xrightarrow{\theta} S^+ \xrightarrow{\mathrm{can}} S^+/pS^+, \tag{3.2.3}$$

defines a p-adic PD-thickening of $\mathrm{Spec}(S^+/pS^+)$. Evaluate our F-crystal $\mathcal{M}$ at it and write $\mathcal{M}(\mathbb{A}_{\mathrm{cris}}(S, S^+))$ for the resulting finite type $\mathbb{A}_{\mathrm{cris}}(S, S^+)$-module. As an element of G_U defines a morphism of the PD-thickening (3.2.3) in the big crystalline site of $\mathcal{X}_0/\mathcal{O}_k$ and $\mathcal{M}$ is a crystal, $\mathcal{M}(\mathbb{A}(S, S^+))$ is endowed naturally with an action of G_U. Similarly, the Frobenius on the crystal $\mathcal{M}$ gives a Frobenius ψ on $\mathcal{M}(\mathbb{A}_{\mathrm{cris}}(S, S^+))$. Set $\mathcal{E}(\mathbb{B}_{\mathrm{cris}}(S, S^+)) := \mathcal{M}(\mathbb{A}_{\mathrm{cris}}(S, S^+))[1/t]$, which is a $\mathbb{B}_{\mathrm{cris}}(S, S^+)$-module of finite type endowed with a Frobenius $\varphi = \psi/p^n$ and an action of G_U.

On the other hand, as $\mathcal{U}$ is small, there exists a morphism $\alpha\colon R^+ \to \mathbb{A}_{\mathrm{cris}}(S, S^+)$ of $\mathcal{O}_k$-algebras, whose composite with the projection $\mathbb{A}_{\mathrm{cris}}(S, S^+) \to S^+$ is the inclusion $R^+ \subset S^+$. For example, consider an étale morphism $\mathcal{U} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\})$. Let (T_i^{1/p^n}) be a compatible system of p^n-th roots of T_i inside $\overline{R}^+ \subset S^+$, and $T_i^\flat$ the corresponding element of $S^{\flat +} := \varprojlim_{x \mapsto x^p} S^+/pS^+$. Then one can take α as the unique morphism of $\mathcal{O}_k$-algebras $R^+ \to \mathbb{A}_{\mathrm{cris}}(S, S^+)$ sending T_i to $[T_i^\flat]$, such that its composite with the projection $\mathbb{A}_{\mathrm{cris}}(S, S^+) \to S^+/pS^+$ is just the natural map $R^+ \to S^+/pS^+$ (such a morphism exists as R^+ is étale over $\mathcal{O}_k\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\}$ and because of (3.2.3); see the proof of Lemma 2.13 for a similar situation). Now we fix such a morphism α. So we obtain a morphism of PD-thickenings from $\mathcal{U}_0 \hookrightarrow \mathcal{U}$ to the one defined by (3.2.3). Consequently we get a natural isomorphism $\mathcal{M}(\mathbb{A}_{\mathrm{cris}}(S, S^+)) \simeq \mathcal{M}(\mathcal{U}) \otimes_{R^+,\alpha} \mathbb{A}_{\mathrm{cris}}(S, S^+)$, whence

$$\mathcal{E}(\mathbb{B}_{\mathrm{cris}}(S, S^+)) \simeq \mathcal{E}(\mathcal{U}) \otimes_{R,\alpha} \mathbb{B}_{\mathrm{cris}}(S, S^+).$$

Using this isomorphism, we define the filtration on $\mathcal{E}(\mathbb{B}_{\mathrm{cris}}(S, S^+))$ as the tensor product of the filtration on $\mathcal{E}(\mathcal{U})$ and that on $\mathbb{B}_{\mathrm{cris}}(S, S^+)$.

Remark 3.21. It is well known that the filtration on $\mathcal{E}(\mathbb{B}_{\mathrm{cris}}(S, S^+))$ does not depend on the choice of α. More precisely, let α' be a second morphism $R^+ \to \mathbb{A}_{\mathrm{cris}}(S, S^+)$ of $\mathcal{O}_k$-algebras whose composite with $\mathbb{A}_{\mathrm{cris}}(S, S^+) \to S^+$ is the inclusion $R^+ \subset S^+$. Fix an étale morphism $\mathcal{U} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\})$. Denote $\beta = (\alpha, \alpha') : R^+ \otimes_{\mathcal{O}_k} R^+ \to \mathbb{A}_{\mathrm{cris}}(S, S^+)$ and by the same notation the corresponding map on schemes, and write $p_1, p_2 : \mathrm{Spec}\, R^+ \times \mathrm{Spec}\, R^+ \to \mathrm{Spec}\, R^+$ the two projections. We have a canonical isomorphism $(p_2 \circ \beta)^*\mathcal{E} \xrightarrow{\sim} (p_1 \circ \beta)^*\mathcal{E}$, as $\mathcal{E}$ is a crystal. In terms of the connection ∇ on $\mathcal{E}$, this gives (cf. [Ber96, 2.2.4]) the following $\mathbb{B}_{\mathrm{cris}}(S, S^+)$-linear isomorphism

$$\eta\colon \mathcal{E}(\mathcal{U}) \otimes_{R,\alpha} \mathbb{B}_{\mathrm{cris}}(S, S^+) \longrightarrow \mathcal{E}(\mathcal{U}) \otimes_{R,\alpha'} \mathbb{B}_{\mathrm{cris}}(S, S^+)$$

sending $e \otimes 1$ to $\sum_{\underline{n} \in \mathbb{N}^d} \underline{N}^{\underline{n}}(e) \otimes (\alpha(\underline{T}) - \alpha'(\underline{T}))^{[\underline{n}]}$, with $\underline{N}$ the endomorphism of $\mathcal{E}$ such that $\nabla = \underline{N} \otimes d\underline{T}$. Here we use the multi-index to simplify the notations, and note that $\alpha(T_i) - \alpha'(T_i) \in \mathrm{Fil}^1\, \mathbb{A}_{\mathrm{cris}}(S, S^+)$ hence the divided power $(\alpha(T_i) - \alpha'(T_i))^{[n_i]}$ is well-defined. Moreover, the series converge since the connection on $\mathcal{M}$ is quasi-nilpotent. Now as the filtration on $\mathcal{E}$ satisfies Griffiths transversality, the isomorphism η is compatible with the tensor product filtrations on both sides. Since the inverse η^{-1} can be described by a similar formula (one just switches α and α'), it is also compatible with filtrations on both sides. Hence the isomorphism η is strictly compatible with the filtrations, and the filtration on $\mathcal{E}(\mathbb{B}_{\mathrm{cris}}(S, S^+))$ does not depend on the choice of α.

Let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf on X, and write as before $V_U(\mathbb{L})$ the $\mathbb{Z}_p$-representation of G_U corresponding to the lisse sheaf $\mathbb{L}|_U$. Following [Fal], we say a filtered convergent F-isocrystal $\mathcal{E}$ *is associated to* $\mathbb{L}$ *in the sense of Faltings* if, for all small open subset $\mathcal{U} \subset \mathcal{X}$, there is a functorial isomorphism:

$$\mathcal{E}\left(\mathbb{B}_{\mathrm{cris}}\left(S, S^+\right)\right) \xrightarrow{\sim} V_U(\mathbb{L}) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{cris}}\left(S, S^+\right) \tag{3.2.4}$$

which is compatible with filtration, G_U-action and Frobenius.

Proposition 3.22. *Keep the notation above. If $\mathcal{E}$ is associated to $\mathbb{L}$ in the sense of Faltings then $\mathbb{L}$ is crystalline (not necessarily associated to $\mathcal{E}$) and there is an isomorphism $\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \simeq \mathcal{E}$ compatible with filtration and Frobenius. Conversely,*

if $\mathbb{L}$ is crystalline and if there is an isomorphism $\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \simeq \mathcal{E}$ of $\mathcal{O}_{X^{\mathrm{an}}}$-modules compatible with filtration and Frobenius, then $\mathbb{L}$ and $\mathcal{E}$ are associated in the sense of Faltings.

Before giving the proof of Proposition 3.22, we observe first the following commutative diagram in which the left vertical morphisms are all PD-morphisms:

$$\begin{array}{ccc} R^+ & \longrightarrow & R^+/pR^+ \\ {\scriptstyle \mathrm{can}}\downarrow & & \downarrow \\ \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S,S^+) & \xrightarrow{\theta_R} & S^+/pS^+ \\ {\scriptstyle \mathrm{can}}\uparrow & & \| \\ \mathbb{A}_{\mathrm{cris}}(S,S^+) & \xrightarrow{\theta} & S^+/pS^+ \end{array}.$$

Therefore, we have isomorphisms

$$\begin{aligned} \mathcal{M}(\mathcal{U}) \otimes_{R^+} \mathcal{O}\mathbb{A}_{\mathrm{cris}}\left(S,S^+\right) &\xrightarrow{\sim} \mathcal{M}\left(\mathcal{O}\mathbb{A}_{\mathrm{cris}}\left(S,S^+\right)\right) \\ &\xleftarrow{\sim} \mathcal{M}\left(\mathbb{A}_{\mathrm{cris}}\left(S,S^+\right)\right) \otimes_{\mathbb{A}_{\mathrm{cris}}(S,S^+)} \mathcal{O}\mathbb{A}_{\mathrm{cris}}\left(S,S^+\right), \end{aligned}$$

where the second term in the first row denotes the evaluation of the crystal $\mathcal{M}$ at the PD-thickening defined by the PD-morphism θ_R in the commutative diagram above. Inverting t, we obtain a natural isomorphism

$$\mathcal{E}(\mathcal{U}) \otimes_R \mathcal{O}\mathbb{B}_{\mathrm{cris}}\left(S,S^+\right) \xrightarrow{\sim} \mathcal{E}\left(\mathbb{B}_{\mathrm{cris}}\left(S,S^+\right)\right) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}\left(S,S^+\right), \tag{3.2.5}$$

where the last tensor product is taken over $\mathbb{B}_{\mathrm{cris}}(S,S^+)$. This isomorphism is clearly compatible with Galois action and Frobenius. By a similar argument as in Remark 3.21 one checks that (3.2.5) is also strictly compatible with the filtrations. Furthermore, using the identification

$$\mathbb{A}_{\mathrm{cris}}\left(S,S^+\right)\{\langle u_1,\ldots,u_d\rangle\} \xrightarrow{\sim} \mathcal{O}\mathbb{A}_{\mathrm{cris}}\left(S,S^+\right), \quad u_i \mapsto T_i \otimes 1 - 1 \otimes [T_i^\flat]$$

we obtain a section s of the canonical map $\mathbb{A}_{\mathrm{cris}}(S,S^+) \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S,S^+)$:

$$s\colon \mathcal{O}\mathbb{A}_{\mathrm{cris}}\left(S,S^+\right) \to \mathbb{A}_{\mathrm{cris}}\left(S,S^+\right) \quad u_i \mapsto 0$$

which is again a PD-morphism. Composing with the inclusion $R^+ \subset \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S,S^+)$, we get a morphism $\alpha_0\colon R^+ \to \mathbb{A}_{\mathrm{cris}}(S,S^+)$ whose composite with the projection $\mathbb{A}_{\mathrm{cris}}(S,S^+) \to S^+$ is the inclusion $R^+ \subset S^+$.

Proof of Proposition 3.22. Now assume that $\mathcal{E}$ is associated with $\mathbb{L}$ in the sense of [Fal]. Extending scalars to $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(S,S^+)$ of the isomorphism (3.2.4) and using the identification (3.2.5), we obtain a functorial isomorphism, compatible with filtration, G_U-action, and Frobenius:

$$V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}\left(S,S^+\right) \xrightarrow{\sim} \mathcal{E}(\mathcal{U}) \otimes_R \mathcal{O}\mathbb{B}_{\mathrm{cris}}\left(S,S^+\right).$$

Therefore, $V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a crystalline G_U-representation (Corollary 3.17), and we get by Lemma 3.15 an isomorphism $\mathcal{E}(\mathcal{U}) \xrightarrow{\sim} \mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{U})$ compatible with filtrations and Frobenius. As such small open subsets $\mathcal{U}$ form a basis for the Zariski topology of $\mathcal{X}$, we find an isomorphism $\mathcal{E} \xrightarrow{\sim} \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ compatible with filtrations and Frobenius, and that $\mathbb{L}$ is crystalline in the sense of Definition 3.11 (Corollary 3.17).

Conversely, assume $\mathbb{L}$ is crystalline with $\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \cong \mathcal{E}$ compatible with filtrations and Frobenius. As in the proof of Corollary 3.17, we have a functorial isomorphism

$$\mathcal{E}(\mathcal{U}) \otimes_R \mathcal{O}\mathbb{B}_{\mathrm{cris}}\left(S, S^+\right) \xrightarrow{\sim} V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}\left(S, S^+\right)$$

which is compatible with filtration, Galois action and Frobenius. Pulling it back via the section $\mathcal{O}\mathbb{B}_{\mathrm{cris}}\left(S, S^+\right) \to \mathbb{B}_{\mathrm{cris}}\left(S, S^+\right)$ obtained from s by inverting p, we obtain a functorial isomorphism

$$\mathcal{E}(\mathbb{B}_{\mathrm{cris}}(S, S^+)) \simeq \mathcal{E}(\mathcal{U}) \otimes_{R,\alpha_0} \mathbb{B}_{\mathrm{cris}}(S, S^+) \xrightarrow{\sim} V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{cris}}(S, S^+),$$

which is again compatible with Galois action, Frobenius and filtrations. Therefore $\mathbb{L}$ and $\mathcal{E}$ are associated in the sense of Faltings. □

Finally we compare Definition 3.11 with its de Rham analogue considered in [Sch13].

Proposition 3.23. *Let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf on X and $\mathcal{E}$ a filtered convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$. Assume that $\mathbb{L}$ and $\mathcal{E}$ are associated as defined in Definition 3.11, then $\mathbb{L}$ is a de Rham in the sense of* [Sch13, Definition 8.3] *. More precisely, if we view $\mathcal{E}$ as a filtered module with integrable connection on X (namely we forget the Frobenius), there exists a natural filtered isomorphism that is compatible with connections:*

$$\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}}$$

Proof. Let $\mathcal{U} = \mathrm{Spf}(R^+) \subset \mathcal{X}$ be a connected affine open subset, and denote U (resp. U^{univ}) the generic fiber of $\mathcal{U}$ (resp. the universal étale cover of U). Let V be a affinoid perfectoid lying above U^{univ}. As $\mathbb{L}$ and $\mathcal{E}$ are associated, there exits a filtered isomorphism compatible with connections and Frobenius

$$\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}} \xrightarrow{\sim} w^{-1}\mathcal{E} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}}.$$

Evaluate this isomorphism at $V \in X_{\mathrm{pro\acute{e}t}}$ and use the fact that the R-module $\mathcal{E}(\mathcal{U})$ is projective (here $R := R^+[1/p]$), we deduce a filtered isomorphism compatible with all extra structures:

$$V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V) \xrightarrow{\sim} \mathcal{E}(U) \otimes_R \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V).$$

Taking tensor product $- \otimes_{\mathcal{O}\mathbb{B}_{\mathrm{cris}}(V)} \mathcal{O}\mathbb{B}_{\mathrm{dR}}(V)$ on both sides, we get a filtered isomorphism compatible with connection:

$$V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}}(V) \xrightarrow{\sim} \mathcal{E}(U) \otimes_R \mathcal{O}\mathbb{B}_{\mathrm{dR}}(V).$$

Again, as $\mathcal{E}(U)$ is a projective R-module and as $\mathcal{E}$ is coherent, the isomorphism above can be rewritten as

$$(\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}})(V) \xrightarrow{\sim} (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}})(V),$$

which is clearly functorial in $\mathcal{U}$ and in V. Varying $\mathcal{U}$ and V, we deduce that $\mathbb{L}$ is de Rham, hence our proposition. □

3.3. **From pro-étale site to étale site.** Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$. For $\mathcal{O} = \mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}[1/p], \mathcal{O}_X^{\mathrm{ur}+}, \mathcal{O}_X^{\mathrm{ur}}$ and a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ with connection, we denote the de Rham complex of $\mathcal{F}$ as:

$$DR(\mathcal{F}) = (0 \to \mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes_{\mathcal{O}} \Omega^1 \xrightarrow{\nabla} \cdots).$$

Let $\overline{w}$ be the composite of natural morphisms of topoi (here we use the same notation to denote the object in $X^{\sim}_{\mathrm{proét}}$ represented by $X_{\overline{k}} \in X_{\mathrm{proét}}$):

$$X^{\sim}_{\mathrm{proét}}/X_{\overline{k}} \longrightarrow X^{\sim}_{\mathrm{proét}} \xrightarrow{w} \mathcal{X}^{\sim}_{\mathrm{ét}}.$$

The following lemma is just a global reformulation of the main results of [AB]. As we shall prove a more general result later (Lemma 5.3), let us omit the proof here.

Lemma 3.24. *Let $\mathcal{X}$ be smooth formal scheme over $\mathcal{O}_k$. Then the natural morphism below is an isomorphism in the filtered derived category:*

$$\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_{\mathcal{O}_k} B_{\mathrm{cris}} \longrightarrow R\overline{w}_*(\mathcal{O}\mathbb{B}_{\mathrm{cris}}).$$

Here $\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_{\mathcal{O}_k} B_{\mathrm{cris}} := \left(\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_{\mathcal{O}_k} A_{\mathrm{cris}}\right)[1/t]$ *with*

$$\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_{\mathcal{O}_k} A_{\mathrm{cris}} := \varprojlim_{n \in \mathbb{N}} \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_k} A_{\mathrm{cris}}/p^n,$$

and $\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_{\mathcal{O}_k} B_{\mathrm{cris}}$ is filtered by the subsheaves

$$\mathcal{O} \widehat{\otimes}_{\mathcal{O}_k} \mathrm{Fil}^r B_{\mathrm{cris}} := \varinjlim_{n \in \mathbb{N}} t^{-n} (\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_{\mathcal{O}_k} \mathrm{Fil}^{r+n} A_{\mathrm{cris}}), \quad r \in \mathbb{Z}.$$

Corollary 3.25. *Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$. Let $\mathbb{L}$ be a crystalline lisse $\widehat{\mathbb{Z}}_p$-sheaf associated with a filtered convergent F-isocrystal $\mathcal{E}$. Then there exists a natural quasi-isomorphism in the filtered derived category*

$$R\overline{w}_*(\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}_{\mathrm{cris}}) \xrightarrow{\sim} DR(\mathcal{E}) \widehat{\otimes}_k B_{\mathrm{cris}}.$$

If moreover $\mathcal{X}$ is endowed with a lifting of Frobenius σ, then the isomorphism above is also compatible with the Frobenii deduced from σ on both sides.

Proof. Using the Poincaré lemma (Corollary 2.16), we get first a quasi-isomorphism which is strictly compatible with filtrations:

$$\mathbb{L} \otimes \mathbb{B}_{\mathrm{cris}} \xrightarrow{\sim} \mathbb{L} \otimes DR(\mathcal{O}\mathbb{B}_{\mathrm{cris}}) = DR(\mathbb{L} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}).$$

As $\mathbb{L}$ and $\mathcal{E}$ are associated, there is a filtered isomorphism $\mathbb{L} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}} \xrightarrow{\sim} w^{-1}\mathcal{E} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}}$ compatible with connection and Frobenius, from which we get the quasi-isomorphisms in the filtered derived category

$$\mathbb{L} \otimes \mathbb{B}_{\mathrm{cris}} \xrightarrow{\sim} DR(\mathbb{L} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}) \xrightarrow{\sim} DR(w^{-1}\mathcal{E} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}). \tag{3.3.1}$$

On the other hand, as $R^j\overline{w}_*\mathcal{O}\mathbb{B}_{\mathrm{cris}} = 0$ for $j > 0$ (Lemma 3.24), we obtain using projection formula that $R^j\overline{w}_*((w^{-1}\mathcal{E} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}})|_{X_{\overline{k}}}) = \mathcal{E} \otimes R^j\overline{w}_*\mathcal{O}\mathbb{B}_{\mathrm{cris}} = 0$ (note that $\mathcal{E}$ is locally a direct factor of a finite free $\mathcal{O}_{\mathcal{X}}[1/p]$-module, hence one can apply projection formula here). In particular, each component of $DR(w^{-1}\mathcal{E} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}})$ is $\overline{w}_*$-acyclic. Therefore,

$$DR(\mathcal{E} \otimes \overline{w}_*\mathcal{O}\mathbb{B}_{\mathrm{cris}}) \xrightarrow{\sim} \overline{w}_*(DR(w^{-1}\mathcal{E} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}})) \xrightarrow{\sim} R\overline{w}_*(DR(w^{-1}\mathcal{E} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}})).$$

Combining this with Lemma 3.24, we deduce the following quasi-isomorphisms in the filtered derived category

$$DR(\mathcal{E}) \widehat{\otimes}_k B_{\mathrm{cris}} \xrightarrow{\sim} DR(\mathcal{E} \otimes \overline{w}_*\mathcal{O}\mathbb{B}_{\mathrm{cris}}) \xrightarrow{\sim} R\overline{w}_*(DR(w^{-1}\mathcal{E} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}})). \tag{3.3.2}$$

The desired quasi-isomorphism follows from (3.3.1) and (3.3.2). When furthermore $\mathcal{X}$ admits a lifting of Frobenius σ, one checks easily that both quasi-isomorphisms are compatible with Frobenius, hence the last part of our corollary. □

Remark 3.26. Recall that G_k denotes the absolute Galois group of k. Each element of G_k defines a morphism of $U_{\overline{k}}$ in the pro-étale site $X_{\text{proét}}$ for any $\mathcal{U} \in \mathcal{X}_{\text{proét}}$ with $U := \mathcal{U}_k$. Therefore, the object $R\overline{w}_*(\mathbb{L} \otimes \mathbb{B}_{\text{cris}})$ comes with a natural Galois action of G_k. With this Galois action, one checks that the quasi-isomorphism in Corollary 3.25 is also Galois equivariant.

Let $\mathcal{E}$ be a filtered convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$, and $\mathcal{M}$ an F-crystal on $\mathcal{X}_0/\mathcal{O}_k$ (viewed as a coherent $\mathcal{O}_{\mathcal{X}}$-module equipped with an integrable connection) such that $\mathcal{E} \simeq \mathcal{M}^{\text{an}}(n)$ for some $n \in \mathbb{N}$ (Remark 3.5). The crystalline cohomology group $H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{M})$ is an $\mathcal{O}_k$-module of finite type endowed with a Frobenius ψ. In the following, the crystalline cohomology (or more appropriately, the rigid cohomology) of the convergent F-isocrystal $\mathcal{E}$ is defined as

$$H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{E}) := H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{M})[1/p].$$

It is a finite dimensional k-vector space equipped with the Frobenius ψ/p^n. Moreover, let $u = u_{\mathcal{X}_0/\mathcal{O}_k}$ be the following morphism of topoi

$$(\mathcal{X}_0/\mathcal{O}_k)^{\sim}_{\text{cris}} \longrightarrow \mathcal{X}^{\sim}_{\text{ét}}$$

such that $u_*(\mathcal{F})(\mathcal{U}) = H^0((\mathcal{U}_0/\mathcal{O}_k)^{\sim}_{\text{cris}}, \mathcal{F})$ for $\mathcal{U} \in \mathcal{X}_{\text{ét}}$. With the étale topology replaced by the Zariski topology, this is precisely the morphism $u_{\mathcal{X}_0/\hat{S}}$ (with $\hat{S} = \text{Spf}(\mathcal{O}_k)$) considered in [BO, Theorem 7.23]. By *loc.cit.*, there exists a natural quasi-isomorphism in the derived category

$$Ru_*\mathcal{M} \xrightarrow{\sim} DR(\mathcal{M}), \tag{3.3.3}$$

which induces a natural isomorphism $H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{M}) \xrightarrow{\sim} \mathbb{H}^i(\mathcal{X}, DR(\mathcal{M}))$. Thereby

$$H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{E}) \xrightarrow{\sim} \mathbb{H}^i(\mathcal{X}, DR(\mathcal{E})). \tag{3.3.4}$$

On the other hand, the de Rham complex $DR(\mathcal{E})$ of $\mathcal{E}$ is filtered by its subcomplexes

$$\text{Fil}^r DR(\mathcal{E}) := (\text{Fil}^r \mathcal{E} \xrightarrow{\nabla} \text{Fil}^{r-1} \mathcal{E} \otimes \Omega^1_{X/k} \xrightarrow{\nabla} \ldots).$$

So the hypercohomology $\mathbb{H}^i(\mathcal{X}, DR(\mathcal{E}))$ has a descending filtration given by

$$\text{Fil}^r \mathbb{H}^i(\mathcal{X}, DR(\mathcal{E})) := \text{Im}\left(\mathbb{H}^i(\mathcal{X}, \text{Fil}^r DR(\mathcal{E})) \longrightarrow \mathbb{H}^i(\mathcal{X}, DR(\mathcal{E}))\right).$$

Consequently, through the isomorphism (3.3.4), the k-space $H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{E})$ is endowed naturally with a decreasing filtration.

Theorem 3.27. *Assume further that the smooth formal scheme $\mathcal{X}$ is proper over $\mathcal{O}_k$. Let $\mathcal{E}$ be a filtered convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$ and $\mathbb{L}$ a lisse $\widehat{\mathbb{Z}}_p$-sheaf on $X_{\text{proét}}$. Assume that $\mathcal{E}$ and $\mathbb{L}$ are associated. Then there is a natural filtered isomorphism of B_{cris}-modules*

$$H^i(X_{\overline{k},\text{proét}}, \mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}_{\text{cris}}) \xrightarrow{\sim} H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{E}) \otimes_k B_{\text{cris}} \tag{3.3.5}$$

which is compatible with Frobenius and Galois action.

Proof. By Corollary 3.25, we have the natural Galois equivariant quasi-isomorphism in the filtered derived category:

$$R\Gamma(X_{\overline{k},\text{proét}}, \mathbb{L}\otimes\mathbb{B}_{\text{cris}}) = R\Gamma(\mathcal{X}, R\overline{w}_*(\mathbb{L}\otimes\mathbb{B}_{\text{cris}})) \xrightarrow{\sim} R\Gamma(\mathcal{X}, DR(\mathcal{E})\widehat{\otimes}_k B_{\text{cris}}).$$

We first claim that the following natural morphism in the filtered derived category is an isomorphism:

$$R\Gamma(\mathcal{X}, DR(\mathcal{E}))\otimes_{\mathcal{O}_k} A_{\text{cris}} \longrightarrow R\Gamma(\mathcal{X}_{\text{ét}}, DR(\mathcal{E})\widehat{\otimes}_{\mathcal{O}_k} A_{\text{cris}}).$$

Indeed, as A_{cris} is flat over $\mathcal{O}_k$, the morphism above is an isomorphism respecting the filtrations in the derived category. Thus to prove our claim, it suffices to check that the morphism above induces quasi-isomorphisms on gradeds. Further filtering the de Rham complex by its naive filtration, we are reduced to checking the following isomorphism for $\mathcal{A}$ a coherent $\mathcal{O}_{\mathcal{X}}$-module:

$$R\Gamma(\mathcal{X}, \mathcal{A})\otimes_{\mathcal{O}_k}\mathcal{O}_{\mathbb{C}_p} \xrightarrow{\sim} R\Gamma(\mathcal{X}, \mathcal{A}\widehat{\otimes}_{\mathcal{O}_k}\mathcal{O}_{\mathbb{C}_p}),$$

which holds because again $\mathcal{O}_{\mathbb{C}_p}$ is flat over $\mathcal{O}_k$. Consequently, inverting t we obtain an isomorphism in the filtered derived category

$$R\Gamma(\mathcal{X}, DR(\mathcal{E}))\otimes_k B_{\text{cris}} \longrightarrow R\Gamma(\mathcal{X}_{\text{ét}}, DR(\mathcal{E})\widehat{\otimes}_k B_{\text{cris}}).$$

In this way, we get a Galois equivariant quasi-isomorphism in the filtered derived category

$$R\Gamma(X_{\overline{k},\text{proét}}, \mathbb{L}\otimes\mathbb{B}_{\text{cris}}) \xrightarrow{\sim} R\Gamma(\mathcal{X}, DR(\mathcal{E}))\otimes_k B_{\text{cris}}.$$

Combining it with (3.3.4), we obtain the isomorphism (3.3.5) verifying the required properties except for the Frobenius compatibility.

To check the Frobenius compatibility, we only need to check that the restriction to $H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{E}) \hookrightarrow H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{E})\otimes_k B_{\text{cris}}$ of the inverse of (3.3.5) is Frobenius-compatible. Let $\mathcal{M}$ be an F-crystal on $\mathcal{X}_0/\mathcal{O}_k$ such that $\mathcal{E} = \mathcal{M}^{\text{an}}(n)$. Via the identification $H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{E}) = H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{M})[1/p]$, the restriction map in question is induced from the following composite of morphisms at the level of derived category:

$$Ru_*\mathcal{M} \xrightarrow{\sim} DR(\mathcal{M}) \longrightarrow DR(\mathcal{M})\widehat{\otimes}_{\mathcal{O}_k} B_{\text{cris}} \simeq DR(\mathcal{E})\widehat{\otimes}_k B_{\text{cris}} \xrightarrow{\sim} R\overline{w}_*(\mathbb{L}\otimes\mathbb{B}_{\text{cris}}),$$

where the first quasi-isomorphism is (3.3.3), and the last morphism is just the inverse in the derived category of the quasi-isomorphism in Corollary 3.25. Let us denote by θ the composite of these morphisms. Let ψ (resp. φ) be the induced Frobenius on $Ru_*\mathcal{M}$ (resp. on $R\overline{w}_*(\mathbb{L}\otimes\mathbb{B}_{\text{cris}})$). One only needs to check that $\varphi\circ\theta = \frac{1}{p^n}\theta\circ\psi$. This can be checked locally on $\mathcal{X}$. So let $\mathcal{U}\subset\mathcal{X}$ be a small open subset equipped with a lifting of Frobenius σ. Thus $\mathcal{M}|_{\mathcal{U}}$ (resp. $\mathcal{E}|_{\mathcal{U}}$) admits naturally a Frobenius, which we denote by $\psi_{\mathcal{U}}$ (resp. $\varphi_{\mathcal{U}}$). Then all the morphisms above except for the identification in the middle are Frobenius-compatible (see Corollary 3.25 for the last quasi-isomorphism). But by definition, under the identification $\mathcal{M}[1/p]|_{\mathcal{U}} \simeq \mathcal{E}|_{\mathcal{U}}$, the Frobenius $\varphi_{\mathcal{U}}$ on $\mathcal{E}$ corresponds exactly to $\psi_{\mathcal{U}}/p^n$ on $\mathcal{M}[1/p]$. This gives the desired equality $\varphi\circ\theta = \frac{1}{p^n}\theta\circ\psi$ on $\mathcal{U}$, from which the Frobenius compatibility in (3.3.5) follows. □

4. Primitive comparison on the pro-étale site

Let $\mathcal{X}$ be a proper smooth formal scheme over $\mathcal{O}_k$, with X (resp. $\mathcal{X}_0$) its generic (resp. closed) fiber. Let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf on $X_{\text{proét}}$. In this section, we will construct a primitive comparison isomorphism for any lisse $\widehat{\mathbb{Z}}_p$-sheaf $\mathbb{L}$ on the pro-étale site $X_{\text{proét}}$ (Theorem 4.3). In particular, this primitive comparison isomorphism also holds for non-crystalline lisse $\widehat{\mathbb{Z}}_p$-sheaves, which may lead to interesting arithmetic applications. On the other hand, in the case that $\mathbb{L}$ is crystalline, such a result and Theorem 3.27 together give rise to the crystalline comparison isomorphism between étale cohomology and crystalline cohomology.

We shall begin with some preparations. The first lemma is well-known.

Lemma 4.1. *Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a projective system of abelian sheaves on a site T, such that $R^j\varprojlim \mathcal{F}_n = 0$ whenever $j > 0$. Then for any object $Y \in T$ and any $i \in \mathbb{Z}$, the following sequence is exact:*

$$0 \longrightarrow R^1\varprojlim H^{i-1}(Y,\mathcal{F}_n) \longrightarrow H^i(Y,\varprojlim \mathcal{F}_n) \longrightarrow \varprojlim H^i(Y,\mathcal{F}_n) \longrightarrow 0.$$

Proof. Let Sh (resp. PreSh) denote the category of abelian sheaves (resp. abelian presheaves) on T, and $\mathrm{Sh}^{\mathbb{N}}$ (resp. $\mathrm{PreSh}^{\mathbb{N}}$) the category of projective systems of abelian sheaves (resp. of abelian presheaves) indexed by $\mathbb{N}$. Let Ab denote the category of abelian groups. Consider the functor

$$\tau\colon \mathrm{Sh}^{\mathbb{N}} \longrightarrow \mathrm{Ab}, \quad (\mathcal{G}_n) \mapsto \varprojlim \Gamma(Y,\mathcal{G}_n).$$

Clearly τ is left exact, hence we can consider its right derived functors. Let us compute $R\tau(\mathcal{F}_n)$ in two different ways.

Firstly, one can write τ as the composite of the following two functors

$$\mathrm{Sh}^{\mathbb{N}} \xrightarrow{\varprojlim} \mathrm{Sh} \xrightarrow{\Gamma(U,-)} \mathrm{Ab}\ .$$

Since the projective limit functor $\varprojlim\colon \mathrm{Sh}^{\mathbb{N}} \to \mathrm{Sh}$ admits an exact left adjoint given by constant projective system, it sends injectives to injectives. Thus we obtain a spectral sequence

$$E_2^{i,j} = H^i(Y, R^j\varprojlim \mathcal{F}_n) \Longrightarrow R^{i+j}\tau(\mathcal{F}_n).$$

By assumption, $R^j\varprojlim \mathcal{F}_n = 0$ whenver $j > 0$. So the spectral sequence above degenerates at E_2 and we get $H^i(Y,\varprojlim \mathcal{F}_n) \simeq R^i\tau(\mathcal{F}_n)$.

Secondly, one can equally decompose τ as follows:

$$\mathrm{Sh}^{\mathbb{N}} \xrightarrow{\alpha} \mathrm{Ab}^{\mathbb{N}} \xrightarrow{\varprojlim} \mathrm{Ab}\ ,$$

where the functor α sends $(\mathcal{G}_n)$ to the projective system of abelian groups $(\Gamma(Y,\mathcal{G}_n))$. Let $I_\bullet = (I_n)$ be an injective object of $\mathrm{Sh}^{\mathbb{N}}$. By [Jan, Proposition 1.1], each component I_n is an injective object of Ab and the transition maps $I_{n+1} \to I_n$ are *split* surjective. Therefore, the transition maps of the projective system $\alpha(I_\bullet)$ are also split surjective. In particular, the projective system $\alpha(I_\bullet)$ is $\varprojlim$-acyclity. So we can consider the following spectral sequence

$$E_2^{i,j} = R^i\varprojlim H^j(Y,\mathcal{F}_n) \Longrightarrow R^{i+j}\tau(\mathcal{F}_n).$$

Since the category Ab satisfies the axiom (AB5*) of abelian categories (*i.e.,* infinite products are exact), $R^i\varprojlim A_n = 0$ ($i \notin \{0,1\}$) for any projective system of abelian

groups $(A_n)_n$. So the previous spectral sequence degenerates at E_2, from which we deduce a natural short exact sequence for each $i \in \mathbb{Z}$

$$0 \longrightarrow R^1 \varprojlim H^{i-1}(Y, \mathcal{F}_n) \longrightarrow R^i\tau(\mathcal{F}_n) \longrightarrow \varprojlim H^i(Y, \mathcal{F}_n) \longrightarrow 0.$$

As we have seen $H^i(Y, \varprojlim \mathcal{F}_n) \simeq R^i\tau(\mathcal{F}_n)$, we deduce the short exact sequence as asserted by our lemma. □

Lemma 4.2. *Let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf on $X_{\overline{k},\mathrm{proét}}$. Then for $i \in \mathbb{Z}$, $H^i(X_{\overline{k},\mathrm{proét}}, \mathbb{L})$ is a $\mathbb{Z}_p$-module of finite type, and $H^i(X_{\overline{k},\mathrm{proét}}, \mathbb{L}) = 0$ whenever $i \notin [0, 2\dim(X)]$.*

Proof. Since all the cohomology groups below are computed in the pro-étale site, we shall omit the subscript "proét" from the notations.

Let $\mathbb{L}_{\mathrm{tor}}$ denote the torsion subsheaf of $\mathbb{L}$. Then our lemma follows from the corresponding statements for $\mathbb{L}_{\mathrm{tor}}$ and for $\mathbb{L}/\mathbb{L}_{\mathrm{tor}}$. Therefore, we may assume either $\mathbb{L}$ is torsion or is locally on $X_{\overline{k},\mathrm{proét}}$ free of finite rank over $\widehat{\mathbb{Z}}_p$. In the first case, we reduce immediately to the finiteness statement of Scholze (Theorem 5.1 of [Sch13]). So it remains to consider the case when $\mathbb{L}$ is locally on $X_{\overline{k},\mathrm{proét}}$ free of finite rank over $\widehat{\mathbb{Z}}_p$. Let $\mathbb{L}_n := \mathbb{L}/p^n\mathbb{L}$. We have the tautological exact sequence

$$0 \longrightarrow \mathbb{L} \xrightarrow{p^n} \mathbb{L} \longrightarrow \mathbb{L}_n \longrightarrow 0,$$

inducing the following short exact sequence

$$0 \longrightarrow H^i(X_{\overline{k}}, \mathbb{L})/p^n \longrightarrow H^i(X_{\overline{k}}, \mathbb{L}_n) \longrightarrow H^{i+1}(X_{\overline{k}}, \mathbb{L})[p^n] \longrightarrow 0. \tag{4.0.6}$$

We shall show that the following natural morphism is an isomorphism:

$$H^i(X_{\overline{k}}, \mathbb{L}) \longrightarrow \varprojlim H^i(X_{\overline{k}}, \mathbb{L}_n). \tag{4.0.7}$$

Indeed, as $\mathbb{L}$ is locally free, using [Sch13, Proposition 8.2], we find $R^j \varprojlim_n \mathbb{L}_n = 0$ for $j > 0$. Moreover, as $H^{i-1}(X_{\overline{k}}, \mathbb{L}_n)$ are finite $\mathbb{Z}/p^n\mathbb{Z}$-modules by [Sch13, Theorem 5.1], $R^1 \varprojlim H^{i-1}(X_{\overline{k}}, \mathbb{L}_n) = 0$. Consequently, by Lemma 4.1, the morphism (4.0.7) is an isomorphism.

In particular, $H^i(X_{\overline{k}}, \mathbb{L}) \xrightarrow{\sim} \varprojlim H^i(X_{\overline{k}}, \mathbb{L}_n)$ is a pro-p abelian group, hence it does not contain any element infinitely divisible by p. Thus, $\varprojlim(H^i(X_{\overline{k}}, \mathbb{L})[p^n]) = 0$ (where the transition map is multiplication by p). From the exactness of (4.0.6), we then deduce a canonical isomorphism

$$\varprojlim \left(H^i(X_{\overline{k}}, \mathbb{L})/p^n\right) \xrightarrow{\sim} \varprojlim H^i(X_{\overline{k}}, \mathbb{L}_n).$$

So $H^i(X_{\overline{k}}, \mathbb{L}) \xrightarrow{\sim} \varprojlim_n H^i(X_{\overline{k}}, \mathbb{L})/p^n$. Consequently the $\mathbb{Z}_p$-module $H^i(X_{\overline{k}}, \mathbb{L})$ is p-adically complete, and it can be generated as a $\mathbb{Z}_p$-module by a family of elements whose images in $H^i(X_{\overline{k}}, \mathbb{L})/p$ generate it as an $\mathbb{F}_p$-vector space. Since the latter is finite dimensional over $\mathbb{F}_p$, the $\mathbb{Z}_p$-module $H^i(X_{\overline{k}}, \mathbb{L})$ is of finite type, as desired. □

The primitive form of the comparison isomorphism on the pro-étale site is as follows.

Theorem 4.3. *Let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf on $X_{\mathrm{proét}}$. There is a canonical isomorphism of B^+_{cris}-modules*

$$H^i(X_{\overline{k},\mathrm{proét}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} B^+_{\mathrm{cris}} \xrightarrow{\sim} H^i(X_{\overline{k},\mathrm{proét}}, \mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}^+_{\mathrm{cris}}) \tag{4.0.8}$$

compatible with Galois action, filtration and Frobenius.

Proof. In the following, all the cohomologies are computed on the pro-étale site, hence we omit the subscript "proét" from the notations.

The proof begins with the almost isomorphism in [Sch13, Theorem 8.4]:

$$H^i(X_{\overline{k}},\mathbb{L})\otimes_{\mathbb{Z}_p} A_{\mathrm{inf}} \xrightarrow{\approx} H^i(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathbb{A}_{\mathrm{inf}}).$$

Therefore, setting

$$\widetilde{A^0_{\mathrm{cris}}} := \frac{A_{\mathrm{inf}}[X_i : i\in\mathbb{N}]}{(X_i^p - a_iX_{i+1} : i\in\mathbb{N})} \quad \text{and} \quad \widetilde{\mathbb{A}^0_{\mathrm{cris}}} := \frac{\mathbb{A}_{\mathrm{inf}}[X_i : i\in\mathbb{N}]}{(X_i^p - a_iX_{i+1} : i\in\mathbb{N})}$$

with $a_i = \frac{p^{i+1}!}{(p^i!)^p}$ and using the fact that $X_{\overline{k}}$ is qcqs, we find the following almost isomorphism

$$H^i(X_{\overline{k}},\mathbb{L})\otimes_{\mathbb{Z}_p}\widetilde{A^0_{\mathrm{cris}}} \xrightarrow{\approx} H^i\left(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\widetilde{\mathbb{A}^0_{\mathrm{cris}}}\right). \tag{4.0.9}$$

On the other hand, from the tautological short exact sequence

$$0\longrightarrow \widetilde{A^0_{\mathrm{cris}}} \xrightarrow{X_0-\xi} \widetilde{A^0_{\mathrm{cris}}} \longrightarrow A^0_{\mathrm{cris}}\longrightarrow 0$$

and the fact that A^0_{cris} is flat over $\mathbb{Z}_p$, we deduce a short exact sequence:

$$0\longrightarrow H^i(X_{\overline{k}},\mathbb{L})\otimes_{\mathbb{Z}_p}\widetilde{A^0_{\mathrm{cris}}} \xrightarrow{X_0-\xi} H^i(X_{\overline{k}},\mathbb{L})\otimes_{\mathbb{Z}_p}\widetilde{A^0_{\mathrm{cris}}} \longrightarrow H^i(X_{\overline{k}},\mathbb{L})\otimes_{\mathbb{Z}_p} A^0_{\mathrm{cris}}\longrightarrow 0.$$

As a result, together with the almost isomorphism (4.0.9), we see that the morphism

$$\alpha_i\colon H^i\left(X_{\overline{k}},\mathbb{L}\otimes\widetilde{\mathbb{A}^0_{\mathrm{cris}}}\right)\longrightarrow H^i\left(X_{\overline{k}},\mathbb{L}\otimes\widetilde{\mathbb{A}^0_{\mathrm{cris}}}\right)$$

induced by multiplication-by-$(X_0-\xi)$ is almost injective for all $i\in\mathbb{Z}$. Similarly, since $\mathbb{A}^0_{\mathrm{cris}}$ is flat over $\widehat{\mathbb{Z}}_p$, the following sequence of abelian sheaves on $X_{\mathrm{proét}}/X_{\overline{k}}$ is exact:

$$0\longrightarrow(\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\widetilde{\mathbb{A}^0_{\mathrm{cris}}})|_{X_{\overline{k}}} \xrightarrow{X_0-\xi} (\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\widetilde{\mathbb{A}^0_{\mathrm{cris}}})|_{X_{\overline{k}}}\longrightarrow(\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathbb{A}^0_{\mathrm{cris}})|_{X_{\overline{k}}}\longrightarrow 0,$$

giving the associated long exact sequence on the cohomology:

$$\cdots\to H^i(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\widetilde{\mathbb{A}^0_{\mathrm{cris}}}) \xrightarrow{\alpha_i} H^i(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\widetilde{\mathbb{A}^0_{\mathrm{cris}}})\to H^i(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathbb{A}^0_{\mathrm{cris}})\to\cdots.$$

As the α_i's are almost injective for all i, the previous long exact sequence splits into short *almost* exact sequences

$$0\longrightarrow H^i(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\widetilde{\mathbb{A}^0_{\mathrm{cris}}}) \xrightarrow{\alpha_i} H^i(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\widetilde{\mathbb{A}^0_{\mathrm{cris}}})\longrightarrow H^i(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathbb{A}^0_{\mathrm{cris}})\longrightarrow 0.$$

Thus the following natural morphism is an almost isomorphism

$$H^i(X_{\overline{k}},\mathbb{L})\otimes_{\mathbb{Z}_p} A^0_{\mathrm{cris}} \xrightarrow{\approx} H^i(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathbb{A}^0_{\mathrm{cris}}),\quad \forall i\geq 0. \tag{4.0.10}$$

To pass to p-adic completion, we remark that, by Lemma 4.2, the $\mathbb{Z}_p$-module $H^i(X_{\overline{k}},\mathbb{L})$ is of finite type and vanishes when $i\notin[0,2\dim X]$. Let N be an integer such that the torsion part $\mathbb{L}_{\mathrm{tor}}$ of $\mathbb{L}$ and for all i the torsion parts $H^i(X_{\overline{k}},\mathbb{L})_{\mathrm{tor}}$ of $H^i(X_{\overline{k}},\mathbb{L})$ are annilated by p^N. For $n>N$ an integer, we claim that the following natural morphism has kernel and cokernel killed by p^{2N}:

$$H^i(X_{\overline{k}},\mathbb{L})\otimes_{\mathbb{Z}_p}(A^0_{\mathrm{cris}}/p^n)\longrightarrow H^i(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathbb{A}^0_{\mathrm{cris}}/p^n), \tag{4.0.11}$$

or equivalently (via the isomorphism (4.0.10)), that the natural morphism below has kernel and cokernel killed by p^{2N}:

$$H^i(X_{\overline{k}},\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}})/p^n\longrightarrow H^i(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathbb{A}^0_{\mathrm{cris}}/p^n). \tag{4.0.12}$$

To see this, consider the following tautological exact sequence

$$\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}}\xrightarrow{p^n}\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}}\longrightarrow\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}}/p^n\longrightarrow 0.$$

Let $\mathbb{K}_n:=\ker(\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}}\xrightarrow{p^n}\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}})$. Then $\mathbb{K}_n$ is isomorphic to $\mathcal{T}or^1_{\mathbb{Z}_p}(\mathbb{L},\mathbb{A}^0_{\mathrm{cris}}/p^n)\simeq\mathcal{T}or^1_{\mathbb{Z}_p}(\mathbb{L}_{\mathrm{tor}},\mathbb{A}^0_{\mathrm{cris}}/p^n)$, thus is killed by p^N. Let $\mathbb{I}_n:=p^n(\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}})\subset\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}}$. So we have two short exact sequences:

$$0\longrightarrow\mathbb{K}_n\longrightarrow\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}}\longrightarrow\mathbb{I}_n\longrightarrow 0,$$

and

$$0\longrightarrow\mathbb{I}_n\longrightarrow\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}}\longrightarrow\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}}/p^n\longrightarrow 0.$$

Taking cohomology one gets exact sequences

$$\ldots\longrightarrow H^i(X_{\overline{k}},\mathbb{K}_n)\longrightarrow H^i(X_{\overline{k}},\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}})\xrightarrow{\gamma_i}H^i(X_{\overline{k}},\mathbb{I}_n)\longrightarrow H^{i+1}(X_{\overline{k}},\mathbb{K}_n)\longrightarrow\ldots$$

and

$$\ldots\longrightarrow H^i(X_{\overline{k}},\mathbb{I}_n)\xrightarrow{\beta_i}H^i(X_{\overline{k}},\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}})\longrightarrow H^i(X_{\overline{k}},\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}}/p^n)\longrightarrow\ldots,$$

which give rise to the exact sequence below:

$$H^i(X_{\overline{k}},\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}})/p^n\longrightarrow H^i(X_{\overline{k}},\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}}/p^n)\longrightarrow\ker(\beta_{i+1})\longrightarrow 0,$$

such that the kernel of the first morphism is killed by p^N. On the other hand, we have the following commutative diagram

$$\begin{array}{ccc} H^{i+1}(X_{\overline{k}},\mathbb{I}_n) & \xrightarrow{\beta_{i+1}} & H^{i+1}(X_{\overline{k}},\mathbb{L}\otimes\mathbb{A}_{\mathrm{cris}}/p^n)\ . \\ \uparrow{\scriptstyle\gamma_{i+1}} & \nearrow{\scriptstyle p^n} & \\ H^{i+1}(X_{\overline{k}},\mathbb{L}\otimes\mathbb{A}_{\mathrm{cris}}) & & \end{array}$$

Note that $\ker(H^{i+1}(X_{\overline{k}},\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}})\xrightarrow{p^n}H^{i+1}(X_{\overline{k}},\mathbb{L}\otimes\mathbb{A}^0_{\mathrm{cris}}))$ is, via the almost isomorphism (4.0.10) for H^{i+1}, almost isomorphic to $\mathrm{Tor}^1_{\mathbb{Z}_p}(H^{i+1}(X_{\overline{k}},\mathbb{L})_{\mathrm{tor}},A^0_{\mathrm{cris}}/p^n)$, hence is killed by p^N. Moreover, $\mathrm{coker}(\gamma_{i+1})$ is contained in $H^{i+1}(X_{\overline{k}},\mathbb{K}_n)$, thus is also killed by p^N. As a result, from the commutative diagram above we deduce that $\ker(\beta_{i+1})$ is killed by p^{2N}, giving our claim.

Now we claim that the following canonical map

$$H^i(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathbb{A}_{\mathrm{cris}})\longrightarrow\varprojlim H^i(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathbb{A}_{\mathrm{cris}}/p^n)\tag{4.0.13}$$

is almost surjective with kernel killed by p^{2N}. Since $\mathbb{L}$ is a lisse $\widehat{\mathbb{Z}}_p$-sheaf, it is locally a direct sum of a finite free $\widehat{\mathbb{Z}}_p$-module and a finite product of copies of $\widehat{\mathbb{Z}}_p/p^m$'s. In particular, $R^j\varprojlim(\mathbb{L}\otimes\mathbb{A}^a_{\mathrm{cris}}/p^n)=0$ whenever $j>0$ as the same holds for the projective system $\{\mathbb{A}^a_{\mathrm{cris}}/p^n\}_n$ (recall (2.1.11)). As a result, by the almost version of Lemma 4.1, we dispose the following almost short exact sequence for each i:

$$0\longrightarrow R^1\varprojlim H^{i-1}(X_{\overline{k}},\mathbb{L}\otimes\mathbb{A}_{\mathrm{cris}}/p^n)\longrightarrow H^i(X_{\overline{k}},\mathbb{L}\otimes\mathbb{A}_{\mathrm{cris}})\longrightarrow\varprojlim H^i(X_{\overline{k}},\mathbb{L}\otimes\mathbb{A}_{\mathrm{cris}}/p^n)\longrightarrow 0$$

On the other hand, the morphisms (4.0.11) for all n give rise to a morphism of projective systems whose kernel and cokernel are killed by p^{2N}:

$$\{H^{i-1}(X_{\overline{k}},\mathbb{L})\otimes_{\mathbb{Z}_p}(A^0_{\mathrm{cris}}/p^n)\}_{n\geq 0}\to\{H^{i-1}(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathbb{A}^0_{\mathrm{cris}}/p^n)\}_{n\geq 0}$$

Therefore $R^1\varprojlim_n H^{i-1}(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathbb{A}^0_{\mathrm{cris}}/p^n)$ is killed by p^{2N}. This concludes the proof of the claim.

Consequently, by the following commutative digram

$$\begin{array}{ccc} H^i(X_{\overline{k}},\mathbb{L})\otimes_{\mathbb{Z}_p} A^a_{\mathrm{cris}} & \xrightarrow{\text{can}} & H^i(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathbb{A}^a_{\mathrm{cris}}) \\ & \searrow^{\varprojlim (4.0.11)} & \downarrow{\scriptstyle (4.0.13)} \\ & & \varprojlim H^i(X_{\overline{k}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathbb{A}_{\mathrm{cris}}/p^n) \end{array}$$

we deduce that the kernel and the cokernel of the horizontal canonical morphism are killed by p^{4N}. On inverting p, we obtain the desired isomorphism (4.0.8).

We still need to check that (4.0.8) is compatible with the extra structures. Clearly only the strict compatibility with filtrations needs verification, and it suffices to check this on gradeds. So we reduce to showing that the natural morphism is an isomorphism:

$$H^i(X_{\overline{k}},\mathbb{L})\otimes_{\mathbb{Z}_p}\mathbb{C}_p(j)\longrightarrow H^i(X_{\overline{k}},\mathbb{L}\otimes\widehat{\mathcal{O}}_X(j)).$$

Twisting, one reduces to $j=0$, which is given by the following lemma. □

Lemma 4.4. *Let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf on $X_{\overline{k},\mathrm{pro\acute{e}t}}$. Then the following natural morphism is an isomorphism:*

$$H^i(X_{\overline{k},\mathrm{pro\acute{e}t}},\mathbb{L})\otimes_{\mathbb{Z}_p}\mathbb{C}_p\xrightarrow{\approx} H^i\left(X_{\overline{k},\mathrm{pro\acute{e}t}},\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\widehat{\mathcal{O}}_X\right),$$

where $\widehat{\mathcal{O}}_X$ is the completed structural sheaf of $X_{\overline{k},\mathrm{pro\acute{e}t}}$ and $\mathbb{C}_p=\widehat{\overline{k}}$.

Proof. The proof is similar to that of the first part of Theorem 4.3. Let $\mathbb{L}_n:=\mathbb{L}/p^n\mathbb{L}$. Using the (finite) filtration $\{p^m\cdot\mathbb{L}_n\}_m$ of $\mathbb{L}_n$ and by induction on m, we get from [Sch13, Theorem 5.1] the following natural almost isomorphisms

$$H^i(X_{\overline{k}},\mathbb{L}_n)\otimes_{\mathbb{Z}_p}\mathcal{O}_{\mathbb{C}_p}\xrightarrow{\approx} H^i\left(X_{\overline{k}},\mathbb{L}_n\otimes_{\widehat{\mathbb{Z}}_p}\widehat{\mathcal{O}}_X^+\right).$$

We need to look at the project limits (with respect to n) of both sides of the previous morphisms. Pick $N\in\mathbb{N}$ such that p^N kills $\mathbb{L}_{\mathrm{tor}}$ and $H^i(X_{\overline{k}},\mathbb{L})_{\mathrm{tor}}$ for all $i\in\mathbb{N}$. For $n\geq N$, there is a tautological exact sequence

$$0\longrightarrow\mathbb{L}_{\mathrm{tor}}\longrightarrow\mathbb{L}\xrightarrow{p^n}\mathbb{L}\longrightarrow\mathbb{L}_n\longrightarrow 0.$$

Splitting it into two short exact sequences and taking cohomology, we obtain exact sequences

$$\ldots\longrightarrow H^i(X_{\overline{k}},p^n\mathbb{L})\longrightarrow H^i(X_{\overline{k}},\mathbb{L})\longrightarrow H^i(X_{\overline{k}},\mathbb{L}_n)\longrightarrow\ldots$$

and

$$\ldots\longrightarrow H^i(X_{\overline{k}},\mathbb{L}_{\mathrm{tor}})\longrightarrow H^i(X_{\overline{k}},\mathbb{L})\longrightarrow H^i(X_{\overline{k}},p^n\mathbb{L})\longrightarrow\ldots,$$

from which we deduce the exact sequence below:

$$H^i(X_{\overline{k}},\mathbb{L})/p^n\longrightarrow H^i(X_{\overline{k}},\mathbb{L}_n)\longrightarrow\ker(H^{i+1}(X_{\overline{k}},p^n\mathbb{L})\longrightarrow H^{i+1}(X_{\overline{k}},\mathbb{L}))\longrightarrow 0.$$

As in the proof of Theorem 4.3, the kernel of the first morphism in the sequence above is killed by p^N while the last abelian group is killed by p^{2N}. Consequently, the kernel and the cokernel of the following natural morphism are killed by p^{2N}:

$$H^i(X_{\overline{k}},\mathbb{L})\otimes\mathcal{O}_{\mathbb{C}_p}/p^n\longrightarrow H^i(X_{\overline{k}},\mathbb{L}_n)\otimes\mathcal{O}_{\mathbb{C}_p}.$$

Passing to projective limits, we find a natural morphism with kernel and cokernel killed by p^{2N}:

$$H^i(X_{\overline{k}},\mathbb{L})\otimes\mathcal{O}_{\mathbb{C}_p}\simeq\varprojlim\left(H^i(X_{\overline{k}},\mathbb{L})\otimes\mathcal{O}_{\mathbb{C}_p}/p^n\right)\longrightarrow\varprojlim\left(H^i(X_{\overline{k}},\mathbb{L}_n)\otimes\mathcal{O}_{\mathbb{C}_p}\right),$$

and $R^1\varprojlim\left(H^i(X_{\overline{k}},\mathbb{L}_n)\otimes\mathcal{O}_{\mathbb{C}_p}\right)\simeq R^1\varprojlim H^i(X_{\overline{k}},\mathbb{L}_n\otimes\widehat{\mathcal{O}}_X^+)$ are both killed by p^{2N}. On the other hand, as $\mathbb{L}$ is a lisse $\widehat{\mathbb{Z}}_p$-sheaf, $R^j\varprojlim(\mathbb{L}_n\otimes\widehat{\mathcal{O}}_X^+)=0$ for $j>0$ since the same holds for $\{\widehat{\mathcal{O}}_X^+/p^n\}_n$; for this, apply [Sch13, Lemma 3.18] to Lemma 4.10 *loc.cit.*. Hence by Lemma 4.1, we deduce a short exact sequence

$$0\longrightarrow R^1\varprojlim H^{i-1}(X_{\overline{k}},\mathbb{L}_n\otimes\widehat{\mathcal{O}}_X^+)\longrightarrow H^i(X_{\overline{k}},\mathbb{L}\otimes\widehat{\mathcal{O}}_X^+)\longrightarrow\varprojlim H^i(X_{\overline{k}},\mathbb{L}_n\otimes\widehat{\mathcal{O}}_X^+)\longrightarrow 0.$$

So we get the following commutative diagram

$$\begin{array}{ccc} H^i(X_{\overline{k}},\mathbb{L})\otimes\mathcal{O}_{\mathbb{C}_p} & \longrightarrow & H^i(X_{\overline{k}},\mathbb{L}\otimes\widehat{\mathcal{O}}_X^+). \\ & \searrow^{\text{iso. up to } p^{2N}-\text{torsion}} & \downarrow{\scriptstyle \text{iso. up to } p^{2N}-\text{torsion}} \\ & & \varprojlim H^i(X_{\overline{k}},\mathbb{L}_n\otimes\widehat{\mathcal{O}}_X^+) \end{array}$$

In particular, the horizontal morphism is an isomorphism up to p^{4N}-torsions. On inverting p, we get our lemma. □

Recall that the notion of lisse $\mathbb{Z}_p$-sheaf on $X_{\text{ét}}$ and lisse $\widehat{\mathbb{Z}}_p$-sheaf on $X_{\text{proét}}$ are equivalent. We finally deduce the following crystalline comparison theorem:

Theorem 4.5. *Let $\mathcal{X}$ be a proper smooth formal scheme over $\mathcal{O}_k$, with X (resp. $\mathcal{X}_0$) its generic (resp. closed) fiber. Let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf on $X_{\text{proét}}$. Assume that $\mathbb{L}$ is associated to a filtered F-isocrystal $\mathcal{E}$ on $\mathcal{X}_0/\mathcal{O}_k$. Then there exists a canonical isomorphism of B_{cris}-modules*

$$H^i(X_{\overline{k},\text{ét}},\mathbb{L})\otimes_{\mathbb{Z}_p}B_{\text{cris}}\xrightarrow{\sim}H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k,\mathcal{E})\otimes_{\mathcal{O}_k}B_{\text{cris}}$$

compatible with Galois action, filtration and Frobenius.

Proof. This is just the composition of the isomorphisms in Themorem 3.27 and Theorem 4.3. □

5. Comparison isomorphism in the relative setting

Let $f\colon\mathcal{X}\to\mathcal{Y}$ be a smooth morphism between two smooth formal schemes over $\mathrm{Spf}(\mathcal{O}_k)$ of relative dimension $d\geq 0$. The induced morphism between the generic fibers will be denoted by $f_k\colon X\to Y$. We shall denote by $w_{\mathcal{X}}$ (resp. $w_{\mathcal{Y}}$) the natural morphism of topoi $X^\sim_{\text{proét}}\to\mathcal{X}^\sim_{\text{ét}}$ (resp. $Y^\sim_{\text{proét}}\to\mathcal{Y}^\sim_{\text{ét}}$). By abuse of notation, the morphism of topoi $X^\sim_{\text{proét}}\to Y^\sim_{\text{proét}}$ will be still denoted by f_k.

Let $\nabla_{X/Y}:\mathcal{O}\mathbb{B}^+_{\text{cris},X}\to\mathcal{O}\mathbb{B}^+_{\text{cris},X}\otimes_{\mathcal{O}_X^{\text{ur}}}\Omega^{1,\text{ur}}_{X/Y}$ be the natural relative derivation, where $\Omega^{1,\text{ur}}_{X/Y}:=w_{\mathcal{X}}^*\Omega^1_{\mathcal{X}/\mathcal{Y}}$.

Proposition 5.1. *(1) (Relative Poincaré lemma) The following sequence of pro-étale sheaves is exact and strict with respect to the filtration giving $\Omega^{i,\text{ur}}_{X/Y}$ degree i:*

$$\begin{aligned}0\longrightarrow\mathbb{B}^+_{\text{cris},X}\widehat{\otimes}_{f_k^{-1}\mathbb{B}^+_{\text{cris},Y}}f_k^{-1}\mathcal{O}\mathbb{B}^+_{\text{cris},Y}\to\mathcal{O}\mathbb{B}^+_{\text{cris},X}&\xrightarrow{\nabla_{X/Y}}\mathcal{O}\mathbb{B}^+_{\text{cris},X}\otimes_{\mathcal{O}_X^{\text{ur}}}\Omega^{1,\text{ur}}_{X/Y}\\&\xrightarrow{\nabla_{X/Y}}\cdots\xrightarrow{\nabla_{X/Y}}\mathcal{O}\mathbb{B}^+_{\text{cris},X}\otimes_{\mathcal{O}_X^{\text{ur}}}\Omega^{d,\text{ur}}_{X/Y}\to 0.\end{aligned}$$

Furthermore, the connection $\nabla_{X/Y}$ is integrable and satisfies Griffiths transversality with respect to the filtration, i.e. $\nabla_{X/Y}(\mathrm{Fil}^i\,\mathcal{O}\mathbb{B}^+_{\mathrm{cris},X}) \subset \mathrm{Fil}^{i-1}\,\mathcal{O}\mathbb{B}^+_{\mathrm{cris},X} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \Omega^{1,\mathrm{ur}}_{X/Y}$.

(2) Suppose the Frobenius on $\mathcal{X}_0$ (resp. $\mathcal{Y}_0$) lifts to a Frobenius σ_X (resp. σ_Y) on the formal scheme $\mathcal{X}$ (resp. $\mathcal{Y}$) and they commute with f. Then the induced Frobenius φ_X on $\mathcal{O}\mathbb{B}^+_{\mathrm{cris},X}$ is horizontal with respect to $\nabla_{X/Y}$.

Proof. The proof is routine (cf. Proposition 2.12), so we omit the detail here. □

For the relative version of the crystalline comparison, we shall need the following primitive comparison in the relative setting.

Proposition 5.2. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a proper smooth morphism between two smooth formal schemes over $\mathcal{O}_k$. Let $\mathbb{L}$ be a lisse $\widehat{\mathbb{Z}}_p$-sheaf on $X_{\mathrm{pro\acute{e}t}}$. Suppose that $R^i f_{k*}\mathbb{L}$ is a lisse $\widehat{\mathbb{Z}}_p$-sheaf on $Y_{\mathrm{pro\acute{e}t}}$ for all $i \geq 0$. Then the following canonical morphism is an isomorphism:*

$$(R^i f_{k*}\mathbb{L}) \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}^+_{\mathrm{cris},Y} \xrightarrow{\sim} R^i f_{k*}(\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}^+_{\mathrm{cris},X}) \tag{5.0.14}$$

which is compatible with filtration and Frobenius.

Proof. Remark first that $R^i f_{k*}\mathbb{L} = 0$ for $i > 2d$ where d denotes the relative dimension of f. To show this, write $\mathbb{L}'$ and $\mathbb{L}''$ the $\mathbb{F}_p$-local system on $X_{\mathrm{pro\acute{e}t}}$ defined by the following exact sequence

$$0 \longrightarrow \mathbb{L}' \longrightarrow \mathbb{L} \xrightarrow{p} \mathbb{L} \longrightarrow \mathbb{L}'' \longrightarrow 0. \tag{5.0.15}$$

We claim that $R^i f_{k*}\mathbb{L}' = R^i f_{k*}\mathbb{L}'' = 0$ for $i > 2d$. Indeed, as $\mathbb{L}'$ is an $\mathbb{F}_p$-local system of finite presentation, it comes from an $\mathbb{F}_p$-local system of finite presentation on $X_{\text{ét}}$, still denoted by $\mathbb{L}'$. By [Sch13, Corollary 3.17(ii)], we are reduced to showing that $R^i f_{k\,\text{ét}*}\mathbb{L}' = 0$ for $i > 2d$, which follows from [Sch13, Theorem 5.1] and [Hub, 2.6.1] by taking fibers of $R^i f_{k\,\text{ét}*}\mathbb{L}'$ at geometric points of Y. Similarly $R^i f_{k*}\mathbb{L}'' = 0$ for $i > 2d$. Then, splitting the exact sequence (5.0.15) into two short exact sequences as in the proof of Theorem 4.3 and applying the higher direct image functor Rf_{k*}, we deduce that the multiplication-by-p morphism on $R^i f_{k*}\mathbb{L}$ is surjective for $i > 2d$. But $R^i f_{k*}\mathbb{L}$ is a lisse $\widehat{\mathbb{Z}}_p$-sheaf on $Y_{\mathrm{pro\acute{e}t}}$ by our assumption, necessarily $R^i f_{k*}\mathbb{L} = 0$ for $i > 2d$. Consequently, we can choose a sufficiently large integer $N \in \mathbb{N}$ such that p^N kills the torsion part of $\mathbb{L}$ and also the torsion part of $R^i f_{k*}\mathbb{L}$ for all $i \in \mathbb{Z}$.

Then, it is shown in the proof of [Sch13, Theorem 8.8 (i)], as a consequence of the primitive comparison isomorphism in the relative setting ([Sch13, Corollary 5.11]), that the following canonical morphism is an almost isomorphism:

$$R^i f_{k*}\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{A}_{\mathrm{inf},Y} \xrightarrow{\approx} R^i f_{k*}(\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{A}_{\mathrm{inf},X}). \tag{5.0.16}$$

With this in hand, the proof of Theorem 4.3 applies and gives the result. Indeed, consider the PD-envelope $\mathbb{A}^0_{\mathrm{cris},X}$ (resp. $\mathbb{A}^0_{\mathrm{cris},Y}$) of $\mathbb{A}_{\mathrm{inf},X}$ (resp. of $\mathbb{A}_{\mathrm{inf},Y}$) with respect to the ideal $\ker(\theta_X\colon \mathbb{A}_{\mathrm{inf},X} \to \widehat{\mathcal{O}_X^+})$ (resp. to the ideal $\ker(\theta_Y\colon \mathbb{A}_{\mathrm{inf},Y} \to \widehat{\mathcal{O}_Y^+})$). Then $\mathbb{A}_{\mathrm{cris},X}$ and $\mathbb{A}_{\mathrm{cris},Y}$ are respectively the p-adic completions of $\mathbb{A}^0_{\mathrm{cris},X}$ and $\mathbb{A}^0_{\mathrm{cris},Y}$. As in the proof of Theorem 4.3, we obtain from (5.0.16) the following canonical almost isomorphism

$$R^i f_{k*}\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{A}^0_{\mathrm{cris},Y} \xrightarrow{\approx} R^i f_{k*}(\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{A}^0_{\mathrm{cris},X}), \tag{5.0.17}$$

from which we deduce that for each n the kernel and the cokernel of the natural morphism below are killed by p^{2N}:

$$R^i f_{k*}\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{A}^0_{\mathrm{cris},Y}/p^n \longrightarrow R^i f_{k*}(\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{A}^0_{\mathrm{cris},X}/p^n). \tag{5.0.18}$$

Recall from the proof of Theorem 4.3 that $R^j \varprojlim(\mathbb{L} \otimes \mathbb{A}^a_{\mathrm{cris},X}/p^n) = 0$ whenever $j > 0$, hence a spectral sequence:

$$E_2^{j,i} = R^j \varprojlim R^i f_{k*}(\mathbb{L} \otimes \mathbb{A}^a_{\mathrm{cris},X}/p^n) \Longrightarrow R^{i+j} f_{k*}(\mathbb{L} \otimes \mathbb{A}^a_{\mathrm{cris},X}).$$

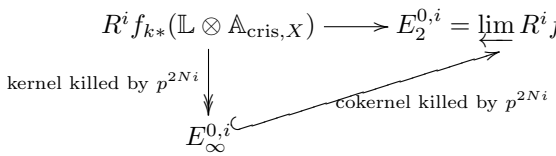
As $R^i f_{k*}\mathbb{L}$ is a lisse $\widehat{\mathbb{Z}}_p$-sheaf, $R^j \varprojlim_n \left(R^i f_{k*}(\mathbb{L}) \otimes \mathbb{A}^0_{\mathrm{cris},Y}/p^n\right) = 0$ for $j > 0$. It follows that $R^j \varprojlim_n R^i f_{k*}(\mathbb{L} \otimes \mathbb{A}^0_{\mathrm{cris},X}/p^n)$ is killed by p^{2N} for any $j > 0$. In particular, $p^{2N} \cdot E_\infty^{j,i} = 0$ whenever $j > 0$. With i fixed, let $\mathbb{H} := R^i f_{k*}(\mathbb{L} \otimes \mathbb{A}^0_{\mathrm{cris},X})$. From the theory of spectral sequences, we know that $\mathbb{H}$ is endowed with a finite filtration

$$0 = F^{i+1}\mathbb{H} \subset F^i\mathbb{H} \subset \ldots F^1\mathbb{H} \subset F^0\mathbb{H} = \mathbb{H},$$

such that $F^q\mathbb{H}/F^{q+1}\mathbb{H} \simeq E_\infty^{q,i-q}$. Therefore, $F^1\mathbb{H}$ is killed by p^{2Ni}. On the other hand, $E_2^{0,i}$ has a filtration of length i

$$E_\infty^{0,i} = E_{2+i}^{0,i} \subset E_{2+(i-1)}^{0,i} \subset \ldots \subset E_2^{0,i} = \varprojlim R^i f_{k*}(\mathbb{L} \otimes \mathbb{A}_{\mathrm{cris},X}/p^n).$$

Since in general $E_{r+1}^{0,i} = \ker(d_r \colon E_r^{0,i} \to E_r^{r,i-r+1})$, it follows that all the successive quotients of the filtration above are killed by p^{2N}. So the inclusion $E_\infty^{0,i} \subset E_2^{0,i}$ has cokernel killed by p^{2Ni}. To summarize, we have a commutative diagram



Hence the natural morphism

$$R^i f_{k*}(\mathbb{L} \otimes \mathbb{A}^a_{\mathrm{cris},X}) \twoheadrightarrow \varprojlim R^i f_{k*}(\mathbb{L} \otimes \mathbb{A}^a_{\mathrm{cris},X}/p^n) \tag{5.0.19}$$

has kernel and cokernel killed by p^{2Ni}. Therefore we deduce like in the proof of Theorem 4.3 that the kernel and the cokernel of the following canonical morphism are killed by p^{2N+2Ni}:

$$R^i f_{k*}\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{A}_{\mathrm{cris},Y} \longrightarrow R^i f_{k*}(\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{A}_{\mathrm{cris},X}). \tag{5.0.20}$$

Inverting p in the above morphism, we obtain the desired isomorphism of our lemma.

It remains to verify the compatibility of the isomorphism (5.0.14) with the extra structures. It clearly respects Frobenius structures. To check the (strict) compatibility with respect to filtrations, by taking grading quotients, we just need to show that for each $r \in \mathbb{N}$, the following natural morphism

$$Rf_{k*}\mathbb{L} \otimes \widehat{\mathcal{O}}_Y(r) \longrightarrow Rf_{k*}(\mathbb{L} \otimes \widehat{\mathcal{O}}_X(r))$$

is an isomorphism: it is a local question, hence it suffices to show this after restricting the latter morphism to $Y_{\overline{k}}$. As $\widehat{\mathcal{O}}_X(r)|_{X_{\overline{k}}} \simeq \widehat{\mathcal{O}}_X|_{X_{\overline{k}}}$ and $\widehat{\mathcal{O}}_Y(r)|_{Y_{\overline{k}}} \simeq \widehat{\mathcal{O}}_Y|_{Y_{\overline{k}}}$, we then reduce to the case where $r = 0$. The proof of the latter statement is similar to that of Lemma 4.4, so we omit the details here. □

For a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{F}$ with an $\mathcal{O}_{\mathcal{Y}}$-linear connection $\nabla\colon \mathcal{F} \to \mathcal{F}\otimes\Omega^1_{\mathcal{X}/\mathcal{Y}}$, we denote the de Rham complex of $\mathcal{F}$ as:

$$DR_{X/Y}(\mathcal{F}) := (\ldots \longrightarrow 0 \longrightarrow \mathcal{F} \xrightarrow{\nabla} \mathcal{F}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^1_{\mathcal{X}/\mathcal{Y}} \xrightarrow{\nabla} \ldots).$$

The same rule applies if we consider an $\mathcal{O}_X^{\mathrm{un}}$-module endowed with an $\mathcal{O}_Y^{\mathrm{un}}$-linear connection etc.

In the lemma below, assume $\mathcal{Y} = \mathrm{Spf}(A)$ is affine and is étale over a torus $\mathcal{S} = \mathrm{Spf}(\mathcal{O}_k\{S_1^{\pm 1},\ldots,S_\delta^{\pm 1}\})$. For each $1 \le j \le \delta$, let $(S_j^{1/p^n})_{n\in\mathbb{N}}$ be a compatible family of p-power roots of S_j. As in Proposition 2.12, set

$$\widetilde{Y} := \left(Y \times_{\mathcal{S}_k} \mathrm{Spa}\left(k\{S_1^{\pm 1/p^n},\ldots,S_\delta^{\pm 1/p^n}\}, \mathcal{O}_k\{S_1^{\pm 1/p^n},\ldots,S_\delta^{\pm 1/p^n}\}\right)\right)_{n\in\mathbb{N}} \in Y_{\mathrm{pro\acute{e}t}}.$$

Lemma 5.3. *Let $V \in Y_{\mathrm{pro\acute{e}t}}$ be an affinoid perfectoid which is pro-étale over $\widetilde{Y}_{\overline{k}}$, with $\widehat{V} = \mathrm{Spa}(R, R^+)$. Let w_V be the composite of natural morphisms of topoi*

$$w_V\colon X^{\sim}_{\mathrm{pro\acute{e}t}}/X_V \longrightarrow X^{\sim}_{\mathrm{pro\acute{e}t}} \xrightarrow{w} \mathcal{X}^{\sim}_{\text{ét}}.$$

Then

(1) for any $j > 0$ and $r \in \mathbb{Z}$, $R^j w_{V*}\mathcal{O}\mathbb{B}_{\mathrm{cris}} = R^j w_{V*}(\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris}}) = 0$; and

(2) the natural morphisms

$$\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \longrightarrow w_{V*}(\mathcal{O}\mathbb{B}_{\mathrm{cris},X})$$

and

$$\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \to w_{V*}(\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris},X}) \quad \text{for all } r \in \mathbb{Z}$$

are isomorphisms. Here $\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) := \left(\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)\right)[1/t]$ with

$$\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V) := \varprojlim\left(\mathcal{O}_{\mathcal{X}}\otimes_A\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)/p^n\right),$$

and

$$\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) := \varinjlim_{n\in\mathbb{N}} t^{-n}\left(\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathrm{Fil}^{r+n}\,\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)\right).$$

In particular, if we filter $\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)$ using $\{\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)\}_{r\in\mathbb{Z}}$, the natural morphism

$$\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \longrightarrow Rw_{V*}(\mathcal{O}\mathbb{B}_{\mathrm{cris},X})$$

is an isomorphism in the filtered derived category.

Proof. (1) Recall that for $j \ge 0$, $R^j w_{V*}\mathcal{O}\mathbb{B}_{\mathrm{cris},X}$ is the associated sheaf on $\mathcal{X}_{\text{ét}}$ of the presheaf sending $\mathcal{U} \in \mathcal{X}_{\text{ét}}$ to $H^i(\mathcal{U}_V, \mathcal{O}\mathbb{B}_{\mathrm{cris},X})$, where $\mathcal{U}_V := \mathcal{U}_k \times_X X_V$. Now we take $\mathcal{U} = \mathrm{Spf}(B) \in \mathcal{X}_{\text{ét}}$ to be affine such that the composition of $\mathcal{U} \to \mathcal{X}$ together with $f\colon \mathcal{X} \to \mathcal{Y}$ can be factored as

$$\mathcal{U} \longrightarrow \mathcal{T} \longrightarrow \mathcal{Y},$$

where the first morphism is étale and that $\mathcal{T} = \mathrm{Spf}(A\{T_1^{\pm 1},\ldots,T_d^{\pm 1}\})$ is a d-dimensional torus over $\mathcal{Y}$. Write $\mathcal{T}_V = \mathcal{T}_k \times_Y V$. Then $\mathcal{T}_V = \mathrm{Spa}(S, S^+)$ with $S^+ = R\{T_1^{\pm 1},\ldots,T_d^{\pm 1}\}$ and $S = S^+[1/p]$. Write also $\mathcal{U}_V = \mathrm{Spa}(\widetilde{S}, \widetilde{S}^+)$. For each $1 \le i \le d$, let $(T_i^{1/p^n})_{n\in\mathbb{N}}$ be a compatible family of p-power root of T_i, and set

$$S_\infty^+ = R^+\{T_1^{\pm 1/p^\infty},\ldots,T_d^{\pm 1/p^\infty}\}, \quad \widetilde{S}_\infty^+ := B\widehat{\otimes}_{A\{T_1^{\pm 1},\ldots,T_d^{\pm 1}\}}S_\infty^+,$$

$S_\infty := S_\infty^+[1/p]$ and $\widetilde{S}_\infty = \widetilde{S}_\infty^+[1/p]$. Then (S_∞, S_∞^+) and $(\widetilde{S}_\infty, \widetilde{S}_\infty^+)$ are two affinoid perfectoid algebras over $(\widehat{\overline{k}}, \mathcal{O}_{\widehat{\overline{k}}})$. Let $\widetilde{\mathcal{U}_V} \in X_{\mathrm{pro\acute{e}t}}$ (resp. $\widetilde{\mathcal{T}_V} \in \mathcal{T}_{k\,\mathrm{pro\acute{e}t}}$) be the

affinoid perfectoid corresponding to $(\widetilde{S}_\infty, \widetilde{S}^+_\infty)$ (resp. to (S_∞, S^+_∞)). So we have the following commutative diagram of ringed spaces

$$\begin{array}{ccccc}
\widehat{\widetilde{\mathcal{U}_V}} = \mathrm{Spf}(\widetilde{S}_\infty, \widetilde{S}^+_\infty) & \xrightarrow{\Gamma} & \widehat{\mathcal{U}_V} = \mathrm{Spa}(\widetilde{S}, \widetilde{S}^+) & \longrightarrow & \mathcal{U} = \mathrm{Spf}(B)\ . \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{\widetilde{\mathcal{T}_V}} = \mathrm{Spa}(S_\infty, S^+_\infty) & \longrightarrow & \widehat{\mathcal{T}_V} = \mathrm{Spa}(S, S^+) & \longrightarrow & \mathcal{T} \\
 & & \downarrow & & \downarrow \\
 & & \widehat{V} = \mathrm{Spa}(R, R^+) & \longrightarrow & \mathcal{Y} = \mathrm{Spf}(A)
\end{array}$$

The morphism $\widetilde{\mathcal{U}_V} \to \mathcal{U}_V$ is a profinite Galois cover, with Galois group Γ isomorphic to $\mathbb{Z}_p(1)^d$.

For $q \in \mathbb{N}$, let $\widetilde{\mathcal{U}_V}^q$ be the $(q+1)$-fold fiber product of $\widetilde{\mathcal{U}_V}$ over $\mathcal{U}_V$. So $\widetilde{\mathcal{U}_V}^q \simeq \widetilde{\mathcal{U}_V} \times \Gamma^q$ is affinoid perfectoid. In particular, for $j > 0$, $H^j(\widetilde{\mathcal{U}_V}^q, \mathcal{O}\mathbb{A}_{\mathrm{cris},X})^a = 0$ (Lemma 2.5) and

$$H^0\left(\widetilde{\mathcal{U}_V}^q, \mathcal{O}\mathbb{A}_{\mathrm{cris},X}\right) = \mathrm{Hom}_{\mathrm{cont}}\left(\Gamma^q, \mathcal{O}\mathbb{A}_{\mathrm{cris},X}(\widetilde{\mathcal{U}_V})\right). \tag{5.0.21}$$

Consequently, the Cartan-Leray spectral sequence associated with the cover $\widetilde{\mathcal{U}_V} \to \mathcal{U}_V$ almost degenerates at E_2:

$$E_1^{q,j} = H^j\left(\widetilde{\mathcal{U}_V}^q, \mathcal{O}\mathbb{A}_{\mathrm{cris},X}\right) \Longrightarrow H^{q+j}(\mathcal{U}_V, \mathcal{O}\mathbb{A}_{\mathrm{cris},X}),$$

giving an almost isomorphism $E_2^{q,0} \approx H^q(\mathcal{U}_V, \mathcal{O}\mathbb{A}_{\mathrm{cris},X})$ for each $q \in \mathbb{Z}$. Using (5.0.21), we see that $E_2^{q,0}$ is the continuous group cohomology $H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris},X}(\widetilde{\mathcal{U}_V}))$. On the other hand, $\mathcal{O}\mathbb{A}_{\mathrm{cris},X}(\widetilde{\mathcal{U}_V}) \approx \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty)$ by Lemma 2.17, so

$$\begin{aligned}
H^q(\mathcal{U}_V, \mathcal{O}\mathbb{B}_{\mathrm{cris},X}) &\simeq H^q(\mathcal{U}_V, \mathcal{O}\mathbb{A}_{\mathrm{cris},X})[1/t] \\
&\simeq H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris},X}(\widetilde{\mathcal{U}_V}))[1/t] \\
&\simeq H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty))[1/t].
\end{aligned}$$

By definition, the last cohomology group is precisely $H^q(\Gamma, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty))$ computed in §6. Namely, from Theorem 6.12 we deduce $H^q(\mathcal{U}_V, \mathcal{O}\mathbb{B}_{\mathrm{cris},X}) = 0$ whenever $q > 0$, and that the natural morphism

$$\mathcal{O}_{\mathcal{X}}(\mathcal{U})\widehat{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V) = B\widehat{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \longrightarrow H^0(\Gamma, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty)) = H^0(\mathcal{U}_V, \mathcal{O}\mathbb{B}_{\mathrm{cris},X})$$

is an isomorphism. Varying $\mathcal{U}$ in $\mathcal{X}_{\text{ét}}$ and passing to associated sheaf, we obtain

$$\mathcal{O}_{\mathcal{X}}\widetilde{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \xrightarrow{\sim} w_{V*}\mathcal{O}\mathbb{B}_{\mathrm{cris},X},$$

where by definition, $\mathcal{O}_{\mathcal{X}}\widetilde{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)$ is the associated sheaf of the presheaf over $\mathcal{X}_{\text{ét}}$ sending $\mathcal{U} \in \mathcal{X}_{\text{ét}}$ to $\mathcal{O}_{\mathcal{X}}(\mathcal{U})\widehat{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V)$.

To concludes the proof of (1) it remains to check that the canonical morphism below is an isomorhpism:

$$\mathcal{O}_{\mathcal{X}}\widetilde{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \longrightarrow \mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V). \tag{5.0.22}$$

Indeed, using $w_{V*}\mathcal{O}\mathbb{A}_{\mathrm{cris},X} = \varprojlim w_{V*}(\mathcal{O}\mathbb{A}_{\mathrm{cris},X}/p^n)$ and Proposition 6.11, we deduce a natural morphism which is injective with cokernel killed by $(1-[\epsilon])^{2d}$:

$$\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V) \longrightarrow w_{V*}\mathcal{O}\mathbb{A}_{\mathrm{cris},X}.$$

Therefore inverting t, we get an isomorphism

$$\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \xrightarrow{\sim} w_{V*}\mathcal{O}\mathbb{B}_{\mathrm{cris},X},$$

from which the desired isomorphism (5.0.22) follows, since the objects on both sides of *loc. cit.* are isomorphic in a natural way to $w_{V*}(\mathcal{O}\mathbb{B}_{\mathrm{cris},X})$.

(2) For $r\in\mathbb{Z}$, define $\mathcal{O}_{\mathcal{X}}\widetilde{\otimes}_A\mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)$ and $\mathcal{O}_{\mathcal{X}}\widetilde{\otimes}_A\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)$ in the same way as is done for $\mathcal{O}_{\mathcal{X}}\widetilde{\otimes}_A\mathbb{B}_{\mathrm{cris},Y}(V)$. Like the proof above, we have the natural isomorphism below

$$\mathcal{O}_{\mathcal{X}}\widetilde{\otimes}_A\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \xrightarrow{\sim} w_{V*}\,\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris},X},$$

and we need to check that the following canonical morphism is an isomorphism:

$$\mathcal{O}_{\mathcal{X}}\widetilde{\otimes}_A\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \longrightarrow \mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V).$$

To see this, remark first that the morphism above is injective since both sides of the morphism above are naturally subsheaves of $\mathcal{O}_{\mathcal{X}}\widetilde{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)\simeq\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)$. For the surjectivity, by taking the graded quotients, we are reduced to showing that the natural morphism below is an isomorphism:

$$\mathcal{O}_{\mathcal{X}}\widetilde{\otimes}_A\mathrm{gr}^r\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \longrightarrow \mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathrm{gr}^r\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V). \tag{5.0.23}$$

But $\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)\approx\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)$, while the latter is a free module over R^+. Furthermore R^+ is almost flat over A by the almost purity theorem (recall that V is pro-étale over $\widetilde{Y}_{\widehat{k}}$ which is an affinoid perfectoid over $(\widehat{\overline{k}},\mathcal{O}_{\widehat{\overline{k}}})$). So, for $\mathcal{U}\in\mathcal{X}_{\text{ét}}$ an affine formal scheme,

$$\mathcal{O}_{\mathcal{X}}(\mathcal{U})\widehat{\otimes}_AR^+=\varprojlim\left((\mathcal{O}_{\mathcal{X}}(\mathcal{U})/p^n)\otimes_A R^+\right)\approx\left(\varprojlim\left((\mathcal{O}_{\mathcal{X}}/p^n)\otimes_A R^+\right)\right)(\mathcal{U}).$$

Inverting p and passing to associated sheaves, we get the required isomorphism (5.0.23), which concludes the proof of our lemma. □

Lemma 5.4. *Let $\mathcal{E}$ be a filtered convergent F-isocrystal on $\mathcal{X}$, $\mathcal{Y}=\mathrm{Spf}\,A$, and $V\in Y_{\mathrm{proét}}$ an affinoid perfectoid which is pro-étale over $\widetilde{Y}_{\widehat{k}}$. The canonical morphism*

$$R\Gamma(\mathcal{X},DR_{X/Y}(\mathcal{E}))\otimes_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)\longrightarrow R\Gamma(\mathcal{X},DR_{X/Y}(\mathcal{E})\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V))$$

is an isomorphism in the filtered derived category.

Proof. Write $\widehat{V}=\mathrm{Spa}(R,R^+)$. To show our lemma, remark first that the morphism above respects clearly the filtration on both sides. Secondly, since $\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)\approx\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)$ while the latter is $\mathcal{I}^2$-flat over A with $\mathcal{I}\subset\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)$ the ideal generated by $([\epsilon]^{1/p^n}-1)_{n\in\mathbb{N}}$ (we refer to Definition 7.1 for this notion and to [Bri, Section 6.3] for the proof of this assertion), for any coherent sheaf $\mathcal{F}$ on $\mathcal{X}$, the kernel and the cokernel of the natural morphism below are killed by some power of $p\mathcal{I}^2$ (Proposition 7.4):

$$H^i(\mathcal{X},\mathcal{F})\otimes_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)\longrightarrow H^i(\mathcal{X},\mathcal{F}\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)).$$

Inverting t (recall that this inverts also p and $[\epsilon]-1$), we get an isomorphism

$$H^i(\mathcal{X},\mathcal{F})\otimes_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)\longrightarrow H^i(\mathcal{X},\mathcal{F}\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)).$$

By some standard devissage, we deduce that the morphism in our lemma is an isomorphism in the derived category. To conclude, we only need to check that the induced morphisms between the graded quotients are all quasi-isomorphisms. As in the proof of Corollary 3.25 we are reduced to checking the almost isomorphism below for any coherent sheaf $\mathcal{F}$ on $\mathcal{X}$:

$$R\Gamma(\mathcal{X},\mathcal{F})\otimes_A \mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)\longrightarrow R\Gamma(\mathcal{X},\mathcal{F}\widehat{\otimes}_A\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)),$$

which is clear since $\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)$ is free over R^+, which is almost flat over A by the almost purity theorem. □

From now on, assume $f\colon\mathcal{X}\to\mathcal{Y}$ is a proper smooth morphism (between smooth formal schemes) over $\mathcal{O}_k$. Its closed fiber gives rise to a morphism between the crystalline topoi,

$$f_{\mathrm{cris}}\colon(\mathcal{X}_0/\mathcal{O}_k)^{\sim}_{\mathrm{cris}}\longrightarrow(\mathcal{Y}_0/\mathcal{O}_k)^{\sim}_{\mathrm{cris}}.$$

Let $\mathcal{E}$ be a filtered convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$, and $\mathcal{M}$ an F-crystal on $\mathcal{X}_0/\mathcal{O}_k$ such that $\mathcal{E}\simeq\mathcal{M}^{\mathrm{an}}(n)$ for some $n\in\mathbb{N}$ (cf. Remark 3.5). Then $\mathcal{M}$ can be viewed naturally as a coherent $\mathcal{O}_{\mathcal{X}}$-module endowed with an integrable and quasi-nilpotent $\mathcal{O}_k$-linear connection $\mathcal{M}\to\mathcal{M}\otimes\Omega^1_{\mathcal{X}/\mathcal{O}_k}$.

In the following we consider the higher direct image $R^if_{\mathrm{cris}*}\mathcal{M}$ of the crystal $\mathcal{M}$. One can determine the value of this abelian sheaf on $\mathcal{Y}_0/\mathcal{O}_k$ at the p-adic PD-thickening $\mathcal{Y}_0\hookrightarrow\mathcal{Y}$ in terms of the relative de Rham complex $DR_{X/Y}(\mathcal{M})$ of $\mathcal{M}$. To state this, take $\mathcal{V}=\mathrm{Spf}(A)$ an affine open subset of $\mathcal{Y}$, and put $\mathcal{X}_A:=f^{-1}(\mathcal{V})$. We consider A as a PD-ring with the canonical divided power structure on $(p)\subset A$. In particular, we can consider the crystalline site $(\mathcal{X}_{A,0}/A)_{\mathrm{cris}}$ of $\mathcal{X}_{A,0}:=\mathcal{X}\times_{\mathcal{Y}}\mathcal{V}_0$ relative to A. By [Ber96, Lemme 3.2.2], the latter can be identified naturally to the open subsite of $(\mathcal{X}_0/\mathcal{O}_k)$ whose objects are objets (U,T) of $(\mathcal{X}_0/\mathcal{O}_k)_{\mathrm{cris}}$ such that $f(U)\subset\mathcal{V}_0$ and such that there exists a morphism $\alpha\colon T\to\mathcal{V}_n:=\mathcal{V}\otimes_A A/p^{n+1}$ for some $n\in\mathbb{N}$, making the square below commute

$$\begin{array}{ccc} U & \hookrightarrow & T \\ {\scriptstyle\mathrm{can}}\downarrow & & \downarrow \\ \mathcal{V}_0 & \hookrightarrow & \mathcal{V}_n \end{array}.$$

Using [Ber96, Corollaire 3.2.3] and a limit argument, one finds a canonical identification

$$R^if_{\mathrm{cris}*}(\mathcal{M})(\mathcal{V}_0,\mathcal{V})\xrightarrow{\sim}H^i((\mathcal{X}_{A,0}/A)_{\mathrm{cris}},\mathcal{M})$$

where we denote again by $\mathcal{M}$ the restriction of $\mathcal{M}$ to $(\mathcal{X}_{A,0}/A)_{\mathrm{cris}}$. Let $u=u_{\mathcal{X}_{A,0}/A}$ be the morphism of topoi

$$(\mathcal{X}_{A,0}/A)^{\sim}_{\mathrm{cris}}\longrightarrow\mathcal{X}_{A\,\text{ét}}^{\sim}$$

such that $u_*(\mathcal{F})(\mathcal{U})=H^0((\mathcal{U}_0/A)^{\sim}_{\mathrm{cris}},\mathcal{F})$ for $\mathcal{U}\in\mathcal{X}_{A\,\text{ét}}$. By [BO, Theorem 7.23], there exists a natural quasi-isomorphism in the derived category

$$Ru_*\mathcal{M}\xrightarrow{\sim}DR_{X/Y}(\mathcal{M}),\tag{5.0.24}$$

inducing an isomorphism $H^i_{\mathrm{cris}}(\mathcal{X}_{A,0}/A,\mathcal{M})\xrightarrow{\sim}\mathbb{H}^i(\mathcal{X}_A,DR_{X/Y}(\mathcal{M}))$. Thereby

$$H^i_{\mathrm{cris}}(\mathcal{X}_{A,0}/A,\mathcal{M})\xrightarrow{\sim}\mathbb{H}^i(\mathcal{X}_A,DR_{X/Y}(\mathcal{M})).\tag{5.0.25}$$

Passing to associated sheaves, we deduce that

$$R^if_{\mathrm{cris}*}(\mathcal{M})_{\mathcal{Y}}=R^if_*(DR_{X/Y}(\mathcal{M})).$$

On the other hand, as $f\colon \mathcal{X} \to \mathcal{Y}$ is proper and smooth, $R^i f_*(DR_{X/Y}(\mathcal{E}))$, viewed as a coherent sheaf on the adic space Y, is the i-th relative convergent cohomology of $\mathcal{E}$ with respect to the morphism $f_0\colon \mathcal{X}_0 \to \mathcal{Y}_0$. Thus, by [Ber86, Théorème 5] (see also [Tsu, Theorem 4.1.4]), if we invert p, the $\mathcal{O}_{\mathcal{Y}}[1/p]$-module $R^i f_*(DR_{X/Y}(\mathcal{E})) \simeq R^i f_*(DR_{X/Y}(\mathcal{M}))[1/p]$, together with the Gauss-Manin connection and the natural Frobenius structure inherited from $R^i f_{\mathrm{cris}*}(\mathcal{M})_{\mathcal{Y}} \simeq R^i f_*(DR_{X/Y}(\mathcal{M}))$, is a convergent F-isocrystal on $\mathcal{Y}_0/\mathcal{O}_k$, denoted by $R^i f_{\mathrm{cris}*}(\mathcal{E})$ in the following (this is an abuse of notation, a more appropriate notation should be $R^i f_{0\,\mathrm{conv}*}(\mathcal{E})$). Using the filtration on $\mathcal{E}$, one sees that $R^i f_{\mathrm{cris}*}(\mathcal{E})$ has naturally a filtration, and it is well-known that this filtration satisfies Griffiths transversality with respect to the Gauss-Manin connection.

Proposition 5.5. *Let $\mathcal{X} \to \mathcal{Y}$ be a proper smooth morphism between two smooth formal schemes over $\mathcal{O}_k$. Let $\mathcal{E}$ be a filtered convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$ and $\mathbb{L}$ a lisse $\widehat{\mathbb{Z}}_p$-sheaf on $X_{\mathrm{pro\acute{e}t}}$. Assume that $\mathcal{E}$ and $\mathbb{L}$ are associated. Then there is a natural filtered isomorphism of $\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}$-modules*

$$R^i f_{k*}(\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}_{\mathrm{cris},X} \widehat{\otimes} f_k^{-1}\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}) \xrightarrow{\sim} w_{\mathcal{Y}}^{-1}(R^i f_{\mathrm{cris}*}(\mathcal{E})) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},Y} \tag{5.0.26}$$

which is compatible with Frobenius and connection.

Proof. Using the relative Poincaré lemma (Proposition 5.1 (1)) and the fact that $\mathbb{L}$ and $\mathcal{E}$ are associated, we have the following filtered isomorphisms compatible with connection:

$$\begin{aligned} & R^i f_{k*}(\mathbb{L} \otimes \mathbb{B}_{\mathrm{cris},X} \widehat{\otimes} f_k^{-1}\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}) \\ \xrightarrow{\sim}\ & R^i f_{k*}(\mathbb{L} \otimes DR_{X/Y}(\mathcal{O}\mathbb{B}_{\mathrm{cris},X})) \\ \xrightarrow{\sim}\ & R^i f_{k*}(DR_{X/Y}(\mathbb{L} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},X})) \\ \xrightarrow{\sim}\ & R^i f_{k*}(DR_{X/Y}(w_{\mathcal{X}}^{-1}\mathcal{E} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},X})) \\ \xrightarrow{\sim}\ & R^i f_{k*}(w_{\mathcal{X}}^{-1} DR_{X/Y}(\mathcal{E}) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},X}). \end{aligned} \tag{5.0.27}$$

On the other hand, we have the morphism below given by adjunction which respects also the connections on both sides:

$$w_{\mathcal{Y}}^{-1} R^i f_*(DR_{X/Y}(\mathcal{E})) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},Y} \longrightarrow R^i f_{k*}(w_{\mathcal{X}}^{-1} DR_{X/Y}(\mathcal{E}) \widehat{\otimes} \mathcal{O}\mathbb{B}_{\mathrm{cris},X}). \tag{5.0.28}$$

We claim that the morphism (5.0.28) is a filtered isomorphism. This is a local question, we may and do assume first that $\mathcal{Y} = \mathrm{Spf}(A)$ is affine and is étale over some torus defined over $\mathcal{O}_k$. Let $V \in Y_{\mathrm{pro\acute{e}t}}$ be an affinoid perfectoid pro-étale over $\widetilde{Y_{\overline{k}}}$. As $R^i f_*(DR_{X/Y}(\mathcal{E})) = R^i f_{\mathrm{cris}*}(\mathcal{E})$ is a locally free $\mathcal{O}_{\mathcal{Y}}[1/p]$-module over $\mathcal{Y}$ and as $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(V)$ is flat over A ([Bri, Théorème 6.3.8]),

$$\left(w_{\mathcal{Y}}^{-1} R^i f_*(DR_{X/Y}(\mathcal{E})) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}\right)(V) \simeq H^i(\mathcal{X}, DR_{X/Y}(\mathcal{E})) \otimes_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V).$$

So we only need to check that the natural morphism below is a filtered isomorphism

$$H^i(\mathcal{X}, DR_{X/Y}(\mathcal{E})) \otimes_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \longrightarrow H^i(X_V, w_{\mathcal{X}}^{-1} DR_{X/Y}(\mathcal{E}) \widehat{\otimes} \mathcal{O}\mathbb{B}_{\mathrm{cris},X}).$$

By Lemma 5.3 one has the following identifications that are strictly compatible with filtrations:

$$\begin{aligned} H^i(X_V, w_{\mathcal{X}}^{-1} DR_{X/Y}(\mathcal{E}) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},X}) &\simeq H^i(\mathcal{X}, Rw_{V*}(w_{\mathcal{X}}^{-1} DR_{X/Y}(\mathcal{E}) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},X})) \\ &\simeq H^i(\mathcal{X}, DR_{X/Y}(\mathcal{E}) \widehat{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)). \end{aligned}$$

Thus we are reduced to proving that the canonical morphism below is a filtered isomorphism:

$$H^i(\mathcal{X}, DR_{X/Y}(\mathcal{E})) \otimes_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \longrightarrow H^i(\mathcal{X}, DR_{X/Y}(\mathcal{E})\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)),$$

which follows from Lemma 5.4.

Composing the isomorphisms in (5.0.27) with the inverse of (5.0.28), we get the desired filtered isomorphism (5.0.26) that is compatible with connections on both sides. It remains to check the Frobenius compatibility of (5.0.26). For this, we may and do assume again that $\mathcal{Y} = \mathrm{Spf}(A)$ is affine and is étale over some torus over $\mathcal{O}_k$, and let $V \in Y_{\mathrm{proét}}$ some affinoid perfectoid pro-étale over $\widetilde{Y}$. In particular, A admits a lifting of the Frobenius on $\mathcal{Y}_0$, denoted by σ. Let $\mathcal{M}$ be an F-crystal on $\mathcal{X}_0/\mathcal{O}_k$ such that $\mathcal{E} = \mathcal{M}^{\mathrm{an}}(n)$ for some $n \in \mathbb{N}$ (Remark 3.5). Then the crystalline cohomology $H^i(\mathcal{X}_0/A, \mathcal{M})$ is endowed with a Frobenius which is σ-semilinear. We just need to check the Frobenius compatibility of composition of the maps below (here the last one is induced by the inverse of (5.0.26)):

$$H^i_{\mathrm{cris}}(\mathcal{X}_0/A, \mathcal{M}) \longrightarrow H^i_{\mathrm{cris}}(\mathcal{X}_0/A, \mathcal{M})[1/p] \xrightarrow{\sim} H^i(\mathcal{X}, DR_{X/Y}(\mathcal{E})) \longrightarrow \left(w_{\mathcal{Y}}^{-1}(R^i f_{\mathrm{cris}*}(\mathcal{E}) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},Y})\right)(V) \longrightarrow H^i(X_V, \mathbb{L} \otimes \mathbb{B}_{\mathrm{cris},X}\widehat{\otimes} f_k^{-1}\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}),$$

which can be done exactly in the same way as in the proof of Theorem 3.27. □

The relative crystalline comparison theorem then can be stated as follows:

Theorem 5.6. *Let $\mathbb{L}$ be a crystalline lisse $\widehat{\mathbb{Z}}_p$-sheaf on X associated to a filtered F-isocrystal $\mathcal{E}$ on $\mathcal{X}_0/\mathcal{O}_k$. Assume that, for any $i \in \mathbb{Z}$, $R^i f_{k*}\mathbb{L}$ is a lisse $\widehat{\mathbb{Z}}_p$-sheaf on Y. Then $R^i f_{k*}\mathbb{L}$ is crystalline and is associated to the filtered convergent F-isocrystal $R^i f_{\mathrm{cris}*}\mathcal{E}$.*

Proof. Let us first observe that the filtration on $R^i f_{\mathrm{cris}*}\mathcal{E}$ is given by locally direct summands (so $R^i f_{\mathrm{cris}*}\mathcal{E}$ is indeed a filtered convergent F-isocrystal on $\mathcal{Y}_0/\mathcal{O}_k$). To see this, one uses [Sch13, Theorem 8.8]: by Proposition 3.23, the lisse $\widehat{\mathbb{Z}}_p$-sheaf $\mathbb{L}$ is de Rham with associated filtered $\mathcal{O}_X$-module with integrable connection $\mathcal{E}$. Therefore the Hodge-to-de Rham spectral sequence

$$E_1^{i,j} = R^{i+j} f_* \left(\mathrm{gr}^i \left(DR_{X/Y}(\mathcal{E})\right)\right) \Longrightarrow R^{i+j} f_*(DR_{X/Y}(\mathcal{E}))$$

degenerates at E_1. Moreover $E_1^{i,j}$, the relative Hodge cohomology of $\mathcal{E}$ in [Sch13, Theorem 8.8], is a locally free $\mathcal{O}_Y$-module of finite rank for all i, j by *loc. cit.* Therefore the filtration on $R^i f_*(DR_{X/Y}(\mathcal{E})) = R^i f_{\mathrm{cris}*}(\mathcal{E})$, which is the same as the one induced by the spectral sequence above, is given by locally direct summands.

To complete the proof, we need to find filtered isomorphisms that are compatible with Frobenius and connections:

$$R^i f_{k*}(\mathbb{L}) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},Y} \xrightarrow{\sim} w_{\mathcal{Y}}^{-1} R^i f_{\mathrm{cris}*}(\mathcal{E}) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}, \quad i \in \mathbb{Z}.$$

By Proposition 5.2 and Proposition 5.5, we only need to check that the natural morphism below is a filtered almost isomorphism compatible with Frobenius and connections:

$$R^i f_{k*}(\mathbb{L} \otimes \mathbb{A}_{\mathrm{cris},X})\widehat{\otimes}\mathcal{O}\mathbb{A}_{\mathrm{cris},Y} \longrightarrow R^i f_{k*}(\mathbb{L} \otimes \mathbb{A}_{\mathrm{cris},X}\widehat{\otimes} f_k^{-1}\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}).$$

The proof of this is similar to that of Proposition 5.2: one just remarks that for each $n \in \mathbb{N}$, $\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}/p^n|_{\widetilde{Y}} \simeq \mathbb{A}_{\mathrm{cris},Y}|_{\widetilde{Y}}\langle w_1, \ldots, w_\delta\rangle/p^n$ hence $\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}|_{\widetilde{Y}}/p^n$ is free over $\mathbb{A}_{\mathrm{cris},Y}/p^n$ with a basis given by the divided powers $w^{[\underline{\alpha}]}$ with $\underline{\alpha} \in \mathbb{N}^\delta$ (recall that for $1 \leq j \leq \delta$, $w_j = S_j - [S_j^\flat] \in \mathcal{O}\mathbb{A}_{\mathrm{cris},Y}|_{\widetilde{Y}}$). □

6. Appendix: geometric acyclicity of $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$

In this section, we extend the main results of [AB] to the setting of perfectoids. The generalization is rather straightforward. Although one might see here certain difference from the arguments in [AB], the strategy and technique are entirely theirs.

Let $f\colon \mathcal{X} = \mathrm{Spf}(B) \to \mathcal{Y} = \mathrm{Spf}(A)$ be a smooth morphism between two smooth affine formal scheme over $\mathcal{O}_k$. Write X (resp. Y) the generic fiber of $\mathcal{X}$ (resp. of $\mathcal{Y}$). By abuse of notation the morphism $X \to Y$ induced from f is still denoted by f.

Assume that $\mathcal{Y}$ is étale over the torus $\mathcal{S} := \mathrm{Spf}(\mathcal{O}_k\{S_1^{\pm 1}, \ldots, S_\delta^{\pm 1}\})$ defined over $\mathcal{O}_k$ and that the morphism $f\colon \mathcal{X} \to \mathcal{Y}$ can factor as

$$\mathcal{X} \xrightarrow{\text{étale}} \mathcal{T} \longrightarrow \mathcal{Y},$$

so that $\mathcal{T} = \mathrm{Spf}(C)$ is a torus over $\mathcal{Y}$ and the first morphism $\mathcal{X} \to \mathcal{T}$ is étale.

Write $C = A\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\}$. For each $1 \leq i \leq d$ (resp. each $1 \leq j \leq \delta$), let $\{T_i^{1/p^n}\}_{n\in\mathbb{N}}$ (resp. $\{S_j^{1/p^n}\}_{n\in\mathbb{N}}$) be a compatible family of p-power roots of T_i (resp. of S_j). As in Proposition 2.12, we denote by $\widetilde{Y}$ the following fiber product over the generic fiber $\mathcal{S}_k$ of $\mathcal{S}$:

$$Y \times_{\mathcal{S}_k} \mathrm{Spa}(k\{S_1^{\pm 1/p^\infty}, \ldots, S_\delta^{\pm 1/p^\infty}\}, \mathcal{O}_k\{S_1^{\pm 1/p^\infty}, \ldots, S_\delta^{\pm 1/p^\infty}\}).$$

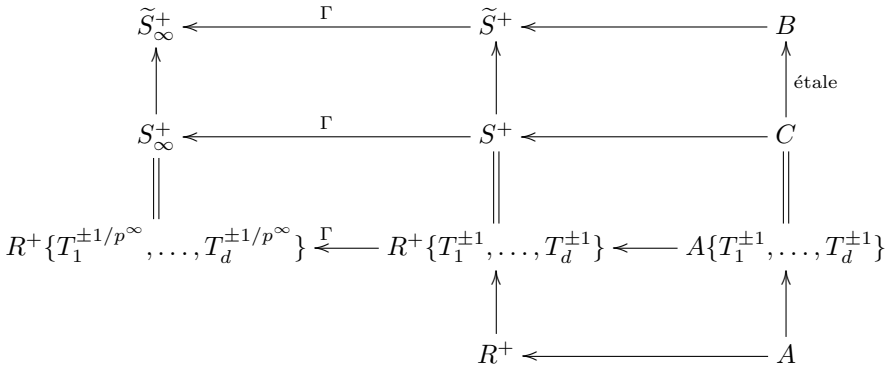
Let $V \in Y_{\text{proét}}$ be an affinoid perfectoid over $\widetilde{Y}_{\overline{k}}$ with $\widehat{V} = \mathrm{Spa}(R, R^+)$. Let $T_V = \mathrm{Spa}(S, S^+)$ be the base change $\mathcal{T}_k \times_Y V$ and $X_V = \mathrm{Spa}(\widetilde{S}, \widetilde{S}^+)$ the base change $X \times_Y V$. Thus $S^+ = R^+\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\}$ and $S = S^+[1/p]$. Set

$$S_\infty^+ = R^+\left\{T_1^{\pm 1/p^\infty}, \cdots, T_d^{\pm 1/p^\infty}\right\}, \quad \widetilde{S}_\infty^+ := B\widehat{\otimes}_C S_\infty^+,$$

$S_\infty := S_\infty^+[1/p]$ and $\widetilde{S}_\infty := \widetilde{S}_\infty^+[1/p]$. Then (S_∞, S_∞^+) and $(\widetilde{S}_\infty, \widetilde{S}_\infty^+)$ are affinoids perfectoids and

$$S_\infty^{\flat +} = R^{\flat +}\left\{(T_1^\flat)^{\pm 1/p^\infty}, \cdots, (T_d^\flat)^{\pm 1/p^\infty}\right\},$$

where $T_i^\flat := (T_i, T_i^{1/p}, T_i^{1/p^2}, \ldots) \in S_\infty^{\flat +}$. The inclusions $S^+ \subset S_\infty^+$ and $\widetilde{S}^+ \subset \widetilde{S}_\infty^+$ define two profinite Galois covers. Their Galois groups are the same, denoted by Γ, which is a profinite group isomorphic to $\mathbb{Z}_p(1)^d$. One can summarize these notations in the following commutative diagramme



The group Γ acts naturally on the period ring $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty)$ and on its filtration $\mathrm{Fil}^r\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty)$. The aim of this appendix is to compute the group cohomology

$$H^q\left(\Gamma, \mathcal{O}\mathbb{B}_{\mathrm{cris}}\left(\widetilde{S}_\infty, \widetilde{S}^+_\infty\right)\right) := H^q_{\mathrm{cont}}\left(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}\left(\widetilde{S}_\infty, \widetilde{S}^+_\infty\right)\right)[1/t]$$

and

$$H^q\left(\Gamma, \mathrm{Fil}^r\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}\left(\widetilde{S}_\infty, \widetilde{S}^+_\infty\right)\right) := \varinjlim_{n \geq |r|} H^q_{\mathrm{cont}}\left(\Gamma, \frac{1}{t^n}\,\mathrm{Fil}^{r+n}\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}\left(\widetilde{S}_\infty, \widetilde{S}^+_\infty\right)\right)$$

for $q, r \in \mathbb{Z}$.

In the following, we will omit systematically the subscript "cont" whenever there is no confusion arising. Moreover, we shall use the multi-index to simplify the notation: for example, for $\underline{a} = (a_1, \ldots, a_d) \in \mathbb{Z}[1/p]^d$, $T^{\underline{a}} := T_1^{a_1} \cdot T_2^{a_2} \cdots T_d^{a_d}$.

6.1. **Cohomology of $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$.** We will first compute $H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S^+_\infty)/p^n)$ up to $(1-[\epsilon])^\infty$-torsion for all $q, n \in \mathbb{N}$.

Lemma 6.1. *For $n \in \mathbb{Z}_{\geq 1}$, there are natural isomorphisms*

$$\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \otimes_{W(R^{\flat +})/p^n} W(S^{\flat +}_\infty)/p^n \xrightarrow{\sim} \mathbb{A}_{\mathrm{cris}}(S_\infty, S^+_\infty)/p^n$$

and

$$\left(\mathbb{A}_{\mathrm{cris}}(S_\infty, S^+_\infty)/p^n \otimes \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n\right)\langle u_1, \ldots, u_d\rangle \xrightarrow{\sim} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S^+_\infty)/p^n,$$

sending u_i to $T_i - [T_i^\flat]$. Here the tensor product in the last isomorphism above is taken over $\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n$.

Moreover, the natural morphisms

$$\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \to \mathbb{A}_{\mathrm{cris}}(S_\infty, S^+_\infty)/p^n, \quad \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S^+_\infty)/p^n$$

are both injective.

Proof. Recall $\xi = [p^\flat] - p$. We know that $\mathbb{A}_{\mathrm{cris}}(S_\infty, S^+_\infty)$ is the p-adic completion of

$$\mathbb{A}^0_{\mathrm{cris}}(S_\infty, S^+_\infty) := W(S^{\flat +}_\infty)\left[\frac{\xi^m}{m!} | m = 0, 1, \cdots\right] = \frac{W(S^{\flat +}_\infty)[X_0, X_1, \cdots]}{(m! X_m - \xi^m : m \in \mathbb{Z}_{\geq 0})}.$$

Note that we have the same expression with R in place of S_∞. We then have

$$\begin{aligned} \mathbb{A}^0_{\mathrm{cris}}(S_\infty, S^+_\infty) &\xleftarrow{\sim} W(S^{\flat +}_\infty) \otimes_{W(R^{\flat +})} \frac{W(R^{\flat +})[X_0, X_1, \cdots]}{(m! X_m - \xi^m | m \in \mathbb{Z}_{\geq 0})} \\ &= W(S^{\flat +}_\infty) \otimes_{W(R^{\flat +})} \mathbb{A}^0_{\mathrm{cris}}(R, R^+). \end{aligned}$$

The first isomorphism follows.

Secondly, as V lies above $\widetilde{Y}_{\overline{k}}$, by Proposition 2.12 we have

$$\mathbb{A}_{\mathrm{cris}}(R, R^+)\{\langle w_1, \ldots, w_\delta\rangle\} \xrightarrow{\sim} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+), \quad w_j \mapsto S_j - [S_j^\flat]$$

where $S_j^\flat := (S_j, S_j^{1/p}, S_j^{1/p^2}, \ldots) \in R^{\flat +}$. Similarly

$$\mathbb{A}_{\mathrm{cris}}(S_\infty, S^+_\infty)\{\langle u_1, \ldots, u_d, w_1, \ldots, w_\delta\rangle\} \xrightarrow{\sim} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S^+_\infty), \quad u_i \mapsto T_i - [T_i^\flat],\ w_j \mapsto S_j - [S_j^\flat].$$

Thus (the isomorphisms below are all the natural ones)

$$\begin{aligned}
\frac{\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)}{p^n} &\xleftarrow{\sim} \left(\frac{\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)}{p^n}\right)\langle u_1,\dots,u_d,w_1,\dots,w_\delta\rangle\\
&\xleftarrow{\sim} \frac{\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)}{p^n}\otimes_{\frac{\mathbb{A}_{\mathrm{cris}}(R,R^+)}{p^n}}\left(\frac{\mathbb{A}_{\mathrm{cris}}(R,R^+)}{p^n}\langle u_1,\dots,u_d,w_1,\dots,w_\delta\rangle\right)\\
&\xrightarrow{\sim} \frac{\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)}{p^n}\otimes_{\frac{\mathbb{A}_{\mathrm{cris}}(R,R^+)}{p^n}}\left(\frac{\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)}{p^n}\langle u_1,\dots,u_d\rangle\right)\\
&\xrightarrow{\sim} \left(\frac{\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)}{p^n}\otimes_{\frac{\mathbb{A}_{\mathrm{cris}}(R,R^+)}{p^n}}\frac{\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)}{p^n}\right)\langle u_1,\dots,u_d\rangle.
\end{aligned}$$

So our second isomorphism is obtained.

Next we prove that the natural morphism $\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n\to\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)/p^n$ is injective. When $n=1$, we are reduced to showing the injectivity of

$$\frac{(R^{\flat+}/(p^\flat)^p)[X_1,X_2,\dots]}{(X_1^p,X_2^p,\dots)}\longrightarrow\frac{(S^{\flat+}/(p^\flat)^p)[X_1,X_2,\dots]}{(X_1^p,X_2^p,\dots)},$$

or equivalently the injectivity of

$$R^{\flat+}/(p^\flat)^p\to S^{\flat+}/(p^\flat)^p=\left(R^{\flat+}/(p^\flat)^p\right)\left[(T_1^\flat)^{\pm1/p^\infty},\dots(T_d^\flat)^{\pm1/p^\infty}\right],$$

which is clear. The general case follows easily since $\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)$ is p-torsion free. One deduces also the injectivity of $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,^+)/p^n\to\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)/p^n$ by using the natural isomorphisms $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n\simeq(\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n)\langle w_1,\dots,w_\delta\rangle$, and $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)/p^n\simeq(\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)/p^n)\langle u_1,\dots,u_d,w_1,\dots,w_\delta\rangle$. This concludes the proof of our lemma. □

Proposition 6.2. *$\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)/p^n$ is free over $\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n$ with a basis give by $\{[T^\flat]^{\underline{a}}|\underline{a}\in\mathbb{Z}[1/p]^d\}$.*

Proof. By Lemma 6.1, $\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)/p^n$ is generated over $\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n$ by elements of the form $[x]$ with $x\in S_\infty^{\flat+}=R^{\flat+}\{(T_1^\flat)^{\pm1/p^\infty},\dots,(T_d^\flat)^{\pm1/p^\infty}\}$. Write $B_n\subset\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)/p^n$ for the $\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n$-submodule generated by elements of the form $[x]$ with $x\in\mathsf{S}:=R^{\flat+}\left[(T_1^\flat)^{\pm1/p^\infty},\dots,(T_d^\flat)^{\pm1/p^\infty}\right]\subset S_\infty^{\flat+}$. We claim that $B_n=\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)/p^n$.

Since $S_\infty^{\flat+}$ is the $p^\flat$-adic completion of S, for each $x\in S^{\flat+}$ we can write $x=y_0+p^\flat x'$ with $x'\in\mathsf{S}$. Iteration yields

$$x=y_0+p^\flat y_1+\cdots+(p^\flat)^{p-1}y_{p-1}+(p^\flat)^px''$$

with $y_i\in\mathsf{S}$ and $x''\in S_\infty^{\flat+}$. Then in $W(S_\infty^{\flat+})$:

$$\begin{aligned}
[x] &\equiv [y_0]+[p^\flat][y_1]+\cdots+[(p^\flat)^{p-1}][y_{p-1}]+[(p^\flat)^p][x''] \mod pW(S_\infty^{\flat+})\\
&\equiv [y_0]+\xi[y_1]+\cdots+\xi^{p-1}[y_{p-1}]+\xi^p[x''] \mod pW(S_\infty^{\flat+}).
\end{aligned}$$

As $\xi\in\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)$ has divided power, $\xi^p=p!\cdot\xi^{[p]}\in p\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)$. So we obtain in $\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)$

$$[x]\equiv[y_0]+\xi[y_1]+\cdots+\xi^{p-1}[y_{p-1}] \mod p\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+).$$

For any $\alpha \in \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n = \mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \otimes_{W(R^{\flat +})/p^n} W(S_\infty^{\flat +})/p^n$, we may write

$$\alpha = \sum_{i=0}^{m} \lambda_i [x_i] + p\alpha', \quad x_i \in S_\infty^{\flat +}, \lambda_i \in \mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n, \alpha' \in \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n.$$

The observation above tells us that one can write

$$\alpha = \beta_0 + p\alpha'', \quad \beta_0 \in B_n, \alpha'' \in \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n.$$

By iteration again, we find

$$\alpha = \beta_0 + p\beta_1 + \cdots p^{n-1}\beta_{n-1} + p^n\tilde{\alpha}, \quad \beta_0, \cdots, \beta_{n-1} \in B_n, \tilde{\alpha} \in \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n.$$

Thus

$$\alpha = \beta_0 + p\beta_1 + \cdots p^{n-1}\beta_{n-1} \in B_n \subset \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n.$$

This shows the claim, i.e. $\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$ is generated over $\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n$ by the elements of the form $[x]$ with $x \in \mathsf{S} = R^{\flat +}[(T_1^\flat)^{\pm 1/p^\infty}, \ldots, (T_d^\flat)^{\pm 1/p^\infty}] \subset S_\infty^{\flat +}$. Furthermore, as for any $x, y \in S_\infty^{\flat +}$

$$[x + y] \equiv [x] + [y] \mod pW(S_\infty^{\flat +}),$$

a similar argument shows that $\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$ is generated over $\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n$ by the family of elements $\{[T^\flat]^{\underline{a}} | \underline{a} \in \mathbb{Z}[1/p]^d\}$.

It remains to show the freeness of the family $\{[T^\flat]^{\underline{a}} | \underline{a} \in \mathbb{Z}[1/p]^d\}$ over $\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n$. For this, suppose there exist $\lambda_1, \cdots, \lambda_m \in \mathbb{A}_{\mathrm{cris}}(R, R^+)$ and distinct elements $\underline{a}_1, \cdots, \underline{a}_m \in \mathbb{Z}[1/p]^d$ such that

$$\sum_{i=1}^{m} \lambda_i [T^\flat]^{\underline{a}_i} \in p^n \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+).$$

One needs to prove $\lambda_i \in p^n \mathbb{A}_{\mathrm{cris}}(R, R^+)$ for each i. Modulo p we find in $\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p$ that

$$\sum_{i=1}^{m} \overline{\lambda_i} \cdot (T^\flat)^{\underline{a}_i} = 0,$$

with $\overline{\lambda_i} \in \mathbb{A}_{\mathrm{cris}}(R, R^+)/p$ the reduction modulo p of λ_i. On the other hand, the family of elements $\{(T^\flat)^{\underline{a}} : \underline{a} \in \mathbb{Z}[1/p]^d\}$ in

$$\begin{aligned} \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p &\simeq \frac{S^{\flat +}/((p^\flat)^p)[\delta_2, \delta_3, \ldots]}{(\delta_2^p, \delta_3^p, \ldots)} \\ &\simeq \frac{R^{\flat +}/((p^\flat)^p)[(T_1^\flat)^{\pm 1/p^\infty}, \ldots, (T_d^\flat)^{\pm 1/p^\infty}, \delta_2, \delta_3, \ldots]}{(\delta_2^p, \delta_3^p, \ldots)} \end{aligned}$$

is free over $\mathbb{A}_{\mathrm{cris}}(R, R^+)/p \simeq \frac{R^{\flat +}/((p^\flat)^p)[\delta_2, \delta_3, \ldots]}{(\delta_2^p, \delta_3^p, \ldots)}$. Therefore, $\overline{\lambda_i} = 0$, or equivalently, $\lambda_i = p\lambda_i'$ for some $\lambda_i' \in \mathbb{A}_{\mathrm{cris}}(R, R^+)$. In particular,

$$\sum_{i=1}^{m} \lambda_i [T^\flat]^{\underline{a}_i} = p \cdot \left(\sum_{i=1}^{m} \lambda_i' [T^\flat]^{\underline{a}_i} \right) \in p^n \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+).$$

But $\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)$ is p-torsion free, which implies that

$$\sum_{i=1}^{m} \lambda_i' [T^\flat]^{\underline{a}_i} \in p^{n-1} \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+).$$

This way, we may find $\lambda_i = p^n\tilde{\lambda}_i$ for some $\tilde{\lambda}_i \in \mathbb{A}_{\mathrm{cris}}(R, R^+)$, which concludes the proof of the freeness. □

Recall that Γ is the Galois group of the profinite cover (S_∞, S_∞^+) of (S, S^+). Let $\epsilon = (\epsilon^{(0)}, \epsilon^{(1)}, \ldots) \in \mathcal{O}_{\widehat{\overline{k}}}^\flat$ be a compatible system of p-power roots of unity such that $\epsilon^{(0)} = 1$ and that $\epsilon^{(1)} \neq 1$. Let $\{\gamma_1, \ldots, \gamma_d\}$ be a family of generators such that for each $1 \leq i \leq d$, γ_i acts trivially on the variables T_j for any index j different from i and that $\gamma_i(T_i^\flat) = \epsilon T_i^\flat$.

Lemma 6.3 ([AB] Lemme 11)**.** *Let $1 \leq i \leq d$ be an integer. Then one has $\gamma_i([T_i^\flat]^{p^n}) = [T_i^\flat]^{p^n}$ in $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$.*

Proof. By definition, $\gamma_i([T_i^\flat]^{p^n}) = [\epsilon]^{p^n}[T_i^\flat]^{p^n}$ in $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)$. So our lemma follows from the fact that $[\epsilon]^{p^n} - 1 = \exp(p^n t) - 1 = \sum_{r\geq 1} p^{nr} t^{[r]} \in p^n A_{\mathrm{cris}}$. □

Let A_n be the $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$-subalgebra of $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$ generated by $[T_i^\flat]^{\pm p^n}$ for $1 \leq i \leq d$. The previous lemma shows that Γ acts trivially on A_n. Furthermore, by the second isomorphism of Lemma 6.1 and by Proposition 6.2, we have

$$\frac{\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)}{p^n} \xrightarrow{\sim} \left(\bigoplus_{\underline{a}\in\mathbb{Z}[1/p]^d\cap[0,p^n)^d} A_n[T^\flat]^{\underline{a}}\right)\langle u_1, \ldots, u_d\rangle, \quad T_i - [T_i^\flat] \mapsto u_i.$$

Transport the Galois action of Γ on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$ to the righthand side of this isomorphism. It follows that

$$\gamma_i(u_i) = u_i + (1 - [\epsilon])[T_i^\flat].$$

Therefore,

$$\gamma_i(u_i^{[n]}) = u_i^{[n]} + \sum_{j=1}^{n} [T_i^\flat]^j (1 - [\epsilon])^{[j]} u_i^{[n-j]}.$$

For other index $j \neq i$, $\gamma_i(u_j) = u_j$ and hence $\gamma_i(u_j^{[n]}) = u_j^{[n]}$ for any n. Set

$$\mathbf{X}_n := \bigoplus_{\underline{a}\in(\mathbb{Z}[1/p]\cap[0,p^n))^d\setminus\mathbb{Z}^d} A_n[T^\flat]^{\underline{a}}, \quad \text{and} \quad \mathbf{A}_n := \bigoplus_{\underline{a}\in\mathbb{Z}^d\cap[0,p^n)^d} A_n[T^\flat]^{\underline{a}}.$$

Then we have the following decomposition, which respects the Γ-actions:

$$\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n = \mathbf{X}_n\langle u_1, \ldots, u_d\rangle \oplus \mathbf{A}_n\langle u_1, \ldots, u_d\rangle.$$

Proposition 6.4 ([AB] Proposition 16)**.** *For any integer $q \geq 0$, $H^q(\Gamma, \mathbf{X}_n\langle u_1, \ldots, u_d\rangle)$ is killed by $(1 - [\epsilon]^{1/p})^2$.*

Proof. Let $A_n[T^\flat]^{\underline{a}}$ be a direct summand of $\mathbf{X}_n$ with $\underline{a} \in (\mathbb{Z}[1/p] \cap [0, p^n))^d \setminus \mathbb{Z}$. Write $\underline{a} = (a_1, \ldots, a_d)$. Clearly we may assume that $a_1 \notin \mathbb{Z}$.

We claim first that cohomology group $H^q(\Gamma_1, A_n[T^\flat]^{\underline{a}})$ is killed by $1 - [\epsilon]^{1/p}$ for any q, where $\Gamma_1 \subset \Gamma$ is the closed subgroup generated by γ_1. As Γ_1 is topologically cyclic with a generator γ_1, the desired cohomology $H^q(\Gamma_1, A_n \cdot [T^\flat]^{\underline{a}})$ is computed using the following complex

$$A_n \cdot [T^\flat]^{\underline{a}} \xrightarrow{\gamma_1 - 1} A_n \cdot [T^\flat]^{\underline{a}}.$$

We need to show that the kernel and the cokernel of the previous map are both killed by $1 - [\epsilon]^{1/p}$. We have $(\gamma_1 - 1)([T^\flat]^{\underline{a}}) = ([\epsilon]^{a_1} - 1)[T^\flat]^{\underline{a}}$. Since $a_1 \notin \mathbb{Z}$, its

p-adic valuation is ≤ -1. Therefore, $1-[\epsilon]^{1/p}$ is a multiple of $1-[\epsilon]^{a_1}$, which implies that $\mathrm{coker}(\gamma-1\colon \mathbf{X}_n \to \mathbf{X}_n)$ is killed by $1-[\epsilon]^{1/p}$. On the other hand, let $x=\lambda\cdot[T^\flat]^{\underline{a}}$ be an element in $\ker(\gamma_1-1\colon A_n\cdot[T^\flat]^{\underline{a}}\to A_n\cdot[T^\flat]^{\underline{a}})$. Then $(\gamma_1-1)(x)=\lambda([\epsilon]^{a_1}-1)[T^\flat]^a$ and necessarily $\lambda([\epsilon]^{a_1}-1)=0$. Since $1-[\epsilon]^{1/p}$ is a multiple of $[\epsilon]^{a_1}-1$, $(1-[\epsilon])\lambda=0$ and thus $(1-[\epsilon]^{1/p})x=0$, giving our claim.

Now for each $1\leq i\leq d$, let

$$E^{(i)}:=\left\{\underline{a}=(a_1,\dots,a_d)\in\left(\mathbb{Z}[1/p]\bigcap[0,p^n)\right)^d : a_1,\dots,a_{i-1}\in\mathbb{Z}, a_i\notin\mathbb{Z}\right\}$$

and

$$\mathbf{X}_n^{(i)}:=\bigoplus_{\underline{a}\in E^{(i)}} A_n[T^\flat]^{\underline{a}}.$$

Then we have the following decompositions which respect also the Γ-action:

$$\mathbf{X}_n=\bigoplus_{i=1}^d\mathbf{X}_n^{(i)},\quad \mathbf{X}_n\langle u_1,\dots,u_d\rangle=\bigoplus_{i=1}^d\mathbf{X}_n^{(i)}\langle u_1,\dots,u_d\rangle.$$

So we are reduced to proving that for each $1\leq i\leq d$, $H^q(\Gamma,\mathbf{X}_n^{(i)}\langle u_1,\dots,u_d\rangle)$ is killed by $(1-[\epsilon]^{1/p})^2$ for all $q\geq 0$. By what we have shown in the beginning of the proof, $H^q(\mathbb{Z}_p\gamma_i,\mathbf{X}_n^{(i)})$ is killed by $1-[\epsilon]^{1/p}$. As γ_i acts trivially on u_j for $j\neq i$, one computes as in [AB, Lemme 15] and then finds that $H^q(\mathbb{Z}_p\gamma_i,\mathbf{X}_n^{(i)}\langle u_1,\dots,u_d\rangle)$ is killed by $1-[\epsilon]^{1/p}$. Finally, one uses the Hochschild-Serre spectral sequence to conclude that $H^q(\Gamma,\mathbf{X}_n^{(i)}\langle u_1\dots,u_d\rangle)$ is killed by $(1-[\epsilon]^{1/p})^2$ for any $q\geq 0$, as desired. □

The computation of $H^q(\Gamma,\mathbf{A}_n\langle u_1,\dots,u_d\rangle)$ is more subtle. Note that we have the following decomposition

$$\mathbf{A}_n\langle u_1,\dots,u_d\rangle=\bigotimes_{i=1}^d(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n)\left[[T_i^\flat]^{\pm1}\right]\langle u_i\rangle,$$

where the tensor products above are taken over $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n$.

We shall first treat the case where $d=1$. We set $T:=T_1, u:=u_1$ and $\gamma:=\gamma_1$. Let $\mathcal{A}_n^{(m)}$ be the A_n-submodule of $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty,S_\infty^+)/p^n$ generated by the $u^{[m+a]}/[T^\flat]^a$'s with $m+a\geq 0$ and $0\leq a<p^n$. Then

$$\mathbf{A}_n\langle u\rangle=A_n\left[[T^\flat]\right]\langle u\rangle=\sum_{m>-p^n}\mathcal{A}_n^{(m)}.$$

Consider the following complex:

$$A_n\left[[T^\flat]\right]\langle u\rangle\xrightarrow{\gamma-1}A_n\left[[T^\flat]\right]\langle u\rangle, \tag{6.1.1}$$

which computes $H^q(\Gamma,\mathbf{A}_n\langle u\rangle)=H^q\left(\Gamma,A_n\left[[T^\flat]\right]\langle u\rangle\right)$.

Lemma 6.5 ([AB] Proposition 20)**.** *The cokernel of* (6.1.1)*, and hence* $H^q\left(\Gamma,\mathbf{A}_n\langle u\rangle\right)$ *for any* $q>0$*, are killed by* $1-[\epsilon]$*.*

Proof. We will proceed by induction on $m>-p^n$ to show that $(1-[\epsilon])\mathcal{A}_n^{(m)}$ is contained in the image of (6.1.1). Note first $\mathcal{A}_n^{(1-p^n)}=A_n\cdot\frac{1}{[T^\flat]^{p^n-1}}$, while

$$(\gamma-1)(U/[T^\flat]^{p^n})=(1-[\epsilon])\frac{1}{[T^\flat]^{p^n-1}}.$$

Thus $(1-[\epsilon])\mathcal{A}_n^{(1-p^n)}$ is contained in the image of (6.1.1). Let $m > -p^n$ be a fixed integer. Then $\mathcal{A}_n^{(m)}$ is generated over A_n by $\frac{u^{[m+p^n-i]}}{[T^\flat]^{p^n-i}}$ for $i = 1, \dots, D := \min(p^n, p^n + m)$. On the other hand, for $u^{[m+a]}/[T^\flat]^a \in A_n\left[[T^\flat]\right]\langle u\rangle$ with $\max(0,-m) \leq a \leq p^n$, we have

$$\begin{aligned}
[\epsilon]^a(\gamma-1)(u^{[m+a]}/[T^\flat]^a) &= \frac{(u+(1-[\epsilon])[T^\flat])^{[m+a]} - [\epsilon]^a u^{[m+a]}}{[T^\flat]^a} \\
&= (1-[\epsilon]^a)\frac{u^{[m+a]}}{[T^\flat]^a} + \sum_{\substack{1\leq i\leq a \\ i\leq m+a}} (1-[\epsilon])^{[i]}\frac{u^{[m+a-i]}}{[T^\flat]^{a-i}} \\
&\quad + \sum_{\substack{a+1\leq i\leq a+p^n \\ i\leq m+a}} (1-[\epsilon])^{[i]}[T^\flat]^{p^n}\frac{u^{[m+a-i]}}{[T^\flat]^{a-i+p^n}} \\
&\quad + \dots
\end{aligned}$$

As for $n \geq 2$, $(1-[\epsilon])^{[n]}$ is a multiple of $1-[\epsilon]$ ([AB, Lemme 18]), and all the terms from the second line of the last equality belong to $\mathcal{A}_n^{(m-rp^n)}$ for $r = 1, 2, \dots$. By induction hypothesis, we may assume that all these terms belong to the image of (6.1.1). Write

$$\alpha_j = \frac{1-[\epsilon]^j}{1-[\epsilon]}, \quad \text{and} \quad \beta_j = \frac{(1-[\epsilon])^{[j]}}{1-[\epsilon]}.$$

Modulo the image of (6.1.1), we find for $\max(0,-m) \leq a < p^n$

$$(1-[\epsilon])\left(\alpha_a\frac{U^{[m+a]}}{[T^\flat]^a} + \frac{U^{[m+a-1]}}{[T^\flat]^{a-1}} + \beta_2\frac{U^{[m+a-2]}}{[T^\flat]^{a-2}} + \dots + \beta_{a-(p^n-D)}\frac{U^{[m+p^n-D]}}{[T^\flat]^{p^n-D}}\right) \equiv 0;$$

and for $a = p^n$, as $[\epsilon]^{p^n} = 1$ in A_{cris}/p^n, modulo the image of (6.1.1) we have

$$(1-[\epsilon])\left(\frac{u^{[m+p^n-1]}}{[T^\flat]^{p^n-1}} + \beta_2\frac{u^{[m+p^n-2]}}{[T^\flat]^{p^n-2}} + \dots + \beta_D\frac{u^{[m+p^n-D]}}{[T^\flat]^{p^n-D}}\right) \equiv 0$$

Therefore, combining these congruences and modulo the image of (6.1.1) we have

$$(1-[\epsilon])\left(\frac{u^{[m+p^n-1]}}{[T^\flat]^{p^n-1}}, \dots, \frac{u^{[m+p^n-D]}}{[T^\flat]^{p^n-D}}\right)\cdot M_n^{(m)} \equiv 0$$

where $M_n^{(m)}$ is the matrix

$$\begin{pmatrix}
1 & a_{p^n-1} & 0 & \dots & 0 \\
\beta_2 & 1 & a_{p^n-2} & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{D-1} & \beta_{D-2} & \beta_{D-3} & \dots & a_{p^n-D} \\
\beta_D & \beta_{D-1} & \beta_{D-2} & \dots & 1
\end{pmatrix} \in \mathbf{M}_D(A_{\mathrm{cris}}/p^n).$$

One can check that this matrix is invertible ([AB, Lemme 19]), so modulo the image of (6.1.1) we have $(1-[\epsilon])\frac{u^{[m+p^n-i]}}{[T^\flat]^{p^n-i}} = 0$ for $1 \leq i \leq D$. In other words, $(1-[\epsilon])\mathcal{A}_n^{(m)}$ is contained in the image of (6.1.1). □

One still needs to compute $H^0(\Gamma, \mathbf{A}_n\langle u\rangle)$. One remarks first that we have the following isomorphism

$$\left(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n\right)\left[T^{\pm 1}\right]\langle u\rangle \xrightarrow{\sim} \mathbf{A}_n\langle u\rangle = \left(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n\right)\left[[T^\flat]\right]\langle u\rangle$$

sending T to $u+[T^\flat]$. Endow an action of Γ on $(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n)[T^{\pm 1}]\langle u\rangle$ via the isomorphism above. So $H^0(\Gamma,\mathbf{A}_n\langle u\rangle)$ is naturally isomorphic to the kernel of the morphism

$$\gamma-1\colon (\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n)[T^{\pm 1}]\langle u\rangle \longrightarrow (\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n)[T^{\pm 1}]\langle u\rangle, \tag{6.1.2}$$

and there is a natural injection

$$C\otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n = (\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n)[T^{\pm 1}] \hookrightarrow H^0(\Gamma,\mathbf{A}_n\langle u\rangle).$$

Lemma 6.6 ([AB] Lemme 29)**.** *The cokernel of the last map is killed by* $1-[\epsilon]$.

Proof. We have $\gamma(u)=T-[\epsilon][T^\flat]=(1-[\epsilon])T+[\epsilon]u$. Therefore

$$\begin{aligned}(\gamma-1)(u^{[m]}) &= ((1-[\epsilon])\,T+[\epsilon]u)^{[m]}-u^{[m]}\\ &= ([\epsilon]^m-1)u^{[m]}+\sum_{j=1}^{m}(1-[\epsilon])^{[j]}T^j[\epsilon]^{m-j}u^{[m-j]}\\ &= (1-[\epsilon])\left(\mu_m u^{[m]}+\sum_{j=1}^{m}\beta_j T^j[\epsilon]^{m-j}u^{[m-j]}\right),\end{aligned}$$

where $\mu_m=\frac{[\epsilon]^m-1}{1-[\epsilon]}$, $\beta_j=\frac{(1-[\epsilon])^{[j]}}{1-[\epsilon]}$ (so $\beta_1=1$). For $N>0$ an integer, consider the matrix

$$G^{(N)}:=\begin{pmatrix} 0 & T & T^2\beta_2 & \dots & T^N\beta_N\\ 0 & \mu_1 & T[\epsilon] & \dots & T^{N-1}\beta_{N-1}[\epsilon]\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & T\beta_1[\epsilon]^{N-1}\\ 0 & 0 & 0 & \dots & \mu_N\end{pmatrix}\in \mathbf{M}_{1+N}(A_{\mathrm{cris},n}[T^{\pm 1}]).$$

Then we have

$$(\gamma-1)(1,u,u^{[2]},\dots,u^{[N]})=(1-[\epsilon])(1,u,u^{[2]},\dots,u^{[N]})G^{(N)}.$$

Write

$$G^{(N)}=\begin{pmatrix} 0 & H^{(N)}\\ 0 & D\end{pmatrix}$$

with $D=(0,\dots,0,\mu_N)$ a matrix of type $1\times N$. On checks that the matrix $H^{(N)}\in \mathbf{M}_N(A_{\mathrm{cris},n}[T^{\pm 1}])$ is invertible.

Now let $\alpha\in(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n)\left[T^{\pm 1}\right]\langle u\rangle$ be an element contained in the kernel of (6.1.2). Write

$$\alpha=\sum_{s=0}^{N}a_s u^{[s]}\in\left(\left(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n\right)\left[T^{\pm 1}\right]\right)\langle u\rangle,\quad a_s\in(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n)[T^{\pm 1}].$$

As α is contained in the kernel of (6.1.2),

$$(\gamma-1)(1,u,\dots,u^{[N]})(a_0,a_1,\dots,a_N)^t=0.$$

It follows that

$$(1-[\epsilon])(1,u,\dots,u^{[N]})\begin{pmatrix} H^{(N)}\tilde{a}\\ \mu_N a_N\end{pmatrix}=0$$

with $\tilde{a}=(a_1,\dots,a_N)$. As the family $\{1,u,\dots,u^{[N]}\}$ is linearly independent, $(1-[\epsilon])H^{(N)}\tilde{a}=0$. Since $H^{(N)}$ is invertible, we deduce $(1-[\epsilon])\tilde{a}=0$. In other words, $(1-[\epsilon])a_i=0$ for $i\geq 1$ and the lemma follows. □

Now we are ready to prove

Proposition 6.7. *For any $d > 0$, $n > 0$ and $q > 0$, $H^q(\Gamma, \mathbf{A}_n\langle u_1, \dots, u_d\rangle)$ is killed by $(1-[\epsilon])^{2d-1}$. Moreover, the natural morphism*

$$C \otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \longrightarrow H^0(\Gamma, \mathbf{A}_n\langle u_1, \dots, u_d\rangle), \quad T_i \mapsto u_i + [T_i^\flat] \tag{6.1.3}$$

is injective with cokernel killed by $(1-[\epsilon])^{2d-1}$.

Proof. Recall that we have the decomposition

$$\mathbf{A}_n\langle u_1, \dots, u_d\rangle = \bigotimes_{i=1}^{d} (\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)\left[[T_i^\flat]^{\pm 1}\right]\langle u_i\rangle.$$

We shall proceed by induction on d. The case $d = 1$ comes from the previous two lemmas. For integer $d > 1$, one uses Hochschild-Serre spectral sequence

$$E_2^{i,j} = H^i(\Gamma/\Gamma_1, H^j(\Gamma_1, \mathbf{A}_n\langle u_1, \dots, u_d\rangle)) \Longrightarrow H^{i+j}(\Gamma, \mathbf{A}_n\langle u_1, \dots, u_d\rangle).$$

Using the decomposition above, the group $H^j(\Gamma_1, \mathbf{A}_n\langle u_1, \dots, u_d\rangle)$ is isomorphic to

$$H^j\left(\Gamma_1, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n\left[[T_1^\flat]^{\pm 1}\right]\langle u_1\rangle\right) \otimes \left(\otimes_{i=2}^{d}(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)\left[[T_i^\flat]^{\pm 1}\right]\langle u_i\rangle\right).$$

So by the calculation done for the case $d = 1$, we find that, up to $(1-[\epsilon])$-torsion, $H^j(\Gamma_1, A_n\langle u_1, \dots, u_d\rangle)$ is zero when $j > 0$, and is equal to

$$(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R+)/p^n)[T_1^{\pm 1}] \otimes \left(\otimes_{i=2}^{d}(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)\left[[T_i^\flat]^{\pm 1}\right]\langle u_i\rangle\right)$$

when $j = 0$. Thus, up to $(1-[\epsilon])$-torsion, $E_2^{i,j} = 0$ when $j > 0$ and $E_2^{i,0}$ is equal to

$$(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R+)/p^n)[T_1^{\pm 1}] \otimes H^i\left(\Gamma/\Gamma_1, \left(\otimes_{i=2}^{d}(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)\left[[T_i^\flat]^{\pm 1}\right]\langle u_i\rangle\right)\right).$$

Using the induction hypothesis, we get that, up to $(1-[\epsilon])^{2(d-1)-1+1}$-torsion, $E_2^{i,0} = 0$ when $i > 0$ and

$$E_2^{0,0} = (\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)[T_1^{\pm 1}, \dots, T_d^{\pm 1}] = C \otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n.$$

As $E_2^{i,j} = 0$ for $j > 1$, we have short exact sequence

$$0 \longrightarrow E_\infty^{q,0} \longrightarrow H^q(\Gamma, \mathbf{A}_n\langle u_1, \dots, u_d\rangle) \longrightarrow E_\infty^{q-1,1} \longrightarrow 0.$$

By what we have shown above, $E_\infty^{q-1,1}$ is killed by $(1-[\epsilon])$ (as this is already the case for $E_2^{q-1,1}$), and $E_\infty^{q,0}$ is killed by $(1-[\epsilon])^{2d-2}$ for $q > 0$ (as this is the case for $E_2^{q,0}$), thus $H^q(\Gamma, \mathbf{A}_n\langle u_1, \dots, u_d\rangle)$ is killed by $(1-[\epsilon])^{2d-1}$. For $q = 0$, $H^0(\Gamma, \mathbf{A}_n\langle u_1, \dots, u_d\rangle) \simeq E_\infty^{0,0} = E_2^{0,0}$. So the cokernel of the natural injection

$$C \otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \longrightarrow H^0(\Gamma, \mathbf{A}_n\langle u_1, \dots, u_d\rangle)$$

is killed by $(1-[\epsilon])^{2d-2}$, hence by $(1-[\epsilon])^{2d-1}$. □

Remark 6.8. With more efforts, one may prove that $H^q(\Gamma, \mathbf{A}_n\langle u_1, \dots, u_d\rangle)$ is killed by $(1-[\epsilon])^d$ for $q > 0$ ([AB, Proposition 21]), and that the cokernel of the morphism (6.1.3) is killed by $(1-[\epsilon])^2$ ([AB, Proposition 30]).

Corollary 6.9. *For any $n \geq 0$ and any $q > 0$, $H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n)$ is killed by $(1-[\epsilon])^{2d}$. Moreover, the natural morphism*

$$C\widehat{\otimes}_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \longrightarrow H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n)$$

is injective, with cokernel killed by $(1-[\epsilon])^{2d}$.

Recall that we want to compute $H^q(\Gamma, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty))$. For this, one needs

Lemma 6.10. *Keep the notations above and assume that the morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is étale; thus $V = \mathrm{Spa}(R, R^+)$ and $X \times_Y V = \mathrm{Spa}(\widetilde{S}, \widetilde{S}^+) = \mathrm{Spa}(\widetilde{S}_\infty, \widetilde{S}^+_\infty)$. The natural morphism*

$$B\widehat{\otimes}_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \longrightarrow \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty)$$

is an isomorphism.

Proof. By Lemma 2.17 we are reduced to showing that the natural map (here $w_j = S_j - [S_j^\flat]$)

$$B\widehat{\otimes}_A \mathbb{A}_{\mathrm{cris}}(R, R^+)\{\langle w_1, \dots, w_\delta\rangle\} \longrightarrow \mathbb{A}_{\mathrm{cris}}(\widetilde{S}, \widetilde{S}^+)\{\langle w_1, \dots, w_\delta\rangle\}$$

is an isomorphism. Since both sides of the previous maps are p-adically complete and without p-torsion, we just need to check that its reduction modulo p

$$B/p \otimes_{A/p} (\mathbb{A}_{\mathrm{cris}}(R, R^+)/p)\langle w_1, \dots, w_\delta\rangle \longrightarrow (\mathbb{A}_{\mathrm{cris}}(\widetilde{S}, \widetilde{S}^+)/p)\langle w_1, \dots, w_\delta\rangle$$

is an isomorphism. Note that we have the following expression

$$(\mathbb{A}_{\mathrm{cris}}(R, R^+)/p)\langle w_1, \dots, w_\delta\rangle \simeq \frac{(R^{\flat +}/((p^\flat)^p))[\delta_m, w_i, Z_{im}]_{1\le i\le \delta, m\in\mathbb{N}}}{(\delta_m^p, w_i^m, Z_{im}^p)_{1\le i\le \delta, m\in\mathbb{N}}},$$

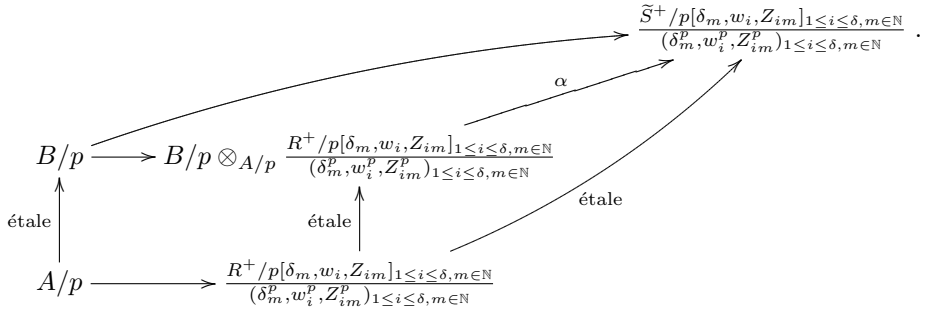
and the similar expression for $(\mathbb{A}_{\mathrm{cris}}(\widetilde{S}, \widetilde{S}^+)/p)\langle w_1, \dots, w_\delta\rangle$, where δ_m is the image of $\gamma^{m+1}(\xi)$ with $\gamma : x \mapsto x^p/p$. One sees that both sides of the morphism above are $p^\flat$-adically complete (in fact $p^\flat$ is nilpotent). Moreover $R^{\flat +}$ has no $p^\flat$-torsion. So we are reduced to showing that the morphism

$$B/p \otimes_{A/p} \frac{(R^{\flat +}/p^\flat)[\delta_m, w_i, Z_{im}]_{1\le i\le \delta, m\in\mathbb{N}}}{(\delta_m^p, w_i^p, Z_{im}^p)_{1\le i\le \delta, m\in\mathbb{N}}} \longrightarrow \frac{(\widetilde{S}^{\flat +}/p^\flat)[\delta_m, w_i, Z_{im}]_{1\le i\le \delta, m\in\mathbb{N}}}{(\delta_m^p, w_i^p, Z_{im}^p)_{1\le i\le \delta, m\in\mathbb{N}}}$$

is an isomorphism. But $R^{\flat +}/p^\flat \simeq R^+/p$ and $\widetilde{S}^{\flat +}/p^\flat \simeq \widetilde{S}^+/p$, so we just need to show that the following morphism is an isomorphism:

$$\alpha : B/p \otimes_{A/p} \frac{(R^+/p)[\delta_m, w_i, Z_{im}]_{1\le i\le \delta, m\in\mathbb{N}}}{(\delta_m^p, w_i^p, Z_{im}^p)_{1\le i\le \delta, m\in\mathbb{N}}} \longrightarrow \frac{(\widetilde{S}^+/p)[\delta_m, w_i, Z_{im}]_{1\le i\le \delta, m\in\mathbb{N}}}{(\delta_m^p, w_i^p, Z_{im}^p)_{1\le i\le \delta, m\in\mathbb{N}}}.$$

To see this, we consider the following diagram



It follows that α is étale. To see that α is an isomorphism, we just need to show that this is the case after modulo some nilpotent ideals of both sides of α. Hence we are reduced to showing that the natural morphism

$$B/p \otimes_{A/p} R^+/p \longrightarrow \widetilde{S}^+/p$$

is an isomorphism, which is clear from the definition. □

Apply the previous lemma to the étale morphism $f\colon \mathcal{X} \to \mathcal{T}$, we find a canonical Γ-equivariant isomorphism

$$B\widehat{\otimes}_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+) \xrightarrow{\sim} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}_\infty^+).$$

In particular, we find

$$H^q(\Gamma, B\widehat{\otimes}_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)) \xrightarrow{\sim} H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}_\infty^+)).$$

Now consider the following spectral sequence

$$E_2^{i,j} = R^i \varprojlim_n H^j(\Gamma, B \otimes_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n) \Longrightarrow H^{i+j}(\Gamma, B\widehat{\otimes}_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+))$$

which induces a short exact sequence for each i:

(6.1.4)

$$\begin{aligned} 0 \longrightarrow R^1 \varprojlim_n H^{i-1}(\Gamma, B \otimes_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty)/p^n) \longrightarrow H^i(\Gamma, B\widehat{\otimes}_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)) \\ \longrightarrow \varprojlim_n H^i(\Gamma, B \otimes_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty)/p^n) \longrightarrow 0. \end{aligned}$$

As B is flat over C, it can be written as a filtered limit of finite free C-modules, and as Γ acts trivially on B, the following natural morphism is an isomorphism for each i:

$$B \otimes_C H^i(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty)/p^n) \xrightarrow{\sim} H^i(\Gamma, B \otimes_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty)/p^n).$$

Therefore, for $i \geq 1$, $H^i(\Gamma, B\otimes_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n)$ is killed by $(1-[\epsilon])^{2d}$ by Corollary 6.9. Moreover, by the same corollary, we know that the following morphism is injective with cokernel killed by $(1-[\epsilon])^{2d}$:

$$C \otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \longrightarrow H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n).$$

Thus the same holds if we apply the functor $B \otimes_C -$ to the morphism above

$$B \otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \longrightarrow B \otimes_C H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n).$$

Passing to limits we obtain an injective morphism

$$B\widehat{\otimes}_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \longrightarrow \varprojlim_n \left(B \otimes_C H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n)\right),$$

whose cokernel is killed by $(1-[\epsilon])^{2d}$, and that

$$R^1 \varprojlim_n \left(B \otimes_C H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n)\right)$$

is killed by $(1-[\epsilon])^{2d}$. As a result, using the short exact sequence (6.1.4), we deduce that for $i \geq 1$, $H^i(\Gamma, B\widehat{\otimes}_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)) \simeq H^i(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}_\infty^+))$ is killed by $(1-[\epsilon])^{4d}$, and that the canonical morphism

$$B\widehat{\otimes}_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \longrightarrow H^0(\Gamma, B\widehat{\otimes}_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)) \simeq H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}_\infty^+))$$

is injective with cokernel killed by $(1-[\epsilon])^{2d}$. One can summarize the calculations above as follows:

Proposition 6.11. *(i) For any $n \geq 0$ and $q > 0$, $H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}_\infty^+)/p^n)$ is killed by $(1-[\epsilon])^{2d}$, and the natural morphism*

$$B \otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \longrightarrow H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}_\infty^+))/p^n$$

is injective with cokernel killed by $(1-[\epsilon])^{2d}$.

(ii) For any $q > 0$, $H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+))$ is killed by $(1-[\epsilon])^{4d}$ and the natural morphism

$$B\widehat{\otimes}_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \to H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}_\infty^+))$$

is injective, with cokernel killed by $(1-[\epsilon])^{2d}$.

From this proposition, we deduce the following

Theorem 6.12. *Keep the notations above. Then* $H^q(\Gamma, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty)) = 0$ *for* $q \geq 1$*, and the natural morphism*

$$B\widehat{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \longrightarrow H^0(\Gamma, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty))$$

is an isomorphism.

Proof. By the previous proposition, we just need to remark that to invert $1-[\epsilon]$ one just needs to invert t, as

$$t = \log([\epsilon]) = -\sum_{n\geq 1}(n-1)!\cdot(1-[\epsilon])^{[n]} = -(1-[\epsilon])\sum_{n\geq 1}(n-1)!\cdot\frac{(1-[\epsilon])^{[n]}}{1-[\epsilon]}.$$

Here, by [AB, Lemme 18], $\frac{(1-[\epsilon])^{[n]}}{1-[\epsilon]} \in \ker(A_{\mathrm{cris}} \xrightarrow{\theta} \widehat{\mathcal{O}_{\overline{k}}})$, hence the last summation above converges in A_{cris}. □

6.2. **Cohomology of** $\mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris}}$**.** We keep the notations at the beginning of the appendix. In this §, for simplicity, we will denote $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty)$ (resp. $\mathrm{Fil}^r \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty)$) by $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ (resp. $\mathrm{Fil}^r \mathcal{O}\mathbb{A}_{\mathrm{cris}}$). This part is entirely taken from [AB, §5], which we keep here for the sake of completeness.

Lemma 6.13. *Let* $q \in \mathbb{N}_{>0}$*, and* n *an integer* $\geq 4d+r$*. The multiplication-by-*t^n *morphism of* $H^q(\Gamma, \mathrm{Fil}^r \mathcal{O}\mathbb{A}_{\mathrm{cris}})$ *is trivial.*

Proof. Let $\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}} := \mathrm{Fil}^r \mathcal{O}\mathbb{A}_{\mathrm{cris}}/\mathrm{Fil}^{r+1}\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. As $\theta(1-[\epsilon])=0$, $\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ is killed by $1-[\epsilon]$. So using the tautological short exact sequence below

$$0 \longrightarrow \mathrm{Fil}^{r+1}\mathcal{O}\mathbb{A}_{\mathrm{cris}} \longrightarrow \mathrm{Fil}^r \mathcal{O}\mathbb{A}_{\mathrm{cris}} \longrightarrow \mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}} \longrightarrow 0$$

and by induction on the integer $r \geq 0$, one shows that $H^q(\Gamma, \mathrm{Fil}^r \mathcal{O}\mathbb{A}_{\mathrm{cris}})$ is killed by $(1-[\epsilon])^{4d+r}$; the $r=0$ case being Proposition 6.11(ii). So the multiplication-by-t^n with $n \geq 4d+r$ is zero for $H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}})$. □

Recall that for $q \in \mathbb{Z}$, we have defined

$$H^q(\Gamma, \mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty)) := \varinjlim_{n\in\mathbb{N}} H^q(\Gamma, t^{-n}\,\mathrm{Fil}^{r+n}\mathcal{O}\mathbb{A}_{\mathrm{cris}}),$$

or, in an equivalent way,

$$H^q(\Gamma, \mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty)) = \varinjlim_{n\in\mathbb{N}} H^q(\Gamma, \mathrm{Fil}^{r+n}\mathcal{O}\mathbb{A}_{\mathrm{cris}}),$$

where for each n, the transition map

$$H^q(\Gamma, \mathrm{Fil}^{r+n}\mathcal{O}\mathbb{A}_{\mathrm{cris}}) \to H^q(\Gamma, \mathrm{Fil}^{r+n+1}\mathcal{O}\mathbb{A}_{\mathrm{cris}})$$

is induced from the map

$$\mathrm{Fil}^{r+n}\mathcal{O}\mathbb{A}_{\mathrm{cris}} \to \mathrm{Fil}^{r+n+1}\mathcal{O}\mathbb{A}_{\mathrm{cris}}, \quad x \mapsto t\cdot x.$$

We have the following easy observation.

Lemma 6.14 ([AB] Lemme 33)**.** *For each* $q \in \mathbb{Z}$*,* $H^q(\Gamma, \mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}^+_\infty))$ *is a vector space over* $\mathbb{Q}_p$*.*

The first main result of this section is the following.

Proposition 6.15 ([AB] Proposition 34)**.** *$H^q(\Gamma, \mathrm{Fil}^r\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}_\infty^+)) = 0$ for any $q > 0$.*

Proof. By Lemma 6.14, the cohomology group $H^q(\Gamma, \mathrm{Fil}^r\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}_\infty^+))$ is a vector space over $\mathbb{Q}_p$. Hence to show the desired annihilation, we just need to show that for any $x \in H^q(\Gamma, t^{-n}\,\mathrm{Fil}^{r+n}\,\mathbb{A}_{\mathrm{cris}})$, there exists $m \gg n$ such that the image of x in $H^q(\Gamma, t^{-m}\,\mathrm{Fil}^{r+m}\,\mathcal{O}\mathbb{A}_{\mathrm{cris}})$ is p-torsion. In view of Lemma 6.13, we are reduced to showing that the kernel of the map

$$H^q(\Gamma, \mathrm{Fil}^{r+1}\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}) \to H^q(\Gamma, \mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}})$$

is of p-torsion, or equivalently, the cokernel of the map

$$H^{q-1}(\Gamma, \mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}) \to H^{q-1}(\Gamma, \mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}})$$

is of p-torsion for any $q \geq 1$. This assertion is verified in the following lemma. □

Lemma 6.16 ([AB] Lemme 35, Lemme 36)**.** *For each $q \in \mathbb{N}$, the cokernel of the map*

$$H^q(\Gamma, \mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}) \to H^q(\Gamma, \mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}) \tag{6.2.1}$$

is of p-torsion.

Proof. Recall that $\tilde{\xi} := \frac{[\epsilon]-1}{[\epsilon]^{1/p}-1}$ is a generator of $\ker(\theta\colon W(\widetilde{S}_\infty^{\flat+}) \to \widetilde{S}_\infty^+)$. Moreover, there exists a canonical isomorphism

$$\mathcal{O}\mathbb{A}_{\mathrm{cris}} \simeq \mathbb{A}_{\mathrm{cris}}(\widetilde{S}_\infty, \widetilde{S}_\infty^+)\{\langle u_1, \ldots, u_d, w_1, \ldots w_\delta\rangle\}.$$

For $\underline{n} = (n_0, n_1, \ldots, n_{d+\delta}) \in \mathbb{N}^{1+d+\delta}$ a multi-index, we set

$$\underline{u}^{\underline{n}} := u_1^{[n_1]} \cdots u_d^{[n_d]}, \quad \underline{w}^{\underline{n}} := w_1^{[n_{d+1}]} \cdots w_\delta^{[n_{d+\delta}]}.$$

In particular $\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ is a free $\widetilde{S}_\infty^+$-module with a basis given by

$$\tilde{\xi}^{[n_0]} \cdot \underline{u}^{\underline{n}} \cdot \underline{w}^{\underline{n}}, \quad \text{where } \underline{n} \in \mathbb{N}^{1+d+\delta} \text{ such that } |\underline{n}| = r.$$

Recall that $\widetilde{S}^+ \subset \widetilde{S}_\infty^+$. Let

$$D_r := \bigoplus_{|\underline{n}|=r} \widetilde{S}^+ \tilde{\xi}^{[n_0]} \cdot \underline{u}^{[\underline{n}]} \cdot \underline{w}^{[\underline{n}]} \subset \mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}.$$

For each $1 \leq i \leq d$, $\gamma_i(u_i) = T_i - [\epsilon][T_i^\flat] = u_i - ([\epsilon]-1)[T_i^\flat] = u_i - ([\epsilon]^{1/p}-1)\tilde{\xi}[T_i]$. Viewed as element in $\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, we have $\gamma_i(u_i) = u_i - (\epsilon^{(1)}-1)\tilde{\xi}T_i$. In particular, $D_r \subset \mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ is stable under the action of Γ. Similarly, one checks that the S^+-submodule $T^{\underline{\alpha}} \cdot D_r \subset \mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ is again stable under the action of Γ.

We first claim that, for any $\underline{\alpha} \in (\mathbb{Z}[1/p]\bigcap[0,1))^d$ such that $\alpha_i \neq 0$ for some $1 \leq i \leq d$, the cohomology $H^q(\Gamma, T^{\underline{\alpha}}D_r/p^hT^{\underline{\alpha}}D_r)$ is killed by $(\epsilon^{(1)}-1)^2$ for any q, h. Using Hochschild-Serre spectral sequence, we are reduced to showing that the cohomology of the complex

$$\cdots \longrightarrow 0 \longrightarrow T^{\underline{\alpha}}D_r/p^hT^{\underline{\alpha}}D_r \xrightarrow{\gamma_i - 1} T^{\underline{\alpha}}D_r/p^hT^{\underline{\alpha}}D_r \longrightarrow 0 \longrightarrow \cdots$$

is killed by $\epsilon^{\alpha_i} - 1$ (as $\alpha_i \neq 0$, $\alpha^{\alpha_i} - 1 | \epsilon^{(1)} - 1$), or in an equivalent way, the cohomology of the complex

$$\cdots \longrightarrow 0 \longrightarrow D_r/p^hD_r \xrightarrow{\epsilon^{\alpha_i}\gamma_i - 1} D_r/p^hD_r \longrightarrow 0 \longrightarrow \cdots$$

is killed by $\epsilon^{\alpha_i}-1$. By definition,

$$\begin{aligned}
&(\epsilon^{\alpha_i}\gamma_i-1)(\tilde{\xi}^{[n_0]}\underline{u}^{[\underline{n}]}\underline{w}^{[\underline{n}]})\\
=\ &\epsilon^{\alpha_i}\tilde{\xi}^{[n_0]}(u_i-(\epsilon^{(1)}-1)\tilde{\xi}T_i)^{[n_i]}\cdot\textstyle\prod_{j\neq i}u_j^{[n_j]}\cdot\underline{w}^{[\underline{n}]}-\tilde{\xi}^{[n_0]}\underline{u}^{[\underline{n}]}\underline{w}^{[\underline{n}]}\\
=\ &(\epsilon^{\alpha_i}-1)\tilde{\xi}^{[n_0]}\underline{u}^{[\underline{n}]}\underline{w}^{[\underline{n}]}+(\epsilon^{(1)}-1)\eta\\
=\ &(\epsilon^{\alpha_i}-1)(\tilde{\xi}^{[n_0]}\underline{u}^{[\underline{n}]}\underline{w}^{[\underline{n}]}+\eta')
\end{aligned}$$

where η and η' are linear combinations of elements of the form $\tilde{\xi}^{[m_0]}\underline{u}^{[\underline{m}]}\underline{w}^{[\underline{m}]}$ with $|\underline{m}|=r$ such that $m_i<n_i$. Note that we have the last equality because $(\epsilon^{\alpha_i}-1)|(\epsilon^{(1)}-1)$. So if we write down the matrix M of $\epsilon^{\alpha}\gamma_i-1$ with respect to the basis $\{\tilde{\xi}^{n_0}\underline{u}^{[\underline{n}]}\underline{w}^{[\underline{n}]}:|n|=r\}$ (the elements of this basis is ordered by the increasing value of the number n_i), then $M=(\epsilon^{\alpha_i}-1)\cdot U$ with U a unipotent matrix. In particular U is invertible and hence the kernel and the cokernel of the map $\epsilon^{\alpha_i}\gamma_i-1\colon D_r/p^hD_r\to D_r/p^hD_r$ are killed by $\epsilon^{\alpha_i}-1$, and hence by $\epsilon^{(1)}-1$.

Now set

$$X:=\bigoplus_{\underline{\alpha}\in(\mathbb{Z}[1/p]\bigcap[0,1))^d\setminus\{(0,\cdots,0)\}}\widetilde{S}^+\cdot T^{\underline{\alpha}}\subset\widetilde{S}^+_\infty.$$

By what we have shown above, $H^q(\Gamma,X\otimes_{\widetilde{S}^+}D_r/p^h)$ is killed by $(\epsilon^{(1)}-1)^2$ for all q,h. Hence a standard use of spectral sequence involving the higher derived functor of $\varprojlim$ shows that $H^q(\Gamma,X\widehat{\otimes}_{\widetilde{S}^+}D_r)$ is killed by $(\epsilon^{(1)}-1)^4$ for all q. On the other hand, as D_r is an $\widetilde{S}^+$-module of finite rank, $X\widehat{\otimes}_{\widetilde{S}^+}D_r=\widehat{X}\otimes_{\widetilde{S}^+}D_r$ and one checks easily that $\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}=D_r\bigoplus(\widehat{X}\otimes_{\widetilde{S}^+}D_r)$. Therefore, the canonical map

$$H^q(\Gamma,D_r)\longrightarrow H^q(\Gamma,\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}})$$

is injective with cokernel killed by $(\epsilon^{(1)}-1)^4$, hence by some power of p.

It remains to show that $H^q(\Gamma,D_r)\subset H^q(\Gamma,\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}})$ is contained in the image of (6.2.1). To see this, we shall use a different basis for the free $\widetilde{S}^+$-module D_r. For each $1\leq i\leq d$, set

$$v_i=\log\left(\frac{[T_i^\flat]}{T_i}\right)=\sum_{m=1}^{\infty}(-1)^{m-1}(m-1)!\left(\frac{[T_i^\flat]}{T_i}-1\right)^{[m]}.$$

Then $v_i\equiv-T_i^{-1}u_i \mod \mathrm{Fil}^2 A_{\mathrm{cris}}$, and $\gamma_i(v_i)=v_i+t$. Moreover, as $t\equiv(\epsilon^{(1)}-1)\tilde{\xi}\ \mathrm{Fil}^2 A_{\mathrm{cris}}$, we see that

$$M_r:=\bigoplus_{|\underline{n}|=r}\widetilde{S}^+(\epsilon^{(1)}-1)^{n_0}\tilde{\xi}^{[n_0]}\underline{u}^{[\underline{n}]}\underline{w}^{[\underline{n}]}=\bigoplus_{|\underline{n}|=r}\widetilde{S}^+t^{[n_0]}\underline{v}^{[\underline{n}]}\underline{w}^{[\underline{n}]}$$

Let $M_r^0:=\bigoplus_{|\underline{n}|=r}\mathbb{Z}_pt^{[n_0]}\underline{v}^{[\underline{n}]}\underline{w}^{[\underline{n}]}$. Then $M_r\simeq\widetilde{S}^+\otimes_{\mathbb{Z}_p}M_r^0$. In particular we find

$$H^q(\Gamma,M_r)\simeq\widetilde{S}^+\otimes_{\mathbb{Z}_p}H^q(\Gamma,M_r^0).$$

On the other hand, let $\widetilde{M_r}$ (resp. $\widetilde{M_r^0}$) be the $B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)$-submodule (resp. the $\mathbb{Z}_p$-submodule) of $\mathrm{Fil}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ generated by $\{t^{[n_0]}\underline{v}^{[\underline{n}]}\underline{w}^{[\underline{n}]}\}_{|\underline{n}|=r}$. Since $\gamma_i(v_i)=v_i+t$, $\widetilde{M_r}$ and $\widetilde{M_r^0}$ are both stable under the action of Γ. Moreover, the natural morphism $\mathrm{Fil}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}\to\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ induces a surjective morphism $\widetilde{M_r}\twoheadrightarrow M_r$. Now

we have the following tautological commutative diagram

$$\begin{array}{ccc}
(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\otimes_{\mathbb{Z}_p} H^q(\Gamma,\widetilde{M}_r^0) & \xrightarrow{\text{sur.}} & \widetilde{S}^+\otimes_{\mathbb{Z}_p} H^q(\Gamma,M_r^0)\ . \\
\downarrow & & \downarrow\simeq \\
H^q(\Gamma,\widetilde{M}_r) & \longrightarrow & H^q(\Gamma,M_r) \\
\downarrow & & \downarrow\beta \\
H^q(\Gamma,\mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}) & \xrightarrow{(6.2.1)} & H^q(\Gamma,\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}})
\end{array}$$

But the morphism β can be decomposed as the composite of two morphisms

$$H^q(\Gamma,M_r)\to H^q(\Gamma,D_r)\hookrightarrow H^q(\Gamma,\mathrm{gr}^r\mathcal{O}\mathbb{A}_{\mathrm{cris}})$$

which is killed by $(\epsilon^{(1)}-1)^{r+4}$. This concludes the proof of our lemma. □

It remains to compute the Γ-invariants of $\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris}}(S_\infty,S_\infty)$. For this, we shall first prove $H^0(\Gamma,\mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}})=B\widehat{\otimes}_A\,\mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)$.

Recall that $\tilde{\xi}:=\frac{[\epsilon]-1}{[\epsilon]^{1/p}-1}$. Put

$$\eta:=\frac{([\epsilon]-1)^{p-1}}{p}\in A_{\mathrm{cris}}[p^{-1}].$$

For each $x\in\ker(\theta\colon\mathcal{O}\mathbb{A}_{\mathrm{cris}}(\widetilde{S}_\infty,\widetilde{S}_\infty^+)\to\widetilde{S}_\infty^+)$, set $\gamma(x):=x^p/p$. One checks that $\gamma(x)\in\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ and $\theta(\gamma(x))=0$.

We have the following observation.

Lemma 6.17 ([AB] Lemme 37)**.** *There exists $\lambda\in\tilde{\xi}W(\mathcal{O}_{\widehat{\overline{k}}^\flat})$ such that $\eta=(p-1)!\tilde{\xi}^{[p]}+\lambda$. In particular, $\eta\in\ker(\theta\colon A_{\mathrm{cris}}\to\mathcal{O}_{\widehat{\overline{k}}})$. Furthermore, $\gamma^\nu(\eta)\in p^{-r}\eta A_{\mathrm{cris}}$ for any integers $0\le\nu\le r$.*

Lemma 6.18 ([AB] page 1011)**.** *There exists isomorphism*

$$\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p\simeq\frac{(\widetilde{S}_\infty^{\flat+}/\tilde{\xi}^p)[\eta_m,w_j,Z_{j,m},u_i,T_{i,m}]_{1\le i,j\le d;m\in\mathbb{N}}}{(\eta_m^p,w_j^p,Z_{j,m}^p,u_i^p,T_{i,m}^p)}$$

where η_m denotes the image of $\gamma^m(\eta)$ in $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p$. We have a similar assertion for $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p$.

Proof. Recall that $\tilde{\xi}$ is a generator of the kernel $\ker(\theta\colon W(S_\infty^{\flat+})\to S_\infty^+)$, hence we have isomorphism

$$\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p\simeq\frac{(S_\infty^{\flat+}/\tilde{\xi}^p)[\delta_m,w_j,Z_{j,m},u_i,T_{i,m}]_{1\le i,j\le d;m\in\mathbb{N}}}{(\delta_m^p,w_j^p,Z_{j,m}^p,u_i^p,T_{i,m}^p)}$$

where δ_m represents the image in $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p$ of the element $\gamma^{m+1}(\tilde{\xi})$. As $\eta=(p-1)!\tilde{\xi}^{[p]}+\lambda$ for some $\lambda\in\tilde{\xi}W(\mathcal{O}_{\widehat{\overline{k}}}^\flat)$, it follows that $\eta_m\in\frac{(S_\infty^{\flat+}/\tilde{\xi}^p)[\delta_\nu]_{0\le\nu\le m}}{(\delta_\nu^p)_{0\le\nu\le m}}\subset\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p$, and $\eta_m^p=0$, giving a morphism of $S_\infty^{\flat+}/\tilde{\xi}^p$-algebras

$$\alpha_m\colon\frac{(S_\infty^{\flat+}/\tilde{\xi}^p)[W_\nu]_{0\le\nu\le m}}{(W_\nu^p)_{0\le\nu\le m}}\longrightarrow\frac{(S_\infty^{\flat+}/\tilde{\xi}^p)[\delta_\nu]_{0\le\nu\le m}}{(\delta_\nu^p)_{0\le\nu\le m}},\quad W_\nu\mapsto\eta_\nu.$$

On the other hand, for any $\nu \in \mathbb{N}$,

$$\gamma^\nu((p-1)!\tilde{\xi}^{[p]}) = \gamma^\nu(\eta - \lambda) = \sum_{i=0}^{p^\nu} a_i \eta^{[p^\nu - i]} \lambda^{[i]}$$

for some coefficients $a_i \in S_\infty^{\flat+}/\tilde{\xi}^p$, and $\lambda^{[i]} \in \frac{(S_\infty^{\flat+}/\tilde{\xi}^p)[\delta_j]_{0\leq j\leq \nu-1}}{(\delta_j^p)_{0\leq j\leq \nu-1}}$ (as $i \leq p^\nu$). By induction on ν, one checks that $\delta_\nu \in \mathrm{Im}(\alpha_m)$, hence α_m is an isomorphism for the reason of length. Taking direct limits, one deduces that the morphism of $S_\infty^{\flat+}/\tilde{\xi}^p$-algebras

$$\alpha\colon \frac{(S_\infty^{\flat+}/\tilde{\xi}^p)[W_m]_{m\in\mathbb{N}}}{(W_m^p)_{m\in\mathbb{N}}} \longrightarrow \frac{(S_\infty^{\flat+}/\tilde{\xi}^p)[\delta_m]_{m\in\mathbb{N}}}{(\delta_m^p)_{m\in\mathbb{N}}}, \quad W_m \mapsto \eta_m$$

is an isomorphism, proving our lemma. □

Lemma 6.19 ([AB] Lemme 38)**.** *For any $r \in \mathbb{N}$, we have $(p^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}) \bigcap \eta\mathcal{O}\mathbb{A}_{\mathrm{cris}} = \sum_{\nu=0}^r p^r\gamma^\nu(\eta)\mathcal{O}\mathbb{A}_{\mathrm{cris}}$.*

Proof. The proof is done by induction on r. The case $r = 0$ is trivial. So we may and do assume that $r > 0$. Let $x \in p^r\mathcal{O}\mathbb{A}_{\mathrm{cris}} \bigcap \eta\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. By induction hypothesis, one can write

$$x = \sum_{\nu=0}^{r-1} p^{r-1}\gamma^\nu(\eta)x_\nu, \quad \text{with } x_\nu \in \mathcal{O}\mathbb{A}_{\mathrm{cris}}.$$

Set $x' := \sum_{\nu=0}^{r-1} \gamma^\nu(\eta)x_\nu$. Then $x' \in p\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ since $x \in p^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. So in $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p$, we have

$$\sum_{\nu=0}^{r-1} \eta_\nu \overline{x_\nu} = 0,$$

where for $a \in \mathcal{O}\mathbb{A}_{\mathrm{cris}}$ we denote by $\overline{a}$ its reduction modulo p and η_ν the image of $\gamma^\nu(\eta)$ in $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p$. Let

$$\Lambda := \frac{(\widetilde{S}_\infty^{\flat+}/\tilde{\xi}^p)[\eta_h, w_j, Z_{jm}, u_i, T_{im}]_{1\leq i\leq d, 1\leq j\leq \delta, m\in\mathbb{N}, h\geq r}}{(\eta_h^p, w_j^p, Z_{jm}^p, u_i^p, T_{im}^p)_{1\leq i\leq d, 1\leq j\leq \delta, m\in\mathbb{N}, h\geq r}}.$$

Then $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p \simeq \Lambda[\eta_m]_{0\leq m\leq r-1}/(\eta_m^p)_{0\leq m\leq r-1}$ by the previous lemma. In particular, $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p$ is a free Λ-module with a basis given by $\{\prod_{m=0}^{r-1}\eta_m^{\alpha_m} : \underline{\alpha} \in \{0,\dots,p-1\}^r\}$. Thus up to modifying the x_ν's for $0 \leq \nu \leq r-2$, we may assume that $\overline{x_\nu} \in \Lambda[\eta_0,\dots,\eta_\nu]/(\eta_0^p,\dots,\eta_\nu^p)$.

Next we claim that $\overline{x_\nu} \in \eta_\nu^{p-1}(\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p)$, which is also done by induction on ν. When $\nu = 0$, as $\overline{x_0} \in \Lambda[\eta_0]/(\eta_0^p)$, we just need to show $\eta_0\overline{x_0} = 0$. But

$$\eta_0\overline{x_0} = -\sum_{h=1}^{r-1} \eta_h\overline{x_h} \in \Lambda[\eta_0,\eta_1,\dots,\eta_{r-1}]/(\eta_0^p,\eta_1^p,\dots,\eta_{r-1}^p),$$

we have necessarily $\eta_0\overline{x_0} = 0$, giving our claim for $\nu = 0$. Let now $\nu \in \{1,\dots,r-1\}$ such that for any $0 \leq h \leq \nu$, $\overline{x_h} \in \eta_h^{p-1}(\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p)$. So $\eta_h\overline{x_h} = 0$ for $0 \leq h \leq \nu$ and

$$\eta_\nu\overline{x_\nu} = -\sum_{l=\nu+1}^{r-1} \eta_l\overline{x_l}.$$

So necessarily $\eta_\nu\overline{x_\nu} = 0$, and hence $\overline{x_\nu} \in \eta_\nu^{p-1}(\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p)$. This shows our claim for any $0 \leq \nu \leq r-1$.

As a result, for any $0 \leq \nu \leq r-1$, we have $x_\nu \in \gamma^\nu(\eta)^{p-1}\mathcal{O}\mathbb{A}_{\mathrm{cris}} + p\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, and hence

$$p^{r-1}\gamma^\nu(\eta)x_\nu \in p^{r-1}\gamma^\nu(\eta)^p\mathcal{O}\mathbb{A}_{\mathrm{cris}} + p^r\gamma^\nu(\eta)\mathcal{O}\mathbb{A}_{\mathrm{cris}}.$$

But $p^{r-1}\gamma^\nu(\eta)^p = p^r\gamma^{\nu+1}(\eta)$, so we get finally $x \in \sum_{\nu=0}^r p^r\gamma^\nu(\eta)\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. □

Recall that we have an injection $B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+) \subset \mathcal{O}\mathbb{A}_{\mathrm{cris}}$ (Proposition 6.11). The key lemma is the following

Lemma 6.20 ([AB] Lemme 39)**.** *Inside $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, we have*

$$(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\bigcap\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}} = \eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)).$$

Proof. The proof of the lemma is divided into several steps. First we remark that, as $B\otimes_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p \hookrightarrow \mathcal{O}\mathbb{A}_{\mathrm{cris}}/p$, we have

$$(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\bigcap p\mathcal{O}\mathbb{A}_{\mathrm{cris}} = p(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)).$$

Secondly, we claim that, to show our lemma, it is enough to prove that

$$(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\bigcap\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}} \subset \eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)) + p\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}}. \tag{6.2.2}$$

Indeed, once this is done, we obtain

$$\begin{aligned}
&(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R+))\bigcap\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}}\\
\subset\ & \eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)) + p\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}}\bigcap(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\\
=\ & \eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)) + p\left(\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}}\bigcap(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\right)\\
\subset\ & \eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)) + p\left(\eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)) + p\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}}\right)\\
\subset\ & \eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)) + p^2\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}}.
\end{aligned}$$

An iteration of this argument shows that

$$(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R+))\bigcap\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}} \subset \eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)) + p^r\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}}$$

for any $r \in \mathbb{N}$. Thus $(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R+))\bigcap\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}} \subset \eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))$ as $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ and $B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)$ are p-adically complete.

Next we claim that for any $r \in \mathbb{N}$ we have

$$(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\bigcap\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}} \subset \eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)) + p^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}. \tag{6.2.3}$$

When $r=0$, this is trivial. Assume now the inclusion above holds for some $r \in \mathbb{N}$. So by Lemma 6.19

$$\begin{aligned}
&(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\bigcap\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}}\\
\subset\ & \eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)) + (B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\bigcap p^r\mathcal{O}\mathbb{A}_{\mathrm{cris}}\bigcap\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}}\\
\subset\ & \eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)) + (B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\bigcap p^r\left(\textstyle\sum_{\nu=0}^r\gamma^\nu(\eta)\mathcal{O}\mathbb{A}_{\mathrm{cris}}\right)\\
\subset\ & \eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)) + p^r\left((B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\bigcap\left(\textstyle\sum_{\nu=0}^r\gamma^\nu(\eta)\mathcal{O}\mathbb{A}_{\mathrm{cris}}\right)\right).
\end{aligned}$$

So we are reduced to showing that

$$\left(\left(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}\left(R,R^+\right)\right)\bigcap\left(\sum_{\nu=0}^r\gamma^\nu(\eta)\mathcal{O}\mathbb{A}_{\mathrm{cris}}\right)\right) \subset p^{-r}\eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)) + p\mathcal{O}\mathbb{A}_{\mathrm{cris}}.$$

Put

$$\Lambda(R,R^+) := \frac{(R^{\flat+}/\tilde{\xi}^p)[w_j, Z_{jm}]_{1\leq j\leq\delta, m\in\mathbb{N}}}{(w_j^m, Z_{jm}^p)_{1\leq j\leq\delta, m\in\mathbb{N}}} \subset \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p,$$

and

$$\Lambda(\widetilde{S}_\infty, \widetilde{S}_\infty^+) := \frac{(\widetilde{S}_\infty^\flat{}^+/\tilde{\xi}^p)[w_j, Z_{jm}]_{1\leq j\leq \delta, m\in\mathbb{N}}}{(w_j^m, Z_{jm}^p)_{1\leq j\leq\delta, m\in\mathbb{N}}} \subset \mathcal{O}\mathbb{A}_{\mathrm{cris}}/p.$$

We claim that the natural morphism $A \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p$ (resp. $B \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}/p$) factors through $\Lambda(R, R^+)$ (resp. through $\Lambda(\widetilde{S}_\infty, \widetilde{S}_\infty^+)[u_i]_{1\leq i\leq d}/(u_i^p)_{1\leq i\leq d}$). Indeed, we consider the reduction modulo p of the morphism θ

$$\overline{\theta}\colon \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p \longrightarrow R^+/p.$$

Then all elements of $\ker(\overline{\theta})$ are nilpotent, and $\overline{\theta}(\Lambda(R, R^+)) = R^+/p$. Recall that A is étale over $\mathcal{O}_k\{S_1^{\pm 1}, \ldots, S_\delta^{\pm 1}\}$ and the composite

$$\mathcal{O}_k\{S_1^{\pm 1}, \ldots, S_\delta^{\pm 1}\} \to A \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p$$

sends S_j to the reduction modulo p of $w_j + S_j^\flat \in \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p$, which factors through $\Lambda(R, R^+)$ and hence so does the morphism $A \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p$ by the étaleness of the morphism $\mathcal{O}_k\{S_1^{\pm 1}, \ldots, S_\delta^{\pm 1}\} \to A$. In a similar way, one sees that the morphism $B \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}/p$ factors through $\Lambda(\widetilde{S}_\infty, \widetilde{S}_\infty^+)[u_i]_{1\leq i\leq d}/(u_i^p)_{1\leq i\leq d}$. In particular, it makes sense to consider the map

$$B \otimes_A \frac{\Lambda(R, R^+)[\eta_m]_{m\in\mathbb{N}}}{(\eta_m^p)_{m\in\mathbb{N}}} \longrightarrow \widetilde{\Lambda} := \frac{\Lambda(\widetilde{S}_\infty, \widetilde{S}_\infty^+)[u_i, \eta_m]_{1\leq i\leq d, m\in\mathbb{N}}}{(u_i^p, \eta_m^p)_{1\leq i\leq d, m\in\mathbb{N}}},$$

which is injective. Now for $x \in (B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+))\bigcap\sum_{\nu=0}^r \gamma^\nu(\eta)\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, its reduction $\overline{x}$ modulo p lies in $\sum_{\nu=0}^r \eta_\nu\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p$. On the other hand, as a module $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p$ is free over $\widetilde{\Lambda}$ and the inclusion $\widetilde{\Lambda} \subset \mathcal{O}\mathbb{A}_{\mathrm{cris}}/p$ is a direct factor of $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p$ as $\widetilde{\Lambda}$-modules, the fact that $\overline{x}$ appears in the image of the morphism above (note that $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p = \Lambda(R, R^+)[\eta_m]_{m\in\mathbb{N}}/(\eta_m^p)_{m\in\mathbb{N}}$) implies that

$$\overline{x} \in \sum_{\nu=0}^r \eta_\nu\widetilde{\Lambda}.$$

Thus

$$\overline{x} \in \sum_{\nu=0}^r \eta_\nu(B\widehat{\otimes}_A\Lambda(R, R^+))[\eta_m]_{m\in\mathbb{N}}/(\eta_m^p)_{m\in\mathbb{N}}$$

as $(B \otimes_A \Lambda(R, R^+))[\eta_m]_{m\in\mathbb{N}}/(\eta_m^p)_{m\in\mathbb{N}}$ and $\widetilde{\Lambda}$ are free over $B \otimes_A \Lambda(R, R^+)$ and over $\Lambda(R, R^+)[u_i]_{1\leq i\leq d}/(u_i^p)_{1\leq i\leq d}$ respectively , with a basis given by $\{\prod_{m\in\mathbb{N}} \eta_m^{\alpha_m} : \underline{\alpha} \in \{0, \ldots, p-1\}^{(\mathbb{N})}\}$. In other words, this shows

$$x \in \sum_{\nu=0}^r \gamma^\nu(\eta)(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)) + p\mathcal{O}\mathbb{A}_{\mathrm{cris}}.$$

By Lemma 6.17, $\gamma^\nu(\eta) \in p^{-r}\eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+))$, we obtained finally

$$x \in p^{-r}\eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)) + p\mathcal{O}\mathbb{A}_{\mathrm{cris}},$$

as desired.

To complete the proof, it remains to show that the inclusions (6.2.3) for all $r \in \mathbb{N}$ implies (6.2.2). For $\nu \geq -1$ an integer, set $\beta_\nu := \prod_{i=0}^\nu \gamma^i(\eta)^{p-1}$. A direct calculation shows $\eta\beta_\nu = p^{\nu+1}\gamma^{\nu+1}(\eta)$ for any $\nu \geq -1$. Let

$$\Lambda := \frac{\Lambda(\widetilde{S}_\infty, \widetilde{S}_\infty^+)[u_i, T_{im}]_{1\leq i\leq d, m\in\mathbb{N}}}{(u_i^p, T_{im}^p)_{1\leq i\leq d, m\in\mathbb{N}}}.$$

Then $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p = \Lambda[\eta_m]_{m\in\mathbb{N}}/(\eta_m^p)_{m\in\mathbb{N}}$. Recall also that

$$\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+) = \Lambda(R,R^+)[\eta_m]_{m\in\mathbb{N}}/(\eta_m^p)_{m\in\mathbb{N}}.$$

For $\nu \geq -1$, let M_ν (resp. N_ν) be the sub-Λ-module (resp. sub-$B\otimes_A\Lambda(R,R^+)$-module) of $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p$ (resp. of $B\otimes_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p$) generated by the family

$$\left\{\prod_{m=0}^{\infty}\eta_m^{\alpha_m} : \underline{\alpha}\in\{0,\dots,p-1\}^{(\mathbb{N})}, \alpha_m = p-1 \text{ for } 0\leq m\leq \nu, \alpha_{\nu+1} < p-1\right\}.$$

Clearly $N_\nu\subset M_\nu$, $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p = \oplus_{\nu\geq -1}M_\nu$, and $B\otimes_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p = \oplus_{\nu\geq -1}N_\nu$. Let $z\in(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\bigcap\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, and write $z=\eta z'$ with $z'\in\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. Let N be an integer such that

$$\overline{z'}\in\bigoplus_{\nu=-1}^{N}M_\nu.$$

By (6.2.3) for $r=N+2$, one can write

$$z=\eta\cdot y+w,\quad \text{with}\quad y\in B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+) \text{ and } w\in p^{N+2}\mathcal{O}\mathbb{A}_{\mathrm{cris}}\bigcap\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}}.$$

By Lemma 6.19, $w = p^{N+2}\sum_{i=0}^{N+2}\gamma^i(\eta)\alpha_i$ with $\alpha_i\in\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. So we find

$$w=\sum_{i=0}^{N+2}p^{N+2-i}p^i\gamma^i(\eta)\alpha_i=\sum_{i=0}^{N+2}p^{N+2-i}\eta\beta_{i-1}\alpha_i.$$

Therefore $z'=y+\sum_{i=0}^{N+2}p^{N+2-i}\beta_{i-1}\alpha_i$ and thus $\overline{z'}=\overline{y}+\overline{\beta_{N+1}}\overline{\alpha_{N+2}}$. But by the definition of the integer N,

$$\overline{z'}\in\bigoplus_{\nu=-1}^{N}M_\nu$$

while

$$\overline{y}\in\bigoplus_{\nu=-1}^{\infty}N_\nu\quad\text{and}\quad\overline{\beta_{N+1}}\overline{\alpha_{N+2}}\in\bigoplus_{\nu>N}M_\nu.$$

So necessarily $\overline{\beta_{N+1}}\overline{\alpha_{N+2}}\in\oplus_{\nu>N}N_\nu\in B\otimes_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p$. In other words, we find $\beta_{N+1}\alpha_{N+2}\in B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)+p\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. So finally

$$z\in\eta(y+\beta_{N+1}\alpha_{N+2})\in\eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))+p\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}},$$

as required by (6.2.2). This finishes the proof of this lemma. □

Corollary 6.21 ([AB] Corollaire 40)**.** *Inside $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, we have*

$$(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\bigcap([\epsilon]-1)^{p-1}\mathcal{O}\mathbb{A}_{\mathrm{cris}}=([\epsilon]-1)^{p-1}(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)).$$

Proof. Let $x\in(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\bigcap([\epsilon]-1)^{p-1}\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. As $p\eta=([\epsilon]-1)^{p-1}$, $x\in(B\widehat{\otimes}_A\mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\bigcap p\mathcal{O}\mathbb{A}_{\mathrm{cris}} = p(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))$. Thus, by Lemma 6.20, we find $x/p\in(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\bigcap\eta\mathcal{O}\mathbb{A}_{\mathrm{cris}}=\eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))$. Therefore $x\in p\eta(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))=([\epsilon]-1)^{p-1}(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))$. □

Finally, we can assemble the previous results to get the main result about the Γ-invariants.

Proposition 6.22 ([AB] Proposition 41)**.** *For each $r \in \mathbb{N}$, the natural injection*

$$\iota_r \colon B\widehat{\otimes}_A \operatorname{Fil}^r \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \longrightarrow H^0(\Gamma, \operatorname{Fil}^r \mathcal{O}\mathbb{A}_{\mathrm{cris}})$$

is an isomorphism.

Proof. We shall begin with the case where $r = 0$. By Proposition 6.11 (ii), the natural morphism $B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \to H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}})$ is injective with cokernel killed by $(1 - [\epsilon])^{2d}$. It remains to show that the latter map is also surjective. Let $x \in H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}})$. So $(1 - [\epsilon])^{2d}x \in B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$. In particular,

$$(1 - [\epsilon])^{2d(p-1)}x \in \big(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)\big) \bigcap (1 - [\epsilon])^{2d(p-1)}\mathcal{O}\mathbb{A}_{\mathrm{cris}}.$$

By Corollary 6.21, the last intersection is just $(1 - [\epsilon])^{2d(p-1)}(B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+))$. In particular, $x \in B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$; note that $1 - [\epsilon]$ is a regular element. This concludes the proof of our proposition for $r = 0$. Assume now $r > 0$ and that the statement holds for $\operatorname{Fil}^{r-1}\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. Then we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B\widehat{\otimes}_A \operatorname{Fil}^r \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) & \longrightarrow & B\widehat{\otimes}_A \operatorname{Fil}^{r-1} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) & \longrightarrow & B\widehat{\otimes}_A \mathrm{gr}^{r-1}\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota_r} & & \downarrow{\scriptstyle \iota_{r-1}} & & \downarrow & & \\
0 & \longrightarrow & H^0(\Gamma, \operatorname{Fil}^r \mathcal{O}\mathbb{A}_{\mathrm{cris}}) & \longrightarrow & H^0(\Gamma, \operatorname{Fil}^{r-1} \mathcal{O}\mathbb{A}_{\mathrm{cris}}) & \longrightarrow & H^0(\Gamma, \mathrm{gr}^{r-1}\mathcal{O}\mathbb{A}_{\mathrm{cris}}) & &
\end{array}$$

One checks easily that the last vertical morphism is injective as $B\widehat{\otimes}_A R^+ = \widetilde{S}^+ \subset \widetilde{S}^+_\infty$. So by snake lemma, that ι_{r-1} is an isomorphism implies that the morphism ι_r is also an isomorphism. This finishes the induction and hence the proof of our proposition. □

Corollary 6.23 ([AB] Corollaire 42)**.** *The natural morphism*

$$B\widehat{\otimes}_A \operatorname{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \longrightarrow H^0(\Gamma, \operatorname{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris}})$$

is an isomorphism, where

$$B\widehat{\otimes}_A \operatorname{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) := \varinjlim_{n \geq 0} B\widehat{\otimes}_A \operatorname{Fil}^{r+n} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$$

with transition maps given by multiplication by t.

7. Appendix: I-flatness and base change

Definition 7.1 (Fontaine, see [AB] Définition 9.2.1)**.** Let A be a ring, B an A-algebra and $I \subset B$ an ideal. We say that B is *I-flat over A* if for any injection of A-modules $u \colon M \to M'$, the B-module $\ker(1 \otimes u \colon M \otimes_A B \to M' \otimes_A B)$ is killed by I.

In the following, let A be a ring, B an A-algebra and $I \subset B$ an ideal.

Lemma 7.2. *The following assertions are equivalent:*

1. *B is I-flat over A;*
2. *for any A-module, the B-module $\operatorname{Tor}_1^A(M, B)$ is killed by I;*
3. *for any A-module and any $i \geq 1$, the B-module $\operatorname{Tor}_i^A(M, B)$ is killed by I.*

Proof. For the equivalence between (1) and (2), see [AB, Proposition 9.2.3 (iii)]. The equivalence between (2) and (3) is standard. □

Proposition 7.3. *Assume A noetherian and that B is I-flat over A. Let X be a finite type separated scheme over $\mathrm{Spec}(A)$ and $\mathcal{F}$ a coherent sheaf on X. Then there exists $N \in \mathbb{N}$ which does not depend on $\mathcal{F}$, such that the canonical morphism*

$$H^i(X, \mathcal{F}) \otimes_A B \longrightarrow H^i(X, \mathcal{F} \otimes_A B)$$

has kernel and cokernel killed by I^N.

Proof. If B is free as an A-module, since X is qcqs, the canonical morphism in our proposition is an isomorphism. In general, by using a free resolution of B (as A-module), we have a natural isomorphism in the derived category:

$$R\Gamma(X, \mathcal{F}) \otimes^L_A B \xrightarrow{\sim} R\Gamma(X, \mathcal{F} \otimes^L_A B),$$

inducing an isomorphism for each $i \in \mathbb{Z}$:

$$H^i(R\Gamma(X, \mathcal{F}) \otimes^L_A B) \xrightarrow{\sim} H^i(X, \mathcal{F} \otimes^L_A B).$$

We claim that the kernel and the cokernel of the natural morphism below are killed by I^{d+1} (here $d := \dim(X)$):

$$H^i(X, \mathcal{F}) \otimes_A B \longrightarrow H^i\left(R\Gamma(X, \mathcal{F}) \otimes^L_A B\right).$$

To see this, consider the spectral sequence

$$E_2^{a,b} = \mathrm{Tor}^A_{-a}(H^b(X, \mathcal{F}), B) \Longrightarrow H^{a+b}\left(R\Gamma(X, \mathcal{F}) \otimes^L_A B\right).$$

The cokernel of the canonical injection

$$\alpha \colon E_\infty^{0,i} \longrightarrow H^i(R\Gamma(X, \mathcal{F}) \otimes^L_A B)$$

has a filtration with successive quotients $E_\infty^{a,i-a}$ with $a < 0$. As X has dimension d, $E_2^{a,i-a} = E_\infty^{a,i-a} = 0$ whenever $i - a \notin [0, d]$. Furthermore, as B is I-flat over A and as $a < 0$, $E_2^{a,i-a}$ hence $E_\infty^{a,i-a}$ is killed by I (Lemma 7.2). It follows that $\mathrm{coker}(\alpha)$ is killed by I^{d+1}. On the other hand, as $E_2^{a,b} = 0$ for $a > 0$, for each $r \geq 2$, one has $E_r^{a,b} = 0$ for $a > 0$ and a tautological exact sequence:

$$E_r^{-r,i+r-1} \longrightarrow E_r^{0,i} \longrightarrow E_{r+1}^{0,i} \longrightarrow 0.$$

Since $E_2^{a,b} = E_r^{a,b} = 0$ for any $b \notin [0, d]$ and any $r \geq 2$, we find $E_r^{-r,i+r-1} = 0$ for $r \geq r_0 := \max\{d + 2 - i, 2\}$. Thus $E_\infty^{0,i} = E_{r_0}^{0,i}$ and the kernel of the surjection

$$E_2^{0,i} \longrightarrow E_{r_0}^{0,i} = E_\infty^{0,i}$$

is annihilated by I^{d+1}. By consequence, the morphism

$$\beta \colon E_2^{0,i} = H^i(X, \mathcal{F}) \otimes_A B \longrightarrow H^i(R\Gamma(X, \mathcal{F}) \otimes^L_A B)$$

has kernel and cokernel killed by I^{d+1}.

Next we claim that the natural morphism

$$H^i(X, \mathcal{F} \otimes^L_A B) \longrightarrow H^i(X, \mathcal{F} \otimes_A B)$$

has kernel and cokernel killed by I^n for some integer n that does not depend on the choice of $\mathcal{F}$. Indeed, we fix a free resolution (concentrated in negative degrees) $B^\bullet$ of B as A-module: so $\mathcal{F} \otimes^L_A B \simeq \mathcal{F} \otimes_A B^\bullet$. Let $\mathcal{K}$ be the kernel of the surjective morphism of complexes $\mathcal{F} \otimes_A B^\bullet \to \mathcal{F} \otimes_A B$. Then

$$H^j(\mathcal{K}) = \begin{cases} 0 & j = 0 \\ \mathrm{Tor}^A_{-j}(\mathcal{F}, B) & j < 0. \end{cases}$$

On the other hand, each abelian sheaf on X for the Zariski topology has a resolution of length d with $R\Gamma(X,-)$-acyclic components (recall that $d=\dim(X)$). Therefore, $H^j(X,\mathcal{K})=H^j(X,\tau_{\geq i-d}\mathcal{K})$ for $j\in\{i,i+1\}$. Applying truncations again on the two-side bounded complex $\tau_{\geq i-d}\mathcal{K}$ and using the fact that $H^j(\tau_{\geq i-d}\mathcal{K})$ is killed by I for all j (Lemma 7.2), we deduce the existence of an integer $n\geq 0$, depending only on i and d, such that $H^j(X,\mathcal{K})=H^j(X,\tau_{\geq i-d}\mathcal{K})$ is annihilated by I^n for $j\in\{i,i+1\}$. So the kernel and the cokernel of the morphism

$$\gamma\colon H^i(X,\mathcal{F}\otimes^L_A B)\longrightarrow H^i(X,\mathcal{F}\otimes_A B)$$

are killed by I^n.

Finally, consider the following commutative square

$$\begin{array}{ccc} H^i(R\Gamma(X,\mathcal{F})\otimes^L_A B) & \xrightarrow{\sim} & H^i(X,\mathcal{F}\otimes^L_A B) \\ \text{iso. up to } I^{d+1}\text{-torsion}\uparrow \beta & & \gamma\downarrow \text{iso. up to } I^n\text{-torsion} \\ H^i(X,\mathcal{F})\otimes_A B & \longrightarrow & H^i(X,\mathcal{F}\otimes_A B) \end{array}$$

It follows that the lower horizontal map has kernel and cokernel killed by I^{n+d+1}. So the integer $N:=n+d+1$ satisfies the property that we need. □

From now on, assume moreover that A and B are p-adically complete.

Proposition 7.4. *Assume A noetherian and that B is I-flat over A. Let $\mathcal{X}\to \mathrm{Spf}(A)$ be a proper morphism between p-adic formal schemes. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$. Then for all i, the kernel and cokernel of the natural morphism*

$$H^i(\mathcal{X},\mathcal{F})\otimes_A B\longrightarrow H^i(\mathcal{X},\mathcal{F}\widehat{\otimes}_A B)$$

are killed by some power of pI.

Proof. Let $\mathcal{G}\subset\mathcal{F}$ be the subsheaf formed by elements killed by some power of p. As $\mathcal{F}$ is coherent and as $\mathcal{X}$ is quasi-compact, $\mathcal{G}$ is killed by a large power of p. Therefore $\mathcal{G}$ and the quotient $\mathcal{F}/\mathcal{G}$ are again coherent sheaves on $\mathcal{X}$. Hence the natural morphism $H^i(\mathcal{X},\mathcal{F})\to H^i(\mathcal{X},\mathcal{F}/\mathcal{G})$ has kernel and cokernel killed by some large power of p, thus so is the induced morphism $H^i(\mathcal{X},\mathcal{F})\otimes_A B\to H^i(\mathcal{X},\mathcal{F}/\mathcal{G})\otimes_A B$. Similarly, the kernel of the surjection $\mathcal{F}\otimes_A B\to(\mathcal{F}/\mathcal{G})\otimes_A B$ is killed by some large power of p, thus so is the kernel of the following surjective morphism of projective systems

$$\left(\frac{\mathcal{F}\otimes_A B}{p^{n+1}}\right)_{n\geq 0}\longrightarrow\left(\frac{(\mathcal{F}/\mathcal{G})\otimes_A B}{p^{n+1}}\right)_{n\geq 0}.$$

In particular, the kernel and the cokernel of the map between the completion $\mathcal{F}\widehat{\otimes}_A B\to(\mathcal{F}/\mathcal{G})\widehat{\otimes}_A B$ are killed by some large power of p. Thus, the same holds for the induced map between the cohomology groups

$$H^i\left(\mathcal{X},\mathcal{F}\widehat{\otimes}_A B\right)\longrightarrow H^i\left(\mathcal{X},(\mathcal{F}/\mathcal{G})\widehat{\otimes}_A B\right).$$

Consequently, in order to prove our proposition, we are reduced to showing that the kernel and the cokernel of the analogous morphism with $\mathcal{F}$ replaced by $\mathcal{F}/\mathcal{G}$ are killed by some power of pI. Therefore, up to replace $\mathcal{F}$ by $\mathcal{F}/\mathcal{G}$, we may and do assume that $\mathcal{F}$ is p-torsion free.

Let $\mathcal{F}_n=\mathcal{F}/p^{n+1}$. By Proposition 7.3, there is some large integer N such that, for all $n\geq 0$, the natural morphisms below have kernel and cokernel killed by I^N:

$$H^i(\mathcal{X},\mathcal{F}_n)\otimes_A B\longrightarrow H^i(\mathcal{X},\mathcal{F}_n\otimes_A B).$$

As the two cohomology groups above vanish when $i \notin [0, \dim(\mathcal{X})]$, enlarging N if necessary, we may assume that the same holds for all $i \in \mathbb{Z}$. Passing to projective limits, we obtain, for each i, a morphism whose kernel and cokernel are killed by I^{2N} thus also by $(pI)^{2N}$:

$$\alpha\colon \varprojlim_n \big(H^i(\mathcal{X}, \mathcal{F}_n) \otimes_A B\big) \longrightarrow \varprojlim_n H^i(\mathcal{X}, \mathcal{F}_n \otimes_A B).$$

On the other hand, as A is neotherian, the projective system $\big(H^{i-1}(\mathcal{X}, \mathcal{F}_n)\big)_{n\geq 0}$ satisfies the (ML) condition ([Gro, Corollaire 3.4.4]), hence so is the induced projective system $\big(H^{i-1}(\mathcal{X}, \mathcal{F}_n) \otimes_A B\big)_{n\geq 0}$. As a result, $R^1 \varprojlim_n (H^{i-1}(\mathcal{X}, \mathcal{F}_n) \otimes_A B) = 0$ and $R^1 \varprojlim_n H^{i-1}(\mathcal{X}, \mathcal{F}_n \otimes_A B)$ is killed by I^N for all i. Moreover using the set of affine open formal subschemes of $\mathcal{X}$ and [Sch13, Lemma 3.18], one checks that $R^1 \varprojlim_n (\mathcal{F}_n \otimes_A B) = 0$. So the natural morphism below is surjective with kernel $R^1 \varprojlim_n H^{i-1}(\mathcal{X}, \mathcal{F}_n \otimes_A B)$ killed by I^N hence also by $(pI)^N$:

$$\beta\colon H^i(\mathcal{X}, \mathcal{F}\widehat{\otimes}_A B) \longrightarrow \varprojlim H^i(\mathcal{X}, \mathcal{F}_n \otimes_A B).$$

Next we claim that the following natural morphism

$$\frac{H^i(\mathcal{X}, \mathcal{F}) \otimes_A B}{p^{n+1}} \longrightarrow H^i(\mathcal{X}, \mathcal{F}_n) \otimes_A B$$

has kernel and cokernel killed by p^a for some large integer a which does not depend on n. To see this, using the following tautological short exact sequence (recall that $\mathcal{F}$ has no p-torsion)

$$0 \longrightarrow \mathcal{F} \xrightarrow{p^{n+1}} \mathcal{F} \longrightarrow \mathcal{F}_n \longrightarrow 0,$$

we obtain a short exact sequence

$$0 \longrightarrow H^i(\mathcal{X}, \mathcal{F})/p^{n+1} \longrightarrow H^i(\mathcal{X}, \mathcal{F}_n) \longrightarrow H^i(\mathcal{X}, \mathcal{F})[p^{n+1}] \longrightarrow 0.$$

As $H^i(\mathcal{X}, \mathcal{F})$ is an A-module of finite type, its A-submodule $H^i(\mathcal{X}, \mathcal{F})_{p\text{-torsion}}$ formed by elements killed by some power of p is finitely generated over A. In particular, there exists $a \in \mathbb{N}$ such that p^a kills $H^i(\mathcal{X}, \mathcal{F})_{p\text{-torsion}}$ and thus $H^i(\mathcal{X}, \mathcal{F})[p^{n+1}]$ for all $n \in \mathbb{N}$. Applying the functor $- \otimes_A B$, we get the exact sequence

$$\begin{aligned}\mathrm{Tor}_1^A(H^i(\mathcal{X}, \mathcal{F})[p^{n+1}], B) &\longrightarrow H^i(\mathcal{X}, \mathcal{F}) \otimes_A B/p^{n+1}\\ \longrightarrow H^i(\mathcal{X}, \mathcal{F}_n) \otimes_A B &\longrightarrow H^i(\mathcal{X}, \mathcal{F})[p^{n+1}] \otimes_A B \longrightarrow 0.\end{aligned}$$

Consequently, for all n, the kernel and the cokernel of the following map are killed by p^a:

$$H^i(\mathcal{X}, \mathcal{F}) \otimes_A B/p^{n+1} \longrightarrow H^i(\mathcal{X}, \mathcal{F}_n) \otimes_A B.$$

Passing to projective limits, one deduces that the morphism

$$\gamma\colon H^i(\mathcal{X}, \mathcal{F}) \otimes_A B = \varprojlim_n \big(H^i(\mathcal{X}, \mathcal{F}) \otimes_A B/p^{n+1}\big) \longrightarrow \varprojlim_n \big(H^i(\mathcal{X}, \mathcal{F}_n) \otimes_A B\big)$$

has kernel and cokernel killed by p^{2a}, hence by $(pI)^{2a}$.

Finally from the commutative diagram below

$$\begin{array}{ccc} H^i(\mathcal{X}, \mathcal{F}) \otimes_A B & \longrightarrow & H^i(\mathcal{X}, \mathcal{F}\widehat{\otimes}_A B) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\ \varprojlim \big(H^i(\mathcal{X}, \mathcal{F}_n) \otimes_A B\big) & \xrightarrow{\alpha} & \varprojlim H^i(\mathcal{X}, \mathcal{F}_n \otimes_A B) \end{array}$$

and the fact that all of the α, β, γ have kernel and cokernel killed by some large power of pI, we obtain that the upper horizontal morphism

$$H^i(\mathcal{X}, \mathcal{F}) \otimes_A B \longrightarrow H^i(\mathcal{X}, \mathcal{F}\widehat{\otimes}_A B)$$

has kernel and cokernel killed by a large power of pI, as wanted. □

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