Master 2, Semestre 9

Algebraic Geometry - TD 1

In this TD, unless said otherwise, the letter k denotes always an algebraically closed field

Exo. 1.

1. Let A be a principal ideal domain, and we consider the polynomial ring $A[X]$ of one variable with coefficients in A. Let $\mathcal{P} \subset A[X]$ be a prime ideal such that $\mathcal{P} \cap A = (0)$. Show that $\mathcal{P}$ is then a principal ideal of $A[X]$.
2. Let $\mathcal{P} \subset k[X, Y]$ be a prime ideal. Show that we are in one of the following two cases :
   (a) $\mathcal{P}$ is a principal ideal.
   (b) $\mathcal{P} = (X - x_0, Y - y_0) \subset k[X, Y]$ for some $x_0, y_0 \in k$.
3. Describe all the *irreducible* closed subsets of $\mathbb{A}_k^2$.
4. Show that the underlying topological space of $\mathbb{A}_k^2$ is *not* the same as the product of two copies of the underlying topological space of $\mathbb{A}_k^1$.

Exo. 2 A topological space X is called *noetherian* if it satisfies the *descending chain condition for closed sets*, namely let

$$Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_i \supseteq \cdots$$

be a chain of closed subsets of X, there is then some integer $r \in \mathbb{Z}_{\geq 1}$ such that

$$Y_r = Y_{r+1} = \cdots .$$

1. Show that if X is a noetherian topological space which is also Hausdorff, then X is a finite set with the discrete topology.
2. Show that a noetherian topological space X is *quasi-compact*, that is, for any open covering $\{U_i | i \in I\}$ of X, on can always find a *finite* subset $J \subset I$ such that
$$\bigcup_{j \in J} U_j = X.$$
3. Show that for a noetherian space X, any closed subset $Y \subset X$ can be written as a finite union of its irreducible closed subsets
$$Y = \bigcup_{i=1}^{n} Y_i$$
such that $Y_i \nsubseteq Y_j$ for $i \neq j$. Moreover this decomposition is unique. The Y_i's are called the *irreducible components* of Y.
4. Show that $\mathbb{A}_k^n$ is a noetherian topological space. Then deduce that the same conclusion holds for an affine algebraic set. In particular, we can apply the previous assertion to $\mathbb{A}_k^n$.

Exo. 3

1. Prove, without using Hilbert's Nullstellensatz, that $I(\mathbb{A}_k^n) = (0)$.
2. Let $Y \subset \mathbb{A}_k^n$ be a subset. Show that $V(I(Y)) = \overline{Y}$.

3. Consider the *twisted cubic curve* :

$$C = \{(t, t^2, t^3) \colon t \in k\} \subset \mathbb{A}^3_k.$$

   Show that C is an irreducible closed subset of $\mathbb{A}^3_k$, and find generators of the ideal $I(C) \subset k[X, Y, Z]$. Same questions for $C' = \{(t^3, t^4, t^5) \colon t \in k\} \subset \mathbb{A}^3_k$.

4. Let $V = V(X^2 - YZ, XZ - X) \subset \mathbb{A}^3_k$. Show that V consists of three irreducible components, and determine the corresponding prime ideals.

5. We identify the space $\mathrm{M}_2(k)$ of 2×2-matrices over k with $\mathbb{A}^4_k$ with coordinates a, b, c, d :

$$\mathrm{M}_2(k) \simeq \mathbb{A}^4_k, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d).$$

   Show that the set of nilpotent matrices is an algebraic subset of $\mathbb{A}^4_k$, and determine its ideal.

Exo. 4 Let k be an algebraically closed field, and A be any k-algebra of finite type. Let $I \subsetneq A$ be an ideal. Show that

$$\sqrt{I} = \bigcap_{I \subset \mathfrak{m} \subset A \text{ maximal}} \mathfrak{m}.$$

What happen if k is *not* algebraically closed ?