COMPOSITION OPERATORS ON ANALYTIC WEIGHTED HILBERT SPACES

K. KELLAY

Abstract. We consider composition operators in the analytic weighted Hilbert space. Various criteria on boundedness, compactness and Hilbert-Schmidt class membership are established.

1. Composition operators on the Hardy space

In this expository paper, we consider composition operators acting on the weighted Hilbert spaces of analytic functions on the unit disc. This paper is based on [5, 6]. A comprehensive study of composition operators in function spaces and their spectral behavior could be found in [3, 11, 15]. See also [4, 8, 9, 13, 14, 16] for a treatment of some of the questions addressed in this paper.

Let \( \varphi \) be an analytic map of the unit disk \( \mathbb{D} \) of the complex plane into itself. We define the composition operator \( C_\varphi \) by

\[
C_\varphi(f) = f \circ \varphi
\]

for \( f \) analytic in \( \mathbb{D} \).

We recall that the Hardy space \( H^2 \) consists of those analytic functions \( f : \mathbb{D} \to \mathbb{C} \) for which the norm

\[
\|f\|_2 = \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_{\mathbb{T}} |f(r\zeta)|^2 |d\zeta| \right)^{1/2}
\]

is finite. Every function \( f \in H^2 \) has non-tangential limits almost everywhere on \( \mathbb{T} \). We denote by \( f(\zeta) \) the non-tangential limit of \( f \) at \( \zeta \in \mathbb{T} \) if it exists.

1.1. Boundedness. It is a consequence of Littlewoods subordination principle [11, 15] that every composition operator \( C_\varphi \) restricts to a bounded operator on \( H^2 \). Furthermore

\[
\|C_\varphi(f)\|_2^2 \leq \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \|f\|_2^2.
\]
1.2. **Compactness.** Shapiro gives in [10] a complete characterization of compact composition operators in the Hardy space in terms of Nevanlinna counting functions.

Let \( \varphi \in \text{Hol}(\mathbb{D}) \) such that \( \varphi(\mathbb{D}) \subset \mathbb{D} \). The Nevanlinna counting function is defined for every \( z \in \mathbb{D} \setminus \{ \varphi(0) \} \) by

\[
N_\varphi(z) = \sum_{\varphi(w)=z} \log \frac{1}{|w|},
\]

where each pre-image \( w \) is counted according to its multiplicity. Note that \( N_\varphi(z) = O(-\log |z|) \). Shapiro’s characterization is as follows:

\[
C_\varphi \text{ is compact in } H^2 \iff \lim_{|z| \to 1} \frac{N_\varphi(z)}{\log(1/|z|)} = 0.
\]

Given \( E \subset \mathbb{T} \), we write \( |E| \) for the Lebesgue measure of \( E \). Note also that if \( C_\varphi \) is compact, then the level set of \( \varphi \)

\[
E_\varphi(1) = \{ \zeta \in \mathbb{T} : |\varphi(\zeta)| = 1 \}
\]

has Lebesgue measure zero. Indeed, suppose that \( E_\varphi(1) \) has positive measure, we will show that \( C_\varphi \) is not compact. Since the sequence \( (z^n)_n \) converges weakly to zero and

\[
\|C_\varphi(z^n)\|_2^2 = \| \varphi^n \|_2^2 = \frac{1}{2\pi} \int_\mathbb{T} |\varphi(\zeta)|^{2n} |d\zeta| \geq \frac{1}{2\pi} \int_{\{\zeta \in \mathbb{T} : |\varphi(\zeta)|=1\}} |\varphi(\zeta)|^{2n} |d\zeta| = \frac{|E_\varphi(1)|}{2\pi} > 0,
\]

\( \|C_\varphi(z^n)\|_2 \) not converge to zero. Hence \( C_\varphi \) is not compact.

1.3. **Hilbert-Schmidt class.** For \( s \in (0, 1) \), the level set \( E_\varphi(s) \) of \( \varphi \) is given by

\[
E_\varphi(s) = \{ \zeta \in \mathbb{T} : |\varphi(\zeta)| \geq s \}.
\]

One can completely describe the membership of \( C_\varphi \) in the Hilbert-Schmidt class in terms of the size of the level sets of the inducing map \( \varphi \). Indeed, \( C_\varphi \) is Hilbert-Schmidt in \( H^2 \) if and only if

\[
\sum_{n \geq 0} \| \varphi^n \|_2^2 = \frac{1}{2\pi} \int_\mathbb{T} \frac{|d\zeta|}{1 - |\varphi(\zeta)|^2} < \infty.
\]

Let \( f \) be a measurable function on \( \mathbb{T} \), the associated distribution function \( m_f \) is given by

\[
m_f(\lambda) = |\{ \zeta \in \mathbb{T} : |f(\zeta)| > \lambda \}|, \quad \lambda > 0.
\]
It then follows that $C_\varphi$ is in the Hilbert-Schmidt class of $H^2$ if and only if
\[
\int_1^\infty \frac{m_{1-|\varphi|^2}^{-1}(\lambda)}{\lambda} d\lambda \lesssim \int_0^1 \frac{|E_\varphi(s)|}{(1-s)^2} ds < \infty.
\]

It was shown by Gallardo–Gonzalez in [9] that there exists a mapping $\varphi : \mathbb{D} \to \mathbb{D}$ such that $C_\varphi$ is compact in $H^2$, and that the level set $E_\varphi(1)$ has Hausdorff measure equal to one. Let $A(\mathbb{D})$ denote the disc algebra. In [6] with El-Fallah, Shabankhah and Youssfi we obtain the following result

**Theorem 1.1.** Let $E$ be a closed subset of $\mathbb{T}$ with Lebesgue measure zero. There exists a mapping $\varphi : \mathbb{D} \to \mathbb{D}$, $\varphi \in A(\mathbb{D})$ such that $C_\varphi$ is a Hilbert-Schmidt operator on $H^2$ and that $E_\varphi(1) = E$.

2. Composition operators on weighted Hilbert spaces of analytic functions

Given a positive integrable function $\omega \in C^2[0,1)$, we extend it by $\omega(z) = \omega(|z|)$, $z \in \mathbb{D}$, and call such $\omega$ a weight function. We denote by $H_\omega$ the weighted Hilbert space consisting of analytic functions $f$ on $\mathbb{D}$ such that
\[
\|f\|^2_{H_\omega} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z) < \infty.
\]
where $dA(z) = dx dy/\pi$ stands for the normalized area measure in $\mathbb{D}$. A simple computation shows that $f(z) = \sum_{n=0}^\infty a_n z^n \in H_\omega$, if and only if
\[
\|f\|^2_{H_\omega} = \sum_{n=0}^\infty |a_n|^2 w_n < \infty,
\]
where $w_0 = 1$ and
\[
\omega_n = 2n^2 \int_0^1 r^{2n-1} \omega(r) dr, \quad n \geq 1.
\]
Let $\alpha > -1$, $\omega_\alpha(r) = (1 - r^2)^\alpha$ and denote $H_{\omega_\alpha}$ by $H_\alpha$. The Hardy space $H^2$ can be identified with $H_1$. The Dirichlet space $D_\alpha$ is precisely $H_\alpha$ for $0 \leq \alpha < 1$ and $H_0$ corresponds to classical Dirichlet space $D$. Finally, the Bergman spaces $A^2_\alpha(\mathbb{D}) := \operatorname{Hol}(\mathbb{D}) \cap L^2(\mathbb{D}, (1 - |z|^2)^\alpha dA(z))$ can be identified with $H_{\alpha+2}$. From now we assume that the weight $\omega$ satisfies the following properties

(\text{W}_1) $\omega$ is non-increasing,
(\text{W}_2) $\omega(r)(1 - r)^{-1+\delta}$ is non-decreasing for some $\delta > 0$,
(\text{W}_3) $\lim_{r \to 1-} \omega(r) = 0$.

Furthermore, we assume that one of the two properties of convexity is fulfilled

(\text{W}_4^{(I)}) $\omega$ is convex and $\lim_{r \to 1-} \omega'(r) = 0$.
(\text{W}_4^{(II)}) $\omega$ is concave.
Such a weight function is called admissible. If $\omega$ satisfies conditions ($W_1$)–($W_3$) and ($W_4^{(I)}$) (resp. ($W_4^{(II)}$)), we shall say that $\omega$ is (I)-admissible (resp. (II)-admissible).

Note that (I)-admissibility corresponds to the case $H^2 \subseteq \mathcal{H}_\omega \subset \mathcal{A}_\alpha^2(\mathbb{D})$ for some $\alpha > -1$, whereas (II)-admissibility corresponds to the case $D \subset \mathcal{H}_\omega \subseteq H^2$. The weight $\omega_0 = 1$ is not an admissible weight, so the results of this section do not apply to the Dirichlet space.

The Nevanlinna counting functions play a key role here (see [7] for recent results on this topic). Let $\varphi \in \text{Hol}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. The generalized counting Nevanlinna function associated to $\omega$ is defined for every $z \in \mathbb{D} \setminus \{\varphi(0)\}$ by

$$N_{\varphi,\omega}(z) = \sum_{\varphi(a) = z, a \in \mathbb{D}} \omega(a).$$

Let us point out two crucial properties the generalized counting Nevanlinna function:

- If $f$ is a positive measurable function on $\mathbb{D}$ and $\varphi$ is a holomorphic self-map on $\mathbb{D}$, then
  $$\int_{\mathbb{D}} (f \circ \varphi)(z) |\varphi'(z)|^2 w(z) dA(z) = \int_{\mathbb{D}} f(z) N_{\varphi,w}(z) dA(z). \quad (1)$$

- If $\omega$ is an admissible weight and $\varphi$ is a holomorphic self-map of $\mathbb{D}$. Then the generalized Nevanlinna counting function satisfies the sub-mean value property, that is for every $r > 0$ and for every $z \in \mathbb{D}$ such that $D(z, r) \subset \mathbb{D} \setminus D(0, 1/2)$
  $$N_{\varphi,\alpha}(z) \leq \frac{2}{r^2} \int_{|\zeta - z| < r} N_{\varphi,\alpha}(\zeta) dA(\zeta). \quad (2)$$

If $\varphi$ is a holomorphic map on the unit disk $\mathbb{D}$ into itself, it is an easy consequence of Littlewood’s subordination principle that the composition operator with $\varphi$, induces a bounded operator $C_\varphi$ on $\mathcal{H}_\omega$ for (I)–admissible weight $\omega$. In [5] with Lefèvre, see also [1], we obtain the following results. For the case of (II)–admissible weight we have

**Theorem 2.1.** Let $\omega$ be a (II)–admissible weight and $\varphi \in \mathcal{H}_\omega$. Then $C_\varphi$ is bounded on $\mathcal{H}_\omega$ if and only if

$$\sup_{|z| < 1} \frac{N_{\varphi,\omega}(z)}{\omega(z)} < \infty \quad (3)$$

The characterization of compactness in the weighted Bergman spaces $\mathcal{H}_{\alpha+2}$ is given by

$$C_\varphi \text{ is compact in } \mathcal{H}_{\alpha+2} \iff \lim_{|z| \to 1^-} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} = +\infty.$$

[10, Theorem 2.3, Corollary 6.11] or [4]. By the classical Julia-Caratheodory theorem (see [11]), this last condition is equivalent to nonexistence of the angular derivative of $\varphi$ at any point $\xi \in \mathbb{T}$. The following theorem obtained in [5] generalizes the previous on Hardy and Bergman spaces.

**Theorem 2.2.** Let $\omega$ be an admissible weight and $\varphi \in \mathcal{H}_\omega$. Then $C_\varphi$ is compact on $\mathcal{H}_\omega$ if and only if

$$\lim_{|z| \to 1^-} \frac{N_{\varphi,\omega}(z)}{\omega(z)} = 0 \quad (4)$$
In particular, Theorem 2.2 asserts that \( C_\varphi \) is compact on \( \mathcal{D}_\alpha := \mathcal{H}_\alpha \) for \( 0 < \alpha < 1 \) if and only if
\[
N_{\varphi, \alpha}(z) := \sum_{\varphi(w) = z} (1 - |w|^2)^\alpha = o((1 - |z|^2)^\alpha).
\]

The characterization of compact composition operators on the Dirichlet spaces \( \mathcal{D}_\alpha \) in terms of Carleson measures for the Bergman spaces \( \mathcal{A}^2_\alpha \) can be found in \([3, 13, 16]\). A positive Borel measure \( \mu \) given on \( \mathbb{D} \) is called a Carleson measure for the Bergman space \( \mathcal{A}^2_\alpha \) if the identity map \( i_\alpha : \mathcal{A}^2_\alpha \rightarrow L^2(\mu) \) is a bounded operator. Such a measure has the following equivalent properties (see \([12]\) Theorem 1.2): A positive Borel measure \( \mu \) is a Carleson measure for \( \mathcal{A}^2_\alpha \) if and only if
\[
\sup_{\lambda \in \mathbb{D}} (1 - |\lambda|^2)^{2+\alpha} \int_{\mathbb{D}} \frac{d\mu(z)}{|1 - \lambda z|^{2(2+\alpha)}} < \infty \iff \sup_{I \subset \mathbb{T}} \mu(S(I)) < \infty,
\]
for any subarc \( I \subset \mathbb{T} \) with arclengh \( |I| \), and the Carleson box
\[
S(I) = \{ z \in \mathbb{D}: z/|z| \in I, \ 1 - |I| \leq |z| < 1 \}.
\]

The measure \( \mu \) is called vanishing (or compact) Carleson measure for \( \mathcal{A}^2_\alpha \) if the identity map \( i_\alpha : \mathcal{A}^2_\alpha \rightarrow L^2(\mu) \) is a compact operator. This happens if and only if
\[
\lim_{|\lambda| \to 1} (1 - |\lambda|^2)^{2+\alpha} \int_{\mathbb{D}} \frac{d\mu(z)}{|1 - \lambda z|^{2(2+\alpha)}} = 0 \iff \lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^{2+\alpha}} = 0.
\]

Therefore, as a consequence of Theorem 2.1 and Theorem 2.2,
\[
C_\varphi \text{ is bounded in } \mathcal{D}_\alpha \iff \sup_{I \subset \mathbb{T}} \frac{1}{|I|^{2+\alpha}} \int_{S(I)} N_{\varphi, \alpha}(z) dA(z) < \infty;
C_\varphi \text{ is compact in } \mathcal{D}_\alpha \iff \lim_{|I| \to 0} \frac{1}{|I|^{2+\alpha}} \int_{S(I)} N_{\varphi, \alpha}(z) dA(z) = 0.
\]

That is \( C_\varphi \) is bounded on \( \mathcal{D}_\alpha \) if and only if \( N_{\varphi, \alpha}(z) dA(z) \) is a Carleson measure for \( \mathcal{A}^2_\alpha \) and \( C_\varphi \) is compact on \( \mathcal{D}_\alpha \) if and only if \( N_{\varphi, \alpha}(z) dA(z) \) is a vanishing Carleson measure for \( \mathcal{A}^2_\alpha \). Note that by the change of variable formula (1),
\[
J_\alpha(\lambda) := (1 - |\lambda|^2)^{\alpha+2} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1 - \lambda \varphi(z)|^{2(\alpha+2)}} dA_\alpha(z)
= (1 - |\lambda|^2)^{2+\alpha} \int_{\mathbb{D}} \frac{N_{\varphi, \alpha}(z) dA(z)}{|1 - \lambda z|^{2(2+\alpha)}}.
\]

Hence, we can also characterize, boundedness and compactness of \( C_\varphi \) on \( \mathcal{D}_\alpha \), \( 0 < \alpha < 1 \), by \( \sup_{\lambda \in \mathbb{D}} J_\alpha(\lambda) < \infty \) and \( \lim_{\lambda \to 1} J_\alpha(\lambda) = 0 \). The Proposition 3.1 is devoted to the case where \( \alpha = 0 \) corresponding to the classical Dirichlet space \( \mathcal{D} \).
3. Composition Operators on the Dirichlet Space

We first give the following result, which was established in a general case in [13, 14, 15]. For the sake of completeness, we give here a simple proof.

**Proposition 3.1.** Let $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\varphi \in D$. Then

(a) $C_\varphi$ is bounded in $D$ $\iff$ $\sup_{\lambda \in \mathbb{D}} (1 - |\lambda|^2)^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1 - \lambda \varphi(z)|^4} dA(z) < \infty$.

(b) $C_\varphi$ is compact in $D$ $\iff$ $\lim_{|\lambda| \to 1} (1 - |\lambda|^2)^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1 - \lambda \varphi(z)|^4} dA(z) = 0$.

**Proof.** (a). We only have to prove the converse; since the necessary condition easily follows from the fact that $z \to (1 - |\lambda|^2)^2 (1 - \bar{\lambda} \varphi(\zeta))$ is bounded (uniformly relatively to $\lambda \in \mathbb{D}$). Without loss of generality, we may assume that $\varphi(0) = 0$. Let $f \in D$

$$\|C_\varphi(f)\|_D^2 = |f(\varphi(0))|^2 + \int_{\mathbb{D}} |\varphi'(z)|^2 |f'(\varphi(z))|^2 dA(z)$$

$$\leq |f(0)|^2 + \int_{\mathbb{D}} |\varphi'(z)|^2 \left( \int_{\mathbb{D}} \frac{|f'(\zeta)|^2}{|1 - \lambda \varphi(z)|^4} dA(\zeta) \right) dA(z)$$

$$= |f(0)|^2 + \int_{\mathbb{D}} |f'(\zeta)|^2 \left( \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1 - \lambda \varphi(z)|^4} dA(z) \right) dA(\lambda)$$

$$\leq c \|f\|_D^2.$$

(b). We only have to prove the converse. Assume that the limit is equal to zero and let $(f_n)_n$ be a sequence which converges weakly to 0 in $D$. Since $f_n' \to 0$ uniformly on compact sets, it follows the proof of part (a) and for $r$ close to 1 that

$$\|C_\varphi(f_n)\|_D^2 - |f_n(0)|^2 \leq \int_{r \mathbb{D}} |f_n'(\zeta)|^2 (1 - |\zeta|^2)^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1 - \lambda \varphi(z)|^4} dA(z) dA(\zeta)$$

$$+ \int_{\mathbb{D} \setminus r \mathbb{D}} |f_n'(\zeta)|^2 (1 - |\zeta|^2)^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1 - \lambda \varphi(z)|^4} dA(z) dA(\zeta) \to 0 \quad \text{as } n \to \infty.$$

For $f \in D$, the Dirichlet integral of $f$ is given by

$$D(f) = \int_{\mathbb{D}} |f'(z)|^2 dA(z).$$

The following result is an immediate consequence of Proposition 3.1.

**Corollary 3.2.** Let $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\varphi \in D$.

(a) If $\sup_{n \geq 1} D(\varphi^n) < \infty$, then $C_\varphi$ is bounded;
(b) If \( \lim_{n \to \infty} D(\phi^n) = 0 \), then \( C\phi \) is compact.

Proof. Both (a) and (b) follow from the following inequality:

\[
(1 - |\lambda|^2)^2 \int_{|\lambda|} \frac{|\phi'(z)|^2}{|1 - \lambda \varphi(z)|^4} dA(z) \leq c (1 - |\lambda|^2)^2 \sum_{n \geq 0} (n + 1)^3 |\lambda|^{2n} \int_{|\lambda|} \frac{|\varphi'(z)|^2 |\varphi^n(z)|^2}{|1 - \lambda \varphi(z)|^4} dA(z) \\
= c (1 - |\lambda|^2)^2 \sum_{n \geq 0} (1 + n) |\lambda|^{2n} D(\varphi^{n+1}) \leq c \limsup_{n \to \infty} D(\varphi^{n+1}).
\]

We are interested herein in describing the spectral properties of the composition operator \( C\phi \), such as compactness and Hilbert-Schmidt class membership, in terms of the size of the level set of \( \varphi \). In order to state our second result, we introduce the notion of logarithmic capacity.

Given a (Borel) probability measure \( \mu \) on \( T \), we define its energy by

\[
I(\mu) = \sum_{n=1}^{\infty} \frac{|\hat{\mu}(n)|^2}{n}.
\]

For a closed set \( E \subset T \), its logarithmic capacity \( \text{cap}(E) \) is defined by

\[
\text{cap}(E) := 1 / \inf \{ I(\mu) : \mu \text{ is a probability measure on } E \}.
\]

Since the Dirichlet space is contained in the Hardy space \( H^2(\mathbb{D}) \), every function \( f \in D \) has non-tangential limits \( f \) almost everywhere on \( T \). In this case, however, more can be said. According to well-known result of Beurling [2], for each function \( f \in D \) then the radial limits \( f(\zeta) = \lim_{r \to 1} f(r \zeta) \) exists q.e on \( T \), that is

\[
\text{cap}(\{ \zeta \in T : f(\zeta) \text{ does not exist} \}) = 0.
\]

The weak–type inequality for capacity [2] states that, for \( f \in D \) and \( t \geq 4 \| f \|_D^2 \),

\[
\text{cap}(\{ \zeta : |f(\zeta)| \geq t \}) \leq \frac{16 \| f \|_D^2}{t^2}.
\]

As a result of this inequality, we see that if \( \liminf \| \phi^n \|_D = 0 \), then \( \text{cap}(E_{\phi}(1)) = 0 \). Indeed, since \( E_{\phi}(1) = E_{\phi^n}(1) \), the weak capacity inequality implies that

\[
\text{cap}(E_{\phi}(1)) = \text{cap}(E_{\phi^n}(1)) \leq 16 \| \phi^n \|_D^2.
\]

Now let \( n \to \infty \), we can conclude. Hence, in particular, if the operator \( C\phi \) is in the Hilbert-Schmidt class in \( D \), \( \sum_{n \geq 0} \| \phi^n \|^2 < \infty \), then \( \text{cap}(E_{\phi}(1)) = 0 \). This result was first obtained by Gallardo-Gutiérrez and González [8] using a completely different method. Theorems 3.3 and 3.4 give quantitative versions of this result. The following theorem is similar to the Theorem 1.1 for the Dirichlet space.
Theorem 3.3. If $C_\varphi$ is a Hilbert-Schmidt operator in $\mathcal{D}$, then
\[ \int_0^1 \frac{\text{cap}(\mathcal{E}_\varphi(s))}{1-s} \log \frac{1}{1-s} \, ds < \infty. \] (5)

The following theorem shows that condition (5) is optimal.

Theorem 3.4. Let $h : [1, +\infty[ \to [1, +\infty[ \to [1, +\infty[$ be a function such that $\lim_{x \to +\infty} h(x) = +\infty$. Let $E$ be a closed subset of $\mathbb{T}$ such that $\text{cap}(E) = 0$. Then there is $\varphi \in A(\mathbb{D}) \cap \mathcal{D}$, $\varphi(\mathbb{D}) \subset \mathbb{D}$ such that :

1. $\mathcal{E}_\varphi(1) = E$;
2. $C_\varphi$ is in the Hilbert–Schmidt class in $\mathcal{D}$;
3. $\int_0^1 \frac{\text{cap}(\mathcal{E}_\varphi(s))}{1-s} \log \frac{e}{1-s} h\left(\frac{1}{1-s}\right) \, ds = +\infty$.

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IMB, Université Bordeaux I, 351 cours de la Libration, 33405 Talence, France
E-mail address: Karim.Kellay@math.u-bordeaux1.fr